String Theory on Calabi-Yau Manifolds: Topics in Geometry and Physics

Thesis by

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Abstract

We study aspects of the geometry and physics of type II string theory compactification on Calabi-Yau manifolds. The emphasis is on non-perturbative phenomena which arise when the compactification manifold develops singularities, and the implications on quantum geometry of the Calabi-Yau spaces. We use both the methods of low energy supergravity and the complementary approach via D brane probes. Applications to the study of four-dimensional $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric gauge theories are considered as well.

Contents

Chapter 1 Introduction

The successes of string theory, that have been highly acclaimed, can be summarized as follows:

- String theory is the only known solution to the problem at the core of modern physics – the incompatibility of quantum field theory and general relativity. The singularities of spacetime are incurable in any theory of gravity based on gravitating point particles.

- It predicts the most important physical principles – general relativity, gauge theory and supersymmetry[*1].

- It is a realization of a very old idea that physics should be determined by mathematical principles alone, with no arbitrary dimensionless parameters.

It is the main purpose of this work to study geometry as it appears in string theory. Since string theory is a theory of quantum gravity, the subject is quantum geometry. The only known formulation of string theory is that of a first quantized theory of relativistic strings, and consequentially string theory is necessarily background dependent. A framework for addressing issues of quantum geometry is study of compactifications of string theory.

We consider string theory on spacetimes of the form $M_n \times X^{10-n}$, where M_n is n-dimensional flat Minkowski space, and X is a compact manifold which will be our laboratory. In picking X we are governed by two considerations: to be worthy of discussion, this background must solve string equations of motion, and second the classical geometry of X must be both complicated enough to be interesting, and simple enough to be tractable. The first requirement is satisfied if X is a Calabi-Yau

[*1] Supersymmetry is believed to be the necessary ingredient for physics beyond the Standard Model.

manifold, and the second if its complex dimension is three [*2].

What is quantum geometry?

String theory has two expansion parameters, the string coupling constant g_s and the string tension $1/\alpha'$, and, consequentially, there are two sources of quantum fluctuations. Quantum mechanics in the conventional sense of the word is governed by g_s which counts the contribution of string loops to scattering amplitudes – expansion in powers of g_s is an expansion in the genus of the Riemann surface which is the string worldsheet. At a given genus, the theory is described by a two-dimensional CFT on the Riemman surface, with a dimensionful coupling constant α', the inverse string tension.

String theory compactifications with $g_s = 0$, described by CFT at genus zero, give rise to classical stringy geometry. The question of how geometry is effected by replacing point particles by strings has been extensively studied in the early '90s. While we will review some of the key results below, let me state them briefly here. The effective coupling constant of the CFT compactification is α'/R^2, where R is some characteristic size in the Calabi-Yau. The correlation functions in the theory will be power series expansions in α'/R^2, so the leading order can be extracted by setting $\alpha' = 0$. This is classical geometry, since in the infinite tension limit string reduces to a point particle, but, the classical quantities will be corrected order by order in stringy fluctuations. This result seems fairly tame, but one finds that some topological quantities on X are not invariants of classical string theory, but are modified by quantum effects. By far the most remarkable result, however is mirror symmetry. Mirror symmetry is an essentially trivial symmetry of the CFT. Its implications on geometry are nothing short of spectacular. It states that there exist pairs of Calabi-Yau manifolds (X, Y) of different topology – their classical geometry is entirely different, which nevertheless yield the same conformal field theory – in stringy geometry they are the same.

Quantum geometry, our subject, is the problem at $g_s \neq 0$. We study geometry and physics which emerge at the point where stringy geometry and quantum mechanics

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[*2] It is the case that state-of-the-art of modern mathematics, algebraic geometry, cannot count beyond three.

meet.

When the coupling is small but non-zero, the perturbation expansion as the sum over surfaces exists, however the series diverges with g_s faster than in field theory. In the lack of microscopic formulation of string theory at arbitrary value of g_s, many aspects of quantum geometry can be studied using effective field theory methods. In this approach the quantum geometry, properties of the compactification manifold X, are determined by the study of the low energy Lagrangian. The Lagrangian describes physics of massless string modes, where the tower of massive string states is integrated out. To the zeroth order in α', the coupling constants of the theory and the masses of BPS states are determined by classical geometry of X. The scalar fields in the Lagrangian are coordinates on the configuration space of the possible choices of X, and their kinetic terms are the metric on the configuration space. The α' and g_s dependence of string theory correct the coupling constants of the low energy theory, and are the quantum gravity corrections to the geometry of X.

It is by now well-known that non-perturbative string theory contains states which do not appear at any order of string perturbation theory, most notably solitons called Dirichlet (D) branes. On one level, compactification of string theory on Calabi-Yau manifolds shows that inclusion of these states is necessary for consistency of string theory. On the other, consequences of including these states on classical and stringy geometry are fairly dramatic: the most basic topological invariants of Calabi-Yau manifolds are not invariants of quantum geometry.

Inclusion of D branes in the theory provides one with new tools to study geometry in string theory. D branes can be used as probes of geometry, so in this way they define quantum geometry: The remarkable fact that D branes have a perturbative string description [*3] allows one to ask what geometry D branes see. As of now, it is not entirely clear how this sort of geometry meshes with quantum string geometry of the type discussed above.

On a different level, particle physics has had beautiful and fruitful connections with geometry – most notable is the relation of YM theory and general relativity to

[*3] Although they are not perturbative string states.

differential geometry. A prominent part of this work is to exploit the essential link between geometry and physics that string theory provides. Mirror symmetry was certainly one great contribution of string theory to mathematics. We will run the arrow in the other direction.

The plan of the work is as follows. In the second chapter we review some basics of the classical geometry of Calabi-Yau manifolds. We discuss $N = 2$ SCFT describing propagation of type II strings on Calabi-Yau manifolds since, in our view, understanding classical string geometry is a prerequisite for the quantum geometry. The aim of the chapter is to serve as an extended introduction for the ones to follow. The key result is mirror symmetry of Calabi-Yau manifolds. In the third chapter we leave the realm of perturbative string theory and discuss Calabi-Yau manifolds from the point of view of effective field theory of string compactification. Singularities in the moduli spaces force one to face the failure of the CFT, but also to find its cure. This leads us to an application which describes work in [1]: we are able to learn about moduli spaces of $\mathcal{N} = 2$ supersymmetric Yang Mills theories from geometry of Calabi-Yau manifolds. In the fourth chapter we study Calabi-Yau orbifolds, spaces of the form $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$ both from the point of view of closed and open string (D brane) conformal field theory. The closed string CFT was introduced in [2, 3], and we develop the open string description [4]. In both cases quantum geometry does not have a classical limit. In the last chapter, we discuss implications of non-perturbative physics on mirror symmetry – the quantum mirror symmetry [5]. Taking the quantum mirror symmetry conjecture seriously, we "derive" mirror pairs of manifolds in a new way. Finally, we resolve some issues in constructions of chiral $\mathcal{N} = 1$ theories in type II string theory using branes constructions ala Hanany-Witten. The last chapter is based on [6].

Chapter 2 Geometry and Physics of Calabi-Yau Manifolds: the view from the worldsheet

2.1 Why Calabi Yau Manifolds?

Consider type II string theory compactified on a spacetime of the form $\mathcal{M}_4 \times X$, where $\mathcal{M}_4$ is flat four-dimensional Minkowski spacetime, and X is a compact six-dimensional manifold. In a theory with dynamical gravity, such as string theory, compactification on $\mathcal{M}_4 \times X$ means a choice of vacuum of the theory of the above form.

Perturbative string theory in this background is given in terms of maps $\phi : \Sigma \to \mathcal{M}_4 \times X$, where Σ is the worldsheet of the string. The string action is given by a product of a free theory describing string propagating on $\mathcal{M}_4$ and a nonlinear sigma model associated with X:

$$S_X = t \int_\Sigma d^2 z [g_{ij} \partial_z \phi^i \bar{\partial}_{\bar{z}} \phi^j + i g_{ij} \psi^i_+ D_{\bar{z}} \psi^j_+ + i g_{ij} \psi^i_- D_z \psi^j_- + \frac{1}{2} R_{ijkl} \psi^i_+ \psi^j_+ \psi^k_- \psi^l_-]. \tag{2.1}$$

Above, $t = 1/4\pi\alpha'$ is the coupling constant, $z, \bar{z}$ are coordinates on Σ, ϕ^i are coordinates on X, so that $\phi^i(z, \bar{z})$ describe the local embedding of Σ into spacetime. ψ_-, ψ_+ are vectors on X (sections of the tangent bundle TX) and from the worldsheet point of view they are left and right moving Majorana-Weyl fermions:

$$D_{\bar{z}} \psi^i_+ = \partial_{\bar{z}} \psi^i_+ + \partial_{\bar{z}} x^j \Gamma^i_{jk} \psi^k_+,$$

and similarly for ψ_-. This action has $(1,1)$ supersymmetry: there is a left and a right moving worldsheet supersymmetry generated by

$$\delta x^i = i\epsilon_- \psi_+^i + i\epsilon_+ \psi_-^i,$$

$$\delta \psi_+^i = -\epsilon_- x^i - i\epsilon_+ \psi_-^j \Gamma^i_{jk} \psi_+^k,$$

$$\delta \psi_-^i = -\epsilon_+ x^i - i\epsilon_- \psi_+^i \Gamma^i_{jk} \psi_-^k.$$

Consistency of string perturbation theory requires the action to be conformaly invariant. In a conformally invariant theory beta function $\beta_{i,j}(g)$ must vanish and at the one loop level, the condition is that X admits a metric whose Ricci tensor vanishes:

$$R_{ij}(g) = 0,$$

which is just the Einstein equation in the vacuum. Generically, there are corrections at all orders in $\frac{\alpha'}{R^2}$ so the solutions are known only in some special cases, for example when the metric g appearing in the action is exactly flat $R_{ijkl} = 0$*1, so and consequentially, no generic solutions to the equations of motion are known.

In cases with more worldsheet supersymmetry the situation improves significantly*2. A general structure of second possible supersymmetry is

$$\delta_f x^i = i\epsilon_- f(x)^i_j \psi_+^j + i\epsilon_+ f(x)^i_j \psi_-^j,$$

$$\delta_f \psi_+^i = -\epsilon_- f(x)^i_j x^j - i\epsilon_+ \psi_-^j \Gamma^i_{jk} \psi_+^k,$$

$$\delta_f \psi_-^i = -\epsilon_+ f(x)^i_j x^j - i\epsilon_- \psi_+^i \Gamma^i_{jk} \psi_-^k,$$

where f is a real tensor on X*3. In order for this to define a supersymmetry trans-

*1 Together with compactness of X this implies $X = T^6$.

*2 Ultimately this will imply more space-time supersymmetry as well.

*3 The left-right symmetric theory in which n such tensors exist has (n,n) supersymmetry on the worldsheet. Often, we will say that such a theory has $N = n$ supersymmetry.

formation f has to satisfy the requirement that [7]:

$$f^i_j f^j_k = -\delta^i_k, \quad D_k f^i_j = 0, \quad \text{and} \quad g_{ij} f^i_k f^j_l = g_{kl}. \tag{2.2}$$

The first condition defines f to be a complex structure, because the existence of f makes X into a complex manifold. Basically, such a tensor always exists if X is a complex manifold [*4]. The more interesting question is whether a given real manifold X is also a complex manifold, that is if one can find a globally defined complex structure f on X. This can be rephrased as a certain topological constraint on X[*5].

The last condition in eq.(2.2) is a condition on the metric g to be Hermitian: in the above choice of complex structure the only non-vanishing components of the metric are $g_{i\bar{j}}$, which together with the fact that the metric is real implies that it is Hermitian in the usual sense. If in addition to being a complex manifold X has a complex structure which is covariantly constant, which is the second condition, X is called a Kähler manifold[*6]. It is common to consider a another quantity built out of g and f, :

$$J = \frac{1}{2} g_{ij} f^i_k d\phi^j \wedge d\phi^k = i g_{i\bar{j}} dz^i \wedge dz^{\bar{j}},$$

which is closed since g and f are covariantly constant, $dJ = \partial_i J_{jk} dx^i \wedge dx^j \wedge dx^k = D_i J_{jk} dx^i \wedge dx^j \wedge dx^k = 0$. The form J is called the Kähler form and being closed J defines a Kähler class

$$[J] \in H^{1,1}(X),$$

which is non trivial since $\frac{1}{3!} J \wedge J \wedge J$ is the volume element on X, and thus J cannot be exact. The fact that the X is Kähler constrains the topology on X – in the given complex structure $H^{1,1}(X) \neq 0$.

[*4] That is, if X is a complex manifold, it can be covered by patches $\{U_i\}$ each of which is the same as $\mathbb{C}^3$ where this identification proves local coordinates on U_i, and such that on all the nonempty overlaps $U_i \cap U_j$, $i \neq j$ there are exist holomorphic changes of coordinates the gluing U_i's together. Now, if one picks a patch with coordinates $z^i, \bar{z}^i = z^{\bar{i}}$ we can take f to be equal i if acting on z^i and $-i$ on $z^{\bar{i}}$ and that in this case f is preserved under holomorphic changes of coordinates, so it is defined globally.

[*5] The constraint is vanishing of Nijenhuis tensor built out of f and its first and second derivatives, and it is topological in the sense that it does not depend on the metric on X.

[*6] It is equivalent to existence of a local function K on X such that $g = \partial\bar{\partial} K$.

In order for the action eq. (2.2), to be in addition conformally invariant X needs to be Ricci. For a Kähler manifold this can be rephrased in an invariant manner: since the only components of the Ricci tensor which are non-zero are $R_{i\bar{j}} = -R_{\bar{j}i}$, the Ricci tensor defines a two form

$$\mathcal{R} = R_{i\bar{j}} dz^i \wedge dz^{\bar{j}},$$

which is closed, $d\mathcal{R} = 0$. The cohomology class of the Ricci form in $H^{1,1}(X)$ is called the first Chern class $c_1(X) = [\mathcal{R}] \in H^{1,1}$, and Ricci flatness is the statement of its triviality,

$$c_1(X) = 0.$$

Def. 1. *A Calabi-Yau manifold is a Kähler manifold with vanishing first Chern class.*

An important consequence of the Calabi-Yau condition is that the holonomy group of the manifold is at most $SU(3)$. On a generic $2d$ real dimensional manifold the holonomy group is $SO(2d)$: upon parallel transport around a closed loop vectors undergo an $SO(2d)$ rotation $\delta v^i = [D_j, D_k]v^i = (R_{jk})^i_p v^p$, with $R_{jk} \in SO(2d)$. On a Kähler manifold, the holonomy is restricted to $U(d)$, since the complex structure is covariantly constant. The trace part of the holonomy, $(R_{ij})^k_k$, is the Ricci tensor and vanishing of this implies that holonomy group is contained in $SU(d)$. Thus, Ricci flatness is equivalent to the statement that the manifold has at most $SU(d)$ holonomy. The holonomy provides classification of threefolds which are Calabi-Yau. Trivial holonomy implies that the manifold is flat – a six torus T^6, and $SU(2) \subset SU(3)$ that it is a product of $K3$ manifold, the unique Calabi-Yau two-fold which is not flat, and T^2. The Calabi-Yau condition severely constrains the topology of X. The fact that the holonomy group is exactly $SU(3)$ implies $h^{1,0} = 0 = h^{2,0}$, and that $h^{3,0} = 1$. The unique holomorphic three-form of $\Omega \in H^{3,0}(X)$, plays an important role, as we will shortly see. Furthermore, if X is connected $h^{0,0} = 1 = h^{3,3}$. The values of $h^{1,1}$ and $h^{2,1}$ are the crudest topological classification of Calabi-Yau threefolds. All in all, after acounting for the usual relations $h^{p,q} = h^{q,p} = h^{3-p,3-q}$, the Hodge diamond of a Calabi-Yau manifold is:

$$
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}
\quad = \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

We should worry that higher loop stringy corrections could render the equation $R_{i\bar{j}} = 0$ physically meaningless. The fact that this is not the case [7] is a consequence of a non-renormalization theorem, and a beautiful theorem due to Yau. The non-renormalization theorem states that starting with a Ricci flat Kähler metric on X, stringy corrections will not change the complex structure on X, nor the Kähler class of the metric. One can furthermore show, that the renormalization does not spoil Ricci-flatness of the metric. S.T.Yau's theorem provides both the existence, and the uniqueness of the metric:

Thm. 1. *If X is a Kähler manifold with a vanishing first Chern class there exists, for a given choice of complex structure on X, a unique Ricci-flat metric.*

2.2 $N = 2$ Superconformal Field Theories

On a Calabi-Yau manifold, the non-linear sigma model action eq.(2.1) is (super)conformally invariant and can be rewritten as:

$$
\begin{aligned}
S = 2t \int_\Sigma d^2 z [\frac{1}{2}(g_{i\bar{j}} \partial_z \phi^i \bar{\partial}_{\bar{z}} \phi^{\bar{j}} + g_{\bar{i}j} \partial_z \phi^{\bar{i}} \bar{\partial}_{\bar{z}} \phi^j) + i g_{i\bar{j}} \psi^i_+ D_{\bar{z}} \psi^{\bar{j}}_+ \\
+ i g_{i\bar{j}} \psi^i_- D_z \psi^{\bar{j}}_- + R_{i\bar{j}l\bar{k}} \psi^i_+ \psi^{\bar{j}}_+ \psi^l_- \psi^{\bar{k}}_-].
\end{aligned}
\tag{2.3}
$$

In general, there are few results in the theory that are explicitly calculable. The problem is evident from the very fact that the coupling functions $g_{i\bar{j}}(\phi)$ – the Calabi-Yau metrics – are not explicitly known. However, $N = 2$ supersymmetry coupled with conformal invariance is quite constraining, and much of the structure of the theory can be inferred from the underlying $N = 2$ superconformal algebra. This is what we turn to next. We will review some basic features of $N = 2$ superconformal field theories, for more details see for example [8, 9]. The standard (left moving) superstring algebra with $N = 1$ supersymmetry is generated by the energy momentum tensor, and its worldsheet superpartner $G(z)$. The $N = 2$ supersymmetry implies the existance of a second supercurrent, and we will denote the two currents $G^{\pm}$. In addition, the left moving theory has a $U(1)$ symmetry under which $\psi_-^i \to e^{i\alpha}\psi_-^i$, and $\psi_-^{\bar{i}} \to e^{i\alpha}\psi_-^{\bar{i}}$ with an associated conserved $U(1)$ current $J = i\psi_-^i i g_{i\bar{j}} \psi_-^j$ [*7].

A generic left moving operator on the world sheet has an expansion

$$\phi(z) = \sum_{n=-\infty}^{\infty} \phi_{n+s} z^{-n-s},$$

where s is the left moving conformal weight of the field. If s is half integer, the sum over n could run over integers or half integers, and this leads to two sub-sectors of the theory: the Neveu-Schwarz sector for $n \in \mathbb{Z} + \frac{1}{2}$, and the Ramond sector for $n \in \mathbb{Z}$. The boundary conditions on ψ are $\psi(e^{2\pi i}z) = e^{-2\pi i s}\psi(z)$, so insertions of fields in the Ramond sector must be accompanied by cuts on the world sheet. The Ramond fields are conserved modulo two, and give fermions in space-time, while the Neveu-Schwarz sector fields yield bosons. The energy momentum tensor has spin two, and its harmonics are commonly denoted by L_m, the $G^{\pm}$ have spin $\frac{3}{2}$, and J has spin 1

[*7] The $N = 1$ theory has the same $U(1)$ symmetry at the classical level, however if X is not Ricci flat the symmetry is spoiled by an anomaly.

spin [8]. In terms of the modes the $N=2$ algebra is

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0},\\
[J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0},\\
[L_m, J_n] &= -nJ_{m+n},\\
[L_m, G_r^{\pm}] &= (\frac{m}{2}-r)G_r^{\pm},\\
[J_m, G_r^{\pm}] &= \pm G_{n+r}^{\pm},\\
\{G_r^-, G_s^+\} &= 2L_{r+s} - (r-s)J_{r+s} + \frac{c}{3}(r^2-\frac{1}{4})\delta_{r+s,0},
\end{aligned} \tag{2.4}$$

where the (anti)commutators we have not written down vanish. The algebra has a very important property. There exists a sequence of one parameter deformations $\mathcal{U}_\theta$ called *spectral flows*, which act by isomorphism on the algebra, and therefore on representations as well,

$$\begin{aligned}
U_\theta L_n U_\theta^{-1} &= L_n + \theta J_n + \frac{c}{6}\theta^2\delta_{n,0},\\
U_\theta J_n U_\theta^{-1} &= J_n + \frac{c}{3}\theta\delta_{n,0},\\
U_\theta G_r^{\pm} U_\theta^{-1} &= G_{r\pm\theta}^{\pm},
\end{aligned}$$

and $\mathcal{U}_\theta : \mathcal{H}_0 \to \mathcal{H}_\theta$. It is possible to find an implicit expression for operator U_θ in terms of the generators of the algebra, but we will not need it here. For $\theta = \frac{1}{2}$, this maps the NS sector to the R sector of the theory [9]. Note that U_θ maps J_n and L_n to affine linear combinations of each other, and thus has essentially trivial action, but it changes the boundary conditions on the supercurrents.

Now let's consider the representations of the algebra eq.(2.4). From above, $[L_0, J_0] = 0$, so we will work in the basis of their eigenvectors $|h, g, * >$, where h is the L_0 eigen-

[8] Spin is the same as the left conformal dimension.

[9] Actually, this is somewhat subtle. One must require that the operator U_θ must be local with respect to the fields in the theory. More precisely, it is already semi local since it is a spin $\frac{1}{2}$ field in the original theory, but its OPE with other fields must introduce no additional non-locality. This turns out to be a constraint on the original theory to contain only integer U(1) charges in the NS sector – this is in fact an additional constraint on the theory we must impose in order to have $N=2$ supersymmetry.

value,the conformal dimension, and the charge q is the eigenvalue of J_0. Commutation relations imply that if the spectrum of L_0 is bounded below, there exists a state $|\phi>$ which is annihilated by all L_m, J_m, for $m>0$ since they act as lowering operators. In the NS sector, it is easy to show that this implies that $G_{m>0}|\phi>=0$ as well, so the NS sector has in fact a unique vacuum. In the Ramond sector the story is different, since there are also zero modes of the supercurrents $G_0^{\pm}$, so Ramond ground states are those which in addition satisfy

$$G_0^+|\phi>=0, \quad G_0^-|\phi>=0.$$

It is conventional to call the ground states primary states, since all other states can be built by raising operators acting on them.

Let us return to the NS sector for a moment. It proves worthwhile to constrain the notion of a primary state to *chiral* primary states which are those that satisfy:

$$G_{-1/2}^+|\phi>=0, \quad G_{1/2}^-|\phi>=0.$$

The second equation is trivial for a primary state, but the reason for writing the condition in this way is that it is manifest that the chiral primary states are images of the Ramond ground states under the spectral flow by $\theta=-\frac{1}{2}$. Similarly, one can define anti-chiral primaries which are annihilated by $G_{-1/2}^-$, and related to Ramond ground states by $\theta=\frac{1}{2}$. The Fock space of the theory is a product of the left and right moving Fock spaces. We will see in the next section that one is able to compute the (R,R) ground states of the Calabi-Yau conformal field theory, since the only information needed is the topology of the space. The remainder of the massless spectrum can be obtained from the (R,R) ground states by spectral flows.

The chiral operators of the $N=2$ conformal field theory form a ring, because the the operator product of any two chiral operators is a chiral operator. To see this, note that for any state $|\lambda>$

$$<\lambda|\{G_{1/2}^-, G_{-1/2}^+\}|\lambda>=<\lambda|2L_0-J_0|\lambda>,$$

and since $G_r^+ = {G_{-r}^-}^\dagger$, the left-hand side is positive semi-definite, so $h_\lambda \geq q_\lambda/2$, for any operator λ. Chiral operators satisfy $G_{-1/2}^+|\lambda>=0$ and saturate the inequality *10. Thus, they correspond to states with $|h = q/2, q, * >$. Now consider the product of two chiral operators $\phi(z), \chi(z)$. On dimensional grounds, the operator product must have the form

$$\phi(z)\chi(w) = \sum_i (z-w)^{h_\phi + h_\chi - h_{\lambda^i}} \lambda^i(z).$$

By charge conservation, $q_\lambda = q_\phi + q_\chi$, but then $h_\lambda \geq \frac{1}{2}(q_\phi + q_\chi) = h_\phi + h_\chi$. Thus we learn that the operator product

$$(\phi\chi)(z) = \lim_{w\to z} \phi(z)\chi(w),$$

is either a chiral primary, or zero.

We will soon be able to show that the number of chiral primaries is finite. Thus, chiral primaries form a finite ring, and the ring is known as the *chiral ring*. Similar considerations apply to anti-chiral ring as well. Finally, there is identical structure for the right movers as well. The full theory is the direct product of the left and right moving fields, so there are four rings $(c,c), (a,c), (c,a), (a,a)$, however the third and the fourth ones are just charge conjugates of the first two. The spectral flows $\mathcal{U}_{\pm 1/2}$ relate the $(c,c), (c,a)$ rings to (R,R) ground-states of the theory, and as we will see in the next section, this provides an identification of these rings with cohomology rings of X.

2.2.1 Non-linear Sigma Model

There is a very beautiful adiabatic argument due to Witten [11] that allows one to compute the (R,R) ground states on any complex manifold, and its specialization applies here as well.

We want to compute the ground states of (2.3). First, observe that the algebra

*10 We haven't shown the converse, but the proof is given in [10].

(2.4) contains the usual $N = 2$ algebra as its sub-sector. By identifying $G_0^{\pm} = Q^{\pm}$, $2L_0 - \frac{c}{12} = H$, one obtains for the right movers

$$\{Q_R^+, Q_R^-\} = H_R, \quad [H_R, Q_R^{\pm}] = 0,$$

and an identical algebra for the left movers. Consequently, the ground states, assuming supersymmetry is not dynamically broken, are zero energy configurations on the world sheet. Classically, any state of non-zero momentum will have non zero energy, so we can limit ourselves to constant configurations on the worldsheet and the action (2.4) reduces to supersymmetric quantum mechanics on X. The canonical anti-commutation relations of the fermions are

$$\{\psi_+^i, \psi_+^j\} = \{\psi_+^{\bar{i}}, \psi_+^{\bar{j}}\} = 0, \quad \{\psi_+^i, \psi_+^{\bar{j}}\} = g^{i\bar{j}}, \tag{2.5}$$

and similarly for ψ_-. After canonical quantization the right-supersymmetry charges are

$$Q_R^+ = \psi_+^i p_i^+, \quad Q_R^- = \psi_+^{\bar{i}} p_{\bar{i}}^+, \tag{2.6}$$

where p_i is the momentum conjugate to $\phi^i(\bar{z})$.

The algebra (2.5) is the usual algebra of fermionic creation and annihilation operators. We may take $\psi_+^i, \psi_-^{\bar{i}}$ to be the operators annihilating the vacuum $|0>$:

$$\psi_+^i|0>= 0, \quad \psi_-^{\bar{i}}|0>= 0,$$

so that states of the theory can be built by acting successively with $\psi_+^{\bar{i}}, \psi_-^i$ on the vacuum. The resulting wave functions are classified by their $U(1)_L \times U(1)_R$ charges, and the state with charge $(r, -s)$ is of the form:

$$b_{i_1 \ldots i_r \bar{j}_1 \ldots \bar{j}_s}(\phi) \psi_-^{i_1} \ldots \psi_-^{i_r} \psi_+^{\bar{j}_1} \ldots \psi_+^{\bar{j}_s} |0> . \tag{2.7}$$

This expression has several remarkable properties. Due to anti-commutation relations of the ψ's the amplitude $b_{i_1\ldots i_r\bar{j}_1\ldots\bar{j}_s}$ is antisymmetric under the interchange of any two i, and any two $\bar{j}$ indices, but apriori, it can have an arbitrary dependence on $\phi \in X$, the bosonic zero-mode. This implies that,effectively, the wave function is an (r,s) differential form on X, $b^{r,s} \in \Omega^{r,s}(X)$. In order for (2.7) to be the ground state of H_R it must be annihilated by the supercharges Q_R^+, Q_R^-. Let's consider the action of the Q_R^- operator in some more detail. It acts on $b_{i_1\ldots i_r\bar{j}_1\ldots\bar{j}_s}$ by taking a derivative with respect to $\bar{j}$ and it adds a fermion $\psi_+^{\bar{j}}$ if the wavefunction does not contain one already. This action is exactly the same as that of $\bar{\partial} = d\bar{z}^i \frac{\partial}{\partial \bar{z}^i}$ on $\Omega^{r,s}(X)$.

$$Q_R^- : \Omega^{r,s} \to \Omega^{r,s+1}.$$

Similarly Q_R^+ is the same as the Hermitian conjugate $\bar{\partial}^\dagger$ of $\bar{\partial}$ [*11] since it takes:

$$Q_R^+ : \Omega^{r,s} \to \Omega^{r,s-1}.$$

The forms in $\Omega^{r,s}(X)$ which are annihilated by both $\bar{\partial}^\dagger$, and $\bar{\partial}$ form a cohomology group $H^{0,s}(X, \wedge^r T^*)$, of $\bar{\partial}$-harmonic forms with coefficients in $\wedge^r T^*$.

We must still ask for these states to be annihilated by the left moving supercharges, since in order to be ground-states of the theory they must be ground-states of the left-moving Hamiltonian as well. By repeating the exercise one finds that the pair of right moving supercharges corresponds to ∂, and $\partial^\dagger$ operators on X. Now, it is a basic result that the ∂,$\bar{\partial}$ and the de-Rham operator cohomologies are isomorphic, in particular their harmonic forms are the same[*12]. The right-moving Hamiltonian $H_R = \{Q_R^+, Q_R^-\}$ is isomorphic to the Laplacian $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$. It is a matter of algebra to show $\Delta_{\bar{\partial}} = \Delta_\partial = \frac{1}{2}\Delta_d$. Thus, we have shown in fact that the Ramond-Ramond ground states correspond to harmonic forms on X, the elements of $H^*(X)$.

Before we leave this subsection, let us note one important fact. Above we have chosen to call ψ_-^i and $\psi_+^{\bar{i}}$ creation operators and ψ_+^i,$\psi_-^{\bar{i}}$ annihilation operators. Whether

[*11] It is the Hermitian conjugate with respect to the natural inner product $<\omega,\eta> = \int_X \omega \wedge *\eta$.
[*12] $H_{\bar{\partial}}^*$ is a refinement of H_d^*, since $d = \partial + \bar{\partial}$, so for example one has $H_d^n = \sum_{n=r+s} H_{\bar{\partial}}^{r,s}$.

we call ψ_+^i or $\psi_+^{\bar{i}}$ a creation operator is clearly arbitrary. However, once this choice is made the choice between ψ_-^i, and $\psi_-^{\bar{i}}$ as the second creation operator is of content.

So consider instead requiring that

$$\psi_+^{\bar{i}}|0>=0, \quad \psi_-^i|0>=0,$$

so that a state of charge $(-r,-s)$ looks like

$$b^{i_1\dots i_r}_{\bar{j}_1\dots\bar{j}_s}(\phi)\psi_{i_1-}\dots\psi_{i_r-}\psi_+^{\bar{j}_1}\dots\psi_+^{\bar{j}_s}|0>.$$

where $\psi_{i-} = g_{i\bar{j}}\psi_-^j$. Following the arguments made above, one finds that there is a ground-state for every generator of $H^{0,s}_{\bar{\partial}}(X,\wedge^r T)$, where $\wedge^r T$ is the r-th exterior power of the tangent bundle of X. It is useful to note that on a Calabi-Yau d-manifold one can use the fact that there is a unique holomorphic d-form Ω, so that the map $\Omega:\wedge^r T\to\wedge^{d-r}T^*$ is an isomorphism of the cohomology groups,

$$H^{0,s}_{\bar{\partial}}(X,\wedge^r T) = H^{0,s}_{\bar{\partial}}(X,\wedge^{d-r}T^*).$$

From the conformal field theory point of view the two choices in computing the (R,R) ground-states differ by the overall sign of the left moving $U(1)$ charge. However in the underying geometric interpretation, the sign flip results in two completely different cohomology groups, $H^{0,s}_{\bar{\partial}}(X,\wedge^r T^*)$ and $H^{0,s}_{\bar{\partial}}(X,\wedge^{d-r}T^*)$, and the exchange of the two will not be a symmetry of a generic Calabi-Yau manifold X.

Rather than picking *one* assignment as a physical one in the confirmal field theory of X, the authors of [10, 12] conjectured that there exist pairs of manifolds X and $\tilde{X}$ which yield the same conformal field theory, but such that the geometrical interpretation of the conformal field theory operators is different in the two compactifications. In particular a given (R,R) ground state that is associated to an generator of $H^{0,s}_{\bar{\partial}}(X,\wedge^r T^*)$ in one interpretation corresponds to a generator of $H^{0,s}_{\bar{\partial}}(\tilde{X},\wedge^{d-r}T^*)$

in the other. This requires that $X, \tilde{X}$ satisfy

$$dim H^{0,s}_{\bar{\partial}}(X, \wedge^r T^*) = dim H^{0,s}_{\bar{\partial}}(\tilde{X}, \wedge^{d-r} T^*).$$

This symmetry of the conformal field theory under the exchange of X and $\tilde{X}$ is known as *mirror symmetry*.

2.3 Moduli Spaces

2.3.1 Families of Calabi-Yau Manifolds

Yau's theor gives the existance of a unique Ricci flat metric on X, provided a choice of complex structure and the Kähler class of the metric. The question we would like to address here is whether the Ricci-flat metric on X is unique. Generically, this is far from being the case. The most general metric deformation $g \to g + \delta g$ is:

$$\delta g = \delta g_{i\bar{j}} dz^i dz^{\bar{j}} + \delta g_{ij} dz^i dz^j + \text{c.c.}, \tag{2.8}$$

requiring that $[\mathcal{R}(g + \delta g)]$ vanishes provides constraints on the $\delta g_{i\bar{j}}$ and $\delta g_{ij} dz$, and we consider them in turn.

