INTERACTIONS OF STRINGS COMPACTIFIED ON ORBIFOLDS

Thesis by
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Dedicated to my parents and
my sister

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ABSTRACT

It is very important to understand the process of string compactification before one can extract any reliable phenomenology. We consider an exactly solvable method of string compactification, in which the internal space is an orbifold. We compute various interaction amplitudes and describe how other amplitudes can be calculated.

Multi-valued string variables are handled by formulating the amplitudes on covering spaces where they become single-valued. In the computation of the amplitudes, stretched string intermediate states give rise to expressions that are non-perturbative from the non-linear sigma model point of view.

We also discuss, in the context of Z orbifold, the "blowing up" of the fixed points of an orbifold, to give rise to a Calabi-Yau manifold. The resulting Calabi-Yau manifold is shown to be non-perturbatively stable.

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1. Introduction

In the last fifty years, quantum field theory has been successfully applied to build and analyze theories that would describe electromagnetism, weak and strong interactions. These theories are further unified in one framework to make up what we call grand unified theories. Although these theories are somewhat successful phenomenologically, it was soon realized that they have too much arbitrariness. Some of the arbitrariness is in the choice of the gauge group, the representations of the matter fields and the large number of undetermined coupling constants. These mean that a grand unified principle is not very predictive.

On the other hand, before the invention of supersymmetry and supergravity there was not a single quantum theory of gravity with any success. Supergravity, for a while, raised hopes that it might be the ultimate theory of quantum gravity and everything else. Supergravity amplitudes were found to be finite in the first few orders of perturbation theory and some versions of it contained a grand unified theory. But this hope didn't last very long. There are strong indications that these theories are also divergent, although in higher orders of perturbation theory [1]. These theories have also problems producing the chiral nature of low-energy physics.

A radically novel approach to unification of gravity is the theory of strings [2]. String theory, over the last few years, has emerged as the only viable candidate for the description of all the forces of the nature [3]. This theory enjoys an enormous amount of consistency and uniqueness. String theory without supersymmetry is consistent only in 26 dimensions. Those which have supersymmetry live in 10 dimensions. There is no freedom in the choice of the fields, and the gauge group is restricted to be either $E_8 \times E_8$ or $SO(32)$ [4]. There are also good indications that the few theories that exist are actually different vacua of the same theory. This theory has no adjustable dimensionless parameters and might well be able to predict the low-energy parameters of the standard model.

A question that immediately comes to mind is: How does string theory choose its vacuum? It is crucial to find a satisfactory answer to this question before one can

make contact with experiments. In particular one would like to understand why we live in four dimensions whereas superstring theories are defined in ten dimensions.

The most popular attempt in this direction has been to consider compactification of strings on six dimensional manifolds. In particular for the $E_8 \times E_8$ heterotic string [5,6,7], compactification on Calabi-Yau manifolds [8] yields a theory in four dimensions which is more or less realistic [9,10,11,12,13]. The main difficulty in considering these compactifications is the complexity of the Calabi-Yau manifolds. Nevertheless, some interesting results have been obtained [14,15,16].

In contrast to the complexity of Calabi-Yau compactification, orbifold compactification [17,18] yields a simple model in which one could directly compute quantities of interest. The simplest orbifolds are a slight generalization of tori. One considers the string propagating in a toroidal background, points of which have been identified by a subgroup G of isometries of the torus. This results in a space with singularities if G has fixed points. The resulting space has G as its holonomy group because tangent vectors that are related by G are also identified. If one wishes to retain supersymmetry one should choose a space whose holonomy group acts trivially on a spinor [8]. In six dimensions, this means that the orbifold group G must be a discrete subgroup of $SU(3)$. One could obtain a Calabi-Yau manifold, if one so pleases, by "blowing up" and resolving the singular points. To lowest order in the non-linear sigma model perturbation, the field theory limit of orbifolds is generally speaking singular [16] (essentially because the norm of some states concentrated near the blown up region goes to zero in the orbifold limit). In such a limit one is inclined not to trust the perturbation theory because in the orbifold limit some couplings go to infinity (as the inverse radius of the blown up region). However, one can discuss the full-fledged string theory (not only the massless fields) relatively easily in the orbifold limit, and so one can see directly whether or not there are any inconsistencies and learn about the "stringy" nature of the compactification.

In this thesis we consider interactions on orbifolds. We will see that the interactions are *not singular*, and are in fact not too difficult to compute. The main technical

difficulty in considering interactions on orbifolds is handling Riemann surfaces with cuts (or more precisely, branch points). In discussing the fermion vertex operator [19], Riemann surfaces with cuts have been considered [20,21]. These cuts correspond to the double-valuedness of the fermionic variables as one goes around the points on the string world sheet from which spacetime fermions are emitted. In discussing interactions on orbifolds one has to consider multi-valuedness of both bosonic and fermionic variables as one goes around points from which "twisted" states are emitted. Twisted states are closed string states which are closed loops up to a group action. It turns out that twisting of the fermionic variables (in the abelian case) could be easily handled [20,21] by bosonizing the fermions in terms of which the twisting is replaced by shifting [18] in the weight lattice of the relevant orthogonal group. Twisting of the bosonic variables is far more difficult and its treatment is our main goal . It is achieved by considering coverings of the world sheet by another surface on which the string variables become single-valued.

The outline of this work is as follows: In sect. 2 we briefly review orbifolds and the general structure of interactions dictated by the group construction of the orbifolds. In sect. 3 we discuss some of the simplest types of interactions on orbifolds. These involve emission of untwisted states from twisted sectors. Even though they are simple, these interactions contain some interesting features, in particular the shifting of the center of mass of the twisted string by the emission of the winding states, and the decay of the winding states into states in the twisted sectors.

To discuss more complicated interactions, the ones involving the emission of twisted states, one will have to consider Riemann surfaces with automorphisms. We will give a elementary introduction to them in sect. 4, emphasizing those aspects which will be relevant from the viewpoint of interactions on orbifolds. Our discussion will be pedagogical and we hope will serve as an introduction to the fascinating subject of Riemann surfaces with automorphisms.

In sect. 5 we discuss the scattering amplitude of four twisted states in a Z_2 bosonic string in detail. This will be the most thoroughly studied example in this work. In

the course of considering this bosonic string interaction we are quite unexpectedly led to a proof of the Jacobi identity. This is the identity that implies the vanishing of the cosmological constant of the superstring at one loop! This result is obtained by computing the standard Koba-Nielsen amplitude in two different ways and demanding equality. It is surprising that bosonic Z_2 twisting has anything to do with superstrings [22].

In sect. 6 we discuss the generalization of these results to more complicated twistings. In sect. 7 we apply these methods to the heterotic strings, and discuss how the relevant vertex operators are modified. As a concrete example we will consider the Z orbifold [17,18] and see that there are non-perturbative corrections to the Yukawa couplings which change the perturbative relations between them. We will also discuss "blowing up" the singularities from the orbifold viewpoint and show that the superpotential will contain non-perturbative terms. We discuss the stability of Calabi-Yau compactification for the Z manifold. In sect. 8 we will present our conclusions.

Also, in the appendix we will discuss the *prime form* and its application in computations in the standard bosonic strings as well as its application to the orbifold interactions.

2. Review of Orbifolds

In this section we will briefly review the basic ideas in constructing orbifolds [17,18]. The type of orbifolds we will discuss are obtained as quotient of the euclidean space by a discrete group. Of course, one could take the quotient of some more complicated space by a discrete isometry group but that is not tractable. If we call the orbifold group G we have to describe its action on the other degrees of freedom of the theory too. But we shall first discuss its restriction to the euclidean group (the symmetry of the X degrees of freedom).

Consider the d dimensional euclidean space R^d. Let's choose a set of d independent vectors. If we divide R^d by the translation group generated by these vectors, we obtain a torus $T^d = R^d/Z^d$. We would like to generalize this construction by allowing some rotations and/or rotations accompanied by translations. This means that we will take as our discrete group elements of the form $g = (\theta, v)$ where θ is a rotation and v is a translation vector. The group with such elements is called a space group. It acts on the points of R^d by

$$x \mapsto gx = \theta x + v. \tag{2.1}$$

It is easy to check that the following relations hold

$$(\theta, v)(\omega, u) = (\theta\omega, v + \theta u)$$

$$(\theta, v)^{-1} = (\theta^{-1}, -\theta^{-1}v) \tag{2.2}$$

$$(\theta, u)(1, v)(\theta, u)^{-1} = (1, \theta v).$$

By dividing the euclidean space by a space group S, we mean that we consider all points x and gx as the same point for all $g \in S$. Another words we identify all the points in a given orbit, and hence the name orbifold. The resulting space $\Omega = R^d/S$ has singularities at the fixed points of the group.

There is an alternative method of constructing this space. We first construct a torus T^d by dividing R^d by the subgroup of S consisting of the elements of the form $(1, v)$ which we call Λ. This subgroup is a translation group and is referred to as the lattice of S. We also consider the point group P which is the subgroup of $O(d)$ (the orthogonal group in d dimensions) made of elements θ such that $(\theta, v) \in S$ for some v. For each $\theta \in P$ there is a unique element (θ, v) corresponding to it up to an element of Λ. This is because

$$(\theta, v)(\theta, u)^{-1} = (\theta, v)(\theta^{-1}, -\theta^{-1}u) = (1, v - u). \tag{2.3}$$

Therefore P has a well defined action on T^d which we call $\overline{P}$. (We use a different notation to keep in mind that unlike P, the action of $\overline{P}$ may involve a shift.) The second way to describe an orbifold is by taking the quotient of the torus T^d by $\overline{P}$. So we can summerize these two constructions in

$$\Omega = R^d/S = \frac{R^d/\Lambda}{\overline{P}} = T^d/\overline{P}. \tag{2.4}$$

The orbifold has P as its holonomy group. It is clear that P is not an arbitrary subgroup of $O(d)$ because it must map T^d onto itself and therefore must have representation matrices with integer entries. In the cases where it is known how to repair the singularities, by a procedure called "blow up", the holonomy group is smoothed into a Lie group containing P as a discrete subgroup.

We also have to discuss how S acts on the gauge degrees of freedom. One choice consistent with modular invariance [18] is to embed the spin connection in the gauge connection. Let's use 16 fermions ψ^I transforming as the 16 dimensional representation of a $SO(16)$ subgroup of E_8 to represent the gauge degrees of freedom. Then we can pick a $SO(6)$ subgroup of $SO(16)$ under which the fermions transform as a 6 dimensional representation plus singlets and consider $g = (\theta, v)$ acting on them by $g\psi = \theta\psi$. If the point group is abelian this can also be easily represented by shifts in the root lattice of E_8.

We will review the general features by considering a particular example. Consider a d dimensional torus T^d, defined by $x^i \sim x^i + R$ with $i = 1, ..., d$. The string Hilbert space of closed strings on this space consists of winding states, corresponding to strings wrapped an arbitrary number of times around each circle. If we parametrize the string by θ running from 0 to 2π, a winding state is specified by an integer vector $W^i = n^i R$ for some integers n^i, such that $X^i(2\pi) = X^i(0) + W^i$. Consider the Z_2 isometry of the torus corresponding to reflection about the origin $x^i \mapsto -x^i$. We construct the Hilbert space of the Z_2 orbifold in two steps: We first project onto the Z_2 invariant subspace of the Hilbert space. This means that we form group invariant states. For instance if we are considering winding states with no momentum and no other excitations we would keep only the invariant states $\frac{1}{\sqrt{2}}(|W\rangle + |-W\rangle)$. The second step in the construction of orbifolds consists of introducing new Hilbert spaces: Since we are thinking of x^i and $-x^i$ as the same point, the twisted string states defined by $X^i(2\pi) = -X^i(0) + W^i$ should also be considered closed, and must be included in the Hilbert space. This is a requirement of modular invariance as we shall now argue. In the path-integral, projection onto the group invariant subspace means we include all boundary conditions in the τ direction with equal weight. Now the modular transformation $\tau \to -\frac{1}{\tau}$ swaps the σ and τ boundary conditions and forces us to include a new Hilbert space [17,18]. This is called the twisted Hilbert space.

The twisted space itself consists of 2^d subsectors. To see this note that the center of mass of the string in the twisted sector should be a fixed point of the transformation, that is $X_{cm} = -X_{cm}$ up to a shift by a vector in the lattice defined by the torus (in this case the standard lattice). In other words $2X_{cm}$ should lie on the lattice. This implies that $X^i_{cm} = Rn^i/2$. But two choices of X_{cm} which are separated by a lattice vector are in fact the same choice and should not be counted more than once. So we can take each n^i to be 0 or 1, giving altogether 2^d choices. Hence, the twisted sector comes in 2^d copies.

We can also use the space group $(\pm 1, n)$ (with n an integer) acting on the d dimensional euclidean space to construct the Z_2 orbifold. The sector twisted by

$(1,n)$ is just the usual winding sector. Since $(1,n)$ is conjugate to $(1,-n)$ we have to take linear combination of states with opposite winding numbers, each projected onto the Z_2 invariant subspace. Also $(-1,n)$ is conjugate to $(-1,n+2k)$. This means that the twisted sector comes in 2^d subsectors, namely each of the d components of n can be an odd or even integer. The correspondence to the first construction should be clear.

This example has the basic features of the general case. We start from the untwisted Hilbert space $\mathcal{H}_1$ and choose a subgroup of the isometries of $\mathcal{H}_1$ and throw away the states in $\mathcal{H}_1$ not invariant under the group action. Then, for each element g in the group, we consider the Hilbert space $\mathcal{H}_g$ corresponding to the string states $X(2\pi) = g \cdot X(0)$ (here X denotes collectively the string variable, and may include fermionic or gauge degrees of freedom in the superstring case). One still has to project the states in $\mathcal{H}_g$ onto group invariant states.

Without going into much detail about the string interactions, we can already conclude that certain couplings vanish: Suppose we have a string in Hilbert space $\mathcal{H}_g$, and another in $\mathcal{H}_h$. When these two strings join they go to the sector $\mathcal{H}_{hg}$.

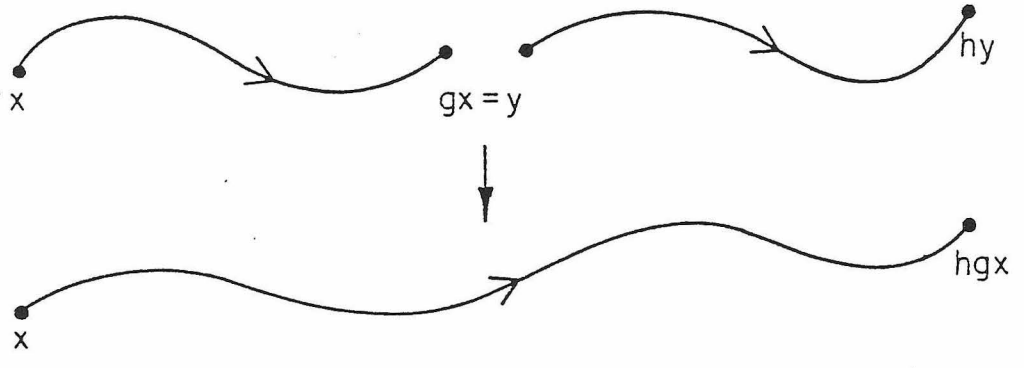

Figure 1. A string from g sector, joins with a string from h sector, to form a string in hg sector.

We can easily see the generalization to more complicated interactions, by reducing it to the interaction involving two string states at a time. If, for brevity, we label a

string state by the group element it is twisted by, we can have a process

$$g_1 g_2 ... g_m \rightarrow h_1 h_2 ... h_n \tag{2.5}$$

only if*

$$\prod_{i=1}^{m} g_i = \prod_{j=1}^{n} h_j. \tag{2.6}$$

Or bringing the outgoing state and viewing it as an incoming state with opposite orientation (implying it corresponds to a Hilbert space with inverse twisting) and setting $g_{m+j} = h_{n-j+1}^{-1}$, we get

$$\prod_{i=1}^{m+n} g_i = 1. \tag{2.7}$$

Since the strings are oriented, if the group is non-abelian the order of the interaction changes the outcome. A string from the sector g_1 interacts with a string from the sector g_2 in two different ways, with the outcomes in sectors $g_1 g_2$ and $g_2 g_1$. These two sectors are conjugate to one another because $g_1 g_2 = g_1 (g_2 g_1) g_1^{-1}$. As discussed in [17,18] we form group invariant states by taking states for each conjugacy class, because the group acts on sectors by taking them to conjugate sectors. Therefore, the two different processes lead to the same group invariant state.

We therefore see that generally speaking the interactions on orbifolds involve transitions between various Hilbert subspaces. In the operator formulation this would imply that the vertex operators for emission of twisted states take states from one Hilbert space into states in another, as is the case for the fermion vertex operator [19]. So the twisting operators will be in general very difficult to construct explicitly. (They have been constructed for a simple case, as we will point out.) It is far simpler to consider the path integral approach, which we will follow in sects. 5, 6 and 7. However, the vertex operators for the emission of untwisted states are not difficult to construct as they do not change the Hilbert space. This will be the topic of our discussion in the next section.

⋆ This result gets modified for non-abelian groups beyond the tree level (see sect. 4).

3. Emission of Untwisted States

The simplest type of interactions on orbifolds involve the untwisted states only. Untwisted states are the states which one already has in the Hilbert space before the construction of orbifolds, i.e., the states of the sector in which string is closed even on the underlying torus of the orbifold. Perhaps we should emphasize that throughout this work we consider only closed oriented strings. The only change one should make in considering orbifolds is to consider group-invariant combinations of such states. Since we already know the vertex operator for the emission of each of these states, we simply take the appropriate linear combination of the corresponding vertex operators, to form a group-invariant vertex operator which describes the emission of a group-invariant untwisted state [23]. Symbolically, if we denote by $V(s)$ the vertex operator of a state s, and by gs the state obtained by the action of the orbifold group element g on the state s, we take

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N} V(g_i s) \tag{3.1}$$

for the group invariant vertex.

As discussed in the last section, the interactions of the twisted states are not so simple, as they involve transition from one Hilbert space to another. An example of this is the vertex operator for the emission of fermions from superstrings [19] which in the RNS formulation [24,25] takes one Hilbert space to another. This was historically [19] one of the difficulties in the construction of the four fermion scattering amplitude. However, the scattering of an arbitrary number of bosons from a fermion is not too difficult to compute, and one does not have to construct fermion-emission vertex operators to discuss it. The reason for this is that the vertex operators for the emission of bosonic states (NS states) do not change the Hilbert space. So in discussing amplitudes involving two fermions, we simply sandwich bosonic vertex operators between incoming and outgoing states from the R sector (this is the sector in which the two-dimensional fermions obey periodic boundary conditions). For instance

the tree-level interaction of two bosons and two fermions is given by

$$_R < 1|V_B(2)V_B(3)|4 >_R . \tag{3.2}$$

We could use the same idea for orbifolds. For the interaction of two twisted states with an arbitrary number of untwisted states, we simply take the twisted states to be the incoming and outgoing states, and construct the vertex operators for the emission of untwisted states in the twisted sector (that is, using twisted string coordinates). The vertex operators will be almost the same as the corresponding vertex operators acting in the untwisted Hilbert space. (For a discussion of these vertex operators from the viewpoint of affine Lie algebras see [26,27,28].)

