Three-Dimensional Gauge Theories
and
Gravitational Instantons
from
String Theory

Thesis by

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To Mikhail Arkadievich Rozenderg

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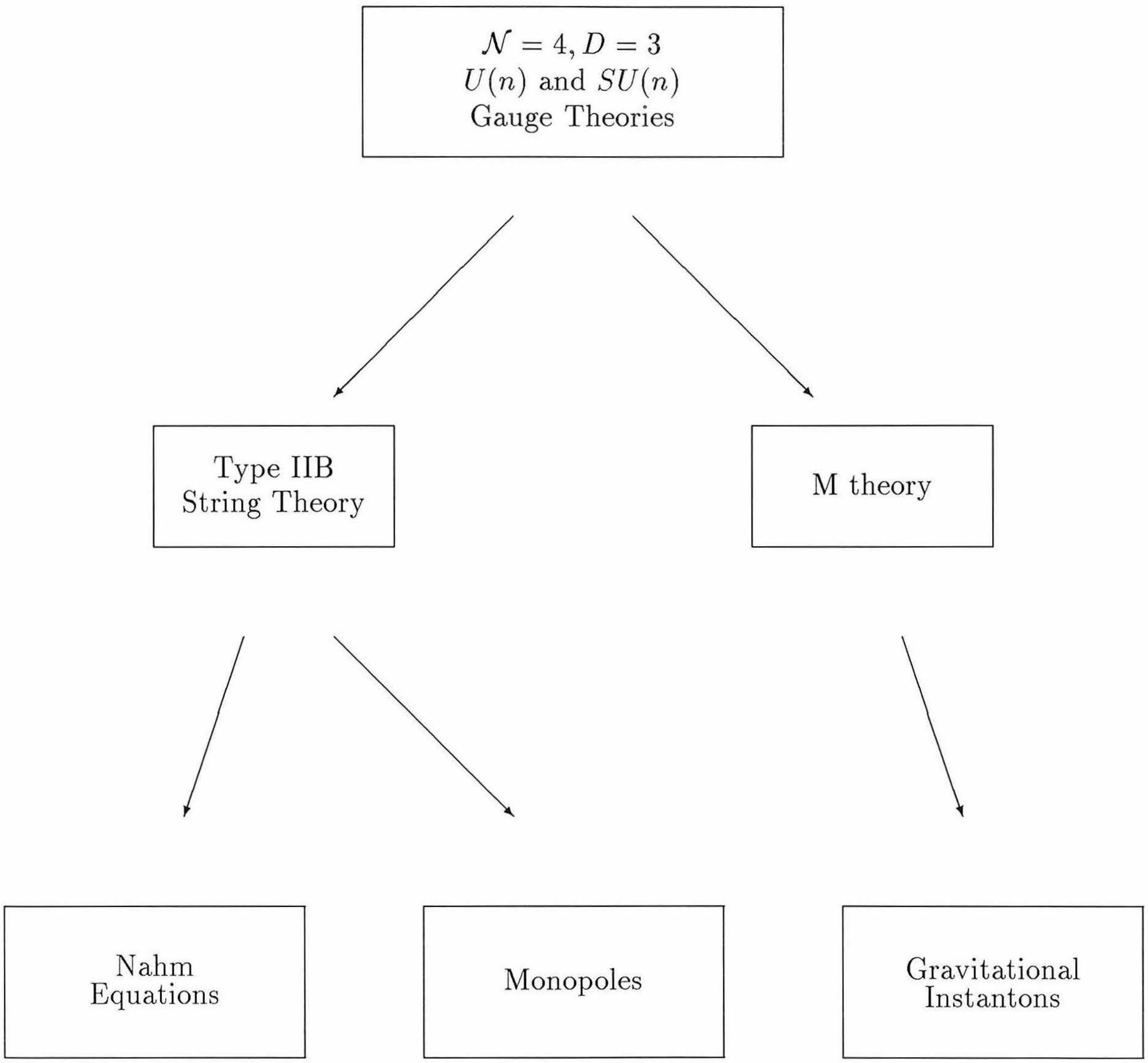
Abstract

Various realizations of gauge theories in string theory allow an identification of their spaces of vacua with gravitational instantons. Also, they provide a correspondence of vacua of gauge theories with nonabelian monopole configurations and solutions of a system of integrable equations called Nahm equations. These identifications make it possible to apply powerful techniques of differential and algebraic geometry to solve the gauge theories in question. In other words, it becomes possible to find the exact metrics on their moduli spaces of vacua with all quantum corrections included. As another outcome we obtain for the first time the description of a series of all D_k-type gravitational instantons.

Contents

List of Figures

Map



$\mathcal{N} = 4, D = 3$
$U(n)$ and $SU(n)$
Gauge Theories
Type IIB
String Theory
M theory
Nahm
Equations
Monopoles
Gravitational
Instantons

Chapter 1 Introduction

This work provides yet another example of how string theory unites seemingly distant physical problems. The central object of our studies is supersymmetric gauge theories in three dimensions. In particular, we shall be interested in their vacuum structure. The other three problems that turn out to be closely related to these gauge theories are:

- Nonabelian *monopoles* of Prasad and Sommerfield, which are solutions of the Bogomolny equation
$$*F = D\Phi$$
(where F is the field-strength of a nonabelian connection $A = A_1 dx^1 + A_2 dx^2 + A_3 dx^3$ and Φ is a nonabelian Higgs field).

- An integrable system of *equations* named after *Nahm*
$$\frac{dT_i}{ds} = \frac{1}{2}\varepsilon_{ijk}[T_j, T_k],$$
for $T_i(s) \in u(n)$. These generalize Euler equations for a rotating top.

- Solutions of the Euclideanized vacuum Einstein equation called *self-dual gravitational instantons*, are four-dimensional manifolds with self-dual curvature tensor
$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}R^{\mu\nu}{}_{\gamma\delta}.$$

Various string and M theory realizations of the gauge theories make the relations between them and these problems manifest. This is schematically represented on the map on page viii.

The relations mentioned allow us to establish the correspondence between particular configurations of monopoles, certain Nahm data and gauge theories. Interpretation in terms of monopoles as well as Nahm data is very convenient, since it allows us to apply (after appropriate modification) well developed methods of modern geometry to describe their moduli spaces completely. Thus these two give us methods and tools. We study these monopoles and Nahm data and use them as two different ways of solving the gauge theories in question.

This also allows us to find the twistor description of two infinite series of gravitational instantons together with their Kähler potentials. One of these was known previously, and the other one is new. This describes all infinite series of self-dual gravitational instantons and, perhaps, ends the classification of them (as the existence of the exceptional ALF instantons is in doubt).

We have to use some geometric constructions in our computations. Among the notions used are twistor and minitwistor spaces, hyperkähler quotients, holomorphic vector bundles, and Ward correspondence. These notions are defined in the text and, for the convenience of the reader, we provide a glossary with short definitions and references at the end.

1.1 Three-Dimensional Gauge Theories with N=4 Supersymmetry

1.1.1 The Theory

$\mathcal{N}=4$ gauge theories we are interested in can be obtained as reductions to three dimensions of six-dimensional gauge theories with minimal supersymmetry. Reduction of a vector multiplet in six dimensions gives a vector multiplet in three dimensions. The vector multiplet of these three-dimensional theories contains a vector A_μ, two two-component Dirac spinors λ and χ, and three real scalars $\phi_i, i=1,2,3$. The scalars can be organized into one real ϕ_1 and one complex $\phi=\phi_2+i\phi_3$. The action for the vector multiplet is given by

$$S_V = \frac{2\pi}{g^2}\int d^3x \mathrm{Tr}(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi_1 D^\mu\phi_1 + D_\mu\phi D^\mu\phi \qquad (1.1)$$

$$+2i\bar{\lambda}\not{D}\lambda + 2i\bar{\chi}\not{D}\chi + 2\bar{\lambda}[\phi_1,\lambda] + 2\bar{\chi}[\phi_1,\chi] + 2i[\phi,\chi]\lambda + 2i\bar{\lambda}[\phi,\bar{\chi}]$$
$$+\frac{1}{4}[\phi,\phi^+]^2 + [\phi_1,\phi^+][\phi_1,\phi]).$$

As usual, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]$, $D_\mu = \partial_\mu + gA_\mu$ and ϕ^+ is the Hermitian conjugate of the field ϕ in the adjoint representation. The square of the coupling constant has dimensions of mass. The matter hypermultiplet (which can also be obtained by reduction of a hypermultiplet in six dimensions) in the fundamental representation of the gauge group consists of two complex scalars q_1 and q_2 and two Dirac spinors ψ_1 and ψ_2 with the action

$$\begin{aligned} S_H \;=\; & \frac{2\pi}{g^2}\int d^3x (D_\mu q_i^+ D^\mu q_i + i\bar{\psi}_i\not{D}\psi_i + \sqrt{2}i\left(q_i^+\lambda\psi_i - \bar{\psi}_i\bar{\lambda}q_i\right) + \bar{\psi}_i\phi_1\psi_i \qquad (1.2)\\ & -q_i^+\phi_1^2 q_i - \frac{1}{8}\left(q_1^+\tau^a q_1 - q_2\tau^a q_2^+\right)^2 - \frac{1}{2}\left(\phi_2\chi q_1 + q_2\chi\psi_1\right) + \frac{1}{2\sqrt{2}}\psi_2\phi\psi_1 + h.c.\\ & -\|\phi q_1\|^2 - \|q_2\phi\|^2 - 2\|q_2 q_1\|^2 + \frac{1}{4}q_1^+[\phi^+,\phi]q_1 - \frac{1}{4}q_2^+[\phi^+,\phi]q_2), \end{aligned}$$

where τ^a form basis of the algebra of the gauge group.

In order to introduce a mass $\vec{m} = (m_1, m_2, m_3)$, one has to add another term to the action

$$\begin{aligned} S_m \;=\; & \frac{2\pi}{g^2}\int d^3x (-m\psi_1\psi_2 - \bar{m}\bar{\psi}_1\bar{\psi}_2 - m_1\bar{\psi}_i\psi_i + \|\vec{m}\|^2 q_i q_i^+ \qquad (1.3)\\ & -q_1^+\left(m_1\chi + \sqrt{2}m_2 q_1 + \sqrt{2}m_3 q_2\right)q_1 - q_2\left(m_1\chi + \sqrt{2}m_2 q_1 + \sqrt{2}m_3 q_2\right)q_2^+), \end{aligned}$$

where $m = m_2 + im_3$ and $\|\vec{m}\|^2 = m_1^2 + m_2^2 + m_3^2$.

1.1.2 Symmetries and Spaces of Vacua

Recall that R-symmetries are global symmetries that rotate the supercharges. Let us describe R-symmetries of these gauge theories. There is an $SU(2)_R$ which is already present as an R-symmetry in six dimensions. Reducing three of the six dimensions, we are left with the $SO(3)$ rotations of these directions as a new R-symmetry. Let $SU(2)_N$ denote the double cover of this $SO(3)$ group. Thus the R-symmetry group of the three-dimensional theories is $SU(2)_R \times SU(2)_N$. The fields ϕ_i as well as the masses m_i are in the $(1,3)$ representation, the fermions λ and χ of the vector multiplets together form a $(2,2)$, the bosons q_1 and q_2 of the hypermultiplet are each in $(2,1)$ and the fermions ψ_1 and ψ_2 of the hypermultiplet are each in $(1,2)$ representations of the R-symmetry group.

The part of the space of vacua in which scalar fields of the vector supermultiplet acquire vevs is called the "Coulomb branch" of the moduli space. (The part in which the hypermultiplet scalars get vevs is referred to as a "Higgs branch.") The last two terms in the vector multiplet action S_V of Eq. (1.1) are the classical potential. We conclude that, classically, for the energy to vanish, all fields ϕ_i should commute with each other, and these directions are flat directions of the classical potential. The general solution is to take each of them to belong to the same Cartan subalgebra. Thus we have $3r$ parameters (where r denotes the rank of the gauge group) on the Coulomb branch corresponding to the vevs of the fields ϕ_i. Classically, at a generic point on the Coulomb branch with nonzero vacuum expectation values of the scalar fields ϕ_i, the gauge group is broken to $U(1)^r$, and we have r massless abelian photons in three dimensions. In three dimensions a vector is dual to a compact scalar, so we can describe these photons by r dual scalar fields. Their vevs define another r coordinates of the moduli space, so the Coulomb branch is $4r$-dimensional with r of the coordinates being periodic.

1.2 Spaces of Vacua and Gravitational Instantons

Supersymmetry constrains both the Higgs and Coulomb branches of the moduli spaces of the three-dimensional theories to be hyperkähler. As the two branches locally have quaternionic structure and are mathematically very similar, often a three-dimensional gauge theory has a dual description in terms of another gauge theory for which the dual Higgs and Coulomb branches are interchanged. This is called mirror symmetry in gauge theory.

In the absence of a Higgs branch, the Coulomb branch is a smooth hyperkähler manifold. If its dimension is four, it provides an example of a self-dual gravitational instanton – a nontrivial solution of the vacuum Einstein equations with Euclidean signature. We shall use this correspondence in order to explicitly describe new gravitational instantons.

1.2.1 Definition

A gravitational instanton is a smooth four-dimensional manifold with a Riemannian metric satisfying the vacuum Einstein equations. A particularly interesting class of gravitational instantons are self-dual gravitational instantons. They have zero action. These are manifolds with self-dual curvature tensor,

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}R^{\mu\nu}{}_{\gamma\delta}, \tag{1.4}$$

where $\varepsilon_{\alpha\beta\mu\nu}$ is an antisymmetric tensor. Self-dual gravitational instantons are hyperkähler, i.e., they are manifolds with holonomy group contained in $SU(2)$. A hyperkähler manifold can be alternatively characterized as a Riemannian manifold admitting three covariantly constant complex structures I, J, K satisfying the quaternion relations

$$IJ = -JI = K, \text{etc.}, \tag{1.5}$$

such that the metric is Hermitian with respect to I, J, and K separately. Covariant constancy of I, J, K implies that three 2-forms $\omega_1 = g(I\cdot,\cdot), \omega_2 = g(J\cdot,\cdot), \omega_3 = g(K\cdot,\cdot)$ are closed. If we pick one of the complex structures, say I, we may regard a hyperkähler manifold as a complex manifold equipped with Kähler metric (with Kähler form ω_1) and a complex symplectic form $\omega = \omega_2 + i\omega_3$.

Hyperkähler four-manifolds arise in several physical problems. For example, compactification of string and M theory on hyperkähler four-manifolds preserves one half of the supersymmetries and provides exact solutions of string theory. Also, M theory compactification on a gravitational instanton describes a nonperturbative object from the point of view of string theory. For example, Taub-NUT gives a D6-brane of type

IIA string theory.

1.2.2 Classification

The only compact hyperkähler four-manifolds are T^4 and K3, but the K3 metric is not known explicitly. In the noncompact case there are several possibilities to consider. There are no nontrivial hyperkähler metrics asymptotically approaching that of $\mathbf{R}^4$, but the situation becomes more interesting if one asks that the metric be only Asymptotically Locally Euclidean (ALE), i.e., that the metric looks like the quotient of $\mathbf{R}^4$ by a finite group of isometries. All such metrics fit into the ADE classification of Kronheimer, which we now briefly explain. Let Γ be a finite subgroup of $SU(2)$. There is a natural correspondence (known as the McKay correspondence [1]) between such Γ's and ADE Dynkin diagrams: an A_k diagram corresponds to the cyclic group $\mathbf{Z}_{k+1}$, a D_k diagram corresponds to the dihedral group $\mathbf{D}_{k-2}$ of order $4k-8$, and the E_6, E_7 and E_8 diagrams correspond to the symmetry groups of tetrahedron, cube, and icosahedron, respectively. Since $SU(2)$ acts on $\mathbf{C}^2$ by the fundamental representation, we may consider quotients $\mathbf{C}^2/\Gamma$ (these quotient spaces are known as Kleinian singularities). Kronheimer showed that resolutions of Kleinian singularities admit ALE hyperkähler metrics, and that all such metrics arise in this way [2, 3]. In the A_k case the metric has been known explicitly for some time: it is the Gibbons-Hawking metric with $k+1$ centers [4]. Kronheimer provided an implicit construction of D_k and E_k ALE gravitational instantons as hyperkähler quotients [2]. We shall present a different solution for the D_k case.

Another interesting class of noncompact gravitational instantons is that of Asymptotically Locally Flat (ALF) manifolds. This means that the metric asymptotically approaches a metric on $(\mathbf{S}^1 \times \mathbf{R}^3)/\Gamma$, where Γ is some finite group. Namely, at infinity the metric on these spaces approaches

$$ds^2 = dr^2 + \sigma_1^2 + r^2(\sigma_2^2 + \sigma_3^2), \tag{1.6}$$

where σ_j are left-invariant one-forms on $\mathbf{S}^3/\Gamma$ for the corresponding finite subgroup

Γ of $SU(2)$.

The only known hyperkähler metric of this sort is the multi-Taub-NUT metric. As a complex manifold the $(k+1)$-center multi-Taub-NUT space is isomorphic to the resolution of $\mathbf{C}^2/\mathbf{Z}_{k+1}$, so we shall call it the A_k-type ALF gravitational instanton.

It should be mentioned that in the limit of infinite radius of the circle at asymptotic infinity in the definition of the ALF space, the ALF space becomes the corresponding ALE space. Kronheimer in [2] studied the topological properties of ALE spaces and showed that the intersections of the two-cycles of the ALE space are given by the corresponding Dynkin diagram. As this is a topological property and, as mentioned above, any ALF space can be continuously deformed to an ALE space, we expect to have the same result for the intersection of the ALF two-cycles. As a matter of fact, the analysis of the twistor spaces of D_k ALF spaces, which we shall present later, verifies this result.

1.2.3 Twistor Spaces

The central object in the calculation that follows is the *twistor space*. Let us recall what it is. As we have mentioned, every hyperkähler manifold has three complex structures: I, J and K. Given three complex structures, there is a whole sphere (S^2) of them, since for any unit three-vector $\vec{n}$, the combination $\mathcal{I}_{\vec{n}} = n_1 I + n_2 J + n_3 K$ is also a complex structure. Now consider a manifold which is constructed as a hyperkähler manifold $\mathcal{M}$ with the sphere of complex structures attached at every point. This is the *twistor space* $\mathcal{Z}$ of $\mathcal{M}$, so $\mathcal{Z} = S^2 \times \mathcal{M}$. Two-sphere S^2 has a complex structure, say I_0, and can be thought of as a complex projective line $\mathbf{P}^1$ parametrized by one complex coordinate ζ. Then the twistor space $\mathcal{Z}$ is a complex manifold with the complex structure given by $(I_0, \mathcal{I}_\zeta)$ at a point $(\zeta, x) \in \mathbf{P}^1 \times \mathcal{M}$.

One can think of the twistor space $\mathcal{Z}$ in two ways. As described above it is an S^2 fibration over $\mathcal{M}$ with $r : \mathcal{Z} \to \mathcal{M}$. Since this is a product space, one can also think of $\mathcal{M}$ fibered over $S^2 = \mathbf{P}^1$ with holomorphic projection $p : \mathcal{Z} \to \mathbf{P}^1$. With respect to this projection, the two sphere $r^{-1}(x)$ above a point in $x \in \mathcal{M}$ is a holomorphic section of the bundle $p : \mathcal{Z} \to \mathbf{P}^1$. More than that, there is an antipodal map on every sphere $r^{-1}(x)$ which defines a *real structure* τ on the twistor space $\mathcal{Z}$. This τ is an antiholomorphic involution on $\mathcal{Z}$ (given by $\zeta \to 1/\bar{\zeta}$).

The last essential piece of data is a holomorphic two-form ω on $\mathcal{Z}$. As mentioned in Section 1.2.1, $\mathcal{M}$ has three covariantly constant closed two-forms ω_1, ω_2 and ω_3. For a complex structure on $\mathcal{M}$ given by some point $\zeta \in S^2$, the form

$$\omega = (\omega_2 + i\omega_3) + 2\zeta\omega_1 - \zeta^2 (\omega_2 - i\omega_3) \tag{1.7}$$

is holomorphic with respect to the complex structure I_ζ. Thus ω is a holomorphic closed two-form on every fiber of $p : \mathcal{Z} \to \mathbf{P}^1$.

Since we are considering holomorphic bundles over $\mathbf{P}^1$, let us recall that any line bundle (i.e., linear bundle with fiber $\mathbf{C}$) over $\mathbf{P}^1$ is characterized by one integer n, denoted $\mathcal{O}(n)$ and has transition function $1/\zeta^n$.

The important theorem we shall make extensive use of (see [5]), states that the holomorphic data of $\mathcal{Z}$ determine $\mathcal{M}$ as a differential manifold. Namely, if $\mathcal{Z}$ is a complex manifold of dimension $2l + 1$ such that

1. $\mathcal{Z}$ admits a holomorphic fibration $p : \mathcal{Z} \to \mathbf{P}^1$,
2. $\mathcal{Z}$ as a bundle has a family of holomorphic sections each with normal bundle being $\oplus_l \mathcal{O}(1)$,
3. there is a holomorphic two-form ω on the fibers which has coefficients in $\mathcal{O}(2)$,
4. $\mathcal{Z}$ has a real structure that respects conditions 1, 2 and 3 and acts on the base $\mathbf{P}^1$ as an antipodal map,

then the space of real (i.e., invariant with respect to the real structure τ) holomorphic sections of $\mathcal{Z}$ is a hyperkähler manifold $\mathcal{M}$ of real dimension $4l$ for which $\mathcal{Z}$ is the twistor space. This correspondence between $\mathcal{M}$ as a differential manifold and its twistor space $\mathcal{Z}$ is a particular case of Ward correspondence.

1.3 String Theory Realizations

Realizing supersymmetric gauge theories as theories on D-branes is very useful for identifying their excitations and spaces of vacua. In certain cases this approach allows one to show that the Coulomb branch of the space of vacua is the same as the moduli space of some self-dual Yang-Mills configurations in an auxiliary gauge theory. For example, as described in Ref. [6], the Coulomb branch of $\mathcal{N} = 4$ supersymmetric $SU(n)$ Yang-Mills theory in three dimensions is the (centered) moduli space of n $SU(2)$ monopoles. There are powerful mathematical methods, such as twistor methods and the ADHM-Nahm construction, developed to describe solutions of the Yang-Mills self-duality equation,

$$F = *F. \tag{1.8}$$

Using these methods one can compute the metric on the space of vacua. In addition, realizing the same theory by different D-brane configurations clarifies the connection between different mathematical constructions and yields nontrivial predictions about the geometry of the space of vacua.

1.3.1 Nahm Equations

In particular, realizing gauge theories as a low energy theory on D3-branes of type IIB string theory finitely stretched between NS5-branes (see Section 2.4) gives interpretation of the theories in Section 1.1.1 as nontrivial reductions of four-dimensional theories on a finite interval. Namely, in this reduction four-dimensional fields have nontrivial dependence on the coordinate of finite length. Requiring that this reduction breaks one half of the supersymmetry determines this dependence. String theory considerations allow us to describe this reduction precisely in terms of Nahm equations, as will be explained in Section 2.2.

1.3.2 Monopoles on Multi-Taub-NUT Space

Analyzing the T-dual description of the above mentioned configuration (Section 2.3), we obtain the description of the Coulomb branch of the three-dimensional theory in question as a moduli space of monopoles on multi-Taub-NUT space, by which we mean self-dual connections on multi-Taub-NUT which are invariant with respect to the triholomorphic $U(1)$ isometry of the multi-Taub-NUT metric in some gauge.

1.3.3 Singular Monopoles

There is yet another way of analyzing the type IIB configuration described in Section 2.4. By looking at the theory on the NS5-branes, we can describe the Coulomb branch of the gauge theory as a moduli space of monopoles with singularities. These can be thought of as a superposition of 't Hooft–Polyakov and Dirac monopoles.

Thus we have various descriptions of the vacua of the three-dimensional gauge theories: via Nahm equations, as monopoles on multi-Taub-NUT and as singular monopoles. Of course all these descriptions are equivalent and give the same moduli space as an answer (which is a rather nontrivial mathematical statement).

The main part of this work is to make use of the correspondences described in order to find the spaces of vacua of $U(n)$ and $SU(n)$ three-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry and k matter multiplets in the fundamental representation. A description in terms of Nahm equations allows for an explicit solution in the case of $SU(2)$ with $k \leq 4$ matter multiplets. This construction is presented in Chapter 3. It gives the corresponding Coulomb branches as hyperkähler quotients (see Appendix A) of well-known manifolds.

For general k the solutions of the Nahm equations are unknown, so we have to use other methods. Namely, we find the twistor spaces (Section 1.2.3) of the Coulomb branches first. This is done by two different methods in Chapters 4 and 5. In order to recover the metric on the Coulomb branch, we either go through the Ward correspondence explicitly, as in Section 6.1, or make use of the Generalized Legendre Transformation (see Appendix B) as in the rest of Chapter 6.