For the mixed index deformation $\delta g_{i\bar{j}}$ one finds that the resulting metric is Ricci flat provided that $\delta J = \delta g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$ is closed, so the inequivalent deformations of this type are classified by the choice of representative

$$[\delta J] \in H^{1,1}(X).$$

This deformation is commonly referred to as the deformation of the Kähler structure on X, and space of all possible choices of $[J]$ is called the moduli space of Kähler structures. We can put coordinates on the moduli space by writing $J = \sum_i J_i e_i$, where e_i forms a basis of $H^{1,1}$. There is a constraint on J which comes from the relationship of J to the metric and the requirement that all the volumes in X are non-negative. It is a basic result of complex differential geometry that for analytic submanifolds $\Sigma_k \in H_{2k}(X, Z)$

$$\mathrm{Vol}(\Sigma_k) = \frac{1}{k!}\int_{\Sigma_k} \wedge^k J,$$

so we must have $\int_{\Sigma_k} \wedge^k J \geq 0$, for all holomorphic complex k cycles. The space of allowed $J_i's$, the classical Kähler structure moduli space, forms a cone called the Kähler cone. The boundaries of the Kähler cone are places where classically some of the Σ_k's shrink to zero volume.

Deformations with pure metric indices are somewhat more complicated, but one can show that one still obtains a Ricci flat metric provided that

$$\Omega_{ijl} g^{l\bar{m}} \delta g_{\bar{m}\bar{k}} dz^i \wedge dz^j \wedge dz^{\bar{k}} \in H^{2,1}(X).$$

The pure index deformation results in a metric which is no longer Hermitian. There always exists a change of variables which will make the metric Hermitian, however such a change of variables cannot be holomorphic, and thus it necessarily changes the complex structure on X [*13].

There is also a natural way to parametrize possible choices of complex structures. We can pick a basis of homology three-cycles, $A_i, B^i \in H_3(X, \mathbb{Z})$, where $i = 0, \ldots h^{2,1}$,

[*13] A somewhat impractical, but easy to visualize way to describe deformations of complex structure is the following. One way to describe a compact complex $d = 3$ manifold is as a holomorphic hypersurface in projective space $X \subset \mathbb{P}^5$,

$$X : f(z) = 0 = \sum_i a_i \prod_{i=1}^{5} z_i^{m_i},$$

where z_i are coordinates on $\mathbb{P}^5$, $z_i \sim \lambda z_i$, $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$. In order for X to be well defined in $\mathbb{P}^5$, the equation must be homogenous of degree n: $f(\lambda z) = \lambda^n f(z)$, and in order for X to be a Calabi-Yau manifold one must have $n = 5$ [13]. We can use coordinates of the embedding space to provide (local) complex coordinates for X, thus defning a choice of complex structure on X. Smooth variations of coefficients a_i's in the defining equation will define a family of manifolds with the same topology as X, however generically there will be no holomorphic change of variables that will map one member of the family to the other.

and the dual basis of $H^3(X)$ consisting of three-forms α^i, β_i, so that

$$\int_{A_i} \alpha^j = \delta^i_j = \int_{B^j} \beta_i,$$

where all other integrals vanish. The holomorphic three-form Ω can be written in this basis as

$$\int_{A_i} \Omega = \phi^i, \quad \int_{B^j} \Omega = \mathcal{F}_j,$$

so that

$$\Omega = \phi^i \alpha_i + \mathcal{F}_i \beta^i.$$

Above, α_i and β^i are really electric magnetic duals of each other, so that ϕ^i can be thought of as coordinates on the complex structure moduli space, and $\mathcal{F}$'s become functions of ϕ's. One can show that this is in fact a complete characterization of the complex structure moduli space: the first-order variation of the holomorphic three-form Ω by definition changes the complex structure, and so produces an $H^{2,1}$ form:

$$\omega_i = \partial_i \Omega \in H^{3,0} \oplus H^{2,1},$$

where $\partial_i = \partial / \partial \phi^i$. The $1 + h^{2,1}$ partial derivatives of Ω give a basis of $H^{3,0} \oplus H^{2,1}$. This can be used to show that there must exist a function $\mathcal{F}(\phi)$, the prepotential, such that

$$\mathcal{F}_i = \partial_i \mathcal{F},$$

which contains all the information about the moduli space of complex structures on X.

We thus obtain a family of Calabi-Yau manifolds which are equivalent to X as topological manifolds, but differ as complex manifolds and by the choice of Kähler class. The classical moduli space of Ricci-flat metrics on X is at least locally

$$\mathcal{M}_{Calabi\,Yau}(X) = \mathcal{M}_{\mathcal{K}} \times \mathcal{M}_{\mathcal{C}}.$$

2.3.2 Families of CFT's

For every point $g(X)$ in the space of metrics on X there exists a conformaly invariant non-linear sigma model which describes string theory on X. Since the space of Calabi-Yau metrics is connected the family of associated CFT's is connected as well. Under a metric deformation $g \to g + \delta g$, the CFT is deformed by

$$S_\Sigma \to S_\Sigma + \frac{1}{8\pi\alpha'}\int_\Sigma x^*(\delta g) + \dots .$$

If δg preserves the Calabi-Yau conditions, the operator $x^*(\delta g)$ preserves $N = 2$ superconformal symmetry– it is exactly marginal. The space of CFTs one can build in this way need not be the same as the space of possible deformations of the metric, because the CFT may have marginal operators that do not have a geometric interpretation.

Most importantly, the NS sector contains a two form B which couples to the worldsheet as the pullback of a two form in space-time $\int x^*(B) = \int d^2 z i B_{i\bar{j}} \partial_z x^i \partial_{\bar{z}} x^{\bar{j}}$ [*14]. Such a coupling preserves conformal invariance provided that B is closed, and furthermore any exact part is irrelevant since the worldsheet is closed, thus all the CFT sees of B is its cohomology class $[B] \in H^{1,1}_{\bar{\partial}}(X)$. One should really be a little more careful. Under $B \to B + \Lambda$, string path integral picks up a phase $e^{2\pi i \int_\Sigma x*(\Lambda)}$, so a transformation with Λ in $H^{1,1}(X, \mathbb{Z})$ leaves the theory invariant. The space of physical B fields is $H^{1,1}_{\bar{\partial}}(X, \mathbb{C})/H^{1,1}_{\bar{\partial}}(X, \mathbb{Z})$.

The full CFT moduli space of Kähler and B field deformations takes a simple form. First, notice that in the presence of the B field we can rewrite the sigma model as

$$\begin{aligned} S = \; & 2t \int_\Sigma \; d^2 z \, [g_{i\bar{j}} \partial\phi^i \bar{\partial}\phi^{\bar{j}} + i g_{i\bar{j}} \psi^i_+ D_{\bar{z}} \psi^{\bar{j}}_+ + i g_{i\bar{j}} \psi^i_- D_z \psi^{\bar{j}}_- + R_{i\bar{j}l\bar{k}} \psi^i_+ \psi^{\bar{j}}_+ \psi^l_- \psi^{\bar{k}}_-] \\ + \; & t \int_\Sigma \; \phi^*(J + iB). \end{aligned} \tag{2.9}$$

[*14] We shall see that string theory contains moduli from the Ramond sector as well, however these couple to the world-sheet in a fairly roundabout manner. In the NS sector, there is also the dilaton ϕ which is universal for any compactification. which enters the string action via a topological term $-2\pi \int d^2 z \phi R_\Sigma$ so counts the Euler characteristic of the worldsheet. String equations of motion are solved if the dilaton is constant, so we will neglect it here.

The natural quantity in the CFT are therefore not the Kähler class J and the B-field separately, but their complex combination $J+iB$. Given a basis $\{e^i\}$ of $H^{1,1}$, the moduli space of complexified Kähler forms can be parametrized as

$$J + iB = \sum_i (J_i + iB_i)e^i, \quad e^i \in H^{1,1}(X).$$

The moduli space of conformal field theories on X is the enlarged moduli space of Calabi-Yau manifolds,

$$\mathcal{M}_{\mathcal{CFT}}(X) = \mathcal{M}_{\mathcal{K}}{}^{\mathbb{C}} \times \mathcal{M}_{\mathcal{C}}.$$

Now, let us look at the Kähler structure moduli space in some more detail.

2.4 Stringy Geometry

The theory can be analyzed exactly in the weak coupling limit of the conformal field theory $\frac{\alpha'}{R^2} \to 0$. In this limit the path integral is dominated by the classical solutions, and since the term $\int_\Sigma J + iB$ is topological in the sense that on a closed worldsheet all of its smooth variations vanish, classical solutions are those that minimize the first line in (2.9). We only need its bosonic piece which is $\int_\Sigma g_{i\bar{j}} \partial_z \phi^i \bar{\partial} \phi^{\bar{j}} = \int_\Sigma |\partial_z \phi^i|^2$, so the solutions to the equations of motion are holomorphic maps

$$\partial_z \phi^i = 0.$$

Consider now the string path integral at genus zero. It can be written as a sum over the homotopy types of maps from the worldsheet, which is a sphere $\Sigma \approx \mathbb{P}^1$ into X, $\phi : \mathbb{P}^1 \to X$[*15]. Such maps are, up to homotopy, classified by their winding number $m = 0, 1, \ldots$ where we take $m = 0$ to correspond to constant maps, mapping the

[*15] Topologically, the projective space $\mathbb{P}^1$ is an sphere S^2, hoverer it is the holomorphic version.

worldsheet to a point in X. The topological piece of the action is therefore

$$\int_{\Sigma} J + iB = \sum_{i} (J_i + iB_i) m^i,$$

where m^i are integers determining the homology class of $\phi(\Sigma)$. In terms of single - valued coordinates on the moduli space

$$q_i = e^{-2\pi t(J_i + iB_i)}, \quad i = 1, \dots h^{1,1},$$

the contribution of a holomorphic map with winding numbers m^i to the string path integral is

$$e^{S[\phi]} = q_1^{m^1} \dots q_{h^{1,1}}^{m^{h^{1,1}}}.$$

In the limit $t \to \infty$ the term that dominates comes from constant maps $m_i = 0$, with exponentially suppressed contributions from higher homotopy classes. Contributions to the action of holomorphic maps are called world sheet instantons.

We have seen that the (c, c) and (a, c) rings of the conformal field theory become associated to the cohomology rings of X, in the sense that there is one to one correspondence of chiral primary fields and the harmonic forms on X. A natural question to ask is whether the map extends beyond just the spectrum to the full ring isomorphism.

For example, one can pick three elements $\omega_i, \omega_j, \omega_k \in H^{1,1}(X)$ and ask to evaluate

$$\int_X \omega_i \wedge \omega_j \wedge \omega_k.$$

In conformal field theory the question is to compute a three-point function $< \phi_i \phi_j \phi_k >$ of observables associated to ω_i. On general grounds one can show that the result must have the form [14]

$$< \phi_i \phi_j \phi_k > = \int_X \omega_i \wedge \omega_j \wedge \omega_k + \sum_{\vec{m}} \frac{\vec{q}^{\vec{m}}}{1 - \vec{q}^{\vec{m}}} \int_{\Sigma} \phi^*_{\vec{m}}(\omega_i) \int_{\Sigma} \phi^*_{\vec{m}}(\omega_j) \int_{\Sigma} \phi^*_{\vec{m}}(\omega_k), \quad (2.10)$$

where $\vec{q}^{\vec{m}} = \prod_l q_l^{m^l}$, and the sum is over all homotopy types of maps $\phi : \mathbb{P}^1 \to X$. We have singled out the classical piece, $m_i = 0, \forall i$, from the instanton sum, which is just the value expected from classical geometry. There is however an infinite instanton sum, which because of the denominator in (2.10), starts to dominate as any of the $q_i = e^{-2\pi t(J_i + iB_i)} \to 1$. Thus, while in classical geometry singularities occur at real codimension one in the moduli space, in quantum theory strong quantum fluctuations occur at codimension two in the moduli space: at least in the case of holomorphic curves we have seen that one must take not only $J_i \to 0$ but also $B_i \to 0$ for the singularities to occur. However, at $B_i = 0 = J_i$ the conformal field theory stops making sense[15].

From the preceding analysis it is clear that world-sheet instanton effects cannot correct the complex structure moduli space, basically because holomorphic maps cannot affect the odd-dimension cycles. Therefore, as far as conformal field theory goes the complex structure moduli space is exact at the tree level. In principle, this only means that stringy corrections are absent, but leaves one to worry about possible loop corrections. However, this will be the subject of the next chapter.

2.5 Mirror Symmetry

We have seen above that a trivial symmetry of the conformal field theory implies the existence of mirror pairs of Calabi-Yau manifolds with $X, \tilde{X}$ satisfying $dim H_{\bar{\partial}}^{r,s}(X) = dim H_{\bar{\partial}}^{d-r,s}(\tilde{X})$, so for $d = 3$ this implies

$$(h^{1,1}(X), h^{2,1}(X)) = (h^{2,1}(\tilde{X}), h^{1,1}(\tilde{X})).$$

The claim is in fact much deeper, since because X and $\tilde{X}$ yield exactly the same conformal field theory, all correlation functions computed from X and $\tilde{X}$ must be the same. Needless to say, this requires miracles to happen from the point of view of geometry. The statement is thus that the moduli space of complex structures on $\tilde{X}$, which is exact at classical level is exactly the same as the quantum-corrected moduli space of Kähler structures on X, and vice versa. Worldsheet instantons come as a relief, of course, giving hope for the sanity of the theory.

We can easily see one immediate prediction of mirror symmetry. We have argued above for the general form of correlation functions of operators ϕ_i, ϕ_j, ϕ_k, (2.10) using their correspondence to elements $H^{1,1}(X)$. In order to compute them one must sum an infinite number of instanton corrections, in particular one must be able to enumerate all possible embeddings of a $\mathbb{P}^1$ into X – counting of "rational curves" is a hard and well known problem in mathematics. Mirror symmetry predicts that ϕ's can just as well be interpreted in terms of geometry of the mirror manifold to correspond to elements $\tilde{\omega}_i, \tilde{\omega}_j, \tilde{\omega}_k \in H^{2,1}(\tilde{X})$, and therefore:

$$< \phi_i \phi_j \phi_k >= \int_{\tilde{X}} \Omega \wedge (\tilde{\omega}_i \wedge \tilde{\omega}_j \wedge \tilde{\omega}_k) \cdot \Omega, \tag{2.11}$$

where Ω is the holomorphic three form [*16]. Mirror symmetry predicts that the two expressions, (2.10) and (2.11) are identical.

More generally, mirror symmetry is a map $\mathcal{M}_{\mathcal{K}}(X) \to \tilde{\mathcal{M}_{\mathcal{C}}}(\tilde{X})$, such that any correlation function on $\mathcal{M}_{\mathcal{K}}(X)$ is a pullback of a correlation function on $\mathcal{M}_{\mathcal{C}}(\tilde{X})$ under the map.

When we discussed the moduli space of complex structures on a Calabi-Yau manifold, we saw that there exists a choice of symplectic coordinates on $\mathcal{M}_{\mathcal{C}}(\tilde{X})$, such that: $i)$ the moduli space is parametrised by the periods

$$z_i = \frac{\int_{\gamma_i} \Omega}{\int_{\gamma_0} \Omega}, \quad i = 1, \ldots, h^{2,1}(\tilde{X})$$

of the holomorphic three-form, and $ii)$ all the correlation functions in $H^3(\tilde{X})$ are determined entirely by the knowledge of the classical prepotential function $\mathcal{F}(z)$.

With this choice of parametrization of the complex structure moduli space, the

[*16] In components, $(\tilde{\omega}_i \wedge \tilde{\omega}_j \wedge \tilde{\omega}_k) \cdot \Omega = (\tilde{\omega}_i)^i \wedge (\tilde{\omega}_j)^j \wedge (\tilde{\omega}_k)^k \Omega_{ijk}$, where $\tilde{\omega}^i$ is the image of $\tilde{\omega} \in H^{2,1}(\tilde{X})$ in $H^1(\tilde{X}, T)$, $\tilde{\omega}^i \sim \Omega^{ijk} \tilde{\omega}_{jk\bar{p}} dz^{\bar{p}}$ under the isomorphism $\Omega : H^{2,1}(\tilde{X}) \to H^1(\tilde{X}, T)$

mirror map is given by

$$J_i + iB_i = \frac{\int_{\gamma_i} \Omega}{\int_{\gamma_0} \Omega}$$

where γ_i, γ_0 are elements of $H_3(\tilde{X})$ appropriately chosen: we know that in the "large radius limit"

$$J_i \to \infty, \quad \forall i,$$

the correlation functions on $\mathcal{M}_{\mathcal{K}}(X)$ reduce to their classical values. For an appropriate choice of γ's there exists a limit in the complex structure moduli space such that the classical correlation functions on $\mathcal{M}_{\mathcal{K}}(X)$ agree with those in $\mathcal{M}_{\mathcal{C}}(\tilde{X})$ [16]. In general, though, the correlation functions are a power series in the z_i's, whose coefficients are integers that count rational curves on X. Of course, the same holds with X and $\tilde{X}$ exchanged.

Chapter 3 Non-Perturbative Phenomena in Calabi-Yau Compactifications

We have seen in the previous chapter that singularities of the conformal field theory are milder, in a sense, than in classical geometry, because they occur only at a real codimension two locus $\mathcal{V}_{\mathcal{S}}$ in the parameter space $\mathcal{M}_{CFT}$, so that one has to tune two parameters rather than one to encounter the singularity – we can always pick a path in the moduli space so as to avoid meeting $\mathcal{V}_{\mathcal{S}}$. In this chapter we wish to discuss what happens when we choose *not* to avoid the singularity. There is nothing wrong with the fact that at a sub-locus $\mathcal{V}_{\mathcal{S}}$ of its parameter space conformal field theory stops making sense – one can simply make a judicious choice not to consider such CFT's. One must, however, face the fact that $\mathcal{M}_{\mathcal{CFT}}$ is also the moduli space of vacua of string theory*[1]. For every marginal operator in the conformal field theory of X which moves one in the space of conformal field theories, there exists a modulus in the low-energy theory parametrizing the corresponding motion among the string vacua, and long wavelength fluctuations of scalar fields will ultimately explore all the vacua of the theory. In this chapter we will leave the microscopic realm to explore the nature of string vacua near $\mathcal{V}_{\mathcal{S}}$.

3.1 View From Low Energies

3.1.1 Massless Fields and Constraints from Supersymmetry

We turn now to some properties of type IIA and type IIB compactification on Calabi-Yau manifolds, partly supplementing the previous section. Consider a vacuum of the theory in which some fraction of the supersymmetry of the Lagrangian is preserved.

*[1] More precisely it is one subset of the string moduli space, as explained earlier, and as we will review below.

In a supersymmetric vacuum, supersymmetry variations of all the fields vanish. Since supersymmetry generators exchange bosonic and fermionic fields, the only nontrivial equations are those requiring that fermions be annihilated by surviving supersymmetries.

In $d = 10$ type IIA and type IIB theory are distinguished by the kind of supersymmetry they have*2. Type IIA theory has non-chiral $(1, 1)$ supersymmetry generated by two ten-dimensional Majorana-Weyl spinors of opposite chirality, denoted by **16** and **16′**, and type IIB theory has chiral $(2, 0)$ supersymmetry The spinors of type II theories are two Majorana-Weyl gravitinos, denoted ψ^i_M, $i = 1, 2$, and a pair of j spinors λ^i, where i labels different chiralities in the type IIA case. Now consider supersymmetry transformations of the fields.

Since $\delta\lambda^i \propto H = dB$, variation of λ^i vanishes using the equations of motion for Calabi-Yau backgrounds we are interested in. The gravitino variation takes the form

$$\delta\psi^i_M = D_M\eta^i, \quad i = 1, 2 \tag{3.1}$$

here M is an index labeling a ten-dimensional vector. The number of supersymmetries unbroken by the background is thus the number of solutions to $\delta\psi^i_M = 0$.

Take the ten-dimensional spacetime to be $\mathcal{M}_4 \times X$, with X a Calabi-Yau manifold. The equation says that η^i must be constant along $\mathcal{M}_4$, and covariantly constant along X. Upon parallel transport along a closed curve on X a spinor field on X is transformed by an element of the holonomy group of X. A spinor on X is a representation of $SO(6) = SU(4)$ and the positive and negative chirality spinors belong to **4** and $\bar{\mathbf{4}}$, respectively. It is a consequence of the Ricci flatness that the holonomy group of X is an $SU(3)$ subgroup of $SU(4)$. Since each **4** and $\bar{\mathbf{4}}$ contain precisely one $SU(3)$ singlet, they each give rise to precisely one covariantly constant spinor on X. Elementary group theory can then be used to show that each of **16** and **16′** contribute a Dirac spinor on $\mathcal{M}_4$. All in all, in compactification of type IIA and type IIB theories on a Calabi-Yau manifold the effective four-dimensional theory has

*2 The conformal field theory of IIA and IIB theory is the same, but they differ by the choice of GSO projection which determines the space-time supersymmetry of the theory.

$\mathcal{N} = 2$ supersymmetry.

Moduli of the compactification are necessarily massless fields on $\mathcal{M}_4$. There are two equivalent approaches to computing the massless sector. We can take the microscopic point of view and consider the product of the free CFT of $\mathcal{M}_4$ with an internal $N = (2,2)$ non-linear sigma model and use the results of the previous chapter to compute the massless spectrum. [*3] Alternatively, we can take the viewpoint that at the end of the day we are considering compactification of type II string theory so that all the states in four-dimensions have their origin in ten [*4].

The massless bosonic spectrum of type II theories in ten-dimensions comes from the (NS, NS) and (R, R) sectors. The (NS, NS) sector fields are the same in both theories, and contains the graviton, g_{MN}, the antisymmetric two-form B_{MN}, and the dilaton ϕ. The (R, R) sector contains antisymmetric $p+1$ form fields $\mathcal{A}^{p+1}$, where p is even in type IIA, $p = 0, 2$, and odd in type IIB, $p = -1, 1, 3$. In addition, as we discussed above there are massless fermions from the (R, NS) and (NS, R) sectors. These fields form a minimal massless multiplet of the $(1,1)$ or$(2,0)$ supersymmetry in ten-dimensions.

As we discussed above, upon compactification on X the marginal operators of the CFT, the zero modes of the metric g and the two-form B give scalars on $\mathcal{M}_4$. These are the $h^{1,1}(X)$ complex scalars from deformations of the complexified Kähler form $J + iB$, and another $h^{2,1}(X)$ complex scalars from deformations of the complex structure on X.

Now consider compactification of a $p+1$ form field A^{p+1}. The equation of motion of a massless form $\mathcal{A}$ is

$$\Delta \mathcal{A} = 0$$

where Δ is the Laplacian on $\mathcal{M}_4 \times X$, $\Delta = dd^* + d^*d = \partial_M \partial^M$ [*5]. Every one of the

[*3] To obtain the final answer one must perform the GSO projection, which we have not discussed there.

[*4] In some cases this point can be very subtle, in particular, for quotient (orbifold) theories. In any event, all claims made here can be checked by explicit CFT computation.

[*5] This is a Lorentz-gauge fixed form, and really the argument is as follows. The equation of motions are $d^\dagger d\mathcal{A} = 0$. On a compact manifold, or for field configurations with sufficiently rapid fall-off at infinity, this is equivalent to the first-order equation $d\mathcal{A} = 0$. Under the same conditions,

$p+1$ indices on $\mathcal{A}^{(p+1)}$ lies either in $\mathcal{M}_4$ or in X, so we can write

$$\mathcal{A}^{p+1} = \sum_n \tilde{\mathcal{A}}^{p+1-n} \wedge \omega_n, \quad \omega_n \in \Omega^n(X),$$

where $\tilde{\mathcal{A}}^{p+1-n}$ is a differential form on $\mathcal{M}_4$, $\tilde{\mathcal{A}}^{p+1-n} \in \Omega^{p+1-n}(\mathcal{M}_4)$. Now, due to the product structure $\mathcal{M}_4 \times X$ the Laplacian Δ is the sum of the operators on $\mathcal{M}_4$ and X,

$$\Delta = \Delta_{\mathcal{M}} + \Delta_X.$$

Thus, $\tilde{\mathcal{A}}_{p+1-n}$ is a massless form on $\mathcal{M}_4$ if and only if Δ_X annihilates ω_n, i.e., if and only if ω_n is harmonic. Harmonic n-forms on X are unique representatives of classes in $H^n(X)$, so

$$\omega_n \in H^n(X).$$

We have, by a different method, come to the result of the previous chapter that massless Ramond-Ramond states correspond to harmonic forms on X.

There are also fields that do not depend on which Calabi-Yau is chosen. The metric on $\mathcal{M}_4$ is free to fluctuate, as is the four- dimensional two form (whose magnetic dual in four-dimensions is a scalar, the axion), and the dilaton.

The only way all these fields can be arranged in multiplets of $\mathcal{N} = 2$ supersymmetry in four dimensions is as follows:

Compactification of type IIA theory on a Calabi-Yau manifold X gives:

- The universal $N = 2$ supergravity multiplet.
- The $h^{1,1}(X)$ vector multiplets.
- The $h^{2,1}(X) + 1$ hyper multiplets, one of which contains the dilaton-axion[*6].

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the all solutions are of the form $\mathcal{A} = \mathcal{A}_0 + d\mathcal{A}_1$, where $\mathcal{A}_0$ is harmonic, from which the result follows.

[*6] The gravity multiplet contains the four-dimensional graviton and graviphoton A^1 plus the superpartners, and it is the generic multiplet in any compactification. The four scalars of the dilaton-axion hypermultiplet are, in addition to the dilaton and the axion the two universal RR scalars coming from the three-form potential on the two universal three-cycles dual to holomorphic and the anti-holomorphic three-forms $\Omega, \bar{\Omega}$. The $h^{1,1}$ vectors come from the RR three-form A^3 in ten-dimensions, the multiplet being completed with the two scalars from the NS sector. The $h^{2,1}$ hypermultiplets contain the 2 NS scalars each, and 2 RR scalars from the three-form.

The moduli space of vector multiplets $\mathcal{M}_{\mathcal{V}}$ is therefore an $h^{1,1}$ complex dimensional manifold, and in fact it is the moduli space of complexified Kähler structures. The hyper multiplet moduli space $\mathcal{M}_{\mathcal{H}}$, on the other hand, is a quaternionic manifold whose "one half" includes the complex structure moduli of the CFT. It is an important consequence of a non-renormalization theorem in $\mathcal{N} = 2$ supersymmetric theories that $\mathcal{M}_{\mathcal{H}}$ and $\mathcal{M}_{\mathcal{V}}$ are decoupled, the metric on the moduli space of vector multiplets is independent of the scalars in hypermultiplets and vice versa [17]. This in particular means that $\mathcal{M}_{\mathcal{V}}$, although corrected by worldsheet instantons, has no corrections, perturbative or not, from the dilaton, because the dilaton sits in a hypermultiplet, The moduli space of hypermultiplets, on the other hand receives dilaton corrections, and is essentially beyond any computational reach at the moment, and we will have really nothing to say about this problem.

Repeating the exercise for type IIB theory on X we find:

- The universal $N = 2$ supergravity multiplet.
- The $h^{1,1}(X) + 1$ hyper multiplets, one of which contains the dilaton-axion.
- The $h^{2,1}(X)$ vector multiplets.[*7]

We see that, just as in type IIA theory, the moduli space of vector multiplets of type IIB is uncorrected by string loops as well. However, since $\mathcal{M}_{\mathcal{V}}$ is associated to choices of complex structures on X, it cannot receive α' corrections either – the classical geometry answer is the complete story.

3.1.2 Mirror Symmetry in Type II String Theory

Now, in the previous chapter we have found that there are mirror pairs of manifolds $(X, \tilde{X})$ which yield isomorphic internal conformal field theories, differing by the overall sign of the left moving $U(1)_L$ charge only. The natural question is what this sign flip does to the spacetime theory. From the microscopic point of view, one can show that

[*7] Here, the two complex scalars in vector multiplet must come from the (NS, NS) sector, while the hypermultiplet scalars have contributions both from the (R, R) and the (NS, NS) states.

flipping the sign of the $U(1)$ charge at the same time exchanges type IIA and type IIB theory. From the low energy point of view we can argue as follows. It is clear that we have two possibilities. Either $IIA(IIB)$ on X is the same as $IIA(IIB)$ on $\tilde{X}$, or it is identical to $IIB(IIA)$ on $\tilde{X}$. Since mirror symmetry requires $h^{1,1}(X) = h^{2,1}(\tilde{X})$ and $h^{2,1}(X) = h^{1,1}(\tilde{X})$, for otherwise generic values of Hodge numbers the first case would require identification of moduli spaces of different dimensions, which is clearly nonsense[*8]. Therefore, given a mirror pair of Calabi-Yau manifolds $(X, \tilde{X})$ type IIA theory compactified on X is the same as type IIB theory compactified on $\tilde{X}$. Actually, this statement is a bit premature. Really all we have shown thus far is that mirror symmetry is a symmetry of perturbative string theory. The fact that the moduli spaces align appropriately, together with perturbative equivalence is certainly a neccesary condition for mirror symmetry to hold non-perturbatively. We will come back to this issue in Chapter 4.

3.1.3 Effective Action and Singularities of Calabi-Yau Manifolds

We now turn to the study of moduli spaces in some more detail. We will consider the vector multiplet moduli space in type IIB theory, as we will have the additional benefit of the result being exact. The vector multiplet moduli $\mathcal{M}_{\mathcal{V}}$ space in type IIB compactification on a Calabi-Yau X is given by the complex structure moduli space of X

$$\mathcal{M}_{\mathcal{V}} = \mathcal{M}_{\mathcal{C}}. \tag{3.2}$$

We can make this correspondence more precise as follows [*9].

There are $h^{2,1}(X)$ abelian vector multiplets of low energy $\mathcal{N} = 2$ supersymmetry

[*8] This argument clearly depends on the fact that we are considering compactification on a Calabi-Yau threefold. In general, in even complex dimension type IIA is mapped to type IIA, and in odd dimension to type IIB.

[*9] The theory is really coupled to gravity, so we are not studying pure gauge theory. However all the moduli come from the scalars in vector multiplets.

on $\mathcal{M}_4$. The complex scalars ϕ^I, $I = 1, \ldots h^{2,1}(X)$ in the vector multiplets come from deformations of the metric on X that preserve Ricci-flatness but change the complex structure on X. At any point in $\mathcal{M}_C$ [*10], with a metric $g_{i\bar{j}}$ such a deformation takes the form $\Omega_{ijk} g^{k\bar{p}} \delta g_{\bar{p}\bar{l}} = (\omega_i)_{ij\bar{l}} \delta\phi^i$, where ϕ^i are coordinates on the complex structure moduli space. In hindsight, ϕ^i will be identified with vector multiplet scalars. One very important thing to keep in mind is that the ϕ^i are really $h^{2,1} + 1$ *homogeneous* coordinates on the moduli space out of which the $h^{2,1}$ physical ones are identified with ϕ^I, and therefrom the difference in notation. The distinction makes all the difference in the world between *local* and *global* supersymmetry in $d = 4$. However in this section we will be sloppy about this since nothing essential to us will be affected by the distinction.

A Calabi-Yau manifold X is a solution to (super)Einstein equations of ten-dimensional type IIB supergravity. The ten-dimensional action expanded in fluctuations $g \to g + \delta g$ about the Ricci-flat solution is quadratic in δg to the leading order. The kinetic term for these fluctuations derived in this way is the Weil-Peterson metric:

$$G(\delta g, \bar{\delta g}) = \frac{1}{V} \int_X d^6x \sqrt{g} g^{i\bar{j}} g^{l\bar{k}} D_\alpha(\delta g_{il}) D^\alpha(\delta g_{\bar{j}\bar{k}}), \tag{3.3}$$

where D_α is the covariant derivative in four-dimensions. This induces a Kähler metric on the space of $\delta\phi^i$'s which, utilizing the special Kähler structure of $\mathcal{M}_C$, can be written as:

$$G_{i,\bar{j}} = \partial_i \partial_{\bar{j}} K$$

where K is given by $K = -\int_X \Omega \wedge \bar{\Omega} = \mathrm{Im}(\bar{\phi}^i \partial_i \mathcal{F})$, and $\mathcal{F}$ is the prepotential on $\mathcal{M}_C$. This structure is familiar from study of low energy effective actions in four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry. The most general form of the effective action in $\mathcal{N} = 1$ superspace is:

$$\mathcal{L}_{eff} = \frac{1}{4\pi} \mathrm{Im}\Big[\int d^4\theta \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi^I} \bar{\Phi}^I + \int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi^I \partial \Phi^J} W_a^I W^{Ja}\Big], \tag{3.4}$$

[*10] Kähler structure is fixed.

where Φ^i is an $\mathcal{N}=1$ chiral multiplet in the $\mathcal{N}=2$ vector multiplet whose scalar component is ϕ^i. We see that the gauge theory prepotential *must* be identified with the prepotential of the Calabi-Yau complex structure moduli space [*11], from which assertion (3.2) follows.

Let us now turn to singularities in the moduli spaces. The Calabi-Yau manifold develops a singularity via degeneration of complex structure when a particular cycle $\gamma \in H_3(X, \mathbb{Z})$ shrinks

$$\phi_C = \int_C \Omega \to 0.$$

Unlike the Kähler deformation case, the geometric meaning of a "shrinking three-cycle" is not so clear. However, one can pick particular homology representatives, the so called "supersymmetric" cycles whose volume is determined by the choice of complex structure, i.e., for which $\mathrm{Vol(C)} = |\int_C \Omega|$.

The singularities in complex-codimension one imply that there is monodromy in circling around the singular locus in the moduli space $\phi_C \to e^{2\pi i}\, \phi_C$ – every cycle $D \in H_3(X)$ undergoes a transformation

$$D \to D + (C \cap D) D.$$

This implies that in the neighbourhood of $\phi_C = 0$,

$$\int_D \Omega = \frac{1}{2\pi i}\,(C \cap D)\, \phi_C\, ln\, \phi_C + \text{regular},$$

in order to transform correctly. In particular, using the special geometry definitions, this implies that the gauge coupling behaves as

$$\tau_{ij} = \partial_i \partial_j \mathcal{F} = \frac{1}{2\pi i}(C \cap B^i)(C \cap B^j)\, \phi_C\, ln\, \phi_C + \text{regular}.$$

This form of running of the gauge coupling is precisely that which results from integrationg out a hypermultiplet of mass $|\phi_C|$, whose charge under i'th $U(1)$ is $C \cap B^i$.