For the sake of simplicity, we will concentrate on bosonic orbifolds obtained by compactifying the 26 dimensional bosonic string on a d-dimensional torus, divided by a discrete subgroup of isometries of the torus. The same ideas are applicable to superstrings. We will consider the Z_2 orbifold discussed in the last section, and comment on generalizations to Z_N orbifolds at the end of the section.

Recall that the Z_2 orbifold was obtained by dividing out T^d by the Z_2 group generated by $x^i \mapsto -x^i$ for $i = 26-d+1, ..., 26$ (the torus being defined by $x^i \sim x^i + R$). In the untwisted sector we have the usual periodic string variables, except that we should allow for the string to wind around the torus, and/or have non-trivial momenta. We expand the string coordinates of the internal dimension as

$$X^i(untwisted) = x^i + p^i\tau + w^i\sigma + \frac{i}{2}\sum_{n\neq 0}\left(\frac{\alpha^i_n}{n}e^{-2in(\tau-\sigma)} + \frac{\tilde{\alpha}^i_n}{n}e^{-2in(\tau+\sigma)}\right), \tag{3.3}$$

where σ runs between 0 and π, $\alpha' = 1/2$, $w^i = Rn^i/\pi$ and $p^i = 2m^i\pi/R$ with n^i and m^i being integers. Let's denote the momenta in the $26-d$ uncompactified dimensions by k. The total right moving number operator is given by

$$N = \sum_0^\infty (\alpha^i_{-n}\cdot\alpha^i_n + \alpha^\mu_{-n}\cdot\alpha^\mu_n) \tag{3.4}$$

with i going over the compactified dimensions and μ going over the transverse space-

time dimensions and similar equation holds for the left moving number operator $\widetilde{N}$. We then have the mass shell condition

$$\frac{1}{4}k^2 = \frac{1}{4}(p^2 + w^2) + N + \widetilde{N} - 2 \tag{3.5}$$

together with the condition of reparametrization invariance which gives $2(N - \widetilde{N}) = p \cdot w$.

We have the following expansion for X^i for the twisted dimensions:

$$X^i(twisted) = x^i_{cm} + \frac{i}{2}\sum_n \left(\frac{\alpha^i_{n+\frac{1}{2}}}{n+\frac{1}{2}} e^{-2i(n+1/2)(\tau-\sigma)} + \frac{\tilde{\alpha}^i_{n-\frac{1}{2}}}{n-\frac{1}{2}} e^{-2i(n-1/2)(\tau+\sigma)} \right), \tag{3.6}$$

and similarly for the left movers. For this sector we get 2^d subsectors corresponding to the choices $x^i_{cm} = Rn^i/2$ with $n^i = 0, 1$. The twisted string is quantized by imposing

$$[\alpha^i_{n+\frac{1}{2}}, \alpha^j_{m-\frac{1}{2}}] = (n + \frac{1}{2})\delta_{n,-m}\delta^{i,j} \tag{3.7}$$

and similarly for the left movers. Note that there are no momenta (terms linear in τ) in the internal dimensions for twisted states. This is because maintaining the condition that the two ends of the string be related by a reflection in twisted directions requires that a twisted string have no net motion in those directions. The zero-point energy for an oscillator with fractional quantum number η is (including both left and right movers) [22,18]

$$\sum_{n=0}^{\infty}(n + \eta) = -\frac{1}{12} + \frac{1}{2}\eta(1 - \eta), \tag{3.8}$$

where the sum is regularized using a ζ-function prescription. Here, we have $24 - d$ oscillators with $\eta = 0$ and d oscillators with $\eta = \frac{1}{2}$ and the zero-point energies sum to $-2 + \frac{d}{8}$. If we label the uncompactified momenta again by k, we get the mass condition

$$\frac{1}{4}k^2 = N + \widetilde{N} + \frac{d}{8} - 2. \tag{3.9}$$

Let us concentrate on the lightest states of the twisted sector. They correspond to states with $N = \widetilde{N} = 0$ with $\frac{1}{4}m^2 = \frac{d}{8} - 2$, and as we mentioned there are 2^d of them

which we denote by $|n^i>$, with $x^i_{cm} = Rn^i/2$, and n^i taking the values 0 or 1. The question which comes to mind is whether there is any physical process by which one can go from the states concentrated about one fixed point to another. The answer is yes: Consider an untwisted string state with a particular winding $w^i = Rn^i_1/\pi$. If this state joins a twisted state in the $|n^i_2>$ sector, we obtain a twisted state in the $|n^i_1 + n^i_2>$ sector (where we should interpret $n^i_1 + n^i_2$ modulo 2). To see this we simply note that the end points of the two string are related to each other by $X(\pi)_1 = X(0)_1 + Rn^i_1$ and $X(\pi)_2 = -X(0)_2 + Rn^i_2$ and $X(\pi)_1 = X(0)_2$, so after joining $X(\pi)_2 = -X(0)_1 + R(n^i_2 - n^i_1)$, which is a string state in $|n^i_1 + n^i_2>$ sector (note that $-n^i_1 = n^i_1$ modulo 2). For an example of this process see fig. 2.

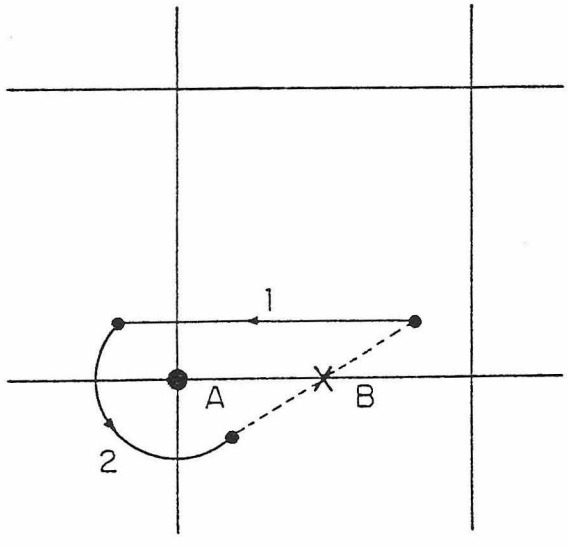

Figure 2. A winding string (1) from the identity sector, joins a twisted string (2) which is expanded around the fixed point A, to form a twisted string expanded around fixed point B.

The question we address is how to represent such a process by a vertex operator. As we noted the first guess is to write the ordinary vertex for the emission of untwisted states, but using twisted string variables. Suppose we were interested in writing the vertex operator for the emission of an untwisted state specified by $w, p, k, N = \widetilde{N} = 0$.

We take as our guess

$$V = g'(p,w)e^{i(p-w)\cdot X_R}e^{i(p+w)\cdot X_L}e^{ik\cdot x}S(w,p) \tag{3.10}$$

(of course we will have to make this vertex operator group invariant by the procedure described at the beginning of this section). Here $S(w,p)$ denotes the shifting operation of the center of mass, i.e., it is a $2^d \times 2^d$ matrix whose action on the center of mass of the twisted sector is given by $S(w^i = n_1^i R/\pi)|n_2^i >= |n_1^i + n_2^i >$. Note that this is really true up to a phase, but let's ignore that and see what happens. For the moment we leave g' undetermined and allow it to depend on (p,w). The string variables in (3.10) are to be written in terms of twisted variables using (3.6). Also, unless otherwise stated, we will take the operators to be normal ordered.

We now compute the scattering amplitude for two untwisted states (w_1,p_1,k_1) and (w_2,p_2,k_2) and two twisted states $|n_3^i,k_3>$ and $|n_4^i,k_4>$. The tree amplitude is easily constructed

$$Amp =< n_3^i,-k_3|V_1\Delta_{\frac{1}{2}}V_2|n_4^i,k_4>, \tag{3.11}$$

where $\Delta_{\frac{1}{2}}$ is the propagator of the twisted string (and the vertex operators are averaged over σ). If we take $w_1^i = Rn_1^i/\pi$ and $w_2^i = Rn_2^i/\pi$, the amplitude vanishes unless

$$n_1 + n_2 + n_3 + n_4 = 0 \qquad (mod\ 2). \tag{3.12}$$

This is easily seen by the action of S on the states. We now use the integral representation of the propagator, do the space-time part of the calculation and obtain

$$Amp = \frac{g_1'g_2'}{4\pi}\int d^2z\,|z|^{-k_1\cdot k_3/2}\,|1-z|^{-k_1\cdot k_2/2}\,|z|^{-(p_1^2+w_1^2)/4}$$

$$\times < V_1(1)V_2(z) >< \tilde{V}_1(1)\tilde{V}_2(\bar{z}) >, \tag{3.13}$$

where the last two terms involve the right and left moving piece of the twisted part

of the vertex, and is to be evaluated in the Fock space of the twisted sector. We have

$$< V_1(1)V_2(z) >= e^{(p_1-w_1)(p_2-w_2)<X(1)X(z)>}, \tag{3.14}$$

where

$$< X(1)X(z) >= \frac{1}{4}\sum_{n=0}^{\infty}\frac{z^{n+1/2}}{n+1/2} = \frac{1}{4}G(z). \tag{3.15}$$

$G(z)$ satisfies the simple equation $\partial_z G = \frac{1}{\sqrt{z}(1-z)}$ which can easily be integrated by the change of variable $\tilde{z} = \sqrt{z}$ to obtain $G(z) = log(\frac{1-\sqrt{z}}{1+\sqrt{z}})$. (This Green function has been considered [29] in connection with off-shell string amplitudes [30,31].) The form of this Green function can be easily understood. We can think of (3.11) as inserting twisted states at the points 0 and ∞ on the world sheet. The twisted states create a cut in the Riemann surface (see next section for more detail), in the sense that the correlation $< X(1)X(z) >$ should come back to minus itself as z goes around the origin. This is easily seen to be true for G, because as z goes around the origin, $\sqrt{z} \mapsto -\sqrt{z}$, so that $G \to -G$.

Finally putting the pieces together we get

$$Amp = \frac{g_1' g_2'}{4\pi}\int d^2z\,|z|^{-k_1k_3/2}\,|1-z|^{-k_1k_2/2}\,|z|^{-(p_1^2+w_1^2)/4}\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{(p_1-w_1)(p_2-w_2)/4}$$

$$\times\left(\frac{1-\sqrt{\bar{z}}}{1+\sqrt{\bar{z}}}\right)^{(p_1+w_1)(p_2+w_2)/4} \times (a\ possible\ phase). \tag{3.16}$$

(We will have to add to this the contribution of vertex operators necessary for constructing Z_2 invariant operators. This could be easily taken care of by noting that the expression above if integrated over all of $\tilde{z}$ space instead of z space, automatically takes care of that. In a sense we have no alternative, because the square roots in the above expression mean that it is not well defined as a function of z. The integrand "knows" that we have to consider group invariant states.)

This amplitude must be symmetric under the exchange of the two untwisted states. In other words the order of the emission shouldn't matter because vertex operators are integrated over all of the world-sheet. This symmetry can be studied by letting $z \to 1/z$. The untwisted part of the amplitude is symmetric but the part involving twisted states picks up $(-1)^{-(w_1 \cdot p_2 + w_2 \cdot p_1)/2}$. Note that $p_i \cdot w_j$ is always an even integer and this phase is well-defined. It is easy to guess the phase that must go along with the amplitude to make it symmetric. In this case it is just $(-1)^{w_1 \cdot p_2/2}$. Later we shall derive this property of twisted vertex operators in a general case.

To determine g_1', g_2', we use factorization. Let $K = k_1 + k_2$, $P = p_1 + p_2$ and $W = w_1 + w_2$. If (W, P, K) satisfy the mass shell condition (3.5)(with $N = \widetilde{N} = 0 = P \cdot W$) we get a pole in the integrand above at $z \sim 1$, corresponding to an on-shell, untwisted intermediate state (see fig. 3).

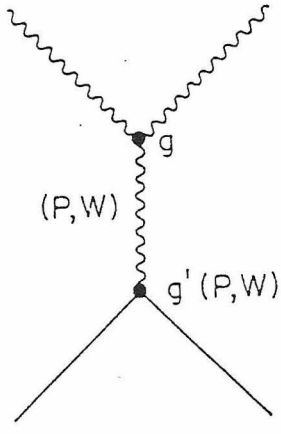

Figure 3. The wiggled lines denote untwisted string states, and solid lines denote the twisted string states. By factorization, the amplitude at resonance is the product of the propagator for the intermediate state (P, W), times the coupling constants g and $g'(P, W)$.

For the residue of the pole we should get $g'(P, W)g$, where g is the coupling in the ordinary vertex operator, coming from the coupling of three untwisted states, and $g'(P, W)$ is the coupling of the untwisted state (P, W) to two unexcited twisted states.

By noting that $\frac{1-\sqrt{z}}{1+\sqrt{z}} \to \frac{(1-z)}{4}$, we can compute the residue of (3.16) and obtain the condition for factorization as*

$$g'(p_1, w_1)g'(p_2, w_2)2^{-(p_1 p_2 + w_1 w_2)} = g'(P, W)g. \tag{3.17}$$

This is easily solved to give (aside from the trivial solution)†

$$g'(p, w) = 2^{-\frac{1}{2}(p^2+w^2)}g. \tag{3.18}$$

We can see the appearance of this term in a different way. Note that when writing vertex operators we have to normal order them to get sensible quantities. For the ordinary vertex operators $g\ e^{ip\cdot X}$ this involves a multiplication by $e^{\frac{1}{4}p^2 \sum \frac{1}{n}}$ to cancel the normal ordering contribution. But in the twisted sector the oscillators are fractional and the normal ordering contributions do not completely cancel this term, leaving us with

$$e^{-\frac{1}{4}(p^2+w^2)\sum(\frac{1}{n-1/2}-\frac{1}{n})}. \tag{3.19}$$

It can be shown that $\sum(\frac{1}{n-1/2} - \frac{1}{n}) = 2\ log2$ (by relating it to the residue of G), and this leaves us with the factor $2^{-(p^2+w^2)/2}$ which is what we found by factorization.

The factor we have found has a simple physical interpretation: Consider a process in which two (unexcited) twisted states from different fixed points join and go to a winding state of the untwisted sector. The amplitude for this process is easily found by sandwiching the vertex operator we have found between the ground states of the twisted sector. This will be equal to $g \cdot 2^{-w^2/2}$. So we see that the amplitude for this process is exponentially suppressed for large distances between the fixed points. This could be physically understood by noting that for two string states from different fixed points to meet, they must stretch, and this will involve an action cost which goes $\sim e^{-aw^2}$ for some constant a.

⋆ The θ integration in computing the factorization of (3.16) gives zero unless $P \cdot W = 0$, which is a condition for reparametrization invariance of the state.

† This solution can also be obtained by constructing vertex operators for bosonic twisting [29].

Note that the joining process of two string states from different fixed points is very "stringy", as it involves an extended state, the stretched string. From the viewpoint of non-linear sigma model, this is a non-perturbative correction, because it goes like $2^{-w^2/2} = e^{-cR^2}$, for some constant c, and the length of the torus R is the inverse of the non-linear sigma model coupling constant. If we denote the sigma model coupling constant by σ then the precise relation is

$$\sigma^2 = \frac{4\pi\alpha'}{R^2}. \tag{3.20}$$

It is amazing that such a simple stringy picture could lead to non-perturbative computations in the language of non-linear sigma models. Similar ideas will lead us to non-perturbative corrections to the Yukawa couplings, as discussed in sect. 7.

We could also turn around the process just described, and consider the decay of a winding state into a pair of twisted states. The winding state becomes unwound in such a process, again with an amplitude which goes like $2^{-w^2/2}$.

We could consider other twistings. For example, if we take the twisted states of the type $e^{2\pi ik/N}$, our considerations go through with the replacement $G(z) = \sum_n \frac{z^{n+k/N}}{n+k/N}$, which is easily summed to give

$$G(z) = log \prod_{n=1}^{N} (1 - e^{-2\pi in/N} z^{1/N})^{e^{2\pi ikn/N}}. \tag{3.21}$$

The coupling g' is easily found by looking at the residue of $e^{-G(z)}$ near $z \sim 1$. The result is

$$g'(p,w) = g\, e^{(p^2+w^2)(\gamma+\psi(\frac{k}{N}))/4}, \tag{3.22}$$

where γ is the Euler's constant and ψ is psi(digamma) function [32].

We shall now discuss the property that the twisted vertex operator (3.10) must have so that the amplitude is Bose symmetric. These vertex operators are required

to commute with each other. Using the twisted propagator of (3.21) we commute two twisted vertex operators and find that S must satisfy

$$S(w_1,p_1)S(w_2,p_2) = S(w_2,p_2)S(w_1,p_1)C(p_1,w_1;p_2,w_2). \tag{3.23}$$

The "structure constants" C are given by

$$C(p_1,w_1;p_2,w_2) = \prod_{n=1}^{N}(-\omega^{-n})^{-\omega^{nk}(\bar{p}_{1L}p_{2L}-\bar{p}_{2R}p_{1R})/4}\ (-\omega^{-n})^{-\omega^{-nk}(\bar{p}_{2L}p_{1L}-\bar{p}_{1R}p_{2R})/4}, \tag{3.24}$$

where $\omega = e^{2\pi i/N}$ and $p_L = p - w$ and $p_R = p + w$. These structure constants also arise in the vertex operator representation of twisted affine algebras [26,27,28]. They also play an important role in the construction of asymmetric orbifolds [33]. The expression for C can be simplified to

$$C(p_1,w_1;p_2,w_2) = e^{i\frac{\pi}{2}(\bar{w}_1p_2+w_1\bar{p}_2+w_2\bar{p}_1+\bar{w}_2p_1)}\ e^{\frac{\pi}{2}cot\frac{\pi k}{N}(\bar{p}_1p_2-p_1\bar{p}_2+\bar{w}_1w_2-w_1\bar{w}_2)}. \tag{3.25}$$

In the $N = 2$ case the second factor vanishes and we obtain the result we previously guessed. An explicit representation for $S(p,w)$ is not necessary but can be constructed [33]. Thus, we see that the generalization to Z_N is also straightforward.

