QUANTUM AND INFLATIONARY COSMOLOGY WITH HIGHER DERIVATIVE GRAVITY

Thesis by

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Posvećeno mojoj majci i očuhu,

uspomeni na moju baku, tatu, nanu i deku,

- koji nikada nisu razumeli šta to radim,

ali su me pustili da idem svojim putem

jer su me voleli.

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ABSTRACT

The subject of this thesis is the description of the Very Early Universe, from the Big Bang to the beginning of the radiation–dominated Friedmann– Robertson–Walker era. We examine a pure gravity inflationary model for the Universe which is based on adding an εR^2 term to the usual gravitational Lagrangian ("improved Starobinsky model"). We find the classical inflationary solution essentially independent of initial conditions. The model has only one free parameter, which is bounded from above by observational constraints on scalar and tensorial perturbations and from below by both the need for standard baryogenesis and the need for galaxy formation. This requires $10^{11} GeV < \varepsilon^{-1/2} < 10^{13} GeV$.

The model is interpreted as a Chaotic Inflationary model, with initial conditions for classical evolution being generated by the quantum fluctuations in metric and curvature in Very Early Universe. We discuss those fluctuations using a particular solution of the Wheeler–De Witt equation and find that the inflationary phase is a highly typical event.

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INTRODUCTION

The inflationary universe model,[1,2] in which the universe has undergone a long period of exponential expansion, has successfully explained many problems in the standard Friedmann cosmology. A particularly attractive feature is that the model provides a mechanism to generate the small-scale density fluctuations in the universe which are needed as seed for galaxy formation.[3,4] They are the zero-point fluctuations of the quantum fields which get pushed into the classical regime by the large expansion.

In the standard picture of inflation this exponential expansion of the universe is driven by the false vacuum energy density of a Higgs field which acts like an effective cosmological constant in the Einstein equations. Many different underlying particle physics theories have been proposed. The most popular of these are the Coleman-Weinberg model,[5] Witten's model with a logarithmic potential,[6] and the N=1 supegravity version of Nanopoulos et al., and Linde.[7]

These proposals, though, are not without their problems. First, one has to typically introduce a scalar "inflaton" field which is postulated especially for the purpose. This makes the whole scenario less plausible in that it is less natural. Second, to achieve a large enough inflation, suitable reheating after the inflation, and to make the material fluctuations small enough to be consistent with observation, relevant couplings or masses in the suggested models all have to be fine–tuned in one way or another. An even more serious problem has been pointed out by Mazenko, Unruh and Wald [8]. A quantum field which is violently fluctuating in its high–temperature symmetric state may not settle into the false vacuum state as the universe cools. This then may invalidate the whole picture of vacuum–energy driven inflation. Although the problem might be circumvented again by fine–tuning the parameters involved,[9] it is reasonable to assert that the idea of inflation is very attractive while the "standard" models which generate the inflationary phase by a false–vacuum energy density are less satisfying.

Is it possible to inflate the universe by a different mechanism? Linde[10] has proposed in his chaotic inflation scenario that the inflation may be a direct result of large fluctuations of quantum fields in the very hot primordial universe. In the Planck regime, a scalar field ϕ will tend to be excited to large values so that its energy density inside some domain will be of order Planck. If ϕ has a very flat potential, i.e., a small "restoring force," it will remain roughly at the fluctuated value for a comparatively long time and hence drive an essentially exponential expansion. Linde has shown that in a $\lambda\phi^4$ theory, there will be a classically tractable sufficient inflation when $\lambda < 10^{-2}$ (for more details see Linde, Ref. [2]). However, two new questions immediately appear which a cosmology based on chaotic inflation must answer: what is the underlying particle model and what determines the initial fluctuations? Without these one has neither a complete nor a realistic model of chaotic inflation. This is one thread leading to the present work.

A second thread leads from the fact that within different frameworks one is repeatedly led to consider an action containing terms of quadratic or higher order in the curvature tensor. We will discuss this point more fully in Section 6. It is anyway important to understand the implication of these higher derivative terms on the evolution of the early universe. In this work we will restrict our attention to terms which are quadratic. They can be written as $\alpha(Riemann)^2+\beta(Ricci)^2+\gamma R^2=\varepsilon R^2+\zeta(Weyl)^2+\eta(Euler)$. When we consider a Robertson–Walker metric (homogeneous and isotropic universe), [11]

$$ds^2=-dt^2+a^2(t)[\frac{dr^2}{1-\kappa r^2}+r^2(d\theta^2+\sin^2\theta d\phi^2)], \tag{1.1}$$

(here $\kappa=+1,-1$, or 0, although, unless otherwise indicated, we will be studying the case $\kappa=0$). This metric is conformally flat so that the Weyl tensor term vanishes. The effective gravitational Lagrangian density yielding the evolution of the universe is then given by

$$L=R+\varepsilon R^2. \tag{1.2}$$

The evolution equation for R determined by (1.2) can be written as

$$\ddot{R} + 3H\dot{R} + \frac{1}{6\varepsilon}R = 0, \tag{1.3}$$

where the dot denotes a coordinate time derivative ($= d/dt$) and H is the Hubble parameter ($H = \dot{a}/a$).

Thus R behaves like a damped harmonic oscillator with the restoring force given by $1/6\varepsilon$. If ε is large, the potential is flat and R takes up the role of the inflation–driving field. The aim of this paper is to study the cosmology based on this model. We show the range of initial data and the allowed value of ε so that inflation can be realized in this curvature–squared model in a manner consistent with observational constraints. We consider now the generic evolution of the universe to be divided into four regimes: (i) There may be a quantum phase in which the universe begins its Lorentzian life, as described in the wave function picture[15] , with some expectation values for the initial conditions but continues with strong fluctuations for some time. The classical evolution only becomes meaningful after fluctuations around the average trajectory have become small. Whether this subsequent classical evolution is applicable to the universe as a whole or just an homogeneous "bubble" part of it (as in Linde's chaotic inflation picture), we expect to be answered by a proper quantum treatment at very early times; (ii) At the start of the classical evolution there will quite generally be an inflationary phase of superluminal expansion in which the Hubble parameter decays linearly in time with small slope; (iii) When the Hubble parameter hits zero and bounces back, the universe goes into an oscillation phase in which it is reheated as material fields are excited by the oscillating geometry; and (iv) There will be a final Friedmann phase in which our now matter–content–dominated model is joined to standard cosmology. We will exhibit and explain the inflationary solution, discuss reheating of the Friedmann universe, and the generation and evolution of scalar and tensor perturbations. These considerations all place constraints on the parameters of the model.

The effect of higher derivative terms on the evolution of the early universe has been studied by many authors. Zeldovich and Pitaevskii[12] have discussed the possibility of avoiding the initial singularity by including the higher order term. Starobinsky[13] has shown that the quantum corrections for a conformally invariant free field will modify the Einstein equations with higher–order terms such that an unstable de Sitter solution will result. Whitt[14] points out that the evolution equation for an $R + \varepsilon R^2$ Lagrangian admits primordial inflation. Hawking and Luttrell[15] have also shown that the wave function of the universe for this Lagrangian is peaked about classical trajectories which exhibit an exponential expansion. In fact, the initial motivation for our work comes from the desire to understand and investigate in detail the inflationary phase displayed in the numerical solution of Hawking and Luttrell's wave function.

Parallel to conducting our discussion directly in the physical spacetime we will make use of the fact that this theory can be rewritten as pure Einstein gravity plus matter in a conformal spacetime. Whitt[14] has shown that by a transformation, $\tilde{g}_{\mu\nu} = (1+2\varepsilon R)g_{\mu\nu}$, we can discuss the theory as Einstein gravity described by $\tilde{g}_{\mu\nu}$ plus a scalar field, R (which is the scalar curvature in the physical space), with minimal coupling to gravity by means of the equation

$$\tilde{R}_{\tilde{\mu}\tilde{\nu}} - \frac{1}{2}\tilde{g}_{\tilde{\mu}\tilde{\nu}}\tilde{R} = 8\pi G \tilde{T}_{\tilde{\mu}\tilde{\nu}}(R), \tag{1.4a}$$

where

$$\tilde{T}_{\tilde{\mu}\tilde{\nu}} = \frac{6\varepsilon^2}{8\pi G(1+2\varepsilon R)^2}\left[\partial_{\tilde{\mu}}R\,\partial_{\tilde{\nu}}R - \tilde{g}_{\tilde{\mu}\tilde{\nu}}\left(\frac{1}{2}\partial^{\tilde{\sigma}}R\,\partial_{\tilde{\sigma}}R + \frac{R^2}{12\varepsilon}\right)\right]. \tag{1.4b}$$

Here, the scalar field, R, can be given an action

$$S[R] = \int d^4x\,\sqrt{-\tilde{g}}\;6\varepsilon^2[1+2\varepsilon]^{-2}\left[\partial_{\tilde{\sigma}}R\,\partial^{\tilde{\sigma}}R + \frac{R^2}{6\varepsilon}\right]. \tag{1.5}$$

In this conformal picture, as we are working with standard Einstein gravity, we already have some known tools which provide for us both insight and a good check on the less familiar behaviour of the full fourth–order model. We will appreciate its full power in evaluating scalar and tensor perturbations.