[*11] One must keep in mind that the theory *is* coupled to gravity.

When the singularity in the moduli space can be understood as resulting from integrating out massless particles it is not dangerous – since including those states in the theory cures the ails. If this also is to be the case in type IIB theory on X, we must find the appropriate particles in the theory.

In our case, the low-energy gauge fields are somewhat peculiar, since they derive from a higher form field, in this case the (R, R) four-form potential reduced on three-cycles in $H_3(X)$, in particular we have

$$\mathcal{A}_i^{(1)} = \int_{B^i} \mathcal{A}^{(4)},$$

so that the object we are looking for is really charged under $A^{(4)}$ – it is the D 3-brane wrapping C [*12]. The D 3-brane is in fact a BPS particle: the mass of the particle is up to a constant proportional to its charge

$$m_C = |\phi_C| = |(C \cap B^i)\phi_i|.$$

One can arrange configurations when there are more cycles C_k, $k = 1, \ldots, n_H$ which vanish, thus more massless hypermultiplets, than there are $U(1)$ factors under which they are charged – $n_H - n_V > 0$. From the point of view of geometry this means that n_H three-cycles obey $n_H - n_V$ homology relations. From the point of view of field theory this allows for another branch of the theory to open up where n_V vector multiplets are massive, having "eaten" n_H scalar's, and which is parametrized by the remaining $n_H - n_V$ massless hypermultiplets. This is simply Higgs mechanism.

Let us now try to interpret the result from the point of view of Calabi-Yau geometry. Before the transition we had $h^{2,1}(X)$ vector multiplets, and $h^{1,1}(X)$ neutral, decoupled hypermultiplets. After the transition the numbers must change:

$$h^{1,1}(X) \to h^{1,1}(X_t) = h^{1,1}(X) + n_H - n_V,$$

[*12] The reader might complain we have singled out B^i – this is not the case since $\int_{A_i} \mathcal{A}^{(4)}$ is the magnetic dual of $\mathcal{A}_i^{(1)}$.

$$h^{2,1}X \to h^{2,1}(X_t) = h^{2,1}(X) - n_V,$$

the topology of X has changed. Quantum geometry allows for completely smooth topology change. This processis called a conifold transition – and we will study it in some detail in chapter 4.

3.2 Application: Geometry for Physics of $\mathcal{N} = 2$ SUSY Gauge Theories

We have seen that singularities in conformal field theory on Calabi-Yau manifolds are cured once non-perturbative effects are taken into account. In the particular example we have treated in the previous section, it was a charged hyper-multiplet which became massless. Is this the only type of singularity that can occur, or can one find massless vector multiplets as well?

If so, there is a clear application that comes to mind. We have seen that the vector multiplet moduli space in a type IIB compactification on a Calabi-Yau manifold is exact at the classical level. Unfortunately, this is not particularly interesting since the gauge group is abelian – $U(1)$ gauge theories are not asymptotically free – so the theory really needs gravity, or string theory, in order to be defined. Massless vector multiplets could, in an appropriate setup, mean that at certain singularities type IIB theory develops enhanced gauge symmetry. In an asymptoticaly free theory one would be able to decouple gravity, and thus obtain exact information about $\mathcal{N} = 2$ supersymmetric gauge theories even in the strong coupling regime.

In fact, the answer is positive.

This is very satisfying from the point of view of a string theorist. The reason is as follows. The fact that one can obtain exact results about the vacuum structure of $\mathcal{N} = 2$ supersymmetric theories is not in itself new, it was pioneered in the work of Seiberg and Witten [18, 19], using field theory arguments. What they observed is that the moduli space of the gauge theory is isomorphic to the moduli space of complex structures on an auxiliary complex curve. This space is highly constrained, and it is governed by a prepotential, a holomorphic function of the moduli of the curve. In particular, the moduli space has the special-geometry structure we have found for the space of complex structures for Calabi-Yau manifolds, and furthermore masses of BPS particles in the theory are given by the periods of the curve, while the period matrix gives the gauge couplings.

The unsatisfactory part of that analysis was that the curve was auxiliary, and surprises of the kind, however pleasant are unsatisfying. Another problem is that no systematic ways of deriving the curves were found for general gauge groups with arbitrary matter content. Although many gauge groups with fundamental matter were analyzed in this [20, 21, 22, 23, 24, 25, 26, 27], it turned out to be rather difficult to generalize these results to theories with matter in any other representation because in these cases the curves encoding the gauge coupling are usually not hyperelliptic, and general Riemann surfaces have more parameters than can be fixed by studying various limits of the gauge theory.

Calabi-Yau manifolds on the other hand provide both the geometric intuition for the work of Seiberg and Witten, and a powerful tool for finding solutions to any gauge theory with essentially any matter content. The complex structure moduli space of the Calabi-Yau manifold, in the limit in which gravity decouples is the moduli space of the gauge theory. The fact that in some cases one obtains a curve is in fact an accident. A more important thing is that the Calabi-Yau moduli space has a local special geometry, associated to local supersymmetry, while the limit $M_P \to \infty$ produces the rigid special geometry of Seiberg and Witten [28]. This geometric approach to solving $\mathcal{N} = 2$ supersymmetric gauge theories was pioneered in [29], and fully developed in [30]. In this application, we will use type IIB string theory compactification on Calabi-Yau manifolds to provide solutions for $SO(N)$ gauge theories with matter in spinor and fundamental representations.

3.2.1 Enhanced Gauge Symmetry in Type II String Theory

In this section, we want to construct ("geometrically engineer", [30]) Calabi - Yau manifolds for type IIB compactification which will give us desired four-dimensional physics.

The problem is as follows. For all of the exactness of complex structure moduli space it is hard, in all but the simplest cases, to develop an intuition for it. Furthermore, the context in which symmetry enhancement is best understood, in compactifications on $K3$, is the one in which type IIB theory is understood the least. We are set on a windy road of string dualities.

Enhanced Gauge Symmetry

As noted above, type IIA obtains enhanced gauge symmetry on $K3$. $K3$ is two complex dimensional Calabi-Yau manifold, and the resulting theory has two six-dimensional supersymmetries[*13]. The singularities $K3$ can develop are extremely constrained. The only non-trivial information about a singularity (apart from some global data), is the intersection form of two-cycles which shrink to zero size at the singularity. These, as it turns out, are in one to one correspondence with ADE classification of Lie algebras – the Cartan matrix of the algebra is (minus) the intersection form of the singularity. Now, we expect to obtain massless BPS saturated particles by wrapping D-branes about two cycles. In type IIA theory, there is a D 2-brane which can wrap a vanishing S^2, to obtain a particle in 6 dimensions[*14]. In type IIB, on the other hand the "smallest" brane that can wrap is a D 3-brane, which leaves a nearly tensionless string in six-dimensions.

What kind of a particle have we obtained in type IIA theory? It is charged under the six-dimensional gauge field obtained from reducing the Ramond-Ramond

[*13]The reason $K3$ carries a special name, is that topologically, it is the only Calabi-Yau two-fold which is not a four torus T^4, in particular it has $h^{1,1} = 20$, and as, the top holomorphic and antiholomorphic forms are unique, $h^{2,0} = 1 = h^{0,2}$

[*14]To be more precise, in order to obtain a BPS saturated particle the S^2 must be a holomorphic curve, $\mathbb{P}^1$ in the $K3$. The constraint that it is an S^2, as opposed to any other Riemman surface comes from the fact that S^2 is the only curve which can shrink to zero size, while preserving Ricci flatness.

three form $\mathcal{A}^{(3)}$ on the S^2. Since the theory has $\mathcal{N} = (1,1)$ supersymmetry in six dimensions, the only BPS multiplets supersymmetry allows are massive vector bosons. Furthermore, the charges determined by the intersection matrix imply that the gauge group (or at least its Lie algebra) is of the associated A, D or E type. The type IIB on $K3$ has chiral $(2,0)$ supersymmetry, with tensor multiplets and tensionless strings about which essentially nothing is known. We will have more to say about the precise identification of the gauge group later.

The reason this is useful for us is that there exist Calabi-Yau manifolds which are $K3$ fibrations. In compactifying type IIA theory on such a Calabi-Yau manifold X, one obtains $\mathcal{N} = 2$ supersymmetric gauge theory with the gauge group given by the type of singularity $K3$ develops, but where the matter content is, at least in principle, determined by how the $K3$ is fibred. Mirror symmetry can be used to sum up the instanton corrections – compactification of type IIB theory on the mirror manifold Y provides an identical theory, but with the benefit that the result is exact at the tree level – it is given by classical geometry of $\mathcal{M}_{\mathcal{C}}(Y)$.

In the following, we will first review the construction of a class of Calabi–Yau threefolds which, when used in type IIA compactification, give rise to $d = 4$, $\mathcal{N} = 2$ $SO(10)$ and $SO(12)$ gauge theories with specific numbers of fundamentals and spinors. We use the toric description of these manifolds to find explicit expressions for the mirror manifolds. A local approximation to the mirror manifolds in the form of ALE fibrations provides the exact solutions for these theories. We also propose generalizations to arbitrary numbers of massive vectors and spinors and perform several consistency checks on our results. The non–simply laced cases $SO(7)$, $SO(9)$ and $SO(11)$ are the subject of the next three subsections. In these cases we slightly modify the conventional method of finding the mirror to obtain the exact solutions in the most convenient form. These modifications are explained in Subsec. 3.3.3. We summarize our results in Subsec. 3.4.

3.2.2 Solving Gauge Theories via Geometry

Type IIA string theory, compactified on a Calabi–Yau threefold that is both an elliptic and a $K3$ fibration, gives rise to an $\mathcal{N}=2$ gauge theory in four dimensions. Such a manifold, locally of the form $T^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, is given by

$$y^2 = x^3 + xf(z_1, z_2) + g(z_1, z_2) \tag{3.5}$$

Above x, y parameterize T^2 (locally) in form of a double cover of a complex plane branched over three points – the roots of the polynomial $x^3+xf(z_1,z_2)+g(z_1,z_2)=0$, and f and g are functions of the base coordinates z_1, z_2. For this equation to define a Calabi–Yau, the functions f and g must be of the form

$$\begin{aligned} f(z_1, z_2) &= \sum_{i=0}^{I} z_1^{8-i} f_{8+n(4-i)}(z_2) \\ g(z_1, z_2) &= \sum_{j=0}^{J} z_1^{12-j} f_{12+n(4-j)}(z_2), \end{aligned} \tag{3.6}$$

where the subscript on the polynomials f and g in the sums indicates their degree in z_2. I and J are the maximum values of i and j such that the degree is not negative. We can view this threefold as an elliptic fibration over the Hirzebruch surface F_n or as a $K3$ fibration over a sphere parameterized by z_2.

We noted above that there is a subtlety in identification of the gauge group in type IIA compactification on $K3$. Historically, the way gauge symmetry enhancement in $K3$ compactifications has been noticed, and which ultimately removes the ambiguity, is that there is a duality to heterotic string compactification on T^4,

$$IIA/K3 = \mathrm{Het}/T^4.$$

Type IIA string theory on a $K3$ fibered Calabi-Yau can be related to a heterotic string compactification by extending the six dimensional duality fibrewise: IIA string theory compactified on Calabi-Yau (3.5) is conjectured to be dual to heterotic $E_8 \times E_8$

string compactified on $K3 \times T^2$ with $12-n$ and $12+n$ instantons embedded in the first and second E_8 [29, 31, 32] and all Wilson lines switched off. The coefficients of the monomials in (3.5) that are proportional to xz_1^4 and z_1^6 correspond to the moduli of the $K3$ and the other terms specify the $E_8 \times E_8$ gauge bundle. The coefficients of terms with lower powers of z_1 define the embedding of $12-n$ instantons in the first E_8 and the remaining terms do the same for the $12+n$ instantons in the second E_8 [31, 32].

For generic choices of the polynomials f and g, the manifold (3.5) is completely smooth, and correspondingly, the instantons break the $E_8 \times E_8$ gauge group of heterotic strings as far as possible. The E_8 with $12+n$ instantons is broken completely (for $n \geq 0$) while the other is broken to some terminal group without matter. This is the case that was studied in [29, 33, 34, 35] for various instanton embeddings.

Here we consider more restrictive instanton embeddings, which result in larger unbroken subgroups of the E_8 with $12-n$ instantons. On the type IIA side such instanton embeddings correspond to choosing Calabi–Yau threefolds that have a more severe singularity in their $K3$ fiber than one would get from the generic choice of polynomials. For example, we can consider the Calabi–Yau defined by setting

$$\begin{aligned} f_{8-2n} &= h_{4-n}^2 \\ g_{12-3n} &= h_{4-n}^3 \\ g_{12-2n} &= q_{6-n}^2 - f_{8-n}h_{4-n} \end{aligned} \tag{3.7}$$

and choosing the coefficients of lower powers of z_1 to vanish. Above, h_{4-n} and q_{6-n} are polynomials in z_2 of the degree indicated by the subscripts. One can use Kodaira's classification to determine the singularity type of the $K3$ fiber. The definitions above ensure that the fiber has a split D_5 singularity [36]. We can make this manifold smooth by blowing up a collection of spheres in the base of the K3, i.e., by modifying its Kähler structure. The intersection forms of these spheres give the entries in the Cartan matrix of the corresponding gauge group ($SO(10)$ for D_5). Compactifying type IIA on a Calabi–Yau with this blown-up $K3$ as a fiber results in a $d=4$ $SO(10)$

gauge theory, where the $SO(10)$ is broken to its Cartan sub-algebra. This happens in the following manner. As stated above, type IIA D 2-branes can wrap around the blow–up spheres in the $K3$ to give rise to a pair of $W^{\pm}$ bosons depending on the orientation of wrapping. Now, from the field theory point of view we expect the mass of the W to be proportional to the vev of the Higgs field. In string theory, this vev is identified with the Kähler structure parameter,

$$\int_{S^i} J + iB.$$

On the other hand, a D 2-brane which wraps the cycle S^i contains exactly such a term in the world-volume action. Taking $\int_{S^i} J + iB \to 0$ corresponds to unhiggsing an $SU(2)$ factor, and the D 2-brane becomes massless. Since the blow–up spheres have intersection forms determined by the singularity type, the corresponding $SU(2)$ factors link up to make the gauge group indicated by the singularity type. Thus it is clear that the Kähler structure moduli are related to the coordinates on the Coulomb branch of the $d = 4$ gauge theories in the type IIA picture. On the heterotic side, these blow–ups correspond to switching on Wilson lines to break the gauge group. However, there are world sheet instanton corrections to the Kähler moduli space, which are related to gauge theory instantons via the duality to heterotic strings [30]. Mirror symmetry provides a way to sum up these corrections.

3.2.3 Mirror Manifolds via Toric Geometry

To find the mirror manifold of the type IIA Calabi–Yau it is convenient to encode its salient properties using toric geometry [37, 38, 39, 40, 41]. Toric variety is a generalization of a weighted projective space to include more coordinates and more $\mathbb{C}^*$ actions. Recall: the only compact Calabi-Yau manifold written as a hypersurface in $\mathbb{C}^4$ is a point, thus a global description of a compact Calabi-Yau manifold requires hypersurfaces in (weighted) projective spaces, and toric varieties serve to add "variety" *15. A four-dimensional toric variety V is described by $4 + N$ homogeneous

*15 This is a pun.

coordinates x^i, which are made "homogeneous" by N $\mathbb{C}^*$ actions

$$x^i \sim \lambda^{q_i^{(a)}} x^i, \quad \lambda \in \mathbb{C}^*, a = 1, \dots N.$$

A Calabi-Yau three-fold X can be described as a hypersurfaces in a toric variety, given by polynomial equations

$$X \subset V: \quad f(x) = 0.$$

In order for the equation to be well defined, $f(x)$ has to scale homogeneously under the $\mathbb{C}^*$ actions, and in order for the equation to define a Calabi-Yau manifold, the weight of f must be the sum of the charges of x's,

$$x^i \to \lambda^{q_i^{(a)}} x^i: \quad f(x) \to \lambda^{\sum q^{(a)i}} f(x).$$

Thus, to specify a Calabi-Yau manifold X, we must give:

- $\mathbb{C}^*$ actions on x^i's,
- Monomials appearing in $f(x)$.

The $\mathbb{C}^*$ actions are encoded in the following way. To every coordinate x^i, associate a vector $v^i \in \mathbb{Z}^4$. There are $N+4$ vectors in 4 dimensional space, giving N relations between them which are taken to be

$$\sum q_i^{(a)} v^i = 0. \tag{3.8}$$

Provided that degenerate solutions are eliminated (solutions which satisfy more constraints than we wish to impose) this determines v^i essentially uniquely as the generators of solutions to (3.8) (the solutions are really rays in $\mathbb{Z}^4$). The resulting set of vectors forms a polyhedron, $\bigtriangledown$ in $\mathbb{Z}^4$. For some perverse reason a glimpse of which we will catch in a moment, the polyhedron has been named the "dual" polyhedron.

To characterize the particular Calabi-Yau manifold X, the second item on the list, we proceed as follows. One basically constructs a most generic Calabi-Yau manifold that can be written down in such a toric variety. It is easy to see that such a space can be written as

$$f = \sum_{\tilde{v}_i \in \triangle} a_i \prod_{v^j \in \bigtriangledown} x_j^{<v^j, \tilde{v}_i>+1},$$

where $\tilde{v}_i$ are vectors forming a polyhedron $\bigtriangledown \in \mathbb{Z}^4$. In order for f to be holomorphic, we need $< v^j, \tilde{v}_i > \geq -1$, which in turn implies that $\triangle$ and $\bigtriangledown$ are dual to each other – the normals to faces of one are vertices of the other. The dual of the "dual" polyhedron is called the "Newton" polyhedron. It is clear that this is a unique completely generic Calabi-Yau manifold that can be written in the toric variety determined by $\bigtriangledown$, essentially because it is the only polynomial which scales properly. Furthermore, varying the coefficients in the defining equation one obtains a whole family of Calabi-Yau manifolds, that is the coefficients a_i are coordinates on the space of complex structures on X.

- $\triangle$ encodes the complex structure of X and $\bigtriangledown$ encodes the Kähler structure on X which is inherited from that of the toric variety.

Mirror symmetry, if it holds, must *exchange* the complex and Kähler moduli, and therefore it must exchange the role of the two polyhedra [41]. That is

- In the mirror manifold Y of X, $\triangle$ that encodes the Kähler structure of Y and $\bigtriangledown$ encodes the complex structure on Y which is inherited from that of the toric variety.

The manifolds we have written down in the previous section in a local form can be encoded in toric terms, and thus we obtain the "dual" and the "Newton" polyhedron for type IIA compactification. The vertices of the dual polyhedron that encode the Kähler structure of the blow–ups on the type IIA side determine the complex structure of the mirror type IIB manifold.

On the type IIB side, the complex structure moduli are vector multiplets and the Kähler structure moduli and the dilaton are hypermultiplets. As on the type

IIA side, the vector moduli space is not corrected by perturbative string effects but on the type IIB side the world sheet instanton corrections are absent as well [30]. Thus the classical description of the complex structure moduli space of the type IIB Calabi–Yau is exact. Since these moduli encode the behavior of the gauge theory, we can read off the exact solutions from the IIB manifold.

Below, we discuss a series of Calabi-Yau manifolds that give rise to $SO(10)$,$SO(12)$ and $SO(7)$,$SO(9)$ and $SO(11)$ gauge groups with spinors and fundamentals. For the cases we are considering here the relevant manifolds and polyhedra were worked out in [36, 42], so we will only summarize the results. In the first two cases we find exact solutions using Batyrev's construction of the mirror [41]. For the non–simply laced cases we slightly modify the construction to simplify the resulting curves. These modifications are explained in Sec. 3.3.3.

3.3 Exact Solutions from Mirror Symmetry

3.3.1 $SO(10)$ with $(4-n)\mathbf{16}+(6-n)\mathbf{10}$

The dual polyhedron for the Calabi–Yau that gives rise to an $SO(10)$ gauge theory with $4-n$ spinors and $6-n$ vectors was constructed in [36][*16]. The derivation there uses Tate's algorithm and a more general form of the defining equation, 3.5, that makes it easier to encode the split or nonsplit property of the singularity. The same polyhedron was also found in [42], using toric arguments only. Using the basis of [42], the dual polyhedron, $\bigtriangledown$, is given by the vertices

$$\begin{array}{lll} \tilde{v}_1=(-1,0,2,3) & \tilde{v}_2=(1,-n,2,3) & \tilde{v}_3=(0,-1,2,3) \\ \tilde{v}_4=(0,0,-1,0) & \tilde{v}_5=(0,0,0,-1) & \tilde{v}_6=(0,0,0,0) \\ \tilde{v}_7=(0,0,2,3) & \tilde{v}_8=(0,1,2,3) & \tilde{v}_9=(0,-2,2,3) \\ \tilde{v}_{10}=(0,-2,1,2) & \tilde{v}_{11}=(0,-1,1,1) & \tilde{v}_{12}=(0,-1,0,1) \\ \tilde{v}_{13}=(0,-1,0,0). & & \end{array} \tag{3.9}$$

[*16] Note that in Refs. [36, 42] the unhiggsing of the E_8 with $12+n$ instantons was studied while we are unhiggsing the E_8 with $12-n$ instantons.

This list of vertices includes all points that do not lie on codimension one facets of the dual polyhedron, i.e., this polyhedron encodes a fully blown–up type IIA manifold. The vertices $\tilde{v}_1, \ldots, \tilde{v}_8$ define the toric variety in which the type IIA manifold is embedded and the remaining vertices correspond to the blow–up spheres needed to repair the D_5 singularity. The vertices of the corresponding Newton polyhedron are

$$\begin{array}{lll} v_1 = (2,1,-1,1) & v_2 = (3,1,1,0) & v_3 = (0,0,1,-1) \\ v_4 = (6,1,1,1) & v_5 = (0,0,-2,1) & v_6 = (6,-6,1,1) \\ v_7 = (-6-6n,-6,1,1) & v_8 = (n-6,1,1,1) & v_9 = (n-3,1,1,0) \\ v_{10} = (n-2,1,-1,1). & & \end{array} \tag{3.10}$$

Note that for $n=4$ the vertices v_1 and v_{10} become identical which allows us to drop one of them.

We can use the information encoded in the dual pair of polyhedra, $\triangle$, ∇, to construct the mirror manifold of our initial Calabi–Yau. Batyrev's construction of the mirror [41] requires that we switch the roles of the two polyhedra. An embedding polynomial defining the mirror manifold is given by

$$W = \sum_j a_j \prod_i x_i^{v_i \cdot \tilde{v}_j + 1} = 0, \tag{3.11}$$

where the x_i are coordinates in a weighted projective space (or more generally in a toric variety). In the cases we consider here there are nine or ten vertices in the Newton polyhedron, corresponding to the same number of coordinates in the hypersurface constraint, Eq. (3.11). We can eliminate some of these coordinates using the $\mathbb{C}^*$ actions that define the identifications of coordinates in the embedding space. Sets of weights for the $\mathbb{C}^*$ actions can be found by looking for sets of five vertices in $\triangle$ such that

$$\sum_i v_i k_i = 0, \tag{3.12}$$

where the coefficients satisfy $k_i \neq 0$. One can use these $\mathbb{C}^*$ actions to set all but five of the coordinates in Eq. (3.11) to one. This gives a description of the Calabi–Yau in some local coordinate patch with one remaining $\mathbb{C}^*$ action. For our purposes it is most convenient to retain x_1, x_2, x_3, x_6, x_7 and set the remaining coordinates to one. This amounts to choosing a patch in which the relevant properties of the Calabi–Yau are described most easily. Using these coordinates we find the following defining equation for the mirror manifold

$$\begin{aligned} W &= x_7^{12+6n} + a_0 x_1^{4-n} x_2^{6-n} x_6^{12+6n} + a_1 x_1 x_2^2 (x_6 x_7)^{12} + a_2 x_1^2 + a_3 x_2 x_3^2 \\ &\quad + a_4 x_1 x_2 x_3 (x_6 x_7) + a_5 x_1^2 x_2^3 (x_6 x_7)^6 + a_6 x_1^3 x_2^4 + a_7 x_2 (x_6 x_7)^{18} \\ &\quad + a_8 (x_6 x_7)^{16} + a_9 x_2 x_3 (x_6 x_7)^9 + a_{10} x_1 (x_6 x_7)^8 + a_{11} x_3 (x_6 x_7)^7. \end{aligned} \tag{3.13}$$

This Calabi–Yau is a $K3$ fibration. We can make this explicit by defining $x_0 = x_6 x_7$ and $\zeta = (x_7/x_6)^{6+3n}$. Using the freedom to rescale x_1, x_2 and x_3 to eliminate three of the coefficients a_i we obtain

$$\begin{aligned} W &= \left(\zeta + a_0 \frac{x_1^{4-n} x_2^{6-n}}{\zeta}\right) x_0^{6+3n} - 2 x_1 x_2^2 x_0^{12} - x_1^2 + x_2 x_3^2 \\ &\quad + a_4 x_1 x_2 x_3 x_0 + a_5 x_1^2 x_2^3 x_0^6 + a_6 x_1^3 x_2^4 + a_7 x_2 x_0^{18} + a_8 x_0^{16} \\ &\quad + a_9 x_2 x_3 x_0^9 + a_{10} x_1 x_0^8 + a_{11} x_3 x_0^7. \end{aligned} \tag{3.14}$$

The first term in this equation describes the base sphere and the remaining terms define a $K3$. Approximating the $K3$ locally as an ALE space, we can bring this expression into a form that is equivalent to a Seiberg-Witten curve. In order to do this, we set $x_0 = 1$ and observe that the first three terms in the $K3$ part give a three–coordinate form of a D_5 singularity located at the origin. The terms with coefficients a_5 and a_6 are irrelevant near the singularity and can be neglected for our present purposes. The remaining terms are the deformations of the D_5 singularity.

The following chain of substitutions brings the singularity into the standard form:

$$\begin{aligned}
x_3 &= y - \frac{1}{2}(a_9 + a_4 x_1) \\
a_8 &= c_1 + \frac{1}{16}\left(8a_{11}a_9 + 4a_{11}a_{10}a_4 - 4a_{10}^2 - a_{11}^2 a_4^2\right) \\
a_7 &= c_2 + \frac{a_9}{8}\left(2a_9 + 2a_{10}a_4 - a_{11}a_4^2\right) \\
a_{10} &= -c_3 + \frac{a_4}{16}\left(8a_{11} + a_4 a_9^2\right) \\
a_9 &= \frac{2}{a_4}c_4 \\
a_{11} &= -2(-c_0)^{1/2}.
\end{aligned} \tag{3.15}$$

Neglecting an irrelevant term proportional to $x_1^2 x_2$ we obtain the standard form of the D_5 singularity after shifting

$$x_1 = x - \frac{1}{8}\left(4c_3 - c_4^2 + 4c_4 z + 8z^2\right) \tag{3.16}$$

and defining $z = x_2$:

$$\begin{aligned}
W = \left(\zeta + a_0 \frac{x_1^{4-n} z^{6-n}}{\zeta}\right) &- x^2 + z^4 + y^2 z - 2(-c_0)^{1/2} y + c_4 z^3 \\
&+ c_3 z^2 + c_2 z + c_1 + \cdots ,
\end{aligned} \tag{3.17}$$

where Eq. (3.16) should be substituted for x_1. The ellipsis denotes contributions from terms that are irrelevant close to the singularity. Neglecting these terms amounts to switching off gravity or conversely taking the field theory limit [33, 34]. The strange choice for the redefinition of a_{11} will become clear below.

This expression is equivalent to a Seiberg-Witten curve for $SO(10)$ with $6-n$ fundamentals and $4-n$ spinors. The coefficients $c_0, \ldots, c_4$ are the gauge invariant coordinates on the moduli space and a_0 can be interpreted as the strong coupling scale of the gauge theory, $a_0 = \Lambda^{2\beta_0}$. The beta function for this $SO(10)$ theory is given by $\beta_0 = 8 - N_f - 2N_s$.

The general method for converting D_n type ALE fibrations into Seiberg-Witten curves was first introduced in [26] to find the curves for $SO(2N)$ gauge groups without matter. Using the same approach we integrate out y from Eq. (3.17) and multiply by z. Absorbing a factor of z into ζ gives

$$W = \left(\zeta + \Lambda^{2\beta_0}\frac{x_1^{4-n}z^{6-n+2}}{\zeta}\right) - x^2 + 2P(z), \tag{3.18}$$

where $P(z)$ is given by

$$P(z) = \frac{1}{2}\left(z^5 + c_4 z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0\right). \tag{3.19}$$

For $n = 4$, x appears only quadratically and can be integrated out trivially. The substitutions $\zeta = y - P(z)$ and $z \to z^2$ result in a double cover version of the curve for $SO(10)$ with two fundamentals

$$y^2 = P^2(z^2) - \Lambda^{12} z^8. \tag{3.20}$$

Note that for the asymptotically free cases, $n = 2, 3, 4$, both x and y appear at most quadratically and can be integrated out. In the cases with one or two spinors of $SO(10)$, $n = 2, 3$, we still obtain a curve but it is no longer hyperelliptic. The $U(1)$ gauge couplings on the Coulomb branch are encoded in the normalized period matrix of this curve. The Seiberg-Witten 1-form needed to evaluate the period matrix, can be derived from the unique holomorphic 3-form, Ω, of the original Calabi–Yau [34].

It is very tempting to modify Eq. (3.17) to allow an arbitrary number of massive spinors and vectors. This can probably be achieved by replacing the fibration over the sphere in Eq. (3.17) according to

$$\zeta + a_0\frac{1}{\zeta}x_1^{4-n}z^{6-n} \to \zeta + a_0\frac{1}{\zeta}\prod_{i=1}^{N_s}(x_1 - m_i^4)\prod_{j=1}^{N_f}(z - m_j^2), \tag{3.21}$$

where the m_i are the masses of the N_s spinors and the m_j are the masses of the N_f

vectors. Using Eq. (3.21) and substituting $\zeta = y - P(z)$, $z \to z^2$ in Eq. (3.18), we get

$$y^2 = x^2\left(y - P(z^2)\right) + P^2(z^2) - \Lambda^{2\beta_0} z^4 \prod_{i=1}^{N_s}(x_1 - m_i^4) \prod_{j=1}^{N_f}(z^2 - m_j^2). \qquad (3.22)$$

The normalized period matrix of this surface encodes the gauge couplings on the Coulomb branch. Here, there is no natural 2-form inherited from Ω, because Eq. (3.22) is generally not a parametrization of a local approximation to a Calabi–Yau. To compute the gauge couplings from this surface, one needs to identify the 2-cycles and construct a suitable 2-form directly.

Our proposal, Eq. (3.21), ensures plausible behavior when either a spinor or a vector is integrated out. Integrating out a vector and a spinor at the same time, we can flow between the theories we obtained from mirror symmetry. To check our solution further, we consider breaking the $SO(10)$ gauge group to $SO(8) \times U(1)$ by giving a large VEV, M, to one component of the $SO(10)$ adjoint. Under this breaking the fundamentals decompose into fundamentals of $SO(8)$ and singlets with $U(1)$ charge. The spinors decompose as $\mathbf{16} \to \mathbf{8}_c^1 \oplus \mathbf{8}_s^{-1}$, where the superscripts denote the $U(1)$ charge [43]. Both the singlets and the two spinor representations of $SO(8)$ acquire a large mass and should drop out from our solution. Taking M to infinity, the piece proportional to $c_4^2 \approx M^4$ will dominate Eq. (3.16). Replacing x_1 by M^4, rescaling Eq. (3.18) by appropriate powers of M and integrating out x reduces it to the $SO(8)$ curve with vector matter only.