4. Riemann Surfaces with Automorphism

In this section we will develop some of the mathematical machinery which is needed for a discussion of interactions involving more than two twisted states of an orbifold. We will see that Riemann surfaces that are themselves orbifolds can be used to construct interactions on orbifolds, and that in order to analyze interactions on orbifolds obtained by dividing out by a group G one should consider surfaces with symmetry group G.

In discussing interactions on orbifolds involving twisted states, it is necessary to consider world sheets with branch points: The string variables are multi-valued as one goes around the points where the vertex operators for the emission of twisted states have been inserted. If we choose radial coordinates (r, θ) around these points, we can locally view $log(r)$ as the "time" variable and θ as the space variable parametrizing the string. The condition that the string be twisted means that $X(r, \theta + 2\pi) = g \cdot X(r, \theta)$ which means that string variables X are multi-valued functions on the world sheet. Multi-valued functions are difficult to deal with. One would like to find a space on which the functions are single-valued. This problem is familiar in complex analysis: the "function" $\sqrt{z}$ is not single-valued on the complex plane, but if we take two sheets of the complex plane, such that circling the origin in the complex plane takes us from one sheet to another, $\sqrt{z}$ becomes a single-valued function which changes sign as we go from one sheet to another. Algebraically this could be accomplished by considering the variable $\tilde{z}^2 = z$, for which our function is simply $\tilde{z}$. This could be viewed more geometrically as follows [34,35]: we can view the compactified complex plane z as a sphere S (see fig. 4).

We make a cut along a line joining 0 to ∞, open up the sphere, so that it becomes a hemisphere, and then glue a copy of itself (the second sheet) to it, with proper care for the orientations along the boundary. The resulting sphere $\tilde{S}$ is to be thought of as the compactified complex plane $\tilde{z}$. We can consider a map from $\tilde{S}$ to S, defined by $z = \tilde{z}^2$. This map is 2 to 1 almost everywhere, because it sends $\tilde{z}$ and $-\tilde{z}$ to the same point. It is 1 to 1 at 0 and ∞ because 0 and ∞ are fixed points of the transformation

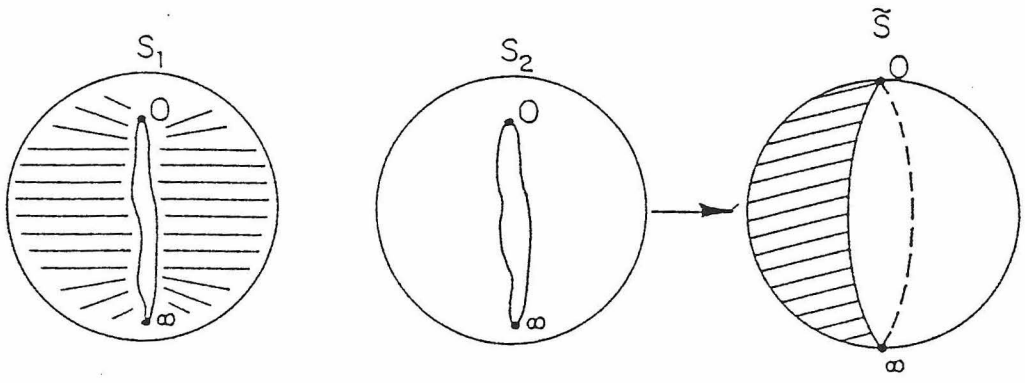

Figure 4. Two Riemann spheres S_1, S_2 with cuts from 0 to ∞ join to form a Riemann sphere $\tilde{S}$.

$\tilde{z} \mapsto -\tilde{z}$. With a view to further generalizations let us introduce some notation: Let the group G denote the two element group acting on $\tilde{z}$ by $\tilde{z} \mapsto \pm\tilde{z}$. Then dividing $\tilde{S}$ by the group action G we have

$$\frac{\tilde{S}}{G} = S. \tag{4.1}$$

This is a concise statement of what we just said in words; namely, if we think of $\tilde{z}$ and $-\tilde{z}$ as the same point, that is nothing but the space S defined by z. If we are interested in double-valued functions on z space, which pick up a sign as we go around the origin, we could equivalently consider single-valued functions on $\tilde{z}$ space, which are odd under $\tilde{z} \mapsto -\tilde{z}$. For instance, if we are interested in the Green function on $\tilde{z}$ space, between 1 and $\tilde{z}$, we would consider $g(\tilde{z}) = log(1 - \tilde{z})$. This is not an odd function of $\tilde{z}$. If we are interested in a double-valued function on the z space, we should make this Green function odd by considering

$$g(\tilde{z}) - g(-\tilde{z}) = log(1 - \tilde{z}) - log(1 + \tilde{z}) = log\left(\frac{1 - \tilde{z}}{1 + \tilde{z}}\right) = log\left(\frac{1 - \sqrt{z}}{1 + \sqrt{z}}\right) \tag{4.2}$$

which is what we found for the Green function in the last section, in the presence of two Z_2 twisted vertex operators located at 0 and ∞. Since S and $\tilde{S}$ are equivalent, we may consider doing all the computations on $\tilde{S}$. We have already noted that double valued functions on S correspond to odd functions on $\tilde{S}$. What about the single

valued functions on S? The single-valued functions are naturally handled on S, but we may still wish to consider them on $\tilde{S}$. It is clear that for this purpose, we will have to consider even functions on $\tilde{S}$. In particular if we were interested in the Green function, we would have to make an even function of $\tilde{z}$, that is

$$g(\tilde{z}) + g(-\tilde{z}) = log(1-\tilde{z}) + log(1+\tilde{z}) = log(1-\tilde{z}^2) = log(1-z). \qquad (4.3)$$

As expected, we recover the ordinary Green function on the z space.

We can easily generalize this argument and obtain what we found in the last section for the case of the scattering of two Z_n twisted states from two untwisted states. One simply constructs a covering of the sphere, by n spheres, each cut along the line $0-\infty$ and glued together like an orange with n pieces (see fig. 5), forming again a sphere.

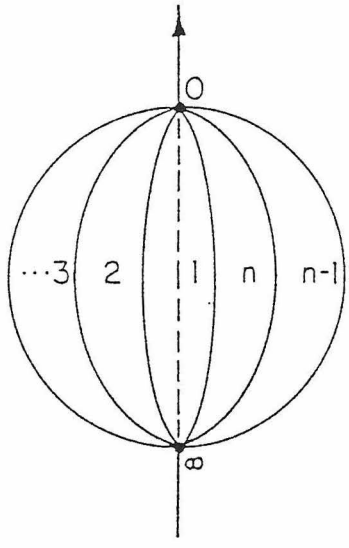

Figure 5. n copies of a sphere cut along the line $0-\infty$ are joined to form another sphere. The Z_n symmetry is generated by a rotation of $\frac{2\pi}{n}$ around the $0-\infty$ axis.

This corresponds to choosing $z = \tilde{z}^n$. The symmetry group will now be generated by α, a $\frac{2\pi}{n}$ rotation of the resulting sphere about the axis joining 0 to ∞. The

condition in the case of Z_2 that a function $\tilde{X}$ be odd in $\tilde{z}$ is now replaced by $\tilde{X}(\alpha\tilde{z}) = e^{2\pi ik/n}\tilde{X}(\tilde{z})$ for a $\frac{k}{n}$ twisting. One can easily construct the Green function in this case and see that it is precisely what we found in the last section.

There are a number of lessons to be learned from these simple examples. Considering a space for which a multi-valued function becomes single-valued is natural, and was in fact the main motivation of Riemann in introducing Riemann surfaces! The Riemann surfaces are the spaces on which multi-valued functions become single-valued. If one is considering multi-valued functions on a Riemann surface Σ, by considering an appropriate Riemann surface $\tilde{\Sigma}$ which covers Σ (generically n *to* 1 in the example of Z_n discussed above), one obtains a single-valued function on $\tilde{\Sigma}$. If z and $\tilde{z}$ correspond respectively to the points on Σ and $\tilde{\Sigma}$, over each point z_0, there corresponds a number of points $\tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_n$. The multi-valued function $f(z)$, corresponds to a single-valued function $\tilde{f}(\tilde{z})$ in the sense that $f(z_0)$ takes any of the values $\tilde{f}(\tilde{z}_1), \tilde{f}(\tilde{z}_2), ..., \tilde{f}(\tilde{z}_n)$. Of course the values of $\tilde{f}$ at the points $\tilde{z}_i$ are not unrelated. In the examples discussed above, they were related to each other by phase multiplications. In the above examples both Σ and $\tilde{\Sigma}$ were topologically a sphere, but in the general case, $\tilde{\Sigma}$ will have a higher genus. To see this, and to prepare for the discussion of the general setup, we will consider some more examples.

Suppose we wanted to compute a scattering process involving four twisted states, each twisted by a Z_2. Take $z_1, ..., z_4$ as the points where we have inserted the twisted states. The twisted string variables pick up a minus sign as they go around the points $z_1, ..., z_4$. To make them single-valued, we consider a two-fold covering of this surface. We take two copies of the surface, and we make a cut in each one along the lines connecting $z_1 - z_2$ and $z_3 - z_4$. We open up the cuts and glue the corresponding cuts on each sheet (see fig. 6).

We obtain a torus! The Z_2 transformation in this case corresponds to a rotation by π along the axis piercing the torus at $z_1, ..., z_4$ as shown in fig. 6. The fixed points of this transformation are precisely $z_1, ..., z_4$, which correspond to the insertion points of the twisted states. Near each of these points the Z_2 transformation acts by

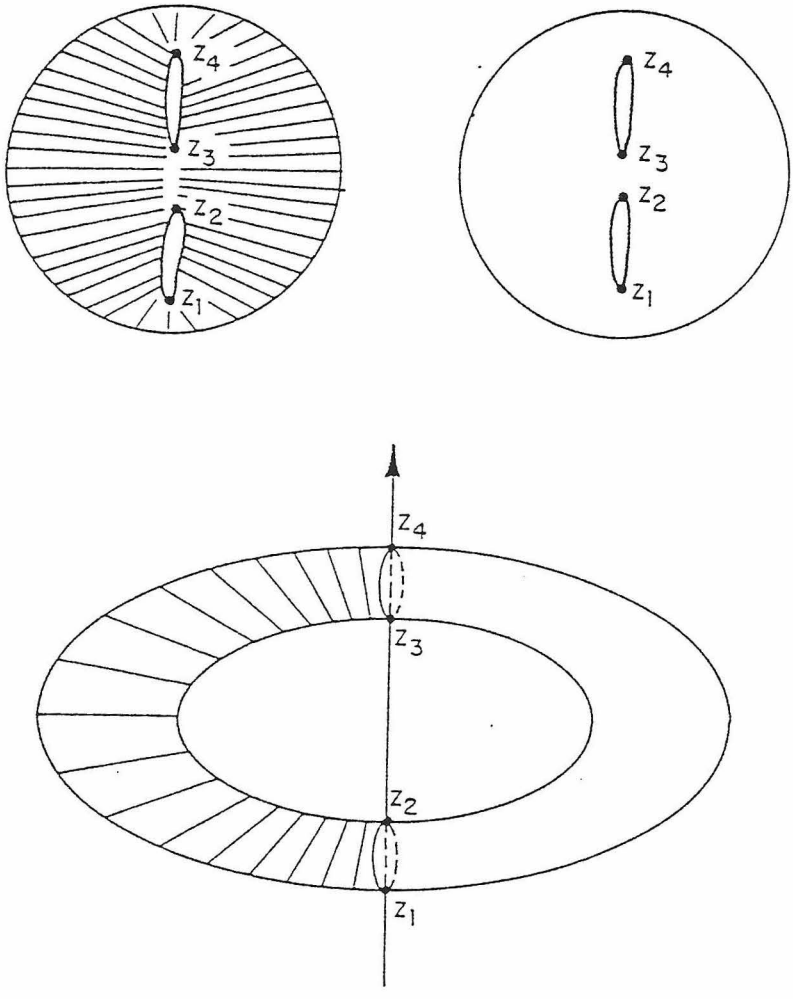

Figure 6. Two copies of a sphere with cuts join to form a torus. The Z_2 symmetry is generated by a rotation by π about the $z_1 - z_4$ axis.

a rotation by π, i.e., $\tilde{z} \mapsto -\tilde{z}$. The space of functions on the torus can be decomposed into two subspaces. The ones odd under the Z_2 transformation and the ones which are even under it. The string coordinates which describe the twisted states come from the functions which are odd under this transformation. The string coordinates which are untwisted correspond to even functions.

This construction could be easily generalized to consider the scattering of four

twisted states in a Z_n orbifold (two with twists $e^{2\pi ik/n}$ and two with $e^{-2\pi ik/n}$). We simply take n copies of the sphere cut as before, and glued as shown in fig. 7. We obtain a surface with $n-1$ handles (i.e., the genus is $n-1$).

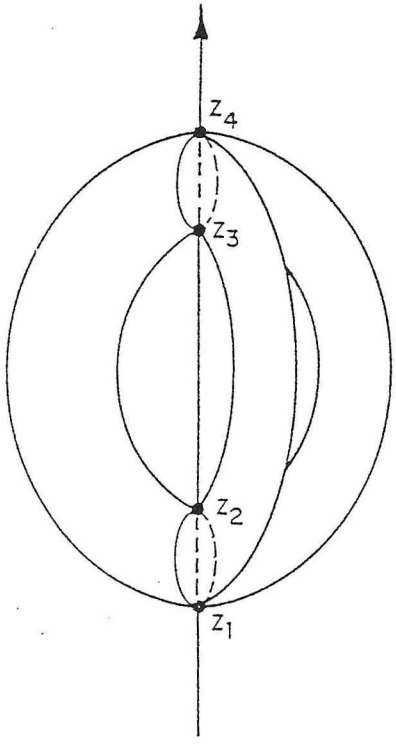

Figure 7. n copies of a sphere with cuts join to form a surface with genus $n-1$ (the $n=3$ case is shown here). The Z_n symmetry is generated by a $2\pi/n$ rotation about the $z_1 - z_4$ axis.

The Z_n in this case is generated by a $\frac{2\pi}{n}$ rotation about the axis piercing the surface at $z_1, ..., z_4$ as shown in fig. 7. Again, one could restrict to functions transforming appropriately under this transformation to obtain the desired multi-valuedness as a function on the sphere.

So we see that to consider interactions on orbifolds we are led to Riemann surfaces with automorphisms, i.e., with some symmetry transformations which takes the surface onto itself. We will now discuss the general setting of Riemann surfaces with automorphisms.

The transformations on the surfaces that we have been considering have been of a special type: They are analytic in the complex parameters, and do not mix z with $\bar{z}$, or in the mathematical language, they preserve the complex structure of the surface. This is necessary for strings. The left moving and right moving string degrees of freedom are described (after a Wick rotation) by analytic and anti-analytic fields. We cannot mix them up if we are to retain the string interpretation. So in considering the covering surface $\tilde{\Sigma}$, we can only divide it out by the group which leaves the complex structure untouched.

The automorphism group for the sphere is simply $SL(2,C)$. Any element of this group has only two fixed points. So let us discuss the situation for the next case, namely the torus. All the tori have a continuous $U(1) \times U(1)$ symmetry, which corresponds to $z \mapsto z + a$ for an arbitrary complex parameter a. This is the symmetry implying that torus has no special points. We would not want to divide out by a continuous symmetry, because we would end up with a lower dimensional space. Dividing by a finite discrete subgroups of $U(1) \times U(1)$ would also be of no use, because they give back a torus (with different periods). What other symmetries are there? Let us parametrize a torus by coordinates (σ_1, σ_2) with the complex coordinate defined by $z = \sigma_1 + \tau\sigma_2$, where τ is a complex parameter and σ_i run from 0 to 1. One would immediately think of modular transformations as symmetries of the torus. These are the $SL(2,Z)$ transformations

$$\sigma_1 \to a\sigma_1 + b\sigma_2 \qquad \sigma_2 \to c\sigma_1 + d\sigma_2 \tag{4.4}$$

with a, b, c, d integers satisfying $ad - bc = 1$. But viewed as transformations on the complex variable z, they are not analytic (for almost all values of τ), except for the transformation $(\sigma_1, \sigma_2) \to (-\sigma_1, -\sigma_2)$, which takes $z \mapsto -z$. This transformation leaves four points of the torus fixed (see fig. 8), so if we divide out the torus by this transformation we get a space with four Z_2 branch points.

This space is a sphere as we could easily see by finding the fundamental domain or noting that the deficit angle from each fixed point is π, giving a total of 4π, which

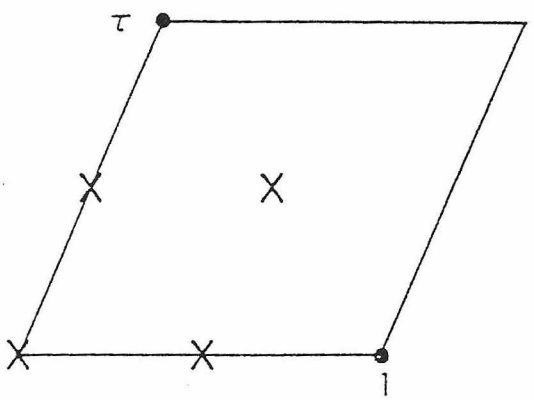

Figure 8. A torus is specified by the complex parameter τ. The Z_2 symmetry of the torus fixes four points of the torus each marked by a cross.

is what the deficit angle should be for a sphere (by the Gauss-Bonnet theorem). So we obtain a sphere with four branch points, which we had already discussed (see fig. 6).

We mentioned above that the modular transformations are not analytic (global diffeomorphisms need not be analytic). If we view the torus with a Euclidean metric and a complex structure compatible with the metric (specified by τ which determines the ratio of the length of the sides and the angle between them (see fig. 8)), the analytic transformations are precisely the ones which leave the metric invariant, i.e., are accomplished by rotations $O(2)$. But the rotations should map the torus to itself and so one can even say more: The rotation should admit an integral representation. This is because we can choose as our basis the vectors generating the lattice, and a transformation preserving the lattice takes each vector to another vector which could be expressed as an integral linear combination of the basis vectors. In particular, the trace of the transformation, which is independent of the choice of the basis, should be an integer. In a complex basis, a rotation could be represented as a diagonal matrix $(\omega,\bar{\omega})$, where ω is a phase. For the trace of this transformation to be an integer we must have $\omega + \bar{\omega} = integer$, which implies that twice the real part of ω is an integer. Since $|\omega| = 1$, the only possibilities are

$$\omega = e^{2\pi ik/4}, \qquad \omega = e^{2\pi ik/6} \tag{4.5}$$

where k is an integer. In the first case we obtain a space with Z_4 symmetry, and the other a space with Z_6 symmetry (the corresponding tori are shown in fig. 9 and fig. 10).