Our final results are twistor spaces of the Coulomb branches of $SU(n)$ $\mathcal{N}=4$ three-dimensional theories with fundamental matter. As an example we work out the Kähler potentials on the Coulomb branch of the $U(1)$ theory with k electrons, which is an A_k ALF gravitational instanton. We also derive the Kähler potential on the Coulomb branch of the $SU(2)$ theory with k fundamentals, which is a D_k ALF gravitational instanton.

Using the same techniques one can find the Kähler potentials in all cases. It is also straightforward to use the same methods to study the SO and Sp gauge groups.

1.4 M Theory Realization

1.4.1 $U(1)$ with Charged Matter $\rightarrow$ M Theory on A_k ALF Space

Since we are interested in three-dimensional theories, it is natural to try to realize them as theories on a collection of three-dimensional (including the time dimension) branes, namely D2-branes of type IIA string theory. A D2-brane has a $U(1)$ theory in its worldvolume, which at low energies is described by the Maxwell lagrangian. The gauge field corresponds to the ground state of an open string with both ends on the D2-brane. Putting n D2-branes together enhances the gauge group to $U(n)$ with new gauge fields coming from strings ending on different D2-branes. This theory has $\mathcal{N}=8$ supersymmetry (16 supercharges).

In order to add matter multiplets to the $U(1)$ theory, we can consider a D2-brane in the background of parallel D6-branes. The presence of the D6-branes breaks half of the supersymmetry. Matter multiplets correspond to ground states of open strings with one end on the D2-brane and another on one of the D6-branes. Since such a multiplet carries one index of the gauge group, it is in the fundamental representation.

Now when the gauge theory is realized in type IIA string theory, we can analyze it from the point of view of M theory. M theory is eleven-dimensional and after the reduction on a circle gives the type IIA string theory. If the circle (say direction x^{10})

is fibered over three (say x^1, x^2, x^3) of the other ten coordinates of M theory in such a way that these three coordinates and the circle form a Taub-NUT space, then in the resulting type IIA string theory, this will correspond to a D6-brane positioned at a point in (x^1, x^2, x^3) where the center of the Taub-NUT is and extending in the other $(x^4, x^5, \ldots, x^9)$ directions.

M theory has two solitonic objects: M2-brane and M5-brane. The D2-brane of type IIA string theory comes from the reduction of the M2-brane, so that the position of the M2-brane on the hidden circle of M theory (x^{10}) is the dual photon of the theory on the D2-brane.

At this point we are in the position to interpret the moduli space of vacua of the $U(1)$ theory with k charged hypermultiplets. It is realized as a theory on a D2-brane parallel to k D6-branes. The vacua are parametrized by vacuum expectation values of the Higgs fields and that of the dual photon. In the M theory picture, the expectation values of the Higgs fields are positions on the M2-brane in the three space transverse to the D6-branes and the vev of the dual photon is its position on the circle. Thus the Coulomb branch of the gauge theory in question is the multi-Taub-NUT space describing this compactification of M theory.

1.4.2 $SU(2)$ with Fundamental Matter $\rightarrow$ M Theory on D_k ALF Space

If we consider type IIA string theory in ten-dimensional space, then it has a $\mathcal{Z}_2$ symmetry generated by $(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_3$, where $(-1)^{F_L}$ changes the sign of all the Ramond states in the left sector, Ω changes the world-sheet parity and $\mathcal{I}_3$ flips the orientation of three of the space-like directions of the target space. The seven-dimensional space left invariant by this transformation constitutes the world-volume of the $O6^-$ orientifold.

In order to realize an $SU(2)$ gauge theory, we consider a D2-brane probe next to an orientifold $O6^-$ of type IIA string theory. In M theory the $O6^-$ is an Atiyah-Hitchin space (which was first studied as a moduli space of two centered [1] $SU(2)$

[1] with the fixed center of mass and common $U(1)$ phase

monopoles in [7]). In order to have k matter multiplets, we introduce k D6-branes parallel to the orientifold. As demonstrated by Sen in [8], M theory compactified on the resulting space has intersections of its two-cycles given by a D_k Dynkin diagram. Thus, interpreting the moduli space of the gauge theory as in the previous section, we obtain the D_k ALF space as the Coulomb branch of this $SU(2)$ gauge theory.

Chapter 2 Type IIB String Theory Realization

2.1 Gauge Theories on D3-branes

Following Ref. [6] we consider configurations of D3, D5 and NS5-branes in IIB string theory, which leave eight supersymmetries unbroken. Let two parallel NS5-branes be some distance d apart in the x^6 direction with worldvolumes parallel to x^0, x^1, x^2, x^3, x^4, x^5. Let n D3-branes stretch between them in the x^6 direction, with other directions of D3's being parallel to x^0, x^1, x^2. This configuration of branes is illustrated in Figure 2.1(a). The theory on the D3-branes reduces to $D = 3$, $\mathcal{N} = 4$ $U(n)$ Yang-Mills theory in the infrared limit. Every such configuration of branes corresponds to a particular vacuum of the Yang-Mills theory. As described in Ref. [6], D3-branes look like $SU(2)$ monopoles in the x^3, x^4, x^5 directions in the theory on the NS5-branes. Vacua of the $U(n)$ Yang-Mills theory on D3-branes are in one-to-one correspondence with charge n monopoles on the NS5-branes. In order to describe the $SU(n)$ Yang-Mills theory (rather than $U(n)$), we should fix the center of mass of the D3-branes. Thus vacua of this theory are given by "centered" monopoles.

Now let us add k D5-branes stretching along $x^0, x^1, x^2, x^7, x^8, x^9$ and positioned outside the NS5-branes at points $\vec{p}_\alpha$ in the (x^3, x^4, x^5) plane (see Figure 2.1(b)). Let each D5-brane be connected by one D3-brane to the NS5-brane closest to it. We shall call these D3-branes *external*, to distinguish them from those connecting the two NS5-branes, which we shall call *internal*. From the point of view of the internal D3-branes, the low-energy theory is a $U(n)$ gauge theory with k matter hypermultiplets in the fundamental representation. Matter comes from the fundamental strings connecting the internal and external D3-branes. The question is what this configuration looks like in the $SU(2)$ theory on the NS5-branes.

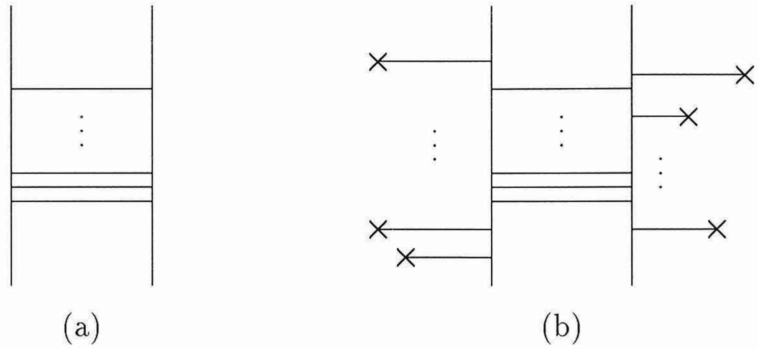


Figure 2.1: (a) Parallel NS5-branes (vertical lines) with D3-branes (horizontal lines) suspended between them. The horizontal direction corresponds to x^6, while the vertical direction corresponds to x^3, x^4, and x^5 collectively. (b) The same, with additional D5-branes (crosses) connected by D3-branes to NS5-branes.

To answer the question we perform S and T duality transformations. First we go to the S dual picture thus exchanging D5 and NS5-branes and leaving the D3-branes unchanged. Then we T dualize along the x^6 direction (after making it periodic) thus turning IIB string theory into IIA string theory, D5-branes into D6-branes, and NS5-branes into an A_{k-1} type ALF space (as explained in Appendix C). Tracing the dualities we have the A_{k-1} ALF metric in $(x^3, x^4, x^5, \hat{x}^6)$, with $\hat{x}^6$ being the compact direction. Four of the directions of the D6-branes are wrapped on this space.

What do the D3-branes turn into after the dualities? If any of the D3-branes were wrapped around x^6, it would turn into a D2-brane located at a point on the A_{k-1} ALF space. As explained in Ref. [9], this D2-brane would look like an instanton in the $U(2)$ theory on the D6-branes. To be more precise, it would be a self-dual $U(2)$ gauge connection on the A_{k-1} ALF space, somewhat resembling Nahm's calorons [10]. Note that the $U(2)$ gauge group is broken down to $U(1) \times U(1)$ by a nontrivial Wilson line at infinity (in the original picture this corresponds to the nonzero distance between the NS5-branes.) Therefore, there may be states in the theory carrying nonzero magnetic charge. The instanton does not have magnetic charge, and neither does the D3-brane wrapped around the x^6 direction in the T-dual description. On the other hand, the internal D3-brane does have a magnetic charge, and therefore corresponds to the monopole solution, by which we mean a self-dual connection carrying magnetic charge.

If there were no D5-branes in the original brane configuration, we would be dealing with self-dual connections on $\mathbf{R}^3 \times \mathbf{S}^1$ rather than on the A_{k-1} ALF space. Then the internal D3-branes would correspond to 't Hooft-Polyakov monopoles localized in $\mathbf{R}^3$, i.e., the four-dimensional self-dual configuration would not depend on the circle coordinate [11]. It is highly plausible that this remains true when D5 branes are present. Indeed, the well-known maxim "Winding is momentum" implies in this case that in the IIA picture nothing depends on the $\hat{x}^6$ direction, since nothing is wound around the T-dual direction in the IIB picture. In the next section we confirm this by exhibiting monopole solutions on the A_{k-1} ALF space which do not depend on the circle coordinate.

2.2 Reduction from $D = 4$

Now we explain how the Nahm equations appear in the context of the string vacuum described above. A set of parallel three-branes has an $\mathcal{N} = 4$ gauge theory on its worldvolume. Consider the world-volume to be in the x^0, x^1, x^2 and x^6 directions. Introduction of a 5-brane that D3-branes end on imposes boundary conditions that break one half of the supersymmetry. The four-dimensional gauge field A_μ, $\mu = 0, 1, 2, 6$ decomposes into a three-dimensional A_j, $j = 0, 1, 2$ and a scalar A_6. The Higgs fields X^A with $A = 3, 4, 5, 7, 8, 9$ that parametrize the positions of the D3-branes in the transverse directions can be separated into $T^i = X^{i+2}$ and $B^i = X^{i+6}$ with $i = 1, 2, 3$. With respect to the $\mathcal{N} = 2$, four-dimensional supersymmetry that survives the $\mathcal{N} = 4$ supermultiplet decomposes into a vector supermultiplet with bosonic fields T^i and A_j and a hypermultiplet with bosonic fields B^i and A_6. As explained in [6] Neumann boundary conditions (which are the case for the configuration described above) imply $A_6 = 0$, $B^i = 0$ at the boundary which makes these fields massive (while Dirichlet boundary conditions would imply $A_j = 0$ at the boundary).

From [12] or [13] the supersymmetry transformations for the gaugino are

$$\begin{aligned}\delta\chi &= \sigma_{\mu\nu}F^{\mu\nu}\epsilon - i\gamma\cdot D\left(\alpha_i T^i + \beta_i B^i i\gamma_5\right)\epsilon \\ &+\frac{1}{2}i\varepsilon^{ijk}\alpha^k g[T^j,T^k]\epsilon + \frac{1}{2}i\varepsilon^{ijk}\beta^k g[B^j,B^k]\epsilon + \alpha^i\beta^j g[T^i,B^j]\gamma_5\epsilon, \qquad (2.1)\end{aligned}$$

with $\{\alpha_i,\alpha_j\} = \{\beta_i,\beta_j\} = -2\delta^{ij}$, and $[\alpha_i,\beta_j] = 0$. All other supersymmetry transformations are proportional to fermionic fields and therefore vanish automatically for a purely bosonic background. Neglecting the B^i and A_6 fields (as they are massive and therefore cannot have vevs), A_j fields (not to break the three-dimensional Lorentz symmetry) and requiring that the gaugino variation be zero for half of supersymmetry transformations implies

$$D_6 T^i + \frac{1}{2}\epsilon^{ijk}[T^j,T^k] = \frac{\partial}{\partial x^6}T^i + \frac{1}{2}\epsilon^{ijk}[T^j,T^k] = 0 \qquad (2.2)$$

which is precisely the Nahm equation.

Thus giving the Higgs fields (corresponding to the directions along the NS5-branes) vacuum expectation values satisfying the Nahm equations leaves half of the supersymmetry unbroken; and considering perturbations near this background that respect this supersymmetry gives the $\mathcal{N}=4$ gauge theory in three dimensions described in Section 1.1.1.

2.3 Monopoles on Multi-Taub-NUT Space

Let us describe precisely what we mean by a monopole on the A_{k-1} ALF space. In coordinates x^1, x^2, x^3, θ the A_{k-1} ALF metric is

$$ds^2 = V dx^i dx^j \delta_{ij} + V^{-1}\left(d\theta + \omega^i dx^i\right)^2, \qquad (2.3)$$

where θ has period 4π and is T-dual to x^6, and

$$V = 1 + \sum_{\alpha=1}^{k}\frac{1}{|\vec{x}-\vec{p}_\alpha|}, \quad \text{grad } V = \text{curl } \vec{\omega}. \qquad (2.4)$$

Note that $\omega = \omega^i dx^i$ is not a globally defined 1-form; rather it is a connection on a nontrivial $U(1)$ bundle and can only be defined patchwise.

A $U(2)$ monopole on this space is a smooth self-dual connection[1] $\hat{\mathbf{A}} = \hat{A}_0 d\theta + \hat{A}_j dx^j$ on a $U(2)$ vector bundle with a nontrivial holonomy (Wilson line) at infinity and nonzero magnetic charge, whose field strength is independent of θ in some local gauge. That is, away from the centers $\vec{x} = \vec{p}_\alpha$ there is a local gauge transformation $g(\vec{x}, \theta)$, such that $\tilde{A}_0 = g^{-1}\hat{A}_0 g - ig^{-1}\partial_\theta g$ and $\tilde{A}_j = g^{-1}\hat{A}_j g - ig^{-1}\partial_j g$ are independent of θ.

As $\hat{\mathbf{A}}$ is smooth and the norm of $\frac{\partial}{\partial\theta}$ vanishes when $\vec{x} \to \vec{p}_\alpha$, one necessarily has $\hat{A}_0(\vec{p}_\alpha) = 0$. $g(\vec{x}, \theta)$ approaches a circle action with integer weights $\ell_\alpha, \ell'_\alpha$ near the centers $\vec{p}_\alpha$. That is in some basis

$$g(\vec{x}, \theta) \to \begin{pmatrix} e^{i\theta\ell_\alpha/2} & 0 \\ 0 & e^{i\theta\ell'_\alpha/2} \end{pmatrix} \tag{2.5}$$

as $\vec{x} \to \vec{p_\alpha}$. Then we easily see that after the above-mentioned gauge transformation the eigenvalues of $\tilde{A}_0$ approach $\ell_\alpha/2, \ell'_\alpha/2$ as $\vec{x} \to \vec{p}_\alpha$.

Since in the new gauge $\mathbf{A}$ does not depend on θ, one may define new fields on $\mathbf{R}^3$

$$\Phi = V\tilde{A}_0, \quad A_i = \tilde{A}_i - \omega_i \tilde{A}_0. \tag{2.6}$$

Kronheimer noticed [14] that these fields satisfy the Bogomolny equation if and only if the initial $\mathbf{A}$ is a self-dual connection on the ALF space. Here Φ is the Higgs field and A_i is the gauge potential on $\mathbf{R}^3$. From Eq. (2.4) it is easy to see that Φ has singularities at $\vec{x} = \vec{p}_\alpha$. Thus monopoles on A_{k-1} ALF space are in one-to-one correspondence with monopoles on $\mathbf{R}^3$ with singular Higgs field

$$\Phi \to \frac{1}{2|\vec{x} - \vec{p}_\alpha|} \operatorname{diag}(\ell_\alpha, \ell'_\alpha) \tag{2.7}$$

near $\vec{x} = \vec{p}_\alpha$ in some gauge. The asymptotic behavior at infinity is the same as for

[1] By this we mean that it has a self-dual curvature $\hat{\mathbf{F}} = d\hat{\mathbf{A}} + \hat{\mathbf{A}} \wedge \hat{\mathbf{A}}$

ordinary monopoles,

$$\Phi \to \mathrm{diag}\left(1 - \frac{n - \sum \ell_\alpha}{2r}, -1 + \frac{n + \sum \ell'_\alpha}{2r}\right). \tag{2.8}$$

We shall call such solutions *singular U(2) monopoles.*

What is the meaning of n in Eq. (2.8)? The rank 2 $U(2)$ bundle decomposes into the sum of eigenspaces of Φ, $E = M \oplus M'$, where M and M' are line bundles. It follows from Eq. (2.7) and Bogomolny equations that upon restriction to a small sphere around $\vec{x} = \vec{p}_\alpha$, the degrees of M and M' are $-\ell_\alpha$ and $-\ell'_\alpha$. In other words, there is a point-like Dirac monopole with charges $-\ell_\alpha, -\ell'_\alpha$ at $\vec{x} = \vec{p}_\alpha$ embedded in the diagonal subgroup of $U(2)$. Similarly, the total magnetic charges of the configuration (the degrees of M and M' restricted to a very large sphere) are $n - \sum \ell_\alpha$ and $-n - \sum \ell'_\alpha$. Let us now focus on the $SU(2)$ subgroup of $U(2)$. Then the total magnetic charge in the $SU(2)$ is $n - \sum(\ell_\alpha - \ell'_\alpha)/2$, while the charge carried by the αth Dirac monopole is $-(\ell_\alpha - \ell'_\alpha)/2$. Therefore, n is naturally interpreted as the number of smooth nonabelian monopoles in the configuration. (Kronheimer [14] calls it nonabelian charge.) One expects that $n \geq 0$, and it can be shown [14] that this is indeed the case.

The Dirac monopole with $\ell = 1, \ell' = 0$ is in fact the reincarnation of the right external D3-brane in the initial brane configuration. This becomes quite obvious if one recalls that the NS5-branes in Figure 2.1(b) correspond to the two $U(1)$ factors in the diagonal subgroup of $U(2)$, and that the end of the D3 brane ending on the NS5-brane from the left (right) carries magnetic charge +1 (-1) in the corresponding $U(1)$. Similarly, the left external D3-brane maps under T-duality to a Dirac monopole with $\ell = 0, \ell' = -1$.

To summarize, the configuration of n internal and k external D3-branes corresponds to a solution of $U(2)$ Bogomolny equations with nonabelian charge n and with Higgs field having k singularities as in Eq. (2.7). Depending on whether the αth D3-brane is right or left, we have $\ell_\alpha = 1, \ell'_\alpha = 0$ or $\ell_\alpha = 0, \ell'_\alpha = -1$.

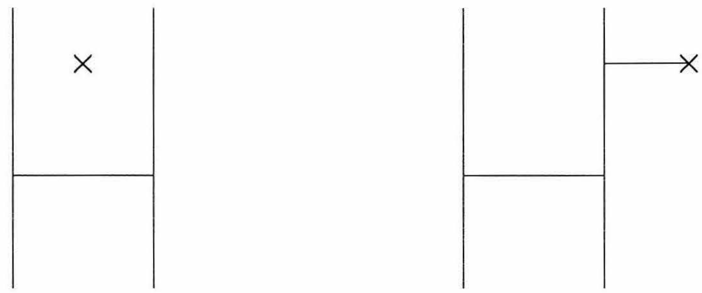


Figure 2.2: Starting with configuration (a) and moving the D5-brane to the right, one gets configuration (b). In both configurations the low-energy gauge theory on the D3-brane has one hypermultiplet.

2.4 Singular Monopoles

2.4.1 Are There Phase Transitions?

The brane configurations in Figures 2.2(a) and 2.3(a) correspond to $U(1)$ gauge theories with one and no charged hypermultiplets, respectively. It was noted in Ref. [6] that the position of the D5-brane in the x^6 direction does not appear as a parameter in the gauge theory. Therefore, one could think that there is a phase transition in the gauge theory when D5 and NS5-branes cross. In fact, as explained in Ref. [6], there is no phase transition because a D3-brane is created when D5 crosses NS5. In the case of the configuration in Figure 2.2(a), moving D5-brane to the right creates a configuration in Figure 2.2(b). The latter corresponds to a $U(1)$ gauge theory with one hypermultiplet, as there are fundamental strings connecting the external and internal D3-branes.

The picture with singular monopoles described in Section 2.3 refers to Figure 2.2(b) on page 20. Figure 2.2(a), however, presents a puzzle: there are no D3-branes connecting D5 with NS5, so the reasoning of Section 2.3 seems to imply that the configuration corresponds to a $U(2)$ monopole on the A_0 ALF space (i.e., the Taub-NUT space) with $n = 1, \ell = \ell' = 0$. It was explained in Section 2.3 that such a monopole is equivalent to a nonsingular monopole on $\mathbf{R}^3$, whose moduli space is $\mathbf{R}^3 \times \mathbf{S}^1$. This is clearly false, since the moduli space of a $U(1)$ gauge theory with one charged hypermultiplet

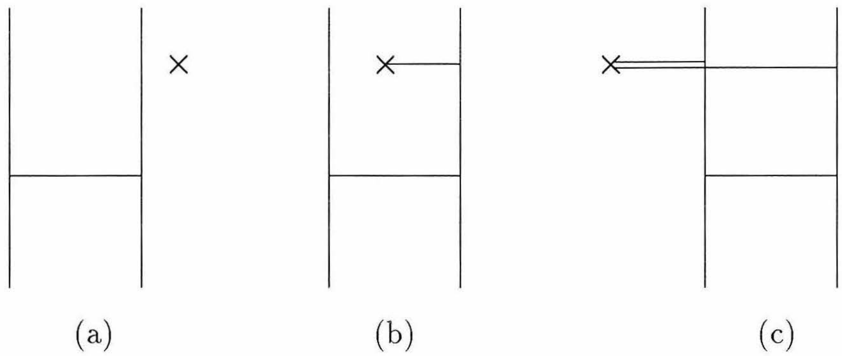


Figure 2.3: Starting with configuration (a) and moving the D5-brane to the left one gets configurations (b) and (c).

is known to be the Taub-NUT space [15, 16], not $\mathbf{R}^3 \times \mathbf{S}^1$. The resolution is that Figure 2.2(a) corresponds to a monopole with $\ell = 1, \ell' = 0$, just as in Figure 2.2(b), but in a singular gauge. As explained in the next section, there is a singular θ-dependent gauge transformation which eliminates the singularity at $\vec{x} = \vec{p}$ but reintroduces one at the monopole core (the singularity is reflected in the "hedgehog" behavior of the Higgs field near the core). In this new gauge the fields are nonsingular at $\vec{x} = \vec{p}$, as expected in a situation like Figure 2(a), with no semi-infinite D3-branes ending on the NS5-branes. Such a "shaggy monopole" has the same moduli space as the normal monopole on the Taub-NUT with $\ell = 1, \ell' = 0$, which in turn is equivalent to a monopole on $\mathbf{R}^3$ with Higgs field diverging near the point $\vec{x} = \vec{p}$ as in Eq. (2.7). The moduli space of the latter is indeed the Taub-NUT space [14].

What happens if one starts with Figure 2.3(a) and moves the D5-brane inside? According to Ref. [6] the final configuration must be that in Figure 2.3(b). One might expect a charged hypermultiplet from strings connecting the internal D3-brane with the newly created one. This again would imply a phase transition when the D5-brane crosses the NS5-brane. Moreover, if we move the D5-brane farther to the left, there will be another D3-brane created (see Figure 2.3(c)) and one might think that two hypermultiplets appear! Is there a phase transition in this case?

2.4.2 Phase Transition and "Shaggy Monopoles"

The interpretation in terms of monopoles on Taub-NUT space helps us to understand what happens to the moduli spaces and to see that there is no phase transition. Namely, Figure 2.3(b) corresponds to a "shaggy monopole" with an additional $\ell = -1, \ell' = 0$ singularity at the center of the Taub-NUT space $\vec{x} = \vec{p}$. The singularities at the monopole core and at $\vec{x} = \vec{p}$ can both be simultaneously eliminated by a gauge transformation (see Section 2.4.2), and we are back to the normal monopole with $\ell = 0, \ell' = 0$. The latter is equivalent to a nonsingular $n = 1$ monopole on $\mathbf{R}^3$, and therefore the moduli space is $\mathbf{R}^3 \times \mathbf{S}^1$, the same as that of the configuration in Figure 2.3(a). In Figure 2.3(c) the D3-branes connecting the D5 with NS5's correspond to a Dirac monopole embedded in the subgroup $U(1)_{cm} \subset U(2) = U(1)_{cm} \times SU(2)$; therefore, they do not influence the $SU(2)$ monopole at all. (The component of the Higgs field in $U(1)_{cm}$ corresponds to the center-of-mass motion of the five-brane.) Thus the moduli space is still $\mathbf{R}^3 \times \mathbf{S}^1$.