3.3.2 $SO(12)$ with $\frac{r}{2}\mathbf{32} + (\frac{4-n-r}{2})\mathbf{32'} + (8-n)\mathbf{12}$

The analysis of the previous subsection can be repeated for $SO(12)$ with r half hypermultiplets in the $\mathbf{32}$, $(4-n-r)$ half hypermultiplets in the $\mathbf{32'}$ and $8-n$ fundamentals. The restrictions on the polynomials f and g in Eq. (3.6) are more complicated for $SO(12)$ than for $SO(10)$ [36], partly because one has the freedom to trade matter fields in the $\mathbf{32}$ for fields in the $\mathbf{32'}$ representation. However, the curve of the $SO(12)$ theories depends only on the total number of fields in the $\mathbf{32}$ and $\mathbf{32'}$, so we will drop this distinction here. Using the vertices of the dual polyhedron given in [42],

$$\begin{array}{lll}
\tilde{v}_1 = (-1,0,2,3) & \tilde{v}_2 = (1,-n,2,3) & \tilde{v}_3 = (0,-1,2,3) \\
\tilde{v}_4 = (0,0,-1,0) & \tilde{v}_5 = (0,0,0,-1) & \tilde{v}_6 = (0,0,0,0) \\
\tilde{v}_7 = (0,0,2,3) & \tilde{v}_8 = (0,1,2,3) & \tilde{v}_9 = (0,-2,2,3) \\
\tilde{v}_{10} = (0,-2,1,2) & \tilde{v}_{11} = (0,-2,0,1) & \tilde{v}_{12} = (0,-1,1,1) \\
\tilde{v}_{13} = (0,-1,0,0) & \tilde{v}_{14} = (0,-1,-1,0), &
\end{array} \tag{3.23}$$

we find for the Newton polyhedron

$$\begin{array}{lll}
v_1 = (2,1,-1,1) & v_2 = (4,1,0,1) & v_3 = (0,0,1,-1) \\
v_4 = (0,0,-2,1) & v_5 = (-6,0,1,1) & v_6 = (6,0,1,1) \\
v_7 = (-6,6,1,1) & v_8 = (-6-6n,-6,1,1) & v_9 = (n-2,1,-1,1) \\
v_{10} = (n-4,1,0,1). & &
\end{array} \tag{3.24}$$

In terms of x_1, x_2, x_3, x_7 and x_8 the hypersurface defining the Calabi–Yau, Eq. (3.11), is given by

$$\begin{aligned}
W \;=\; & \left(\zeta + a_0 \frac{x_1^{4-n} x_2^{8-n}}{\zeta}\right) x_0^{6+3n} - 2x_1 x_2^3 x_0^{12} - x_1^2 x_2 + x_3^2 + a_4 x_1 x_2 x_3 x_0 \\
& + a_5 x_1^2 x_2^4 x_0^6 + a_6 x_1^3 x_2^5 + a_7 x_2^2 x_0^{18} + a_8 x_2 x_0^{16} + a_9 x_0^{14} + a_{10} x_2 x_3 x_0^9 \\
& + a_{11} x_3 x_0^7 + a_{12} x_1 x_0^6,
\end{aligned} \tag{3.25}$$

where we defined $x_0 = x_7 x_8$ and $\zeta = (x_8/x_7)^{6+3n}$ and rescaled the coordinates to eliminate the coefficients of the first three terms defining the fiber. The terms with coefficients a_5 and a_6 are again irrelevant near the singularity. Making the substitutions

$$\begin{aligned}
x_3 &= x - \frac{1}{2}\left(a_{11} + a_{10} x_2 + a_4 x_1 x_2\right) \\
x_1 &= y - \frac{1}{4}\left(a_{11} a_4 + a_{10} a_4 x_2 + 4x_2^2\right) \\
x_2 &= z
\end{aligned} \tag{3.26}$$

and neglecting an irrelevant piece proportional to $x_1^2 x_2^2$ brings Eq. (3.25) into the form

$$W = \left(\zeta + a_0 \frac{x_1^{4-n} z^{8-n}}{\zeta}\right) + x^2 + z^5 - y^2 z + 2(c_0)^{1/2} y + c_5 z^4 + c_4 z^3 + c_3 z^2 + c_2 z + c_1 + \cdots . \quad (3.27)$$

In this expression, x_1 is given by

$$x_1 = y - \frac{1}{8}\left(4c_4 - c_5^2 + 4c_5 z + 8z^2\right). \quad (3.28)$$

We can identify a_0 with the strong coupling scale of the $SO(12)$ gauge theory: $a_0 = \Lambda^{2\beta_0}$. The β-function for this theory is given by $\beta_0 = 10 - N_f - 2N_s$, where N_s counts the number of half hypermultiplets in the spinor representation of $SO(12)$. One can check that for $n = 4$ Eq. (3.27) reduces to the known curve for $SO(12)$ with four fundamentals [25, 26]. In the asymptotically free cases, $n = 2, 3, 4$, this expression reduces to a curve, because both x and y appear at most quadratically.

Again we conjecture that Eq. (3.27) can be modified to accommodate N_s spinors with masses m_i and N_f vectors with masses m_j by the following substitution

$$\zeta + a_0 \frac{1}{\zeta} x_1^{4-n} z^{8-n} \to \zeta + a_0 \frac{1}{\zeta} \prod_{i=1}^{N_s} (x_1 - m_i^4) \prod_{j=1}^{N_f} (z - m_j^2). \quad (3.29)$$

As in the $SO(10)$ case, this results in an expression that shows the expected behavior under adjoint breaking of the $SO(12)$ to $SO(10)$. The substitution above also ensures that spinors and vectors can be integrated out consistently.

3.3.3 $SO(7)$ with $(3-n)\mathbf{7} + (8-2n)\mathbf{8}$

The $SO(7)$ theory with $3-n$ fundamentals and $8-2n$ spinors differs from the theories we considered above in several respects. It is our first example of a non–simply laced group. Unlike in the previous cases, the $K3$ part of the Calabi–Yau cannot have a singularity of a type that corresponds to the gauge group, since a $K3$ can only have ADE type singularities. Thus we should expect some mixture of fiber and base coordinates even if there is no matter in the theory. The second difference is that the $SO(7)$ theory makes sense only for $n = 2, 3$. For $n = 4$, the fiber of the type IIA manifold cannot have a semisplit D_4 singularity [36], which would give rise to an $SO(7)$ gauge theory. Thus we cannot consider the case without spinors to compare to known results. Apart from that, it will turn out that the most convenient representation of the $SO(7)$ curve requires a slight modification of Batyrev's construction of the mirror.

The polar polyhedron giving rise to the $SO(7)$ gauge theory is defined by the vertices

$$\begin{array}{lll} \tilde{v}_1 = (-1, 0, 2, 3) & \tilde{v}_2 = (1, -n, 2, 3) & \tilde{v}_3 = (0, -1, 2, 3) \\ \tilde{v}_4 = (0, 0, -1, 0) & \tilde{v}_5 = (0, 0, 0, -1) & \tilde{v}_6 = (0, 0, 0, 0) \\ \tilde{v}_7 = (0, 0, 2, 3) & \tilde{v}_8 = (0, 1, 2, 3) & \tilde{v}_9 = (0, -2, 2, 3) \\ \tilde{v}_{10} = (0, -1, 1, 1) & \tilde{v}_{11} = (0, -1, 0, 1) & \end{array} \tag{3.30}$$

and the corresponding Newton polyhedron is given by

$$\begin{array}{lll} v_1 = (4, 2, 0, 1) & v_2 = (0, 0, -2, 1) & v_3 = (0, 0, 1, -1) \\ v_4 = (6, 2, 1, 1) & v_5 = (6, -6, 1, 1) & v_6 = (-6 - 6n, -6, 1, 1) \\ v_7 = (2n - 6, 2, 1, 1) & v_8 = (2n - 4, 2, 0, 1). & \end{array} \tag{3.31}$$

Using Eq. (3.11) and setting $x_4 = x_7 = x_8 = 1$, we find the defining equation of the Calabi–Yau

$$\begin{aligned} W \; = \; & \left(\zeta + a_0 \frac{x_1^{8-2n}}{\zeta}\right) x_0^{6+3n} + x_1^2 x_0^{12} + x_1 x_2^3 + x_2^2 + a_4 x_1 x_2 x_3 x_0 + a_5 x_1^4 x_0^6 \\ & + a_6 x_1^6 + a_7 x_0^{18} + a_8 x_3 x_0^9 + a_9 x_2^2 x_0^8. \end{aligned} \tag{3.32}$$

The $K3$ part of this expression can be transformed into the standard form of the classical piece of the $SO(7)$ curve using coordinate redefinitions as in the previous subsections. This results in an expression of the form

$$W = \left(\zeta + a_0 \frac{x_1^{8-2n}}{\zeta}\right) + x^2 + y^2 + z^6 + c_3 z^4 + c_2 z^2 + c_1 + \cdots , \tag{3.33}$$

where x_1 is some function of x, z and the Casimirs c_i. In this format there is no obvious way to identify the powers of the fiber coordinates that multiply the coordinate of the lower sphere with the number of matter fields.

This problem can be circumvented by replacing the Calabi–Yau, Eq. (3.32), with another Calabi–Yau that encodes the same field theory information. Recall that on the IIB side, the field theory information is encoded in the complex structure moduli, which in turn determine the period integrals over the three cycles of the Calabi–Yau. The Kähler structure moduli determine the integrals over two cycles but do not affect the integrals over the three cycles. Thus we can modify the Kähler structure of our manifold without changing the information about the gauge theory.

One way of seeing that the information encoded in the complex structure is invariant under changes of the Kähler structure is provided by the $\bigtriangledown$-hypergeometric system of partial differential equations (see, e.g., [44] for details). The period integral over the three cycles of the Calabi–Yau is given by

$$\Pi_k(a) = \int_{\gamma_k} \frac{1}{W(a,x)} \prod_p \frac{dx_p}{x_p}, \tag{3.34}$$

where $W(a,x)$ is a hypersurface constraint such as Eq. (3.32), x_p are the coordinates of the embedding space and a denotes the set of complex structure moduli. The period integrals satisfy a set of differential equations

$$\mathcal{D}_l \Pi_k = 0, \quad \mathcal{Z}_\alpha \Pi_k = 0, \tag{3.35}$$

where the differential operators are given by

$$\mathcal{D}_l = \prod_{l_i>0}\left(\frac{\partial}{\partial a_i}\right)^{l_i} - \prod_{l_i<0}\left(\frac{\partial}{\partial a_i}\right)^{-l_i}, \quad \mathcal{Z}_\alpha = \sum_i \tilde{v}_{i,\alpha} a_i \frac{\partial}{\partial a_i}, \quad \mathcal{Z}_0 = \sum_i a_i \frac{\partial}{\partial a_i} + 1. \tag{3.36}$$

Here, $\tilde{v}_{i,\alpha}$ denotes the α component of the i-th vector in the dual polyhedron and the vectors l define relations between the vertices $\tilde{v}_i$

$$\sum_i \tilde{v}_i l_i = 0, \quad \sum_i l_i = 0. \tag{3.37}$$

One can check that the hypersurface constraints obtained by Batyrev's construction satisfy these relations.

However, this does not exhaust the list of hypersurface constraints that satisfy Eq. (3.35). One can find many additional manifolds by solving these equations directly. In this approach, one does not need the information encoded in the Newton polyhedron. This reflects the fact that all of the information on the behavior of the gauge theory is contained in the dual polyhedron. Different solutions to Eqs. (3.35) will describe different Calabi–Yau manifolds but they will all have the same period integrals over the three cycles and therefore they encode the same gauge theory.

We can easily find other hypersurface constraints which satisfy Eqs. (3.35) by adding points to the Newton polyhedron that lie in its convex hull. Using the coordinates corresponding to these points to parametrize the hypersurface constraint guarantees that the resulting Calabi–Yau has the same period integrals as Eq. (3.32). Adding the vector $v_9 = (n-2, 1, -1, 1)$ to the Newton polyhedron and using the coordinates associated to v_8, v_9, v_3, v_5 and v_6 we find the hypersurface constraint

$$\begin{aligned} W &= \left(\zeta + a_0 \frac{x_8^{8-2n} x_9^{4-n}}{\zeta}\right) x_0^{6+3n} - x_8^2 x_9 x_0^{12} + 2x_8 x_9^2 + x_3^2 + a_4 x_3 x_8 x_9 x_0 \\ &\quad + a_5 x_8^4 x_9^2 x_0^6 + a_6 x_8^6 x_9^3 + a_7 x_0^{18} + a_8 x_3 x_0^9 + a_9 x_9 x_0^8. \end{aligned} \tag{3.38}$$

Setting $x_0 = 1$, neglecting the terms with coefficients a_5 and a_6, and substituting

$$\begin{aligned} x_3 &= x - \frac{1}{2}(a_8 + a_4 x_8 x_9) \\ x_8 &= y + x_9 - \frac{1}{4} a_4 a_8 \\ x_9 &= z \end{aligned} \qquad (3.39)$$

we find after redefining the complex structure parameters

$$W = \left(\zeta + a_0 \frac{x_8^{8-2n} z^{4-n}}{\zeta}\right) + x^2 + z^3 - y^2 z + c_2 z^2 + c_1 z + c_0 + \cdots, \qquad (3.40)$$

where $x_8 = y + z + c_2/2$. We can identify a_0 with $\Lambda^{2\beta_0}$ and the c_i with the Casimirs of $SO(7)$. Since we cannot choose n to eliminate all spinors, we cannot compare this curve directly to known results. However, higgsing $SO(7)$ to $SO(5)$ as in Sec. 3.3.1, we obtain the expected curve for $SO(5)$ with $3-n$ fundamentals. If we modify Eq. (3.40) to allow arbitrary numbers of spinors and vectors with arbitrary masses by replacing

$$\zeta + a_0 \frac{1}{\zeta} x_8^{8-2n} z^{4-n} \rightarrow \zeta + a_0 \frac{1}{\zeta} z \prod_{i=1}^{N_s} (x_8 - m_i^2) \prod_{j=1}^{N_f} (z - m_j^2), \qquad (3.41)$$

we can integrate out all spinors in Eq. (3.40). Then x and y can be integrated out trivially and substituting $z \rightarrow z^2$, we find the double cover version of the $SO(7)$ curve with $3-n$ fundamentals [25, 27]. Unlike in the previous cases, we can write Eq. (3.40) as a curve only for $n = 3$.

3.3.4 $SO(9)$ with $(4-n)\mathbf{16} + (5-n)\mathbf{9}$

In this section we repeat the analysis of the previous sections for a class for Calabi–Yau manifolds that lead to an $SO(9)$ gauge theory with $5-n$ vectors and $4-n$

spinors. The toric description of these manifolds is given by the vertices

$$\begin{aligned} \tilde{v}_1 &= (-1,0,2,3) & \tilde{v}_2 &= (1,-n,2,3) & \tilde{v}_3 &= (0,-1,2,3) \\ \tilde{v}_4 &= (0,0,-1,0) & \tilde{v}_5 &= (0,0,0,-1) & \tilde{v}_6 &= (0,0,0,0) \\ \tilde{v}_7 &= (0,0,2,3) & \tilde{v}_8 &= (0,1,2,3) & \tilde{v}_9 &= (0,-2,2,3) \\ \tilde{v}_{10} &= (0,-2,1,2) & \tilde{v}_{11} &= (0,-1,1,1) & \tilde{v}_{12} &= (0,-1,0,1) \end{aligned} \tag{3.42}$$

of the dual polyhedron. The Newton polyhedron consists of the vertices

$$\begin{aligned} v_1 &= (2,1,-1,1) & v_2 &= (6,2,1,1) & v_3 &= (0,0,1,-1) \\ v_4 &= (6,-6,1,1) & v_5 &= (0,0,-2,1) & v_6 &= (-6-6n,-6,1,1) \\ v_7 &= (2n-6,2,1,1) & v_8 &= (n-2,1,-1,1). \end{aligned} \tag{3.43}$$

Using these vectors and Eq. (3.11), we can write down the mirror. It is convenient to use the $\mathbb{C}^*$ actions to set all coordinates except x_1, x_2, x_3, x_4 and x_6 to one. Defining $x_0 = x_4 x_6$ and $\zeta = (x_4/x_6)^{6+3n}$ we get

$$\begin{aligned} W &= \left(\zeta + a_0 \frac{x_1^{4-n} x_2^{12-2n}}{\zeta}\right) x_0^{6+3n} + 2x_1 x_2^4 x_0^{12} - x_1^2 + x_3^2 \\ &\quad + a_4 x_1 x_2 x_3 x_0 + a_5 x_1^2 x_2^6 x_0^6 + a_6 x_1^3 x_2^8 + a_7 x_2^2 x_0^{18} + a_8 x_0^{16} \\ &\quad + a_9 x_2 x_3 x_0^9 + a_{10} x_1 x_0^8. \end{aligned} \tag{3.44}$$

For $SO(9)$, Batyrev's construction gives a description of the mirror in which the matter content of the theory is visible in the fibration over the lower sphere. The terms with coefficients a_5 and a_6 are irrelevant near the singularity. We can transform the fiber into the standard form for an $SO(9)$ theory by making the following substitutions

$$\begin{aligned} x_3 &= x - \frac{1}{2}\left(a_4 x_1 x_2 + a_9 x_2\right) \\ x_1 &= y + \frac{1}{4}\left(2a_{10} - a_4 a_9 x_2^2 + 4x_2^4\right). \\ x_2 &= z. \end{aligned} \tag{3.45}$$

Neglecting an irrelevant term of the form $x_1^2x_2^2$ and renaming the coefficients, we find

$$W = \left(\zeta + a_0\frac{x_1^{4-n}z^{12-2n}}{\zeta}\right) + x^2 - y^2 + c_0 + c_1z^2 + c_2z^4 + c_3z^6 + z^8 + \cdots, \qquad (3.46)$$

where

$$x_1 = y + \frac{1}{8}\left(4c_2 - c_3^2 + 4c_3z^2 + 8z^4\right). \qquad (3.47)$$

It is straightforward to check that for $n = 4$ this curve agrees with the curves in [25, 27], once one identifies the c_i with the gauge invariant polynomials that parametrize the Coulomb branch and sets $a_0 = \Lambda^{2\beta_0}$.

Again, the substitution

$$\left(\zeta + a_0\frac{x_1^{4-n}z^{12-2n}}{\zeta}\right) \to \zeta + a_0\frac{1}{\zeta}z^2\prod_{i=1}^{N_s}(x_1 - m_i^4)\prod_{j=1}^{N_f}(z^2 - m_j^2) \qquad (3.48)$$

presumably results in a solution of the theory with arbitrary numbers of massive vectors and spinors. Repeating the checks as in Sec. 3.3.1, we find consistent behavior.

3.3.5 $SO(11)$ with $(\frac{4-n}{2})\mathbf{32} + (7-n)\mathbf{11}$

For $SO(11)$ with $4 - n$ half hypermultiplets in the spinor representation and $7 - n$ vectors we can repeat the steps that provided the curve for $SO(7)$. The polar polyhedron is given by the vertices

$$\begin{array}{lll} \tilde{v}_1 = (-1,0,2,3) & \tilde{v}_2 = (1,-n,2,3) & \tilde{v}_3 = (0,-1,2,3) \\ \tilde{v}_4 = (0,0,-1,0) & \tilde{v}_5 = (0,0,0,-1) & \tilde{v}_6 = (0,0,0,0) \\ \tilde{v}_7 = (0,0,2,3) & \tilde{v}_8 = (0,1,2,3) & \tilde{v}_9 = (0,-2,2,3) \\ \tilde{v}_{10} = (0,-2,1,2) & \tilde{v}_{11} = (0,-2,0,1) & \tilde{v}_{12} = (0,-1,1,1) \\ \tilde{v}_{13} = (0,-1,0,0) & & \end{array} \qquad (3.49)$$

and the corresponding Newton polyhedron is defined by

$$
\begin{array}{llll}
v_1 = (2,1,-1,1) & v_2 = (6,1,1,1) & v_3 = (0,0,1,-1) & \\
v_4 = (0,0,-2,1) & v_5 = (6,-6,1,1) & v_6 = (-6-6n,-6,1,1) & \quad (3.50) \\
v_7 = (n-6,1,1,1) & v_8 = (n-2,1,-1,1). & &
\end{array}
$$

Using these polyhedra, we can write down the mirror Calabi–Yau but as in the $SO(7)$ case there is no choice of coordinates in which the fibration over the lower sphere has a simple interpretation in terms of the number of fundamentals and spinors. However, we can add the vector $v_9 = (n-4,1,0,1)$ to the Newton polyhedron and use x_8, x_9, x_3, x_5, x_6 with $x_0 = x_5 x_6$ and $\zeta = (x_5/x_6)^{6+3n}$ to parametrize the Calabi–Yau

$$
\begin{aligned}
W &= \left(\zeta + a_0 \frac{x_8^{4-n} x_9^{8-n}}{\zeta}\right) x_0^{6+3n} + 2x_8 x_9^3 x_0^{12} - x_8^2 x_9 + x_3^2 + a_4 x_3 x_8 x_9 x_0 + a_5 x_8^2 x_9^4 x_0^6 \\
&\quad + a_6 x_8^3 x_9^5 + a_7 x_9^2 x_0^{18} + a_8 x_9 x_0^{16} + a_9 x_0^{14} + a_{10} x_3 x_9 x_0^9 + a_{11} x_3 x_0^7. \qquad (3.51)
\end{aligned}
$$

Near the singularity we can neglect the terms with coefficients $a_{5,6}$. Substituting

$$
\begin{aligned}
x_3 &= x - \frac{1}{2}\left(a_{11} + a_{10} x_9 + a_4 x_8 x_9\right) \\
x_8 &= y - \frac{1}{4}\left(a_{11} a_4 + a_{10} a_4 x_9 - 4x_9^2\right) \qquad (3.52) \\
x_9 &= z
\end{aligned}
$$

into the defining equation of the Calabi–Yau gives

$$
W = \left(\zeta + a_0 \frac{x_8^{4-n} z^{8-n}}{\zeta}\right) + x^2 + z^5 - zy^2 + c_5 z^4 + c_4 z^3 + c_3 z^2 + c_2 z + c_1 + \cdots, \qquad (3.53)
$$

where

$$
x_8 = y - \frac{1}{8}\left(c_5^2 - 4c_4 - 4c_5 z - 8z^2\right). \qquad (3.54)
$$

For $n = 4$ we can integrate out y trivially. Substituting $z \to z^2$, Eq. (3.53) reduces to the double cover version of the curve for $SO(11)$ with three fundamentals [25, 27]. In the other two asymptotically free cases $n = 2, 3$, we also obtain a curve but it is not hyperelliptic. Presumably we can obtain an exact solution for any number of massive vectors and spinors by substituting

$$\left(\zeta + a_0 \frac{x_8^{4-n} z^{8-n}}{\zeta}\right) \to \zeta + a_0 \frac{1}{\zeta} z \prod_{i=1}^{N_s} (x_8 - m_i^4) \prod_{j=1}^{N_f} (z - m_j^2). \qquad (3.55)$$

Again, our solution passes the tests given in Sec. 3.3.1.

3.4 Summary and Concluding Remarks

We have obtained exact solutions to $\mathcal{N} = 2$ supersymmetric $SO(N)$ gauge theories for $N = 10, 12$ and $N = 7, 9, 11$ with massless matter in the spinor and the fundamental representation. We gave a description of the Coulomb branch of these theories in terms of ALE spaces fibered over a sphere.

These solutions were obtained by compactifying type IIA string theory on Calabi–Yau threefolds with singular $K3$ fibers. The singularity type of the $K3$ determines the gauge group of the $d = 4$ gauge theory and the duality to heterotic strings compactified on $K3 \times T^2$ can be used to determine the charged matter content of the theory. Mirror symmetry relates the Calabi–Yau for type IIA compactification to a different Calabi–Yau that gives rise to the same field theory when type IIB string theory is compactified on it. The exact solutions can be extracted from this mirror Calabi–Yau.

This approach provides exact solutions for the gauge theories listed above with specific matter contents. We proposed some generalizations of these results to arbitrary numbers of massive spinors and vectors and verified that our solutions are consistent under adjoint breaking and integrating out matter fields. Unfortunately, the list of asymptotically free $SO(N)$ theories with spinors is not exhausted by the cases we have studied. For $SO(8)$ there is no toric description of the corresponding type IIA and IIB

Calabi–Yau manifolds and the higher rank groups $SO(N), N = 13, 14, 15, 16$ cannot be obtained from compactifying type IIA on a Calabi–Yau threefold or conversely from breaking the adjoint of E_8 on the heterotic side.

The results presented may ultimately provide some insights into how to construct matter representations other than fundamentals and two index tensors from branes. In principle it should be possible to find a brane configuration corresponding to the theories we analyzed here by studying an M-theory 5-brane wrapped on $R^4 \times \Sigma$ where Σ is the curve encoding the gauge couplings on the Coulomb branch.

Since our solutions agree with known field theory results, in the cases where these are available, one can view the results of this paper as further confirmation of mirror symmetry and the duality between type IIA and heterotic strings.

Chapter 4 CFT's of $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$ Orbifolds

4.1 Calabi-Yau via D Brane Probes

In the first three chapters we have studied exclusively closed string theory on Calabi-Yau manifolds. This is not the only possible approach to study string theory on Calabi-Yau manifolds. Type II string theory contains, in addition to "fundamental" strings in terms of which string perturbation theory is formulated, extended objects of various sorts which carry charges under the (R, R) and (NS, NS) gauge potentials. These solitons must be included for consistency of the theory, and furthermore various string dualities – or simply conifold transitions of Chapter 2, in the case of (R, R) solitons – exchange the fundamental string states with the non-perturbative states. Because of this, it has been clear for some time now that strings themselves are not the fundamental objects in the theory, and that the concept of fundamental degrees of freedom of a theory is itself ill-defined in string theory at strong string or sigma-model coupling.

The one advantage of "fundamental" strings is that they provide tools for computations. However, even that is true only in a limited sense. The reason is that (R, R) solitons are D branes, so they do have a perturbative string description at weak string coupling: in terms of CFT with Dirichlet boundary conditions – open string CFT for short. Because of that it makes sense to ask for a description of Calabi-Yau compactification in terms of D branes.

There are several things to keep in mind. Since open string CFT does not contain gravity, the question of D brane geometry is formulated in the background of type II string compactification on a Calabi-Yau X. Second, since D branes are extended objects, D brane configurations on X are classified by $H_*(X, \mathbb{Z})$, in other words, D branes must wrap cycles in X. The geometry a D brane sees will depend on which element in $H_*(X, \mathbb{Z})$ it wraps – the geometry is the space of all D brane configurations in the same homotopy class.The D brane configuration that naturally probes the geometry of X is a single D brane which is pointlike in X – corresponding to the generator of $H_0(X,\mathbb{Z})$, since its classical moduli space is all of X, and it is this configuration we will concentrate on in this Chapter.

It is an important question what the relationship between the open and closed string theory approaches is. Open string CFT is complementary to the closed string theory in the sense that short distances in one theory corresponds to long distances in the other*1. In the cases studied thus far, the two descriptions seem to match smoothly onto each other *2, although there have been few studies that go beyond just looking at the topological properties of $\mathcal{M}$.

What we turn next to is a study of open string theory CFT on a certain class of Calabi-Yau spaces where closed string theory is known to exhibit some peculiar and little understood features.

4.2 Introduction to Orbifolds (with Discrete Torsion)

There exists a special class of Calabi-Yau manifolds, called orbifolds, whose CFTs are exactly solvable – this is a marked exception in the Calabi-Yau world. The price to pay for the simplicity of the CFT is that the spaces themselves are singular, so in this sense they are only cousins of Calabi-Yau manifolds. One is in for a surprise, however, since the conformal field theory on orbifolds turns out to be perfectly well behaved, and furthermore the spectrum of the theory happens to be exactly the same as that on the *blowup* of the orbifold *3. What is surprising about this is that from

*1 This is a direct consequence of the fact that a loop of open string stretched between two D-branes can be interpreted as a closed string, tree level interaction.

*2 The firm ground on this issue is really established only on Calabi-Yau two-folds, in the ALE limit [45]. On Calabi-Yau threefold it also seems to be the case, modulo some important technicalities which are beyond the scope of this work. For more detail consult [46]

*3 Blowing up is a means of resolving singularities of manifolds by cutting the singularity out and replacing the singular locus by a holomorphic cycle of appropriate dimension. One "adds" a cycle of codimension one in the space transverse to the singular locus, which is a natural way to repair the singularity if the complex structure is not to be disturbed. The allowed singularities must be such the standpoint of classical geometry, an orbifold is too singular to be a manifold and consequently it is hard to even define what one means by cohomology of the space. So, somehow string theory "knows" that the space can be resolved, and the CFT orbifold is simply a special point in the moduli space of CFTs associated to the resolved space where the CFT becomes solvable and the geometry singular. Once this is accepted it does not come as a surprise, in the view of stringy Kähler geometry, that singular geometry may not imply singularity of the conformal field theory. We have seen that a CFT has non-geometric parameters, the B fields, and turning on these moduli is sufficient to remove the singularity from the conformal field theory.

This seems to be a nice and consistent picture which entails the assumption that non-linear sigma models are ultimately based on classical geometry, and that in the point particle limit stringy geometry will agree with classical geometry. As noted above, this picture seems to be supported by D-brane probes as well.

However, in the work of Vafa and Witten from 1994 [3], counter- examples to this notion have been constructed in the language of closed string CFT, and they are known as orbifolds with discrete torsion. Basically it turns out that apart from the "conventional" orbifold CFT discussed above, there exist other CFT's which can be associated to the same geometric space, and they are labeled by the choice of discrete torsion. The theories with torsion do not share the nice properties of "conventional" orbifold CFT we talked about above, in that they contain singularities which cannot be resolved, but remain as regions in which stringy effects are always large. This is particularly puzzling since, at least from the classical geometry, one would expect the theory to probe the smooth neighboring vacua, but the CFT simply fails to see them.

It is an interesting question, therefore, whether similar ambiguities in defining orbifold theory CFT arise for D-brane probes as well. The answer to this turns out to be positive, and furthermore D brane geometry agrees perfectly with that of closed string CFT, but not with classical geometry.

---

that one can repair them by keeping the first chern class trivial. Another way to say this, is that such a singularity can be obtained by deforming Kähler structure of a smooth Calabi-Yau manifold.

4.3 Closed String CFT on Orbifolds

An orbifold is a quotient of a smooth manifold $\mathcal{M}$ by a discrete isometry group Γ. The quotient is taken in the usual sense of identifying the points on the orbits of the Γ action, a point $x \in \mathcal{M}$ being identified with $gx \in \mathcal{M}$. If the Γ action has fixed sets, the quotient space will have singularities. The reason is as follows. The action of Γ on $\mathcal{M}$ lifts to an action of the tangent bundle of $\mathcal{M}$, denoted by $T\mathcal{M}$. It does so in such a way that

$$x \xrightarrow{g} gx : \quad T_x \xrightarrow{g} T_{gx},$$

for every x in $\mathcal{M}$ and every g in Γ. If, however, x is fixed under g, so that $x = gx$, then g maps T_x to itself. This map must be a rotation of the vectors in T_x, since Γ acts as an isometry and so its action must be norm-preserving. Because Γ is discrete, the quotient singularities produced by the identifications are conical, deficit angle singularities.

The singular set of $\mathcal{M}/\Gamma$ is a union of spaces $S_g = \{x|gx = x\}$ fixed by elements $g \in \Gamma$ [*4]. Another fact that will be important to us is that the quotient space is Calabi-Yau. One will recall that $\mathcal{M}$ is Calabi-Yau manifold if and only if it is a Kähler manifold with a unique nowhere vanishing holomorphic d-form $\Omega^{d,0}$. Then, $\mathcal{M}/\Gamma$ is Calabi-Yau as well provided the action of Γ preserves Ω.

4.3.1 Generalities of "Ordinary" Orbifolds

Field theory on $\mathcal{M}/\Gamma$ is simply defined in terms of truncation of the theory on $\mathcal{M}$ to Γ invariant states. In string theory however quotient theory will have states which do not come from $\mathcal{M}$. Since open and closed string theories behave quite differently in this respect, (although, as we will see shortly the difference is only at a superficial level), we will concentrate on the closed string theory first.

In closed string theory the new states go under the name "twisted" string states. The name derives from the fact that those states come from quantization of strings

[*4] It is not necessary that $\mathcal{M}$ be smooth, but in this section we will assume so, since we consider quotient singularities only.

which are closed on $\mathcal{M}$ only up to the Γ action, and thus are created by fields with twisted boundary conditions

$$\Theta(\sigma + 2\pi) = g \circ \Theta(\sigma), \quad g \in \Gamma.$$

Classically, the massless string states are constant configurations on the world-sheet so the above equation implies that twisted strings of zero mass must propagate only on the fixed set S_g.

Now, the problem of computing the massless spectrum of superstring theory on smooth, compact complex manifolds has a well-known solution – the Ramond-Ramond ground states are determined by topological data only, their number is given by the dimension of $H^*(X)$. Since S_g is smooth for every g in Γ, the twisted string states are in one to one correspondence with the generators of $H^*(S_g)$ [*5]. More precisely, the correspondence is given by the following formula

$$H^{p,q}_{\Gamma}(S_g) \to H^{p+s,q+s}(\mathcal{M}/\Gamma).$$

The shift in the Hodge numbers comes about because the assignment of the cohomology groups $H^{p,q}$ to (R, R) states comes via the (p, q) charge of the Ramond- Ramond fields acting on the vacuum under the left and the right-moving $U(1)$ current on the world sheet. In the orbifold, however, the vacuum itself carries non-zero $U(1)$ charge which is computed in [47] (in fact most of the introduction follows this paper), with the result that $g : z^{\alpha} \to e^{2\pi i \theta_{\alpha}} z^{\alpha}$, where $0 \leq \theta_{\alpha} < 1$,then

$$s_g = \sum_{\alpha} \theta_{\alpha}. \tag{4.1}$$

It should be clear that the twisted states too must be projected to those that are Γ invariant, so

$$H(S_g) \quad \to \quad H_{\Gamma}(S_g).$$

[*5] One can in fact show that S_g is a Kähler submanifold (basically the Kähler form on S_g is given by the pullback of the Kähler form on on $\mathcal{M}$, and such is preserved by action of Γ [47]).

4.3.2 A $\mathbb{Z}_2 \times \mathbb{Z}_2$ Example

The simplest Calabi-Yau threefold which has an orbifold singularity is a space which is (locally) $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. If we take $z_\alpha, \alpha = 1, 2, 3$ to be the choice of complex coordinates on $\mathbb{C}^3$ than $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts by:

$$\begin{aligned} g &: \quad (z_1, z_2, z_3) \to (z_1, -z_2, -z_3), \\ h &: \quad (z_1, z_2, z_3) \to (-z_1, z_2, -z_3), \\ gh &: \quad (z_1, z_2, z_3) \to (-z_1, -z_2, z_3). \end{aligned}$$

The fixed set of Γ consists of $S_g = (z_1, 0, 0)$, $S_h = (0, z_2, 0)$, and $S_{gh} = (0, 0, z_3)$. These are three curves of singularities, in the neighborhood of each of which $\mathcal{M}/\Gamma$ looks like $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$, and which intersect over a point $(0, 0, 0)$. The complex manifold $\mathbb{C}^3$ has a unique holomorphic three form, $\Omega^{3,0} = dz_1 \wedge dz_2 \wedge dz_3$ which survives the quotient since Γ flips the sign of coordinates pairwise. Now let us consider the string spectrum [*6]. The fixed set of g is just a copy of $S_g = \mathbb{C}$ parametrized by z_1. The total cohomology of $\mathbb{C}$ is generated by

$$1, \ dz_1, \ d\bar{z}_1, \ dz_1 \wedge d\bar{z}_1,$$

which belong to $H^{0,0}$, $H^{1,0}$, $H^{0,1}$ and $H^{1,1}$ respectively. Since $h : z_1 \to -z_1$, only 1 and $dz_1 \wedge d\bar{z}_1$ are invariant under Γ, so the contribution to the stringy cohomology of the orbifold of g-twisted states is $h^{1,1} = 1 = h^{2,2}$, and zero otherwise. There are two more elements similar to g, so we find that

$$h^{1,1} = 3, \quad h^{2,1} = 0,$$

on the orbifold in string theory. This is a remarkable result. The point is that

[*6] For our methods, as outlined above, we really need to consider compact spaces. It suffices to think about $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ as a piece of a compact manifold, $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$.This space has 64 fixed points, the neighborhood of each of which looks like our space. Alternatively, one can consider compactly supported cohomology on S_g in order to obtain normalizable ground states. We will be loose about this point.

defining cohomology of a singular space is ordinarily fairly hard, and to get something reasonable, one has to do so using "simplicial" rather than de-Rham cohomology. What we find here is that string theory anticipates the cohomology of the *resolved* orbifold since the Hodge numbers the CFT computes correspond precisely to what one would have obtained by *blowing* up the orbifold. This seems to be a generic behavior of string theory on orbifolds.