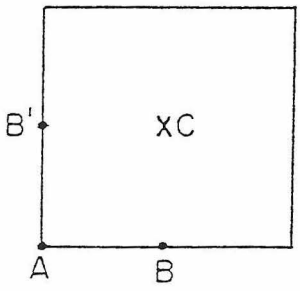

Figure 9. The symmetries of Z_4 torus is generated by a rotation by $\frac{\pi}{2}$. It fixes A and C and interchanges B with B'. We get three orbits (A), (B), and (C).

The Z_2 symmetry which is common to all tori is also seen here. $Z_6 = Z_3 \times Z_2$ and Z_4 includes Z_2 as a subgroup. These are the *only* tori with extra automorphisms. In a sense Riemann surfaces with (extra) automorphisms are rare objects. They become even less frequent for higher genus surfaces. It is very instructive to see what orbifolds we obtain by dividing out these two tori, which we denote Z_6 and Z_4, by some subgroups of their automorphism groups.

If we divide by the Z_2 subgroup for either of the cases, we obtain what has already been discussed. For Z_4 this is the only proper subgroup. But for Z_6 we could consider the subgroup Z_3.

Under the Z_3 transformation, the Z_6 orbifold has three fixed points (see fig. 9), so the space obtained by dividing the Z_6 torus by Z_3 will have three branch points. The resulting space will be flat everywhere except at three points each with curvature singularity corresponding to a deficit angle of $\frac{4\pi}{3}$. So altogether we get $3 \cdot (\frac{4\pi}{3}) = 4\pi$, as the net deficit angle and thus the resulting space is a sphere. Going around the branch points on the sphere corresponds to going around the fixed points on the torus by $\frac{2\pi}{3}$. So we could use this torus for discussions of the interactions of three twisted states of a Z_3 orbifold (each state coming from a sector twisted by $e^{2\pi i/3}$).

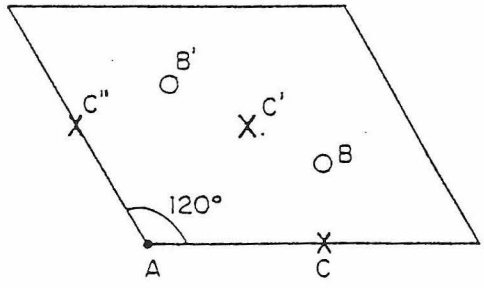

Figure 10. The symmetries of Z_6 torus is generated by a rotation by $\pi/3$(denoted by α). α and α^{-1} fix only A, α^2 and α^{-2} fix A, B, B'. α^3 fixes A,C,C',C''. α mixes B with B', and C with C' and C''. The fixed points form three orbits (A),(B), and (C).

Now suppose we wanted to divide out the Z_6 torus by the entire Z_6 group. To see how this goes we will have to look at the fixed points of the Z_6 transformations more carefully. Let us denote the generator of Z_6 by α. Under α and α^{-1} the torus has only one fixed point, the origin. Under α^2 and α^{-2} the torus has the three fixed points already discussed. Under α^3 we have the four fixed points of the Z_2 case discussed before. Under the action of Z_6 the fixed points form three distinct orbits shown in fig. 9. So there are only three classes of fixed points. The resulting space will have only three branch points, one for each class of fixed points.

We can compute the Euler characteristic of the orbifold resulting from dividing out by Z_6 by using the Lefschetz fixed point theorem. According to the Lefschetz theorem, the Euler characteristic of the resulting space is the average of the Euler characteristic of the subspaces left invariant by each group transformation:

$$\chi = \frac{1}{N}\sum_g \chi(g), \tag{4.6}$$

$\chi(g)$ being the Euler characteristic of the subspace left invariant by g, and N is the order of the group. In our case we obtain $\chi = \frac{1}{6}(0+1+3+4+3+1) = 2$, so the resulting space is again a sphere. The sphere is branched over three points. We can find what branching each corresponds to by finding out how many times going around

each of the points on the sphere corresponds to going around once on the torus. In this way we find out that the branching numbers are $2, 3$ and 6, corresponding to an interaction in the Z_6 orbifold of states twisted by $e^{(2\pi i)/2}, e^{(2\pi i)/3}$ and $e^{(2\pi i)/6}$ respectively. One can similarly consider dividing the Z_4 torus by Z_4. We obtain, once again, a sphere with three branch points, with branching numbers $2, 4$ and 4.

Let us see how these ideas work in the general case. Consider a Riemann surface $\tilde{\Sigma}$ having a discrete group G consisting of N elements as its automorphism group. We would like to consider dividing out $\tilde{\Sigma}$ by the action of G. Suppose P is the fixed point of one of the group transformations. Let us consider the little group of the point P, that is the subgroup H of G which leaves the point P invariant. By a theorem in mathematics [35] this is a cyclic group. To get a feeling for this theorem, note that we can always choose coordinates such that the action of the group is locally represented by $z \mapsto \omega z$, where ω is a phase. We can choose the element g in the little group which corresponds to the smallest such phase. This will generate a cyclic group of order n if $\omega^n = 1$. Now suppose that there is another group element h which has P as a fixed point. Then by our assumption about ω, it must be that for some m, hg^m acts as identity in the neighborhood of P, and by analyticity must be the identity everywhere. So we see that $h = g^{-m}$, and therefore the little group is generated by g. We can assign a number ν to each fixed point, corresponding to the order of the cyclic group which fixes it. Under the action of the group G, the point P forms an orbit, whose order is $|G|/|H| = N/\nu$. After dividing out by G all the points in a given orbit correspond to a single point.

The fact that G does not act freely implies that the resulting space has singularities at the fixed points. So the resulting space is an orbifold! But the singularities are only curvature singularities, and the orbifold is topologically a Riemann surface. We do not have to "blow up" the singularities to obtain a manifold. This is why the Lefschetz formula needs no modification in the computation of the Euler characteristic (otherwise we would have to use a formula such as the one discussed in [17]). This is particular to two dimensions. For higher dimensional spaces, we would have to resolve the singularities to obtain a smooth manifold. The fact that for the inter-

actions on orbifolds one has to consider a two dimensional world sheet orbifold which is topologically just a Riemann surface may be ultimately the reason why orbifold singularities of the imbedding space are harmless.

Let us find the Euler characteristic χ of $\Sigma = \tilde{\Sigma}/G$. Let P_i^k denote the fixed point set of G, where i runs over the set of orbits, and k runs over the points in a given orbit. For each orbit labeled by i we assign ν_i as the order of the little group of each point in the orbit (the order of the little group of each point in the orbit is independent of the point chosen, because all the little groups for points in a given orbit are conjugate). The index k therefore runs from 1 to $\frac{N}{\nu_i}$. Let us denote by $\tilde{\chi}$ the Euler characteristic of $\tilde{\Sigma}$. We first delete the fixed points of G, of which there are $\sum_i \frac{N}{\nu_i}$. Then G acts freely on the remainder of the space and we get, by dividing out by G, a space with Euler characteristic

$$\frac{\tilde{\chi} - \sum_i \frac{N}{\nu_i}}{N}. \tag{4.7}$$

We then add one point for each deleted orbit, which means adding $\sum_i 1$ to the Euler characteristic. This gives the space Σ. Noting that the Euler characteristic is $2-(2\cdot genus)$, and denoting by $\tilde{g}$ and g the genera of $\tilde{\Sigma}$ and Σ respectively, we obtain

$$2\tilde{g} - 2 = N(2g-2) + N\sum_i (1 - \frac{1}{\nu_i}). \tag{4.8}$$

This result is known as the Riemann-Hurwitz relation. It could have also been derived using the Lefschetz fixed point theorem.

Now we are in a position to state our strategy for computing orbifold interactions. Suppose we are interested in computing the interactions on an orbifold obtained by dividing a space by the group G. We would like to compute the interaction of states twisted by some g_i emitted from points z_i of Σ at g^{th} loop order. We will then have to find another Riemann surface $\tilde{\Sigma}$ such that

$$\frac{\tilde{\Sigma}}{G} = \Sigma \tag{4.9}$$

and, in addition, each point z_i is covered by some points of $\tilde{\Sigma}$, whose little group is

the cyclic group generated by g_i or an element conjugate to it. In other words, if we start from any point Q near an insertion point z_i, and pick, arbitrarily, a point $\tilde{Q}$ in the covering space $\tilde{\Sigma}$ covering Q, close to the point P_i covering z_i, then as we go around a loop about z_i, the point we have chosen to cover Q moves continuously and comes back to a point in $\tilde{\Sigma}$ covering Q, which is not necessarily $\tilde{Q}$. The two points must be related by a group transformation which fixes the point P_i, and is in fact g_i (In the non-abelian case, other choices of the covering of the point Q would give us a group element conjugate to g_i, as is familiar from a gauge transformation of the parallel transport $Pe^{i\int Adx}$ in the Yang-Mills theory. The fact that we cannot assign an element of the group to each insertion point of a twisted operator, but only a conjugacy class, suggests that the interactions "know" that in the non-abelian case the states are formed in conjugacy classes of the group). In the non-abelian case the interaction law (2.3) gets modified beyond the tree level. To see this we represent the Riemann surface by a polygon with certain identifications. We consider n twisted states $g_1, g_2, ..., g_n$, and a base point P (see fig. 10). As the loop starting from P encircles the insertion points of the twisted states in order and comes back to itself, the covering point comes back, multiplied by $g_1 g_2 ... g_n$. If we were considering interactions on a sphere we could pull the loop over the other side of the sphere and obtain the group law (2.3).

But for Riemann surfaces of higher genus we may get a contribution from the boundaries of the polygon. The string variables need not be periodic at the boundary, i.e., as we go around non-trivial cycles on the surface they may be transformed by a group element. Let us denote the canonical cycles by a_i, b_i, and the corresponding twists on the fields by $\phi(a_i), \phi(b_i)$, then the modified rule for interaction is that

$$\prod_{k=1}^{n} g_k = \prod_{i=1}^{g} [\phi(a_i), \phi(b_i)] \tag{4.10}$$

with proper care of the orderings ([ ] denotes the group commutator and so for the abelian case we get back the group law (2.3)). Therefore, non-abelian orbifolds have the additional complication that various sectors mix beyond tree-level. For example,

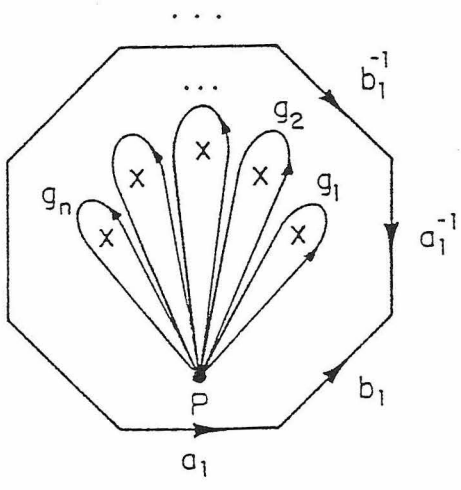

Figure 11. A Riemann surface could be represented by a polygon with identifications of the sides. A loop starting from P and encircling the insertion points of the twisted states $g_1, g_2, \ldots, g_n$ could be deformed to go around the boundary of the polygon.

the untwisted sector will mix at one-loop level with all the sectors that are twisted by a commutator element. In fact, sectors belong to G/C_G, where C_G is the commutator subgroup of G. If the group is simple the interactions between all sectors are allowed for sufficiently high order in the string loop expansion, because each product of group elements can be written as the product of commutators. This suggests that there may be something special about orbifolds constructed using simple groups. It may be an interesting way to get a hierarchy of couplings. For example, suppose we are interested in computing the strength of coupling between three states coming from sectors g_1,g_2, and g_3. If $g_1g_2g_3 = 1$, these states interact at the tree-level of the string theory. However, if $g_1g_2g_3 = [h,k]$ for some group elements h and k which don't commute, then they can only interact at one-loop level and the strength of their coupling is suppressed by $\kappa = e^{-<\Phi>}$ where Φ is the dilaton field. The corresponding suppression at g-loop is κ^g.

To complete the picture, we have to discuss how the multi-valued string variables X^μ on Σ "lift" to single-valued variables $\tilde{X}^\mu$ on $\tilde{\Sigma}$. We note that the group G is a symmetry of the theory and therefore acts on string variables. So the natural transformation law for $\tilde{X}^\mu$ is

$$\tilde{X}^\mu(g \cdot \tilde{z}) = (g \cdot \tilde{X})^\mu(\tilde{z}). \tag{4.11}$$

This transformation law, together with the assumption of how $\tilde{\Sigma}$ covers Σ, implies that the variables X^μ have the appropriate multi-valuedness about each insertion point of a twisted state. If we were interested in ordinary single-valued string variables on Σ, we would have required on $\tilde{\Sigma}$ that

$$\tilde{X}^\mu(g \cdot \tilde{z}) = \tilde{X}^\mu(\tilde{z}). \tag{4.12}$$

The action for the string variables $\tilde{X}$ will be the same as the usual action except that we will have to divide it by N, because $\int_{\tilde{\Sigma}} = N \int_{\Sigma}$, which follows because $\tilde{\Sigma}$ covers Σ generically N times. So we have for the action

$$S = \frac{1}{2\pi N} \int_{\tilde{\Sigma}} \partial \tilde{X} \overline{\partial} \tilde{X}. \tag{4.13}$$

The genus of the covering space could be computed from (4.8) just knowing the order of the group, and the order of the string loop expansion (g), and the knowledge of which twisted states one would like to consider (this gives us ν_i). In contrast to the simplicity of the computation of the genus of the covering space, a simple representation of the covering space $\tilde{\Sigma}$ does not in general exist. For computations in string theory, one would like to have a concrete realization of the covering surface. It turns out that Riemann surfaces can be given relatively simple algebraic representations and it will be convenient in discussions of interactions on orbifolds to use such a representation. Independently of the question of interactions on orbifolds, an algebraic view of Riemann surfaces may allow reformulation and generalizations of string theories (see below). We will therefore give a brief description of Riemann surfaces as *algebraic curves.*

Before considering the more general definition of an algebraic curve, we shall consider an example. Let us go back to the interaction of four twisted states in a Z_2

orbifold. Let us denote the four insertion points by $z_1, ..., z_4$. We saw that there was a cover of the Riemann sphere which is 2 to 1 everywhere, except over the branch points, over which it is 1 to 1. Consider an analytic "function" $w(z)$, satisfying

$$w^2 = \prod_i (z - z_i). \tag{4.14}$$

Let us consider the set of points (z, w) in C^2 satisfying (4.14) and let us denote it by $\tilde{\Sigma}$. $\tilde{\Sigma}$ is a complex, one dimensional subspace of a two dimensional complex space. It is called an algebraic curve. $\tilde{\Sigma}$ is in fact a Riemann surface (after compactification by including the points at infinity).

For each point z on the complex plane, except for z_i, there are two values of w, differing by a minus sign, satisfying (4.14). For $z = z_i$, only one value of w satisfies (4.14), namely $w = 0$. Therefore, $\tilde{\Sigma}$ covers the sphere in a 2 to 1 fashion, except over the points z_i, for which it is 1 to 1 (the projection is given by $(z, w) \mapsto z$). We have therefore recovered our previous space, and $\tilde{\Sigma}$ must in fact be a torus. $\tilde{\Sigma}$ has clearly a Z_2 symmetry given by $(z, w) \mapsto (z, -w)$, with four fixed points $(z_i, 0)$. We can think of w and z as coordinate choices for the surface $\tilde{\Sigma}$. Away from the branch points, z serves as a good coordinate, and around the branch points w is a good coordinate. Also, we can think of $\tilde{\Sigma}$ as the space on which w becomes single-valued.

$\tilde{\Sigma}$ is a torus and so we should be able to identify a choice of two canonical cycles a and b which intersect at one point. Let us use the z coordinates. Consider the two cycles drawn in fig. 12.

For each point z on the cycle we have two possible choices of w, i.e., two points of $\tilde{\Sigma}$. Pick a point of each cycle and choose one of the two possible values of w, i.e., choose a point on $\tilde{\Sigma}$ over that point. As we go around the cycles drawn, the value of w changes continuously and comes back to itself. This means that the cycles we have drawn on the z plane are actually closed loops of $\tilde{\Sigma}$. If we had chosen cycles to enclose an odd number of branch points, they would not have corresponded to closed cycles of $\tilde{\Sigma}$. $w = \sqrt{\prod_i (z - z_i)}$ would go to minus itself in going around such a loop.

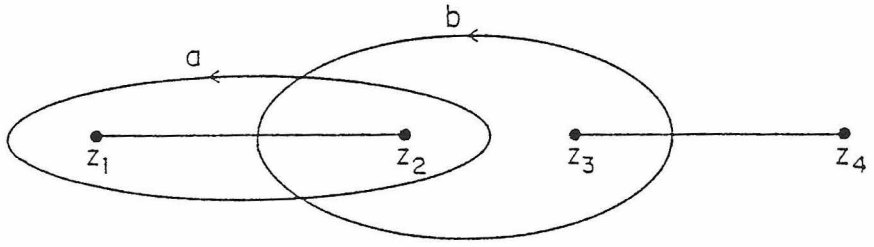

Figure 12. The two cycles a and b correspond to the canonical cycles of the torus. Even though they meet at two points in the projection of the torus on the z plane, they meet only at one point on the torus.

The two cycles a and b have two intersection points on the z-plane, but as cycles in $\tilde{\Sigma}$ they intersect only at one point: If we assign the same value of w at one of their intersection points, it is easy to see that at the other intersection point in the z plane the value of w differs by a minus sign, and corresponds to two different points of $\tilde{\Sigma}$.

One may wonder what is to be gained from this representation of the surface? It turns out that all the objects of interest could be easily written down and computed in this representation of the surface (of course for the case of torus we have other simpler methods, but in general this representation may be the simplest to work with). For example consider the holomorphic one-form on the torus (the holomorphic one-forms are analytic differential forms of the type $f(z)dz$ without any poles). It is not too difficult to write down the holomorphic one-form of this surface. In fact it is simply given by $\omega = \frac{dz}{w}$. To see this, we have to make sure that it has no singularities. One might at first sight think that it has singularities at $w = 0$, but this is not so because there z is not a good coordinate. At such points $dz \sim wdw$ and so $\omega \sim dw$ and we get no singularities. There is one other potential source of singularity at $z = \infty$. Defining $z' = (\frac{1}{z})$, we see that $dz' = -\frac{dz}{z^2}$ and $w \sim z^2$ near $z = \infty$, so $\omega \sim dz'$, and thus behaves nicely at $z' = 0$.