Apparently, the naive counting of string modes fails in situations like those in Figures 2.3(b) and 2.3(c): in these cases there are no stable fundamental string states connecting the internal D3-branes with the D3-branes stretched between the D5 and NS5-branes.

Similar arguments apply when there is more than one D3 and/or D5-brane.

In order to define "*shaggy monopoles*" and give their interpretation in terms of D-branes, first let us look at self-dual $U(2)$ connections on $\mathbf{R}^3 \times \mathbf{S}^1$. We can think of these connections as living on the world-volume of two coincident D6-branes wrapped around $\mathbf{R}^3 \times \mathbf{S}^1$. Let R_A be the radius of $\mathbf{S}^1$, and let θ be the coordinate along it. We consider the connections with fixed second Chern class and fixed conjugacy class of holonomy around $\mathbf{S}^1$ at infinity. If the asymptotic eigenvalues of A_0 (the θ-component of the gauge field) are μ_1 and μ_2, the holonomy at infinity is conjugate to

$$W = \begin{pmatrix} e^{2\pi i R_A \mu_1} & 0 \\ 0 & e^{2\pi i R_A \mu_2} \end{pmatrix}. \tag{2.9}$$

Let's assume that $\mu_1 > \mu_2$, for definiteness. After one T-dualizes along the θ direction, $2\pi\mu_1$ and $2\pi\mu_2$ are interpreted as the x^6 positions of the D5-branes (we set the string scale α' to 1). Now with two D5-branes at points $x^6 = 2\pi\mu_1$ and $x^6 = 2\pi\mu_2$, there are two ways to stretch a D3-brane between them. After T-duality in the x^6 both types of stretched D3-branes turn into $U(2)$ monopoles on $\mathbf{R}^3 \times \mathbf{S}^1$, but with different asymptotic eigenvalues of A_0. Namely, the eigenvalues are μ_1, μ_2 in one case and $\mu_2 + 1/R_A, \mu_1$ in the other [11]. Both configurations are in fact 't Hooft-Polyakov monopoles, with A_0 playing the role of the Higgs field; consequently, they are θ-independent.

Of course the eigenvalues of A_0 can be changed by a θ-dependent gauge transformation. For example, to change the eigenvalues from $\mu_2 + 1/R_A, \mu_1$ to μ_1, μ_2, one has to use the following gauge transformation:

$$g = \exp\left(\frac{i\theta}{2R_A}\left[\frac{A_0 - \frac{1}{2}\mathrm{tr}A_0}{\|A_0 - \frac{1}{2}\mathrm{tr}A_0\|} - 1\right]\right), \tag{2.10}$$

where we defined $\|\phi\|^2 = \frac{1}{2}\mathrm{tr}\phi^2$. Simultaneously this transformation makes A_i θ-dependent and singular at every point $\vec{q}$ where the traceless part of A_0 vanishes. One can think of such a point as the center of the monopole (there is just one such point for a single smooth 't Hooft-Polyakov monopole). The gauge transformation Eq. (2.10) creates a "hedgehog" at the monopole center, in the sense that the eigenvalues of the Higgs field approach $1/R_A, 0$ as $\vec{x} \to \vec{q}$, but the direction of the Higgs field in the $u(2)$ algebra depends on the direction of approach. This singular and θ-dependent $U(2)$ connection on $\mathbf{R}^3 \times \mathbf{S}^1$ is what we call a "shaggy monopole." Naturally, the moduli space of a "shaggy monopole" is the same as the moduli space of a regular monopole from which it was obtained by a gauge transformation.

We would like to stress that if only D3-branes of one type are present, one can always use the gauge in which the corresponding gauge configuration is θ-independent. But if D3-branes stretched both ways are present, then θ-dependence cannot be removed by a gauge transformation. In particular, the "Wilson-line instanton" of Ref. [11] is θ-dependent.

Now we consider a monopole of magnetic charge 1 on the A_{k-1} ALF space. Let the asymptotic radius of the compact direction be R_A. As in the case of $\mathbf{R}^3 \times \mathbf{S}^1$, we want to fix the holonomy at infinity. Monopoles on the A_{k-1} ALF space can be obtained from smooth BPS monopoles on $\mathbf{R}^3$ with asymptotic eigenvalues of the Higgs field at infinity either μ_1, μ_2 or $\mu_2 + 1/R_A, \mu_1$, the holonomy being the same. Let us choose the latter possibility. Performing the change of variables as in Eq. (2.6) we get a θ-independent connection $\tilde{A}$ on the A_{k-1} ALF space. Since $V^{-1}(\vec{p}_\alpha) = 0$ for all α, the traceless part of $\tilde{A}_0$ vanishes not only at the monopole center, but also at the k centers of the A_{k-1} ALF space. Now suppose we want to change the asymptotic eigenvalues of $\tilde{A}_0$ to μ_1, μ_2. To this end we perform the gauge transformation as in Eq. (2.10). The new connection will be singular at the monopole center, as well as at $\vec{x} = \vec{p}_\alpha, \alpha = 1, \ldots, k$. Thus a smooth monopole on the A_{k-1} ALF space with all ℓ_α equal to zero is gauge-equivalent to a "shaggy monopole" which has singularities at the monopole core and at the centers of the A_{k-1} ALF space.

On the other hand we can start with a θ-independent monopole on the A_{k-1} ALF space which has $\ell_\alpha = 1, \ell'_\alpha = 0,\ \alpha = 1, \ldots, k$.. After the gauge transformation inverse to that in Eq. (2.10), it turns into a connection which has $\hat{A}_0(\vec{p}_\alpha) = 0,\ \alpha = 1, \ldots, k$, and a "hedgehog" in the monopole center. Thus we can trade the singularities at the centers of the A_{k-1} ALF space for a similar singularity at the monopole center by means of a singular gauge transformation.

Chapter 3 Solving Gauge Theories by Solving Nahm Equations ($D_k, k \leq 4$)

3.1 Moduli Spaces of Singular Monopoles

There are several approaches to finding metrics on monopole moduli spaces. The most direct one is to use the Nahm transform [10]. In principle, this should yield an isometry between the monopole moduli space and the space of solutions of Nahm equations. So far the details have been worked out only for nonsingular $SU(2)$ monopoles [17]. The idea of the minitwistor approach [18, 19] is to encode the monopole data in terms of an algebraic curve in $T\mathbf{P}^1$ (the tangent bundle of $\mathbf{P}^1$). This curve is then reinterpreted as a spectral curve of Nahm equations, in the spirit of Refs. [20, 19]. This program has been realized for nonsingular monopoles of all classical groups in Ref. [21]. This approach only allows one to prove that the moduli spaces of monopoles and Nahm data are diffeomorphic. There is a natural hyperkähler metric on the space of Nahm data, so it is very plausible that these manifolds are in fact isometric. Here we adopt a less rigorous approach, regarding singular $U(2)$ monopoles as a limit of nonsingular $SU(3)$ monopoles. Therefore, we can construct the moduli spaces in question by considering a certain limit of Nahm equations for $SU(3)$ monopoles.

Let us recall what Nahm equations for $SU(3)$ monopoles look like according to Ref. [21]. In the case of maximal breaking to $U(1) \times U(1)$, which is all we need, $SU(3)$ monopoles are labeled by a pair of nonnegative integers (n, k). The two cases of $n > k$ and $n < k$ are very similar. Only in the matching conditions (iii) one should interchange the superscripts 1 and 2. For $n < k$ the Nahm data consist of two quadruplets

$$\left(T_0^{(\lambda)}(s), T_1^{(\lambda)}(s), T_2^{(\lambda)}(s), T_3^{(\lambda)}(s)\right), \ \lambda = 1, 2, \tag{3.1}$$

with the first quadruplet defined for $s \in (0,1)$, and the second one defined for $s \in (1,\mu)$. $T_i^{(1)}$ and $T_i^{(2)}$, $i = 0,\ldots,3$ take values in $u(n)$ and $u(k)$, respectively. It is very convenient to combine the functions $T_i^{(\lambda)}, i = 0,\ldots,3,\ \lambda = 1,2$, into two quaternions

$$T^{(\lambda)} = T_0^{(\lambda)} + e_1 T_1^{(\lambda)} + e_2 T_2^{(\lambda)} + e_3 T_3^{(\lambda)},\ \lambda = 1,2, \tag{3.2}$$

with e_i being the quaternion units. (In what follows we shall denote the real (T_0) and imaginary ($T - T_0$) parts of quaternions by the symbols Re and Im respectively, and think of the purely imaginary quaternions as three-component vectors.) Thus one can think of $T^{(1)}$ and $T^{(2)}$ as two functions $T^{(1)}(s) : (0,1) \to u(n) \otimes \mathbf{H}$ and $T^{(2)}(s) : (1,\mu) \to u(k) \otimes \mathbf{H}$. They must satisfy a number of constraints [21]:

(i) Both functions satisfy Nahm equations

$$\frac{dT_i}{ds} + [T_0, T_i] = \frac{1}{2}\varepsilon^{ijk}\,[T_j, T_k]\,,\ i = 1,2,3. \tag{3.3}$$

(ii) Re $T^{(1)}(s)$ and Re $T^{(2)}(s)$ extend smoothly to $[0,1]$ and $[1,\mu]$, respectively. Im $T^{(1)}(s)$ has a simple pole at $s = 0$. The residue is an n-dimensional irreducible representation of $su(2)$. Im $T^{(2)}(s)$ has a simple pole at $s = \mu$ with a residue which is a k-dimensional irreducible representation of $su(2)$.

(iii) Im $T^{(1)}(s)$ extends smoothly to $(0,1]$. In the neighborhood of $s = 1$ Im $T^{(2)}(s)$ has the following form

$$T_i^{(2)}(s) = \left(\begin{array}{cc} \rho_i/(s-1) + O(1) & O\left((s-1)^{(k-n-1)/2}\right) \\ O\left((s-1)^{(k-n-1)/2}\right) & T_i^{(1)}(1) + O(s-1) \end{array} \right), \quad i = 1,2,3. \tag{3.4}$$

Here $\rho_i = \rho(i\sigma_i/2), i = 1,2,3$, where ρ is a $(k-n)\times(k-n)$ irreducible representation of $su(2)$ and σ_i are Pauli matrices.

The set of all Nahm data satisfying conditions (i)-(iii) is invariant with respect to gauge transformations which act in a more-or-less obvious manner: the gauge group is $U(n)$ on the interval $[0,1]$ and $U(k)$ on the interval $[0,\mu]$. To preserve the condition

(iii) and the residues of $T^{(1)}, T^{(2)}$ one must also require that at $s = 0$ and $s = \mu$ the gauge transformations reduce to identity, and at $s = 1$ the "right" gauge group $U(k)$ reduces to the "left" $U(n)$.

For $n > k$ the constraints are the same, with the roles of $T^{(1)}$ and $T^{(2)}$ interchanged. For $n = k$ the Nahm data include, in addition, a quaternionic vector $a \in \mathbf{H}^n$, and the condition (iii) is replaced by the following one:

(iii′) Im $T^{(1)}$ and Im $T^{(2)}$ extend smoothly to $(0, 1]$ and $[1, \mu)$, respectively, so that

$$\operatorname{Im} T^{(2)}_{AB}(1) - \operatorname{Im} T^{(1)}_{AB}(1) = \frac{i}{2} a_{(A} e_1 \bar{a}_{B)} + \frac{1}{2} a_{[A} \bar{a}_{B]}, \; A, B = 1, \ldots, n. \tag{3.5}$$

(The parentheses and brackets denote symmetrization and antisymmetrization, respectively.)

In this case the gauge group $U(n)$ acts also on a from the right, $a_A \to a_B \, g(1)_{BA}$, where $g(s)$ is a gauge transformation.

The space of all Nahm data modulo gauge transformations is diffeomorphic to the space of all (n, k) monopoles [21]. There is a natural hyperkähler metric on the space of Nahm data, and therefore it is expected that the two spaces are isometric.

To see the metric on the equivalence classes of Nahm data, notice that the gauge group acts triholomorphically on the flat infinite-dimensional hyperkähler manifold consisting of *all* pairs $\left(T^{(1)}, T^{(2)}\right)$ satisfying (ii) and (iii), except that now the lower right corner of Im $T^{(2)}(1)$ need not be equal to Im $T^{(1)}(1)$. (For $n = k$ one must consider instead the space of all triplets $\left(T^{(1)}, T^{(2)}, a\right)$ such that $T^{(1)}$ and $T^{(2)}$ satisfy (ii) and extend smoothly to $s = 1$, and $a \in \mathbf{H}^n$.) The Nahm equations can be interpreted as moment map equations for gauge transformations which are identity at $s = 1$. The boundary conditions for Nahm data at $s = 1$ can be interpreted as moment map equations for the action of the residual gauge group, which is the group of all gauge transformations modulo those which are the identity at $s = 1$. (This group is $U(min(n, k))$.) Thus one can use the hyperkähler quotient construction of Hitchin et al. [5] to construct a hyperkähler metric on the space of Nahm data modulo gauge transformations.

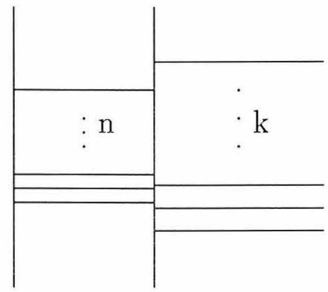


Figure 3.1: The brane configuration corresponding to singular $U(2)$ monopoles can be obtained as a limit of that corresponding to regular $SU(3)$ monopoles.

To obtain $U(2)$ n-monopoles with k singularities of the type $\ell = 1, \ell' = 0$ one should take the limit $\mu \to \infty$ of (n, k) $SU(3)$ monopoles, fixing the positions of k of them which become infinitely heavy. In this limit $SU(3)$ is broken down to $U(2)$ at a very high scale, so that $(0, 1)$ monopoles become point-like Dirac monopoles, while $(1, 0)$ monopoles remain smooth. It is easy to see that the magnetic charges carried by Dirac monopoles also come out right. The corresponding brane configuration is shown in Figure 3.1. From the above description it is clear that in this limit $T^{(2)}$ becomes a function defined on $(1, +\infty)$ and satisfying (i) and (iii) (or (iii′) if $n = k$). A natural boundary condition at $+\infty$ is to require that $\lim_{s\to+\infty} T^{(2)}(s)$ exists in some gauge, and that Re $T^{(2)}(+\infty) = 0$. Then the Nahm equations imply that the matrices $T_i^{(2)}(+\infty)$, $i = 1, 2, 3$, commute and can be reduced to a diagonal form Im $T^{(2)}(+\infty) = \mathrm{diag}(\vec{p}_1, \ldots, \vec{p}_k)$ for some $\vec{p}_\alpha \in \mathbf{R}^3$. It remains to understand what the positions of infinitely heavy monopoles are. The Nahm data just described depend on k vectors $\vec{p}_\alpha$, $\alpha = 1, \ldots, k$. It is therefore tempting to identify them as the positions of k infinitely heavy monopoles, i.e., singularities of the Higgs field. This identification can be justified by recalling the physical meaning of Nahm data [22]. The variable s is interpreted as the coordinate x^6 along the horizontal direction in Figure 3.1. The matrices $T_i^{(2)}$ describe the transverse coordinates of semi-infinite D3-branes. For generic values of s, the three matrices $T_i^{(2)}$ do not commute, so the notion of the transverse position of a given D3-brane is not defined. However, at $s = +\infty$ the matrices do commute, and their eigenvalues $\vec{p}_\alpha$ have the meaning of the D3-branes' asymptotic coordinates in the (x^3, x^4, x^5) plane. From Section 2.1 we know that these asymptotic coordinates are precisely the positions of the singularities of the Higgs field.

3.2 Examples

In this section we illustrate the above construction by the examples of $U(1)$ and $SU(2)$ gauge theories with k massive fundamentals. In what follows $\vec{p}_\alpha$, $\alpha = 1, \ldots, k$ are hypermultiplet masses. Let us also recall that $n = 1$ (one monopole) corresponds to a $U(1)$ gauge theory, and $n = 2$ (two monopoles) corresponds to a $U(2)$ or $SU(2)$ gauge theory.

3.2.1 $n = 1$, k Arbitrary

The Nahm data consist of an $\mathbf{H}$-valued function $T^{(1)}(s)$ on $[0, 1]$ and a $u(k) \otimes \mathbf{H}$-valued function $T^{(2)}(s)$ on $(1, +\infty)$. $T^{(2)}$ has a simple pole at $s = 1$, and the matching condition (iii) of Section 3.1 is satisfied. The boundary conditions at $s = +\infty$ are $\mathrm{Re}\, T^{(2)}(+\infty) = 0$, $\mathrm{Im}\, T^{(2)}(+\infty) = \mathrm{diag}(\vec{p}_1, \ldots, \vec{p}_k)$.

In this case we expect, from field theory, that the moduli space is the A_{k-1} ALF space [15]. To obtain this result in our setup we would have to solve $k \times k$ Nahm equations on a half-line. Alas, we do not know how to do it directly. Fortunately, there is a way to find the moduli space of Nahm equations without actually solving them [23]. First one splits the three Nahm equations into one complex and one real equation. This amounts to picking a complex structure out of a $\mathbf{P}^1$ of available complex structures. The complex equation is invariant under the complexification of the gauge group $G^{\mathbf{C}}$. Donaldson proved that the space of solutions of the complex equation modulo $G^{\mathbf{C}}$ is the same as the space of solutions of all three equations modulo G. Thus it suffices to solve the complex equation to determine the moduli space of Nahm equations as a complex manifold. (This is similar to how one computes the moduli space of supersymmetric gauge theories: instead of solving both D and F-flatness conditions, one solves only the F-flatness conditions modulo the complexification of the gauge group.) Solving the complex equation is easy since it is locally trivial. We refer the reader to Ref. [23] for details, and to Refs. [24, 25, 26] for some applications of this technique. In our case the moduli space turns out to be isomorphic to a hypersurface in $\mathbf{C}^3$ specified by the equation

$$xy = \prod_{\alpha=1}^{k}(z - p_\alpha), \tag{3.6}$$

where p_α is the "complex" part of $\vec{p}_\alpha$, $p_\alpha = p_{\alpha 1} + ip_{\alpha 2}$. This is the complex structure of the resolution of A_{k-1} singularity.

To find the metric one needs to know all three complex structures, however. To this end one has to vary the arbitrary complex structure we picked in the beginning and see how variables in Eq. (3.6) change [26]. The result is rather simple to describe: p_α are fixed real sections of $\mathcal{O}(2)$, z is the coordinate in the fiber of the line bundle $T\mathbf{P}^1 = \mathcal{O}(2)$ over the $\mathbf{P}^1$ of complex structures, while x and y take values in the line bundles $L(k)$ and $L^{-1}(k)$ over $T\mathbf{P}^1$, respectively. (In the notation of Ref. [19] $L^x(k)$ is a line bundle over $T\mathbf{P}^1$ with a transition function $\zeta^{-k}e^{x\eta/\zeta}$, where ζ is the coordinate on $\mathbf{P}^1$ and η is the coordinate in the tangent space at ζ, $d\zeta(\eta) = 1$.) With such identification Eq. (3.6) describes the twistor space of the A_{k-1} ALF space [27].

3.2.2 $n = 2,\ k = 1$

The Nahm data consist of a $u(2)\otimes\mathbf{H}$-valued function $T^{(1)}(s)$ on $(0, 1]$ and a $u(1)\otimes\mathbf{H}$-valued function $T^{(2)}(s)$ on $[1, +\infty)$. $T^{(1)}$ has a simple pole at $s = 0$ with residue $e_1\rho_1 + e_2\rho_2 + e_3\rho_3$, where $\rho_i = i\sigma_i/2$, $i = 1, 2, 3$. The boundary conditions at $s = +\infty$ are $\mathrm{Re}\ T^{(2)}(+\infty) = 0$, $\mathrm{Im}\ T^{(2)}(+\infty) = \vec{p}$. At $s = 1$ the matching condition (iii) is satisfied.

It proves convenient to perform the hyperkähler quotient in two steps. First we take the quotient with respect to gauge transformations which are the identity at $s = 1$. This amounts to solving Nahm equations on two intervals separately and finding their moduli spaces of solutions. For $s \in (1, +\infty)$ Nahm equations just tell us that Im $T^{(2)}$ is independent of s and equal to $\vec{p}$, i.e., the moduli space is just a point. Solving Nahm equations for 2×2 matrices on $(0,1)$ is also elementary, since the equations can be reduced to those of the Euler top. In fact, for $s \in (0,1]$ the moduli space of solutions with boundary behavior as described above has been investigated by Dancer [25]. It turns out that the moduli space is a 12-dimensional hyperkähler manifold M^{12} of the form $\mathbf{R}^3 \times \mathbf{S}^1 \times M^8$, where M^8 is also hyperkähler and irreducible. M^8 admits a triholomorphic action of $SU(2)$.

Second, we take the quotient with respect to the $U(1)$ group "living" at $s = 1$. This $U(1)$ is a subgroup of $U(2)$ which is the group of all gauge transformations modulo those which reduce to identity at $s = 1$. More concretely, the $U(1)$ acts on $\mathbf{R}^3 \times \mathbf{S}^1 \times M^8$ as follows: it rotates the $\mathbf{S}^1$, and it acts on M^8 as a maximal torus of the triholomorphic $SU(2)$ mentioned in the end of the previous paragraph. The boundary condition (iii) implies that the level of the quotient is $2\vec{p}$. The net result is an 8-dimensional hyperkähler manifold depending on $\vec{p}$ as a parameter. It is the moduli space of two monopoles with a fixed singularity at $\vec{x} = \vec{p}$ and corresponds to the $U(2)$ gauge theory with one massive fundamental hypermultiplet.

If one wishes to obtain the Coulomb branch of the $SU(2)$ theory with the same matter content, one should perform a further $U(1)$ hyperkähler quotient (i.e., pass to the centered monopole moduli space). This $U(1)$ is easily identified: it acts on $\mathbf{R}^3 \times \mathbf{S}^1$ by rotating the $\mathbf{S}^1$. The level of the quotient is simply the position of the monopoles' center of mass. It can always be absorbed into $\vec{p}$, so we can set it (i.e., the level) to zero. Performing this $U(1)$ quotient rids the M^{12} of the $\mathbf{R}^3 \times \mathbf{S}^1$ factor. Thus we conclude that the moduli space of the $SU(2)$ gauge theory with one fundamental is the hyperkähler quotient of M^8 by $U(1)$ at the level $2\vec{p}$. This is exactly the four-dimensional manifold constructed by Dancer in Ref. [25] and proposed in Ref. [16] as a candidate for the Coulomb branch. Moreover, we showed above that $\vec{p}$ should be identified as the mass of the hypermultiplet. This agrees with Ref. [16] where it was suggested that the level of the quotient should be *twice* the mass of the fundamental.

3.2.3 $n = 2,\ k = 2.$

In this case the Nahm data consist of a $u(2) \otimes \mathbf{H}$-valued function $T^{(1)}(s)$ on $(0, 1]$, a $u(2) \otimes \mathbf{H}$-valued function $T^{(2)}(s)$ on $[1, +\infty)$, and a quaternionic vector $a \in \mathbf{H}^2$. Both functions satisfy Nahm equations. $T^{(1)}$ has a simple pole at $s = 0$ with residue $e_1\rho_1 + e_2\rho_2 + e_3\rho_3$, where $\rho_i = i\sigma_i/2,\ i = 1, 2, 3$. The boundary conditions at $s = +\infty$ are $\mathrm{Re}\, T^{(2)}(+\infty) = 0,\ \mathrm{Im}\, T^{(2)}(+\infty) = \mathrm{diag}(\vec{p}_1, \vec{p}_2)$. At $s = 1$ the matching condition (iii$'$) is satisfied.

Again we split the calculation in two steps. The solution of Nahm equations for $s \in (0, 1)$ is the same as before. To solve the equations on $(1, +\infty)$, we split $T^{(2)}$ into a part proportional to the identity matrix and a traceless matrix. The equations for the "identity" part simply say that $Tr\left(\mathrm{Im}\, T^{(2)}\right)$ is independent of s and equal to $\vec{p}_1 + \vec{p}_2$. The equations for the traceless part can be solved in terms of hyperbolic functions. After one performs the quotient with respect to the $U(2)$ gauge group which degenerates to the identity at $s = 1$ and to $U(1) \times U(1)$ at $s = +\infty$, one gets a four-dimensional moduli space M_{EH}. Its metric can be computed to be the two-center Gibbons-Hawking (or Eguchi-Hanson) metric with $|\vec{p}_1 - \vec{p}_2|$ being the distance between the centers. Actually, this is a particular case of Kronheimer's construction of hyperkähler metrics on the coadjoint orbits of a complex group G [24]. Kronheimer's construction also uses Nahm equations, and for $G = SL(2, \mathbf{C})$ coincides with ours. (The coadjoint orbit here happens to be isomorphic to $T\mathbf{P}^1$ as a complex manifold.) The Eguchi-Hanson metric admits a triholomorphic action of $SU(2)$.