4.4 Discrete Torsion

It turns out that the orbifold CFT of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ is not unique but admits a generalization by turning on discrete torsion which really means adding certain discrete phases to the string path integral. The non-trivial phases can be introduced if $H^2(\Gamma, U(1)) \neq 0$, equivalently when there exist maps $\Gamma \times \Gamma \to \mathbb{Z}$ which are antisymmetric and whose image is not trivial. For $\Gamma = \mathbb{Z}_m \times \mathbb{Z}_n$, $H^2(\Gamma, U(1)) = \mathbb{Z}_r$, where $r = \gcd(m,n)$, so there are r possible choices of torsion. We briefly describe how this is done, following [3] closely. Consider strings propagating on a patch of some (possibly compact) manifold, biholomorphic to $\mathbb{C}^3/\Gamma$, $\Gamma = Z_m \times Z_n$. The CFT is constructed in terms of maps $\Sigma \to \mathbb{C}^3/\Gamma$, where Σ denotes a world-sheet of closed string. Alternatively, one considers maps to $\mathbb{C}^3$, and those which are closed up to Γ action can be described in terms of world-sheets twisted by elements of Γ. At genus one for example, let Σ be a quotient of the $\sigma_1 - \sigma_2$ plane by $\sigma_1 \to \sigma_1 + 1$, $\sigma_2 \to \sigma_2 + 1$. Twisted maps will include twists both along the σ_1 and σ_2 directions.

The inclusion of discrete torsion can then be described as follows. Pick an integer $p = 0, \ldots, r-1$, and let $\zeta = e^{2\pi i/r}$. The contribution to the one-loop path integral of the world-sheet twisted by $g^a h^b$ along σ_1, and $g^{a'} h^{b'}$ along σ_2 is weighted by an additional phase

$$\epsilon(g^a h^b, g^{a'} h^{b'}) = \zeta^{p(ab' - ba')}. \tag{4.2}$$

It was shown in [2] that for every choice of p, there exists a unique generalization of this to higher genus surfaces. The effect of torsion is to change the transformation law of the $g^{a'}h^{b'}$ twisted states under $g^a h^b$, by addition of the phase (4.2), and this changes the notion of Γ invariance in the orbifold.

"Ordinary" orbifold CFT corresponds to no torsion, $p = 0$, but there are $r - 1$ closed string theories with torsion one can define. As we will see, while an orbifold theory without torsion has a "good" geometric interpretation, an orbifold theory with torsion does not.

4.4.1 $\mathbb{Z}_2 \times \mathbb{Z}_2$ Theory with Torsion

Before we go on to do the general case, let us return to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ example.

We recall that g-twisted states corresponded to 1, dz_1, $d\bar{z}_1$, $dz_1 \wedge d\bar{z}_1$. Now,

$$\epsilon(g^a, h^b, g) = (-1)^b,$$

so that Γ invariant states are those transforming as (-1) under h. These are precisely dz_1 and $d\bar{z}_1$, so that this time it is $H^{1,0}(S_g)$ and $H^{0,1}(S_g)$ that survive the quotient. Taking into the account the other two elements of Γ and the shifts in the cohomology labels, the $H^{1,0}(S_g)$ orbifold with torsion has

$$h^{1,1} = 0, \quad h^{2,1} = 3.$$

Generators of $H^{2,1}(\mathcal{M}/\Gamma)$ correspond to deformations of complex structure of the orbifold. Complex structure of $\mathbb{C}^3/\Gamma$ is given in terms of Γ invariant monomials on $\mathbb{C}^3$, modulo any relations between them. Here, Γ invariant monomials are $x_i = z_i^2$, $i = 1, 2, 3$, and $y = z_1 z_2 z_3$, which satisfy one relation:

$$y^2 = x_1 x_2 x_3.$$

By projecting onto $x_i = const$, the space contains three curves of singularities of the form

$$y^2 \propto x_i x_j.$$

The three elements of $H^{2,1}(\mathcal{M}/\Gamma)$ can be thought of as deforming each curve of singularities (from the orbifold point of view, this seems as a natural interpretation, since the twisted states which give rise to the deformations are supported there. However, one should be cautious since the concept of locality when it comes to deformations of the complex structure is obscure.) Such a deformation could for example look like

$$y^2 = x_1 x_2 x_3 + b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2.$$

This resolves each curve of singularities, however, one clearly has a conifold singularity left at the origin: upon adding the deformations the $x_1 x_2 x_3$ term becomes irrelevant and can be neglected. What is left over is an equation of the conifold. So upon turning on the deformations present in string theory, we have found that we cannot completely resolve the singularities. From the mathematical standpoint there is no obstruction to deforming the conifold away via

$$y^2 = b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 + c,$$

however in string theory this deformation is absent, leaving a stable conifold singularity. This singularity is not a singularity in the CFT, unlike the conifold treated in [48], but is simply a region where stringy effects are large due to concentrated curvature. The conifold theory obtained above is smooth: it is a deformation of the orbifold CFT which does not have singularities, and the deformations we employed are not expected to introduce singularities. This manifests itself here precisely by the impossibility of turning on the offending deformation c. One final note: in the case of an orbifold without torsion, the spectrum we computed corresponded to the blowup of the orbifold. There, unlike in this case, no additional singularity at the intersection was found – basically the reason is that for a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, resolving singularities in codimension two automatically resolves the singularities at codimension three, as can be easily seen torically.

4.4.2 General $\mathbb{Z}_m \times \mathbb{Z}_n$ Case

Take the orbifold group $\Gamma = \mathbb{Z}_m \times \mathbb{Z}_n$ to act as

$$\begin{aligned} g: &\quad (z^1, z^2, z^3) \to (z^1, e^{\frac{2\pi i}{m}} z^2, e^{-\frac{2\pi i}{m}} z^3), \\ h: &\quad (z^1, z^2, z^3) \to (e^{\frac{2\pi i}{n}} z^1, z^2 e^{-\frac{2\pi i}{n}} z^3). \end{aligned}$$

Torsion depends only on the ratio of p and r in the eq.(4.2), so let $q = gcd(p, r)$. Using the formula (4.2),

$$\epsilon(g^a h^b, g^{\frac{r}{q}a'} h^{\frac{r}{q}b'}) = 1,$$

for all a, a', b, b'. Let us denote

$$m = rs, \quad n = rt, \quad gcd(s, t) = 1,$$

and put $\tilde{g} = g^{\frac{r}{q}}$ and $\tilde{h} = h^{\frac{r}{q}}$. From above, we see that a subgroup

$$\tilde{\Gamma} = \mathbb{Z}_{qs} \times \mathbb{Z}_{qt} \subset \Gamma$$

generated by $\tilde{g}$, and $\tilde{h}$ is completely unaffected by torsion. Thus in a spectrum of the Γ orbifold with p units of torsion, we will find a complete twisted sector of a $\mathbb{C}^3/\tilde{\Gamma}$ orbifold.

This contribution is as follows. In the "ordinary" orbifold, the CFT spectrum agrees with the spectrum on the *blowup* of the orbifold. The cohomology of this orbifold can be computed by toric methods, with the following result:

- The $\mathbb{C}^3/\tilde{\Gamma}$ orbifold has three curves of $\mathbb{Z}_{qs}$, $\mathbb{Z}_{qt}$, and $\mathbb{Z}_q$ singularities. Blowing them up contributes $qs - 1$, $qt - 1$ and $q - 1$ to $h^{1,1}$. (Roughly, each singularity is of the form $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_*$. Blowing up replaces the singular curve by a chain of S^2's fibered over $\mathbb{C}$, which contributes to h_4, and by the dual-of-the-dual to $h^{1,1}$.)

- Since $\tilde{g}^a\tilde{h}^b : (z_1, z_2, z_3) \to (e^{\frac{2\pi i}{qs}a}z_1, e^{\frac{2\pi i}{qt}b}z_2, e^{-\frac{2\pi i}{q^2st}(qta+qsb)}z_3)$, the origin is fixed by elements of the form $\tilde{g}^a\tilde{h}^b$, where

$$0 < a, \quad 0 < b,$$

$$qta + qsb < q^2st,$$

a simple counting shows that there are $\frac{1}{2}(q^2st - qs - qt - q + 2)$ such elements, and they contribute to $h^{1,1}$: the fixed set is a point whose cohomology consists of constant functions contributing to $H^{0,0}$ and this assignment gets shifted by 1 following (4.1). Poincaré duality requires the same contribution to $h^{2,2}$, which comes from the remainder of elements (a, b) fixing the origin which satisfy instead:

$$q^2st < qta + qsb < 2q^2st.$$

It is easy to see that no other elements of Γ can contribute to the $H^{1,1}$ cohomology of the $\mathbb{C}^3/\Gamma$ orbifold. For, curves of singularities contribute to $H^{1,1}$ and $H^{2,2}$ via 1 and $dz \wedge d\bar{z}$, and these are always invariant under all other elements of Γ, so either torsion is non-trivial, $p \neq 0 \bmod r$, and they are projected out, or torsion is trivial and they have already been accounted for. Thus,

$$h^{1,1} = \frac{1}{2}(q^2st + qs + qt + q - 4).$$

Let us now turn to $H^{2,1}$. The contribution to these group elements can *only* come from the curves of singularities, from sectors twisted by g^a, h^a or $g^{as}h^{-at}$, and so elements generating them are always of the form dz and $d\bar{z}$. Consider, for example g^a twisted states, which propagate along $S_g = (z_1, 0, 0)$. According to the above discussion, we are instructed to keep states that transform as ζ^{ap} under h, which is only possible if $n = r$ and $ap = \pm 1 \bmod r$. The second equation is the statement that $q = gcd(p, r) = 1$, and if in addition $n = r$ then, for every choice of sign, there is only one solution for a in $\mathbb{Z}_n$. The choice of sign in effect picks out one of the dz or $d\bar{z}$'s as the "invariant" element of the fixed set cohomology group. Similar considerations for other elements can be used to show:

- If $m = n = r$ where $r = gcd(m,n)$, that is if $\Gamma = \mathbb{Z}_r \times \mathbb{Z}_r$, and if in addition $q = gcd(r,p) = 1$, then $h^{2,1} = 3 = h^{1,2}$.
- In all other cases, $h^{2,1} = 0 = h^{1,2}$.

4.5 Interpretation of the Moduli

We have computed above the massless twisted sector states of the $\Gamma = \mathbb{Z}_m \times \mathbb{Z}_n$ orbifold CFT with p units of torsion. The twisted sector states are associated to exactly marginal operators which can be used to deform the orbifold CFT to a nearby CFT describing string propagation of a (partially) smoothed out space. In the $\mathbb{Z}_2 \times \mathbb{Z}_2$ example, we have seen that, turning on or off torsion has an interpretation of picking a smoothing of the singularities in target space, but that the theory with torsion can achieve this only partially. We can now try to repeat the exercise for more general orbifolds.

We have seen that the $\mathbb{C}^3/\Gamma$ orbifold contains as a sub-sector the fields needed to resolve the $\mathbb{C}^3/\tilde{\Gamma}$ orbifold. A reasonable interpretation of this, as is easy to see torically is that the partial resolution of the orbifold using these states produces a space containing $N = qs \cdot qt$ orbifold singularities all of the type $H = \mathbb{Z}_{\frac{r}{q}} \times \mathbb{Z}_{\frac{r}{q}}$, not all of which are independent, and which cannot be blown up any further in string theory. Another way to say this is that the space is a quotient of $\mathbb{C}^3/\tilde{\Gamma}$ by H. The above calculation shows further that, if in fact $N > 1$, there are no additional states preventing resolution of singularities. It is important to note that, in that case, the singularities are *not* isolated.

If, on the other hand $N = 1$, which corresponds to $m = n = r$, and $q = 1$, string theory provides exclusively a resolution via deformation of the complex structure, with a single element resolving each curve of singularities. In this case, we may argue as follows. The complex structure of a $\mathbb{Z}_r \times \mathbb{Z}_r$ orbifold is given by

$$y^r = x_1 x_2 x_3, \tag{4.3}$$

as before, we can expect on the basis of stringy derivation of the spectrum one element of $h^{2,1}$ resolving each curve of singularities, and leaving *isolated* singularities. The curves are of the form

$$y^r \propto x_i x_j,$$

and, without disturbing the rest of the space, one can argue that the resolution takes the form

$$y^r \propto x_i x_j + x_k^2.$$

This leaves a conifold singularity at the origin, a singularity that requires $r - 1$ parameters to be resolved, which would replace $y^r \to \prod_{\alpha=0}^{r}(y - a_\alpha)$. And which in this case are missing. We could also ask what happens in the case of our partial resolutions.

We will see in the next section that similar phase ambiguities arise in the open string sector as well.

4.6 Open Strings on Orbifolds

In string theory there is a complementary way to study singular spaces, and that is by using D-brane probes. In the language of D-branes, the spacetime itself arises indirectly, as the moduli space of the gauge theory living on the D-brane world volume. To describe a Calabi-Yau threefold, it is necessary to use D p-brane probes with $p \leq 3$. For concretness we will henceforth set $p = 3$, the other cases being related to it via dimensional reduction. The world volume theory must have $\mathcal{N} = 1$ supersymmetry and the singularities in the moduli spaces are resolved by turning on FIayet-Iliopoulos parameters or by deformations of the superpotential.

For a general X there is no known prescription of how to determine precisely the world volume gauge theory, with the only requirement really being that the moduli space of a single D-brane on X should in fact be X itself. With this requirement alone we can say very few things. First, N D branes at a smooth point in the Calabi-Yau space will neccesarily be described, at energies $E \ll \frac{R}{\alpha'}, \frac{1}{\sqrt{\alpha'}}$, by $U(N)$ gauge theory on the world volume, with effective $\mathcal{N} = 4$ supersymmetry. When the curvatures are large *and* substringy there must exist an effective gauge theory description of the "compactification" manifold, as long as the probe itself is small enough to be a "good" probe of geometry. Beyond this one must aproach the problem on a case by case basis. Exceptions to this are orbifolds $X \approx \mathbb{C}^3/\Gamma$[*7], where a simplification occurs because the theory on $\mathbb{C}^3/\Gamma$ is a quotient of the theory on $\mathbb{C}^3$[*8]. Studies of $\mathbb{C}^2/\mathbb{Z}_m$, and $\mathbb{C}^3/\mathbb{Z}_m$ showed that in both cases the stringy constructions provide a physical realization of such concepts as Hyper-Kähler quotients and symplectic quotient constructions respectively. It is then interesting to ask if the same phenomena we have found in the closed string theory on $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$ will persist in the open string theory as well.

First, let us briefly review the general construction of D-branes probing orbifolds. Throughout we will mostly keep the discussion at the level of low-every effective field theory on the world volume. The theory of D 3-branes on $\mathbb{C}^3/\Gamma$ is defined as a truncation of the theory on $\mathbb{C}^3$, where only Γ invariant configurations are kept in the quotient.

What does Γ invariance mean? Forgetting for the moment the non-Abelian nature of the theory (or more properly, thinking about open string CFT with boundary conditions), we can associate a Chan-Paton factor i to a D-brane at $z(i) \in \mathbb{C}^3$, then the Γ action on $\mathbb{C}^3$ translates into

$$g \circ z(i) = z'(\gamma(g)i), \quad \forall g \in \Gamma.$$

[*7] In general $\mathbb{C}^3$ can be replaced with some other Ricci-flat three-dimensional manifold $\mathcal{M}$ admitting a symmetry Γ which is useful if the probe theory on $\mathcal{M}$ is known. Recently, this was done for the case when $\mathcal{M}$ is a conifold, [49].

[*8] It is true in fact that the knowledge of the $\mathbb{C}^3/\Gamma$ theory allows one to describe all other singualrities which are toric[50]. The unsolved problem is what to do for Calabi-Yau hypersurface singularities.

A Γ invariant configuration of D branes must then consist of orbits of Γ action on $\mathbb{C}^3$, a generic orbit in this case consisting of $|\Gamma|$ points on $\mathbb{C}^3$.

Thus, to describe N D 3-branes on $\mathbb{C}^3/\Gamma$ we must start with a $d = 4$, $\mathcal{N} = 4$ supersymmetric $U(N|\Gamma|)$ gauge theory, as an effective open string theory of $N|\Gamma|$ branes on the covering space. The action of Γ on the open string CFT induces the action on the effective bosonic degrees of freedom of the form

$$g \; : \; A \rightarrow \gamma(g) A \gamma(g)^{-1}, \tag{4.4}$$

$$Z^i \rightarrow \gamma(g) \left(g \circ Z^i \right) \gamma(g)^{-1}, \tag{4.5}$$

where Z^i are the scalar fields whose diagonal pieces parametrize the position of the D-branes on the covering space, and $\gamma(\Gamma)$ is an embedding of the orbifold group

$$\gamma : \Gamma \rightarrow U(N|\Gamma|).$$

At the end of the day, if Γ is a subgroup of $SU(3)$*9 as in the case we are interested in, the quotient theory will have $\mathcal{N} = 1$ supersymmetry in $d = 4$.

Let's now take $\Gamma = \mathbb{Z}_m \times \mathbb{Z}_n$, so that Γ is generated by two elements g, and h, satisfying $g^m = 1$, $h^n = 1$, $gh = hg$. $\mathbb{C}^3/\Gamma$ is then a quotient of $\mathbb{C}^3$ by

$$g: \quad (z^1, z^2, z^3) \rightarrow (e^{\frac{2\pi i}{m}} z^1, z^2, e^{-\frac{2\pi i}{m}} z^3)$$

$$h: \quad (z^1, z^2, z^3) \rightarrow (z^1, e^{\frac{2\pi i}{n}} z^2, e^{-\frac{2\pi i}{n}} z^3).$$

Orbits of Γ are generated by g, h, so a D-brane at a generic point in $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$ must have mn preimages on $\mathbb{C}^3$. It is convenient to label D-branes with a biindex (i, α) where i, α naturally label points on the orbits generated by g, h respectively, so that $i \in \{0, \ldots, m-1\}$ and $\alpha \in \{0 \ldots, n-1\}$ *10.

*9 Γ is the holonomy group of the orbifold. If $\Gamma \in SU(3)$ upon resolution of singularities in the orbifold the holonomy becomes (at most) $SU(3)$.

*10 There are also N-valued indices labeling distinct physical branes, but since we will consider here only the generic orbits of Γ, whatever goes through for a single brane holds for any number of them. We will thus set $N = 1$ for clarity of the text.

The orbifold fixes the action of Γ on the single open string states, which is inherited from Γ action on $\mathbb{C}^3$, but we must still pick an action on the Chan-Paton factors. There is a natural "geometric" choice of action which can be described as follows. Since a generic orbit of Γ is just a copy of Γ[*11], action of Γ on its orbit is an action of Γ on itself which gives, by definition, the regular representation, $\gamma(g)_{i\alpha,j\beta} = g^i\delta_{i,j}\delta_{\alpha,\beta}$, and $\gamma(h)_{i\alpha,j\beta} = \delta_{i,j}h^\alpha\delta_{\alpha,\beta}$. With this definition of the Γ projection, N D branes on the orbifold are described by a quiver $\prod_{i,\alpha} U(N)_{i,\alpha}$ gauge theory with chiral matter in bifundamental representations, and a superpotential which is a reduction of the $\mathcal{N}=4$ superpotential $\mathcal{W} = TrZ^1[Z^2, Z^3]$.

4.7 Open String Orbifolds with Torsion

What we described above is not the only way to define the theory. As was argued in [45] to obtain a consistent theory it is necessary that the group relations are satisfied up to phase factors only, which means that $\gamma(\Gamma)$ need not be a representation of Γ. However, since the action of $\gamma(\Gamma)$ on the fields is in the adjoint, it will nevertheless be well defined. This requirement in our case states that

$$\begin{aligned} \gamma(g)^m &\propto id, \\ \gamma(h)^n &\propto id, \\ \gamma(g)\gamma(h) &= \epsilon(g;h)\gamma(h)\gamma(g), \end{aligned}$$

must hold in order to satisfy the group relations.

The overall phases can always be removed by rescaling of the γ's, but the phase ϵ is interesting. It follows from the first two relations above that $\epsilon(g;h)^m = 1 = \epsilon(g;h)^n$, and if we denote by r the greatest common divisor of m and n, then ϵ must be an r-th root of unity. So if we let $\zeta = \exp(2\pi i/r)$, then

$$\epsilon(g;h) = \zeta^p, \quad p = 0, \ldots, r-1.$$

[*11] There are also smaller orbits, the origin for example. This is not a generic orbit.

This is the same cocycle appearing in the closed string theory.

The choice of p determines a "representation" of Γ but with

$$\gamma_p(g)\gamma_p(h) = \zeta^p \gamma_p(h)\gamma_p(g) \tag{4.6}$$

The "conventional orbifold" corresponds to $p = 0$, but, as was the case in the closed string sector, there are $r-1$ other choices we can make, and they will result in different theories on $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$.

It is now simple to find $\gamma_p(g), \gamma_p(h)$ that will provide us with representation of the algebra (4.6). To describe the action of Γ on a generic orbit, i.e. forbidding any additional constraints other than what is implied by group relations, one should ask that forgetting about the $\mathbb{Z}_n$ action, $\gamma_p(g)$ is a copy of the regular representation of $\mathbb{Z}_m$ and that the analogous statement holds for $\gamma_p(h)$. With this requirement, the solution is unique. We can always pick a basis in which, for example, $\gamma(g)$ is diagonal, so let's take $\gamma_p(g)_{i\alpha,j\beta} = g^i \delta_{i,j}\delta_{\alpha,\beta}$, this is just the regular representation of $\mathbb{Z}_m$. Then the action of h will include a twist action on g representations, $\gamma_p(h)_{i\alpha,j\beta} = \delta_{i,j+sp} h^\alpha \delta_{\alpha,\beta}$.

It is now easy to compute the states surviving the projection. Let's work out the gauge group on the quotient first. As the gauge fields live on space transverse to the orbifold, an element $g^A h^B \in \Gamma$ acts as
$A_\mu \to \gamma_p(g^A h^B) A_\mu \gamma_p(g^A h^B)^{-1}$, i.e.,

$$g^A h^B : (A_\mu)_{i\alpha,j\beta} \to g^{A(i-j)}\ h^{B(\alpha-\beta)}\ (A_\mu)_{(i-spB)\alpha,(j-spB)\beta}. \tag{4.7}$$

First note that since $\zeta^{r/q} = 1$, $\tilde{g} = g^{\frac{r}{q}}$ and $\tilde{h} = h^{\frac{r}{q}}$ generate an *ordinary* algebra corresponding to $\tilde{\Gamma} = \mathbb{Z}_{qs} \times \mathbb{Z}_{qt}$. To take this into account, it is convenient to set

$$i = I + \tilde{i}qs, \quad \alpha = A + \tilde{\alpha}qt$$

$$0 \leq I \leq sq-1, \quad 0 \leq A \leq tq-1, \quad 0 \leq \tilde{i},\ \tilde{\alpha} \leq \frac{r}{q} - 1,$$

in terms of which the invariant fields of the $\tilde{\Gamma}$ subgroup of Γ are of the form $A^{I,A}\tilde{i}\tilde{\alpha};\tilde{j},\tilde{\beta}$.

For the ease of notation we have written, for example $A_{I;I} \equiv A^I$. The projection by g requires only fields with $\tilde{i} = \tilde{j}$ to be kept in the quotient. Finally, projection by h sets

$$A^{I,A,\tilde{i}}_{\tilde{\alpha};\tilde{\beta}} = \zeta^{\tilde{\alpha}-\tilde{\beta}} A^{I,A,\tilde{i}-\frac{p}{q}}_{\tilde{\alpha};\tilde{\beta}}. \tag{4.8}$$

It may be helpful to collect the results obtained thus far: Given a $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$ orbifold with $r = gcd(m,n)$, there are r sectors of discrete torsion labeled by p, $0 \leq p \leq r-1$. If we denote the greatest common divisor of p and r by q, the gauge group on the world-volume of a single D 3-brane on the orbifold (to be precise the gauge group when the D brane is sitting at the point of maximal symmetry, the origin), is

$$\prod_{I=0}^{sq-1} \prod_{A=0}^{tq-1} U(\frac{r}{q})_{I,A} \ , \tag{4.9}$$

where $m = rs, n = rt$. It is just as straightforward to work out what happens to scalars in the $\mathcal{N} = 4$ vector multiplet. Since Γ acts on $\mathbb{C}^3$ by (with an obvious abuse of notation)

$$g^A h^B (z^1, z^2, z^3) = (h^B z^1, g^A z^2, g^{-A} h^{-B} z^3),$$

the Higgs fields surviving the quotient must satisfy

$$Z^i = \gamma(g^A h^B)(g^a h^b \circ Z)^i \gamma(g^a h^b)^{-1}.$$

Consider Z^1 first, Projecting with $\tilde{\Gamma} \subset \Gamma$, $(Z^1)^I_{A,\tilde{i}\tilde{\alpha};A+1,\tilde{j}\tilde{\alpha}}$ are kept with the understanding that the index A is now $\mathbb{Z}_{qt}$-valued. Finally, g sets $\tilde{i} = \tilde{j}$, and projection by h requires:

$$(Z^1)^{I,\tilde{i}}_{A,\tilde{\alpha};A+1,\tilde{\beta}} = \zeta^{\tilde{\alpha}-\tilde{\beta}+\eta(A)} (Z^1)^{I,\tilde{i}-\frac{p}{q}}_{A,\tilde{\alpha};A+1,\tilde{\beta}}, \tag{4.10}$$

where $\eta(A) = \delta_{A,qt-1}$. For Z^2 we find

$$(Z^2)^A_{I,\tilde{i},\tilde{\alpha};I+1,\tilde{i}+\epsilon,\tilde{\beta}} = \zeta^{\tilde{\alpha}-\tilde{\beta}}(Z^2)^A_{I,\tilde{i}-\frac{p}{q},\tilde{\alpha};I+1,\tilde{i}+\epsilon(I)-\frac{p}{q},\tilde{\beta}}, \tag{4.11}$$

$\epsilon(I) = \delta_{I,qs-1}$, and for Z^3:

$$(Z^3)_{I+1,A+1,\tilde{i}+\epsilon,\tilde{\alpha};I,A,\tilde{i},\tilde{\beta}} = \zeta^{\tilde{\alpha}-\tilde{\beta}-\eta(A)}(Z^3)_{I+1,A+1,\tilde{i}+\epsilon(I)-\frac{p}{q},\tilde{\alpha};I,A,\tilde{i}-\frac{p}{q},\tilde{\beta}} \tag{4.12}$$

with η, ϵ as defined above.

The fields coming Z^i's live in fundamental representation of the gauge group whose index they carry. For example, $(Z^1)^{I,\tilde{i}}_{A,\tilde{\alpha};A+1,\tilde{\beta}}$ transforms in $(\square,\bar{\square})$ under $U(\frac{r}{q})_{I,A,\tilde{i}} \times U(\frac{r}{q})_{I,A+1,\tilde{i}}$ however, all the gauge groups for fixed I, A are identified in the quotient, as described above, up to a gauge transformation. We can solve the constraint equations (4.8),(4.10), (4.11),(4.12) by putting:

$$\begin{aligned}
&(A_\mu)^{I,A,\tilde{i}}_{\tilde{\alpha};\tilde{\beta}} \equiv \zeta^{(\tilde{\alpha}-\tilde{\beta})\tilde{i}v}(A_\mu)^{I,A}_{\tilde{\alpha};\tilde{\beta}},\\
&(Z^1)^{I,\tilde{i}}_{A,\tilde{\alpha};A+1,\tilde{\beta}} \equiv \zeta^{(\tilde{\alpha}-\tilde{\beta}+\eta(A))\tilde{i}v}(Z^1)^{I,A}_{\tilde{\alpha};\tilde{\beta}},\\
&(Z^2)^A_{I,\tilde{i},\tilde{\alpha};I+1,\tilde{i}+\epsilon(I),\tilde{\beta}} \equiv \zeta^{(\alpha-\beta)\tilde{i}v}(Z^2)^{I,A}_{\alpha;\beta},\\
&(Z^3)_{I+1,A+1,\tilde{i}+\epsilon(I),\tilde{\alpha};I,A,\tilde{i},\tilde{\beta}} \equiv \zeta^{(\tilde{\alpha}-\tilde{\beta}-\eta(A))\tilde{i}v}(Z^3)^{I,A}_{\tilde{\alpha};\tilde{\beta}},
\end{aligned}$$

where v is a number defined by $q = ur + vp$, which of course, is not unique but is defined only up to $\frac{r}{q}$, as it should be.

The theory contains a superpotential which is a reduction

$$\mathcal{W} \to \mathcal{W}_\Gamma,$$

of the $\mathcal{N} = 4$ superpotential $\mathcal{W} = Tr(Z^1[Z^2, Z^3])$ to the Γ-invariant fields. We can compute $\mathcal{W}_\Gamma$ in terms of the reduced fields to find

$$\mathcal{W}_\Gamma = \sum_{I,A,\tilde{i}} \sum_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}} (Z^1)^{I,A}_{\tilde{\alpha};\tilde{\beta}}[(Z^2)^{I,A+1}_{\tilde{\beta};\tilde{\gamma}}(Z^3)^{I.A}_{\tilde{\gamma};\tilde{\alpha}} - \zeta^{v(\tilde{\alpha}-\tilde{\beta}+\eta)\epsilon}(Z^3)^{I-1,A}_{\tilde{\beta};\tilde{\gamma}}(Z^2)^{I-1,A}_{\tilde{\gamma};\tilde{\alpha}}].$$

It seems natural at this point to redefine

$$(Z^2)^{I,A}_{\tilde{\alpha};\tilde{\beta}} \to (\tilde{Z}^2)^{I,A}_{\tilde{\alpha};\tilde{\beta}} = (Z^2)^{I,A}_{\tilde{\alpha};\tilde{\beta}}\zeta^{-v\tilde{\beta}\epsilon(I)},$$

and

$$(Z^3)^{I,A}_{\tilde{\alpha};\tilde{\beta}} \to (\tilde{Z}^3)^{I,A}_{\tilde{\alpha};\tilde{\beta}} = (Z^3)^{I,A}_{\tilde{\alpha};\tilde{\beta}}\zeta^{v\tilde{\alpha}\epsilon(I)}.$$

After the redefinition, the superpotential is

$$\mathcal{W}_\Gamma = \frac{r}{q}\sum_{I,A} Tr\{(Z^1)^{I,A}[(\tilde{Z}^2)^{I,A+1}(\tilde{Z}^3)^{I,A} - \zeta^{v\eta(A)\epsilon(I-1)}(\tilde{Z}^3)^{I-1,A}(\tilde{Z}^2)^{I-1,A}]\},$$

where the factor of $\frac{r}{q}$ out front comes from the sum over $\tilde{i}$. It is easy to show that the redefinitions not only simplify the superpotential,but are necessary in order for the matter fields to have canonical kinetic terms.

4.8 Moduli Spaces

We have computed above the open string theory description of orbifolds with discrete torsion. We have found that, just like in the closed string theory, there are ambiguities present in defining the theory on $\mathbb{Z}_m \times \mathbb{Z}_n$ orbifolds. More precisely, we have found that there are exactly $r = gcd(m,n)$ orbifold gauge theories one can define, differing by either the matter content or the choice of the superpotential. The basic physical requirement that all the theories must satisfy is that the vacuum moduli space be an $\mathbb{Z}_m \times \mathbb{Z}_n$ orbifold. Thus, if we have made no mistakes, all the above gauge theories, for any choice of torsion, should have exactly the same vacuum moduli space. We will now check that this is so.