In Section 2 we consider the classical evolution of a flat ($\kappa = 0$) Robertson–Walker universe under the influence of an R^2 term in the effective Lagrangian. In Section 3 we then treat in greater detail the exit from the inflationary phase, the reheating of the universe, and the sub–sequent joining to Friedmann behaviour. Next, in Sections 4 and 5, we estimate the generation of gravitational waves and scalar perturbations in the model. In Section 6 we display some present constraints on and possible origins for ε. Finally, conclusions are presented in Section 7.

Throughout this work we use units in which $\hbar = c = k_B = 1$. We measure all quantities in Planck units so that the gravitational constant, G, is equal to 1 ${l_{Pl}}^2$ (where l_{Pl} denotes the Planck length).

II.

CLASSICAL EVOLUTION

We begin discussion of the universe and its evolution at the time when it emerges from the Planck era. The universe would then be filled with relativistic particles of violently fluctuating energy density and its space–time geometry, too, would be violently fluctuating. However, a region not too big compared to the Planck size could be approximately isotropic and homogeneous and could then be described by the Robertson–Walker metric (1.1). For simplicity we consider only the case $\kappa = 0$. We follow the evolution of this small region with the classical equations of motion derived from the Lagrangian density (1.2).

It is straightforward to write down the field equation for the effective gravitational Lagrangian density (1.2) with a cosmological constant term and matter field terms added:[14,15]

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}+2\varepsilon[R(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R)+R_{;\kappa\lambda}(g^{\kappa\lambda}g_{\mu\nu}-\delta^{\kappa}_{\mu}\delta^{\lambda}_{\nu})] \qquad (2.1)$$

$$=8\pi G T_{\mu\nu}$$

For the most part in this work we will set $\Lambda = 0$ and we will always use a perfect cosmological fluid expression for $T_{\mu\nu}$

$$T_{\mu\nu}=(\rho+p)u_\mu u_\nu+\rho g_{\mu\nu}, \tag{2.2}$$

where $p = \rho/3$ (a relativistic equation of state) and $\vec{u} = \partial/\partial t$ (comoving 4–velocity). It is simple to verify that the left–hand side of (2.2) is divergence–free so that energy–momentum conservation is still given by

$$T^{\mu\nu}{}_{;\nu}=0, \tag{2.3a}$$

which implies

$$\rho \sim \frac{1}{a^4}, \tag{2.3b}$$

as in the standard Einstein cosmology.

There are only two nonvacuous field equations. The $t-t$ component of (2.1) can be written as

$$\dot{R} = \frac{1}{12}\frac{R^2}{H} - RH - \frac{H}{2\varepsilon} + \frac{4\pi}{3}\frac{G}{\varepsilon}\frac{\rho}{H}, \tag{2.4}$$

and the contraction of (2.1) gives

$$\ddot{R} + 3H\dot{R} + \frac{1}{6\varepsilon}R = 0. \tag{2.5}$$

The relations of R and H to the scale factor $a(t)$ are given by

$$R = 6\dot{H} + 12H^2 \tag{2.6}$$

and

$$H = \dot{a}/a. \tag{2.7}$$

Equations (2.4), (2.5), (2.6), and (2.7) are then a complete set for describing the classical evolution of the universe.

Next we notice that with ρ given by (2.3b), Eq. (2.4) is the first integral of (2.5). Therefore the system we have left is equivalent to a third order differential equation in the scale factor $a(t)$. We set the time coordinate origin so that our analysis begins at t = 0, which is the time the classical evolution begins to make sense. A complete set of initial conditions for the system is then given by ρ_i, a_i, H_i, and R_i (the subscript i will be used to denote quantities at t = 0). We first assume for simplicity the matter term on the right–hand side of Eq (2.4) to be negligible (that is, $\rho_i \approx 0$); we shall insert its contribution at a latter point. Now the initial size, a_i, of the small homogeneous domain does not enter the dynamical equations and it relates coordinate length to physically measured length at $t = 0$ (the equation for $a(t)$ is trivially integrated in terms of $H(t)$). We will take ε to be a free parameter since before appeal to a higher theory it can be regarded as a new fundamental constant subject to experimental verification. So, one way to phrase the question that this paper addresses is: what are the allowed ranges of ε and the initial data, H_i and R_i, so that the non–Einstein term will produce a sensible inflation, give sufficient expansion to solve the horizon and flatness problems, command an exit from the inflationary phase, yield a reheating temperature high enough not to thwart standard baryogenesis but low enough to avoid the GUT phase transition and its associated monopole problem, and finally deliver the correct material and gravitational perturbation spectrum and magnitude?

We study first the classical evolution by means of equations (2.4)–(2.7). To ensure the classical validity of the evolution we will think of H_i and R_i as being both less than or of order of the Planck scale. We may combine equations (2.4) and (2.6) to derive a master equation for the classical evolution with zero matter content:

$$\ddot{H}-\frac{1}{2}\frac{1}{H}\dot{H}^2+3H\dot{H}+\frac{1}{12\varepsilon}H=0. \tag{2.8}$$

The remaining dependence on the parameters H_i, R_i, and ε can then be discussed as follows:

(A) $\varepsilon > 0$, $R_i > 0$, and $H_i > 0$.

We will show that this is the only case that will be of interest; therefore we will consider it in detail:

(i) First, we look at the case where R starts at roughly its maximum value, that is $\dot{R}(t=0) = 0$. Then Eq. (2.4) relates R_i and H_i by

$$R_i=6H_i^2\left[1+\sqrt{1+\frac{1}{6\varepsilon H_i^2}}\right]. \tag{2.9}$$

The typical behaviour of $H(t)$ for this case is shown in Fig. 1. There is a long phase in which H decreases linearly in time with a small slope. This slope may be estimated from Eq. (2.8). For $\varepsilon \geqq 1$ and $H \geqq 1/6\sqrt{6\varepsilon}$ we have

$$\dot{H} \approx -\frac{1}{36\varepsilon}. \tag{2.10}$$

Hence the total expansion in the scale factor of the universe after this linear near–de Sitter phase is given by

$$a(t_{H\approx 0}) = a_i e^{18\varepsilon H_i^2}. \tag{2.11}$$

To obtain a cosmologically significant expansion, say a factor of e^{75} (cf. Linde, Ref 2.), we see that we need only to have $\varepsilon H_i^2 \geqq 4.2$, a perfectly natural value in our picture. This explicitly is the sought–for inflation in the model. When H finally gets small, as shown in Fig. 1, it switches from the linearly decaying phase into a damped oscillation. This oscillation will be seen to reheat the universe.

(ii) What if $R_i \gg 6H_i^2(1+\sqrt{1+(1/6\varepsilon H_i^2)})$? From equations (2.4) and (2.6) it is clear that both R and H will increase rapidly:

$$R \approx \frac{1}{12}\frac{R_i^2}{H_i}t \tag{2.12a}$$

and

$$H \approx \frac{1}{6}\int R\,dt \approx \frac{1}{144}\frac{R_i^2}{H_i}t^2. \tag{2.12b}$$

Therefore $12H^2$ will catch up with R at t_m:

$$t_m \approx 5.2 \left(\frac{H_i}{R_i^2} \right)^{\frac{1}{3}} \tag{2.13}$$

and

$$H_m \equiv H(t_m) \approx 0.2 \left(\frac{R_i^2}{H_i} \right)^{\frac{1}{3}}. \tag{2.14}$$

Then by Eq. (2.6), $\dot{H}$ will change sign and then go into the linear decaying phase of the previous case (i). The total expansion accumulated during the initial rapidly rising period is negligible:

$$\int H dt \approx \frac{1}{400}.$$

We can thus perfectly well regard $H(t_m)$ and $R(t_m)$ as the initial values from which the linear phase begins.

(iii) If $R_i \ll 6H_i^2(1+\sqrt{1+(1/6\varepsilon H_i^2)})$, then

$$\dot{R} \approx -\frac{H}{2\varepsilon} \tag{2.15a}$$

and

$$\dot{H} \approx -2H^2. \tag{2.15b}$$

Both H and R will fall rapidly. For a typical value of H there will not be sufficient inflation before it bounces at zero. The universe will go into the oscillation phase without having been inflated.

(B) $H_i < 0$.

From Eq. (2.8) we can see that as $H \to 0$, $\dot{H}$ must also go to zero so that $\dot{H}^2/H$ is finite. Therefore, $\ddot{H}$ is negative if H approaches zero on the negative side. Thus when H hits zero it will bounce back and remain negative (on the other hand, a positive H will remain positive for the same reason). For the case $H_i<0$ the domain in consideration will always be contracting until it collapses back to the Planck regime.

(C) $R_i < 0, H_i > 0$.

From Eq. (2.6) H will be decreasing rapidly as long as R is negative. Since H has to remain positive as argued in case (B), R will have to cross zero and become positive. Again, typically the total expansion in the initial period will be negligible and we arrive back at case (A).

(D) $\varepsilon < 0$.

From Eq. (2.5) we see that when ε is negative we have an anti-restoring force. Indeed, it is easy to see that when H_i is positive, the solution will go into a linearly increasing form asymptotic to a slope

$$\dot{H} = -\frac{1}{36\varepsilon} > 0,$$

which is physically unacceptable. When H_i is negative $H(t)$ will be decreasing and will not be interesting as described under case (B).