The second step is to take the hyperkähler quotient of

$$M^{12} \times \mathbf{H}^2 \times M_{EH} \tag{3.7}$$

with respect to $U(2)$. This residual $U(2)$ is the quotient of all gauge transformations by those which reduce to the identity at $s = 1$. The subgroup $U(1) \subset U(2) = U(1) \times SU(2)$ acts on $M^{12} = \mathbf{R}^3 \times \mathbf{S}^1 \times M^8$ by rotating $\mathbf{S}^1$, and on $a \in \mathbf{H}^2$ by right multiplication by $\exp(e_1\phi)$. The matching condition (iii$'$) means that the $U(1)$ quotient should be performed at level $\vec{p}_1 + \vec{p}_2$. The $SU(2)$ subgroup acts on the M^8

part of M^{12} and on the Eguchi-Hanson space M_{EH}. It also acts on a by the right multiplication $a \to a\, g^t$.

The quotient manifold is an 8-dimensional hyperkähler manifold depending on $\vec{p}_1$ and $\vec{p}_2$ as parameters. It is the moduli space of two monopoles with two fixed singularities at $\vec{p}_1$ and $\vec{p}_2$ and corresponds to the $U(2)$ gauge theory with two massive fundamental hypermultiplets.

To obtain the Coulomb branch of the $SU(2)$ theory we must "center" the monopoles, as in the previous example. The position of the center of mass can be set to zero without loss of generality. As earlier, "centering" monopoles is achieved by taking a $U(1)$ quotient. This procedure eliminates the $\mathbf{R}^3 \times \mathbf{S}^1$ factor of M^{12}. Then we need to compute the hyperkähler quotient of

$$M^8 \times \mathbf{H}^2 \times M_{EH} \tag{3.8}$$

by $U(2) = U(1) \times SU(2)$, where $U(1)$ acts only on $\mathbf{H}^2$ by right multiplication, and $SU(2)$ acts on all three factors. The level of the quotient is $\vec{p}_1 + \vec{p}_2$. Since $U(1)$ acts so simply, we can perform the quotient with respect to it explicitly (see, e.g., Ref [28]). The final result is that the Coulomb branch of the $SU(2)$ theory with two fundamentals is the hyperkähler quotient of $M^8 \times M'_{EH} \times M_{EH}$ with respect to $SU(2)$. Both M'_{EH} and M_{EH} are the two-center Gibbons-Hawking (Eguchi-Hanson) spaces with distances between the centers $\vec{p}_1 + \vec{p}_2$ and $\vec{p}_1 - \vec{p}_2$ respectively.

3.2.4 $n = 2,\ k = 3$

It is convenient to slightly change our point of view and regard two $U(2)$ monopoles with three singularities as a limit of $SU(4)$ $(1, 2, 2)$ monopoles, rather than the limit of $SU(3)$ $(2, 3)$ monopoles. The limit is such that $(1, \ , \ )$ and $(\ , \ , 2)$ monopoles become infinitely heavy. The corresponding brane construction is shown in Figure 3.2. The Nahm data consist of three functions $T^{(0)} : (-\infty, 0] \to u(1) \otimes \mathbf{H}$, $T^{(1)} : [0, 1] \to u(2) \otimes \mathbf{H}$, $T^{(2)} : [1, +\infty) \to u(2) \otimes \mathbf{H}$ and a quaternionic vector $a \in \mathbf{H}^2$. The boundary conditions at $s = \pm\infty$ are $\mathrm{Re}\ T^{(0)}(-\infty) = \mathrm{Re}\ T^{(2)}(+\infty) = 0$, $\mathrm{Im}\ T^{(0)}(-\infty) =$

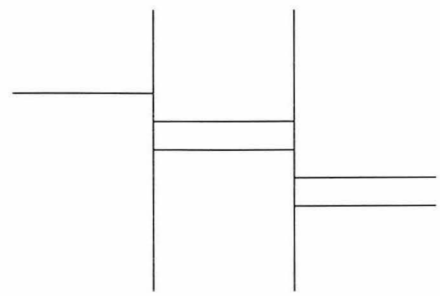


Figure 3.2: A $U(2)$ 2-monopole with three singularities is a limit of a regular $(1,2,2)$ $SU(4)$ monopole.

$\vec{p}_1$, $\mathrm{Im}\ T^{(2)}(+\infty) = \mathrm{diag}(\vec{p}_2, \vec{p}_3)$. Here $\vec{p}_\alpha$ are the positions of the singularities of the Higgs field. The matching condition at $s=0$ is

$$\mathrm{Im}\ T^{(1)}(0)_{2,2} = \mathrm{Im}\ T^{(0)}(0). \tag{3.9}$$

The matching condition at $s=1$ is

$$\mathrm{Im}\ T^{(2)}_{AB}(1) - \mathrm{Im}\ T^{(1)}_{AB}(1) = \frac{i}{2} a_{(A} e_1 \bar{a}_{B)} + \frac{1}{2} a_{[A}\bar{a}_{B]},\ A, B = 1, 2. \tag{3.10}$$

The advantage of this point of view is that we already know what the moduli spaces of solutions of Nahm equations look like for $s \in (-\infty, 0)$ and $s \in (1, +\infty)$: in the first instance it is a point, and in the second instance it is a Eguchi-Hanson space M_{EH} with distance between the centers $|\vec{p}_2 - \vec{p}_3|$. For $s \in (0,1)$ we now have to analyze the space of solutions of 2×2 Nahm equations with nonsingular boundary behavior. Luckily, this has been done by Dancer [29]. The moduli space is a 16-dimensional hyperkähler manifold N^{16} which has the form $\mathbf{R}^3 \times \mathbf{S}^1 \times N^{12}$. N^{12} is hyperkähler and irreducible. It admits a triholomorphic $SU(2)_L \times SU(2)_R$ action. (We call these two $SU(2)$'s $SU(2)_L$ and $SU(2)_R$ because they originate from the action of the residual gauge group at $s=0$ and $s=1$.)

If we perform the hyperkähler quotient in two steps, as before, on the first step we get $N^{16} \times \mathbf{H}^2 \times M_{EH}$. On the second step we take the hyperkähler quotient with respect to $U(1) \times U(2)$. $U(1)$ acts on the N^{12} part of N^{16} by a maximal torus of $SU(2)_L$, and the level of this quotient is $2\vec{p}_1$. $U(2) = U(1) \times SU(2)$ acts as follows: its $U(1)$ subgroup acts only on $\mathbf{H}^2$ by right multiplication by $\exp(e_1\phi)$, while its $SU(2)$ subgroup acts on all three factors, the action on N^{16} being that of $SU(2)_R$. The resulting manifold is the Coulomb branch of the $U(2)$ gauge theory with three hypermultiplets. "Centering" the monopole moduli space we get the following description of the Coulomb branch of the $SU(2)$ theory with three hypermultiplets: it is a hyperkähler quotient of

$$N^{12} \times M'_{EH} \times M_{EH} \tag{3.11}$$

with respect to $U(1) \times SU(2)$, where M'_{EH} and M_{EH} are Eguchi-Hanson spaces with distances between the centers $|\vec{p}_2 + \vec{p}_3|$ and $|\vec{p}_2 - \vec{p}_3|$, respectively. Here $U(1)$ acts only on N^{12} by a maximal torus of $SU(2)_L$, and the level is $2\vec{p}_1$. $SU(2)$ acts on all three factors, the action on N^{12} being that of $SU(2)_R$.

3.2.5 $n = 2,\ k = 4$

Similarly to the previous example, we regard two monopoles with four singularities as a limit of $SU(4)$ $(2,2,2)$ monopoles. The limit is such that $(2, \ , \ )$ and $(\ , \ ,2)$ monopoles become infinitely heavy. The corresponding brane configuration is shown in Figure 3.3.

We do not spell out in detail the manipulations with hyperkähler quotients, since they are almost the same as in the previous example. We just give the result for the Coulomb branch of the $SU(2)$ gauge theory with four fundamental hypermultiplets: it is a hyperkähler quotient of

$$M_{EH} \times M'_{EH} \times N^{12} \times M''_{EH} \times M'''_{EH} \tag{3.12}$$

with respect to $SU(2)_L \times SU(2)_R$, where, as the names suggest, $SU(2)_L$ acts on

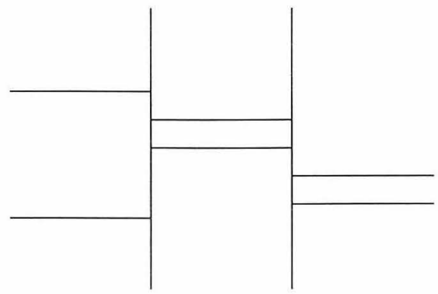


Figure 3.3: A $U(2)$ 2-monopole with four singularities is a limit of a regular $(2,2,2)$ $SU(4)$ monopole.

M_{EH}, M'_{EH}, N^{12}, and $SU(2)_R$ acts on N^{12}, M''_{EH}, M'''_{EH}. The spaces M_{EH}, M'_{EH}, M''_{EH}, and M'''_{EH} are Eguchi-Hanson spaces with distances between the centers $|\vec{p}_1+\vec{p}_2|$, $|\vec{p}_1-\vec{p}_2|$, $|\vec{p}_3+\vec{p}_4|$, and $|\vec{p}_3-\vec{p}_4|$. Arguments from M theory [15] or field theory [16] show that this space is an ALF gravitational instanton of type D_4. Thus we have a rather simple construction of such a space.

3.3 Comparison of Complex Structures of Three and Four-Dimensional Theories

Our description of the metrics on the Coulomb branches is implicit. This is hardly a drawback, since an explicit formula would be horribly complicated (see, e.g., Ref. [25] where the metric corresponding to $n=2,\ k=1$ is discussed). The only exception is the case of $U(1)$ gauge theories where the moduli space is of the multi-Taub-NUT form. For general n it is possible to give an explicit description of the Coulomb branches as *complex manifolds.* We recall that hyperkähler manifolds have three different complex structures $I,\ J,\ K$, but we shall concentrate on just one of them, say I. We shall limit the discussion of the complex structures to the cases studied in the previous section.

Computing the complex structures allows us to perform some checks of the metrics. It follows both from M theory [15] and field theory [16] considerations that the Coulomb branch of $SU(2)$ gauge theory with k fundamentals is an ALF gravitational instanton of type D_k. We shall be able to see that indeed our manifolds are resolutions of D_k singularities, at least for $k \leq 4$. A more detailed check can be performed by comparison with the Seiberg-Witten solutions of the corresponding $\mathcal{N} = 2$ $SU(2)$ gauge theories in four dimensions. Recall that in four dimensions the description of the Coulomb branch of an $\mathcal{N} = 2$ $SU(2)$ gauge theory involves a complex torus fibered over a complex plane [30]. The plane is the moduli space of the theory, while the torus is an auxiliary object whose τ-parameter is the low-energy gauge coupling. Upon compactification to three dimensions the total space of this fibration becomes the moduli space of the corresponding three-dimensional $\mathcal{N} = 4$ theory [16]. Moreover, it was argued [16] that the complex structure remains the same as in four dimensions (of course, after compactification the moduli space grows another two complex structures which we disregard in this section.) We shall see this quite explicitly below.

To compute the complex structure on the moduli space of Nahm equations, we followed the approach of Donaldson [23] (see Section 3.2.1). Above we described the moduli spaces of some $SU(2)$ gauge theories as hyperkähler quotients of known manifolds. By first finding the complex structures of these manifolds and then performing the quotient, we find the complex structures on the Coulomb branches presented below.

3.3.1 $SU(2)$ Theory with One Fundamental Hypermultiplet

The complex structure is given by

$$y^2 = x^2 z + 2px + 1, \tag{3.13}$$

where p is the "complex" part of the hypermultiplet mass parameter $\vec{p}$. The Seiberg-Witten solution [30] of this theory gives the complex structure on the moduli space:

$$y^2 = x^2(x-u) + 2mx + 1, \tag{3.14}$$

where m is the hypermultiplet mass in four dimensions. Obviously, the two complex structures agree after on sets $z = x - u,\ p = m$ in Eq. (3.13).

3.3.2 $SU(2)$ Theory with Two Hypermultiplets

The complex structure is given by

$$y^2 = x^2 z - z + 2xp_1p_2 - (p_1^2 + p_2^2). \tag{3.15}$$

The Seiberg-Witten solution (see [30]) is

$$y^2 = (x^2-1)(x-u) + 2xm_1m_2 - (m_1^2 + m_2^2). \tag{3.16}$$

Eq. (3.15) agrees with Eq. (3.16) if one sets $z = x - u,\ p_1 = m_1,\ p_2 = m_2$.

3.3.3 $SU(2)$ Theory with Three Hypermultiplets

The complex structure is given by

$$y^2 = x^2 z - z^2 - z(p_1^2 + p_2^2 + p_3^2) + 2xp_1p_2p_3 - (p_1^2p_2^2 + p_1^2p_3^2 + p_2^2p_3^2). \tag{3.17}$$

The Seiberg-Witten solution (as in [30]) is

$$\begin{aligned} y^2 &= x^2(x-u) - (x-u)^2 - (x-u)(m_1^2 + m_2^2 + m_3^2) + 2xm_1m_2m_3 \\ &\quad -(m_1^2m_2^2 + m_2^2m_3^2 + m_1^2m_3^2). \end{aligned} \tag{3.18}$$

Eqs. (3.17) and (3.18) agree if one sets $z = x - u,\ p_1 = m_1,\ p_2 = m_2,\ p_3 = m_3$.

3.3.4 $SU(2)$ Theory with Four Hypermultiplets

The complex structure is given by

$$\begin{aligned} y^2 &= x^2 z - z^3 + z^2\left(S_2 - \frac{S_1^2}{2}\right) - z\left(\frac{3S_1^4}{32} - \frac{3S_1^2 S_2}{8} + \frac{S_2^2 + S_1 S_3}{4} + \frac{S_4}{2}\right) \\ &+x\left(\frac{S_1^4}{32} - \frac{S_1^2 S_2}{8} + \frac{S_1 S_3}{4} - \frac{S_4}{2}\right) - \frac{S_1^6}{128} + \frac{3S_1^4 S_2}{64} - \frac{S_1^2 S_2^2 + S_1^3 S_3}{16} \\ &+\frac{S_1 S_2 S_3 - S_3^2}{8} + \frac{S_2 S_4}{4}, \end{aligned} \tag{3.19}$$

where S_ℓ is an elementary symmetric polynomial in $p_1, \ldots, p_4$ of degree ℓ, i.e., $S_1 = p_1 + p_2 + p_3 + p_4, \ldots, S_4 = p_1 p_2 p_3 p_4$. One immediately sees that this is indeed a resolution of D_4 singularity. To compare with the Seiberg-Witten solution in $D = 4$, it is best to think of $SU(2)$ theory with four flavors as a special case of $Sp(2n)$ theory with $2n+2$ flavors. Specializing the formula for the $Sp(2n)$ curve given in Ref. [31] to $n = 1$, we get the following hyperelliptic curve:

$$xy^2 = \left(x(x-u) + g\prod_{l=1}^{4} m_l\right)^2 - g^2 \prod_{l=1}^{4}(x - m_l^2). \tag{3.20}$$

Eqs. (3.19) and (3.20) agree if in Eq. (3.20) one sets $m_l = ip_l/\sqrt{2g},\ l = 1, \ldots, 4$, and makes the following substitution:

$$\begin{aligned} y &\longrightarrow \frac{y}{\sqrt{g}}, \\ x &\longrightarrow \frac{1}{2g}\left(x - z + \frac{S_2}{2} - \frac{S_1^2}{4}\right), \\ u &\longrightarrow \frac{1}{2}\left(x + 3z - \frac{S_2}{2} + \frac{S_1^2}{4}\right) + \frac{1}{2g}\left(x - z + \frac{S_2}{2} - \frac{S_1^2}{4}\right). \end{aligned} \tag{3.21}$$

The above quoted results for the complex structures of the elliptic fibrations which give the solutions to Seiberg-Witten theories were derived by methods completely different from ours. Thus the above comparisons provide a nontrivial check of our descriptions of the moduli spaces.

Chapter 4 Twistor Description of the Coulomb Branch using Nahm Equations

4.1 Nahm Equations

As explained in Section 2.2 (see also [22] and [32]) the configuration of Figure 3.1 can be described by the following Nahm data satisfying the Nahm equations.

Our Nahm data will be a set of four functions T_0, T_j, $j = 1,2,3$ of real coordinate $s \in [-d, \infty)$ taking values in $u(2)$ for $s < 0$ and in $u(k)$ for $s > 0$. The matching condition at $s = 0$ is

$$T_j(s>0) = \begin{pmatrix} -i\rho_j/2s + O(1) & s^{(k-3)/2}p_j + O(s^{(k-1)/2}) \\ -s^{(k-3)/2}\bar{p}_j^T + O(s^{(k-1)/2}) & T_j(0-) + O(s) \end{pmatrix} \tag{4.1}$$

where ρ_j are $k-2$ dimensional representations of Pauli sigma matrices σ_j, p_j are $(k-2) \times 2$ matrices. At $s = -d$ we require $T_j(s) = -\frac{i}{2}\frac{\sigma_j}{s+d} + O(s+d)$, and at $s \to \infty$ $T_j \to diag(x_1^{(j)}, x_2^{(j)}, \dots, x_k^{(j)})$. The parameter d is the distance between the NS5-branes of Figure 3.1 and its inverse will give the radius of the circle at asymptotic infinity of the ALF space.

There is a natural action of a group G of gauge transformations on these Nahm data. G is parametrized by functions $g(s)$ valued in $U(2)$ for $s \in [-d, 0]$ and in $U(k)$ for $s > 0$ and satisfying $g = 1$ at $s = -d$, $g \to 1$ as $s \to \infty$, and

$$g(0+) = \begin{pmatrix} 1 & 0 \\ 0 & g(0-) \end{pmatrix}. \tag{4.2}$$

We subject the Nahm data to the Nahm equations

$$\frac{dT_i}{ds} + [T_0, T_i] = \frac{1}{2}\epsilon_{ijk}[T_j, T_k]. \tag{4.3}$$

The moduli space of such Nahm data modulo gauge transformations is the hyperkähler manifold $\mathcal{M}$ we are interested in.

As a matter of fact, if we consider all quadruplets T_0, T_1, T_2, T_3 valued in $u(2)$ on $[-d, 0)$ and in $u(k)$ on $[0, \infty)$ we can define the norm as

$$||T||^2 = Tr \int_{-d}^{\infty} \left(T_0^2 + T_1^2 + T_2^2 + T_3^2\right) ds. \tag{4.4}$$

The metric corresponding to this norm is hyperkähler. This becomes obvious if we think of a quadruplet $T_0, \ldots, T_3$ as a matrix of quaternions

$$T = T_0 + iT_1 + jT_2 + kT_3.$$

Then the Nahm equations are moment maps for the action of G

$$T_0 \to g^{-1}T_0 g + g^{-1}dg/ds,\ T_j = g^{-1}T_j g. \tag{4.5}$$

We construct $\mathcal{M}$ in the following way. Let us first consider the quotient of the Nahm data on the interval $[0, \infty)$ modulo the gauge transformations with $g(0) = 1$ and call the resulting manifold $\mathcal{M}_+$. Then consider the quotient of the Nahm data on $[-d, 0]$ modulo the gauge transformations with $g(0) = g(-d) = 1$ and call this manifold $\mathcal{D}^{12}$. We can ease the $g(-d) = 1$ condition and consider the gauge transformations with $SU(2)$ part of $g(-d)$ equal to the identity. This gives a triholomorphic action of a $U(1)$ on $\mathcal{D}^{12}$ of $U(1)$ gauge transformations "localized" at $s = -d$. The group of all gauge transformations modulo those with $g(0) = 1$ is $U(2)$. This $U(2)$ can be thought of as "localized" at $s = 0$. It acts triholomorphically on both $\mathcal{M}_+$ and $\mathcal{D}^{12}$. The manifold $\mathcal{M}$ is the hyperkähler quotient of $\mathcal{M}_+ \times \mathcal{D}^{12}$ by the $U(1) \times U(2)$ action.

4.2 The Complex Structure of $\mathcal{M}$

Following Donaldson [23] we define

$$\alpha = \frac{1}{2}(T_0 - iT_1), \ \beta = \frac{1}{2}(T_2 + iT_3). \tag{4.6}$$

Then the Nahm equations can be written as a pair of equations. We shall need only one of them called the "complex" equation:

$$\frac{d\beta}{ds} + 2[\alpha, \beta] = 0. \tag{4.7}$$

One can show that the space of Nahm data satisfying Nahm equations modulo the group G of gauge transformations is the same as the space of solutions of the complex equation Eq. (4.7) modulo the complexified group of gauge transformations $G^{\mathbf{C}}$. This will allow us to describe $\mathcal{M}$ as a complex manifold.

First let us look at $\mathcal{M}_+$. Let $a = s\alpha, b = s\beta, s = e^t$; then

$$\frac{db}{dt} + 2[a, b] = b \tag{4.8}$$

is equivalent to the complex equation Eq. (4.7). Fixing the gauge at $t \to -\infty$ so that a is independent of t we have the solution of Eq.(4.8)

$$a = \begin{pmatrix} -\rho_1/4 & 0 \\ 0 & 0 \end{pmatrix}, b = e^t \begin{pmatrix} e^{\rho_1 t/2} & 0 \\ 0 & 1 \end{pmatrix} \hat{b} \begin{pmatrix} e^{-\rho_1 t/2} & 0 \\ 0 & 1 \end{pmatrix} \tag{4.9}$$

with the matrix $\hat{b}$ independent of t. Comparing with the boundary conditions at $t \to -\infty$ following from Eq. (4.1), we find

$$\alpha = \begin{pmatrix} -\frac{1}{4s}\rho_1 & 0 \\ 0 & 0 \end{pmatrix}, \beta = \begin{pmatrix} -\frac{i}{4s}(\rho_2 + i\rho_3) + A & (-2is)^{(k-3)/2}\tilde{p} \\ (-2is)^{(k-3)/2}\tilde{q} & B \end{pmatrix} \tag{4.10}$$

with $[\rho_1, A] = 0$, $\rho_1\tilde{p} = (k-3)\tilde{p}$, $\tilde{q}\rho_1 = -(k-3)\tilde{q}$. It follows that A is a diagonal

$(k-2)\times(k-2)$ matrix (in the basis in which ρ_1 is diagonal). We call its eigenvalues $A_1,\ldots,A_{k-2}$. Now recall that $\tilde{p}$ and $\tilde{q}$ are $(k-2)\times 2$ and $2\times(k-2)$ matrices. Letting a vector v be the highest weight of the representation ρ (that is $\rho_1 v = (k-3)v$) with $|v|=1$, we see that the two columns of $\tilde{p}$ are $p_1 v$ and $p_2 v$ where $p_1, p_2 \in \mathbf{C}$. So $\tilde{p}$ is parametrized by a vector $p \in \mathbf{C}^2$ with components (p_1, p_2). Similarly, $\tilde{q}$ can be parametrized by a vector $q \in \mathbf{C}^2$.

Boundary conditions at $s \to +\infty$ restrict the eigenvalues of β to be $z_a = (x_a^{(2)} + ix_a^{(3)})/2$. Thus $\det(\beta - z) = 0$ has roots z_a. The computation of the determinant yields

$$\det(\beta - z) = \det(B-z)\left(\det(A-z) - p^T(B-z)^{-1}q\right). \tag{4.11}$$

Turning our attention to $\mathcal{D}^{12}$, the solutions of Nahm equations on $s \in [-d, 0]$ modulo gauge transformations with precisely the right boundary conditions were described by Dancer [26]. As a complex manifold $\mathcal{D}^{12}$ is a set of pairs (B, w) with B being a 2×2 matrix and $w \in \mathbf{C}^2$, such that w and Bw are linearly independent.

So far we described the solutions of the complex equation Eq. (4.7) modulo complexified gauge transformations $g(s)$ which are the identity at $s=0$. Every such solution is given by a set (A, B, p, q, w) where A is a diagonal matrix with eigenvalues $A_1,\ldots,A_{k-2}$, B is a 2×2 matrix, and $p, q, w \in \mathbf{C}^2$. Now we have to take a quotient by the complexified groups $U(1)^{\mathbf{C}} = \mathbf{C}^*$ and $U(2) = \mathbf{C}^* \times SL(2,\mathbf{C})$. The moment map of the $U(1)^{\mathbf{C}}$ is TrB, so we put $TrB = 0$.