The vacuum moduli space is the space of solution of F and D flatness conditions, modulo gauge transformations. F-flatness conditions are $\delta\mathcal{W}_\Gamma/\delta(Z^i)^{I,A}=0$ so:

$$\begin{aligned}
\frac{\delta\mathcal{W}_\Gamma}{\delta(Z^1)^{A,I}} &= (\tilde{Z}^2)^{I,A+1}(\tilde{Z}^3)^{I,A}-\zeta^{v\eta(A)\epsilon(I-1)}(\tilde{Z}^3)^{I-1,A}(\tilde{Z}^2)^{I-1,A},\\
\frac{\delta\mathcal{W}_\Gamma}{\delta(\tilde{Z}^2)^{A,I}} &= (\tilde{Z}^3)^{I,A-1}(Z^1)^{I,A-1}-\zeta^{v\eta(A)\epsilon(I)}(Z^1)^{I+1,A}(\tilde{Z}^3)^{I,A},\\
\frac{\delta\mathcal{W}_\Gamma}{\delta(\tilde{Z}^3)^{A,I}} &= (Z^1)^{I,A}(Z^2)^{I,A+1}-\zeta^{v\eta(A)\epsilon(I)}(\tilde{Z}^2)^{I,A}(Z^1)^{I+1,A}.
\end{aligned} \tag{4.13}$$

And vanishing of D-terms requires $D_\Gamma=\sum_i[Z^i,Z^{i\dagger}]_\Gamma=0$, that is:

$$\begin{aligned}
D_{I,A}=0 &= (Z^1)^{I,A}(Z^{1\dagger})^{I,A}-(Z^{1\dagger})^{I,A-1}(Z^1)^{I,A-1}+(Z^2)^{I,A}(Z^{2\dagger})^{I,A}\\
&- (Z^{2\dagger})^{I-1,A}(Z^2)^{I-1,A}+(Z^3)^{I-1,A-1}(Z^{3\dagger})^{I-1,A-1}-(Z^{3\dagger})^{I,A}(Z^3)^{I,A}.
\end{aligned}$$

We can solve the equations by putting

$$(Z^1)^{I,A}_{\tilde{\alpha}\tilde{\beta}}=z^1\zeta^{-\eta(I)v\tilde{\alpha}}\delta_{\tilde{\alpha},\tilde{\beta}};\quad (Z^2)^{I,A}_{\tilde{\alpha}\tilde{\beta}}=z^2\delta_{\tilde{\alpha},\tilde{\beta}-\epsilon};\quad (Z^3)^{I,A}_{\tilde{\alpha}\tilde{\beta}}=z^3\zeta^{\eta v(\tilde{\alpha}-\epsilon)}\delta_{\tilde{\alpha},\tilde{\beta}+\epsilon}.$$

Above, as before $\epsilon=\epsilon(I)=\delta_{I,qs-1}$, and $\eta=\eta(A)=\delta_{A,qt-1}$. There is unbroken diagonal $U(1)$ gauge symmetry, under which all the mater-fields are neutral. In addition, there can be unbroken discrete gauge symmetries. What this means is that there can be identifications on the space of solutions to (4.13),(4.14) which are induced by the gauge redundancies. In this case, the group of discrete gauge transformations which respect the equations (4.13,4.14) is generated by:

$$(\gamma_g)^{I,A}_{\tilde{\alpha},\tilde{\beta}}=g^{-I}\zeta^{-\tilde{\alpha}}\delta_{\alpha,\beta+1},\quad (\gamma_h)^{I,A}_{\tilde{\alpha},\tilde{\beta}}=h^{-A}\delta_{\tilde{\alpha},\tilde{\beta}+1},$$

which acts on the Z^i's precisely as the orbifold group does, for example

$$(\gamma_h)^{I,A}(Z^1)^{I,A}(\gamma_h^{-1})^{I,A+1}=h(Z^1)^{I,A},$$

and similarly for the others. Since the D- and F- flatness conditions are invariant under this action, so are their solutions. The moduli space is thus precisely the orbifold

$\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$, as anticipated. At the generic, smooth point in the moduli space, the effective gauge theory is a $U(1)$ gauge theory with $\mathcal{N} = 4$ supersymmetry. The singularities in the classical moduli spaces correspond to points of partial gauge symmetry restoration, the most singular point being the origin where the full symmetry (4.9) is recovered.

4.8.1 Resolution of Singularities in the Moduli Spaces

The singularities of manifolds can be resolved by deformations of the complex structure or blow-ups, as we have seen when we discussed the closed string theory description of the $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$ orbifold. The open string counterpart of this is the possibility of smoothing the singularities in the moduli spaces via deformations of the superpotential, or by turning on FI parameters. The deformations of the superpotential have the effect of changing the complex structure of the moduli space. The reason is that the space of solutions to F- and D-flatness conditions modulo gauge transformations is precisely equivalent, as shown in [51], to setting F terms to zero and dividing by the complexified gauge group. This in turn produces the description of the moduli space as a variety parameterized by gauge invariant polynomials modulo relations with additional constraints from the vanishing of the F-terms. Deforming the F terms will then modify the relations between gauge invariant monomials. The D-flatness conditions, on the other hand represent the moduli space in terms of a symplectic quotient, and this is directly related to the blowing up procedure, as we will see shortly. There are two separate cases to consider, one in which only deformations of the complex structure of the moduli space are allowed, and the other when singularities can only be blown up.

4.8.1.a. **Case $qst = 1$.**

When $qs = qt = 1$, there are deformations of the theory via $\mathcal{W} \to \mathcal{W} + \Delta\mathcal{W}$ which preserve supersymmetry and resolve singularities of the moduli space. The unique gauge invariant deformation of the superpotential is given by

$$\Delta\mathcal{W} = \sum \xi_i Tr Z^i.$$

This deformation preserves the $U(1)^3$ R -symmetry under which the superpotential has charge one, provided that ξ's are assigned appropriate charges. As shown in [52], the moduli space is a deformation of the $\mathbb{Z}_r \times \mathbb{Z}_r$ orbifold, (for $qs = qt = 1$, this is the same as $m = n = r$),

$$y^r = x_1 x_2 x_3,$$

to

$$y^r + \sum_{i=0}^{r-2} a_i y^i = x_1 x_2 x_3 - b_1 x_1 - b_2 x_2 - b_3 x_3 + 2b_0,$$

where $y = TrZ^1 Z^2 Z^3$, $x_i = Tr(Z^i)^r$. Above, $b_0 = 2(b_1 b_2 b_3)^{1/2}$, and all the coefficients are suitable functions of ξ_1, ξ_2, ξ_3, computed in [52]. It is easy to show that the space has $n-1$ conifold singularities located at $x_i = \sqrt{\frac{b_j b_k}{b_i}}$, and $r-1$ roots of $P_r(y) = y^r + \sum_{i=0}^{r-2} a_i y^i = 0$, and $\partial_y P_r = 0$ (for this to be so the polynomial P must clearly be very special, and one can in fact show that there are roots of the form $y = cos\frac{\pi k}{r}$, $\quad k = 1, \ldots, r-1$, in agreement with closed string theory.

4.8.1.*b*. **Case** $qst \neq 1$.

When the number of gauge groups, $qs \times qt \neq 1$, there is a possibility of turning on Fayet-Iliopoulos parameters. We can deform the D-flatness conditions (4.14) for the (I, A)-th gauge group via:

$$D_{I,A} = \xi_{I,A}, \tag{4.14}$$

$$0 \leq I \leq qs - 1, \quad 0 \leq A \leq qt - 1.$$

This will, as we are about to show, resolve the singularities of the moduli space $\mathbb{C}^3/\mathbb{Z}_{rs} \times \mathbb{Z}_{rt}$ albeit only partially when torsion, $p \neq 0$, is turned on. To solve the vacuum constraints eq.(4.13),(4.14), it is instructive to note that upon projection

$U(r/q)_{I,A}$ to the center $U(1)_{I,A}$ for all I, A, the F and D flatness conditions reduce to that appropriate for an ordinary $\tilde{\Gamma} = \mathbb{Z}_{qs} \times \mathbb{Z}_{qt}$ orbifold. We can take advantage of this by putting:

$$(Z^1)^{I,A}_{\tilde{\alpha}\tilde{\beta}} = (z^1)^{I,A}\zeta^{-\eta(A)v\tilde{\alpha}}\delta_{\tilde{\alpha},\tilde{\beta}}, \quad (4.15)$$
$$(Z^2)^{I,A}_{\tilde{\alpha}\tilde{\beta}} = (z^2)^{I,A}\delta_{\tilde{\alpha},\tilde{\beta}-\epsilon(I)}, \quad (4.16)$$
$$(Z^3)^{I,A}_{\tilde{\alpha}\tilde{\beta}} = (z^3)^{I,A}\zeta^{\eta(A)v\tilde{\beta}}\delta_{\tilde{\alpha},\tilde{\beta}+\epsilon(I)}. \quad (4.17)$$

Using this ansatz, when eq.(4.13),(4.14) are written in terms of the $\mathbb{C}$-valued variables $(z^i)^{I,A}$, they reduce to the D and F flatness conditions appropriate for the resolution of $\mathbb{C}^3/\tilde{\Gamma}$ orbifold, where the blowup parameters of the orbifold are related to the $\xi_{I,A}$'s. This of course cannot be all, since with or without torsion we have set out to study $\Gamma = \mathbb{Z}_{rs} \times \mathbb{Z}_{rt}$ orbifold.

There is an additional quotient we must take into account, and it derives from the fact that the identifications we have made in eq.(4.15) are not unique, but there is an additional action of the unbroken discrete subgroup of the diagonal $U(r/q)$ on the space of solutions to eq.(4.13),(4.14) which is equivalent to taking a quotient by $H = \mathbb{Z}_{r/q} \times \mathbb{Z}_{r/q}$. The discrete gauge symmetry group is generated by $\gamma(\tilde{g}) = \delta_{\tilde{\alpha},\tilde{\beta}+1}$, and $\gamma(\tilde{h}) = \zeta^{-\tilde{\alpha}}\delta_{\tilde{\alpha},\tilde{\beta}}$, and since gauge symmetries reflects redundancy of description,$\gamma(\tilde{g})$ and $\gamma(\tilde{h})$ act via identifications on the fields:

$$\tilde{g} : ((z^1)^{I,A}, (z^2)^{I,A}, (z^3)^{I,A}) \sim ((z^1)^{I,A}, \zeta^{\eta(A)}(z^2)^{I,A}, \zeta^{-\eta(A)}(z^3)^{I,A}),$$
$$\tilde{h} : ((z^1)^{I,A}, (z^2)^{I,A}, (z^3)^{I,A}) \sim (\zeta^{\epsilon(I)}(z^1)^{I,A}, (z^2)^{I,A}, \zeta^{-\epsilon(I)}(z^3)^{I,A}). \quad (4.18)$$

The moduli space is thus a quotient $\tilde{\mathcal{M}}/H$, with $\tilde{\mathcal{M}}$ a generic blowup of $\mathbb{C}^3/\tilde{\Gamma}$ by the group $H = \mathbb{Z}_{r/q} \times \mathbb{Z}_{r/q}$.

Moduli spaces for $qs \times qt > 1$ as toric varieties

There is an explicit and efficient way to describe the moduli space with arbitrary FI parameters and it is as follows*[12].

Let us first aim for a description of $\tilde{\mathcal{M}}$, which is the solution of the "reduced" F- and D- flatness conditions of the blowup of $\mathbb{C}^3/\tilde{\Gamma}$. The F-flatness conditions are given by:

$$\begin{aligned} 0 &= (z^2)^{I,A+1}(z^3)^{I,A} - (z^3)^{I-1,A}(z^2)^{I-1,A}, \\ 0 &= (z^3)^{I,A-1}(z^1)^{I,A-1} - (z^1)^{I+1,A}(z^3)^{I,A}, \\ 0 &= (z^1)^{I,A}(z^2)^{I,A+1} - (z^2)^{I,A}(z^1)^{I+1,A}. \end{aligned} \tag{4.19}$$

Not all of the equations (4.19) are independent, but one can show that they can be boiled down to $2|\tilde{\Gamma}|-2$ equations of the type $monomial = monomial$ in $3|\tilde{\Gamma}|$ variables $(z^i)^{I,A}$, where $|\tilde{\Gamma}| = qs \cdot qt$.

The space of solutions to (4.19) form what is called an affine toric variety. In algebraic geometry it is well known that all the information about a variety $\mathcal{M}$ is encoded in the space of functions which are well defined on $\mathcal{M}$, that is all rational functions without poles on $\mathcal{M}$. These functions form a ring, $R(\mathcal{M})$, and a way to describe $R(\mathcal{M})$ is as follows. Since $\mathcal{M}$ is a variety, by definition, there exists a set of polynomials F_i in $\mathbb{C}^n$, such that $\mathcal{M}$ is given by equations of the form $F_i(z_1, \ldots, z_n) = 0$. Then any polynomial $g \in R(\mathbb{C}^n)$ restricts to a polynomial on $\mathcal{M}$, by regarding g as a function of points in $\mathcal{M}$, and this map is a ring homomorphism. The kernel of this homomorphism consists of polynomials that vanish over all points in $\mathcal{M}$, so $R(\mathcal{M}) = R(\mathbb{C}^n)/I_{\mathcal{M}}$. The ideal $I_{\mathcal{M}}$ is generated by F_i. In our case, the ideal $I_{\mathcal{M}}$ is generated by relations between monomials in $\mathbb{C}^n$, with $n = 3|\tilde{\Gamma}|$, and it is in this special case when the variety is given in terms of monomial relations that one obtains an "affine toric variety." The virtue of affine toric variety, is that one has the ability to take the quotient directly. The efficient way to describe the quotient is as follows.

*[12] This section is fairly technical and serves to support in detail the very intuitive picture above. There are no new physical results, so the reader uninterested in detail may just skip this section.

Let us assign a vector $\vec{m^i}$ to each variable z_{m^i}, $i = 1, \dots n$, where the vector space structure is inherited from $z^\alpha_m z^\beta_{m'}$ mapping to $\alpha\vec{m} + \beta\vec{m}'$. The equations (4.19) can then be written as $2|\tilde{\Gamma}| - 2$ linear relations among $3|\tilde{\Gamma}|$ vectors. Explicitly, we can pick a basis for the space of solutions consisting of $|\tilde{\Gamma}| + 2$ vectors (monomials) and we will denote them as $\vec{e^\alpha}$, $\alpha = 1, \dots, |\tilde{\Gamma}| + 2$. All $\vec{m}$'s can be expanded in terms of the basis vectors as $\vec{m}^i = \sum_\alpha m^i_\alpha \vec{e^\alpha}$, with coefficients m_α determined via the relations (4.13). The ring of polynomials well defined on the space of solutions to F-flatness conditions is generated by monomials $(\{\prod_\alpha z_\alpha^{m^i_\alpha}\})$ where $i = 1, \dots, 3|\tilde{\Gamma}|$, and α runs over the basis vectors, $\alpha = 1, \dots, |\tilde{\Gamma}| + 2$. The toric geometry encodes the ring structure in the following manner. Let $\mathbb{M} \sim \mathbb{Z}^{n+2}$ be a lattice generated by $\{\vec{e^\alpha}\}$. The collection of $n = 3|\tilde{\Gamma}|$ vectors $\vec{m}^i$ belong to $\mathbb{M}$, and define a cone $M_+ \in \mathbb{M}$ consisting of all the vectors which can be written as linear combinations of the vectors $\vec{m^i}$ with non-negative integral coefficients. The result that we need here is that all the coefficients m^i_α are in fact integral*[13]. The utility of the lattice construction lies in part in the fact that there is a one-to-one correspondence between monomials in $R(\mathcal{M})$ and vectors in M_+, with the algebra of the functions on $\mathcal{M}$ encoded in the geometry of M_+.

To obtain the vacuum moduli space we must, in addition to F flatnes constraints eq.(4.19), satisfy D flatness conditions, eq.(4.14), modulo gauge equivalence. It is well known that, instead of setting F and D-terms to zero and dividing by the gauge group, we can set F terms to zero and divide by the complexified gauge group. Dividing by the complexified gauge group is accomplished by working with invariant monomials and in terms of the toric construction this amounts to picking out a sub-fan $\tilde{M}_+ \subset M_+$.of monomials defined on the space of solutions to F flatness conditions that are invariant under the complexified gauge transformations. Following the original argument, generators of $\tilde{M}_+$ form a basis of the algebra of functions of the underlying moduli space, and from it we can reconstruct the variety $\tilde{\mathcal{M}}$ itself: simply, the relations among the generators of $\tilde{M}_+$ determine the ideal $I(\tilde{\mathcal{M}})$.

The shortcoming of the above approach is that only the complex structure of $\tilde{\mathcal{M}}$ is

*[13] The coefficients can in fact take values of $0, \pm 1$ only. To see this, consider any given point on the space of solutions. Under rescaling $(z^i)^{I,A} \to t(z^i)^{I,A}$, of one of the coordinates of the point, the others must be rescaled by one of the $t^0, t^{\pm 1}$ in order to still solve the F-flatness conditions.

explicitly determined, and we loose any reference to the Fayet-Illiopolous parameters. The affine toric varieties have the property that the complex structure of the variety determines the space of possible Kähler structures on it. The way to describe the Kähler structure of the affine toric variety is to write it as a symplectic quotient, and this is what we will do next.
Consider the dual lattice $\mathbb{N}$ of $\mathbb{M}$, in the sense that $\mathbb{N} = Hom(\mathbb{M}, \mathbb{Z})$. The collection of vectors $\vec{v} \in \mathbb{N}$ such that

$$< \vec{m}, \vec{n} > \geq 0, \forall \vec{m} \in \mathbb{M},$$

form a dual cone N_+ of M_+, and let its generators be denoted $\vec{v}^i$, $i = 1, \ldots q$. Since the dual lattice is n dimensional, there will be $q - n$ relations between the $\vec{v}_i$'s,

$$\sum_i Q_i^{(a)} \vec{v}^i = 0, \quad a = 1, \ldots, q - n.$$

The toric variety is described as a solution to $q - n$ equations:

$$\sum_i Q_i^{(a)} |x_i|^2 = \eta_a, \tag{4.20}$$

modulo $q - n$ $U(1)$ actions $x_i \to e^{iQ_i^a \theta_a} x_i$, $\theta_a \in \mathbb{R}$. It is both striking and natural that the equations take the form of D-flatness conditions of an Abelian gauge theory. It requires a little bit of thought to see that the cone M_+ we constructed above contains gauge invariant monomials (more precisely, the monomials that are invariant under the *complexified* gauge group) of this auxiliary gauge theory: any monomial of the form $\prod_i x_i^{<\vec{v}_i, \vec{m}>}$, with $\vec{m} \in M_+$ is invariant under the complexified gauge group, and the association $z_\alpha \leftrightarrow \vec{e}^\alpha$ induces a "change of basis" to the old variables:

$$z_\alpha = \prod_i x_i^{<\vec{v}^i, \vec{e}^\alpha>}.$$

The above gives an equivalent description of the variety built from M_+ but we still need to take into account the physical $D-$flatness conditions (4.14). This can be accomplished as follows: The z_α's carry charges under $|\tilde{\Gamma}|-1$ physical $U(1)$'s, where we have factored out the center of mass $U(1)$, so that $z_\alpha \sim e^{iq_\alpha^A\theta_A} z_\alpha$. If we let $\vec{q}^A$ have an expansion $\vec{q}^A = \sum_i q_i^A \vec{v}^i$, this induces an action $x_i \sim e^{iq_i^A\theta_A} x_i$, and therefore an additional set of D-flatness conditions on the x's:

$$\sum_i q_i^{(A)} |x_i|^2 = \xi_A, \tag{4.21}$$

and this of course induces additional relations $\sum_i q_i^{(A)} \vec{v}^i = 0, \quad A = 1, \ldots, k-1$. We should really note that only ξ_A are physical parameters determined by FI terms, while η_i do not arise from the parameters in the Lagrangian, and therefore the only natural values they can take are zero. The vacuum moduli space is thus given as a set of solutions to eq.(4.20,4.21), with all $\eta_i = 0$, modulo the enlarged gauge equivalence as defined above.

Let us now get back to the problem at hand. We considered a $\Gamma = \mathbb{Z}_{rs} \times \mathbb{Z}_{rt}$ orbifold with p units of torsion such that $qs \vee qt \neq 1$, with $q = gcd(p, r)$. We found that the D-brane probe theory admits a "projection" to an ordinary $\tilde{\Gamma} = \mathbb{Z}_{qs} \times \mathbb{Z}_{qt}$ orbifold without torsion and its resolution, but with additional identifications under $H = \mathbb{Z}_{r/q} \times \mathbb{Z}_{r/q}$. The construction of the resolution of singularities in the ordinary orbifold theory was reviewed above, and the question is what does the additional H action mean geometrically. Since the H action is non-trivial only on $z^i_{I,A}, I = qs-1, A = qt-1, \ i = 1,2,3$, it is natural to pick these to correspond to three of the basis elements $\vec{e}^\alpha$ of $\mathbb{M}$. The quotient picks out a sub-lattice $\mathbb{M}_H$ of H-invariant monomials in $\mathbb{M}$, and thereby the dual lattice $\mathbb{N}_H$ of $\mathbb{M}_H$, which always satisfies $\mathbb{N} \subset \mathbb{N}_H$. What this means is the following. It is the usual practice in toric geometry to use fans in $\mathbb{N}$ lattices as visualization aids. Replacing a lattice $\mathbb{N}$ by a finer lattice in which it is contained, the fan N_+ in $\mathbb{N}$ maps to a fan in $\mathbb{N}_H$, and viewed in $\mathbb{N}_H$, N_+ has some points "missing." In the spirit of orbifolding it is clear that this should not only signal singularities, but also the possibility of *resolution* of singularities by adding the missing points back in. This clearly adds more vectors $\vec{v}$ to ${N_+}^H$, and thereby more equations of the type (4.21), and more FI parameters.

However, it can happen that physically there is no possibility of altering anything – this really amounts to frozen orbifold singularities, and this is exactly what happens here. To see how this works, let us take the simplest non-trivial example – a $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold with $p = 1$ units of torsion.

Example: $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold with $p = 1$.

From above, for $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$ and $p = 1$, we find $\tilde{\Gamma} = \mathbb{Z}_2$, and $H = \mathbb{Z}_2 \times \mathbb{Z}_2$. The gauge theory associated to an ordinary $\mathbb{C}^3/\mathbb{Z}_2$ orbifold is $U(1) \times U(1)$, with matter fields whose charges are: x_0, z_1 in $(+1, -1)$, x_1, z_0 in $(-1, +1)$, and two gauge singlets y_0, y_1[*14]. The theory has a superpotential $W = x_0 y_1 z_0 - x_0 z_0 y_0 + x_1 y_0 z_1 - x_1 z_1 y_1$, which reproduces the F- flatness conditions, application of (4.19) to $\tilde{\Gamma} = \mathbb{Z}_2$,

$$x_0 z_0 = x_1 z_1,$$

$$y_0 = y_1.$$

We can "solve" this via monomial vectors $m_i \in \mathbb{M}$

$$\begin{aligned} x_0 \sim \vec{m}^0 = (1, 0, -1, 1), \quad y_0 \sim \vec{m}^2 = (0, 1, 0, 0), \quad z_0 \sim \vec{m}^4 = (0, 0, 1, 0), \\ x_1 \sim \vec{m}^1 = (1, 0, 0, 0), \quad y_1 \sim \vec{m}^3 = (0, 1, 0, 0), \quad z_1 \sim \vec{m}^5 = (0, 0, 0, 1), \end{aligned}$$

from which the gauge invariant monimials correspond to a subfan $\tilde{M}_+$ of the fan M_+ spanned by $\vec{m}_i$, which is generated by:

$$\begin{aligned} x_0 x_1 \sim \tilde{\vec{m}}^0 = (2, 0, -1, 1), \qquad z_0 z_1 \sim \tilde{\vec{m}}^1 = (0, 0, 1, 1), \\ x_1 z_1 \sim \tilde{\vec{m}}^2 = (1, 0, 0, 1)\ , \qquad y_1 \sim \tilde{\vec{m}}^3 = (0, 1, 0, 0). \end{aligned}$$

[*14] To make contact with the notation above, put $m = 2, n = 4$, $x_0 = Z^1_{0,1}$, $x_1 = Z^1_{1,0}$, $y_0 = Z^2_{0,0}$, $y_1 = Z^2_{1,1}$, $z_0 = Z^3_{1,0}$, $z_1 = Z^3_{0,1}$.

Note that $\vec{m}^i$'s acctually span only three-dimensional space. We can now find the dual cone $\tilde{N}_+ \in \tilde{\mathbb{N}}$ to consist of:

$$\vec{n}_0 = (0,0,0,1), \qquad \vec{n}_1 = (0,0,-1,1), \quad \vec{n}_2 = (0,0,1,1),$$
$$\vec{n}_3 = (0,1,0,0), \qquad \vec{n}_4 = (\frac{1}{2},0,\frac{1}{2},-\frac{1}{2}).$$

The first four vectors generate the toric diagram of $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$, and the fourth plays no role*15. Now, H acts on this via (4.18):

$$h_1 : (x_0, x_1, y_0, y_1, z_0, z_1) \rightarrow (x_0, x_1, y_0, -y_1, z_0, -z_1),$$

$$h_2 : (x_0, x_1, y_0, y_1, z_0, z_1) \rightarrow (x_0, -x_1, y_0, y_1, z_0, -z_1),$$

which implies that the lattice $\mathbb{M}$ must be reduced to the H invariant lattice $\mathbb{M}_H$, by picking only those vectors $\vec{m} \in \mathbb{M}$ which have integer inner product with $\vec{h}_1 = (0,\frac{1}{2},0,\frac{1}{2})$ and $\vec{h}_2 = (\frac{1}{2},0,0,\frac{1}{2})$. This in terms of the $\tilde{\mathbb{N}}$ lattice this precisely means that we must add $\vec{h}_1, \vec{h}_2$ to the lattice, thereby obtaining $\mathbb{N}_H$. The additional vectors that exist in $\mathbb{N}_H$, but not in $\tilde{N}_+$ are are:

$$\vec{n}_5^H = (0,0,\frac{1}{2},1), \quad \vec{n}_6^H = (0,0,\frac{1}{2},1), \quad \vec{n}_7^H = (0,\frac{1}{2},0,\frac{1}{2}),$$
$$\vec{n}_8^H = (0,\frac{1}{2},\frac{1}{2},\frac{1}{2}), \quad \vec{n}_9^H = (0,\frac{1}{2},-\frac{1}{2},\frac{1}{2}).$$

The effect of this is twofold:

- The fact that they exist in $\mathbb{N}_H$ but not in $\tilde{N}_+$ states that the toric variety associated to $\tilde{N}_+ \subset \mathbb{N}_H$ is singular.
- The toric variety of $\tilde{N}_+$ is a quotient by H of the blowup of $\mathbb{C}^3/\tilde{\Gamma}$, and it differs from the blowup of $\mathbb{C}^3/\Gamma$ precisely by the absence of blowup modes coming from $\{\vec{n}^H\}$.

*15 More precisely, the fourth vector is an anomaly due to the fact we are describing a space which is really a hyper-Kähler quotient, $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ via a symplectic quotient, and will be absent in more complicated examples, with either $q > 1$, or $s, t > 1$. The sole purpose of $\vec{n}_4$ is to state that M_+ lives in three dimensional space orthogonal to $\vec{n}_4$.

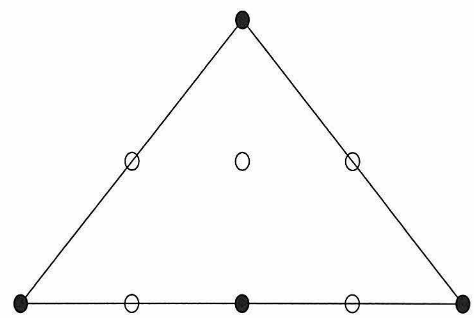

Fig.4.1. Toric diagram of the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold with one unit of torsion. The "empty" dots correspond to missing blowup modes.

4.9 Discussion

We have seen that orbifolds with torsion can be defined in both the open and closed string CFT, and furthermore in both theories precisely the same stable singularities appear. The fact that topologically distinct resolutions of orbifolds exist, corresponding to whether $qst = 1$, or not, is not in itself surprising. Recently it has been shown in classical geometry[53] that $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$ orbifolds do admit topologically distinct resolutions. In the $\mathbb{Z}_2 \times \mathbb{Z}_2$ example worked out there, one does find both the Kähler and complex structure deformations of the orbifold, much like we found for the two $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold CFTs. However, from the point of view of classical geometry, in the case of complex structure deformed orbifold no obstruction to deforming the singularity at the origin, and this is unlike what we found in string theory. We have shown above that, in a generic case, one has in fact whole curves of stable singularities, which are resolvable classicaly but not in the CFT. The reason for stable singularities in string theory is not known. This is connected to the fact that, for different reasons, we do not really understand the meaning of orbifolds in the theory. Even more puzzling is an aspect which we have not discussed thus far in the text. Namely, D brane probes of orbifolds carry information about the behavior of M theory on orbifolds – rather their smooth counterparts in M theory. The reason they have to be smooth is well understood – there are no B fields in M theory on Calabi-Yau manifolds. However, the of torsion, as defined here, is a thoroughly stringy effect. What is its meaning in M theory? The only certain thing is that singularities of the geometry cannot be singularities in M theory, simply because they are not singularities of type II string theory either.

Finally, [53] also shows that there are still other resolutions of orbifolds which we have not found in CFT. This is not so puzzling, since it is plausible that these spaces are resolutions of geometric orbifolds, but are not in any way connected to any string CFT orbifold. This is probably true, but it would be nice to understand precisely what the distinction is.

The work on this topic is in progress and we hope to report the results elsewhere.

Chapter 5 Mirror Symmetry, Brane Configurations and Branes at Singularities

5.1 Background: Quantum Mirror Symmetry

Mirror symmetry was originally proposed as a symmetry of the conformal field theory: there exist pairs of Calabi-Yau manifolds $(\mathcal{M}, \mathcal{W})$ such that the conformal field theory of $\mathcal{M}$ is identical to the conformal field theory of $\mathcal{W}$ [12, 54]. A natural question is whether mirror symmetry extends beyond CFT to a symmetry of the full string theory. On a different level, one would like to have an explicit way of identifying mirror pairs, as opposed to the implicit proposition given above. Recently, under the assumption that mirror symmetry holds non-perturbatively– that mirror pairs $(\mathcal{M}, \mathcal{W})$ satisfy the stronger property,"quantum mirror symmetry"– the authors of [5] provided an explicit geometric interpretation of the mirror map.

The argument goes as follows. If mirror symmetry is a quantum symmetry it must be respected by BPS states of the theory. In compactifying type II theory on a Calabi-Yau space $\mathcal{M}$ one has to include among other things D p-branes wrapping homology cycles in $\mathcal{M}$. There exist special cycles in Calabi-Yau manifolds, called supersymmetric cycles, which have the property that a D brane wrapping such a cycle preserves some fraction of supersymmetry – it is BPS. In the special case when the number of spatial dimensions of a D brane coincides with the dimension of a supersymmetric cycle the brane wraps, one obtains a particle in the four-dimensional theory. Supersymmetric cycles $[C] \in H_*(\mathcal{M})$ generally admit deformations to nearby supersymmetric cycles, so one is dealing with moduli spaces M_C of such objects. In addition, a D brane of type II theory carries a $U(1)$ gauge-field on its world volume, so M_C must be enlarged by moduli coming from the deformation of the $U(1)$ bundle on C. We will call the enlarged moduli space $\tilde{M}_C$. BPS states themselves are obtained by computing the ground states of the supersymmetric quantum mechanics on $\tilde{M}_C$.

The quantum mirror symmetry, as explained in [5], implies not only that the BPS spectra of type IIA theory on $\mathcal{M}$ and type IIB theory on the mirror $\mathcal{W}$, are equal, but that equality of the full theories including interactions requires the equality of the moduli spaces of these objects as well.

The application of this idea leads to the following [5, 55]. Type IIA theory contains D p-branes for p even, so in particular it contains D 0-branes. The brane has no spatial extent, so the moduli space of a D 0-brane on $\mathcal{M}$ is $\mathcal{M}$ itself. What is the mirror object in type IIB theory on $\mathcal{W}$? Since type IIB theory contains only p-odd branes, the only brane that can wrap anything in a Calabi-Yau manifold to give a particle is a D 3-brane. In order for it to be supersymmetric it must wrap a supersymmetric cycle C in $\mathcal{W}$ of real dimension three, and such cycles in a Calabi-Yau manifold are called *special lagrangian*. Thus, quantum mirror symmetry implies

$$\mathcal{M} = \tilde{M}_C.$$

There is a theorem in mathematics [55] which says that the dimension of the moduli space M_C of a special Lagrangian cycle C is $b_1(C)$. As the moduli space of flat bundles on C has dimension $b_1(C)$ as well, we find that the complex dimension of $\tilde{M}_C$ is $b_1(C)$, so that it must be $b_1(C) = 3$.

Now fixing a point C in M_C and varying the bundle one obtains a real torus $T^{b_1(C)} = T^3$, which itself turns out to be a supersymmetric cycle on $\tilde{M}_C = \mathcal{M}$. We learn therefore that $\mathcal{M}$ is a fibration by special Lagrangian tori T^3 over the base $B = M_C$,

$$\mathcal{M} \xrightarrow{\pi} B, \quad \pi^{-1}(x) = T^3.$$

Exchanging the roles of $\mathcal{M}$ and $\mathcal{W}$, that is wrapping D 3-branes on fibers of $\mathcal{M}$ one concludes that $\mathcal{W}$ is a T^3 fibration by special Lagrangian cycles as well, but that fibers of $\mathcal{W}$ are the T-dual tori $\hat{T}^3$ *1. Finally the bases of the two fibrations must be

*1 The moduli space of flat connections on a torus is a dual torus which has the inverse metric of the original one. The operation T-duality.

equal as well, since the moduli space of all fibers of a fibration is just the base of the fibration. Reversing the roles of $\mathcal{M}$ and $\mathcal{W}$, it follows that

$$\mathcal{W} \xrightarrow{\hat{\pi}} B, \quad \hat{\pi}^{-1}(x) = \hat{T}^3.$$

Note that we must allow for the possibility of fibers degenerating in some locus in the base space*2, and this is what we turn next to.

In the rest of the Chapter we discuss T-duality and mirror symmetry for type II string theory on a particular class of Calabi-Yau spaces following [6]. In our point of view, the geometry will mainly serve as a background, and we will study the gauge theories living on D-branes that probe or wrap cycles in the manifolds, along the lines of the general discussion above. Perhaps, before we go on, it wouldbe helpful to summarize our results. In the course of the discussion we show how using T-duality alone given a manifold $\mathcal{M}$, in the cases we study, one can explicitly derive its mirror $\mathcal{W}$, and furthermore that the result agrees with what one obtains using conventional tools of toric geometry. To be precise, we really do a variant of mirror symmetry called local mirror symmetry, but this makes our results no less general. Finally, we apply our results to correct string theory constructions of $\mathcal{N} = 1$ field theories from brane configurations [56, 57, 58]. We also obtain a brane-based realization of non-abelian conifold transitions, along the lines of [59].

5.2 Introduction to Mirror Symmetry, Brane Configurations and Branes on Singularities

Consider a D3 brane probing a Calabi-Yau manifold $\mathcal{M}$. At a smooth point in $\mathcal{M}$ the tangent space is $\mathbb{R}^6$ and the D3 brane will have $N = 4$ supersymmetry on its world volume. To get something more interesting, we have to consider Calabi-Yau manifolds with singularities. Since we are only interested in the local physics near the singularity, the manifold is a singular, non-compact Calabi-Yau space which can,

*2 To our knowledge this has not been studied in the literature.

if one desires, be viewed as having a completion to a compact Calabi-Yau.