We conclude that (i) ε has to be positive to give a finite period of inflation (note that tachyonic solutions would also exist if ε were negative[16]); (ii) to study the inflation we only have to study the case with positive H_i. The inflation occurs during a period when H decreases linearly with a slope $-1/36\varepsilon$. The total expansion factor in this phase is given by Eq. (2.11) (with H_i replaced by H_m in the case of (Aii) or (Aiii)); (iii) the linearly decaying $H(t)$ will bounce into an oscillation phase when it approaches zero. These descriptions of the evolution have been verified numerically.

Now we return to consider the contribution of the material term which we neglected in Eq. (2.4). By Eq. (2.3b) the energy density ρ of the relativistic particles evolves inversely proportional to a^4. It is then clear that once the inflationary era begins ρ will be quickly red–shifted away. Thus by Eq. (2.4) the effect of ρ on the evolution is just to give R an initial kick. That is, if ρ_i is large while H_i and R_i are of order 1, then R will quickly rise to

$$\left[\frac{16\pi}{\varepsilon}\rho_i\right]^{\frac{1}{2}}$$

in a short time. The subsequent evolution is then given by case (Aii).

It is nice to see the inflationary solution also by considering the conformal picture. In the conformal picture the classical background consists of gravity described by a scale factor $\tilde{a}(\tilde{t})$ and a spatially homogeneous scalar field $R(\tilde{t})$. They evolve according to

$$\frac{d^2R}{d\tilde{t}^2}-\frac{2\varepsilon}{1+2\varepsilon R}\left[\frac{dR}{d\tilde{t}}\right]^2+3\tilde{H}\frac{dR}{d\tilde{t}}+\frac{R(\tilde{t})}{6\varepsilon(1+2\varepsilon R)}=0 \tag{2.16}$$

and

$$\tilde{H}^2=\frac{\varepsilon^2}{(1+2\varepsilon R)^2}\left[(\frac{dR}{d\tilde{t}})^2+\frac{R^2}{6\varepsilon}\right], \tag{2.17}$$

where $d\tilde{t}=(1+2\varepsilon R)^{1/2}dt$. It is easy to see that there is a consistent solution for $\varepsilon R \gg 1$:

$$\tilde{H}=\frac{1}{2\sqrt{6\varepsilon}} \tag{2.18}$$

and

$$R(\tilde{t})=R_i-\frac{\tilde{t}}{3\varepsilon\sqrt{6\varepsilon}}. \tag{2.19}$$

Transforming back we find a linearly decreasing Hubble parameter as discussed above. The fact that in the conformal picture one has a solution as nice as de Sitter makes the prospect for further analysis very promising.

From now on we consider only the case (A) above since the other cases either lead back to it or are uninteresting and we will refer to the inflated region as "the universe." In the linear phase, we have by comparing terms in Eq. (2.8)

$$\left|\frac{1}{2}\frac{1}{H}\dot{H}^2\right| \ll \left|3H\dot{H}\right|. \tag{2.20}$$

As H decreases and becomes small the inequality sign will eventually flip and we will go over to the oscillatory phase. Equation (2.8) then becomes

$$\ddot{H}-\frac{1}{2}\frac{1}{H}\dot{H}^2+\frac{1}{12\varepsilon}H=-3H\dot{H}\approx 0. \tag{2.21}$$

If one neglects the $3H\dot{H}$ term in Eq. (2.21), the solution is given easily by

$$H(t)=\text{Const.}\times\cos^2\omega t, \tag{2.22}$$

where

$$\omega\equiv\frac{1}{\sqrt{24\varepsilon}}.$$

To do better in approximation and in particular to obtain the damping for the amplitude we have to include the presently neglected term. We do this by substituting a form for $H(t)$ which is $H=f(t)\cos^2\omega t$ and then finding $f(t)$ in the approximation that the damping is slow $\dot{f}^2/f\approx 0$, $\ddot{f}\approx 0$. The initial value of f is determined by matching on to the linear phase–

that is, requiring the two terms in (2.20) to be equal at $t = t_{os}$, the time the oscillation phase begins. When this has been accomplished we determine the following approximate analytic form for the whole classical evolution of the universe in the abscence of matter fields:

$$H(t) \approx \begin{cases} H_m - \dfrac{1}{36\varepsilon}(t-t_m) & t_m < t \leqq t_{os} \\ \left[\dfrac{3}{\omega}+\dfrac{3}{4}(t-t_{os})+\dfrac{3}{8\omega}\sin 2\omega(t-t_{os})\right]^{-1} \cos^2\omega(t-t_{os}) & t_{os} \leqq t, \end{cases} \tag{2.23}$$

where $\dot{R}(t_m) = 0$, $\omega \equiv (1/\sqrt{24\varepsilon})$, and $t_{os} = 36\varepsilon H_m + t_m - (1/(2\omega)) \approx 36\varepsilon H_m$. A simple approximate solution for $a(t)$ in the oscillatory phase can be obtained by integrating the H averaged over a few cycles:

$$a(t) \approx \begin{cases} a_m \, e^{H_m(t-t_m)+\frac{t_m}{72\varepsilon}(2t-t_m)-\frac{t^2}{72\varepsilon}} & t_m < t \leqq t_{os} \\ a_{os}\left[1+\dfrac{\omega(t-t_{os})}{4}\right]^{2/3} & t_{os} \leqq t, \end{cases} \tag{2.24}$$

where $a_{os} \equiv a_m \exp(18\varepsilon {H_m}^2 - 1/12)$.

In the oscillation phase R is essentially $6\dot{H}$ (cf. Eq. (2.6)) so that we have

$$R(t) \approx \begin{cases} 6\left[2{H_m}^2 - 1/36\varepsilon - \dfrac{H_m}{9\varepsilon}(t-t_m) + \dfrac{2}{36\varepsilon^2}(t-t_m)^2\right] & t_m < t < t_{os} \\ -6\left[\dfrac{3}{\omega}+\dfrac{3}{4}(t-t_{os})+\dfrac{3}{8\omega}\sin 2\omega(t-t_{os})\right]^{-1} \omega \sin 2\omega(t-t_{os}) & t_{os} < t. \end{cases} \tag{2.25}$$

Notice that $a(t)$ and $H(t)$ are matched at $t = t_{os}$ whereas $R(t)$ is not; otherwise we would have had an exact solution. It is important that the oscillatory phase depends only on the parameter ε for size and shape; the oscillatory solution has no dependence on the initial conditions except in the time the phase begins (at $t_{os} \approx 36\varepsilon H_m$). Eq. (2.24) shows that the scale factor expands like a matter–dominated universe: $a(t) \propto t^{2/3}$, as in the post–inflationary phase of the Starobinsky model [13] where it is known as the "scalaron" phase.

III.

REHEATING OF THE UNIVERSE DURING THE INFLATION/FRIEDMANN INTERPHASE

These oscillations will excite the material fields and reheat the universe. To estimate the reheating, we consider the simple case of a scalar field ϕ satisfying

$$g^{\mu\nu}\phi_{;\mu\nu}=0, \tag{3.1}$$

The energy density of the scalar particles produced can be easily determined. Let

$$\phi=\int d^3k\,(\hat{a}_k u_k+\hat{a}_k{}^{+}u_k{}^{*}) \tag{3.2a}$$

and

$$u_k(x,t)=\frac{1}{(2\pi)^{3/2}}\frac{1}{a}\chi_k(t)e^{ikx}, \tag{3.2b}$$

where $\hat{a}_k$ and $\hat{a}_k{}^{+}$ are the usual annihilation and creation operators. In terms of the conformal time $\eta \equiv \int_0^t a^{-1}dt$, χ_k satisfies[17]

$$\frac{d^2\chi_k}{d\eta^2}+k^2\chi_k=V\chi_k, \tag{3.3a}$$

where

$$V \equiv \frac{1}{6} a^2 R. \tag{3.3b}$$

As we shall see, the typical wavenumber k which enters our calculation is much bigger than one, whereas V is of order one at early times ($\eta \sim 0$). Therefore, the wave is essentially living on a flat background at early times and the positive frequency mode is then given by

$$\chi_k{}^{(i)} \approx \left[\frac{1}{\sqrt{2k}} e^{-ik\eta}\right]. \tag{3.4}$$

Now we follow Zeldovich and Starobinsky[18] and rewrite (3.3) as an integral equation:

$$\chi_k(\eta) = \chi_k{}^{(i)} + \frac{1}{k}\int_0^\eta V(\eta')\sin(k\eta - k\eta')\chi_k(\eta')d\eta'. \tag{3.5}$$

For a first–order iteration, we substitute $\chi_k{}^{(i)}$ in the integrand of (3.5) for $\chi_k(\eta')$. At asymptotically late times the universe will be flat again and the positive frequency mode function is again given by (3.4). Hence the Bogoliubov coefficient describing the particle production is given by

$$\beta_{kk'} = \delta_{kk'}\frac{-i}{2k}\int_0^\infty V(\eta')e^{-2ik\eta'}d\eta', \tag{3.6}$$

and the coordinate energy density $\vec{p}\cdot(\partial/\partial\eta)$ (where $p \equiv$ momentum per unit comoving volume) is given by

$$p_\eta = \frac{\pi}{(2\pi)^3}\int_0^\infty d\eta \int_0^\infty d\eta' V(\eta)V(\eta')\int_0^\infty dk \left[k\, e^{2ik(\eta'-\eta)}\right]. \tag{3.7}$$