We could also ask about the modular parameter τ defining the torus. That could easily be obtained by

$$\tau = \frac{\int_b \omega}{\int_a \omega}. \tag{4.15}$$

One can use this expression to get a relation between crossed ratios of z_i and τ (in the next section we will get this relation also by a different method).

It is now not too difficult to write down a representation of the interaction of any number of twisted Z_2 states. One simply takes as many branch points as there are twisted states and modifies (4.14) in the obvious way to get a representation of the surface with Z_2 symmetry branched over the desired point (this is called a hyperelliptic surface, the elliptic case being the torus). One could also do an analogous construction for scatterings on Z_N orbifolds. If we are interested in the scattering of states twisted by $e^{2\pi i m_1/N}, ..., e^{2\pi i m_k/N}$ we simply consider the surface represented by $w^N = \prod_i (z - z_i)^{m_i}$.

Unfortunately a general method for writing down a simple representation for a Riemann surface suitable for an arbitrary interaction on an arbitrary non-abelian orbifold does not exist. Nevertheless, by an existence theorem, a suitable surface does always exist. But it is clear that once the representation is written down, one could construct the objects of interest, for instance the holomorphic one forms and the period matrix (which will be of use in sect. 6).

To see how the suitable surfaces of some complicated orbifold groups can be represented, consider the surface defined by $w^7 = z(z-1)^2$. This rather simple looking algebraic relation gives rise [35] to a Riemann surface of genus 3 with a non-abelian automorphism group of order 168! If we divide out this Riemann surface by its automorphism group we obtain a sphere, with branch points at $0, 1$ and ∞ (corresponding to certain elements of order $2, 3$ and 7). This surface could be used for the interaction of three twisted states of any orbifold obtained by dividing out by the 168 group corresponding to certain group elements of order $2, 3$ and $7^\star$.

$\star$ This group, 168, is also encountered in the classification of orbifolds with $SU(3)$ holonomy [18]. It is the only simple subgroup of $SU(3)$. It turns out that this group plays a special role for Riemann surfaces with automorphisms: It can easily be shown, using (4.8), that the order of

Having seen these examples, it seems appropriate to discuss what an algebraic curve is more generally. Consider a polynomial in two variables z and w of the form

$$P(z,w) = a_n(z)w^n + ... + a_1(z)w + a_0(z), \tag{4.16}$$

where $a_i(z)$ are polynomials in z. Consider the subspace of (z,w) satisfying $P(z,w) = 0$. This defines in general a singular subspace of C^2 (a Riemann surface which may be potentially degenerate or self intersecting as it sits in C^2). One of the most important theorems in the theory of Riemann surfaces [34,35] is that every Riemann surface can be represented in this way! One is tempted to view this purely algebraic formulation of Riemann surfaces as significant for string theory: In the path integral formulation of point particle quantum mechanics one sums over all paths $x(t)$, where x and t are real variables. One is tempted to think of string theory algebraically as an extension of this to complex variables. We simply view the z as the "complex time" variable and w as playing the role of x. It is not correct to view w as a single-valued function of z, but actually $x(t)$ in the point particle path integral is not to be thought of as a single-valued function either. In point particle theory one should sum over *all* paths, and the paths which go backwards in time do not correspond to single-valued functions of t[†].

Before continuing our discussion of interactions on orbifolds, it may be useful to discuss the meaning of surfaces with automorphisms for string theory. Let us recall

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the automorphism group of a Riemann surface of genus $g \geq 2$ is bounded by $84(g-1)$ [35]. This upper bound is first achieved for $g = 3$, giving the group 168 and the surface discussed above. It is a fact that this upper bound is achieved for infinitely many values of g. Each time this happens, we get a group G, with a normal subgroup H, such that G/H is the 168 group. So this group is in some sense universal for surfaces with maximal symmetry. It is tempting to conjecture a relation between this fact of Riemann surface theory, with the correct orbifold choice of space time. (We can choose the jacobian lattice of this genus three surface, which is a three dimensional complex torus, and consider the group 168 acting on it through the jacobian map (see sect. 6) to construct an orbifold which may be interesting phenomenologically.)

† This suggests a generalization of string theories: Riemann surfaces with a finite genus correspond, as discussed above, to polynomials $P(z,w)$ of finite degree. These correspond to very special paths from the point of view of point particles. We could consider an *arbitrary* analytic function $F(z,w)$ which would correspond to an arbitrary motion of string world sheet w parametrized by z. This will in general give rise to an infinite genus Riemann surface. Unfortunately the mathematics of such surfaces is not well understood.

that we could label an arbitrary torus by the complex parameter τ, with $Im\,\tau > 0$. Under modular transformations we have $\tau \to \frac{a\tau+b}{c\tau+d}$. Modular transformations act on the points of the upper half plane. Some modular transformations leave some points fixed. For example $\tau \to -\frac{1}{\tau}$ fixes $\tau = i$ and $\tau \to -\frac{1}{\tau+1}$ fixes $\tau = e^{2\pi i/3}$. What is the meaning of a fixed point of a modular transformation? A fixed point of modular transformation corresponds to a surface with a global diffeomorphism which does not change its complex structure, or in other words, it is a Riemann surface with automorphisms. The two values of τ we just mentioned (and the ones related to them by modular transformations) correspond to the only tori with automorphisms (apart from the Z_2 automorphism which is common to all tori). Dividing out the upper half plane by the action of $PSL(2,Z)$ gives rise to the moduli space of tori, which could be represented by the fundamental domain ($-\frac{1}{2} \leq Re\,\tau \leq \frac{1}{2}$, $Im\,\tau > 0$, and $|\tau| \geq 1$) with the sides properly identified. Since modular transformations do not act freely, moduli space is an orbifold, the fixed points of the modular transformation giving rise, in this case, to conical singularities at $\tau = i, e^{2\pi i/3}$. (Sometimes it is convenient to compactify the moduli space and include $\tau = \infty$, which will also give rise to a conical singularity.)

The moduli space of Riemann surfaces for larger values of genus is also an orbifold having singularities corresponding to surfaces with extra automorphisms. This is rather peculiar: To discuss interactions for orbifolds, we will have to consider surfaces with automorphisms, which are therefore themselves singular points of the moduli space which is itself an orbifold!

5. Twisted Interactions

In sect. 3 we saw how any amplitude involving at most two twisted states could be computed. It was shown that one can construct amplitudes involving two twisted states by considering vertex operators for emission of untwisted states acting on the twisted Hilbert space. Working in the twisted sector saved us from having to write a complicated vertex operator for the emission of a twisted state (which takes one Hilbert space to another)*. In this section we will follow a path integral approach in computing amplitudes involving more than two twisted states. We will first discuss some general features of twisted operators and following that we work out in detail some amplitudes involving four Z_2 twisted states.

Let us denote a twist operator for the emission of the ground state of a twisted sector g by $\Lambda_g(z)$. This means that as the string variables $X(w)$ (which we shall take to be bosonic in this section) are transported around the point z, they come back to themselves transformed by g. In the operator approach this would take us from the ground state of the untwisted sector to the sector twisted by g^{-1}. The first thing we would like to know about this operator is its conformal weight. The simplest way to compute this is to note that since it takes the untwisted ground state to the twisted ground state, the conformal weight is simply the difference in the energy of the ground states. If we label the eigenvalues of g by $e^{2\pi i\eta_i}$ (and its complex conjugates), as (3.8) shows the energy of the twisted ground state is higher by

$$\delta = \frac{1}{2}\sum \eta_i(1-\eta_i), \tag{5.1}$$

and this is therefore the conformal weight of Λ_g (η_i's are between 0 and 1). However, since we would like to have a path integral approach, it would be nice to see this

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$\star$ In some very special cases in which the underlying torus corresponds to the weight lattice of a Lie group, the problem of constructing twist operators may be simplified by relating it to a shifted lattice of the group [27] and representing the twisted vertex operators by the standard vertex operators of the shifted lattice.

result also using operator product expansions. Let T denote the energy momentum tensor. We would like to show that

$$T(z)\Lambda_g(0) \sim \frac{\delta}{z^2}\Lambda_g(0) \ . \tag{5.2}$$

Since this is a local property, to establish this it suffices to consider the complex plane. We note that $\Lambda_g(0)$ has the effect of making X multi-valued. We recall that T is defined by a limiting procedure [36] (with appropriate normalization)

$$T(z) = lim_{w\to z}(\partial X(z)\partial X^*(w) - \frac{1}{(z-w)^2}). \tag{5.3}$$

In the absence of twist operators the expectation value of T is zero. In the presence of twist operators the correlation $\partial X(z)\partial X^*(w)$ changes, and the leading singularity does not vanish anymore. To establish (5.2), it is sufficient to check it for one eigenvalue at a time, say the eigenvalue $\eta = k/n$ (we are assuming that the order of g is n). Let $\omega = e^{-2\pi i/n}$. From the results of sect. 3 (or the last section), we get

$$T(z)\Lambda(0) \sim lim_{w\to z}\partial_z\partial_w log\ [\frac{(\tilde{z}-\omega\tilde{w})^{\omega^{-k}}(\tilde{z}-\omega^2\tilde{w})^{\omega^{-2k}}\cdots}{(\tilde{z}-\omega\tilde{w})(\tilde{z}-\omega^2\tilde{w})\ldots}]\Lambda(0) \tag{5.4}$$

where $z = \tilde{z}^n$ and $w = \tilde{w}^n$ (in the language of last section we are going to the n-fold covering of the sphere). Using the identity

$$\sum_{l=1}^{n-1}\frac{\omega^l(\omega^{-lk}-1)}{(1-\omega^l)^2} = \frac{1}{2}k(n-k), \tag{5.5}$$

we obtain (5.2).

In considering excited states of the twisted sectors we will have to consider other operators, for instance $\frac{\partial X(z)}{\partial z^{k/n}}\Lambda_\eta(z)$. By the method we just discussed, it could be easily seen that this operator has conformal weight $\frac{1}{2}\eta(1-\eta)+\frac{k}{n}$. This is what one would expect, because the power of z in the derivative, instead of the standard value of one, is

$\frac{k}{n}$. It is possible again to view this as the change in the energy eigenvalue. Let us now construct the vertex operators for the emission of twisted states in bosonic orbifolds. The lightest state of a twisted sector is specified by the choice of the uncompactified momentum k (note that there are no momenta in the twisted directions in a twisted Hilbert space, there is instead a discrete choice of the center of mass), such that $k^2/4 = -2 + 2\delta$. This suggests considering the vertex operator $e^{ikX(z)}\Lambda_g(z)$ which has conformal weight $-k^2/8 + \delta = 1$ as the vertex operator for the emission of the ground state of the twisted sector. Similarly, for the vertex operator for emission of excited states (in the twisted or untwisted directions) of the twisted sector, we should consider operators with gradient terms, for example of the form $e^{ikX}(\partial_{\tilde{z}^k}X)\Lambda_g$. Once again, these would correspond to vertex operators with the correct weight.

Before considering a concrete example, we will make a brief remark about the center of mass of the twisted string. The vertex operator we have discussed corresponds to the emission of a twisted state whose center of mass was at the origin ($X = gX$ at the branch points, implying that $X_{cm} = 0$). This could always be done locally on the world sheet by a choice of the origin of the torus. But globally, if we are considering the interaction of several twisted states, the *relative* positions of the center of mass cannot be changed by a shift of the coordinate system. This could be handled in the path integral by decomposing X into a background classical piece and a quantum fluctuation, in such a way that X_{cl} is a stationary point of the action $\partial\bar{\partial}X_{cl} = 0$, and that X_{cl} at the insertion points of the twisted states correspond to the center of mass of the twisted state.

The main difficulty in computing the amplitudes in an orbifold interaction involves the computation of $< \Lambda_{g_1}...\Lambda_{g_l} >$. An operator approach is rather difficult*.

$\star$ In [29,30], one may find the explicit construction of the vertex operator Λ_g when $g^2 = 1$, and the computation of correlation of four Z_2 twist operators. These works were motivated by attempts to construct off-shell string amplitudes. In these references the conformal weight of the twist operator was found to be 1 only when there were 16 twisted directions. This number is the same as the number of Z_2 twistings one has to do to get a twisted sector whose ground state is massless [22] , and so the conformal weight of the twist operator is the same as that of the twist operator multiplied by e^{ikX}. From this point of view the fact that twisting 16 directions gives rise to a conformal weight 1 operator is not surprising.

Instead we will consider the path integral approach, which we have outlined in the last section. This approach has the advantage that it easily generalizes to more complicated orbifolds.

Let us go back to our bosonic Z_2 orbifold with d compact dimensions. Let us consider the interaction of four twisted states with $k_i^2/4 = d/8 - 2$. The amplitude is given by

$$\begin{aligned} Amp &= g^2 \int < e^{ik_1 X(z_1)} \Lambda_{Z_2}(z_1) ... e^{ik_4 X(z_4)} \Lambda_{Z_2}(z_4) > \\ &= g^2 \int < e^{ik_1 X(z_1)} ... e^{ik_4 X(z_4)} >< \Lambda_{Z_2}(z_1) ... \Lambda_{Z_2}(z_4) > . \end{aligned} \tag{5.6}$$

(the last equality follows from the fact that k_i has components in untwisted directions only). If we fix the points z_1, z_2, z_4 to correspond to $0, 1, \infty$ and multiply by the factor $|z_1 - z_2|^2 |z_2 - z_4|^2 |z_4 - z_1|^2$ to fix the $SL(2, C)$ freedom, we get

$$Amp = g^2 \int \frac{d^2 z_3}{4\pi} |1 - z_3|^{-k_2 k_3/2} |z_3|^{-k_1 k_3/2} F(z_3), \tag{5.7}$$

where $F(z_3)$ is independent of the momenta k_i and contains the information about the correlation of twist operators.

How can we compute $F(z_3)$? The main difficulty in computing F is the twist correlation functions. Let us follow the approach suggested in the last section. Namely let us formulate our strings not on the sphere with branch points, but consider instead a space $\tilde{\Sigma}$ which is a 2 to 1 covering of the sphere everywhere except at the four branch points, and such that the multi-valued functions on the sphere become single-valued on $\tilde{\Sigma}$. In this case $\tilde{\Sigma}$ is a torus (see fig. 8). The points on $\tilde{\Sigma}$ which cover the insertion points of the twisted operators are the midpoints of the torus. The coordinates X which are twisted on the sphere correspond to odd functions on the $\tilde{\Sigma}$, odd with respect to the automorphism $\tilde{z} \mapsto -\tilde{z}$, and the untwisted coordinates, i.e., single-valued functions on the sphere, are described by even functions on the torus. A particular value of z_3 corresponds to a torus with a particular modular parameter τ

(we shall give an explicit expression for $z_3(\tau)$ below). So on $\tilde{\Sigma}$ we don't integrate over the insertion points of the vertex operators; they are fixed at the midpoints of the torus. However, we will integrate over the moduli (τ) of the torus. This corresponds to integration on Σ of the insertion point of the vertex operator at z_3 (the other three insertion points being held fixed by $SL(2,C)$ invariance at 0,1 and ∞).

To compute the twisted four point amplitude, we will have to integrate over τ, with twisted coordinates corresponding to odd functions on the torus. Unfortunately there is a complicated measure factor $d^2\tau f(\tau)$ which should correspond to $d^2z_3F(z_3)$. However, we will use the following trick: If the even functions on the torus are the same as ordinary single-valued functions on the sphere, what prevents us from computing the standard four point Koba-Nielsen amplitude on the torus, restricted to even functions? In fact, strange as it sounds, so far as the functions are concerned there is no distinction between them and we can use the torus for the *standard tree level* computation. If $d^2\tau f_t(\tau), d^2\tau f_u(\tau)$ represent respectively the measure of integration for twisted and untwisted strings respectively, they should correspond by a change of integration variables to $d^2z_3F(z_3)$ and d^2z_3 (the respective measure of integrations on the sphere). This means that

$$F(z_3) = \frac{f_t(\tau)}{f_u(\tau)} \tag{5.8}$$

So all we will have to do in order to compute $F(z_3)$ is to see how the measure of integration changes on the torus when we consider the ordinary string interactions as opposed to the twisted string interactions.

A simple representation of the $\tilde{\Sigma}$ covering the sphere, is given by

$$w^2 = z(z-1)(z-z_3) \tag{5.9}$$

(the covering is branched over 0, 1, and ∞). The abelian differential is $\omega = \frac{dz}{w}$, and

as discussed in the last section

$$\tau = \frac{\int_b \omega}{\int_a \omega} \tag{5.10}$$

(see fig. 12, with the appropriate choices of z_i). This gives τ in terms of z_3 as

$$\tau = \frac{iK'(\sqrt{z_3})}{K(\sqrt{z_3})}, \tag{5.11}$$

where K and K' are the complete elliptic integrals [37]. However, we will derive an expression for z_3 in terms of τ in a different way below. Let's choose the identifications $\tilde{z}_1 = \tau/2$, $\tilde{z}_2 = 1/2$, $\tilde{z}_3 = (1+\tau)/2$, and $\tilde{z}_4 = 0$, with the midpoints of the torus (fig. 8).