The Calabi Yau manifolds we will study the most have hyperquotient singularities that can be obtained as orbifolds of the well-known conifold singularity $\mathcal{C}$, and so are of the form $\mathcal{C}/\Gamma$, where Γ is a discrete symmetry group a conifold admits. One of the reasons we are interested in these types of spaces is that the theory on the probe is known. Recently the gauge theory of D3 brane at a conifold singularity was derived in [49]. The theory of a D3 brane on $\mathcal{C}/\Gamma$ is then defined as a quotient of the theory on $\mathcal{C}$ by Γ[*3].

We actually need to be a little bit more precise about the meaning of singularities in string theory. The Calabi-Yau singularities often have topologically distinct resolutions, so the singularity can be obtained via degenerations of spaces of different topology, the conifold singularity $\mathcal{C}$ being the simplest example of the phenomenon. In the case of the conifold, we could either deform the complex structure of the space or its Kähler structure to obtain a smooth manifold. Now, in string theory, deformations of the Kähler structure are complexified by addition of parameters that are not geometric, the B-field fluxes, so that even when discussing a geometry which is singular, the conifold for example, we have to specify the means of its smoothing. The D3 brane theory constructed in [49] is the theory on the Kähler side of the conifold. Taking a quotient of this theory by Γ the resulting theories have the same property.

Locally, complex and Kähler structure moduli spaces decouple. Thus, if we are interested in a neighborhood in $\mathcal{M}$ that develops a singularity through degeneration of its Kähler structure, we can take the complex structure to be smooth, and therefore trivial. Canonically, mirror symmetry acts by exchange of complex and Kähler structure. If $(\mathcal{M}, \mathcal{W})$ form a mirror pair, it is the complex structure of $\mathcal{W}$ that will

[*3]The discussion of a D3 branes on 6-dimensional singularities is very closely related to the by now famous AdS/CFT correspondence. Superconformal $N=2$ or $N=1$ (and also $N=0$) gauge theories can be constructed as the duals of supergravity on $AdS_5 \times X^5$, where X^5 is a certain five-dimensional (Einstein) manifold. First, for the case of D3 branes on six-dimensional orbifold singularities $\mathcal{O} = \mathbb{R}^6/\Gamma$, where Γ is a discrete group, X^5 is given by S^5/Γ, as discussed in [60, 61]. The corresponding orbifold gauge theory can be calculated using string perturbation theory. The conifold singularities were later obtained in [49], where for the simplest conifold the corresponding Einstein space X^5 is the homogeneous space $T^{1,1} = (SU(2) \times SU(2))/U(1)$. Further conifold type of singularities were recently discussed in [62].

be interesting.

The mirror geometries of the singularities will be constructed precisely in the spirit of [5], namely by performing T-duality transformations around three isometric directions of the geometric singularity. The singularities we are interested in have a toric description so we can equivalently [63] apply the local mirror map in the toric language [64]. The first point of view will be more useful for us, since it will allow us to follow the action of mirror symmetry on D-branes. Since the mirror symmetry acts in the space transverse to the D3 branes, the IIB gauge theory of D3 branes probing the space $\mathcal{M}$ will be mirror to an identical gauge theory but now due to IIA D6 branes wrapping a 3-cycles in $\mathcal{W}$. The "mirror" of the D3 brane at a smooth point will be a D6 brane wrapping T^3. What will be the mirror of a D3 brane at the singular point? The mirror D6 brane will wrap a three cycle which is still a special Lagrangian, but is now a degenerate three cycle which is homologous to the fiber at a generic point.

As is known for some time [65, 66], the geometric orbifold or conifold singularities are T-dual to a certain number of Neveu-Schwarz (NS) 5-branes. This T-duality can be used [67, 58, 68, 62, 69, 70] to transform the D3 branes probing a singularity into a pure brane configuration of intersecting NS branes and D branes of the Hanany-Witten type [71, 56]. It is this fact that we will systematically explore here.

In our case manifold $\mathcal{M}$ has three isometries on which T-duality T_{mirror} can be performed to obtain $\mathcal{W}$. We can write the mirror transform T_{mirror} as a composition of two dualities T_U and T_V, such that starting with a singularity $\mathcal{M}$ and acting with T_{mirror} on that space, we will first dualize to a certain brane configuration and subsequently further to the mirror geometry $\mathcal{W}$. From the brane point of view (taking NS5 branes to have $x^{0,1,2,3}$ as common directions and extend along $x^{4,5}$ and $x^{8,9}$) so we will call $T_U = T_6$ and $T_V = T_{48}$. From these two differently oriented NS branes we can build boxes or intervals, respectively, each giving rise to a pair of (T_U, T_V) dual mirror geometries. As is well known [72], one can suspend D4 branes on the intervals, and D5 branes on the boxes to obtain four-dimensional gauge theories on the D brane world volumes. T-duality will map these either to probe D3 branes or the D6 branes wrapping three-cycles of the mirror geometry. The resulting field theory should be the same in all the T-dual realizations.

Using these relations we can derive the rules that govern which gauge theory is encoded in a given brane setup.

The chapter is organized as follows. In the next section we will introduce the relevant geometries, namely the conifold singularities and the orbifold singularities and their generalizations. In the fourth section we will discuss the gauge theories that appear on the D3 brane probes. In section five we will introduce the T-dual brane setups – T-duality by T_U or T_V respectively – and will discuss the T-duality without the probe. Putting together the two T-duality transformations we will see the mirror geometries emerging. In section six we than incorporate the D3 brane probes. We will find that the brane box is the natural dual of the blowup of the orbifolded conifold and of the deformed generalized conifold. In order to incorporate this result we have to modify the brane box rules of Hanany and Zaffaroni [56]. Their gauge theories reappear in a special corner of moduli space. Our new construction makes some aspects of the box rules more transparent. In section seven we will wrap up by considering some related issues. We will show that by the same transformation T_{468} mirror symmetry can be defined for brane setups as well, turning 2-cycles into 3-cycles. We will show how to put both, the box and the interval together in one picture. This way we obtain a domain wall in an $N = 1$ 4d gauge theory that lifts to M-theory via a G_2 3-cycle as in [73].

5.3 The Geometries

5.3.1 Conifold

The simplest isolated singularity that a three-dimensional Calabi-Yau manifold can develop is the conifold:

$$\mathcal{C}: \quad xy - uv = 0 \tag{5.1}$$

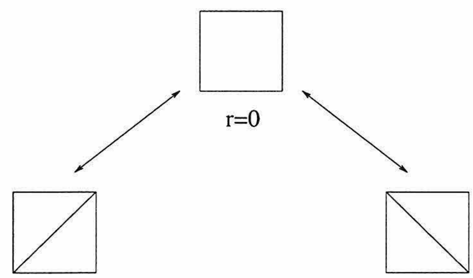

Fig.5.1. Two small resolutions of the conifold are related by a flop.

The singularity is located at $x = y = u = v = 0$ where the manifold fails to be transverse: $f = xy - uv = 0$, $\partial_i f = 0$ have a common solution there. There are two ways of smoothing the singularity, resulting in topologically distinct spaces.

- The so called small resolution – replacing the singular point by a $\mathbb{CP}^1$, thereby changing the Kähler structure. The resulting space has $h^{1,1} = 1$, $h^{2,1} = 0$.
- By deformation of the defining equation, thereby changing the complex structure. After the deformation, $h^{1,1} = 0$, $h^{2,1} = 1$.

a. Small Resolution

There are many ways in which one can exhibit the small resolution of the conifold. The one particularly well suited for our purposes is as follows. One can solve equation (5.1) by simply putting

$$x = A_1 B_1, \quad y = A_2 B_2, \quad u = A_1 B_2, \quad v = A_2 B_1, \tag{5.2}$$

where $A_i, B_j \in \mathbb{C}^4$. There clearly is a redundancy in this identification, since for any $\lambda \in \mathbb{C}^*$, taking $A_i \to \lambda A_i, B_j \to \lambda^{-1} B_j$ maps to the same point of the conifold. We can remedy this as follows. We will think about $\mathbb{C}^*$ as $\mathbb{R}_+ \times S^1$, that is we will put $\lambda = Re^{i\theta}$, with $R > 0$. Take a quotient by $\mathbb{R}_+$ first, by picking R to set

$$|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = 0. \tag{5.3}$$

To obtain a space isomorphic to the conifold we started with we must still divide by the $S^1 = U(1)$.

One can obtain a more physical interpretation of what we have done, which stems from observation that the description of the conifold we have come up with above is precisely that of a Higgs branch of a particular linear sigma model. It corresponds to a theory with four real supercharges, gauge group $U(1)$ with four matter fields A_i, B_j with charges $+1$ and -1, respectively, and no superpotential. The D-flatness conditions are then given by equation (5.3). This is of course not a new construction [74, 49].

Turning on the FI parameter r will modify the D-flatness conditions to

$$|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = r. \tag{5.4}$$

We have three cases to consider here.

a.) For $r = 0$ we have a singular manifold the conifold.

b.) For $r > 0$, the origin $A_i = 0 = B_j$ of the conifold is replaced by a sphere of size $|A_1|^2 + |A_2|^2 = r$*4. From the point of view of geometry, turning on the FI parameter [74] is naturally interpreted as blowing up a sphere of size r.

c.) For $r < 0$, from the point of view of b) the Kähler class is negative. We do still have a smooth manifold, because now the origin is replaced by $|B_1|^2 + |B_2|^2 = r$.

The manifolds in b.) and c.) are topologically distinct – they are related by a flop transition (see Fig.5.1).

b. Deformation

In addition to the smoothings we discussed above, conifold singularity can be smoothed out by keeping the Kähler structure fixed but modifying the defining equation. For

*4 This is an S^3 which quotiented by the $U(1)$ produces the two sphere which replaces the origin.

this it suffices to change the complex structure to:

$$xy - uv = \epsilon.$$

As long as $\epsilon \neq 0$, the conifold singularity has been removed. By examining the equation in detail, one can show that the origin was replaced by an S^3.

5.3.2 More General Singularities

We are now more or less in place to introduce toric geometry, as a tool for treating more complicated singularities.

We will use the language of linear sigma models to put the discussion on a more physical basis [74, 75]. We are constructing a linear sigma model whose moduli space will be a Calabi-Yau manifold $\mathcal{M}$. First, the number of independent FI parameters, or equivalently the number of $U(1)$ factors, will equal $h^{1,1}(\mathcal{M})$ (unless stated otherwise, by $\mathcal{M}$ we mean the manifold obtained by smoothing out the singularity). It is this number, and the charges of various matter multiplets that toric geometry must encode.

A toric diagram consists of $d+n$ vectors $\{\vec{v}_i\}$ in a lattice $\mathbb{N} = \mathbb{Z}^d$. Every vector $\vec{v}_i$ corresponds to a matter multiplet in our sigma model which we will call x_i. To describe a toric variety homeomorphic to other than flat space we need $n > 0$. Since $\mathbb{N}$ is d-dimensional, there are n relations between the $d+n$ vectors which we will write in the form

$$\sum_{i=1}^{d+n} Q_i^a \vec{v}_i = 0, \quad a = 1, \ldots, n. \tag{5.5}$$

It is clear that the Q's should be interpreted as the charges of the matter fields under the n $U(1)$'s. As a consequence, the D-flatness conditions will read

$$\sum_{i=1}^{d+n} Q_i^a |x_i|^2 = r_a, \quad a = 1, \ldots, n. \tag{5.6}$$

$\mathcal{M}$ is a space of solutions to (5.6), up to the identifications imposed by gauge symmetry. Or, instead of setting D-terms to zero and dividing by the gauge group, we could have taken a quotient by the complexified gauge group $x_i \to \lambda^{Q_i^a} x_i$, $a = 1, \ldots, n$, where $\lambda \in \mathbb{C}^*$, and express the moduli space as the space of gauge invariant polynomials in x's, modulo any relations between them. This is the language of eq.(5.1).

There is one slight simplification that occurs when $\mathcal{M}$ is a (non-compact) Calabi-Yau manifold. Namely, $\mathcal{M}$ is a Calabi-Yau if and only if there exists a vector $\vec{h} \in \mathbb{M}$, where $\mathbb{M}$ is the dual lattice of $\mathbb{N}$, such that

$$< \vec{h}, \vec{v}_i > \;=\; 1, \quad \forall \vec{v}_i,$$

i.e. if and only if all the vectors $\vec{v}_i$ live on a hyperplane a unit distance away from the origin of $\mathbb{N}$. Therefore in all of our examples toric singularities can be described by planar diagrams, only.

Hyperquotient Singularities

As is well known, one can obtain more complicated geometries by taking a quotient of the simpler ones by a properly chosen group action. Dividing $\mathbb{C}^n$ by a discrete symmetry group Γ we obtain orbifolds with quotient singularities. Hyperquotient singularities are quotients of a hypersurface singularities, $\mathcal{C}$ for example. Both can be treated easily in the language of toric geometry. First however, we must find an appropriate symmetry group of our manifold. Clearly, any action $x_i \to \lambda_i x_i, |\lambda| = 1$ leaves the manifold (5.6) invariant. There are $n + d$ coordinates x_i, and so a $U(1)^{n+d}$ acts on them. The symmetry group, however, is $U(1)^{n+d}/U(1)^n = \exp(2\pi i \mathbb{Z}^d)$, because the gauge symmetry group is $U(1)^n$. There is a natural way to encode the action of $U(1)^d = T^d$ on the toric variety as follows. To any element $\vec{n} \in \mathbb{Z}^d$, we can associate an element of $U(1)^d$ via $x_i \to e^{i n_i \theta} x_i$, where $\vec{n} = \sum n^i \vec{v}_i$. The coefficients n^i are defined up to $\sum Q_i^a \vec{v}_i = 0$, since $x_i \sim e^{i\theta Q_i^a} x_i$ for $a = 1, \ldots n - d$.

So far, our lattice was integral. Now suppose we refine the lattice by adding a vector in $\mathbb{Q}^d$, for example $\vec{q} = \frac{1}{r}(a_1, \ldots, a_d)$. As long as the lattice was integral the torus action was well defined. With the addition of $\vec{q}$, it will be so only after we induce additional identifications on the x_i's. Namely, if we write $\vec{q}$ as $\vec{q} = \sum \frac{q_i}{r}\vec{v}_i$, (mod $\sum Q_i \vec{v}_i$), the identification we need is:

$$x_i \sim e^{\frac{2\pi i q_i}{r}} x_i.$$

Another way to express the action of the quotient is in terms of gauge invariant monomials. For an $\mathbb{N}$ an integral lattice, any $\mathbb{C}^*$ invariant monomial is of the form

$$x^m = \prod x_i^{<\vec{v}_i, \vec{m}>},$$

so the space of $\mathbb{C}^*$ invariant monomials is just the dual lattice $\mathbb{M}$ of $\mathbb{N}$. Actually we want a bit less, since a) only the positive powers should appear, so we only want those $\vec{m}$'s that satisfy $< \vec{m}, \vec{v}_i > \geq 0$, $\forall i$, and b) we only want the independent ones, which are the generators of the group of invariant monomials.

By adding a vector $\vec{q}$ to the lattice $\mathbb{N}$ and thereby generating a finer lattice $\bar{\mathbb{N}}$, only a subset of monomials in $\mathbb{M}$ will be kept in $\bar{\mathbb{M}}$. The monomials in $\bar{\mathbb{M}}$ are precisely those $\vec{m} \in \mathbb{M}$, for which $< \vec{q}, \vec{m} > \in \mathbb{Z}$. Alternatively, $\bar{\mathbb{M}}$ is produced from $\mathbb{M}$ via identification induced on the monomials in $\mathbb{M}$ by

$$x^m \sim e^{2\pi i < \vec{q}, \vec{m} >} x^m.$$

In any event, we should now be ready to produce new spaces. We are up to producing orbifolds of the conifold, $\mathcal{C}/\Gamma$. Let us take $\Gamma = \mathbb{Z}_k \times \mathbb{Z}_l$. So, start with our conifold $\mathcal{C}$, defined by four vectors $\vec{v}_{1,2,3,4} \in \mathbb{N}$ as before, but refine the lattice to $\mathbb{N}'$ by adding two vectors, $\vec{e}_k = (\frac{1}{k}, 0, 0)$, and $\vec{e}_l = (0, \frac{1}{l}, 0)$. The resulting toric diagram (cf. Fig.5.2) "looks" the same as that for the conifold $\mathcal{C}$, except for the fact that it lives in a finer lattice. This, as explained above, results in the following identifications:

$$\vec{e}_k = \frac{1}{k}(\vec{v}_2 - \vec{v}_1),$$

we find that the quotient acts by

$$A_1 \sim e^{-\frac{2\pi i}{k}} A_1, \quad B_1 \sim e^{\frac{2\pi i}{k}} B_1, \quad A_2 \sim A_2, \quad B_2 \sim B_2,$$

and similarly for $\vec{e}_l$,

$$\vec{e}_l = \frac{1}{l}(\vec{v}_3 - \vec{v}_1),$$

the quotient is by

$$A_1 \sim e^{-\frac{2\pi i}{l}} A_1, \quad B_1 \sim B_1, \quad A_2 \sim A_2, \quad B_2 \sim e^{\frac{2\pi i}{l}} B_2.$$

Equivalently on $xy = uv$, we identify $x \sim x, y \sim y, u \sim e^{-\frac{2\pi i}{k}} u, v \sim e^{\frac{2\pi i}{k}} v$, and $x \sim e^{-\frac{2\pi i}{l}} x, y \sim e^{\frac{2\pi i}{l}} y, u \sim u, v \sim v$. In terms of Γ invariant coordinates $x' = x^l, y' = y^l, z = xy, u' = u^k, v' = v^k, w = uv$, the defining equation of the conifold becomes simply $z = w$. Taking into account that not all the invariant monomials are independent, the $\Gamma = \mathbb{Z}_k \times \mathbb{Z}_l$ orbifolded conifold, after obvious renaming of variables becomes:

$$\mathcal{C}_{k,l}: \quad xy = z^l, \quad uv = z^k. \tag{5.7}$$

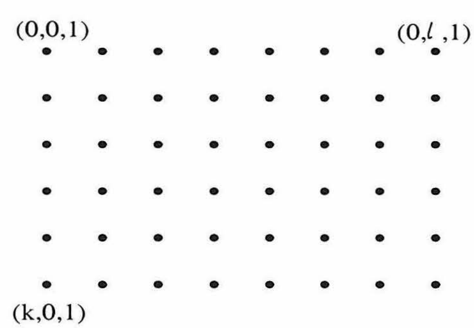

Fig.5.2. (Blowup of) orbifolded conifold $\mathcal{C}_{k,l}$.

a. Blowing Up

Toric geometry has equipped us with a means of blowing up the singularity. First let's look at the orbifold $\mathcal{C}_{kl}$. There are still only four vectors defining the diagram which were inherited from the conifold. There is a single relation between them, and thus a single Kähler class but this is insufficient to smooth out $\mathcal{C}_{k,l}$. However, due to the fact that the lattice is finer, there exist lattice points within the rectangle, these are all the points $\vec{v}_{i,j} = (i, j, 1)$, $0 \leq i \leq k$, $0 \leq j \leq l$. We can add these points to the toric diagram. In the language of linear sigma models, the effect is to add more matter fields, but also more $U(1)$ factors, and thus more FI parameters. Clearly, the resolved manifold will have $h^{1,1}(\mathcal{C}_{k,l}) = (k+1)(l+1) - 3$, which is the total number of linearly dependent vectors within the diagram. (Points outside the diagram can be added as well. However they will not contribute to the resolution of the singularity, but only modify it by irrelevant pieces.) We will not try to specify the precise region in the Kähler structure moduli space where the resolution lives, which would correspond to picking a triangulation of the toric diagram, because we will not need this piece of information. It is clear there will be very many different such regions, and they are all related by flops.

Finally, starting from the orbifolded conifold $\mathcal{C}_{k,l}$, with k, l sufficiently large, by performing partial resolutions we can obtain essentially any other toric singularity[*5]. The basic fact to note is that adding or subtracting one of the boundary points of the diagram changes $h^{1,1} \to h^{1,1} - 1$. The right interpretation of this is that we are probing the region of the Kähler structure moduli space where the four cycle associated to this point in the toric diagram becomes large enough that it in fact becomes irrelevant to the local physics – the associated vectors can be dropped altogether.

We will provide some more examples of the spaces we will explicitly use and introduce some terminology.

Starting from an orbifold of $\mathcal{C}$ by $\mathbb{Z}_k$,

$$\mathcal{C}_k: \quad xy = z^k, \quad uv = z, \tag{5.8}$$

or equivalently $xy = (uv)^k$, which has $h^{1,1}(\mathcal{C}_k) = 2k - 1$, by partial resolution we can

[*5] These singularities have been introduced in the physics literature for the description of gauge theories in [76].

obtain the generalization of a conifold,

$$\mathcal{G}_{kl}: \quad xy = u^k v^l \tag{5.9}$$

with only $k+l-1$ Kähler structure deformations. Clearly, in this notation $\mathcal{C}_k \equiv \mathcal{G}_{kk}$.

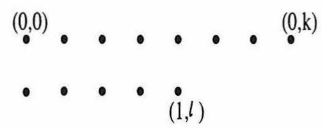

Fig.5.3. (Blowup of) generalized conifold $\mathcal{G}_{kl}$.

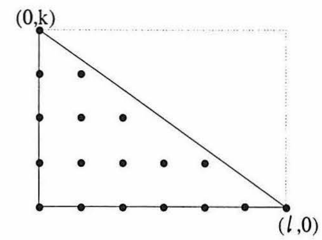

Fig.5.4. Toric diagram of the $\mathbb{Z}_k \times \mathbb{Z}_l$ orbifold $\mathcal{O}_{kl}$.

The conventional orbifold $\mathcal{O}_{kl} = \mathbb{C}^3/\mathbb{Z}_k \times \mathbb{Z}_l$ can be found in the Kähler structure moduli space of the orbifolded conifold $\mathcal{C}_{kl}$, the toric diagram of the orbifold being contained in that of the orbifolded conifold. One way to see that the Fig.5.4 is a toric diagram of $\mathbb{C}^3/\mathbb{Z}_k \times \mathbb{Z}_l$ is to use the fact that it can be obtained starting from a toric diagram containing just three vectors $\vec{v}_1 = (0,0,1), \vec{v}_2 = (1,0,1), \vec{v}_3 = (0,1,1)$ in an integral lattice $\mathbb{N}$, which gives a toric variety homeomorphic to flat space, and then refine the lattice to $\mathbb{N}'$, as in Fig.5.4. The map from the toric variety in $\mathbb{N}$ to the one living in $\mathbb{N}'$ is one-to-one provided one includes discrete identifications on the three matter fields $A_i, i = 1,2,3$,

$$A_1 \sim e^{-\frac{2\pi i}{l}} A_1, \quad A_2 \sim e^{\frac{2\pi i}{l}} A_2, \quad A_3 \sim A_3,$$

and

$$A_1 \sim e^{-\frac{2\pi i}{k}} A_1, \quad A_2 \sim A_2, \quad A_3 \sim e^{\frac{2\pi i}{k}} A_3.$$

As before, the number of Kähler structure deformations is just the number of independent points in the toric diagram, and this number will clearly depend on whether (k,l) are coprime or not, since the number of points on the diagonal is $gcd(k,l)+1$.

b. Deformations

- The orbifolded conifold $\mathcal{C}_{kl}$: $\quad xy = z^k, \quad uv = z^l$ can be deformed to a smooth space by modifying the defining equation as:

$$xy \;=\; \prod_{i=1}^{k}(z - w_i) \qquad uv \;=\; \prod_{j=1}^{l}(z - w'_j). \tag{5.10}$$

One of these parameters can be set to 1 by shifting z, so we are left with $k+l-1$ parameters. This gives $h^{2,1}(\mathcal{C}_{kl}) = k + l - 1$.

- The generalized conifold $\mathcal{G}_{kl}$: $\quad xy = u^k v^l$ can be deformed into

$$xy = \sum_{i,j=0}^{k,l} m_{ij} u^i v^j. \tag{5.11}$$

This time we see $h^{2,1}(\mathcal{G}_{kl}) = (k+1)(l+1) - 3$ complex structure deformations m_{ij}: by shifting u, v, we can eliminate two of the parameters, and another one by rescaling the defining equation.

Mirror Symmetry

Toric geometry is well adopted to discussing mirror symmetry as well. We will review it here very briefly, only. Mirror symmetry exchanges the Kähler structure parameters with the complex structure parameters. Now, to understand the mirror map, we first need to know something about the complex structure moduli space. How is the complex structure encoded in the equation of the manifold? The answer is as follows: the coefficients of the monomials appearing in the defining equation are coordinates on the complex structure moduli space. What they parameterize are the "sizes" of various three-cycles (i.e., the periods of the holomorphic three form) and the metric on the moduli space. The periods, (and therefore the metric – the moduli space has special geometry structure) can be derived directly as a solution to a system of differential equations. The main point is that the differential equations depend solely on the relationships between the monomials in the defining equation and nothing else.

Given the toric manifold, relations between the vectors in the toric diagram of $\mathcal{M}$ (we are assuming a completely smooth space here, with all the possible blowups performed),

$$\sum_{i=1}^{n+d} Q_i{}^a \vec{v}_i, \quad a = 1, \dots n$$

map to relationships between the monomials in the defining equation of $\mathcal{W}$, the mirror of $\mathcal{M}$, given by

$$\mathcal{W}: \quad \sum_i a_i m_i = 0, \tag{5.12}$$

where a_i are coefficients, and m_i monomials, the monomials must satisfy

$$\prod_{i=1}^{n+d} m_i^{Q_i^a} = 1, \quad a = 1, \dots n. \tag{5.13}$$

Any solution to these equations (and in general there are more than one) will represent the same complex structure (by decoupling of complex and Kähler moduli spaces). Note that there are $n+d$ monomials with n relations between them. Together with the hypersurface equation, this gives a $d-1$ dimensional manifold, but the homogeneity of the monomial relations will allow us to remove one more. The mirror will naively have $d-2$ dimensions. This is not a problem, rather an artifact of the fact that local mirror symmetry is encoding all the information about the complex structure of the mirror, and nothing but. One can fix the "dimensionality" of the local mirror by adding quadratic pieces, as this will not influence the complex structure moduli space.

Let us briefly show how this works for the two examples we are going to be concerned with in this work [*6], $\mathcal{G}_{kl}$ and $\mathcal{C}_{kl}$. Consider first the blowup of $\mathcal{C}_{kl}$. We want to interpret the same diagram Fig.5.2 as defining the complex structure of the

[*6] These examples and many more along these lines have been recently analyzed in great detail in [77].

mirror. Assigning the vector $(i, j, 1)$ to a monomial $u^i v^j$, eq.(5.13) is satisfied for all the relations. The defining equation for the mirror of $\mathcal{C}_{kl}$ hence becomes according to eq.(5.12):

$$\sum_{i,j=0}^{k,l} m_{ij} u^i v^j = 0$$

or after adding the irrelevant quadratic piece xy

$$xy = \sum m_{ij} u^i v^j$$

which is nothing but the deformation of $\mathcal{G}_{kl}$.

Having established that the deformation of $\mathcal{G}_{kl}$ is mirror to the blowup of $\mathcal{C}_{kl}$ we can find another dual pair by following our geometries through a conifold transition. We should find that the blowup of $\mathcal{G}_{kl}$ is the mirror of the deformation of $\mathcal{C}_{kl}$. Let us see how this works. As above we read off the mirror to be

$$\prod_{i=1}^{k}(z - w_i) + t\prod_{j=1}^{l}(z - w'_j) = 0.$$

Because t appears only linearly this encodes the same complex structure as

$$\prod_{i=1}^{k}(z - w_i) = uv, \quad \prod_{j=1}^{l}(z - w'_j) = xy$$

which is indeed the deformation of $\mathcal{C}_{kl}$ as presented in eq.(5.10).

5.4 The Gauge Theories

Having introduced the geometric background spaces, we will now discuss the corresponding gauge theories if one adds M D3 branes with world volume transverse to the non-compact manifolds. The corresponding gauge group for the orbifolded conifold $\mathcal{C}_{kl}$, eq.(5.7), is given by the following $N = 1$ supersymmetric gauge theory:

$$SU(M)^{kl} \quad \times \quad SU(M)^{kl} \tag{5.14}$$

with matter fields $(A_1)_{i+1,j+1;I,J}$, $(A_2)_{i,j;I,J}$, $(B_1)_{i,j;I,J+1}$, $(B_2)_{i,j;I+1,J+1}$. We label the gauge groups with $i, I = 1 \ldots k$ and $j, J = 1 \ldots l$. All the matter fields are bifundamental under the gauge groups indicated by the indices. The $\mathbb{Z}_k$ orbifolded conifold arises as the special case $l = 1$. In addition there will be a quartic superpotential

$$\begin{aligned} W &= \sum_{i,j} (A_1)_{i+1,j+1;I,J}(B_1)_{i,j;I,J+1}(A_2)_{i,j+1;I,J+1}(B_2)_{i,j+1;I+1,J+1} \\ &- \sum_{i,j} (A_1)_{i+1,j+1;I,J}(B_1)_{i,j;I+1,J}(A_2)_{i+1,j;I+1,J}(B_2)_{i+1,j;I+1,J+1}. \end{aligned} \quad (5.15)$$

The other singularity, the generalized conifold in eq.(5.9), corresponds to

$$SU(M)^{k+l}$$

with bifundamental matter according to Uranga's rules [62] and quartic superpotentials.

Finally consider M D3 branes on a transversal orbifold singularity $\mathcal{O}_{kl}$. They give rise to an

$$SU(M)^{kl} \quad (5.16)$$

gauge theory with 3 types of chiral bifundamental multiplets $H_{i,j;i+1,j}$, $V_{i,j;i,j+1}$ and $D_{i+1,j+1;i,j}$ in each gauge group and a cubic superpotential

$$W = \sum_{i,j} H_{i,j;i+1,j} V_{i+1,j;i+1,j+1} D_{i+1,j+1,i.j} - \quad (5.17)$$

$$\sum_{i,j} V_{i,j;i,j+1} H_{i,j+1;i+1,j+1} D_{i+1,j+1;i,j}.$$

This way the orbifold gauge theories will have $3M$ matter fields per gauge group and cubic superpotentials, leaving us with a finite $\mathcal{N} = 1$ theory. The conifold gauge theories have $2M$ matter fields per gauge group and quartic superpotentials. These theories are non-finite but flow to a fixed line parameterized by a marginal operator in the IR.

5.5 The T-dual Brane Setups and Mirror Symmetry

In this section we would like to discuss the brane configurations that are T-dual to the singularities introduced in section 5.3. Specifically, we are interested in two different T-duality transformations: the first duality, which we will reffer to as T_U duality, was recently discussed by Uranga [62] and by Dasgupta and Mukhi [69]. The dual brane picture consists of NS and rotated NS' 5-branes. The D3 branes probing the singularities, which we study in the next section become D4 branes after the $T_U = T_6$ duality transformation which live on the compact interval in x^6.

Second we perform a T-duality along the compact directions x^4 and x^8, $T_{48} = T_4 T_8$. This maps the singularities again to NS and NS' branes, where now the D3 probes become D5 branes which fill the compact brane box in the $x^4 - x^8$ spatial directions. This T-duality was first introduced in [66] and for a special point in moduli space used by [58] to study D3 branes on orbifold singularities. We will henceforth refer to it as T_V.

These T-dualities are very useful in the sense that they allow us to read off the gauge groups on the D3 brane world volume according to some very intuitive graphic rules encoded in the brane configuration. While for the orbifold a perturbative string calculation is also available to get the gauge group, for the more general singularities discussed here, one would have to rely on a technique in [49].

Combining the two, that is doing T_{468}, we actually perform a local mirror symmetry transformation. We will see explicitly, that T_{mirror} takes a geometry $\mathcal{W}$ into its mirror geometry $\mathcal{M}$. The gauge theory of a D3 brane probing $\mathcal{W}$ has to be identical to that on a D6 brane wrapping a 3-cycle in $\mathcal{M}$.

This should correspond to the mirror transformations for Calabi-Yau spaces, which are the compact counterparts of our non-compact singularities. These compact CY's are assumed T^3-fibrations (with T^3 a special Lagrangian submanifold of the CY) and the mirror transformations acts as the inversion of the volume of the T^3. Obviously, this T^3 corresponds to our three directions x^4, x^6 and x^8, on which the mirror symmetry acts.

5.5.1 The Brane Setup

Before we embark on the discussion let us briefly recall the basic brane setup. There are two configurations we are going to consider. One is the standard HW [71, 78] type of brane setup, where D4 branes are stretched in between NS and NS' branes, with the former living along 012345 and the latter along 012389, the rotation being necessary in order to break SUSY from 8 to 4 supercharges. In order to obtain a supersymmetric theory from D4 branes on the interval all the NS and NS' branes have to be at the same position in the 7 direction. Separations along the 7 direction would be interpreted as FI terms or baryonic branches in the gauge theory and effectively leads to a breaking of the gauge group. Similarly, we should require all the NS branes to have the same position in 89 and all the NS' branes to have the same position in 45 space. They are separated along the 6 direction building the intervals, along which the D4 branes (living in 01236) stretch.

The second kind of brane setup we are going to consider are the so-called brane boxes [56], which are a straightforward generalization of the interval theories. The brane box is a rectangle bounded by NS and NS' branes with a D5 brane suspended on it. This can be achieved by the same NS and NS' branes as above but now all branes have to be located at the same 67 position, closing the intervals. We can open up the boxes by separating the NS and NS' branes along their 48 directions (unfortunately this way we differ from the notation in [56], where the boxes were taken to live in the 46 space. This is necessary, since it is crucial for us, that box and interval can be realized by the same set of NS and NS' branes). We still want to keep the 5 and 9 positions equal in order to preserve supersymmetry of the suspended probes. Deformations along these directions are again FI terms in the gauge theory, which are reinterpreted as baryonic branches after freezing out the diagonal $U(1)$s.