Note that prior to the inflation $V=(1/6)a^2R$ is many orders of magnitude less than its value during the oscillating phase. Also V becomes small after the universe goes into the radiation–dominated Friedmann phase (cf. Eq. (3.17) below). Thus we can drop the surface terms in evaluating (3.7) and arrive at

$$p_\eta = \frac{1}{8}\frac{1}{(2\pi)^2}\int_0^\infty d\eta \frac{dV}{d\eta}\int_0^\infty d\eta' \frac{V(\eta')}{\eta'-\eta}. \tag{3.8}$$

We restrict attention to a case where $V(\eta) = F(\eta)\sin(k'\eta)$ and the amplitude $F(\eta)$ for the oscillation is only slowly varying in time, which is the case for our present model. Then with $k'\eta \gg 1$ Eq.(3.8) gives approximately the energy production rate

$$\frac{dp_\eta}{d\eta} \approx \frac{1}{32\pi}k'F^2(\eta)\cos^2 k'\eta \tag{3.9}$$

$$\approx \frac{k'a^4}{1152\pi}\bar{R}^2. \tag{3.10}$$

Here $\bar{R}$ denotes the scalar curvature (2.25) with a $\pi/2$ phase shift in the oscillating factor and the scale factor $a(t)$ is given by (2.24). The proper energy density, $\rho \equiv \frac{1}{a^3}\vec{p}\cdot(\partial/\partial t)$, is determined by

$$\frac{d\rho}{dt} = -4\rho H + \frac{1}{a^5}\frac{dp_\eta}{d\eta} = -4\rho H + \frac{\omega \bar{R}^2}{1152\pi}, \tag{3.11}$$

where $\omega = k'/a = 1/\sqrt{24\varepsilon}$ is the angular frequency of the oscillation in proper time and is given by (2.23).

When the final term in (3.11) vanishes at late times we have $d(\rho a^4)/dt = 0$ as radiation, having an equation of state $p = (1/3)\rho$ should give. When the $\overline{R}^2$ term is nonzero the equation of state is modified. The pressure of the particles is determined by equations (2.3a) and (3.11) to be

$$p=\frac{1}{3}\rho-\frac{\omega}{1152\pi}\frac{\overline{R}^2}{H}. \tag{3.12}$$

The complete field equations with the back reaction of the particle generation included can be estimated by putting this p and ρ (Eq. (3.11) and (3.12)) back into the field equations.[19] The $t-t$ part of Eq. (2.1) becomes

$$H^2+2\varepsilon\left[H\dot{R}-\frac{1}{12}R^2+RH^2\right] = \frac{8\pi}{3}G\frac{N}{a^4}\int_{t_{os}}^{t}\frac{\omega}{1152\pi}\overline{R}^2a^4dt \tag{3.13}$$

$$= \frac{8\pi}{3}G\rho_{\text{matter}}(t),$$

and the trace of Eq. (2.1) gives

$$\ddot{R}+3H\dot{R}+\frac{1}{6\varepsilon}R = \frac{4\pi GN}{\varepsilon}\left[\frac{\omega\overline{R}^2}{1152\pi H}\right], \tag{3.14}$$

where we have inserted a factor N which denotes the number of fields that can be excited by the cosmological oscillation (since massless conformal fields will not be excited, this N will be less than the total number of particles in the theory).

The right–hand side of Eq. (3.13), $8\pi GN\rho_{\text{matter}}/3$, can be estimated using (2.24) and (2.25). Not too long after the universe has come into the oscillation phase, say at $t-t_{os} \sim 10/\omega \approx 10\sqrt{24\varepsilon}$, we have $\rho \approx 6\times10^{-7}N/\varepsilon^2$, which corresponds to a reheating temperature of

$$T_r \approx 3\times10^{-2}/\sqrt{\varepsilon} = 4\times10^{17}\,\text{GeV}\left[\frac{\varepsilon}{1{l_{Pl}}^2}\right]^{-1/2}. \tag{3.15}$$

If ε is not too much bigger than G this particle production timescale may be shorter than the thermalization of the particle content. Still, the reheating temperature, T_r, is a useful characterization of the reheating energy (we will, however, show that ε must be indeed large). If this temperature were higher than the GUT phase transition temperature, we would be left with the monopole problem. If this temperature were too cool then baryogenesis may no longer go through. We will return to this point shortly.

When $t-t_{os} \gg 1/\omega$, the time dependence of ρ_{matter} is given by

$$\rho_{\text{matter}}(t) \approx \frac{3}{5}\frac{32}{1152\pi}\frac{N\omega^3}{(t-t_{os})}.$$

If we now neglect the back–reaction, H^2 at late times is given by (2.23) to be

$$H^2 \sim \frac{4}{9}\frac{1}{(t-t_{os})^2}. \tag{3.16}$$

Hence at $(t-t_{os}) \approx 1200\ \varepsilon^{3/2}/GN$ the term on the right–hand side of (3.11) will be comparable with H^2 and the matter produced will begin to have a significant dynamic effect on the evolution of the universe. The solution of Eq. (3.13) gradually goes over to a radiation–dominated Friedmann expansion with

$$H \propto \frac{1}{2t}\ , R=0\ ,\ a \propto t^{1/2}\ ,\ \text{and}\ \rho \propto 1/t^2. \tag{3.17}$$

However, the transition from the oscillation phase to the radiation–dominated phase will be slow even after $8\pi G\rho_{\text{matter}}/3$ is comparable to H^2, as a numerical integration of Eq. (3.13) shows. We estimate the time it takes for the Friedmann phase to begin by taking roughly 10 times this value, so that the time the Friedmann phase begins is given by $t_F \geqq t_{os} + 12000\ \varepsilon^{3/2}/GN$. The energy density will then be

$$\rho(t_F) \leqq 4\times10^{-9} GN^2/\varepsilon^3. \tag{3.18}$$

And the Friedmann Universe thus begins with the temperature

$$T_F \leqq 1\times10^{17}\ \text{GeV}\left(\frac{\varepsilon}{1\ l_{Pl}{}^2}\right)^{-3/4} N^{1/4}. \tag{3.19}$$

Notice that the ways T_r and T_F depend on ε are different. It is clear that any constraint on T_F will not be significant. There are important constraints on T_r, however. It must be higher than 10^{10-12} GeV so that gauge and Higgs bosons can be created and baryogenesis can proceed in the usual way, but lower than any GUT phase transition temperature $\sim 10^{16}$ GeV so that the monopole problem can be avoided.[2] Eq. (3.15) then requires ε to be in the range

$$10^3\, l_{Pl}{}^2 < \varepsilon < 10^{15-12}\, l_{Pl}{}^2. \tag{3.20}$$

These bounds will be tightened when we consider perturbations generated in the inflationary phase. We summarize the classical evolution of the universe as follows:

(i) A homogeneous and isotropic region near the Planck time with a Hubble parameter H_m will expand with a linearly decreasing H for a total expansion factor $\sim \exp(18\varepsilon H_m{}^2)$.

(ii) Particles will be created during the oscillation phase. The total expansion factor during this time will be

$$\exp\left[\int_{t_{os}}^{t_{os}+12000\varepsilon^{3/2}/GN} H\,dt\right] \approx 70\left[\frac{\varepsilon}{NG}\right]^{2/3}.$$

(iii) The universe will then go over to a radiation–dominated Friedmann phase with the temperature given by Eq. (3.19). To red–shift this to the present value of 3 ^0K we must have an expansion factor

$$\left[\frac{T_F}{3\,{}^oK}\right] \approx \left[5\times10^{29}\left(\frac{G}{\varepsilon}\right)^{\frac{3}{4}} N^{1/4}\right].$$

Therefore, the total expansion since the Planck era is obtained by multi– plying the expansion factors under (i), (ii), and (iii) and it should be greater than the present horizon size, where $1/H_0 \sim 10^{55}\, l_{Pl}$. This requires in terms of the expansion factor

$$e^{18\varepsilon H_m{}^2} \geqq 2\times10^{25}\left(\frac{H_m}{1\ l_{Pl}{}^{-1}}\right) \qquad (3.21)$$

(the dependence on N is very weak, so we have set $N\approx100$ as a typical value). The expansion factor is depends very sensitively on εH_m^2, so that unless the initial parameter, H_m, is fine–tuned, the left–hand side of Eq. (3.21) is very likely to be very much bigger than 10^{25}. We thus expect to have much more inflation than is necessary.

IV.