For the torus $\tilde{\Sigma}$ the string action is given (for the path integral derivation of one loop amplitudes see [38])

$$\frac{1}{2} \cdot \frac{1}{2\pi} \int_{\tilde{\Sigma}} d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X \partial_\beta X, \tag{5.12}$$

where

$$h = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \quad \tau = \tau_1 + i\tau_2. \tag{5.13}$$

Note that as mentioned in section 4 we have divided the action by a factor of 2 to compensate the fact that $\tilde{\Sigma}$ covers Σ twice. In both the twisted theory (compactification on a d dimensional torus divided out by Z_2) and the ordinary bosonic string (corresponding to $d = 0$) there are untwisted coordinates ($26 - d$ of them) which are allowed to have non-trivial momenta. Therefore, in the computation of the scattering amplitudes on $\tilde{\Sigma}$ we will have to compute the Green functions between the midpoints $\tilde{z}_i$ for even functions. The Green functions for odd variables vanish at midpoints of the torus ($x(m) = -x(m)$ implies that $x(m) = 0$), so the ordinary Green function on the torus between fixed points is the same as the Green function restricted to even

functions, except that it should be multiplied by 2, because of the factor of 1/2 in (5.12). So we have the factor

$$G = \prod_{i<j} (\chi_{ij})^{-k_i k_j} \tag{5.14}$$

with χ given by [19,38]

$$\chi(\tilde{z}_i, \tilde{z}_j) = e^{-\pi (Im(\tilde{z}_i - \tilde{z}_j))^2/\tau_2} \left| \frac{2\pi\theta_1(\tilde{z}_i - \tilde{z}_j)}{\theta_1'(0)} \right| . \tag{5.15}$$

To put it in a form suitable for comparison with (5.7), we use momentum conservation to write

$$G = (\chi_{34}\chi_{12})^{k^2} \left(\frac{\chi_{13}\chi_{24}}{\chi_{12}\chi_{34}} \right)^{-k_1 k_3} \left(\frac{\chi_{23}\chi_{14}}{\chi_{12}\chi_{34}} \right)^{-k_2 k_3} \tag{5.16}$$

where $k^2/4 = -2 + d/8$. Since the points are on the midpoints of the torus, we can have the following relations between the various θ functions: $\theta_1(\frac{1}{2}|\tau) = \theta_2(0|\tau)$, $\theta_1(\frac{\tau}{2}|\tau) = ie^{-i\pi\tau/4}\theta_4(0|\tau)$, and $\theta_1(\frac{1}{2}+\frac{\tau}{2}|\tau) = e^{-i\pi\tau/4}\theta_3(0|\tau)$. Using these we can write G as

$$G = \left| \frac{2\pi\theta_3}{\theta_1'} \right|^{-16+d} \left| \frac{\theta_2^4}{\theta_3^4} \right|^{-k_1 k_3/2} \left| \frac{\theta_4^4}{\theta_3^4} \right|^{-k_2 k_3/2} . \tag{5.17}$$

We are evaluating the θ's at $(0|\tau)$. This is the only part of the amplitude where the momenta enter. So demanding equality for the momentum dependence of (5.17) with (5.7), we get the identifications (using the fact that z_3 is a holomorphic function of τ as seen from (5.11) , and up to a phase)

$$z_3 = \frac{\theta_2^4}{\theta_3^4}, \qquad 1 - z_3 = \frac{\theta_4^4}{\theta_3^4} \tag{5.18}$$

In fact the phase must be one, and we could in this way "derive" the Jacobi identity $\theta_2^4 + \theta_4^4 - \theta_3^4 = 0$, because $z_3 + (1 - z_3) = 1$ (we could presumably fix the phase by

doing an analogous computation for open strings). This identity of Jacobi is the one which guarantees that there are equal number of bosons and fermions at each mass level in the superstring theories. It is remarkable that we can *derive* it (up to a phase) by a computation in bosonic strings! Not only that, but we can also use it to express z_3 in terms of τ.

We see that the prefactor in (5.17) is different for $d = 0$ and $d \neq 0$, and so we get a contribution to $F(z_3)$, depending on whether we have d twistings, or no twistings:

$$F(z_3) = \left| \frac{2\pi\theta_3}{\theta_1'} \right|^d \cdot R. \tag{5.19}$$

The rest of the contribution R to F comes from two sources: One comes from noticing that in the path integral measure we have to integrate over the coordinates, and this gives rise to determinants. Depending on whether we are discussing twisted theory or the standard theory, we should integrate over odd functions as opposed to even functions (in the d twisted directions). This gives rise to the ratio of determinant of laplacian restricted to odd and even functions

$$\left(\frac{det_o \Delta}{det_e \Delta} \right)^{-d/2}. \tag{5.20}$$

Actually we should delete the zero mode in the determinant over even functions, and replace it with integration over the zero modes of X giving us a contribution of $\tau_2^{-d/2}$ to F [38]. The non-zero eigenvalues occur with the same multiplicity for even and odd functions, as could be easily seen by explicit enumeration of eigenfunctions. Therefore, the ratio of determinants is 1 and does not contribute to F.

The last contribution to F comes from remembering that in the twisted sector we still have 2^d different choices for the center of mass. It is not quite correct to say that the twisted string variables vanish on the midpoints, because all we should require is that they be mapped to a center of mass of a twisted sector. Two twisted strings coming from different fixed points could join, to go to an untwisted string with winding

and momentum, and then decay to two twisted strings with potentially different centers of mass. So we have topologically distinct possibilities for interactions. In the language of path integrals we decompose X into $X = X_{cl} + X_q$, where X_{cl} is a stationary point of the action ($\partial\overline{\partial} X_{cl} = 0$) and X_q represents a quantum fluctuation about this background. We should further require X_{cl} to satisfy $X_{cl}(-\sigma_1, -\sigma_2) = -X_{cl}(\sigma_1, \sigma_2)$ and map the midpoints of the torus to the centers of mass of the desired twisted states. By demanding that the origin of the torus get mapped to the twisted sector at $X_{cm} = 0$, we have

$$X^i_{cl} = R(k^i \sigma_1 + l^i \sigma_2) \tag{5.21}$$

and the centers of mass of the twisted strings correspond to $0, kR/2, (k+l)R/2, lR/2$, with k, l integers. So we get a contribution to F, which we may call the instanton contribution, as

$$\sum_{k,l} e^{-S} = \sum_{k,l} e^{-\frac{R^2}{4\pi}|k-\tau l|^2/\tau_2}. \tag{5.22}$$

The sum over k, l depends upon the scattering process being computed. If k_0 and l_0 denote two integers which map the world sheet to the desired centers of mass, so do $k = 2m + k_0, l = 2n + l_0$ for arbitrary integers m, n, as they correspond to translates of the center of mass by a lattice vector. Now we use the Poisson resummation formula

$$\sum_{m=-\infty}^{\infty} e^{-am^2+2abm} = \frac{\sqrt{\pi}}{a} e^{ab^2} \sum_{m=-\infty}^{\infty} e^{-\pi^2 m^2/a - 2i\pi bm}, \tag{5.23}$$

on m with $a = R^2/(\pi\tau_2)$ and $b = [(2n + l_0)\tau_1 - k_0]/2$ to make contact with operator formulation. So the sum is over arbitrary m, n and we finally get for (5.7) (replacing z_3 by z)*

$$Amp \sim \frac{g^2}{4\pi} \int d^2z \, |z|^{-k_1 k_3/2} \, |1-z|^{-k_2 k_3/2} \left| \frac{2\pi\theta_3}{\theta_1'} \right|^d$$

---

* The X_q contribution to our measure also appears in the operator approach of [29,30].

$$\times\left(\frac{1}{R^d}\sum_{m,n} e^{i\pi m k_0}\, q^{(W-P)^2/8}\ \bar{q}^{(W+P)^2/8}\right) \tag{5.24}$$

where $q = e^{i\pi\tau}$, $W = \frac{R(2n+l_0)}{\pi}$, and $P = \frac{2m\pi}{R}$. The overall normalization can be fixed, as usual, by demanding factorization. Note that if all four states come from the same fixed point, i.e., $k_0 = l_0 = 0$, the amplitude should be dual, i.e., it should be invariant under $z \to 1 - z$. This corresponds to $\tau \to \frac{-1}{\tau}$ as can be seen from (5.18). It is easy to see that (5.24) is invariant under this transformation.

We can study factorization by letting $z \to 0$. In this limit from (5.18) we see that $z \sim 16q$, and we get a pole at $\frac{1}{4}k^2 = \frac{1}{4}W^2 + \frac{1}{4}P^2 - 2$ with $W \cdot P = 0$, corresponding to an intermediate state of the untwisted sector with spacetime momentum $k = k_1 + k_3$, internal momentum P and winding W ($N = \widetilde{N} = 0$). The residue of (5.24) is seen to be

$$res. \sim \frac{g^2}{R^d}\, 2^{-(P^2+W^2)}. \tag{5.25}$$

with the P, W dependence of the residue coming from the instanton sum. This agrees with what we found in sect. 3 for the coupling between two twisted states with an untwisted state (after including a $1/\sqrt{R}^d$ factor in the definition of the vertex operators in sect. 3 which is necessary for the emission of normalized states). This is a strong check on the validity of our approach.

Finally let us briefly discuss what happens for the interactions of four excited states of the twisted sector, and in particular the excited states corresponding to $N = \widetilde{N} = \frac{1}{2}$. As discussed at the beginning of this section, the vertex operator will now involve a term of the form $\frac{\partial X^i}{\partial\sqrt{z}} = \frac{\partial X^i}{\partial \tilde{z}}$. The amplitude will be

$$Amp \sim \frac{g^2}{4\pi}\int d^2z\, |1-z|^{-k_2 k_3/2}\, |z|^{-k_1 k_3/2}\left(\frac{1}{\tau_2}\right)^{d/2}\left(\frac{2\pi\theta_3}{\theta_1'}\right)^{d+8}\mathcal{K}, \tag{5.26}$$

with

$$\mathcal{K} = \sum_{inst.} e^{-S_{cl}} < \partial X^{i_1}\bar{\partial}X^{j_1}...\partial X^{i_4}\bar{\partial}X^{j_4} > . \tag{5.27}$$

The quantity $\mathcal{K}$ is straightforward, but tedious, to evaluate. It could be computed

by noting that each X can be decomposed into a classical piece and a quantum piece (which gives the correlation $< \partial X_q^i \partial X_q^j >$ between various fixed points), and $\mathcal{K}$ will involve both quantum contractions and the gradient of the classical solution X_{cl}. Again we have checked that this amplitude is dual and factorizes correctly.

6. Generalizations

Our general approach follows the method used in the last section for constructing the interaction for four Z_2 twisted states. Namely, we will consider an appropriate covering space $\tilde{\Sigma}$ whose automorphism group contains the orbifold group G, and such that $\tilde{\Sigma}/G$ gives the world sheet Σ with appropriate branching points. As discussed in sect. 4, choosing the string variables to transform equivariantly with respect to the action of G on $\tilde{\Sigma}$ (4.11) gives us string variables on Σ with appropriate multi-valuedness, whereas the ordinary (untwisted) string variables transform as (4.12), corresponding to single-valued functions on Σ. Changing the moduli of $\tilde{\Sigma}$ corresponds to moving the branch points on Σ, and so integration over insertion points of twisted states could be accomplished by integration over moduli of $\tilde{\Sigma}$. This will involve a complicated measure of integration over moduli. Once this measure is found one can, by a change of variables of integration, choose the insertion points of states on Σ as the integration variables. The effect of twisted coordinates on Σ will be simply a jacobian factor in the integration over insertion points.

To find this measure of integration over Σ, we can follow the approach of the previous section: We see what *changes* in the contribution to the path integral over $\tilde{\Sigma}$ when we consider the contribution of the twisted (satisfying (4.11)), relative to the standard strings (untwisted string satisfying (4.12)). Since we know what the contribution of the untwisted string is to the path integral on Σ (i.e., the standard bosonic amplitudes), for consideration of twisted string amplitudes all we have to do is to correct it by the ratio of the contribution of twisted strings to untwisted strings in the path integral on $\tilde{\Sigma}$.

In the last section we saw that this ratio is given in terms of the Green function on $\tilde{\Sigma}$ between the fixed points of transformation and the ratio of the determinant of laplacian restricted to twisted and untwisted variables. (In the previous section this latter contribution did not play a crucial role, because in that case the determinants of the laplacian for even and odd functions on the torus are the same.)

Aside from the jacobian factor just discussed, there was a factor in the measure

which represented lattice shifts which are ignored in the transformation law (4.11). We wrote $X = X_{cl} + X_q$ where the classical piece satisfies the string equation (i.e., it is harmonic) and was arranged so that the fixed points of $\tilde{\Sigma}$ get mapped to the fixed points of the orbifold, corresponding to the choice of the twisted sector chosen (i.e, the choice of the center of mass of the twisted state). The treatment of the classical contribution gave rise to a sum over inequivalent instanton configurations.

In this section we will discuss how these considerations generalize. We begin with the Green function between fixed points. Suppose we wanted to compute the scattering of N twisted states in an orbifold with $|G| = n$. We take these states to have momenta $k_1, k_2, k_\alpha, k_N \quad 3 \leq \alpha \leq N-1$ in the uncompactified directions. The momenta-dependent integrand in the amplitude is

$$G = \prod_{i<j} (\chi_{ij})^{-\frac{n}{2} k_i \cdot k_j} \tag{6.1}$$

with χ evaluated between the fixed points. Now using momentum conservation we can express k_N and $k_1 \cdot k_2$ in terms of $k_1 \cdot k_\alpha, k_2 \cdot k_\alpha$ and $k_\alpha \cdot k_\beta$. Then we obtain

$$G = \prod_{\alpha<\beta} \left(\frac{\chi_{\alpha\beta}\chi_{1N}\chi_{2N}}{\chi_{12}\chi_{\alpha N}\chi_{\beta N}} \right)^{-\frac{n}{2} k_\alpha \cdot k_\beta} \left(\frac{\chi_{1\alpha}\chi_{2N}}{\chi_{12}\chi_{\alpha N}} \right)^{-\frac{n}{2} k_1 \cdot k_\alpha} \left(\frac{\chi_{2\alpha}\chi_{1N}}{\chi_{12}\chi_{\alpha N}} \right)^{-\frac{n}{2} k_2 \cdot k_\alpha}$$

$$\times \quad \left[\prod_{\alpha} (\chi_{\alpha N})^{\frac{n}{2} k^2} (\chi_{12})^{\frac{n}{4}(N-2)k^2} (\chi_{1N}\chi_{2N})^{\frac{n}{4}(4-N)k^2} \right] \tag{6.2}$$

Now it is easy to read off the Green function correction factor to the measure as the last three parentheses with k^2 replaced by Δk^2, where Δk^2 is the change in $(mass)^2$ from the untwisted case. For the tree level computation, identification of this amplitude with the Koba-Nielsen amplitude allows us to express the Koba-Nielsen integration variables as

$$z_\alpha = \left(\frac{\chi_{1\alpha}\chi_{2N}}{\chi_{12}\chi_{\alpha N}} \right)^n \tag{6.3}$$

and furthermore obtain various identities involving χ's between fixed points. In the appendix it is shown that χ can be expressed (up to a volume form) in terms of θ

functions. In fact this method of obtaining relations between various θ functions and expressing the branch points in terms of the period matrix is known to mathematicians (for a quite readable discussion with many examples for the case of hyperelliptic surfaces see chap. 7 of [35]).

The construction of the Green function in the appendix requires mainly a knowledge of the holomorphic one-forms ω^i. An explicit representation of the Riemann surfaces, such as the one described by algebraic functions discussed in sect. 4, would allow us to construct ω^i explicitly and therefore give a simple way of computing the Green function between any two points and, in particular, between the points fixed by the action of some automorphism.

Another factor necessary for constructing interaction amplitudes involves the ratio of determinants of the laplacian restricted to (vector valued) functions transforming as (4.11) to those transforming as (4.12). Unfortunately we do not have a way to approach this problem in general. For the simple cases we have considered the two are equal (see last section, and sect. 7), but we do not expect this to be true in general. This question seems a rather difficult mathematical problem in general (in a similar context [40], Selberg's zeta function was used to construct such ratios).

Finally the last ingredient needed for constructing orbifold amplitudes is the sum over the classical backgrounds. This comes from the different choices of the center of mass of the twisted sectors. From the viewpoint of the non-linear sigma model it contains the non-perturbative corrections. Fortunately it is relatively simple to construct the contribution of these instantons to the interaction amplitude. One simply looks for harmonic maps from $\tilde{\Sigma}$ to the orbifold, such that the fixed points of the automorphisms of $\tilde{\Sigma}$ corresponding to insertion points of the twisted states get mapped to the choices of the center of mass of the twisted state. This could be done with the same method [40] used in constructing the lattice sum needed for bosonization of the fermions for an arbitrary number of loops.

The essential point is that there is a map from a Riemann surface of genus g to C^g given by choosing a base point P_o, and mapping $P \to \int_{P_o}^{P} \omega^i$. The integral depends

on the choice of the path of integration, the difference of two paths being an integer combination $m_i a^i + n_i b^i$ (see the appendix for notation). So strictly speaking we get a map of a Riemann surface to

$$J = C^g / L_\Omega, \tag{6.4}$$

where L_Ω is the lattice generated by $Z^g + \Omega Z^g$. J is called the jacobian variety of the surface. It is a g-dimensional complex torus. We can think of the Riemann surface as sitting in the jacobian variety via this map. An automorphism of $\tilde{\Sigma}$ gives rise to an automorphism of $J(\tilde{\Sigma})$. Any global diffeomorphism of the surface is represented by an element of $Sp(2g, Z)$, the symplectic modular group, which acts on the period matrix by

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1} \tag{6.5}$$

with D, C, B, A denoting the blocks of a matrix in $Sp(2g, Z)$. If a global diffeomorphism is in fact an automorphism, this implies that the period matrix is not changed under the symplectic group, i.e., $\Omega = (A\Omega + B)(C\Omega + D)^{-1}$. To see this note that under modular transformations $\omega^i \to \left[\omega(C\Omega + D)^{-1}\right]^i$ (this is obtained by demanding that the holomorphic one forms transform in such a way that the $\int_{a^i} \omega^j$ remain δ^i_j). If we represent the element of the automorphism group and its action by g, then $g\omega^j$ is still a holomorphic one form (this would not be true for arbitrary global diffeomorphisms which mix holomorphic and antiholomorphic indices) with the proper normalization on ga^i ($\int_{a^i} \omega^j = \int_{ga^i} g\omega^j$, because g is a diffeomorphism). So we have $g\omega = \omega(C\Omega + D)^{-1}$. By the same argument, the integration of $g\omega$ over the transformed b cycles (given by $Ba + Ab$) does not change, and we must have

$$\Omega = \int_{gb} g\omega = \int_{Ba+Ab} \omega(C\Omega + D)^{-1} = (A\Omega + B)(C\Omega + D)^{-1}. \tag{6.6}$$

We will now discuss how an automorphism is represented on the points of the jacobian. We denote the points of the jacobian by z (a complex g-dimensional vector).

We consider the image gP of the point P under the action of g. This is represented in the jacobian by

$$z(gP) = \int_{P_o}^{gP} \omega = \int_{g^{-1}P_o}^{P} g^{-1}\omega = (\int_{g^{-1}P_o}^{P_o} + \int_{P_o}^{P})\omega(-C^t\Omega + A^t)^{-1}. \tag{6.7}$$

If we define $\Delta_o(g) = \int_{g^{-1}P_o}^{P_o} \omega(-C^t\Omega + A^t)^{-1}$, we get

$$z(gP) = z(P)(-C^t\Omega + A^t)^{-1} + \Delta_o(g). \tag{6.8}$$

This could be used to define the action of g on the whole jacobian torus (if we choose P_o to be a fixed point of the automorphism, $\Delta_o(g)$ would be zero).