5.5.2 Deformations and Blowups

As mentioned above, it is important to distinguish whether we want to study the deformation or the blowup of the singularity under investigation. The corresponding parameters should have an interpretation in the brane picture as well. If the dual is 'pure brane', i.e., consists only of branes in flat space, this interpretation will be solely in terms of NS brane positions and, as will be established later, on brane shapes. Otherwise some of the parameters encode blowups of the non-trivial background geometry. Even though the latter description of the probe may still be useful, e.g., in order to read off the gauge group and matter content, we would like to focus in the rest of our discussion on the case in which the dual is 'pure brane'. Let us forget for a moment about the D brane probes altogether. That is, we want to study the map of the singular geometry into a configuration of NS branes, as pioneered in [66]. Actually, it turns out to be easier to start with the NS brane configurations, where it is clear what we mean by the 4, 6 and 8 directions. Performing T_{48} and T_6, respectively,we will find two different geometries, which have to be the local mirrors of each other. By construction, these are precisely the geometries that have a pure brane dual (we started out with a pure brane setup!). We will find the following relations, as indicated in Fig.5.8 in the summary at the end of this paper:

- The blowup of the generalized conifold is T_U dual to NS branes separated along 67 (the interval). These are in turn T_V dual to the mirror, the deformation of the orbifolded conifold.
- Similarly the blowup of the orbifolded conifold will T_V dualize into a box and then T_U dualize in the mirror, the deformation of the generalized conifold.

Indeed these two transformations are related by a conifold transition, that is bringing together the NS branes on the interval and then separating them along 4589 instead corresponds to blowing down the 2-cycles and opening up the 3-cycles of the deformed conifold (and vice versa for the orbifolded conifold).

5.5.3 The Brane Box, Blowup of the Orbifolded and Deformation of the Generalized Conifold

Let $m_i = (x^8, x^9)$, $m'_j = (x^4, x^5)$ positions of the k NS and l NS' branes respectively in $x^{4,5,8,9}$, and $w_i = (x^6, x^7)$, $w'_j = (x^6, x^7)$ the positions in the other two directions. Let us start with a "brane box", that is we set all the w_i and w'_j to zero. T-dualizing the brane box along $x^{4,8}$ we obtain a manifold we call $\mathcal{M}$ and T-dualizing along x^6 we obtain $\mathcal{W}$. The resulting geometries are related by $T_{468} = T_{mirror}$.

- $\mathbf{T_V = T_{48}}$: The T-dual space $\mathcal{M}$ is a $\mathbb{Z}_k \times \mathbb{Z}_l$ orbifolded conifold

$$\mathcal{C}_{k,l}: \quad xy = z^l, \quad uv = z^k$$

as in (5.7), where k, l are numbers of NS and NS' branes. This is a double $\mathbb{C}^*$ fibration over the z plane, that is the space has 2 U(1) isometries used in T duality. The x^4, x^8 separations of the branes must map into B-fluxes through 2-cycles of the T-dual space. We must therefore identify m_i, m'_j as deformations of the *Kähler* structure. Deformations of the Kähler structure cannot change the complex structure, so the m_i and m'_j will not be visible in the defining equations. Having identified m_i, m'_j as the Kähler structure parameters, w_i and w'_j are identified as complex structure parameters. But they are frozen, since turning them on would destroy the box structure.

For definiteness take IIB theory on $\mathcal{C}_{kl}$. T_V duality takes us back to type IIB with NS branes. In this case Kähler structure parameters, that is the 2-sphere sizes, sit in hypermultiplets. The other 3 scalars in this multiplet are the NS-NS B-flux the RR B-flux and the RR 4-form-flux through the sphere. The latter is a 2-form in 4d, which can be dualized into a scalar. The 2-sphere size and the NS-NS B-flux are the complexified Kähler parameter, which map to m_i and m'_j under T_V. In the brane box the two other scalars come from Wilson lines of the NS-world volume gauge fields in 45 and 89, which pair up in hypermultiplets with m_i and m'_j respectively.

Note, however, that we have a puzzle. The orbifolded conifold $\mathcal{C}_{kl}$ has, as we have found from the toric description,

$$(k+1)(l+1)-3 = kl+k+l-2$$

Kähler structure parameters m_{ij}, which can be turned on to smooth out $\mathcal{C}_{kl}$. Only $k+l-2$ have been realized in terms of the (relative) brane positions m_i and m'_j.

So where are the kl hypermultiplets in the brane box skeleton? They sit at the kl intersections! Strings stretching from NS to NS' give rise to precisely these hypermultiplets [*7].

Turning on vevs for the two scalars corresponding to 2-sphere sizes and NS-NS fluxes resolves the intersection of the NS and NS' into a smooth object, a little 'diamond'. For non-zero B-fields this diamond will open up in the 48 plane, for 2-sphere sizes in the 59 plane. This interpretation will become more suggestive after discussing T_U on this configuration and once we start discussing the D3 brane probes.

In the geometry the 2-spheres give rise to strings from wrapping D3 branes around them. How do we see them in the NS5 box skeleton? The D3 branes on the $k+l-2$ spheres from the curves of singularities correspond to (fractional) D3 branes living in the boxes (or better in whole stripes). The additional kl strings must now correspond to D3 branes *in the diamonds*. We will indeed see that the diamonds allow for such a configuration.

Of course the same story can be repeated in type IIA. Here the diamonds will correspond to matter on the intersection of type IIA NS5 branes, this time sitting in a vector multiplet. Again the 2 scalars correspond to the kl sizes and B-fluxes of the corresponding 2-spheres. Instead of the two additional scalars in the hypermultiplet this time we see a vector from the RR 3-form on the

[*7] They are Strominger's D3 brane on the vanishing 3-sphere in the geometry (remember that we only consider blowups, so the 3-spheres are fixed at zero size).

sphere. In the brane language the Wilson lines of the NS5 gauge field have to be substituted by Wilson lines of the (2,0) 2-form field, again giving rise to vectors.

- $\mathbf{T_U} = \mathbf{T_6}$, T-duality to $\mathcal{W}$. What happens now is as follows. Since we did a T_6 duality, x^6 separations will become the B-fields. Thus, now the w_i (which had to be put to zero since we are discussing a box) parameters are *Kähler* structure deformations, while the non-zero m_{ij} now should show up as complex structure deformations.

  The dual geometry should be a single $\mathbb{C}^*$ fibration. This will be described by an equation whose parameters, the complex structure deformations, must be m_{ij}. Let us first study the situation where the vevs of the hypers living at the intersections are zero. In this case the $\mathbb{C}^*$ fibration must degenerate over the NS and NS' positions m_i, m'_j, but in an independent way, since the branes are orthogonal – it must contain two curves of singularities A_{m-1}, and A_{n-1} corresponding to NS and NS' branes. There is one such equation for generic values of m_i's

$$\mathcal{W}: \quad uv = \prod_{i=1}^{k}(z - m_i)\prod_{j=1}^{l}(w - m'_j).$$

  The curve contains kl conifold singularities located at $z = m_i$ and $w = m'_j$ corresponding to the fact that all the hypermultiplets at the intersections were turned off.

  Let us jump ahead and realize $\mathcal{W}$ directly as the mirror of $\mathcal{M}$. Performing the local mirror map we obtain:

$$\mathcal{W}: \quad uv = \sum_{i=0}^{k}\sum_{j=0}^{l} m_{ij} z^i w^j.$$

  By now the T-dual interpretation of this more general space should be clear. It describes a single NS brane wrapping a curve

$$\Sigma: \quad 0 = \sum_{i=0}^{k} \sum_{j=0}^{l} m_{ij} z^i w^j.$$

The smoothing out of the intersections corresponds to the diamonds. For example one intersecting NS and NS' brane is described by $zw = 0$. Turning on the hypermultiplet corresponds to smoothing this out to $zw = m_{00}$ e.g., as discussed in [79] for the related case of intersecting D7 branes. Indeed the resulting smooth curve has a non-vanishing circle of radius $(m_{00})^{1/2}$ as can be seen by writing it as $x^2 + y^2 = m_{00}$ and restrict oneself to the real section thereof, for example[*8]. This is precisely what we need: we can suspend a D3 brane as a soap bubble on the NS skeleton, its boundary being given by the circle. The tension of the resulting string is given by the area of the disk and hence is proportional to m_{00} as expected from the dual geometry $\mathcal{M}$ (where the size of the 2-sphere was also proportional to m). In $\mathcal{W}$ the same string will be given by a D4 brane on the vanishing 3-sphere.

In the same way we can T-dualize any singularity that can be represented as a toric variety into a generalized box of NS branes, with a certain amount of diamonds frozen.

5.5.4 Going to the Interval: the Conifold Transition

We can derive a second T-dual triple of geometry, T-dual brane setup and mirror geometry, by studying T_U and T_V on the interval theory. Note that the interval theory can be directly obtained from the box by brane motions. First we move all the NS and NS' branes on top of each other, setting all m_{ij} to zero, closing all the boxes and diamonds. This is the conifold point. Now we see that we have the choice to open up the intervals, by turning on the w_i and w'_j.

We can follow this transition in the geometry as well. Let us see what it does to $\mathcal{M}$. For one we have shrunk all the 2-spheres to zero size, putting us at the most singular point of the geometry. In addition we have put all the B-fields to zero. So

[*8] We are very grateful to M. Bershadsky for very helpful discussions on this point.

we are really sitting at the real codimension 2 locus of Kähler moduli space, where the closed string CFT description goes bad [74]. This is once more the conifold point. From there we can deform the singularity by turning on 3-spheres to obtain $\mathcal{M}_T$ and this is precisely what corresponds to turning on the w_i and w'_j in the brane picture. This is a (non-abelian) conifold transition [59]. We went from the blowup of the orbifolded conifold $\mathcal{C}_{kl}$ to its deformation. Let us see that T_V still works. The w_i, w'_i must now be identified with complex structure deformations. The geometry has to have a $\mathbb{C}^* \times \mathbb{C}^*$ fibration which degenerates over those points. This leads us to

$$xy = \prod_{i=1}^{k}(z - w_i)$$

$$uv = \prod_{j=1}^{l}(z - w'_i)$$

as the T-dual geometry.

Last but not least we can study the effect on $\mathcal{W}$. In going to $\mathcal{W}_T$, the mirror of $\mathcal{M}_T$, we this time send all the 3-spheres to zero size and then turn on blowup modes, taking us from the deformed generalized conifold $\mathcal{G}_{kl}$ to its blowup.

5.6 Probing the Mirror Geometries

5.6.1 Introducing the Probe: Elliptical Models

As a next step we want to introduce M D3 brane probes on top of our geometry. This way we break the supersymmetry down to 4 supercharges and get interesting $N = 1$ 4d gauge theories. The deformation parameters of the singularity appear as parameters in the gauge theory, the moduli space of the gauge theory describes the motion of M D3 branes on the singular space. These probe theories have received a lot of attention recently. They give rise to conformal field theories and have a dual AdS description.

In principle we could take any of the four geometries we introduced, compactify type IIB on it and then put a D3 brane probe on top of the singularity. The two situations we are going to study are M D3 branes on the blowup of the generalized conifold $\mathcal{G}_{kl}$ (on $\mathcal{W}_T$) and M D3 branes on the blowup of the orbifolded conifold $\mathcal{C}_{kl}$ (on $\mathcal{M}$).

Performing our two T-dualities T_U and T_V we will find two different realizations of each of the probe theories. The background geometry will transform precisely as we discussed in the last section. This way

- M D3 brane probes of the blowup of the generalized conifold $\mathcal{W}_T$ are T_U dual to D4 branes on an interval defined by w_i and w'_j and T_{mirror} to D6 branes wrapping 3-cycles in $\mathcal{M}_T$
- M D3 brane probes of the blowup of the orbifolded conifold $\mathcal{M}$ are T_V dual to D5 branes on a box defined by m_{ij} and T_{mirror} to D6 branes wrapping 3-cycles in $\mathcal{W}$.

We will have to deal with what is usually referred to as elliptical models in the literature [80, 58]. That is the 6 direction of the interval or the 48 direction of the box are actually compact, leaving no room for semi-infinite branes. All D-brane groups will then be gauged.

5.6.2 The Generalized Conifold and the Interval

First we would like to consider the gauge theory on the world volume of M D3 brane probes on the blowup of a generalized conifold singularity*9. This gauge theory is given e.g., in [62] and can be read off most easily in the dual brane setup we are about to describe. In the last section we showed that this geometry is T_U dual to NS and NS' branes on a circle, forming intervals with 67 separations given by w_i and w'_j, all the m_{ij} being zero. As utilized in [62, 69] this means that the M D3 brane probes turn into an elliptical model with M D4 branes wrapping the circle. It is straightforward to read off the gauge theory from this according to the standard HW

*9 Similar setups have been discussed recently in [81].

rules. Of course it agrees perfectly with the one obtained from applying a standard orbifold procedure directly on the conifold gauge theory of [49].

There is yet another realization of the same gauge theory. Performing the whole $T_{mirror} = T_{468}$ we can turn $\mathcal{W}_T$, the blowup of the generalized conifold on which we originally put the D3 brane probes, into $\mathcal{M}_T$, the deformation of the orbifolded conifold. The theory with which we have to compare is that on the mirror of the D3 probe, that is a D6 brane wrapping SUSY 3-cycles in $\mathcal{M}_T$. But this is precisely the situation discussed in [82]. The parameters w_i and w'_i in $\mathcal{M}_T$, given by (5.10) determine the loci in the z plane where the $\mathbb{C}^* \times \mathbb{C}^*$ fibration degenerates. As found in [82] in order to have a BPS state the w_i and w'_i have to align along a line in the z plane. Since the $S^1 \times S^1$ fibration degenerates over w_i and w'_i, we can regard this fibration over the interval between neighboring w_i and w'_i as a 3-cycle. In [82] it was shown that this 3-cycle is S^3 and $S^2 \times S^1$ respectively, depending on whether neighboring points are a w, w' pair or both w (both w'). In the former case one obtains a quartic superpotential, in the latter case an $N = 2$ like setup. Obviously this yields the same gauge theory as the D3 probe on $\mathcal{W}_T$ and the D4 brane on the interval.

5.6.3 D5 Branes on the Box: the Modified Box Rules

The second theory we would like to consider are M D3 branes on an $\mathbb{Z}_k \times \mathbb{Z}_l$ orbifolded conifold. As shown above, the geometry dualizes under T_V into brane boxes where the NS5 brane skeleton wraps the curve $\sum_{i,j=0}^{k,l} m_{ij} z^i w^j$. $k + l - 2$ of the m_{ij} parameters can be associated to brane positions, while the other kl parameters correspond to diamonds, that is the hypermultiplets sitting at the NS NS' intersections, whose vev smoothes out the singular intersections.

The probe D3 branes turn into D5 branes living on these boxes and diamonds. Again this should in principle be a very useful duality in the sense that we can read off the associated gauge theories by using some analogue of the HW rules. In addition some information about the corresponding quantum gauge theory should be obtainable by lifting the setup to M-theory.

In order to understand our rules it is best to start with the easiest example, the conifold $\mathcal{C}$, eq.(5.1), itself. The dual description just is that of a single NS and NS' brane on a square torus, as depicted in the upper left corner of Fig.5.5. The

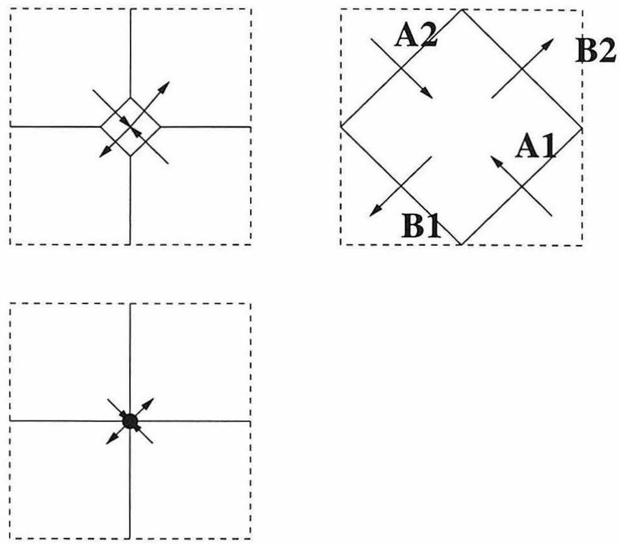

Fig.5.5. Upper left : the box with generic B-value;
Upper right: maximal B-value;
Lower left : Taking B to 0 sending one gauge coupling to infinity.

conifold has one blowup parameter, corresponding to the one diamond sitting at the intersection. As long as we keep the size of the 2-sphere zero, the B-flux through the sphere will correspond to the size of the diamond. As we have argued in the last section, the curve describing the diamond actually supports a non-trivial S^1 on which the D5 brane can end, so the gauge theory will have two group factors, $SU(M) \times SU(M)$. The inverse gauge couplings are proportional to the area of the corresponding faces. There is a special point, when the diamond has the same area as the other gauge group, that is the diamond occupies half of the torus. In this case we know that we have to recover the standard conifold gauge theory of [49]. This can easily be implemented using the simple brane rules specified in the upper right corner. We have to demand, that half of the matter multiplets we would naively expect are projected out. The orientation of the arrows seems quite arbitrary. Indeed we will

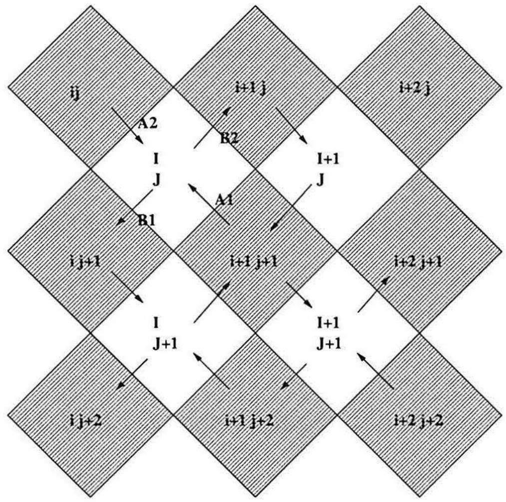

Fig.5.6. The diamond rules at the point of maximal B-fields.

see that the orientation can be changed and that this corresponds to performing flop transitions in the dual geometry. Indeed one can easily establish that these rules also are capable of realizing more complicated setups. Generically, the gauge theory on the $\mathbb{Z}_k \times \mathbb{Z}_l$ orbifolded conifold has a $SU(M)^{kl} \times SU(M)^{kl}$ gauge group. In our picture the gauge group factors will correspond to the kl diamonds and the kl boxes respectively. Again it is easiest to compare at the point, where all gauge couplings are equal. In this case, both the diamonds as well as the boxes degenerate to rhombes, as pictured in Fig.5.6, where we denoted them as filled and unfilled boxes. Generalizing our A and B fields from above we will find that the matter fields transform as (where the two sets of kl gauge groups are indexed by small and capital letters respectively)

$$\begin{array}{ll} (A_1)_{i+1,j+1;I,J} & (\Box_{i+1,j+1}, \bar{\Box}_{I,J}) \\ (A_2)_{i,j;I,J} & (\Box_{i,j}, \bar{\Box}_{I,J}) \\ (B_1)_{I,J;i,j+1} & (\bar{\Box}_{i,j+1}, \Box_{I,J}) \\ (B_2)_{I,J;i+1,j} & (\bar{\Box}_{i+1,j}, \Box_{I,J}) \end{array}$$

which are exactly the rules expected [62]. This proposal can also easily deal with the situation of non-trivial identifications on the torus as discussed in [58]. In addition there will be quartic superpotential for every closed rectangle, the relative sign being given by the orientation

$$\begin{aligned} W &= \sum_{i,j} (A_1)_{i+1,j+1;I,J}(B_1)_{I,J;i,j+1}(A_2)_{i,j+1;I,J+1}(B_2)_{I,J+1;i+1,j+1} - \\ & \sum_{i,j} (A_1)_{i+1,j+1;I,J}(B_2)_{I,J;i+1,j}(A_2)_{i+1,j;I+1,J}(B_1)_{I+1,J;i+1,j+1}. \end{aligned}$$

We do not expect that this picture changes when we take the sizes of box and diamond to differ. We will still see the A and B fields. Only the relative couplings will change and no new fields or interactions appear, since they certainly don't in the dual geometry. The singular conifold points correspond to the situations where diamonds close. From the field theory point of view this just means that we take the corresponding gauge coupling to infinity. As in the standard HW situation with only parallel NS branes this corresponds to a strong coupling fixed point with possibly enhanced global symmetry if several NS branes coincide.

Another interesting question to consider is to ask ourselves what happens when we blow up the spheres to finite size. This now should correspond to some mode of the diamond that "rotates" it away out of the 48 plane into the 59 plane. According to common lore this should correspond to a FI term in the gauge theory. We will no longer be able to support a D5 brane stretched inside the diamonds in a supersymmetric fashion, independent of their size (that is the B-field)[*10]. Since we expect that the center of mass $U(1)$'s are frozen out as in [80], the FI term will be reinterpreted as usual as a baryonic branch. Especially there should exist a baryonic branch along which we reduce to the orbifold gauge theory.

Indeed as shown in [62] the gauge theories we described here do have such a baryonic branch. Giving a vev to (say) all the A_2 fields will break each $SU(M)_{ij}$ $\times$

[*10] This is very similar to what happens on the interval: blowing up a sphere corresponds to moving off an NS brane in the 7 direction. Since in order to preserve supersymmetry branes are only allowed to stretch along the 6 direction this effectively reduces the number of gauge groups (the number of intervals) by one. The 6 position of the brane we moved away (the B-field on the blown up sphere) does not affect the massless matter content anymore.

$SU(M)_{IJ}$ pair down to its diagonal $SU(M)_{ab}$ subgroup. The remaining massless fields after the Higgs mechanism are

$$\begin{array}{lcll} D_{a+1,b+1;a,b} & = & (A_1)_{a+1,b+1;A,B} & (\Box_{a+1,b+1}, \bar{\Box}_{a,b}) \\ H_{a,b;a,b+1} & = & (B_1)_{A,B;a,b+1} & (\Box_{A,B}, \bar{\Box}_{a,b+1}) \\ V_{a,b;a+1,b} & = & (B_2)_{A,B;a+1,b} & (\Box_{A,B}, \bar{\Box}_{a+1,b}) \end{array}$$

with the remaining superpotential:

$$\begin{array}{rcl} W & \sim & \sum_{a,b} D_{a+1,b+1;a,b} H_{a,b;a,b+1} V_{a,b+1;a+1,b+1} - \\ & & \sum_{a,b} D_{a+1,b+1;a,b} V_{a,b;a+1,j}, H_{a+1,b;a+1,b+1} \end{array}$$

which are precisely the box rules of [56], as claimed. Note that the diagonal D fields are not special at all, they arise just from the fundamental A, B degrees of freedom of the generalized box.

A small complication arises once we consider situations that are more involved

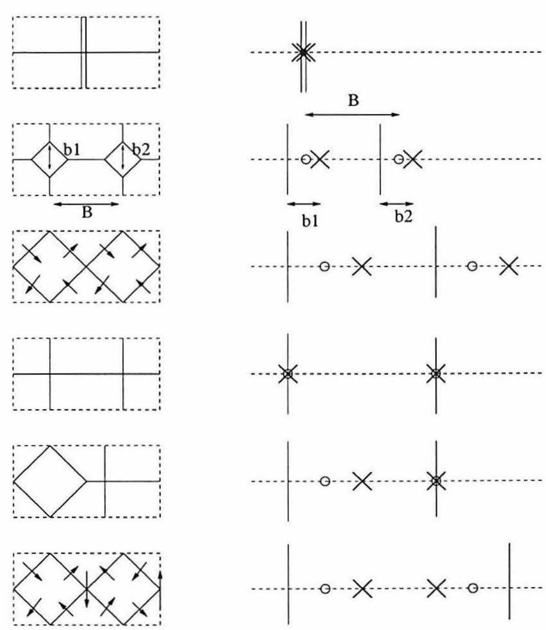

Fig.5.7. Diamonds do have an orientation.

than the conifold. For simplicity let us study the case of the $\mathbb{Z}_2$ orbifolded conifold. Since this can as well be thought of as the $\mathcal{G}_{22}$ generalized conifold, it has an interval dual as well as a box dual. Both of them are displayed in Fig.5.7 for various values of the B-fields. The gauge group is $SU(M)^4$. We should see 3 B-fields governing the relative sizes of the gauge couplings. According to our scenario this will correspond to one relative brane position B and the sizes of two diamonds b_1 and b_2. In the interval picture $b_{1,2}$ will be the distance between $\mathrm{NS}_{1,2}$ and $\mathrm{NS'}_{1,2}$ while B is the distance between the center of masses of the two NS NS' pairs, denoted as circles in Fig.5.7. Take the circle to have circumference 2 and the torus to have sides 2 and 1. Since B-fields (=inverse gauge couplings) are length on the interval and areas on the torus, in these units the area of a given gauge group on the torus should have the same numerical value as the corresponding length on the circle (total area=total length=2). The third picture in Fig.5.7 shows $B = 1$ $b_1 = b_2 = 1/2$. Both sides have 4 gauge groups of size 1/2.

It is easy to identify in both theories the point where all gauge couplings are equal, the point where all B-fields are zero (the most singular point) and the point where the setup looks like two separated conifolds. Similarly, for all positive values of the b_i and of B we can read off the gauge theory from the diamonds, just using the standard A and B fields, representing the diamonds as rhombes of area b_i. However from the interval it is clear, that we can also pass an NS' brane through an NS brane, performing Seiberg duality on the gauge theory and simultaneously changing the sign of one of the b_i fields [62, 83]. If we set $b_1 = b_2 = -1/2$ the picture looks the same as for $b_1 = b_2 = 1/2$. The overall sign does not matter. However the sixth picture of Fig.5.7 shows a setup where the signs of the b_i differ. We should assign our diamonds an orientation in order to be able to address this issue. This orientation assigns whether the A or the B fields point outward or inward, the other doing the opposite. The rules we have introduced are valid for the case that all orientations are equal. The situation with opposite orientations is slightly more complicated. The rules can be determined by comparing with the interval. Whenever the arrows point around the closed rectangle we write down a quartic superpotential. If diamonds with different orientation touch, we will have to introduce additional 'meson' fields with cubic superpotential (see the 6th picture in Fig.5.7). Since this inversion of orientation should correspond to Seiberg duality in the field theory, we basically found this way a realization of $N = 1$ dualities in the box and diamond picture! It would be clearly interesting to pursue this point further, for example by studying theories with orientifolds. This may give us a hint of a brane realization of Pouliot like dualities [84] and spinors, since it is easy to realize the magnetic side of these theories in the box and diamond picture using orientifolds.

Last but not least we should be able to see the same gauge groups in the third T-dual realization as well, that is from D6 branes wrapping the 3-cycles of the deformed generalized conifold geometry (5.11)

$$xy = \sum_{i,j=1}^{k,l} m_{ij} u^i v^j$$

in the same spirit as above following [82]. It would be interesting to work this out and see if some properties of the gauge theory can be better understood in this language.

5.7 Mirror Branes and Domain Walls

5.7.1 The Mirror Branes

The D3 brane probe we have been considering so far maps to a D4 brane on the interval and a D5 brane in the box respectively. We identified the corresponding gauge theories above. For a special subclass of models we were considering we can actually perform both. These geometries are those whose toric diagram is given by two rows of k points. Viewing them as $\mathbb{Z}_k$ orbifolded conifolds $\mathcal{C}_k$, they (or better their blowup) turn into a box with 1 NS' and k NS under T_V. We can as well describe them as a $\mathcal{G}_{kk}$ generalized conifold and hence T_U dualize them into an interval with k NS and k NS' branes. According to our philosophy these two ways of realizing the gauge theory should actually be mirror to each other! We turned one HW setup into its 'mirror branes'.

Now we can try to solve these gauge theories via the lift to M-theory. Interestingly enough, the intervals lift via SUSY 2-cycles in $\mathbb{R}^6$ while the boxes lift via SUSY 3-cycles [85] in $\mathbb{R}^6$. So for every 3-cycle we should find a dual 2-cycle encoding the same information and vice versa.

Putting Together Intervals and Boxes

Above we obtained an $N = 1$ $d = 4$ gauge theory from intervals in type IIA and boxes in type IIB setups respectively. Of course we can as well build a box in type IIA or an interval in type IIB in order to obtain odd dimensional gauge theories with 4 supercharges. The singular point should correspond to having all NS branes coinciding.

We can do both together, that is put branes on the box and the interval simultaneously, provided we put in enough NS branes so that we can open up both a box and an interval. From the dual geometry point of view this corresponds to considering manifolds with both complex and Kähler deformations turned on simultaneously. An interesting example is type IIA with NS 012345, NS' 012389, D4 01236, D4 01248. It is easy to convince oneself that this now lifts to M-theory via a SUSY 3-cycle in G_2[*11]. That is, we now break another half of the SUSY, leaving us with 2 unbroken supercharges, or $N = 1$ in $d = 3$. Note that this gauge theory actually only lives on the boxes, since the interval theory is 4d while the box theory is 3d. Things become more interesting if we compactify the x^3 direction. In this case both the interval and the box give 3d gauge theories.

These brane setups fit nicely into the framework of brane cubes. These also lead to 2 supercharges. They lift via G_2 and $SU(4)$ 4-cycle respectively and are dual to probes on $SU(4)$ and G_2 orbifolds. Now we have a 3rd kind of brane setup in this league, which lifts via G_2 3-cycle and should probably also be dual to probes on a G_2 singularity.

Note that from the point of view of the four-dimensional theory on the D4 branes

[*11] This is a supersymmetric cycle in $\mathbb{R}^7$ which is calibrated using a G_2 invariant three-form.

on the interval, the D4 branes on the box look like domain walls (they are localized in x^3). This is nice, since Witten argued [73] before that domain walls in $d = 4$, $N = 1$ gauge theory should be associated to M5 branes on G_2 3-cycles.

5.8 Summary

Let us briefly summarize the main results of the chapter. For two classes of non-compact (complex 3-dimensional) Calabi-Yau spaces we constructed the T-dual NS brane configurations. Specifically blowups (resp. deformations) of orbifolded conifold singularities, denoted by $\mathcal{C}_{kl}$, are T_V dual to boxes (resp. intervals) of NS branes, whereas blowups (resp. deformations) of generalized conifold singularities, called $\mathcal{G}_{kl}$, are T_U dual to intervals (resp. boxes) of NS branes. Since the composition of T_U and T_V corresponds to a T-duality with respect to three isometrical $U(1)$ directions of $\mathcal{M}$ (resp. $\mathcal{W}$), it should not come as a surprise that $\mathcal{C}_{kl}$ and $\mathcal{G}_{kl}$ are actually mirror pairs. The Kähler (resp. complex structure) parameters of the geometric singularities correspond to positions of the NS branes in the dual brane picture. Moreover the conifold transition for the non-compact Calabi-Yau spaces $\mathcal{C}_{kl}$ or $\mathcal{G}_{kl}$ via shrinking 2-cycles and blowing up 3-cycles precisely corresponds to the transition between the box and interval theory or vice versa, by first moving all NS branes on top of each other and then removing them into different directions. All this is summarized in Fig.5.8 below.

Constructing gauge theories from branes, the geometric singularities as well as the NS brane configurations serve as backgrounds, which are probed by a certain number of D branes. We have seen that the "mirror map" does not change the corresponding gauge theories. At the conifold point some of the gauge couplings go to infinity.

In order to establish the duality between conifold singularities and brane boxes we had to generalize the concept of brane boxes by also including brane diamonds. We formulate rules for deriving the matter content of the gauge theories living on boxes and diamonds. Along a baryonic branch of the gauge theory, which corresponds to partially resolving the conifolds $\mathcal{C}_{kl}$ to the orbifold singularities $\mathcal{O}_{kl}$, we recovered the

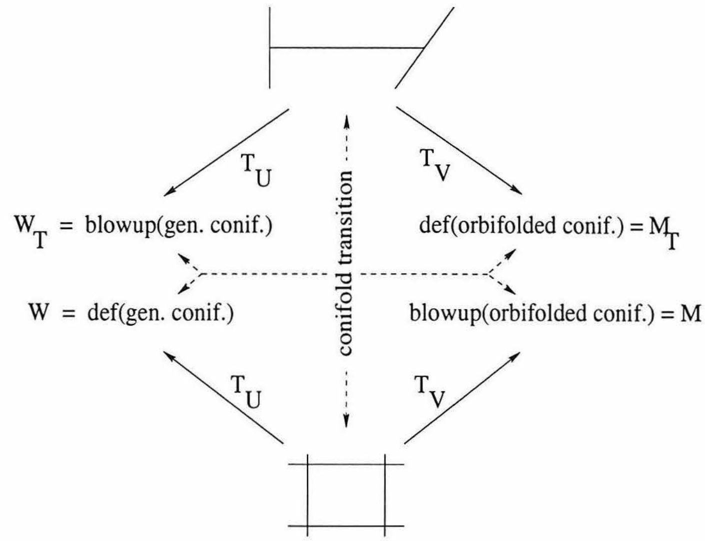

Fig.5.8. The proposed picture.

orbifold gauge theories from our general rules.

Blowups (or deformations) of certain geometries, namely $\mathcal{C}_{1k} \equiv \mathcal{G}_{kk}$, allow both for a dual brane box as well as for a dual interval description. It follows that the corresponding gauge theory on the interval and on the brane box are mirror to each other. This observation could be useful for the investigation of the non-perturbative quantum dynamics of these kind of $N = 1$ gauge theories: namely for every supersymmetric 2-cycle which describes the dynamics of the interval theory embedded in M-theory, there should exist a mirror supersymmetric 3-cycle for the brane box theory also embedded in M-theory. It would be interesting to work out this mirror map between 2- and 3-cycles explicitly. Moreover one could expect that due to quantum corrections the physics of the gauge theories at the conifold point is not as singular as in the classical description we have discussed. Finally, it would be also interesting to relate the brane constructions of $N = 1$ supersymmetric gauge theories, considered here, to the geometric engineering approach, where various branes are wrapped around non-trivial cycles of Calabi-Yau 4-folds or manifolds of G_2 holonomy.

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