An element g of the $SL(2,\mathbf{C})$ acts on these data as

$$B \to gBg^{-1}, w \to gw, p^T \to p^T g^{-1}, q \to gq, \tag{4.12}$$

and two $\mathbf{C}^*$ actions of $\mathbf{C}^*_\lambda$ and $\mathbf{C}^*_\kappa$ can be represented by

$$B \to B, w \to w, p^T \to p^T\lambda^{-1}, q \to \lambda q \quad , \tag{4.13}$$

$$B \to B, w \to \kappa w, p^T \to p^T, q \to q. \tag{4.14}$$

The $\mathbf{C}^*_\kappa$ action and the fact that w and Bw are linearly independent allows to put

$$w^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Bw = 1.$$

Now we can define $SL(2, \mathbf{C})$ invariants x_1, x_2, y_1, and y_2 by

$$p^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = x_1 w^T B^T + x_2 w^T, \tag{4.15}$$

$$q = -y_1 Bw + y_2 w. \tag{4.16}$$

$x_1, x_2, y_1, y_2, \det B$ and $A_1, \ldots, A_{k-2}$ form the full set of invariants with respect to $SL(2, \mathbf{C})$.

An element λ of the residual $\mathbf{C}^*$ acts by $B \to B, x_i \to \lambda^{-1} x_i, y_i \to \lambda y_i$. Thus the invariants of $\mathbf{C}^* \times GL(2, \mathbf{C})$ are given by

$$\begin{aligned} \eta_2 &= -\det B \\ \Psi_1 &= x_2 y_2 \\ \Psi_2 &= x_1 y_1 \\ \Psi_3 &= x_1 y_2 \\ \Psi_4 &= x_2 y_1, \end{aligned} \tag{4.17}$$

and A_b, $b = 1, \ldots, k-2$ with relations

$$\Psi_1 \Psi_2 = \Psi_3 \Psi_4 \tag{4.18}$$

$$\left(z_a^2 - \eta_2\right) \prod_{b=1}^{k-2} (A_b - z_a) + z\left(\Psi_3 - \Psi_4\right) + \Psi_1 - \Psi_2 \eta_2 = 0 \tag{4.19}$$

for $a = 1, \ldots, k$.

Let us denote symmetric polynomials of order m by S_m. For example $S_1(z) =$

$z_1 + z_2 + \ldots + z_k$, and $S_{k-2}(A) = A_1 A_2 \ldots A_{k-2}$. One can rewrite Eqs. (4.18,4.19) as

$$\begin{aligned}
S_1(z) &= S_1(A) \\
S_2(z) &= S_2(A) - \eta_2 \\
S_3(z) &= S_3(A) - \eta_2 S_1(A) \\
&\vdots \\
S_m(z) &= S_m(A) - \eta_2 S_{m-2}(A) \\
&\vdots \\
S_{k-2}(z) &= S_{k-2}(A) - \eta_2 S_{k-4}(A) \\
\Psi_1 &= \eta_2 \Psi_2 + F(\eta_2) \\
\Psi_4 &= \Psi_3 + G(\eta_2)
\end{aligned}$$

$$\eta_2 \Psi_2^2 + \Psi_2 F(\eta_2) = \Psi_3^2 + \Psi_3 G(\eta_2), \tag{4.20}$$

where polynomials $F(\eta_2)$ and $G(\eta_2)$ are given by

$$\begin{aligned}
F(\eta_2) &= S_k(z) + S_{k-2}(z)\eta_2 + \ldots + S_{k-2l}(z)\eta_2^l + \ldots \\
G(\eta_2) &= S_{k-1}(z) + S_{k-3}(z)\eta_2 + \ldots + S_{k-1-2l}(z)\eta_2^l + \ldots .
\end{aligned} \tag{4.21}$$

The above equations define our complex manifold $\mathcal{M}$ as a subvariety in $\mathbf{C}^{k+3}$. From Eqs. (4.20,4.21) one can see that $\mathcal{M}$ develops a D_k-type singularity when all $z_a = 0$.

4.3 The Twistor Space of $\mathcal{M}$

In the previous section we described $\mathcal{M}$ as a complex variety after picking a particular complex structure on the Nahm data. In reality there is a whole sphere of such complex structures. Thus to obtain the twistor space of $\mathcal{M}$ we need to trace the dependence on the choice of complex structure.

Parametrizing the sphere of complex structures by ζ we define

$$\begin{aligned}
\alpha^0 &= \frac{1}{2}\left(T_0 + iT_1 + \zeta\left(T_3 - iT_2\right)\right), \\
\beta^0 &= \frac{i}{2}\left(T_3 + iT_2 + 2i\zeta T_1 + \zeta^2\left(T_3 - iT_2\right)\right)
\end{aligned} \tag{4.22}$$

for $\zeta \neq \infty$ and

$$\begin{aligned}
\alpha^1 &= \frac{1}{2}\left(T_0 - iT_1 - \frac{1}{\zeta}\left(T_3 + iT_2\right)\right), \\
\beta^1 &= \frac{1}{2}\left(T_2 + iT_3 - 2\frac{1}{\zeta}T_1 + \frac{1}{\zeta^2}\left(-T_2 + iT_3\right)\right)
\end{aligned} \tag{4.23}$$

for $\zeta \neq 0$. Both pairs (α, β) satisfy the complex Nahm equation

$$\frac{d\beta}{ds} + 2[\alpha, \beta] = 0. \tag{4.24}$$

The relation between them is given by

$$\alpha^1 = \alpha^0 + \frac{i}{\zeta}\beta^0, \quad \beta^1 = \frac{1}{\zeta^2}\beta^0. \tag{4.25}$$

Following the same steps as in the Section 4.2 and fixing the gauge so that $s\alpha$ is constant, we obtain

$$\begin{aligned}
\alpha^0 &= \begin{pmatrix} \frac{1}{4s}\left(\rho_1 - 2\zeta\rho_+\right) & 0 \\ 0 & 0 \end{pmatrix}, \\
\beta^0 &= \begin{pmatrix} \frac{i}{2s}\left(\rho_- + \zeta\rho_1 - \zeta^2\rho_+\right) + e^{-\zeta\rho_+}A^0(\zeta)e^{\zeta\rho_+} & s^{(k-3)/2}e^{-\zeta\rho_+}p^0(\zeta) \\ s^{(k-3)/2}q^0(\zeta) & B^0(\zeta) \end{pmatrix}
\end{aligned} \tag{4.26}$$

and

$$\alpha^1 = \begin{pmatrix} -\frac{1}{4s}\left(\rho_1 + \frac{2}{\zeta}\rho_-\right) & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.27}$$

$$\beta^1 = \begin{pmatrix} -\frac{i}{2s}\left(\rho_+ - \frac{1}{\zeta}\rho_1 - \frac{1}{\zeta^2}\rho_-\right) + e^{-\rho_-/\zeta}A^1(\zeta)e^{\rho_-/\zeta} & s^{(k-3)/2}p^1(\zeta) \\ s^{\frac{k-3}{2}}q^1(\zeta)e^{\frac{\rho_-}{\zeta}} & B^1(\zeta) \end{pmatrix}.$$

In these formulas $\rho_+ = \frac{1}{2}(\rho_2 + i\rho_3), \rho_- = \frac{1}{2}(\rho_2 - i\rho_3)$,

$$[A^0(\zeta), \rho_1] = 0,\ [A^1(\zeta), \rho_1] = 0 \tag{4.28}$$

$$\rho_1 p_0(\zeta) = -(k-3)p_0(\zeta),\ q_0(\zeta)\rho_1 = (k-3)q_0(\zeta) \tag{4.29}$$

$$\rho_1 p_1(\zeta) = (k-3)p_0(\zeta),\ q_1(\zeta)\rho_1 = -(k-3)q_1(\zeta). \tag{4.30}$$

Equation (4.25) relates (α^0, β^0) and (α^1, β^1), but in a gauge different from that in Eqs. (4.26,4.27). Namely the gauge transformation that makes $s\alpha^1 = s\left(\alpha^0 + \frac{i}{\zeta}\beta^0\right)$ independent of s is

$$g = e^{-\frac{2i}{s\zeta}\beta^0} e^{-\frac{1}{\zeta}\rho_- - \rho_1 + \zeta\rho_+}. \tag{4.31}$$

Comparing the two expressions for β^1 we can find the transition functions between (A^0, p^0, q^0, B^0) and (A^1, p^1, q^1, B^1). If we let

$$R = e^{\rho_-/\zeta} e^{-\zeta\rho_+} e^{\rho_-/\zeta} \tag{4.32}$$

then

$$A^1(\zeta) = RA^0R^{-1}, p^1 = \frac{1}{\zeta^2}Rp^0, q_1 = \frac{1}{\zeta^2}q^0R^{-1}, B^1 = \frac{1}{\zeta^2}B^0. \tag{4.33}$$

Thus we conclude that the variables in Eq. (4.12) are sections of the following bundles:

$$\begin{gathered} B \in Mat(2) \otimes \mathcal{O}(2), A_b \in \mathcal{O}(2), \\ p \in \mathcal{O}(k-1) \times \mathcal{O}(k-1), q \in \mathcal{O}(k-1) \times \mathcal{O}(k-1). \end{gathered} \tag{4.34}$$

Above $Mat(2)$ denotes the space of 2×2 matrices and $\mathcal{O}(n)$ is the line bundle on $\mathbf{P}^1$ with the transition function ζ^{-n}. Furthermore, from Dancer's analysis it follows that $w^1 = \zeta e^{2idB^0/\zeta} w^0$. From the above one can find the dependence of $\Psi_1, \ldots, \Psi_4$ on ζ. This completely determines the twistor space $\mathcal{Z}$ of $\mathcal{M}$.

It turns out that the Ψ_i are not taking values in any nice fibrations, but one can define combinations of them that do. Let $\eta = \sqrt{-\eta_2}$, and define

$$\begin{aligned} \mu &= \Psi_1 + \eta^2\Psi_2 - i\eta(\Psi_3 - \Psi_4), \\ \nu &= \Psi_1 + \eta^2\Psi_2 + i\eta(\Psi_3 - \Psi_4), \\ \rho &= \Psi_1 - \eta^2\Psi_2 + i\eta(\Psi_3 + \Psi_4), \\ \xi &= \Psi_1 - \eta^2\Psi_2 - i\eta(\Psi_3 + \Psi_4). \end{aligned} \tag{4.35}$$

These are two-valued functions of ζ as $\eta(\zeta) = \sqrt{-\eta_2(\zeta)}$, but they have simple transformation laws:

$$\tilde{\mu} = \zeta^{-2k}\mu, \ \tilde{\nu} = \zeta^{-2k}\nu, \tag{4.36}$$

$$\tilde{\rho} = \zeta^{-2k}e^{2d\eta/\zeta}\rho, \tag{4.37}$$

$$\tilde{\xi} = \zeta^{-2k}e^{-2d\eta/\zeta}\xi. \tag{4.38}$$

Then from equations (4.20)

$$\mu = \prod_{a=1}^{k}\left(z_a(\zeta) + i\eta\right), \tag{4.39}$$

$$\nu = \prod_{a=1}^{k}\left(z_a(\zeta) - i\eta\right), \tag{4.40}$$

and

$$\rho\,\xi = \prod_{a=1}^{k}\left(z_a^2(\zeta) + \eta^2\right). \tag{4.41}$$

From this description of $\mathcal{Z}$ one can see that it is exactly the twistor space of the centered moduli space of two monopoles with k singularities described in Ref. [33]. In the direct image sheaf construction of Ref. [33] $\rho_0\xi_0, \rho_1\xi_1, \rho_0\xi_1$, and $\rho_1\xi_0$ correspond to Ψ_1, Ψ_2, Ψ_3, and Ψ_4. This establishes the isometry between the moduli space $\mathcal{M}$ and the centered moduli space of singular monopoles of nonabelian charge two.

The twistor space $\mathcal{Z}$ is nothing but $\mathbf{P}^1 \times \mathcal{M}$, so the fiber of $\mathcal{Z}$ over ζ is our manifold $\mathcal{M}$ with the complex structure determined by ζ. In order to recover $\mathcal{M}$ as a Riemannian manifold we need to pick two complex coordinates on $\mathcal{M}$ that depend on

ζ holomorphically. For example, locally we can pick (η, ρ) or (η, ξ) as such coordinates. There is a natural two-form $\omega = 2d\eta \wedge d\xi/\xi = -2d\eta \wedge d\rho/\rho$ on $\mathcal{Z}$. It is degenerate along the ζ direction and satisfies $\tilde{\omega} = \zeta^{-2}\omega$. Let us rewrite it as

$$\omega = d\eta \wedge d\log\frac{\xi}{\rho}. \tag{4.42}$$

Then we can introduce a new coordinate

$$\chi = \frac{1}{\eta}\log\frac{\rho}{\xi} = \frac{1}{\eta}\log\frac{\Psi_1 - \eta^2\Psi_2 - i\eta(\Psi_3 + \Psi_4)}{\Psi_1 - \eta^2\Psi_2 + i\eta(\Psi_3 + \Psi_4)} \tag{4.43}$$

which does not depend on the choice of the branch of the square root in $\eta = \sqrt{-\eta_2}$. In terms of η_2 and χ, $\omega = d\eta_2 \wedge d\chi$. Now we have to find χ as a function on ζ.

Chapter 5 Twistor Description of the Coulomb Branch using Monopoles

5.1 Regular Monopoles

To describe a monopole with singularities, we shall use the minitwistor space approach of Hitchin (see [18]) modified in [34]. Our main objects of interest will be the space $\mathbf{T}$ of lines in R^3, the spectral curve S, and holomorphic bundles over $\mathbf{T}$ and over S. Every line γ in a three-dimensional space is $\{\vec{x} \mid \vec{x} = \vec{u}t + \vec{v}\}$ where $\vec{u}$ is a unit vector and $\vec{v}$ is orthogonal to $\vec{u}$ and t is a real coordinate along the line. So the space of oriented lines $\mathbf{T}$ is the tangent space (with the tangent space parametrized by $\vec{v}$) to a unit sphere (parametrized by $\vec{u}$). We can introduce complex coordinates (ζ, η) on $\mathbf{T}$. ζ is the complex coordinate on the sphere and $\eta\frac{\partial}{\partial\zeta}$ is a holomorphic tangent vector. Now the set of lines passing through any given point (x_1, x_2, x_3) is given by the section

$$\eta = (-x_1 + ix_2) + 2x_3\zeta + (x_1 + ix_2)\zeta^2 \tag{5.1}$$

of the bundle $\mathbf{T}$. Inversion of orientation of all lines maps $\mathbf{T}$ onto itself and gives a real structure on $\mathbf{T}$. Every holomorphic section of $\mathbf{T}$ is given by $a\zeta^2 + 2b\zeta + c$ for some complex numbers a, b and c. Let us recall that the real structure τ acts by $(\zeta, \eta) \to (-1/\bar{\zeta}, -\eta)$, so the section is real (i.e., it is mapped into itself by the action of τ) when it is of the form (5.1). So we can think of points in R^3 as real sections of $\mathbf{T}$.

Now we describe a bundle E over $\mathbf{T}$ that encodes all the information about the monopole configuration. For every line γ the equation

$$\nabla_{\vec{u}}s - i\Phi s = 0, \tag{5.2}$$

with the covariant derivative $\nabla_{\vec{u}} = \vec{u} \cdot \vec{D}$ along the line, has two solutions. This space of solutions is the fiber E_γ of the bundle E. The condition for this bundle to be holomorphic is equivalent to the Bogomolny equation for the connection A_i and Higgs field Φ.

There are natural subbundles L^+ and L^- in E. L^+ consists of solutions to equation (5.2) that decay as $t \to +\infty$, and L^- of those decaying as $t \leftarrow -\infty$. The points (ζ, η) of $\mathbf{T}$ above which $L^+_{(\zeta,\eta)} = L^-_{(\zeta,\eta)}$ form the spectral curve S. In other words the points of the spectral curve S are lines in R^3 for which equation (5.2) has a solution decaying at large $|t|$.

The spectral curve corresponding to a monopole of charge N is given by equation $F(\eta, \zeta) = \eta^N + a_1(\zeta)\eta^{N-1} + \ldots + a_N(\zeta) = 0$ and is real in $\mathbf{T}$. The spectral curve determines the monopole solution up to gauge transformations.

For future use let us define another bundle L over $\mathbf{T}$. It has a one-dimensional fiber given by solutions of $\partial_{\vec{u}} s - is = 0$. In terms of L, L^+ and L^- are isomorphic to $L(-N)$ (that is the bundle L twisted by $\mathcal{O}(-N)$) and $L^*(-N)$ (bundle L^* twisted by $\mathcal{O}(-N)$).

5.2 Singular Monopoles

To construct the moduli space of singular $U(2)$ monopoles, we shall use a version of Ward correspondence due to Hitchin [18]. The set of all oriented straight lines $\mathbf{T}$ in $\mathbf{R}^3$ has a natural complex structure, as it is the tangent bundle of the projective line. $\mathbf{T}$ can be covered by two patches $V_0(\zeta \neq \infty)$ and $V_1(\zeta \neq 0)$ with coordinates (η, ζ) and $(\eta', \zeta') = (\eta/\zeta^2, 1/\zeta)$. For any point $\vec{x} \in \mathbf{R}^3$ the set of all oriented straight lines through x sweeps out a projective line $P_x \in \mathbf{T}$; thus there is a holomorphic map $P_x : \mathbf{P}^1 \to \mathbf{T}$. The reversal of the orientation of lines in $\mathbf{R}^3$ is an antiholomorphic map $\tau : \mathbf{T} \to \mathbf{T}$ satisfying $\tau^2 = id$. It is called the real structure of $\mathbf{T}$. For any $\vec{x}$ it acts on P_x as the antipodal map. Thus P_x is a real holomorphic section of $\mathbf{T}$.

For any straight line in $\mathbf{R}^3$

$$\gamma = \{\vec{x} | \vec{x} = \vec{u}t + \vec{v}, \vec{u}\cdot\vec{u} = 1, \vec{u}\cdot\vec{v} = 0\}$$

let

$$\begin{aligned}\gamma_+ &= \{\vec{x} | \vec{x} = \vec{u}t + \vec{v}, t > R\}, \\ \gamma_- &= \{\vec{x} | \vec{x} = \vec{u}t + \vec{v}, t < -R\},\end{aligned} \tag{5.3}$$

where R is a positive number greater than any $|\vec{p}_\alpha|$. Now we define two complex rank 2 vector bundles E^+ and E^- over $\mathbf{T}$:

$$\begin{aligned}E^+ &= \{s \in \Gamma(\gamma_+, E) \,|D_\gamma s = i\Phi s\}, \\ E^- &= \{s \in \Gamma(\gamma_-, E) \,|D_\gamma s = i\Phi s\}.\end{aligned} \tag{5.4}$$

From Bogomolny equations it follows, as in Ref. [18], that these bundles are holomorphic. The real structure τ on $\mathbf{T}$ can be lifted to an antilinear antiholomorphic map

$$\sigma : E^+ \to (E^-)^*.$$

Thus every solution of $U(2)$ Bogomolny equations maps to a pair of holomorphic rank two bundles on $\mathbf{T}$ interchanged by the real structure.

Let P_x denote the real section corresponding to $\vec{x}$, and P_α the real section corresponding to $\vec{p}_\alpha$. Let P be the union of all P_α. If γ does not pass through any $\vec{p}_\alpha$, any solution s can be continued from γ_+ to γ_-. This defines a natural identification of the fibers E^+_γ and E^-_γ. This identification is determined everywhere on $\mathbf{T}bfP$ (i.e., on the minitwistor space $\mathbf{T}$ excluding P). Therefore, we have an isomorphism

$$h : E^+|_{\mathbf{T}\backslash P} \to E^-|_{\mathbf{T}\backslash P}, \tag{5.5}$$

where $B_{\mathbf{T}\backslash P}$ denotes restriction of the bundle B to its part above the points of the base $\mathbf{T}$ that do not belong to P.

For nonsingular monopoles h extends to an isomorphism over the whole $\mathbf{T}$; therefore, the Ward correspondence maps a nonsingular monopole to a holomorphic bundle over $\mathbf{T}$. In the present case h or h^{-1} may have singularities at P, and the Ward correspondence maps a singular monopole into a triplet (E^+, E^-, h). This triplet satisfies a certain triviality constraint which we now proceed to formulate.

For any $\vec{x}$ distinct from all $\vec{p}_\alpha$, the intersection $P_x \bigcap P$ consists of an even number of points. For a generic $\vec{x}$ the cardinality of $Q_x = P_x \bigcap P$ is $2k$. For any $\vec{x}$ we can arbitrarily split Q into two sets of equal cardinality Q_x^+ and Q_x^- and construct a vector bundle E_x over P_x by gluing together E^+ restricted to $P_x \backslash Q_x^+$ and E^- restricted to $P_x \backslash Q_x^-$, with the transition function h. (Of course, E_x depends on the splitting.) The triviality constraint is that for any $\vec{x}$ there is a splitting $Q_x = Q_x^+ \bigcup Q_x^-$ continuous in $\vec{x}$ such that E_x is trivial.

Now we state the Ward correspondence between singular $U(2)$ monopoles and twistor data. There is a bijection[1] between singular monopoles modulo gauge transformations and pairs (E^+, E^-) of holomorphic rank 2 bundles over $\mathbf{T}$ equipped with an isomorphism Eq. (5.5) satisfying the following conditions:

(a) For any $\vec{x} \neq p_\alpha$ there is a splitting $Q_x = Q_x^+ \bigcup Q_x^-$ such that E_x is trivial.

(b) In the vicinity of each point of P, there exist trivializations of E^+ and E^- such that h takes the form

$$h = \begin{pmatrix} 1 & 0 \\ 0 & \prod_\alpha (\eta - P_\alpha(\zeta)) \end{pmatrix}, \tag{5.6}$$

so that h extends across P to a morphism $E^+ \to E^-$.

(c) The real structure τ on $\mathbf{T}$ lifts to an antilinear antiholomorphic map $\sigma : E^+ \to (E^-)^*$.

Let us explain where (a) and (b) come from. The condition (b) arises from studying the behavior of the solutions of the equation $D_\gamma s = i\Phi s$ as γ approaches p_α .

[1]The injectivity of the Ward correspondence can be shown by a straightforward modification of the argument in Ref. [18]. We conjecture surjectivity by analogy with the nonsingular case.

Details can be found in Ref. [14]. (There the $SU(2)$ case was analyzed, but the extension to $U(2)$ is straightforward.) To demonstrate (a) it is sufficient to exhibit a holomorphic trivialization of E_x. Take any $\vec{x} \neq \vec{p}_\alpha, \alpha = 1, \dots, k$ and recall that P_x consists of all straight lines γ passing through $\vec{x}$. To obtain a holomorphic section of E_x, pick a vector v_1 in the fiber of B over $\vec{x}$ and take it as an initial condition for the equation $D_\gamma s = i\Phi s$ at $t = 0$. Integrating it forward and backward in t and varying γ yields sections of E^+ and E^- related by h. It is easy to check that they are holomorphic and thereby combine into a holomorphic section s_1 of E_x. To get a section s_2 of E_x linearly independent from s_1, just pick a vector v_2 linearly independent from v_1 and repeat the procedure. (This argument has to be modified if there is a straight line γ passing through $\vec{x}$ and $\alpha, \beta \in \{1, \dots, k\}$ such that $\vec{p}_\alpha$ and $\vec{p}_\beta$ lie on γ and $\vec{x}$ separates them. In this case one of the vectors v_1, v_2 has to be varied, $v_i \sim \zeta^{-1}$, as one varies γ.)

We now want to encode the twistor data in an algebraic curve $S \subset \mathbf{T}$, in the spirit of Ref. [18]. We denote by $L^x(m)$ a line bundle over $\mathbf{T}$ with the transition function $\zeta^{-m} e^{-x\eta/\zeta}$ from V_0 to V_1. Let L_1^+ be a line subbundle of E^+ which consists of solutions of $D_\gamma s = i\Phi s$ bounded by $const \cdot \exp(-\mu_1 t) t^n$ as $t \to +\infty$. Similarly, a line bundle $L_1^- \subset E^-$ consists of solutions bounded by $const \cdot \exp(-\mu_2 t) t^{-n'}$. The line bundles L_2^+ and L_2^- are defined by

$$L_2^+ = E^+ / L_1^+, \qquad L_2^- = E^- / L_1^-.$$

As in Ref. [18] the asymptotic conditions on the Higgs field can be used to show that $L_{1,2}^+$ and $L_{1,2}^-$ are holomorphic line bundles, and that the following isomorphisms hold:

$$L_1^+ \simeq L^{\mu_1}(-n),\ L_2^+ \simeq L^{\mu_2}(n'),\ L_1^- \simeq L^{\mu_2}(-n'),\ L_2^- \simeq L^{\mu_1}(n).$$

Consider a composite map

$$\psi : L_1^+ \to E^+ \to E^- \to L_2^-,$$

where the first arrow is an inclusion, the second arrow is h, and the third arrow is a natural projection. We may regard ψ as an element of $H^0(\mathbf{T}, \mathcal{O}(2n))$. Let us define the spectral curve S to be the zero level of ψ. S is in the linear system $\mathcal{O}(2n)$. Arguments identical to those in Ref. [18] can be used to prove that S is compact and real (i.e., $\tau(S) = S$).