We have seen, therefore, that the action of the group on the surface could be represented on its jacobian. A fixed point of the action of the group element on the Riemann surface is easily seen (by the definition of the jacobian map) to be represented in the jacobian by a fixed point of the group action on the jacobian. It is now fairly easy to write down the classical solution for instanton solutions relevant for orbifolds. We want $\partial\overline{\partial}X = 0$, which implies that it could be written as (we take X to be a real variable)

$$X(P) = \int_{P_o}^{P} v \cdot \omega + c.c. \tag{6.9}$$

(v has two suppressed indices, one running from 1 to g, and the other a space-time index). Viewed as a function on the jacobian, this could be written as

$$X(z) = v \cdot z + c.c. \tag{6.10}$$

Given the action of g on z, the requirement to satisfy (4.11) puts restrictions on the vector v. Furthermore, we should choose v so that the fixed points of the transformation on the jacobian get mapped to the desired centers of mass of the twisted orbifold

states. The action of an instanton is easily computed. One uses $\int \omega\overline{\omega} = 2Im\Omega$ to compute

$$\int \partial X \overline{\partial} X = 4\overline{v} Im\Omega v. \tag{6.11}$$

We see that we do not need to know much about the Riemann surface itself to construct the relevant instanton contribution. The knowledge of the period matrix, together with the representation of the automorphism group on the jacobian torus, is essentially sufficient to construct the instanton contribution. This contribution is some kind of theta function of the jacobian lattice. (For more discussion on the representation of automorphism on the holomorphic forms and for restrictions on the representation coming from a fixed point formula (Eichler trace formula), see [35].) Gradients of the instanton in the vertex operator are also easily handled, because we have an explicit solution of the instanton (6.10).

7. Interactions on Heterotic Orbifolds

In this section we will focus on the interactions of strings on orbifolds obtained by dividing out the heterotic string by a subgroup of its symmetries. The group which is divided out acts on the bosonic degrees of freedom (X), as well as RNS fermionic (ψ) and gauge degrees of freedom represented by fermions (λ). In constructing vertex operators for twisted states we will have in our vertex operator the twist operators $\Lambda_B \Lambda_{RNS} \Lambda_G$, where the twist operators twist the bosonic, RNS and gauge degrees of freedom, respectively. This means that as the string variables (X, ψ, λ) are transported on the world sheet around the insertion point of the twist operators, they get rotated by $(\phi_1(g), \phi_2(g), \phi_3(g))$, where $\phi_i(g)$ represents the respective representations of the twisting group element g on the string degrees of freedom.

The bosonic piece of the vertex operators that we shall write below is handled the way we have described in the previous sections. The new feature is the treatment of twisted fermions. But here, bosonization comes to rescue, at least for abelian orbifolds, to simplify the computations. In fact, bosonization has been of great computational use in obtaining the amplitudes involving the fermion vertex operators [20,21]. The fermion vertex operators twist the fermionic degrees of freedom by changing the boundary conditions by a multiplication by -1.

In bosonization, we replace the fermionic degrees of freedom by bosons which have their momenta on a lattice. For the ten RNS fermionic degrees of freedom, we use five bosonic degrees of freedom H^i which have non-trivial momenta lying on the weight lattice of $SO(10)$ (after Wick rotation), where the physical states correspond to the subspace of the lattice consisting of vector representations and one chirality of spinor representations. The gauge degrees of freedom are described by sixteen bosons F^i, with non-trivial momenta on the $E_8 \times E_8$ (or the $SO(32)$) lattice. If we twist the fermionic degrees of freedom by $e^{2\pi i v \cdot J}$ (where J corresponds to a set of Cartan generators), this corresponds to shifting in the lattice by v (for details of how this works, together with examples, see [18]). This means that the momenta p on the lattice should be replaced by $p - v$ to construct the physical states of the twisted

sectors. Therefore, there is not much difficulty in handling vertex operators for these twistings: We simply replace the momenta by the shifted momenta in the standard vertex operator and multiply by the appropriate operator cocycle to get the twisted vertex. (This representation of the twisting groups works only for abelian groups. In the non-abelian case, we will have to formulate the whole string theory, and not just the bosonic part, on the covering space $\tilde{\Sigma}$.)

To illustrate the idea we will consider the example of the Z orbifold discussed in [17,18] (the generalization to other orbifolds will be clear). Let us briefly recall what the Z orbifold is. One starts with the product of three tori (the Z_6 torus shown in fig. 9), and divides out by the Z_3 symmetry of the tori acting simultaneously on all three tori. In this way we obtain an orbifold with 27 singularities corresponding to $3\times3\times3 = 27$ fixed points of the Z_3 action. The action of the Z_3 group on RNS fermions is determined (up to the $(-1)^F$ factor) by the fact that they transform just like X under the action of the euclidean group. The action of Z_3 on the gauge degrees of freedom is somewhat arbitrary (apart from the requirement of level matching which comes from modular invariance [41]) but there is one choice, consistent with modular invariance, which corresponds to identifying the gauge connection of one of the E_8's with the spin connection. For example, if we take the Z_3 element to be $e^{2\pi i(J_{56}+J_{78}-2J_{9,10})/3}$, we represent it by the shift vector $(0,0,\frac{1}{3},\frac{1}{3},-\frac{2}{3})$ on the $SO(10)$ weight lattice, and $(\frac{1}{3},\frac{1}{3},-\frac{2}{3},0,0,0,0,0)$ in one of the E_8's.

The massless states of the Z orbifold correspond to $N = 1$ supergravity coupled to an $E_6 \times SU(3) \times E_8$ super Yang-Mills theory coming from the identity (untwisted) sector, together with some matter fields. The matter fields from the identity sector are 3 fields transforming as $(27,3)$ and 10 fields in the singlet representation of $E_6\times SU(3)$. The matter fields from the twisted sectors are, from each of the 27 fixed points, a $(27,1)$ and 3 fields in $(1,\overline{3})$ representation.

One could obtain a Calabi-Yau manifold by "blowing up" and resolving the 27 singularities [8]. In this case the particle content of the theory is expected to be $N = 1$ supergravity coupled to $E_6 \times E_8$ super Yang-Mills with 36 fields in the (27)

representation of E_6, and 37 singlet fields. Nine of these singlets correspond to the freedom in choosing different sizes and angles of the tori, 27 of them correspond to the size of the blown up regions, and one is the axion which is always present in Calabi-Yau compactifications. The relation between this particle content, and that of Z orbifold, is not difficult to see. It is essentially the Z orbifold with the residual $SU(3)$ gauge symmetry completely broken. The only subtlety comes in the 27×3 twisted fields in the $(1, \overline{3})$ representation. The relation with the blown up theory suggests that for each of the 27 fixed points, all the nine states $3(1, \overline{3})$, except for the one corresponding to the freedom in the choice of the size of the blown up region, should get mass. We will see how this results from considering interactions on orbifolds. Relations between the Yukawa couplings of the E_6 generations for general Calabi-Yau compactifications [14,15], and in particular for the smooth version of the Z orbifold [16], have been obtained by topological considerations. These relations hold to all orders in the perturbation expansion of the relevant non-linear sigma model [42]. But we will see that these relations get modified by non-perturbative effects.

Without much thought, one could immediately rule out some Yukawa couplings: By the group law either all three states come from the identity sector, or all three come from the twisted sector (one twisted sector gives rise to states which are anti-particle of the other twisted sector, so considering interactions between the identity sector, a twisted sector, and the inverse twisted sector does not give rise to Yukawa couplings between generations). The vertex operators for the emission of untwisted states are the same as in the standard theory [7], except that one has to form Z_3 invariant vertex operators. Let's denote by $\Phi^i_{\alpha,a}$ the matter superfields of the untwisted sector, with i a holomorphic coordinate, α a label for the (27) of E_6, and a a label for the (3) of $SU(3)$. The group invariant Yukawa coupling is

$$\Phi^i_{\alpha,a}\Phi^j_{\beta,b}\Phi^k_{\gamma,c}\ \epsilon_{ijk}\ d^{\alpha\beta\gamma}\ \epsilon^{abc}, \tag{7.1}$$

here d is the totally symmetric invariant tensor of E_6 and ϵ's are totally antisymmetric. We would like to compare this with topological considerations in [16] which we

shall briefly review. There is an element of the fourth homology class H_4 associated with each untwisted generation which we call T_{ij}, with $i,j \in 1,2,3$. The Yukawa couplings were shown to be proportional to the intersection matrices of these surfaces $g(T_{ij}, T_{kl}, T_{mn})$, and it was argued that these vanish unless $ij \neq kl \neq mn$. This is precisely what we found considering orbifolds. Therefore, we see that untwisted states interactions give rise to Yukawa coupling relations which are in agreement with the topological considerations, as there are no non-perturbative corrections to the field theory results.

However, the picture changes for the twisted sectors. It is easy to rule out the twisted couplings of the type $(27,1)(1,\overline{3})(1,\overline{3})$ or $(27,1)(27,1)(1,\overline{3})$, by the conservation of gauge quantum numbers. The only allowed ones are $(27,1)(27,1)(27,1)$ and $(1,\overline{3})(1,\overline{3})(1,\overline{3})$. Without any computations, one could easily see that there will be non-perturbative corrections. If we start with a twisted string state with a given center of mass, it could join with string states whose center of masses are at other fixed points. But to do this, as discussed in sect. 3, the strings have to stretch, and so the amplitudes go like e^{-aR^2}, where R is the distance between the fixed points. This is a non-perturbative (because of (3.20)) correction in the language of sigma models. But one would expect that perturbative analysis could see the Yukawa couplings only between the $(27,1)$ states coming from the *same* fixed point, for which case there is no need for the strings to stretch in order to meet. This is in fact what is seen to happen in the non-linear sigma model perturbation [16].

To compute twisted amplitudes we will write down the vertex operators for the emission of these states. The vertex operators could be written as the tensor product of the vertex operators of the left movers (containing gauge degrees of freedom) and right movers (containing the RNS degrees of freedom). The right movers' vertex operator for the twisted states is common to both $(27,1)$ and $(1,\overline{3})$ states. It corresponds to the emission of either a four dimensional scalar or spinor. We will use the powerful formalism developed in [21] for writing down covariant vertex operators. The scalars come from the NS sector of the twisted Hilbert space, and so we will use the vertex operator for the emission of the vectors, which as viewed from four

dimensions correspond to scalars. It is simplest to write the vector vertex operators with ghost number -1. (In writing the vertex operators we will ignore the cocycle operator part of it, as they contribute a phase to an amplitude that we can determine by demanding the obvious symmetries from the amplitude.) The V_{-1} vertex operator for the untwisted state is [21] :

$$V_{-1} = e^{i\alpha\cdot H} e^{ikX} e^{-\phi}, \tag{7.2}$$

(we are using units different from sect. 3, in order to agree with the conventions of [21]), where α is chosen to correspond to a weight of the form $(0,0,0,0,\pm 1)$ (there are ten such possibilities corresponding to the polarizations of a 10 dimensional vector). We recall that the operator $e^{q\phi}$ where ϕ is the bosonized spinorial ghost has ghost number q and conformal weigth $-\frac{1}{2}q(q+2)$. Therefore, the conformal weight of this vertex operator which is the sum of the conformal weights of each piece is $\alpha^2/2 - k^2/2 + 1/2 = 1/2 + 0 + 1/2 = 1$. The massless scalar state in the twisted Hilbert space comes from $p = (0,0,0,0,-1)$, so that $\alpha_{sc} = p - v = (0,0,-\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$. Now the vertex operator for the emission of the twisted scalar is

$$V_{-1}^{sc}(k) = e^{i\alpha_{sc}\cdot H} \Lambda\ e^{ikX} e^{-\phi}, \tag{7.3}$$

where Λ twists the six bosonic coordinates, and k is the four dimensional momentum. The conformal weight of Λ can be calculated using (5.1) and is $6(1/4)(1/3)(2/3) = 1/3$ and the conformal weight of $e^{i\alpha_{sc}\cdot H}$ is $\alpha_{sc}^2/2 = 1/6$, so the conformal weight of (7.3) is also 1. This is not an accident. The computation we just did for the difference of the conformal weight of (7.3) and (7.2) is the same as the computation one does for computing the difference in $(mass)^2$ of the twisted scalar and the ordinary vector in RNS. Therefore, even though we have presented the vertex operator in the context of this example, we will always end up with conformal weight 1 vertex operator.

The discussion for the spinors is also simple. The ordinary weights of the spinors correspond to $(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$ with an even number of $-$ signs. The vertex

operator for the emission of the spinors with $-\frac{1}{2}$ ghost charge in the untwisted theory could be written as

$$V_{-1/2} = e^{i\alpha\cdot H} e^{ikX} e^{-\phi/2}. \tag{7.4}$$

This has conformal weight $5/8 + 0 + 3/8 = 1$. To write down the vertex operator for the emission of the twisted spinors, we note that the two four dimensional spinors have weights $(\mp\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ giving us $\alpha_{sp}^{\pm} = (\mp\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. So we write down for the twisted vertex operator

$$V_{-1/2}^{sp^{\pm}}(k) = e^{i\alpha_{sp}^{\pm}\cdot H} \Lambda e^{ikX} e^{-\phi/2}. \tag{7.5}$$

Once again, this operator has conformal weight $7/24 + 1/3 + 0 + 3/8 = 1$. The computation of the difference in the conformal weights of (7.5) and (7.4), is the same as the computation of the change of $(mass)^2$ of the spinor, i.e., the change in the zero point energy of the fermions $(7/24 - 5/8)$ plus that of bosons (the twist operator with conformal weight $1/3$). These two cancel to leave the conformal weight unchanged. So the construction of the vertex operators we have described applies to arbitrary twistings and gives a vertex operator with conformal weight one.

Having described the right hand part of the twisted vertex operators, we move on to the left movers. Let us first discuss the $(27, 1)$ states. On the E_8 lattice shifted by $(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, 0, 0, 0, 0, 0)$ there are 27 states $\tilde{\alpha}_l$ with $\tilde{\alpha}_l^2/2 = 2/3$, which correspond to the 27 of E_6. The vertex operators for these are

$$V(l, k) = e^{i\tilde{\alpha}_l\cdot F} \Lambda e^{ikX}. \tag{7.6}$$

The conformal weight is again $2/3 + 1/3 + 0 = 1$. This could also be similarly understood as the change of the contributions of zero point energies in the computation of the mass of this state.

To write down the vertex operator for the $(1, \overline{3})$ states we will have to recall how they arose in the twisted sector. There were three $(1, \overline{3})$ states for each of the 27 fixed

points which came about by applying $\tilde{\alpha}^i_{-1/3}$ to the $\overline{3}$ ground state ($\tilde{\alpha}^i_{-1/3}$ is any of the three bosonic $1/3$ twisted spacetime degrees of freedom, with i a holomorphic index). So we expect our vertex operator to have $\frac{\partial X^i}{\partial \bar{z}^{1/3}}$. On the shifted E_8 lattice there are three states $\tilde{\beta}_a$, with $\tilde{\beta}^2_a/2 = 1/3$, which correspond to $\overline{3}$ of $SU(3)$. So we write down the vertex operator

$$V(i,a,k) = e^{i\tilde{\beta}_a \cdot F} \Lambda \partial_{\bar{z}^{1/3}} X^i e^{ikX}. \tag{7.7}$$

Again the conformal weight is $1/3 + 1/3 + 1/3 + 0 = 1$, as expected.

In computations of the amplitudes, we obtain product of correlation functions involving separately X, F, and H. The computations of the F and H correlation functions are trivial since they are simply the ordinary bosonic amplitudes. The conservation of momenta in the F and H lattices implies that we form $E_6 \times SU(3)$ and $SO(10)$ invariant (i.e., zero net momentum) amplitudes. The computations involving the X variables are just the type of computations we have been discussing in the previous sections.

Consider the Yukawa couplings of the Z orbifold. From the right movers we get $< V^{sc}_{-1} V^{sp^+}_{-1/2} V^{sp^-}_{-1/2} >$ (note that the $SO(10)$ momentum is conserved ($\alpha_{sc} + \alpha_{sp^-} + \alpha_{sp^+} = 0$). For the left movers we get either $< V(l_1)V(l_2)V(l_3) >$ or $< V(i_1,a_1)V(i_2,a_2)V(i_3,a_3) >$, for $(27,1)(27,1)(27,1)$ and $(1,\overline{3})(1,\overline{3})(1,\overline{3})$ respectively.

The F and H integrations in both cases give rise to contracting the indices to give singlet states. The interesting piece involves the X integration. The relevant covering space $\tilde{\Sigma}$ for this computation is the Z_6 torus (shown in fig. 9). This will involve a sum over different background classical configurations (the instantons), similar to the computation done for four Z_2 twisted states. The classical solution has to be chosen to satisfy the appropriate transformation properties under the Z_3 symmetry of the torus (4.11), and also to map the fixed points of the two dimensional torus to the center of mass of the twisted sectors whose interactions we are computing.

To enumerate the 27 states, it is convenient to introduce some notation. Let L

denote the 6 dimensional lattice of the Z orbifold. Let us denote the Z_3 rotation by θ. A sector could be denoted by (θ, v) which means that $x_f = \theta x_i + v$. The sectors defined by (θ, v) and $(\theta, v + (1-\theta)l)$, for arbitrary $l \in L$, define the same center of mass of the twisted state (shifted by a lattice vector). The inequivalent shift vectors correspond to $L/(1-\theta)L$, which are in one to one correspondence with the 27 choices for the center of mass of the twisted state.

The space $L/(1-\theta)L$ is isomorphic with the group $Z_3 \times Z_3 \times Z_3$. To see this we take one vector with the shortest lengths from each of the three lattices (of which L is a product) v^i. Any element of $L/(1-\theta)L$ can be expressed as a linear combination of these three vectors, and it is easy to see that $3v^i$ is an element of $(1-\theta)L$, and so it is trivial in $L/(1-\theta)L$. This establishes the above isomorphism, with the three v^i as the generators of $Z_3 \times Z_3 \times Z_3$. Each center of mass of the twisted state could be represented by an element $m_1 v^1 + m_2 v^2 + m_3 v^3$, where m_i are integers modulo three.

In discussing interaction of the twisted sectors, we should require that the product of the group elements be the identity. The group we have considered was implicitly the point group Z_3. But actually if we view the Z orbifold as obtained by dividing out by a space group (see [18]) we should require the product of the group elements of the space group to be one. This will imply, in particular, that in a process involving the twisted states with their center of mass defined by m_i^α, there is a constraint

$$\sum_\alpha m_i^\alpha = 0 \qquad (mod\ 3). \tag{7.8}$$

In computing Yukawa couplings, we should therefore choose three centers of mass (among the 27 possibilities) which satisfy (7.8). The choice of any two fixed points therefore fixes the third choice by (7.8) (this is obvious from the geometrical picture of strings joining, and we encourage the reader to translate (7.8) into such a picture). In fact, because of the translation invariance of the torus only the relative center of masses of the two joining strings is relevant for the computation of the amplitudes.