GRAVITATIONAL WAVE GENERATION

It is crucial to study the generation of gravitational waves in the model since it is well known that inflation close to the Planck time tends to yield excessive gravitational wave generation.[20] In the transverse–traceless gauge, a gravitational wave can be expressed in terms of a scalar amplitude h. For a wave with wavenumber k the metric can be written as

$$ds^2 = -dt^2 + a^2(t)[\delta_{ij} + he_{ij}]dx^i dx^j, \tag{4.1}$$

where $i,j = 1,2,3$ and e_{ij} is the polarization tensor satisfying both the transverse condition $e_{ij}k^j = 0$ and the traceless condition $e^i_i = 0$. The field equation (2.1) then reduces to

$$\ddot{h} + [3H + \frac{1}{6}\frac{\varepsilon R^2}{(1+2\varepsilon R)H}]\dot{h} - \frac{1}{a^2}\partial_i^2 h = 0. \tag{4.2}$$

The second term in the bracket is due to the presence of the εR^2 term in the gravitat,ional Lagrangian. Other than this term $h(t)$ satisfies the same equation as an ordinary scalar field in a Robertson–Walker background. Since Eq. (4.2) is second order in the spacetime derivatives, the quantization can proceed in the usual way. We construct S from which (4.2) can be derived:

$$S=\int d^4x\sqrt{-g}L\,, \tag{4.3a}$$

where

$$L=(1+2\varepsilon R)g^{\mu\nu}\partial_\mu h\partial_\nu h \tag{4.3b}$$

(here we use the background metric of equation (1.1) (with $\kappa=0$) to compute the quantities $\sqrt{-g}$, $g_{\mu\nu}$, and R). The quantization condition is then

$$[h(t,x),\frac{\partial L}{\partial\dot{h}}(t,y)]=iG\frac{\delta^3(x-y)}{a^3}. \tag{4.4}$$

For L given by Eq. (4.3b) we have

$$\left[h(t,x),\dot{h}(t,y)\right]=iG\frac{\delta^3(x-y)}{a^3(1+2\varepsilon R)} \tag{4.5}$$

(note that the additional factor of $1/(1+2\varepsilon R)$ in the normalization enters because of the εR^2 term). It is straightforward to check that the evolution equation preserves this commutation relation.

If h is composed of modes of more than one wave vector, it is written as

$$h(t,x)=\int d^3k\left[\hat{a}_k\, h_k\, e^{ikx}+\hat{a}_k{}^+ h_k{}^*\, e^{-ikx}\right], \tag{4.6}$$

with the creation and annihilation operators satisfying the usual relations:

$$\left[\hat{a}_k, \hat{a}_{k'}^{+}\right] = \delta^3(k-k') \text{, etc.} \qquad (4.7)$$

Then equations (4.4) and (4.5) determine the normalization for (4.6):

$$h_k \dot{h}_k^{*} - h_k^{*} \dot{h}_k = \frac{iG}{(2\pi)^3 a^3 (1+2\varepsilon R)}. \qquad (4.8)$$

The evolution equation for h_k is then

$$\ddot{h}_k + \left[3H + \frac{2\varepsilon \dot{R}^2}{H(1+2\varepsilon R)}\right]\dot{h}_k + \frac{k^2}{a^2} h_k = 0. \qquad (4.9)$$

Now we consider a wave with wave–length equal to or smaller than the present horizon size, $1/H_0$. If the expansion factor in the linear phase is much greater than the minimum requirement (3.21) (cf. also text following (3.21)), the wave number k of these waves will be much greater than 1. On the other hand, the term inside the brackets in equation (4.9) is of order 1 as $t \to 0$ and is thus negligible compared to k/a. Once again we are considering a wave evolving on essentially a flat background. Thus, the initial mode function can be chosen as

$$h_k = h_k^{(i)} e^{-ik\int \frac{dt}{a}}. \qquad (4.10a)$$

And the normalization $h_k^{(i)}$ is determined by Eq. (4.5) to be

$$|h_k{}^{(i)}| = \frac{\sqrt{G}}{\sqrt{2k}\,(2\pi)^{3/2} a(1+2\varepsilon R)^{1/2}}.$$

In the linear phase $a(t)$ is rapidly increasing so that the wave is soon well outside the horizon (i.e., $k \ll aH$) and the third term in Eq. (4.9) becomes negligible so that h_k approaches a constant. This constant can be estimated by extrapolating (4.10) to the horizon crossing time. h_k then remains at this value until it finally reenters the horizon in the Fried–mann phase. This "freezing out" of the gravitational waves often goes by the name of amplification[21] since it is amplification above the adiabatic behaviour (Eq. (4.10)). The amplitude of the gravitational wave of wave number k at reentry is thus given by

$$A_k = (2\pi k)^{3/2} |h_k(t_{hc})| = \frac{\sqrt{G} H(t_{hc})}{\sqrt{2}(1+2\varepsilon R(t_{hc}))^{1/2}}, \tag{4.11}$$

where t_{hc} denotes the initial horizon crossing time in the linear phase. At that time $\dot{H} \sim -1/36\varepsilon$, so we have

$$\varepsilon R(t_{hc}) \sim 12\varepsilon H^2(t_{hc}). \tag{4.12}$$

We assume waves which reenter the horizon at late times have left the horizon during the inflationary epoch so that $2\varepsilon R(t_{hc}) \gg 1$ and

$$A_k \approx \frac{1}{\sqrt{2}} \frac{\sqrt{G}}{\sqrt{24\varepsilon}}. \tag{4.13}$$

Notice that the spectrum is flat. Comparing to the $\Delta T/T$ limit for the microwave anisotropy[20] we have

$$A_k \approx \left|\frac{\Delta T}{T}\right| \leq \sqrt{7}\times 10^{-4} \tag{4.14}$$

or

$$\varepsilon \geqq 3\times 10^5\, l_{Pl}{}^2, \tag{4.15}$$

which tightens up the bound (3.20) somewhat. Unlike usual inflationary models, it turns out that the microwave measurements constrain not the value of $H(t_{hc})$ but rather the value of ε. This is due to the fact that the quantization condition (4.5) is modified by the curvature–squared coupling.

In the conformal picture we arrive at the result quite easily due to the fact that the background is de Sitter. Note that the conformal transformation maps backgrounds, but leaves the perturbations unchanged: $A = \tilde{A}$, so we have by conventional means

$$A_k = \tilde{A}_k \approx \sqrt{4\pi G}\tilde{H}, \tag{4.16}$$

which leads to $\varepsilon > 7\times 10^6\, l_{Pl}{}^2$, agreeing with the above limit (4.15) to the order of approximation we are using.

Note that in this picture one matches the amplitude at $\tilde{a}\tilde{H} = k$, while the true perturbation crosses the physical horizon at $aH = k$. However, the difference between the two is $O(\dot{R}/R)$ so that with the same accuracy that one has obtained the de Sitter solution one can safely evaluate the perturbation at $\tilde{a}\tilde{H} = k$.

A comparison between the two pictures sheds more light in understanding why the final result does not depend on H_{hc} as in the usual case. In the standard calculation one can estimate the amplitude of the wave by requiring that the expectation value of the total energy of waves within the horizon equals the zero point energy of quantum fluctuations [21], $E = (1/2)\omega = (1/2)(k/a)$:

$$\frac{1}{H^3} < \rho > = E. \tag{4.17}$$

The amplitude of the wave at the horizon crossing is obtained by extrapolating this relation to t_{hc}, which gives $A \propto H_{hc}$. Now in conformal space where the gravity is pure Einstein and the stress tensor for gravitational waves has the usual form, one imposes

$$\frac{1}{\tilde{H}^3} < \tilde{\rho} > = \tilde{E}. \tag{4.18}$$

However, this relation is not conformally covariant as $\tilde{H} \approx \Omega^{-1/2} H$, $\tilde{E} = \Omega^{-1/2} E$, and $\tilde{\rho} = \Omega^{-1}\rho$ (here Ω is the conformal factor $= (1+2\varepsilon R)$). So, in terms of the physical H and R this relation reads

$$\frac{1}{H^3} < \rho > = \frac{E}{\Omega}. \tag{4.19}$$

Since $\Omega = (1+2\varepsilon R) \approx 24\varepsilon H_{hc}^2$ the Hubble parameter drops from the final answer.

V.

SCALAR PERTURBATIONS

As is usual in inflationary models, rather stringent constraints on the model parameters arise from present observational limits on scalar perturbations. In our model scalar perturbations are generated by quantum fluctuations in the scalar curvature around background values. A major obstacle to evaluating these fluctuations is that we are dealing with a fourth–order gravity in which the quantization is not easy. We thus avoid the problem by working in the conformal picture. In the conformal picture there is a neat separation of the degrees of freedom and the background is de Sitter, so that our result is easily obtained. From action (1.5) we obtain a field equation which is full of nonlinearities. However, we may make use of the fact that during the inflationary epoch εR is large and the field equation reduces to

$$\frac{d^2(\delta R)}{d\tilde{t}^2}+3\tilde{H}\frac{d(\delta R)}{d\tilde{t}}-\tilde{a}^{-2}(\partial_i^2\delta R)=0. \qquad (5.1)$$

That is, δR evolves like a minimally coupled scalar field. However, it really is not one, as can be seen by its stress tensor. We may use the stress–energy tensor given by Eq. (1.4b) to find the background energy density and pressure during this expansion phase (when the matter content is negligible):

$$\tilde{\rho}=\tilde{T}_{\tilde{t}\tilde{t}}=\frac{1}{64\pi G\varepsilon}\frac{1}{\left[1+\frac{1}{2\varepsilon R}\right]^2}\left[1+6\varepsilon\left[\frac{1}{R}\frac{dR}{d\tilde{t}}\right]^2\right] \tag{5.2a}$$

and

$$\tilde{p}=\frac{\tilde{T}_{\tilde{x}\tilde{x}}}{\tilde{a}^2}=\frac{-1}{64\pi G\varepsilon}\frac{1}{\left[1+\frac{1}{2\varepsilon R}\right]^2}\left[1-6\varepsilon\left[\frac{1}{R}\frac{dR}{d\tilde{t}}\right]^2\right] \tag{5.2b}$$