Consider now a map $\phi : \det E^+ \to \det E^-$ induced by h. By virtue of Eq. (5.6) the zero level of ϕ is precisely P. We shall assume in what follows that S does not contain any of P_α as components. Physically this corresponds to the requirement that none of the nonabelian monopoles was located at $\vec{x} = \vec{p}_\alpha$. For simplicity we shall also assume that $S \bigcap P$ consists of $2nk$ points (this is a generic situation).

The construction here bears a close resemblance to that in Ref. [21], where nonsingular monopoles for all classical groups were constructed. According to Ref. [21], the spectral data for nonsingular $SU(3)$ monopoles with magnetic charge (k, n) include a pair of spectral curves S_1, S_2 in the linear systems $\mathcal{O}(2n), \mathcal{O}(2k)$. Our S and P are analogs of S_1 and S_2. The condition that $S \bigcap P$ consists of $2nk$ points is analogous to the requirement in Ref. [21] that monopoles be generic. (This resemblance is not a coincidence: if we consider an $SU(3)$ gauge theory broken down to $SU(2) \times U(1)$ by a large vev of an adjoint Higgs field, the (k, n) monopoles of $SU(3)$ reduce to singular monopoles of $SU(2) \times U(1)$ with nonabelian charge n and abelian charge k. In this limit the spectral data of Ref. [21] must reduce to ours.)

Since $L_1^+|_S = \ker \psi|_S$, we have a well-defined holomorphic map $\rho : L_2^+|_S \to L_2^-|_S$ induced by h. There is also a holomorphic map $\xi : L_1^+|_S \to L_1^-|_S$ induced by h. Thus we have natural elements $\rho \in H^0\left(S, L^\mu(k)\right)$ and $\xi \in H^0\left(S, L^{-\mu}(k)\right)$. It also easily follows from the definition that $\rho \otimes \xi = \phi|_S$, and therefore the divisors of both ρ and ξ are subsets of $S \bigcap P$. ρ and ξ are interchanged by real structure, and therefore the same is true about their divisors. It follows that the divisors of ρ and ξ are disjoint and have equal cardinality. Thus we can define the spectral data for a generic singular monopole to consist of

(i) A spectral curve S, which is a real compact curve in the linear system $\mathcal{O}(2n)$ such that $S \bigcap P$ consists of $2nk$ disjoint points.

(ii) A splitting $S \bigcap P = Q^+ \bigcup Q^-$ into sets of equal cardinality interchanged by τ.

(iii) A section ρ of $L^{\mu}(k)|_S$ with divisor Q^+ and a section ξ of $L^{-\mu}(k)|_S$ with divisor Q^-. ρ and ξ are interchanged by real structure.

The condition (iii) is a constraint on S. It implies that ρ and ξ satisfy

$$\rho\xi = \prod_{\alpha}(\eta - P_{\alpha}(\zeta)). \tag{5.7}$$

For nonsingular monopoles it reduces to the requirement that $L^{\mu}|_S$ be trivial, as in Ref. [18]. As a consequence of (iii), $L^{2\mu}|_S\,[Q_- - Q_+]$ is trivial.

Recall that the spectral data for nonsingular $SU(2)$ monopoles satisfy an additional constraint, the "vanishing theorem" of Ref. [19]. It says that $L^{z\mu}(n-2)$ is nontrivial for $z \in (0,1)$. A natural guess for the analogue of this condition in our case is

(iv) $L^{z\mu}(n-2)\,[-Q_+]$ is nontrivial for $z \in (0,1)$.

We already mentioned a close connection of the spectral data for singular $U(2)$ monopoles and those for nonsingular $SU(3)$ monopoles [21] with the largest Higgs vev set to $+\infty$. Consequently, one can obtain the condition (iv) from the "vanishing theorem" of Ref. [21] by taking the appropriate limit. A direct derivation of (iv) should also be possible.

Arguments very similar to those in Ref. [18] show that the spectral data determine the singular monopole uniquely. A natural question is if there is a one-to-one correspondence between singular $U(2)$ monopoles and spectral data defined by (i-iv). The answer was positive for nonsingular $SU(2)$ monopoles [19], so it is highly plausible that the same is true in the present case. Presumably a proper proof of this can be achieved by converting the spectral data into solutions of Nahm equations, and then reconstructing singular monopoles by an inverse Nahm transform. We leave this as a problem for the future.

5.3 Twistor Space for Singular Monopoles

Having established the correspondence between singular $U(2)$ monopoles and algebraic data on $\mathbf{T}$, we now proceed to construct the twistor space (in the sense of Penrose) for the monopole moduli space. We follow the method of Ref. [7]. For fixed $\zeta = \zeta_0$ every point in Z_n yields a spectral curve S which intersects the fiber of $\mathbf{T}$ over ζ_0 at n points. Thus we have a projection

$$Z_n \to \oplus_{j=1}^{n} \mathcal{O}(2j) = Y_n.$$

Concretely, if S is given by $\eta^n + \eta_1\eta^{n-1} + \cdots + \eta_n = 0$, the corresponding point in Y_n is $(\eta_1, \ldots, \eta_n)$. Now consider an n-fold cover of Y_n

$$X_n = \left\{(\eta, \eta_1, \ldots, \eta_n) \in \mathcal{O}(2) \oplus Y_n | \eta^n + \eta_1\eta^{n-1} + \cdots + \eta_n = 0\right\}.$$

There are two natural projections $\pi_1 : X_n \to \mathbf{T}$ and $\pi_2 : X_n \to Y_n$. Using these projections, we get a rank n bundle V^+ over Y_n as a direct image sheaf $V^+ = \pi_{2*}\pi_1^* L^\mu(k)$. Similarly, we get a rank n bundle $V^- = \pi_{2*}\pi_1^* L^{-\mu}(k)$. For any point in Z_n we have a section ρ of $L^\mu(k)|_S$ and a section ξ of $L^{-\mu}(k)|_S$. Therefore, there is an inclusion $Z_n \subset V^+ \oplus V^-$. To describe this inclusion more concretely, we must rewrite the condition (iii) in terms of sections of $V^\pm$. The result is as follows. Let U be a $2n+1$-dimensional subvariety in $\mathbf{C}^{3n+1}$ with coordinates $(\zeta, \eta_1, \ldots, \eta_n, \rho_0, \ldots, \rho_{n-1}, \xi_0, \ldots, \xi_{n-1})$ defined by

$$\begin{aligned}&(\rho_0 + \rho_1\eta + \cdots + \rho_{n-1}\eta^{n-1})(\xi_0 + \xi_1\eta + \cdots + \xi_{n-1}\eta^{n-1}) = \prod_\alpha(\eta - P_\alpha(\zeta)),\\ &\text{mod } \eta^n + \eta_1\eta^{n-1} + \cdots + \eta_n = 0. \end{aligned} \tag{5.8}$$

Take two copies of U and glue them together over $\zeta \neq 0, \infty$ by

$$\begin{aligned}&\tilde{\zeta} = \zeta^{-1}, \\ &\tilde{\eta}_j = \zeta^{-2j}\eta_j, \; j = 1, \ldots, n,\end{aligned} \tag{5.9}$$

$$\tilde{\rho}_0 + \tilde{\rho}_1\eta + \cdots + \tilde{\rho}_{n-1}\eta^{n-1} = e^{-\mu\eta/\zeta}\zeta^{-k}(\rho_0 + \rho_1\eta + \cdots + \rho_{n-1}\eta^{n-1}),$$
$$\tilde{\xi}_0 + \tilde{\xi}_1\eta + \cdots + \tilde{\xi}_{n-1}\eta^{n-1} = e^{\mu\eta/\zeta}\zeta^{-k}(\xi_0 + \xi_1\eta + \cdots + \xi_{n-1}\eta^{n-1}),$$

all modulo $\eta^n + \eta_1\eta^{n-1} + \cdots + \eta_n = 0$. The resulting $2n+1$-dimensional variety is Z_n, the twistor space of singular monopoles with nonabelian charge n.

To reconstruct the hyperkähler metric from the twistor space, one has to find a holomorphic section of $\Lambda^2 T_F^* \otimes \mathcal{O}(2)$, where T_F^* is the cotangent bundle of the fiber of Z_n. Upon restriction to any fiber of Z_n, this section must be closed and nondegenerate. An obvious choice (the same as in Ref. [7]) is

$$\omega = 4\sum_{j=1}^{n} \frac{d\rho(\beta_j) \wedge d\beta_j}{\rho(\beta_j)}, \tag{5.10}$$

where β_j are roots of $\eta^n + \eta_1\eta^{n-1} + \cdots + \eta_n = 0$.

Chapter 6 Coulomb Branch Metric from its Twistor Space

6.1 Explicit Construction for $U(1)$ Theories

In this section we shall concentrate on the case of one monopole with k singularities. As explained in Section 2.1, in the IIB picture this corresponds to having one internal and k external D3-branes. The spectral curve is therefore a sphere. In the notation of Section 1.3.3, sections ρ and ξ map it into the space

$$Z = \left\{(x, y) \in L(k) \oplus L^*(k) \mid xy = \prod(\eta - P_\alpha)\right\}.$$

Also the image of the spectral curve under this map $\hat{S}$ is real with respect to the real structure induced by the change of orientation of lines in R^3 described in Section 5.1. Thus our spectral data consist of a real section of Z. In the rest of this section we show that Z is the twistor space of an A_{k-1} ALF space with $V = 2 + \sum_{\alpha=1}^{k} 1/r_\alpha$ following Hitchin [35], concluding that the moduli space is the above-mentioned ALF space.

We need to find solutions to

$$xy = \prod_{\alpha=1}^{k} (\eta - P_\alpha). \tag{6.1}$$

As the spectral curve is real, $\eta(\zeta) = a\zeta^2 + 2b\zeta - \overline{a}$. For $P_\alpha(\zeta) = a_\alpha\zeta^2 + 2b_\alpha\zeta - \overline{a_\alpha}$ let u_α and v_α be the roots of $\eta(\zeta) - P_\alpha(\zeta)$. Then

$$u_\alpha = \frac{-(b - b_\alpha) + \Delta_\alpha}{a - a_\alpha}, \qquad v_\alpha = \frac{-(b - b_\alpha) - \Delta_\alpha}{a - a_\alpha}, \tag{6.2}$$

where

$$\Delta_\alpha = \sqrt{(b - b_\alpha)^2 + (a - a_\alpha)(\overline{a} - \overline{a}_\alpha)}. \tag{6.3}$$

Then the solutions of equation (6.1), valued in appropriate bundles, are

$$x = Ae^{-b - a\zeta} \prod_{\alpha=1}^{k} (\zeta - u_\alpha), \quad y = Be^{b + a\zeta} \prod_{\alpha=1}^{k} (\zeta - v_\alpha) \tag{6.4}$$

with $AB = \prod (a - a_\alpha)$. From the reality condition

$$A\overline{A} = \prod (b - b_\alpha + \Delta_\alpha). \tag{6.5}$$

Tangent vectors to the space of all holomorphic sections (not only real ones) are (a^t, b^t, c^t, A^t) with $\eta(\zeta) = a\zeta^2 + 2b\zeta + c$. Two sections are null separated if they intersect. Thus the vector is null if there is a ζ such that

$$\eta^t(\zeta) = a^t\zeta^2 + 2b^t\zeta + c^t = 0, \tag{6.6}$$

$$\frac{x^t(\zeta)}{x(\zeta)} = \frac{A^t}{A} - b^t - a^t\zeta - \sum \frac{u^t_\alpha}{\zeta - u_\alpha} = 0. \tag{6.7}$$

Relating the variation u^t_α of the root u_α to a^t and b^t, we find from (6.6) and (6.7)

$$\zeta \left(\frac{2A^t}{A} + 2b^t - a^t \sum \frac{u_\alpha}{\Delta_\alpha} \right) + \left(2 + \sum \frac{1}{\Delta_\alpha} \right) = 0. \tag{6.8}$$

Defining

$$V = 2 + \sum \frac{1}{\Delta_\alpha}, \quad D = \sum \frac{u_\alpha}{\Delta_\alpha}, \tag{6.9}$$

and substituting ζ found from (6.8) into (6.6), we obtain the condition for the vector to be null

$$\left(\frac{2A^t}{A} + (2 - V)\, b^t - \Delta a^t \right)^2 + V^2 \left(a^t c^t - \left(b^t\right)^2 \right) = 0. \tag{6.10}$$

This defines the conformal structure on the space of sections.

It follows from condition (6.5) that for a real section

$$Re\left(\frac{2A^t}{A}\right) = b^t(V-2) + Re\left(Da^t\right). \tag{6.11}$$

So the conformal structure on real sections is given by

$$V^2\left(a^t\overline{a}^t + \left(b^t\right)^2\right) + \left(Im\left(\frac{2A^t}{A} - Da^t\right)\right)^2. \tag{6.12}$$

To fix the conformal factor of the metric, we compare the volume given by the metric with the volume form $\omega = dy \wedge d\eta \wedge d\overline{y} \wedge d\overline{\eta}/|y|^2$ inherited from C^3. We obtain

$$(ds)^2 = V\left(a^t\overline{a}^t + \left(b^t\right)^2\right) + V^{-1}\left(Im\left(\frac{2A^t}{A} - Da^t\right)\right)^2. \tag{6.13}$$

Recalling the interpretation (5.1) of real sections of $\mathbf{T}$, we have

$$(ds)^2 = V d\vec{x}\cdot d\vec{x} + V^{-1}\left(d\tau + \vec{\omega}\cdot d\vec{x}\right)^2, \tag{6.14}$$

where $V = 2 + \sum 1/|\vec{x} - \vec{p}_\alpha|$ and grad $V =$ curl $\vec{\omega}$.

Thus Z is indeed the twistor space of A_{k-1} ALF space with metric (6.14). An alternative way to see this is by following the construction of Gibbons and Rychenkova [28] as in section 3.6 of [36] tracing the dependence on the choice of complex structure.

6.2 Kähler Potential via Generalized Legendre Transform

6.2.1 Moduli Space M_1 of the $n = 1$ Monopole ($U(1)$ Theories Revisited)

Specializing the formulas of Chapters 4 and 5 to $n = 1$, we get that the twistor space Z_1 is a hypersurface in the total space of $L^{\mu}(k) \oplus L^{-\mu}(k)$

$$\rho_0 \xi_0 = \prod_{\alpha=1}^{k} (\eta - P_\alpha(\zeta)), \tag{6.15}$$

where $\rho_0 \in L^{\mu}(k), \xi_0 \in L^{-\mu}(k)$, and $\eta \in \mathcal{O}(2)$. Let us recall that μ was the value of the monopole Higgs field at infinity. Obviously, for fixed ζ this is a resolution of $\mathbf{C}^2/\mathbf{Z}_k$, so the corresponding hyperkähler metric is an A_{k-1} gravitational instanton. In fact, it is well known what the metric is: it is the multi-Taub-NUT metric with k centers. In the remainder of this section we rederive this result using the Legendre transform method of Refs. [5, 37]. This will serve as a warm-up for the discussion of D_k ALF metrics in the next section.

First we find the real sections of the twistor space Z_1. This amounts to solving Eq. (6.15) with ρ_0, ξ_0, and η now regarded as holomorphic sections of the appropriate bundles. Recalling that $\eta = a\zeta^2 + b\zeta - \bar{a}$ and $P_\alpha(\zeta) = a_\alpha \zeta^2 + 2b_\alpha \zeta - \bar{a}_\alpha$ with $b_\alpha \in \mathbf{R}$, one gets in the patch V_0

$$\begin{aligned} \rho_0 &= A e^{-b - a\zeta} \prod_{\alpha=1}^{k} (\zeta - u_\alpha), \\ \xi_0 &= B e^{b + a\zeta} \prod_{\alpha=1}^{k} (\zeta - v_\alpha), \end{aligned}$$

with $AB = \prod (a - a_\alpha)$. Here

$$u_\alpha = \frac{-(b - b_\alpha) + \Delta_\alpha}{a - a_\alpha},$$

$$v_\alpha = \frac{-(b-b_\alpha)-\Delta_\alpha}{a-a_\alpha}.$$

Since the real structure must interchange ρ_0 and ξ_0, we get

$$A\overline{A} = \prod (b - b_\alpha + \Delta_\alpha)\,. \tag{6.16}$$

Thus we have a family of solutions to Eq. (6.15) parametrized by Re a, Im a, b, and Arg A.

Having found the real sections, we compute the Kähler potential. The twistor space Z_1 is fibered over $\mathbf{P}^1$ with an intermediate projection

$$Z_1 \to \mathcal{O}(2) \to \mathbf{P}^1. \tag{6.17}$$

In the above ζ and η are coordinates on the base and the fiber of $\mathcal{O}(2)$, respectively. The holomorphic 2-form $\omega \in \Lambda^2 T^* \otimes \mathcal{O}(2)$ is given by

$$\omega = 2d\eta \wedge \frac{d\rho}{\rho} \tag{6.18}$$

in the $\zeta \neq \infty$ patch. For $\zeta \neq \infty$ we can choose $\eta(\zeta)$ and $\chi = \log \frac{\rho}{\xi}$ as two coordinates on the moduli space M_1 holomorphic with respect to the complex structure defined by ζ, and for $\zeta \neq 0$ the corresponding coordinates are denoted by $\eta'(\zeta)$ and χ'. The coordinates in the two patches are related by

$$\eta' = \eta/\zeta^2, \chi' = \chi - 2\mu\eta/\zeta.$$

The second equation here follows from ρ_0 and ξ_0 being sections of $L^\mu(k)$ and $L^{-\mu}(k)$. That is

$$\rho_0' = \zeta^{-k} e^{-\mu\eta/\zeta} \rho_0, \xi_0' = \zeta^{-k} e^{\mu\eta/\zeta} \xi_0. \tag{6.19}$$

In terms of these coordinates

$$\omega = d\eta \wedge d\chi = \zeta^2 d\eta' \wedge d\chi'. \tag{6.20}$$

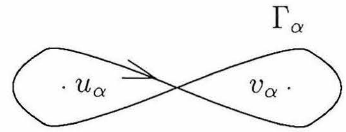


Figure 6.1: The contour Γ_α enclosing u_α and v_α.

Following [37] we define an auxiliary function $\hat{f}$ and a contour C by the equation

$$\oint_C \frac{d\zeta}{\zeta^j}\hat{f} = \oint_0 \frac{d\zeta}{\zeta^j}\chi + \oint_\infty \frac{d\zeta}{\zeta^j}\chi' = \left(\oint_0 + \oint_\infty\right)\frac{d\zeta}{\zeta^j}\chi - 2\mu\oint_\infty \frac{d\zeta}{\zeta^{j+1}}\eta \tag{6.21}$$

for any integer j. Here and in what follows the integrals $\oint_0$ and $\oint_\infty$ are taken along small positively oriented contours around the respective points. This implies in the first of the integrals that the contour runs counterclockwise, while in the second one it runs clockwise. Substituting an explicit expression for χ, we find

$$\oint_C \frac{d\zeta}{\zeta^j}\hat{f} = \sum_\alpha \oint_{\Gamma_\alpha} \frac{d\zeta}{\zeta^j}\log\left(\eta(\zeta) - P_\alpha(\zeta)\right) + 2\mu\oint_0 \frac{d\zeta}{\zeta^{j+1}}\eta. \tag{6.22}$$

Here Γ_α is a figure-eight-shaped contour enclosing u_α and v_α (see Figure 6.1).

We define a function $G(\eta,\zeta)$ by $\partial G/\partial\eta = \hat{f}$. According to Ref. [37] the Legendre transform of the Kähler potential is given by

$$F(a,b) = \frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta^2}G(\eta,\zeta). \tag{6.23}$$

Using Eq. (6.22) we find

$$F(a,b) = \frac{\mu}{2\pi i}\oint_0 \frac{d\zeta}{\zeta^3}\eta^2 + \sum_{\alpha=1}^{k}\frac{1}{2\pi i}\oint_{\Gamma_\alpha}\frac{d\zeta}{\zeta^2}\left(\eta - P_\alpha\right)\log\left(\eta - P_\alpha\right). \tag{6.24}$$

The Kähler potential K is the Legendre transform of F:

$$K(a,\bar{a},t,\bar{t}) = F - b\left(t+\bar{t}\right), \quad \frac{\partial F}{\partial b} = t + \bar{t}. \tag{6.25}$$

It is a well known fact that the metric corresponding to Eq. (6.24) is the multi-Taub-NUT metric with k centers [37, 5]. This is in agreement with string theory predictions [38].

6.2.2 Moduli Space of the Centered $n = 2$ Monopole ($SU(2)$ Gauge Theories)

Twistor space Z_2^0 of centered $n = 2$ monopole

For $n = 2$ the moduli space M_2 is eight-dimensional and admits a triholomorphic $U(1)$ action. We define the centered moduli space M_2^0 to be the hyperkähler quotient of M_2 with respect to this $U(1)$ (at zero level). The $U(1)$ action on M_2 lifts to a $\mathbf{C}^*$ action on Z_2. It acts by $\rho_j \to \lambda\rho_j$, $\xi_j \to \lambda^{-1}\xi_j$. The corresponding moment map is η_1, as can be easily seen from the expression for ω. Thus Z_2^0, the twistor space of M_2^0, is the $\mathbf{C}^*$ quotient of the subvariety $\eta_1 = 0 = \tilde{\eta}_1$ in Z_2. We first investigate one coordinate patch of Z_2^0. Let us denote $\psi_1 = \rho_0\xi_0, \psi_2 = \rho_1\xi_1, \psi_3 = \frac{1}{2}(\rho_0\xi_1 + \rho_1\xi_0), \psi_4 = \frac{1}{2}(\rho_0\xi_1 - \rho_1\xi_0)$. The variables ψ_i are invariant with respect to $\mathbf{C}^*$ action and satisfy

$$\begin{aligned}
&\psi_1\psi_2 = \psi_3^2 - \psi_4^2, \\
&\psi_1 - \eta_2\psi_2 + 2\sqrt{-\eta_2}\psi_3 = \prod_\alpha(\sqrt{-\eta_2} - P_\alpha(\zeta)), \\
&\psi_1 - \eta_2\psi_2 - 2\sqrt{-\eta_2}\psi_3 = \prod_\alpha(-\sqrt{-\eta_2} - P_\alpha(\zeta)).
\end{aligned} \tag{6.26}$$

These equations define a three-dimensional subvariety U^0 in $\mathbf{C}^6$ with coordinates $(\zeta, \eta_2, \psi_1, \ldots, \psi_4)$. Geometric invariant theory tells us that Z_2^0 can be obtained by gluing together two copies of U^0 over $\zeta \neq 0, \infty$. The transition functions can be computed from Eq. (5.9):

$$\begin{aligned}
\tilde{\zeta} &= \zeta^{-1}, \\
\tilde{\eta}_2 &= \zeta^{-4}\eta_2, \\
\tilde{\psi}_1 &= \frac{\zeta^{-2k}}{2}\left(\psi_1 - \eta_2\psi_2 + \cos\gamma(\psi_1 + \eta_2\psi_2) - 2\psi_4\sqrt{\eta_2}\sin\gamma\right),
\end{aligned} \tag{6.27}$$

$$
\begin{aligned}
\tilde{\psi}_2 &= \frac{\zeta^{4-2k}}{2\eta_2}\left(-(\psi_1 - \eta_2\psi_2) + \cos\gamma(\psi_1 + \eta_2\psi_2) - 2\psi_4\sqrt{\eta_2}\sin\gamma\right), \\
\tilde{\psi}_3 &= \zeta^{2-2k}\psi_3, \\
\tilde{\psi}_4 &= \frac{\zeta^{2-2k}}{2}\left((\psi_1 + \eta_2\psi_2)\frac{\sin\gamma}{\sqrt{\eta_2}} + \psi_4\cos\gamma\right),
\end{aligned}
$$

where $\gamma = 2\mu\sqrt{\eta_2}/\zeta$.