For the $(27,1)^3$ interaction, the classical sum is easily performed to give

$$L_{Yukawa} \sim g \sum_{w\in(1-\theta)L+n_iv^i} e^{\frac{-\sqrt{3}}{24\pi}|w|^2}\psi^+_{m^1}\psi^-_{m^1+n}\phi_{-2m^1-n} \tag{7.9}$$

with implicit E_6 group index contractions. (The proportionality constant is independent of radius and is best found by considering a four point amplitude and factorizing it in different ways. Apart from factors of 2 and π, we expect this to give us, as for the Z_2 case, the Green function between the fixed points of the Z_6 torus, and the ratio of appropriate determinants which is again equal to one for this case.) We see that for $n \neq 0$ we get Yukawa couplings between different sectors, which are non-perturbative (the largest term in (7.9)$\sim e^{-cd^2}$ where d is the distance between the fixed points).

For the $(1,\overline{3})^3$ couplings, the gradient term in (7.7) modifies the computation. There is no contribution from the quantum fluctuations because $< \overline{\partial}X^i\overline{\partial}X^j >= 0$, when i and j are holomorphic indices since we need to contract a holomorphic index with an antiholomorphic index to get a non-zero result. However, the gradient terms do contribute to the instanton sum to give us*

$$L_{Yukawa} \sim g \sum_{w\in(1-\theta)L+n_iv^i} (w^i)^3 e^{\frac{-\sqrt{3}}{24\pi}|w|^2}\psi^{+,i}_{m^1,a}\psi^{-,i}_{m^1+n,b}\phi^i_{-2m^1-n,c}\,\epsilon^{abc} \tag{7.10}$$

(we are viewing w as a complex vector and have written down the tangent indices i and $SU(3)$ indices a,b,c of the fields). In this sum we have all three tangent indices take the same value, since otherwise the instanton sum would vanish. This is because each two dimensional sublattice of L has a Z_3 symmetry $x^i \mapsto e^{2\pi i/3}x^i$, which is left unbroken by the construction of the orbifold (in fact there will be a Z_6 symmetry which preserves one fixed point and exchanges the other two in each torus). The term with $n = 0$ is absent from (7.10) because of the symmetry of contractions (or alternatively because in this case there is actually a Z_6 symmetry).

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* This is in agreement with the results of [43] which showed that for topologically trivial configurations, there is no contribution to the F term involving the E_6 singlet fields.

In the superfield language the Yukawa coupling of $(1,\overline{3})^3$ comes from the F-term

$$\Phi^i_{m^1,a}\Phi^i_{m^2,b}\Phi^i_{-m^1-m^2,c}\,\epsilon^{abc}. \tag{7.11}$$

Now we are in a position to discuss the 'blowing up' of the 27 singularities from the orbifold point of view. The blowing up simply corresponds to giving an expectation value to $\phi^i_{m,a}$ maintaining supersymmetry, i.e., finding flat directions in the superpotential. Let us blow up each of the 27 fixed points independently, i.e., we probe for flat directions in which we allow the ϕ's from all fixed points except for one to be turned off. Consider for instance the ϕ from one fixed point m, and give it an expectation value $<\phi^i_{m,a}>= M^i_a$. This will be allowed only if the D terms in the potential energy, which appear because ϕ is non-singlet under $SU(3)$, vanish. This implies that $M^i_a\overline{M}^b_i = \lambda^2\delta^b_a$, and therefore we can set $M^i_a = \lambda\delta^i_a$ by an $SU(3)$ gauge rotation. This expectation value breaks $SU(3)$ completely. If we want to blow up other fixed points as well, the F-term (7.11) is easily seen to imply that all the expectation values are diagonal in i and a, and if we want each fixed point independently blown up, each expectation value should be proportional to δ^i_a (again from the vanishing of the D terms). So the most general ansatz for independent blow up is

$$<\phi^i_{m,a}>= \lambda_m\delta^i_a \tag{7.12}$$

(we have taken the holomorphic indices of the internal space to run from 1 to 3). It is easily checked that the potential and superpotential vanish with this ansatz for arbitrary λ_m. That the expectation values are of the form δ^i_a suggests that we are identifying the space connection (i being a holomorphic tangent index) with the gauge index (a denoting the $\overline{3}$ of $SU(3)$). In this way we can interpret λ_m as the size of the blown up region. It is also easy to see from the D terms that at each fixed point all but one of these fields obtain masses proportional to $|\lambda_m|$.

What about higher terms in the superpotential? To probe the superpotential further we need the boson vertex operator with zero ghost number V_0^{sc}, in order to

have the correct net ghost number in the amplitudes. We use the machinery developed in [21] to find V_0^{sc} from the operator product expansion of j_{BRST} with ξV_{-1}^{sc}. The result is (for $k = 0$)

$$V_0^{sc} = \sum_i (\partial_{z^{1/3}} X^{\bar{i}} e^{i\beta_i \cdot H})\Lambda \tag{7.13}$$

with $\beta_1 = (0, 0, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$, and β_2 and β_3 are permutations of this in the last three coordinates. Since $\beta_i^2/2 = 1/3$ we see that this vertex operator has, as expected, conformal weight one.

Let us see if any corrections to the superpotential will destablize the ansatz (7.12). Let us consider the contribution of the E_6 singlets coming from the twisted sector. To probe the superpotential for terms of the form Φ^{3n} we will have to consider

$$< V_{-1/2}^{sp^+} V_{-1/2}^{sp^-} V_{-1}^{sc} V_0^{sc} \cdots V_0^{sc} >< V(i_1, a_1) \cdots V(i_{3n}, a_{3n}) > \tag{7.14}$$

with V_0^{sc} appearing $3n - 3$ times. $SO(10)$ symmetry requires that the H momenta add up to zero and the $SU(3)$ symmetry of the shifted E_8 lattice requires that we form $SU(3)$ singlets by using ϵ^{abc}. Let us see if the superpotential vanishes with the ansatz (7.12). Because of the $SU(3)$ symmetry, we must have equal numbers of $SU(3)$ indices $1, 2$, and 3, in any potentially non-vanishing term of the superpotential. The ansatz (7.12) implies that we have an equal number of the tangent indices $i = 1, 2, 3$ if the superpotential is not to vanish. This means that the relevant term in (7.14) will contain n of each of the gradients $\overline{\partial} X^1, \overline{\partial} X^2$, and $\overline{\partial} X^3$. The $SO(10)$ lattice sum, on the other hand, implies that there are $n - 1$ of each of the gradients $\partial X^{\bar{1}}, \partial X^{\bar{2}}$, and $\partial X^{\bar{3}}$ (this follows from the fact that $\sum m_i \beta_i = 0$ implies that $m_1 = m_2 = m_3$). This means that we have one net unpaired gradient $\overline{\partial} X^i$ for each i, and so the amplitude will vanish by the Z_3 symmetry of each of the three tori. By a similar argument one can show that the gradient of the superpotential also vanishes with the ansatz (7.12) . (There *are* corrections to the superpotential, however, which do not destablize (7.12) .) We therefore conclude that the Calabi-Yau compactification in this case is non-perturbatively stable at the tree-level of string theory.

It would be interesting to consider other orbifolds and formulate what blowing up of singular points is more generally, and to see whether or not it gives rise to stable Calabi-Yau compactifications. In the cases that the orbifold singularities can not be blown up this might actually suggest a modified blowing up procedure. We wish to emphasize, however, that the dynamics does not seem to *necessitate* blowing up of the fixed points, and we could consider the orbifolds as they are without running into any problems.

8. Conclusions

We have seen how to compute interactions on orbifolds. The complexity of dealing with multi-valued bosonic variables is handled by considering the string variables on a Riemann surface covering the world sheet of the string, in such a way that a multi-valued function on the world sheet becomes single-valued. With this method we computed some simple interactions and sketched the steps involved in the general situation. The most interesting twisted interactions involve the twisted states from different fixed points stretching and joining. This is completely stringy and would be missed by a field theory analysis. Also, for the heterotic strings we found that the Yukawa couplings receive non-perturbative corrections which modify the topological relations between them. We discussed, in the context of the Z orbifold, the meaning of blowing up and resolving the singularities. The blowing up of the singularities breaks $SU(3)$ symmetry completely, and gives rise to the massless states expected from the field theory of Calabi-Yau compactification. Using covariant vertex operators, we then showed that the non-perturbative corrections do not generate any terms in the superpotential that would destablize Calabi-Yau compactification. We demonstrated this by showing that the generated superpotential and potential energy vanish for the Calabi-Yau ansatz.

Also, the mathematics of the Riemann surfaces with automorphisms, which is needed for computations of orbifold interactions, seems to be a fascinating subject in its own right (one is tempted to conjecture that there may be a relation between the automorphisms of Riemann surfaces and the choice of the correct orbifold). We also saw that some interesting identities (in particular, the Jacobi identity) could be derived by demanding equality between two different ways of computing Koba-Nielsen amplitudes, one on the sphere, and the other on the covering of the sphere by a Riemann surface with an automorphism.

We conclude by noting that explicit computations of interactions on orbifolds are not very difficult (it would be interesting, however, to work out some more examples in detail, particularly those of non-abelian orbifolds). This is in contrast to the

more general case of the Calabi-Yau compactification for which even the metric is not explicitly known. We feel that orbifolds are a particularly "stringy" method of compactification. But even if one wishes to consider Calabi-Yau compactifications, orbifolds could be used as starting points of a perturbation expansion for some of them. In this setting, the mathematical procedure of "blowing up" singularities has a simple physical interpretation in terms of some fields acquiring expectation values.

The work described here is based on the material in [44]. Interactions on orbifolds have also been considered in [45]. The correlation functions for Z_n twists are also computed in [46]. Also [47] obtains similar results for the non-perturbative corrections to the Yukawa couplings and the superpotential in the type $(2,2)$ Calabi-Yau compactifications.

APPENDIX: Prime Forms

Ordinarily one thinks of Green functions as functions, i.e., invariant under coordinate transformations. We will argue that from the point of view of string theory the exponential of the Green function for the ordinary laplacian is more conveniently viewed as a differential, not as a "function". In fact we will compute the exponential of the Green "function" for an arbitrary number of loops in string theory in terms of prime forms [48] (where it is shown how the abelian differentials of the third kind could be written in terms of prime forms). Of course the Green functions could be written in terms of abelian differentials of the third kind [49]. But then one will have to make subtractions at coincident points. Our main new point here is that going through abelian differentials of the third kind is unnecessary, and the prime form is itself the correct subtracted exponential of a Green "function". Knowing the integration measure over the moduli, which has been recently obtained in [50,51] will complete the computation of an arbitrary scattering of tachyons (in the bosonic string).

Before discussing what the prime forms are, let us recall how the Green function appears in bosonic string. For an arbitrary process one considers

$$\int \prod_i \sqrt{g(z_i)} d^2 z_i < V_1(z_1) ... V_N(z_N) >, \qquad (A.1)$$

where $V_i(z_i)$ is the vertex for the emission of the state i. Let us consider the scattering of N tachyons with $p_i^2/4 = -2$. The vertex operator is given by $V_i(z_i) = e^{ip_i \cdot X(z_i)}$. The first thing that is puzzling about (A.1) is that the amplitude seems to depend on the choice of the metric of the surface through $\sqrt{g(z_i)}$. So the theory does not seem, naively, conformally invariant. The second thing that is puzzling is that after computing (A.1) we get

$$Amp = \int \prod_i \sqrt{g(z_i)} d^2 z_i \prod_{i,j} e^{-p_i G(z_i, z_j) p_j / 4} \qquad (A.2)$$

which has divergent Green functions at coincident points (the $i = j$ terms in the

product). These two puzzles cancel each other [52,53] : To define the Green functions at coincident points we have to subtract the leading divergence which, in a given coordinate system goes as $G(p,q) \rightarrow_{p\rightarrow q} log(z(p) - z(q))$. But there is no coordinate independent way of carrying out the subtraction. If we have a metric we can carry out the subtraction by defining

$$\tilde{G}(p,p) = lim_{q\rightarrow p}(G(p,q) - log\, d(p,q)) \qquad (A.3)$$

where $d(p,q)$ is the distance between p and q. This subtraction process also depends on the metric and is not conformally well defined (i.e., it cannot be done without a reference to a metric). The net result of what appears in the integrand of the amplitude, however, is independent of the metric. If we rescale the metric by $g_{z\bar{z}} \rightarrow e^{2\phi} g_{z\bar{z}}$, then $e^{-p_i^2 \tilde{G}(z_i,z_i)/4} \rightarrow e^{p_i^2\phi/4} e^{-p_i^2\tilde{G}(z_i,z_i)/4}$. Therefore, for $p_i^2/4 = -2$ we see that $\sqrt{g(z_i)}e^{-p_i^2\tilde{G}(z_i,z_i)/4}$ is invariant under conformal rescaling of the metric and so (A.2) is well defined. This suggests that there may be a nicer way of writing (A.2) which makes no reference to any choice of metric. We will see that this is indeed possible, and one does not have to go through any complicated process of subtraction of leading singularities to obtain the final metric independent form of the amplitude. The subtraction of the leading singularity has made the vertex operator which classically is conformally invariant, a conformal object with weight $(1,1)$.

For tree amplitudes, (A.2) is easily computed on the complex plane, giving us the standard Koba-Nielsen amplitude

$$Amp = \int \prod_i d^2 z_i \prod_{i<j} |z_i - z_j|^{-p_i p_j/2} \qquad (A.4)$$

(of course, divided by the $SL(2,C)$ volume factor). We will rewrite this amplitude in a form more convenient for generalization:

$$Amp = \int \prod_{i<j} \left| \frac{z_i - z_j}{\sqrt{dz_i}\sqrt{dz_j}} \right|^{-p_i p_j/2} . \qquad (A.5)$$

to see the equivalence we note that by momentum conservation

$$\prod_{j\neq i}\left|\sqrt{dz_i}\right|^{p_ip_j/2}=\left|\sqrt{dz_i}\right|^{-p_i^2/2}=\left|\sqrt{dz_i}\right|^4=d^2z_i. \qquad (A.6)$$

This demonstrates the equality of (A.5) with (A.4). We see that the integration measure can be absorbed as part of the definition of the exponential of the Green function. Moreover, if we consider the vertex operator for any excited state, the exponential part of the vertex e^{ipX} contributes to the measure a factor $(d^2z)^{-p^2/8}$, i.e., it has conformal weight $(-p^2/8,-p^2/8)$ which is the correct weight of the exponential part of a vertex operator (in our units). The rest of the weight (to make it a (1,1) form) comes from the rest of the vertex operator. For instance, for the massless states $\partial X\overline{\partial}X$ has manifestly conformal weight $(1,1)$ and the differentials used in the definition of the exponential of the Green function will have no contribution to the measure $(-p^2/8=0)$.

How can we generalize this to higher loops? We will have to find a suitable generalization of $E(z_i,z_j)=\frac{(z_i-z_j)}{\sqrt{dz_i}\sqrt{dz_j}}$. The main properties that we will require E to have is that it be a holomorphic differential of weight $(-\frac{1}{2},-\frac{1}{2})$ (to guarantee the correct conformal weight of the vertex operator) and that it have a zero of first order only for coincident points (corresponding to a logarithmic divergence of the Green function at coincident points). It turns out that such a differential exists, and is much studied in the mathematics literature [48,54]. It is called the prime form.

The prime form is given in terms of theta functions. To write down the prime form we will make use of holomorphic one forms. At g loops there are g holomorphic one forms which we denote by ω^i. Theta functions come with characteristics (for more details see [48]). Consider an odd characteristic theta function given by

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\Omega)=\sum_{n\in Z^g}e^{i\pi(n+\alpha)\Omega(n+\alpha)^t+2\pi i(n+\alpha)(z+\beta)^t}, \qquad (A.7)$$

with α and β half integral g-dimensional vectors with $4\alpha\cdot\beta=odd$. An odd theta function is odd under $z^k\rightarrow -z^k$ and thus vanishes at $z^k=0$. An odd theta function

is called non-singular if not all its first derivatives vanish at $z^k = 0$ (a non-singular odd theta function always exists). The prime form is defined by

$$E(p,q) = \frac{\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\int_p^q \omega^i)}{\sqrt{\partial_i \vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](0)\omega^i(p)}\sqrt{\partial_j \vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](0)\omega^j(q)}}. \tag{A.8}$$

(This is independent of which odd spin structure one chooses [54].) The prime form would be the right candidate for what should go into the integrand of the path integral, except that it is not single-valued. If a^i and b^i denote the canonical cycles ($\int_{a^j} \omega^i = \delta^i_j$, $\int_{b^j} \omega^i = \Omega^i_j$ the period matrix), as we move the points p or q around an a^i cycle the prime form does not change, but as we go around the b^i cycle they pick up a prefactor. This could be easily compensated by considering

$$\chi(p,q) = e^{-\pi(Im\int_p^q \omega)(Im\Omega)^{-1}(Im\int_p^q \omega)}|2\pi E(p,q)| \tag{A.9}$$

(this prefactor corresponds to our freedom in redefining the Green function by addition of holomorphic one forms [49], and the factor of 2π is inserted for convenience). $\chi(p,q)$ is indeed single-valued and is the correct expression which should go in the integrand. For instance the interaction amplitude for tachyons will be

$$Amp = \int \prod_{i<j} \chi(z_i, z_j)^{-p_i p_j/2}. \tag{A.10}$$

Note that the integration measure is absorbed in χ. ((A.10) will be the full amplitude after integration over the moduli, corresponding to Riemann surfaces with different complex structures.) It is easy to check that the χ constructed above, agrees with the direct computations done at one-loop [19,38] (It is also possible to use the prime form to construct abelian differentials of the third kind, construct the Green function out of the abelian differentials, subtract the leading singularity, make the Green function single-valued, and obtain what we found for χ. We could in addition compute the correlation functions between the derivatives of string coordinates using this Green function.).

Since we have an explicit expression for the Green functions, we can compute the Koba-Nielsen tree amplitude on higher genus Riemann surfaces with automorphisms (similar to what was done in sect. 5) in terms of theta functions. Demanding equality with the standard amplitude then gives us not only the relations between the variables in the Koba-Nielsen integration and the moduli of $\tilde{\Sigma}$, but also give some non-trivial identities among theta functions for surfaces with automorphisms. The only subtlety is that in the integration on $\tilde{\Sigma}$ we have replaced the integration over the insertion-points, by integration over the moduli, so that we really need a function for this case. This could be done by multipling the Green function χ by an appropriate measure factor (which we suspect to be $\sum \omega^i (Im\Omega)^{-1}_{ij} \overline{\omega}^j$, as is the case for the Z_2 computation).

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