For a scalar wave perturbation of wavenumber k, we can find the linear and quadratic corrections to the energy density:

$$\delta\tilde{\rho}=\delta\tilde{\rho}^{(1)}+\delta\tilde{\rho}^{(2)}, \tag{5.3a}$$

where, in particular, to leading order in $\frac{1}{\varepsilon R}$ we have

$$\delta\tilde{\rho}^{(2)}=\frac{3}{16\pi G}\left[\frac{k}{\tilde{a}}\right]^2\left[\frac{\delta R}{R}\right]^2. \tag{5.3b}$$

Now we proceed in determining the mean square quantum fluctuations of δR (i.e., for waves much shorter than the horizon) from the fact that their energy is just the zero–point energy. That is,

$$\frac{1}{\tilde{H}^3}<\delta\tilde{\rho}^{(1)}+\delta\tilde{\rho}^{(2)}>=\tilde{E}=\frac{1}{2}\frac{k}{\tilde{a}}. \tag{5.4}$$

We evaluate Eq. (5.4) using (5.3) for scales much shorter than the horizon. The expectation value $<\delta\tilde{\rho}^{(1)}>$ is zero and we obtain

$$<\delta R^2> \doteq \frac{8\pi G}{3}\left(\frac{k}{\tilde{a}}\right)^2 R^2. \tag{5.5}$$

Finally, we extrapolate this to the horizon crossing of the fluctuation, where it is physically matched to the classical post–horizon–crossing amplitude by $|\delta R_{hc}|^2 = 2<\delta R^2>$, so

$$|\delta R_{hc}| \doteq \frac{1}{3}\left(\frac{2\pi G}{\varepsilon}\right)^{1/2} R_{hc} \doteq 4\left(\frac{2\pi G}{\varepsilon}\right)^{1/2} H_{hc}{}^2. \tag{5.6}$$

Now we may determine the metric potential, $\tilde{A}$, due to a classical wave of amplitude $|\delta R_{hc}|$ using the "time–lag" method of Guth and Pi:[3]

$$\tilde{A} \sim \frac{\delta(\tilde{a}^2)}{\tilde{a}^2} = 2\tilde{H}\,\delta\tilde{t} = 2\tilde{H}\,\left|\frac{dR}{d\tilde{t}}\right|^{-1} |\delta R_{hc}|. \tag{5.7}$$

If we now plug (5.6) into (5.7) we obtain

$$\tilde{A} \sim \frac{2}{3}\left(\frac{2\pi G}{\varepsilon}\right)^{1/2} (18\varepsilon H_{hc}{}^2). \tag{5.8}$$

We stress that this is the asymptotic value of the metric perturbation at the end of the inflationary phase and therefore gives the magnitude of the inhomogeneities in the subsequent Friedmann evolution.

Alternatively, we proceed more cautiously using the gauge–invariant formalism of Brandenberger and Kahn.[4] We neglect the effect of sources outside the horizon so that we may use a quantity, $\tilde{\zeta}$, as a conserved gauge–invariant expression between horizon crossings:

$$\tilde{\zeta}=2/3\frac{\left[\tilde{\Phi}_H+\tilde{H}^{-1}\frac{d\tilde{\Phi}_H}{d\tilde{t}}\right]}{(1+\tilde{p}/\tilde{\rho})}+\tilde{\Phi}_H\left[1+\frac{2}{9}\left(\frac{k}{\tilde{a}\tilde{H}}\right)^2\frac{1}{(1+\tilde{p}/\tilde{\rho})}\right], \tag{5.9}$$

where $\tilde{\Phi}_H$ is now a gauge–invariant metric potential given by

$$\tilde{\Phi}_H=4\pi G\,\tilde{a}^2\nabla^{-2}\left[\tilde{T}_{\tilde{t}\tilde{t}}^{(1)}-3\tilde{a}\frac{d\tilde{a}}{d\tilde{t}}\nabla^{-2}\tilde{T}_{\tilde{t}}{}^{\tilde{j}}{}_{,\tilde{j}}{}^{(1)}\right]. \tag{5.10}$$

Here ∇^{-2} is the inverse Laplacian and $\tilde{T}_{\tilde{\mu}\tilde{\nu}}{}^{(1)}$ is the first–order perturbation in the stress–energy. We may calculate from (1.4b) to leading order in $\frac{1}{\varepsilon R}$ (that is, during the inflationary epoch after the horizon crossing so that the wave is fully classical)

$$\tilde{T}_{\tilde{t}\tilde{t}}^{(1)}=\delta\tilde{\rho}^{(1)}\approx\frac{1}{64\pi G\varepsilon}\frac{1}{\varepsilon R}\frac{\delta R}{R}. \tag{5.11a}$$

And from the stress–energy (1.4b) we find, again to leading order (this term is the same order as the first, contrary to Brandenberger and Kahn[4])

$$\tilde{T}_{\tilde{t}}{}^{\tilde{j}}{}_{,\tilde{j}}{}^{(1)}\approx\left(\frac{k}{\tilde{a}}\right)^2\left(\frac{dR}{d\tilde{t}}\right)\delta R. \tag{5.11b}$$

We have then at the horizon crossing of Eq. (5.9)

$$\tilde{\zeta}_{hc}=\left[\frac{20}{3}\frac{\delta\tilde{\rho}^{(1)}}{(\tilde{\rho}+\tilde{p})}+\frac{2\tilde{H}^{-1}\frac{d\delta\tilde{\rho}^{(1)}}{d\tilde{t}}}{(\tilde{\rho}+\tilde{p})}+\frac{3\delta\tilde{p}^{(1)}}{\tilde{\rho}}\right], \tag{5.12}$$

where $\delta\tilde{\rho}^{(1)}$ is now calculated in (5.11a) from the classical amplitude $|\delta R_{hc}|$ in Eq. (5.6). And we may find $\tilde{\zeta}_{hc}$ by putting (5.11a,b) into (5.12):

$$\tilde{\zeta}_{hc}=39\varepsilon\,|\delta R_{hc}|=\frac{26}{3}\left(\frac{2\pi G}{\varepsilon}\right)^{1/2}(18\varepsilon H_{hc}{}^{2}). \tag{5.13}$$

This fixes $\tilde{\zeta}$ at the initial horizon crossing, which quantity is roughly conserved until reentry. At the reentry of the scale of interest, the universe will be in a matter–dominated Friedmann phase ($\tilde{p}=0$) and we may use the Friedmann equation at reentry, $\tilde{H}^2=(8/3)\pi G\tilde{\rho}$, to find

$$\tilde{\Phi}_H(\tilde{t}_{\text{reentry}})=\frac{3}{2}\frac{\delta\tilde{\rho}^{(1)}}{\tilde{\rho}}. \tag{5.14}$$

We may now drop the tildes at reentry since during this late phase the conformal factor is ≈1. We have

$$\zeta_{\text{reentry}}\approx\frac{35}{9}\Phi_H(t_{\text{reentry}})\approx\tilde{\zeta}_{hc}. \tag{5.15}$$

And finally the metric potential after reentry is

$$\Phi_H(t_{\text{reentry}})\approx\frac{78}{35}\left(\frac{2\pi G}{\varepsilon}\right)^{1/2}(18\varepsilon H_{hc}{}^{2}). \tag{5.16}$$

We see that $\Phi_H(t_{\rm reentry}) \sim \tilde{A}$ (here $\tilde{A}$ is given by Eq. (5.8)) to within numerical factors. In the $H_{hc}{}^2$ factor we have some weak–scale dependence in the perturbation spectrum. In fact, the spectrum is scale–invariant up to a logarithmic term as in the case of standard inflation. We calculate this dependence in the following way – at both the initial and final horizon crossings we have in the physical space

$$a_{hc}H_{hc} = k = a_{\rm reentry}H_{\rm reentry}. \qquad (5.17)$$

We plug into this our evolution law (2.23)–(2.24), assuming, of course, that the initial horizon crossing occurs during the linear inflationary phase of the model, and we obtain

$$H_{hc} = H_0\left(\frac{k_{\rm reentry}}{k_0}\right)e^{18\varepsilon H_{hc}{}^2}(3.3\times10^{31})\left(\frac{\varepsilon}{G}\right)^{-1/12}N^{-5/12}, \qquad (5.18)$$

where H_0 is the Hubble parameter today (we use $H_0 = 50$ km/sec Mpc^{-1} = $9\times10^{-56}\, l_{Pl}{}^{-1}$ and k_0 is the scale which crosses the horizon today). From this equation we may directly exhibit the logarithmic scale dependence of the perturbations:

$$\frac{\tilde{A}_2}{\tilde{A}_1} \doteq 1 - \frac{1}{18\varepsilon H_{hc}{}^2}\ln\left(\frac{k_2}{k_1}\right). \qquad (5.19)$$