From this explicit description of Z_2^0, one can see that for any ζ the fiber of Z_2^0 is a resolution of the D_k singularity. Indeed, combining Eqs. (6.26) we see that the fiber of U_0 over ζ is a hypersurface in $\mathbf{C}^3$ given by

$$
\psi_4^2 + \eta_2\psi_2^2 + \psi_2\mathcal{Q}(\eta_2) - \mathcal{R}(\eta_2)^2 = 0, \tag{6.28}
$$

where $\mathcal{Q}(\eta_2), \mathcal{R}(\eta_2)$ are polynomials in η_2 defined by

$$
\begin{aligned}
2\mathcal{Q}(\eta_2) &= \prod_\alpha(\sqrt{-\eta_2} - P_\alpha(\zeta)) + \prod_\alpha(-\sqrt{-\eta_2} - P_\alpha(\zeta)), \\
4\sqrt{-\eta_2}\mathcal{R}(\eta_2) &= \prod_\alpha(\sqrt{-\eta_2} - P_\alpha(\zeta)) - \prod_\alpha(-\sqrt{-\eta_2} - P_\alpha(\zeta)).
\end{aligned}
$$

Furthermore, these formulas imply that if all points $\vec{p}_1, \ldots, \vec{p}_k$ are distinct, the manifold M_2^0 is a smooth complex manifold in any of its complex structures. Since the 2-form ω is smooth as well, we conclude that M_2^0 is a smooth hyperkähler manifold. The smoothness of M_2^0 is also in agreement with string theory predictions. Indeed, as explained in Ref. [38], the space M_2^0 is the Coulomb branch of $\mathcal{N} = 4, D = 3$ $SU(2)$ gauge theory with k fundamental hypermultiplets, with $\vec{p}_\alpha$ being hypermultiplet masses. When $\vec{p}_\alpha$ are all distinct, the theory has no Higgs branch, and therefore the Coulomb branch is smooth everywhere. When some masses become equal, the Higgs branch emerges, and the Coulomb branch develops an orbifold singularity at the point where it meets the Higgs branch. Thus we expect that when some of $\vec{p}_\alpha$ coincide, or equivalently, when some of ℓ_α are bigger than 1, the manifold M_2^0 has orbifold singularities.

In Ref. [33] the same manifold Z_2^0 arose as the twistor space of the moduli space of a system of ordinary differential equations (so called Nahm equations). This is of course a consequence of a general correspondence between solutions of Bogomolny equations and Nahm equations [19, 21].

Real holomorphic sections of Z_2^0

The discussion of Section 5.2 implies that a real holomorphic section of the uncentered twistor space Z_2 is a triplet (S, ρ, ξ), where S is the spectral curve in $\mathbf{T}$ given by $\eta^2 + \eta_1\eta + \eta_2 = 0$, ρ and ξ are holomorphic sections of $L^{\mu}(k)|_S$ and $L^{-\mu}(k)|_S$ satisfying the condition (iii) of Section 5.2. Then, as explained in Section 5.3, the real sections of Z_2^0 are obtained by setting $\eta_1 = 0$ and modding by the $\mathbf{C}^*$ action $\rho \to \lambda\rho, \xi \to \lambda^{-1}\xi$. In this subsection we find the explicit form of the real holomorphic sections of Z_2^0.

The curve $\eta^2 + \eta_2 = 0$ is either elliptic or a union of two $\mathbf{CP}^1$'s. The former case is generic, while the latter occurs at a submanifold of the moduli space. Intuitively the latter case corresponds to the situation when the two nonabelian monopoles are on top of each other. It suffices to consider the elliptic case.

By an $SO(3)$ rotation

$$\zeta = \frac{a\tilde{\zeta} + b}{-\bar{b}\tilde{\zeta} + \bar{a}}, \qquad \eta = \frac{\tilde{\eta}}{(-\bar{b}\tilde{\zeta} + \bar{a})^2}, \qquad |a|^2 + |b|^2 = 1, \tag{6.29}$$

we can always bring the elliptic curve $\eta^2 = -\eta_2(\zeta)$ to the form

$$\tilde{\eta}^2 = 4k_1^2\left(\tilde{\zeta}^3 - 3k_2\tilde{\zeta}^2 - \tilde{\zeta}\right), k_1 > 0, k_2 \in \mathbf{R}. \tag{6.30}$$

It follows that the discriminant $\Delta > 0$, and therefore the lattice defined by the curve S is rectangular. We denote this lattice 2Ω and its real and imaginary periods by 2ω and $2\omega'$, respectively.

We parametrized S by five real parameters: the Euler angles of the $SO(3)$ rotation and a pair of real numbers k_1 and k_2. We shall see in a moment that the condition (iii) imposes one real constraint on them, so we shall obtain a four-parameter family of real sections, as required.

To write explicitly a section of $L^{\mu}(k)|_S$, we shall use the standard "flat" parameter on the elliptic curve u defined modulo 2Ω, in terms of which $\tilde{\eta} = k_1\mathcal{P}'(u), \zeta = \mathcal{P}(u) + k_2$. Here $\mathcal{P}(u)$ is the Weierstrass elliptic function. In terms of u the real structure acts by $u \to -\overline{u} + \omega + \omega'$.

A section of $L^\mu(k)|_S$ can be thought of as a pair of functions on S f_1, f_2 such that f_1 is holomorphic everywhere except $\zeta = \infty$, f_2 is holomorphic everywhere except $\zeta = 0$, and for $\zeta \neq 0, \infty$ $f_2(\zeta) = \zeta^{-k}\exp(-\mu\eta/\zeta) f_1(\zeta)$. The point $\zeta = \infty$ corresponds to two points $u_\infty, -u_\infty$ on S defined by $\mathcal{P}(u_\infty) + k_2 = \overline{a}/\overline{b}$. Furthermore, condition (iii) implies that the divisor of f_1 is Q_+. Let us recall that $Q_+ \bigcup Q_- = \bigcup_\alpha Q_\alpha$, where $Q_\alpha = S \bigcap P_\alpha, \alpha = 1, \ldots, k$. Thus Q_α consists of solutions of a system of two equations $\eta = P_\alpha(\zeta), \eta^2 = -\eta_2(\zeta)$. Obviously, this defines four points on the elliptic curve S. Because of real structure, these four points split into two pairs whose members are interchanged by τ. Q_+ includes one point from each pair (for all α), Q_- includes the rest.[1] Let us denote the "flat" coordinates of points in Q_+ by $u_\alpha, u'_\alpha, \alpha = 1, \ldots, k$, and those in Q_- by $v_\alpha, v'_\alpha, \alpha = 1, \ldots, k$. By definition, $v_\alpha = -\overline{u}_\alpha + \omega + \omega' (\text{mod } 2\Omega), v'_\alpha = -\overline{u}'_\alpha + \omega + \omega' (\text{mod } 2\Omega)$. We fix the mod$2\Omega$ ambiguity by requiring that $u_\alpha, u'_\alpha, v_\alpha, v'_\alpha$ be in the fundamental rectangle of 2Ω. In this notation a section of $L^\mu(k)|_S$ is given by

$$f_1 \sim \exp\left(-\mu k_1(\zeta_W(u + u_\infty) + \zeta_W(u - u_\infty)) + Cu\right) \prod_\alpha \frac{\sigma(u - u_\alpha)\sigma(u - u'_\alpha)}{\sigma(u - u_\infty)\sigma(u + u_\infty)}. \quad (6.31)$$

Here $\zeta_W(u)$ and $\sigma(u)$ are Weierstrass quasielliptic functions (we denote Weierstrass ζ-function by $\zeta_W(u)$ to avoid confusion with the affine coordinate ζ on the $\mathbf{P}^1$ of complex structures), and C is a constant. Similarly, a section of $L^{-\mu}(k)|_S$ with the divisor Q_- is represented by a pair of functions g_1, g_2 related by $g_2(\zeta) = \zeta^{-k}\exp(\mu\eta/\zeta) g_1(\zeta)$. Explicitly g_1 is given by

$$g_1 \sim \exp\left(\mu k_1(\zeta_W(u + u_\infty) + \zeta_W(u - u_\infty)) + Du\right) \prod_\alpha \frac{\sigma(u - v_\alpha)\sigma(u - v'_\alpha)}{\sigma(u - u_\infty)\sigma(u + u_\infty)}, \quad (6.32)$$

where D is another constant. In general f_1 and g_1 are quasiperiodic with periods 2ω

[1] Of course, there is a 4^m-fold ambiguity involved in the splitting $Q = Q_+ \bigcup Q_-$. It can be fixed, in principle, by the comparison with the known asymptotic behavior of the metric.

and $2\omega'$. The condition (iii) is equivalent to asking that f_1 and g_1 be doubly periodic. One can see that the latter can be achieved by adjusting C and D if and only if

$$\begin{aligned} 2\mu k_1 + \sum_\alpha (u_\alpha + u'_\alpha) &\in 2\Omega, \\ 2\mu k_1 - \sum_\alpha (v_\alpha + v'_\alpha) &\in 2\Omega. \end{aligned} \tag{6.33}$$

Recalling that k_1 is real and positive, we conclude that there exist integers m, m', p, p' and a real number $x \in (0, 2\omega]$ such that $\sum_\alpha (u_\alpha + u'_\alpha) = -x + 2m\omega + 2m'\omega', \sum_\alpha (v_\alpha + v'_\alpha) = x + 2p\omega + 2p'\omega'$. Then Eqs. (6.33) together with the condition (iv) imply

$$2\mu k_1 = x. \tag{6.34}$$

Then for f_1 and g_1 to be doubly periodic, one has to set

$$C = 2m\zeta_W(\omega) + 2m'\zeta_W(\omega'), D = 2p\zeta_W(\omega) + 2p'\zeta_W(\omega').$$

Let us notice for future use that

$$\begin{aligned} \log f_1(u+\omega) - \log f_1(u) &= -2\pi i m', \\ \log f_1(u+\omega') - \log f_1(u) &= 2\pi i m, \\ \log g_1(u+\omega) - \log g_1(u) &= -2\pi i p', \\ \log g_1(u+\omega') - \log g_1(u) &= 2\pi i p. \end{aligned} \tag{6.35}$$

Eq. (6.34) is a transcendental equation on k_1, k_2, and the $SO(3)$ rotation required to bring S to the standard form Eq. (6.30). It reduces the number of real parameters in the equation of the curve from 5 to 4. Thus we have a four-parameter family of real sections of Z_2^0.

The Kähler potential of the centered $n = 2$ moduli space

Having found a four-parameter family of real holomorphic sections of Z_2^0, we now would like to compute the corresponding hyperkähler metric. Since Z_2^0 has an intermediate holomorphic projection on $\mathcal{O}(4)$, we can use the method of Ref. [37] to write down the Legendre transform of the Kähler potential. The existence of the projection is equivalent to saying that η_2 is a holomorphic coordinate on Z_2^0. The holomorphic 2-form ω in the patch $\zeta \neq \infty$ can be written as

$$\omega = d\eta_2 \wedge d \sum_{branches} \frac{1}{\eta} \log \frac{f_1}{g_1} \equiv d\eta_2 \wedge d\chi.$$

Here f_1 and $\eta = \sqrt{-\eta_2}$ are regarded as double-valued functions of $\zeta \in \mathbf{P}^1 \backslash \{\infty\}$ (i.e., $\zeta \neq \infty$), and the sum is over the two branches of the cover $S \to \mathbf{P}^1$. Similarly, in the patch $\zeta \neq 0$ we can write

$$\omega' = d\eta_2' \wedge d\chi'.$$

On the overlap we have the relations

$$\omega' = \zeta^{-2}\omega, \quad \eta_2' = \zeta^{-4}\eta_2, \quad \chi' = \zeta^2\chi - 2\mu\zeta. \tag{6.36}$$

Following Ref. [37], we would like to find a (multi-valued) function $\hat{f}(\eta, \zeta)$ and a contour C on the double cover $S \to \mathbf{P}^1$ such that

$$\oint_C \frac{d\zeta}{\zeta^j} \hat{f}(\eta, \zeta) = \sum_{branches} \left(\oint_0 \frac{d\zeta}{\zeta^{j-2}} \chi + \oint_\infty \frac{d\zeta}{\zeta^j} \chi' \right)$$

for any integer j. Here the contours of integration on the RHS are small positively oriented loops around $\zeta = 0$ and $\zeta = \infty$. To find $\hat{f}$ we substitute the explicit expressions for χ and χ' and rewrite the integral on the RHS as an integral in the u-plane. Then the RHS becomes

$$\oint \frac{du}{k_1} \zeta(u)^{-j+2} \log \frac{f_1(u)}{g_1(u)} + 4\mu \oint_0 \frac{d\zeta}{\zeta^{j-1}}, \tag{6.37}$$

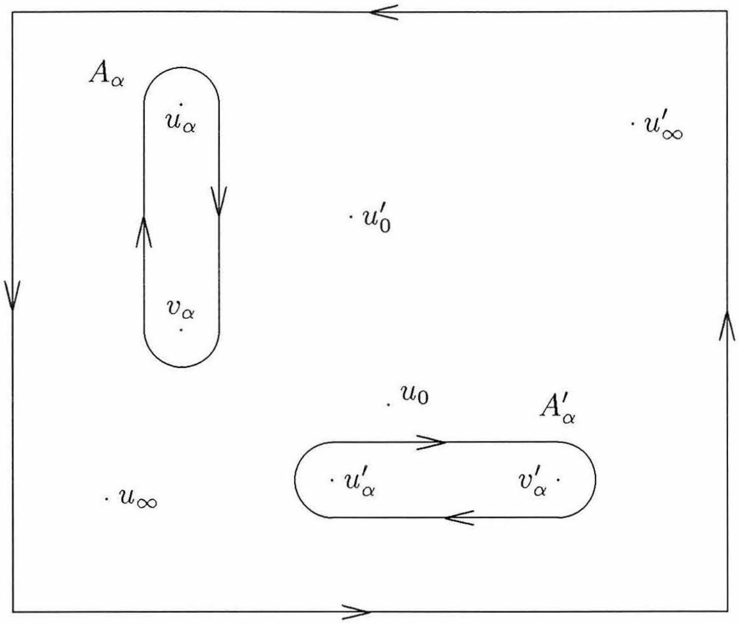


Figure 6.2: Integration contours in Eq. (6.38). Only one of the contours A_α and one of the contours A'_α are shown.

where the contour in the first integral consists of four small positively oriented loops around four preimages of the points $\zeta = 0$ and $\zeta = \infty$ in the fundamental rectangle of the lattice 2Ω. We denote these points $u_0, u'_0 = 2(\omega + \omega') - u_0, u_\infty, u'_\infty = 2(\omega + \omega') - u_\infty$. Besides these four points the only other branch points of $\log f_1(u)/g_1(u)$ in the fundamental rectangle are $u_\alpha, u'_\alpha, v_\alpha, v'_\alpha, \alpha = 1, \ldots, k$. As for $\zeta(u)$, it is elliptic. Then we can rewrite Eq. (6.37) as

$$\oint_{bdry} \frac{du}{k_1} \zeta(u)^{-j+2} \log \frac{f_1(u)}{g_1(u)} + \sum_\alpha \oint_{A_\alpha + A'_\alpha} \frac{du}{k_1} \zeta(u)^{-j+2} \log \frac{f_1(u)}{g_1(u)} + 4\mu \oint_0 \frac{d\zeta}{\zeta^{j-1}}, \qquad (6.38)$$

where the contour in the first integral runs along the boundary of the fundamental rectangle, while A_α and A'_α enclose the pairs of points u_α, v_α and u'_α, v'_α, respectively (see Figure 6.2). Using Eqs. (6.35) the integral over the boundary can be simplified to

$$-2\pi i \oint_{(m-p, m'-p')} \frac{du}{k_1} \zeta(u)^{-j+2},$$

where the contour $(m-p, m'-p')$ winds $m-p$ times around the real cycle and $m'-p'$

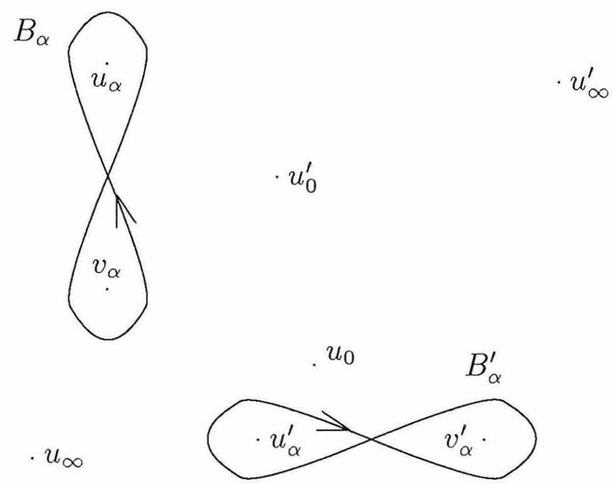


Figure 6.3: The contours B_α and B'_α.

times around the imaginary cycle. Recalling the explicit form of $f_1(u)$ and g_1, we can rewrite the integral over $A_\alpha + A'_\alpha$ as

$$\oint_{B_\alpha + B'_\alpha} \frac{du}{k_1} \zeta(u)^{-j+2} \log \sigma(u-u_\alpha)\sigma(u-u'_\alpha)\sigma(u-v_\alpha)\sigma(u-v'_\alpha), \tag{6.39}$$

where the contours B_α and B'_α are figure-eight-shaped contours shown in Figure 6.3. On the other hand, it can be easily seen that

$$\eta(u) - P_\alpha(\zeta(u)) \sim \frac{\sigma(u-u_\alpha)\sigma(u-u'_\alpha)\sigma(u-v_\alpha)\sigma(u-v'_\alpha)}{\sigma(u-u_\infty)^2\sigma(u+u_\infty)^2}.$$

Since neither u_∞ nor u'_∞ are enclosed by the contour $B_\alpha + B'_\alpha$, the integral Eq. (6.39) is equal to

$$\oint_{B_\alpha + B'_\alpha} \frac{du}{k_1} \zeta(u)^{-j+2} \log(\eta(u) - P_\alpha(\zeta(u))).$$

Collecting all of this together, we get

$$\oint_C \frac{d\zeta}{\zeta^j}\hat{f}(\eta,\zeta) = -2\pi i \oint_{(m-p,m'-p')} \frac{d\zeta}{\eta}\zeta^{-j+2}$$
$$+\sum_\alpha \oint_{C_\alpha + C'_\alpha} \frac{d\zeta}{\eta}\zeta^{-j+2}\log(\eta - P_\alpha(\zeta)) + 4\mu \oint_0 \frac{d\zeta}{\zeta^{j-1}}. \quad (6.40)$$

Here all the functions are regarded as functions on the double cover of the ζ-plane, and the contours C_α, C'_α are the images of B_α, B'_α under the map $u \mapsto \zeta$. We now define a function $G(\eta,\zeta)$ by $\partial G/\partial\eta = -2\eta\zeta^{-2}\hat{f}$. According to Ref. [37] the Legendre transform of the Kähler potential is given by

$$F = \frac{1}{2\pi i}\oint \frac{d\zeta}{\zeta^2} G(\eta,\zeta).$$

Hence we can read off F:

$$F = -\frac{1}{2\pi i}\oint_0 d\zeta \frac{4\mu\eta^2}{\zeta^3} + \oint_{(m-p,m'-p')} d\zeta \frac{2\eta}{\zeta^2} \quad (6.41)$$
$$-\sum_\alpha \frac{1}{2\pi i}\oint_{C_\alpha + C'_\alpha} \frac{d\zeta}{\zeta^2} 2(\eta - P_\alpha(\zeta))\log(\eta - P_\alpha(\zeta)).$$

F may be regarded as a function of the coefficients of $\eta_2(\zeta) = z + v\zeta + w\zeta^2 - \overline{v}\zeta^3 + \overline{z}\zeta^4$. Since w is real, F depends on 5 real parameters. These parameters are subject to one transcendental constraint expressed by Eq. (6.34). (Alternatively one can rewrite this constraint as $\frac{\partial F}{\partial w} = 0$.) Thus we may think of w as an implicit function of z and v. The Kähler potential $K(z,\overline{z},u,\overline{u})$ is the Legendre transform of F:

$$K(z,\overline{z},u,\overline{u}) = F(z,\overline{z},v,\overline{v},w) - uv - \overline{uv}, \quad \frac{\partial F}{\partial v} = u, \frac{\partial F}{\partial \overline{v}} = \overline{u}.$$

Eq. (6.41) agrees with a conjecture by Chalmers [39].

We already saw in Section 6.2.2 that M_2^0 is a resolution of D_k singularity. Now we can check that it is ALF. To this end we take the limit $k_1 \to +\infty$. Eq. (6.34) implies that in this limit $\omega \to \infty$, while ω' stays finite. Thus the curve S degenerates: $\eta_2(\zeta) \to -(P(\zeta))^2$, where $P(\zeta)$ is a real section of $\mathbf{T}$. It is easy to see that in this limit F reduces to the Taub-NUT form (see Section 6.2.1):

$$F \sim \frac{1}{2\pi i} \oint_0 d\zeta \left(-\frac{4\mu P^2}{\zeta^3} + K \frac{P \log P}{\zeta^2} \right), \tag{6.42}$$

where K is an integer depending on the limiting behavior of $u_\alpha, u'_\alpha, v_\alpha, v'_\alpha$. Therefore, asymptotically the metric on M_2^0 has the Taub-NUT form. With some more work it should be possible to compute the integer K as well.

Note also that if we set $\mu = 0$, then the metric becomes ALE. Kronheimer proved [3] that the D_k ALE metric is essentially unique. Thus we have obtained the Legendre transform of the Kähler potential for the D_k metrics of Ref. [2]. It would be interesting to obtain a similar representation for the E_k ALE metrics.

Chapter 7 Conclusions

We have studied $\mathcal{N}=4$ supersymmetric $U(n)$ and $SU(n)$ gauge theories in three dimensions with matter in the fundamental representation. Realizing some of these gauge theories as theories on D2-branes of type IIA string theory, we saw their Coulomb branches to be gravitational instantons. On the other hand realization of these theories in type IIB string theory uncovered the relation with monopoles and Nahm equations. By identifying their Coulomb branches with moduli spaces of monopoles, we have found the twistor spaces of the Coulomb branches. Considering the moduli space of solutions of Nahm equations provided an independent way of constructing these twistor spaces. We verified in the known cases that the resulting distinguished complex structures of the Coulomb branches coincide with those of four-dimensional $\mathcal{N}=2$ theories. This provided a nontrivial check of the correspondences used.

From the twistor data one can extract the Kähler potentials. For the cases of $U(1)$ and $SU(2)$ gauge theories, we worked out the Kähler potentials on the Coulomb branches. These two spaces provide examples of A_k and D_k gravitational instantons as predicted by the type IIA string theory picture. This describes all infinite series of gravitational instantons. The existence of exceptional (E_6, E_7 and E_8) ALF spaces is still an open question. It remains to be seen whether the methods presented here can shed some light on whether they exist. Note, that if E-type ALF spaces do exist, then M theory compactification on them would give new nonperturbative objects of type IIA string theory for which there is no other evidence.

We would like to note that using the same method one can obtain the Kähler potential for all Coulomb branches of $U(n)$ and $SU(n)$ gauge theories with fundamental matter. Our results are easily generalized to the gauge group being a product of SU groups with matter multiplets in bifundamental representation as well as to the cases of SO and Sp gauge groups. The case of an SU gauge theory with one adjoint multiplet should also be included in the above list.

$\mathcal{N}=4$ three-dimensional gauge theories can be thought of as limits of $\mathcal{N}=2$ four-dimensional theories on $\mathbf{R}^3 \times S^1$ with the radius of the S^1 sent to zero. We have found the three-dimensional theories to be related to the integrable system of Nahm equations. These are differential equations in one real parameter. Analogous methods allow one to relate theories on $\mathbf{R}^3 \times S^1$ to systems of integrable equations on a torus, which are generalizations of the Hitchin system. This illuminates the relation between the Seiberg-Witten solutions of gauge theories and integrable systems.

Applying the methods described here one can find the moduli spaces of the four-dimensional gauge theories, which will give examples of hyperkähler manifolds with two compact directions.

Chapter 8 Glossary

ADHM-Nahm construction A construction of subbundle of a trivial bundle such that the induced connection has self-dual curvature and is valued in an appropriate group. See [40] and [41].

ADE classification Simply-laced semisimple Lie algebras, Kleinian singularities, finite subgroups of $SU(2)$ as well as self-dual gravitational instantons fall into A_k, D_k and E_6, E_7, E_8 types. See [1] for the former three of these and [2] for the latter.

Algebraic curve A Riemann surface, a complex manifold of one complex dimension.

ALE space A four-dimensional Riemannian manifold with the self-dual curvature which is asymptotically $\mathbf{R}^4/\Gamma$, with $\Gamma \in SU(2)$.

ALF space A four-dimensional Riemannian manifold with the self-dual curvature which is asymptotically $\mathbf{R}^3 \times S^1/\Gamma$, with $\Gamma \in SU(2)$.

Atiyah-Hitchin space A moduli space of $SU(2)$ monopoles of charge two with the fixed center of mass. See [7].

BPS monopole A pair of a gauge connection A and an adjoint scalar Higgs field Φ in three dimensions satisfying the Bogomolny equation

$$*F = D\Phi,$$

where F is the field strength of A, D is a covariant differential, and $*$ denotes the Hodge dual.

Bogomolny equation see **BPS monopole**.

Centered moduli space of monopoles A moduli space of monopoles with the fixed center of mass and common $U(1)$ phase. Defined as a hyperkähler quotient of the total monopole moduli space with respect to the action of the $U(1)$ changing the common phase.

Chern class A topological invariant of a bundle. The total Chern class $c(F)$ is defined by

$$c(F) = \det\left(1 + \frac{iF}{2\pi}\right),$$

where F is the curvature of the bundle.