We note that Eq. (5.18) for a given scale of observational interest completely fixes the horizon–crossing Hubble parameter in terms of the model parameter ε. That is, the metric potential, $\tilde{A}$, given by Eq. (5.8), again for a given scale, is only dependent on ε. Scales which are inside the horizon today are bounded by the microwave anisotropy limit[20] so that $\tilde{A} \leqq \sqrt{7}\times10^{-4}$ and $k_{\mathrm{reentry}}/k_0 = 1$. We have

$$H_{hc}(k_0)\approx5\times10^{-6}\, l_{Pl}{}^{-1} \text{ and } \varepsilon>8\times10^{10}\, l_{Pl}{}^{2}. \tag{5.20}$$

If we want this primordial spectrum of density fluctuations to be a successful seed for galaxy formation and we use a standard value for the scalar perturbation amplitude of $\sim 10^{-4}$, then essentially our bound in (5.20) would change into an equality. If, however, we choose a different scenario[22] that is less constraining in which $\tilde{A} > 10^{-6}$ for scales $k_{\mathrm{reentry}}/k_0 \approx 150$, we have

$$H_{hc}(k_{\mathrm{cluster}})\approx2\times10^{-8}\, l_{Pl}{}^{-1} \text{ and } \varepsilon<5\times10^{15}\, l_{Pl}{}^{2}. \tag{5.21}$$

The bound (5.20) tightens up (3.20) considerably, although this number is only to be taken as very rough. Notice also that $18\varepsilon(H_{hc}(k_0))^2 \approx 52$ so that the early evolution for $H(t) > H_{hc}(k_0) \sim 5\times10^{-6}\, l_{Pl}{}^{-1}$ is irrelevant to all present observation. Putting it another way, with initial conditions of order Planck our model predicts that the universe has been expanded something like 2×10^{12} e–foldings so that the observable part of the universe will be the same for many future generations.

The scales which cross the horizon at $H_{hc} > H_b \equiv 1/(12\sqrt{2\pi G\varepsilon})$ have perturbations bigger than 1 today. From Eq. (5.20),

$H_b \lessequal 10^{-3}\, l_{Pl}^{-1}$.

If $H_m > H_b$, it simply means that one has, at scales much larger than the present horizon, fluctuations which cannot be treated in linear theory. Of course, H_m can as well be less than H_b, it is bounded below only by $H_{hc}(k_0)$. The requirement that the perturbations are small at the initial horizon crossing so that the use of perturbation theory is justified leads to only a very weak constraint on ε, well within our other bounds.

Interestingly, all these numbers tell us that there is one characteristic mass scale present in the theory as $H_{hc}(k_0) \sim \varepsilon^{-1/2} \sim 10^{-6}\, l_{Pl}^{-1}$. Perturbations in an inflationary model with a massive scalar inflaton have been considered by Halliwell and Hawking,[23] using the full wave function formalism. They found that compatability with observation restricts this mass to be less than 10^{14} GeV. As we have seen, the scalar curvature does obey an equation for a massive scalar field of mass $\sim 1/\sqrt{6\varepsilon}$. So we see that despite the unusual self–couplings present in the εR^2 theory, the physical analogy works remarkably well.

Finally, from equations (4.13) and (5.8) the neat result follows that the contribution to the microwave anisotropy of the scalar fluctuations overpowers that from gravitational waves by a factor $18\varepsilon(H_{hc}(k_0))^2 \sim 52$.

This is the reason that the bound ε is much tighter from considering scalar perturbations.

VI.

PRESENT BOUNDS ON ε AND POSSIBLE ORIGINS

It may seem that the condition $\varepsilon > 10^{11}\, l_{Pl}{}^2$ places a very large unnatural limit on ε, which in terms of Planck units it does. We would like to point out that in terms of any presently measured curvature this is really quite small.

We can manipulate the field equation (2.1) in the usual way to get

$$\Lambda - 4\pi G(\rho_S + 3p_S) = 3H_0{}^2(\sigma_0 - q_0), \qquad (6.1)$$

where

$$\rho_S \equiv \frac{3\varepsilon}{4\pi G}\left(\frac{R^2}{12} - \dot{R}H - RH^2\right) \qquad (6.2a)$$

and

$$p_S \equiv \frac{\varepsilon}{4\pi G}\left(\ddot{R} + 2\dot{R}H + \frac{R^2}{12} - RH^2 - \frac{R\kappa}{a^2}\right). \qquad (6.2b)$$

This is the usual equation which is used to set a limit on the cosmological constant Λ in terms of the presently observed H_0 (the Hubble parameter), σ_0 (density parameter), and q_0 (deceleration parameter). If we assume

$\Lambda = 0$ we thus obtain a cosmological limit on ε:

$$\varepsilon \leqq 10^{120}\, l_{Pl}{}^2. \tag{6.3}$$

Similarly one can consider a limit on ε by asserting that εR is small in all horizon–exterior curvatures encountered presently in our universe. We may use for R typically M/r^3 and go to the gravitational radius of a black hole. Then $\varepsilon R \ll 1$ requires only

$$\varepsilon \ll 10^{77} \left(\frac{M}{M_{sun}}\right)^2 l_{Pl}{}^2. \tag{6.4}$$

This, of course, is a bit of a swindle because a black hole is also a solution of εR^2 gravity[14] so that $R = 0$ and ε will have no effect. We conclude, though, that $\varepsilon = 10^{11}\, l_{Pl}{}^2$ in terms of any presently encounterable curvature is very small.

We have not as yet addressed the question of the origin of the ε term. Basically, there are three ways that one might imagine it arising. First, it may be that the full fourth–order theory should be postulated as fundamental. Such a form is naturally suggested if one thinks about gravity as the gauge theory of the Poincare group.[24] Furthermore, the εR^2 terms in the field equations violate the strong energy condition so that the initial singularity might be avoided.[12] It has also been shown that such a theory is renormalizable.[24] And the long–standing objection that it is nonunitary might not be true.[25] Secondly, it may be a remnant from some more fundamental theory. For instance, in superstring theory the Lagrangian of the point–particle limit of the 10–dimensional full string theory contains the following terms[26] :

$$R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}+aR^{\mu\nu}R_{\mu\nu}+bR^2,$$

where a and b are some constants. After compactification this leads to

$$L=R+(\frac{a+1}{3}+b)\frac{GV_6}{\phi}R^2, \tag{6.5}$$

where V_6 is the compactified volume of the six "other" dimensions and ϕ is the vacuum expectation value of a scalar field known as the dilaton. We see that this might directly give us an εR^2 behaviour even classically in the Lagrangian with a completely determined ε. However, the highly preferred values[27] for a and b are $a=-4$, $b=1$ and then $\varepsilon=0$ at the classical level and there is no R^2 term in superstring theory.

Nevertheless, ε should also be expected to arise in a third way as a quantum effective action correction to the bare theory. Here, the specific fields will contribute to its value. Indeed, this is the approach of Starobinsky.[13] As a quantum correction term ε would be given by

$$\varepsilon\sim G\ln(\frac{\Lambda_{\mathrm{high}\,cutoff}}{\Lambda_{\mathrm{low}\,cutoff}}) \tag{6.7}$$

and we would again be forced to consider a more complete theory to fix ε.

VII.

CONCLUSION

We thus conclude that at a classical level a cosmology based on the $R + \varepsilon R^2$ Lagrangian generically has an inflationary phase with a linearly decreasing Hubble parameter. The total number of expansion e–foldings during this phase is $\sim 18\varepsilon {H_m}^2$ (if $\dot{R}_i = 0$, then $t_i = 0 = t_m$). After the linear decaying phase $H(t)$ bounces off zero and the universe goes into an oscillatory phase. The total expansion is sufficient to solve the horizon and flatness problems if $18\varepsilon {H_m}^2 > 75$. At the classical level this is a natural and consistent model that relies solely on a modified gravity for its dynamics. Here, the quadratic correction to the Hilbert–Einstein action would be expected to be present somewhat independently of the specific form of the matter Lagrangian (although a value for ε must necessarily come from a higher theory).

The post–inflation oscillatory phase yields a maximal reheating temperature which is small:

$$T_r \approx 1.2\times10^{12}\ \mathrm{GeV}\left(\frac{\varepsilon}{10^{11}{l_{Pl}}^2}\right)^{-1/2},$$

in any case very much below any expected GUT phase transition, so that the monopole problem is avoided by the εR^2–driven expansion. Standard baryogenesis still may go through at this temperature but the details of this on the non–standard background will require further attention. Finally, there is a joining to a Friedmann phase at a temperature

$$T_F \leqq 6\times10^8\, GeV \left(\frac{\varepsilon}{10^{11}} \right)^{-3/4} N^{1/4},$$

when the evolution goes over to a radiation–dominated expansion.

Gravitational waves and scalar perturbations both yield bounds on the parameters of the model when we must set them small so as not to disturb the isotropy of the microwave background. The bound from gravitational waves is $\varepsilon > 10^6\, l_{Pl}{}^2$ with no restriction on H_{hc} as would occur for the standard inflationary scenario. This spectrum of gravitational waves is scale–invariant. However, the scalar perturbations give the much tighter bound of $\varepsilon \geqq 10^{11}\, l_{Pl}{}^2$, and this in turn implies that the perturbation scale which reenters the horizon today must cross the horizon at $H_{hc}(k_0) \sim 10^{-6}\, l_{Pl}{}^{-1}$, that is, at a late stage of the extremely long linear phase. The spectrum of scalar perturbations has only logarithmic dependence on the scale. If one wants baryogenesis to proceed in the usual way there is an upper bound $\varepsilon < 10^{15}\, l_{Pl}{}^2$. A similar bound follows from a comparison between galaxy formation and the microwave anisotropy in models of galaxy formation with cold dark matter.[22] However, these considerations both carry their own difficulties so that we place somewhat less emphasis here on the upper bound. Our condition of sufficient inflation requires that $H_m > 10^{-5} l_{Pl}^{-1}$; that is, we find that our model would work for essentially all reasonable initial conditions. We thus conclude that the εR^2 model satisfies all requirements for a realistic inflationary model as long as ε is large enough.

To investigate the very early phase we have attempted a preliminary wave function calculation by solving the Wheeler–de Witt equation to WKB approximation subject to a tunneling boundary condition in the manner of Vilenkin.[28] We thus obtain peak values for the wave function assuming a closed ($\kappa = +1$) universe of $<a> \sim .056\, l_{Pl}$, $<R> \sim 3800\, l_{Pl}^{-2}$, and $<H> \sim 18\, l_{Pl}^{-1}$ independent of ε (the details of that calculation will be reported in subsequent work). We interpret these as typical of the tunneling values for the universe into the Lorentzian/classically allowed regime. Also, the peak is not very strong so that these numbers end up only as bounds. That is, we might say

$$R_i \gtrsim 4000\, l_{Pl}^{-2},$$

$$a_i \sim .06\, l_{Pl} \left[\frac{R_i}{4000\, l_{Pl}^{-2}}\right]^{-1/2},$$

and

$$H_i \sim 20\, l_{Pl}^{-1} \left[\frac{R_i}{4000\, l_{Pl}^{-2}}\right]^{1/2},$$

(and $t_i = t_m = 0$). These numbers are sufficiently distant from the horizon crossing of interesting perturbations that the wave function offers no conflict with our lower bound on H_m. We thus find the classical evolution to be generally independent of initial conditions. The one remaining question is whether or not there will be a long quantum gap separating the tunneling point from the onset of the classical model. That is, are quantum fluctuations large for an extended period during early times? This of course must be answered by the wave function itself. Also, after doing this further calculation we can determine whether the inflated portion of our present universe is the whole universe or only a fluctuated bubble part of it as in Linde's chaotic inflation picture. We note now only that the initial parameters preferred above indicate that the tunneled universe is strongly quantum.

The complete analysis of the initial conditions for the classical evolution (within the spatially homogeneous model), will be presented elsewhere. It is sufficient to say here that the classical inflationary phase appears as a typical phase in the very early universe, right after the Big Bang.

At the end we should mention the original Starobinsky model [13], whose quantum cosmology has been studied recently [28]. In this model there is a trace anomaly induced term in the Lagrangian, in addition to Eq.(1). This term dominates the very early evolution of the universe.

Consequently, the behavior of the quantum fluctuations in metric and curvature is rather different. However, at later times, the extra term can be neglected and it is during that phase that scales within the presently observable universe leave the inflationary horizon. The correct expressions for the scalar and tensorial perturbations, which agree with ours, were given by Starobinsky [29,30]. Regretfully, we were not familiar with this work when we started our investigation. Our analysis of the scalar and tensorial perturbations is explicit and fully gauge–invariant. We have also shown that the reheating phase can be characterized with two very different temperatures, which invites some interesting speculations about the dynamics of reheating, baryon number production and possible production of the extended structures.

A recent review of the relationship between the original Starobinsky model and the $R+R^2$ model is given in Ref. 31, where the later one has been named the "improved Starobinsky model". In addition to these arguments we would like to add two more. It is still not completely clear what to do with the infinities that occur in computing quantum corrections in curved space time. The trace anomaly term that is present in the original Starobinsky model has an open cut–off whose interpretation precludes straightforward use of this Lagrangian. Further, no matter how we look at the quantum gravity, the R^2 term appears to be present. It is this possibility, to be able to say something about the ε from some fundamental theory like superstrings, that makes this model especially promising.

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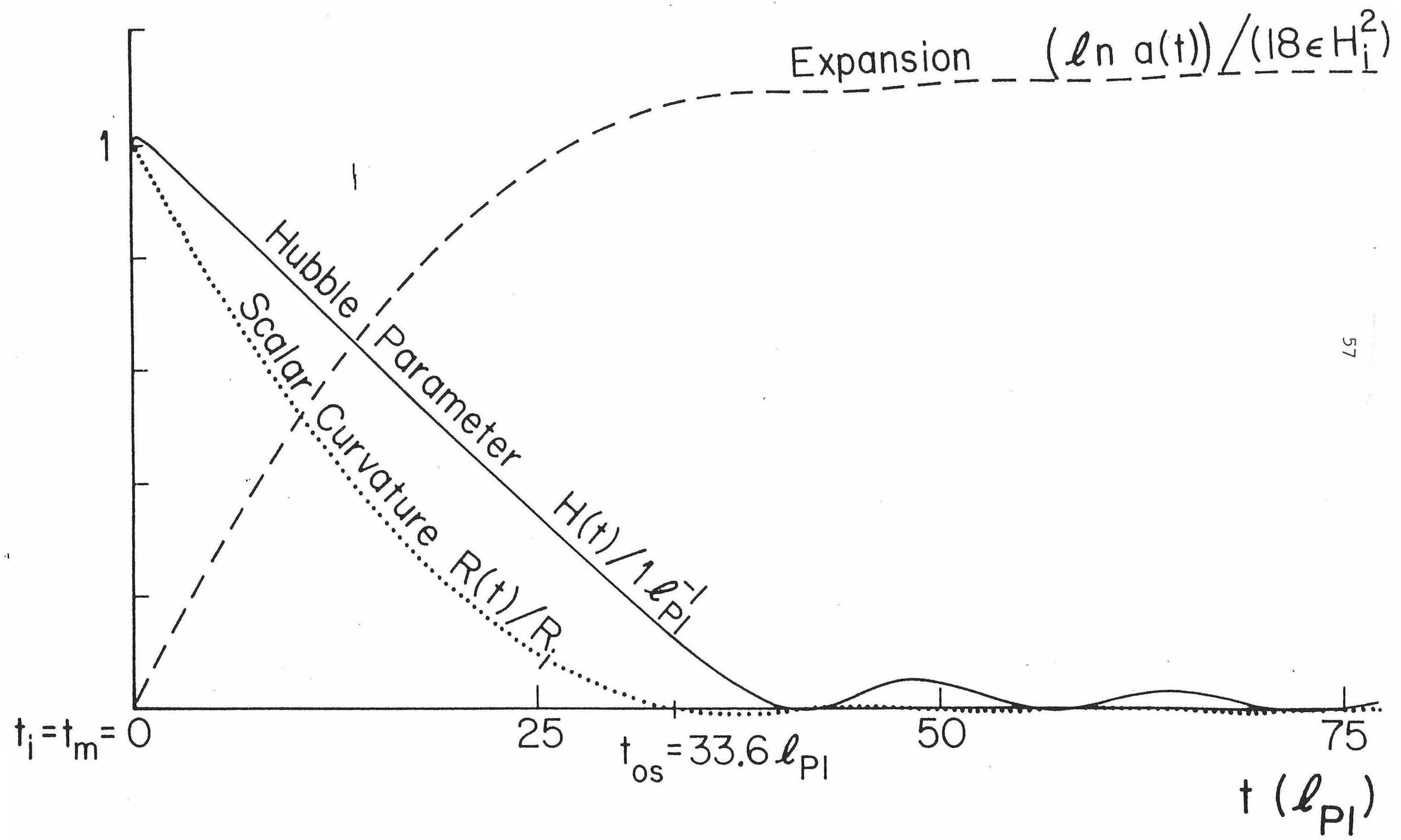
Figure Caption

FIG. 1. A model cosmology for $\varepsilon = 1$, $H_i = 1\ l_{Pl}{}^{-1}$, and $\dot{R}_i = 0$ (corresponding to the case (Ai) of the text so that $R_i \approx 12.5\ l_{Pl}{}^{-2}$), showing typical behaviour of the Hubble parameter ($H(t)/1\ l_{Pl}{}^{-1}$), the normalized scalar curvature ($R(t)/R_i$), and the inflation–normalized number of expansion e–foldings ($\ln a(t)/18\varepsilon H_i^2$). This plot has been generated from a numerical integration of the field equations (2.4)–(2.7) with zero initial matter content. The Hubble parameter displays a clean separation between the linear inflationary phase and the subsequent oscillation phase at $t_{os} = 36\varepsilon H_m - (1/(2\omega)) \approx 33.6\ l_{Pl}$ (cf. Eq. (2.23)). The slight initial rise in $H(t)$ is real since at the start $\dot{H} = (1/6)(R - 12H^2) > 0$. For models with a much higher value of the parameter ε (we are observationally constrained to $\varepsilon > 10^{11} l_{Pl}{}^2$) the linear phase is stretched out to a shallow slope and the subsequent oscillations are correspondingly reduced in both amplitude and frequency.


Expansion (ℓn a(t)) / (18ϵH_i^2)
1
Hubble Parameter H(t)/1ℓ_Pl^-1
Scalar Curvature R(t)/R_i
t_i=t_m=0
25
t_os=33.6ℓ_Pl
50
75
t (ℓ_Pl)