Coadjoint orbit There is a natural adjoint action of a group of its algebra. It induces an action of the group on the space dual to the algebra. This action is called coadjoint. See, e.g., [42].

Complex structure An automorphism of the tangent bundle which is linear on the fibers and squares to -1.

Coulomb branch A branch of the space of vacua with broken gauge symmetry.

D and F flatness conditions See [43].

Dirac monopole A $U(1)$ valued solution of the Bogomolny equation defined everywhere except for one point (the center of the Dirac monopole).

D-pbrane A manifold of space-like dimension p in the target space of a string theory which serves as a Dirichlet boundary condition for the world-sheet embedding.

Dynkin diagram A diagram encoding the Cartan matrix of a simple Lie algebra. See, e.g., [1].

D_k-type singularity A singularity of the form $x^2 + y^2 z + z^{k-1}$.

Eguchi-Hanson space An nontrivial example of an ALE gravitational instanton. This is a space with the metric

$$ds^2 = V^{-1}\left(d\theta + \vec{\omega}\cdot d\vec{x}\right)^2 + V d\vec{x}^2,$$

where $V = 1/\|\vec{x}\|$ and $\mathrm{grad}V = \mathrm{curl}\vec{\omega}$. See [44].

Gravitational Instanton A four-dimensional Riemannian manifold with the Ricci tensor proportional to the metric.

Higgs branch A branch of the moduli space of vacua on which scalar fields of a hypermultiplet acquire vevs.

Higgs field A scalar field in some (usually the adjoint) representation of the gauge group.

Holonomy To every closed path on a Riemannian manifold we can assign a linear map of the tangent space at its original point which takes a vector to its parallel transport along the path. This is the holonomy corresponding to the path. A set of all holonomies has a natural group structure.

Hyperkähler manifold A manifold with three complex structures I, J and K related by $IJ = K$.

Hyperkähler quotient A modification of symplectic quotient. See Appendix A.

Hypermultiplet A supermultiplet containing matter fields. See, e.g., [45].

Kähler form For a manifold with a metric g and a complex structure I, the kähler form is $\omega = g(I\cdot, \cdot)$.

Kähler manifold A Riemannian manifold with a covariantly constant complex structure.

Kähler potential A function K on a Kähler manifold such that the Kähler form $\omega = d\bar{d}K$.

K3 manifold A compact four-dimensional hyperkähler manifold which is not a T^4.

Level of a quotient The value of the moment map on the subspace to be divided by the symmetry group. See Appendix A.

Line bundle A linear bundle with one-dimensional fibers.

M theory An unknown eleven-dimensional quantum theory which is a supergravity in the low energy limit and type IIA string theory when compactified on a circle. See, e.g., [46], [47].

Maximal torus A maximal abelian subgroup of a group.

Moduli space A space of solutions of some equations or of some theory. It often automatically carries some extra structure.

Moment map For a group G (with an algebra g) of symmetries of a symplectic manifold $\mathcal{M}$, it is a function $\mu : \mathcal{M} \to g^*$ such that

$$(\mu(y), X) = H_X(y),$$

where $X \in g$ is a vector field on $\mathcal{M}$, $dH_X = \omega(X, \cdot)$, and y is a point of $\mathcal{M}$. See Appendix A.

Nahm equations

$$\frac{dT_1}{ds} = [T_2, T_3],$$

and permutations of the indices (1,2,3). See [10].

Nahm data Solution of Nahm equation satisfying certain boundary conditions. See, e.g., Chapter 4.

NS5-brane A solitonic brane of string theory carrying a Neveau-Schwarz charge.

Orientifold plane A subspace of the target space left fixed by the action of the target-space part $\mathbf{Z}_2$ of the $(-1)^{F^L} \cdot \Omega \cdot \mathbf{Z}_2$ gauged symmetry. $\mathbf{Z}_2$ reverses the orientation of three of the target space coordinates, and Ω reverses the parity of the world-sheet.

Quasiperiodic function A function on a complex plane which changes by an additive piece when acted on by a particular $\mathbf{Z}^2 \in \mathbf{C}$. See, e.g., [48].

R-symmetry A symmetry acting on fermions. See [43].

Real structure An antiholomorphic involution. See Section 1.2.3.

Real section A section which is mapped onto itself by the real structure.

S duality A duality exchanging the strong and weak coupling limits.

Self-dual connection A connection with the self-dual curvature.

Shaggy monopole See Section 2.4.2.

Spectral curve A set in the minitwistor space consisting of lines on which the scattering problem has a bound state.

Spectral data All data necessary in the minitwistor formulation in order to describe a monopole. See Section 5.1.

Symplectic structure A closed non-degenerate two-form.

't Hooft-Polyakov monopole An $SU(2)$ valued one monopole solution of the Bogomolny equation.

Twisted bundle A tensor product of a bundle with some other standard bundle.

Taub-NUT space A nontrivial example an ALF gravitational instanton. This is a space with the metric

$$ds^2 = V^{-1}\left(d\theta + \vec{\omega}\cdot d\vec{x}\right)^2 + V d\vec{x}^2,$$

where $V = 1 + 1/\|\vec{x}\|$ and $\mathrm{grad}V = \mathrm{curl}\vec{\omega}$.

Triholomorphic Holomorphic with respect to three complex structures.

Twistor Space The space of all complex structures as a complex manifold. See Section 1.2.3.

Type IIA string theory A superstring theory with two supercharges of opposite chiralities on the world-sheet. It is nonchiral. See [49].

Type IIB string theory A superstring theory with two supercharges of the same chirality on the world-sheet. It is chiral. See [49].

Vector multiplet A supermultiplet containing a vector field.

Ward correspondence A correspondence between differential and holomorphic data.

Wilson loop

$$\mathrm{Pexp}\left(\oint_{\Gamma} A_i dx^i\right).$$

Appendix A Hyperkähler Quotient

Here we review the hyperkähler quotient construction. For a good introduction to the subject see [50] and [51].

Let us first define symplectic quotient. Having a symplectic manifold $(\mathcal{M}, \omega)$ (that is an even-dimensional manifold with a closed non-degenerate two-form ω), for every vector field X that preserves the symplectic structure

$$\mathcal{L}_X \omega = 0 \tag{A.1}$$

(where $\mathcal{L}_X$ is a Lie derivative along the field X), there is a generating function H_X. That is

$$\omega(X, \cdot) = dH_X.$$

Then for any continuous group of symmetries G acting on $\mathcal{M}$ in such a way that G preserves ω, there is a function μ on $\mathcal{M}$ valued in g^* the dual of the algebra g of G. Namely for every vector field $X \in g$

$$(\mu(y), X) = H_X(y) \tag{A.2}$$

for every point $y \in \mathcal{M}$. This map $\mu : \mathcal{M} \to g^*$ is called a *moment map*. If we restrict our attention to the submanifold $\mu^{-1}(A)$ of $\mathcal{M}$ (where $A \in g^*$ is invariant with respect to the coadjoint action of G), we find that G is a symmetry of $\mu^{-1}(A)$ and the restriction of the symplectic form ω to $\mu^{-1}(A)$ is G-invariant and degenerate (see [50] for the proof) on the orbits of G. Thus $\mu^{-1}(A)/G$ is a symplectic manifold called the *symplectic quotient* of $\mathcal{M}$ with respect to the group G.

For example, for a mechanical system reduction to the center of mass coordinates can be thought of as a symplectic reduction of the total phase space of the system with respect to the group of translations. In this case $\omega = \sum_i q_i \wedge p_i$ (where q_i are coordinates and p_i are conjugate momenta), $\mu = \sum_i p_i$ and $A = 0$.

In the case of a hyperkähler manifold, there are three different symplectic structures, thus for any group G that respects the hyperkähler structure we have three moment maps that can be united in one three-component moment map

$$\mu \rightarrow g^* \otimes \mathbf{R}^3.$$

Now a hyperkähler manifold $\mu^{-1}(A)/G$ is called a *hyperkähler quotient*, where $A \in g^* \otimes \mathbf{R}^3$ is G-invariant. A is called the *level* of the quotient.

Appendix B Legendre Transform

We briefly outline here the Generalized Legendre Transformation of [5] closely following [37]. Consider a twistor space $\mathcal{Z}$ with an intermediate projection $p : \mathcal{Z} \rightarrow \mathcal{O}(2n) \rightarrow \mathbf{P}^1$. (The cases we are particularly interested in are $n = 1$ and $n = 2$ corresponding to the $U(1)$ and $SU(2)$ theories.) We can consider $\mathbf{P}^1$ to be covered by two open charts U and V, one containing $\zeta = 0$ and the other $\zeta = \infty$. The projection p defines a coordinate $\eta \in \mathcal{O}(2n)$ on the fiber. Suppose we can find another coordinate on the fiber χ such that the two-form ω described in Eq.(1.7) is given by

$$\omega = d\eta_1 \wedge d\chi_1 = \zeta^2 d\eta_2 \wedge \chi_2, \tag{B.1}$$

(recall that, as explained in Section 1.2.3, the form ω has coefficients in $\mathcal{O}(2)$) where the indices 1 and 2 denote the two charts of $\mathbf{P}^1$. We can trivialize the twistor space $\mathcal{Z}$ considered as a bundle over the $\mathbf{P}^1$ in each chart. Say

$$\eta_1(\zeta) = z + v\zeta + w_2\zeta^2 + \ldots + w_{2n-2}\zeta^{2n-2} + (-1)^{n-1}\bar{v}\zeta^{2n-1} + (-1)^n \bar{z}\zeta^{2n} \tag{B.2}$$

$$\chi_1(\zeta) = u + t\zeta + O(\zeta^2) \tag{B.3}$$

and

$$\chi_2(\zeta^{-1}) = \tilde{u} + \tilde{t}\zeta^{-1} + O(\zeta^{-2}). \tag{B.4}$$

η_2 is given by $\zeta^{-2n}\eta_1$. Then (z, u) define the trivialization near $\zeta = 0$. Expanding the form ω in powers of ζ and comparing to Eq. 1.7 we find the Kähler form

$$\omega_1 = dz \wedge dt + dv \wedge du. \tag{B.5}$$

Now if we define a function $\hat{f}$ and a contour C such that

$$\oint_C \frac{d\zeta}{\zeta^m}\hat{f} = \oint_0 \frac{d\zeta}{\zeta^{m+2-2n}}\chi_1 + \oint_\infty \frac{d\zeta}{\zeta^m}\chi_2 \tag{B.6}$$

and a function $G(\eta_1, \zeta)$ such that $\partial G/\partial\eta_1 = \zeta^{2-2n}\hat{f}$, we can construct yet another function F of coefficients of η_1 by

$$F(z, v, w_2, \ldots, w_{2n-2}) = \frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta^2}G(\eta_1, \zeta). \tag{B.7}$$

It is easy to verify that from these definitions it follows that

$$\frac{\partial F}{\partial w_a} = 0 \tag{B.8}$$

for $2 \leq a \leq 2n-2$ and that

$$\frac{\partial F}{\partial z} = t, \frac{\partial F}{\partial v} = u, \frac{\partial F}{\partial \bar{v}} = (-1)^n\tilde{u}, \frac{\partial F}{\partial \bar{z}} = (-1)^{n+1}\tilde{t}. \tag{B.9}$$

So that $\omega_1 = dz \wedge d\left(\frac{\partial F}{\partial z}\right) + dv \wedge du$.

Now performing a Legendre transformation on F with respect to the coordinates v and $\bar{v}$ we obtain the function

$$K(z, \bar{z}, u, \bar{u}) = F(z, \bar{z}, v, \bar{v}, w_a) - uv - \bar{u}\bar{v}. \tag{B.10}$$

Then using

$$\frac{\partial K}{\partial z} = \frac{\partial F}{\partial z}, dv = -d\left(\frac{\partial K}{\partial u}\right) \tag{B.11}$$

we find the Kähler form

$$\omega_1 = dz \wedge d\left(\frac{\partial K}{\partial z}\right) - d\left(\frac{\partial K}{\partial u}\right) \wedge du \tag{B.12}$$

and see that the function K is the Kähler potential. As

$$\omega_1 = \frac{\partial^2 K}{\partial z \partial \bar{z}} dz \wedge d\bar{z} + \frac{\partial^2 K}{\partial z \partial \bar{u}} dz \wedge d\bar{u} + \frac{\partial^2 K}{\partial u \partial \bar{z}} du \wedge d\bar{z} + \frac{\partial^2 K}{\partial u \partial \bar{u}} du \wedge d\bar{u}. \tag{B.13}$$

Appendix C T Duality and NS5-brane

We would like to establish that an NS5-brane of type IIB string theory becomes a Taub-NUT space when T duality is performed in one of the directions orthogonal to its world-volume. Let us consider M theory on a space with two periodic coordinates (say x^9 and x^{10}) forming circles S^1_α and S^1_β such that S^1_α is fibered over three other directions (say x^1, x^2 and x^3) forming a Taub-NUT space, while the S^1_β is fibered trivially. $S^1_\alpha \times S^1_\beta$ is a torus and M theory on a torus is known to reproduce type IIB string theory with the τ-parameter of the torus giving the string coupling constant (see [47]). More than that, the $SL(2, \mathbf{Z})$ duality symmetries of type IIB theory corresponds to the $SL(2, \mathbf{Z})$ modular transformations of the torus $S^1_\alpha \times S^1_\beta$ as explained in [47].

NS5 and D5-branes are interchanged by S duality of type IIB string theory, which (i.e., S duality) corresponds to $\tau \to -1/\tau$ element of the $SL(2, \mathbf{Z})$. Now consider the S^1_α to be the M theory circle and shrink the sizes of both circles S^1_α and S^1_β to zero. In this case shrinking the size of the S^1_α we obtain type IIA theory compactified on a circle S^1_β. The now hidden S^1_α being fibered nontrivially gives Ramond-Ramond fields corresponding to the off-diagonal terms in the eleven-dimensional metric. Via Kaluza-Klein reduction we find these fields having a source positioned in the (x^1, x^2, x^3) space where the center of the Taub-NUT was. This source is the D6-brane in type IIA string theory. Its world-volume is along the other directions one of which is the circle S^1_β. If we shrink S^1_β, type IIB theory T dual (with duality along the x^{10}) description emerges. In the T dual picture the D6-brane of type IIA theory turns into a D5-brane of type IIB theory.

S duality interchanges the D5 and NS5-branes on one hand and corresponds to interchanging the S^1_α and S^1_β circles on the other hand. Thus after S duality S^1_β is the circle of M theory, so we have type IIA string theory compactified on a Taub-NUT space. Shrinking the circle of the Taub-NUT takes us to the dual type IIB picture in which this is an NS5-brane as stated in the beginning of this paragraph. Therefore, we find that T dual of an NS5-brane is a Taub-NUT space (with duality in the direction orthogonal to the NS5-brane world volume).

Appendix D Dancer Manifold

The hyperkähler manifolds $\mathcal{M}^{12}$ and $\mathcal{M}^{8}$ studied by Dancer in [25] and [26] are defined as follows.

Consider four $u(2)$ valued functions T_0, T_j, $j = 1, 2, 3$ of a real variable $s \in [0, 1]$ such that T_0 is analytic on $[0, 1]$, T_j analytic on $(0, 1]$ and

$$T_j(s) = -\frac{i\sigma_j}{2s} + O(s), \tag{D.1}$$

where σ_j are Pauli matrices, subject to the Nahm equations

$$\frac{dT_i}{ds} + [T_0, T_i] = \frac{1}{2}\epsilon_{ijk}[T_j, T_k] \tag{D.2}$$

modulo the gauge group G_0 of analytic $U(2)$-valued gauge transformations which are the identity at $s = 0, 1$. This defines the hyperkähler manifold $\mathcal{M}^{12}$. The fact that it is 12-dimensional can be easily seen from the Dancer description reviewed below.

If we denote the group of analytic $U(2)$ gauge transformations which are the identity at $s = 0$ by G, then the manifold $\mathcal{M}^{12}$ has a triholomorphic action of $U(2) = G/G_0$.The manifold $\mathcal{M}^{8}$ is a hyperkähler quotient of $\mathcal{M}^{12}$ by the $U(1)$ which is the center of this $U(2)$ group.

As described by Dancer in [26] $\mathcal{M}^{12}$ as a complex manifold is given by a pair (B, w) of complex 2×2 matrix and a two-vector such that w and Bw are linearly independent. An element g of the $U(2)$ group acts by

$$B \to gBg^{-1}, \qquad w \to gw. \tag{D.3}$$

Let ζ be a parameter on the projective line $\mathbf{P}^1$. Then the twistor space of $\mathcal{M}^{12}$ is described by functions of ζ (B^0, w^0) holomorphic in the neighborhood of $\zeta = 0$ and

(B^1, w^1) holomorphic in the neighborhood of $\zeta = \infty$ with

$$\begin{aligned} B^1 &= \zeta^{-2} B^0 && \text{(D.4)} \\ w^1 &= \zeta e^{2B^0/\zeta} w^0. && \text{(D.5)} \end{aligned}$$

We can pick new local coordinates $\gamma_1, \gamma_2, p_1, p_2, q_1, q_2$ on $\mathcal{M}^{12}$, where γ_1, γ_2 are eigenvalues of B,

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = Bw - \gamma_2 w, \qquad \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = Bw - \gamma_1 w. \tag{D.6}$$

These have transition functions

$$\begin{aligned} \gamma_1^1 &= \frac{1}{\zeta^2}\gamma_1^0 && \text{(D.7)} \\ \gamma_2^1 &= \frac{1}{\zeta^2}\gamma_2^0 && \text{(D.8)} \\ \begin{pmatrix} p_1^1 \\ q_1^1 \end{pmatrix} &= e^{2\gamma_1/\zeta}\begin{pmatrix} p_1^0 \\ q_1^0 \end{pmatrix} && \text{(D.9)} \\ \begin{pmatrix} p_2^1 \\ q_2^1 \end{pmatrix} &= e^{2\gamma_2/\zeta}\begin{pmatrix} p_2^0 \\ q_2^0 \end{pmatrix}. && \text{(D.10)} \end{aligned}$$

In terms of these coordinates the two-form $\omega \in \Lambda^2 T^* \otimes \mathcal{O}(2)$ is given by

$$\omega = -\frac{1}{\gamma_1 - \gamma_2} d\gamma_1 \wedge d\gamma_2 + d\gamma_1 \wedge \theta_1 + d\gamma_2 \wedge \theta_2 + (\gamma_1 - \gamma_2) d\theta_1, \tag{D.11}$$

where θ_1 and θ_2 are

$$\theta_1 = \frac{q_1 dp_2 - p_1 dq_2}{p_1 q_2 - p_2 q_1} \qquad \theta_2 = \frac{p_2 dq_1 - q_2 dp_1}{p_1 q_2 - p_2 q_1}. \tag{D.12}$$

Now since $\mathcal{M}^8$ is a hyperkähler quotient of $\mathcal{M}^{12}$, the twistor space of $\mathcal{M}^8$ is a symplectic quotient of the twistor space of $\mathcal{M}^{12}$ by the action of the complexified group $U(1)^{\mathbf{C}} = \mathbf{C}^*$. An element $\lambda \in \mathbf{C}^*$ acts by $p_i \to \lambda p_i, q_i \to \lambda q_i$. The corresponding moment map is $\gamma_1 + \gamma_2$. So we put $TrB = \gamma_1 + \gamma_2 = 0$ (picking the zero level of the moment map) and $p_1 q_2 - p_2 q_1 = \gamma_1$ (fixing the $\mathbf{C}^*$ gauge). Then in terms of $\gamma = \gamma_1, p_2, q_1, q_2$, the two-form ω on the twistor space of $\mathcal{M}^8$ is the restriction of ω^{12} and is given by

$$\omega = 2d\gamma \wedge \log \frac{p_1}{q_2} + d(q_1 p_2) \wedge d \log \frac{p_1 p_2}{q_1 q_2}. \tag{D.13}$$

Let us introduce $\eta_2 = \gamma^2 = -\det B$, $\eta_1 = q_1 p_2$, $\chi_2 = \frac{1}{\sqrt{\eta_2}} \log \frac{p_1}{q_2}$, and $\chi_1 = \log \frac{p_1 p_2}{q_1 q_2}$ as local coordinates on the twistor space of $\mathcal{M}^8$. Then the transition functions are given by

$$\tilde{\eta}_2 = \frac{1}{\zeta^4}\eta_2 \tag{D.14}$$

$$\tilde{\eta}_1 = \frac{1}{\zeta^2}\eta_1 \tag{D.15}$$

$$\tilde{\chi}_2 = \zeta^2 \chi_2 + 4\zeta \tag{D.16}$$

$$\tilde{\chi}_1 = \chi_1. \tag{D.17}$$

Now we are in a position to use the generalized Legendre transform method of [52] (also see [37]). The auxiliary functions $\hat{f}_1, \hat{f}_2$ are given by

$$\oint_{C_1} \frac{d\zeta}{\zeta^j} \hat{f}_1 = \oint_0 \frac{d\zeta}{\zeta^j} \chi_1 + \oint_\infty \frac{d\zeta}{\zeta^j} \tilde{\chi}_1 = \oint_{\Gamma_1} \frac{d\zeta}{\zeta^j} \chi_1, \tag{D.18}$$

$$\oint_{C_2} \frac{d\zeta}{\zeta^j} \hat{f}_2 = \oint_0 \frac{d\zeta}{\zeta^{j-2}} \chi_2 + \oint_\infty \frac{d\zeta}{\zeta^j} \tilde{\chi}_2 = \oint_{\Gamma_2} \frac{d\zeta}{\zeta^{j-2}} \chi_2 + 4 \oint_\infty \frac{d\zeta}{\zeta^{j-1}} - 2\pi i k \oint_{real\ period} \frac{d\zeta}{\zeta^{j-2}} \frac{1}{\sqrt{\eta_2}}, \tag{D.19}$$

where the contours Γ_1, Γ_2 surround the two pairs of roots of p_1 and q_2 (these are solutions of $2\sqrt{\eta_2} + \eta_1 = 0$), and Γ_1 also surrounding the roots of p_2 and q_1 (these are solutions of $\eta_1 = 0$). Recalling the definitions of χ_1, χ_2 and the relations

$$p_1 q_2 = 2\sqrt{\eta_2} + \eta_1 \tag{D.20}$$

$$p_2 q_1 = \eta_1 \tag{D.21}$$

we rewrite eqs. (D.18), (D.19) as

$$\oint_{C_1} \frac{d\zeta}{\zeta^j}\hat{f}_1 = \left(\oint_C + \oint_\Gamma\right)\frac{d\zeta}{\zeta^j}(\log(2\eta_2+\eta_1)+\log\eta_1) \tag{D.22}$$

$$\oint_{C_2} \frac{d\zeta}{\zeta^j}\hat{f}_2 = \oint_C \frac{d\zeta}{\zeta^{j-2}}\frac{1}{\sqrt{\eta_2}}\log(2\sqrt{\eta_2}+\eta_1) - 4\oint_0 \frac{d\zeta}{\zeta^{j-1}} - 2\pi i k \oint_{real\ period}\frac{d\zeta}{\zeta^{j-2}}\frac{1}{\sqrt{\eta_2}} \tag{D.23}$$

with the figure-eight shaped contour C surrounding pairs of conjugate roots of $2\eta_2+\eta_1$ and figure-eight shaped contour Γ surrounding the two roots of η_1.

We can think of η_2 as defining an elliptic curve $S : t^2 = \eta_2(\zeta)$ covering the $\mathbf{P}^1$ parametrized by ζ twice. Thanks to eqs. (D.9),(D.10) functions p_1, p_2 define a section of $L^{-2}|_S$ and functions q_2, q_1 define a section of $L^2|_S$. This picture is very close to the one in [33] and leads us to the same conclusion that $\log\frac{p_1}{q_2}$ picks up $4\pi i$ when we move around the imaginary cycle of S. This gives $k = 2$ in the above formula.

Now we define function $G(\eta_1, \eta_2, \zeta)$ such that $\partial G/\partial\eta_j = \hat{f}_j/\zeta^{\deg\eta_j - 2}$. And for the Legendre transform of the Kähler potential $F = \frac{1}{2\pi i}\oint\frac{d\zeta}{\zeta^2}G(\eta_1,\eta_2,\zeta)$ we find

$$\begin{aligned} F = {} & \frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta^2}(2\sqrt{\eta_2}+\eta_1)(\log(2\sqrt{\eta_2}+\eta_1)-1) - \\ & \frac{1}{2\pi i}\oint_0 \frac{4\eta_2}{\zeta^3}d\zeta + \frac{1}{2\pi i}\oint_\Gamma \frac{d\zeta}{\zeta^2}\eta_1(\log\eta_1 - 1) - \qquad (D.24) \\ & 2\oint_{realperiod}\frac{d\zeta}{\zeta^2}\sqrt{\eta_2}. \qquad (D.25) \end{aligned}$$

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