I - MODE LOCKING AND ULTRASHORT LASER PULSES BY A REFRACTIVE INDEX NONLINEARITY

II - A THEORETICAL STUDY OF OPTICAL WAVE PROPAGATION THROUGH A RANDOM MEDIUM AND ITS APPLICATION TO OPTICAL COMMUNICATION

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Je dédie ces pages arides et passionantes

à Gertie

à mes parents

à toute ma famille

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PART I: MODE LOCKING AND ULTRASHORT LASER PULSES BY A REFRACTIVE INDEX NONLINEARITY

Jean-Pierre Raymond Henri Laussade

ABSTRACT

A new method for locking the longitudinal modes of a laser resonator and generating ultrashort pulses of light has been found. The cavity modes are coupled together when a medium possessing a refractive index nonlinearity is placed inside the cavity.

A theoretical study is presented which analyzes the mode structure of a laser resonator containing a cell filled with an anisotropic molecular liquid. It is found that under certain conditions the energy exchange between the modes gives rise to a mode locked spectrum and to the attendant generation of ultrashort pulses of light ($\sim 10^{-11}$sec for a ruby laser, $\sim 10^{-12}$sec for a Nd^{3+} glass laser).

An experimental investigation is reported. The presence of ultrashort pulses in the output of a Q-switched ruby laser is observed when a liquid cell containing nitrobenzene or α-chloronaphthalene is placed inside the cavity.

PART II: A THEORETICAL STUDY OF OPTICAL WAVE PROPAGATION THROUGH A RANDOM MEDIUM AND ITS APPLICATION TO OPTICAL COMMUNICATION

Jean-Pierre Raymond Henri Laussade

ABSTRACT

In this report we are interested in a theoretical study of wave propagation in a randomly turbulent medium and the application of the results to the evaluation of optical communication systems through the atmospheric turbulence.

We first derive a power series expansion solution for the wave function $u(\vec{x})$ of a wave propagating through a medium with a random index of refraction. The average wave function $\overline{u(\vec{x})}$ and the correlation function $\overline{u(\vec{x}_1)\, u^*(\vec{x}_2)}$ are calculated in terms of the correlation function of the index of refraction, the only assumption being that the wavelength of the wave is much smaller than the smallest size of the turbulence. The intensity correlation function $\overline{I(\vec{x}_1)\, I(\vec{x}_2)}$ is investigated and recent experimental results concerning the behavior of the intensity fluctuations are discussed.

Next, the performances of two schemes of optical communication through the random atmospheric turbulence are compared: (a) heterodyne detection, (b) video communication. It is found that for long propagation paths and strong turbulences, scheme (b) is preferable to scheme (a). This is due to the cancellation of the phase fluctuations between "reference" and "signal" parts of the beam in the video communication scheme.

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I. MODE LOCKING AND ULTRASHORT LASER PULSES BY A REFRACTIVE INDEX NONLINEARITY

INTRODUCTION

The invention of the laser has provided many physicists in many areas of research with a useful research tool.

The field of nonlinear optics is one of the most privileged in this regard. The intense optical electric fields which are present inside and outside a laser cavity have been successfully used to study and obtain a better comprehension of the nonlinear behavior of many materials. Intensive and fruitful research has been performed on optical nonlinear phenomena such as stimulated Raman scattering, stimulated Brillouin scattering, stimulated Rayleigh scattering. The nonlinear properties of certain crystals have been used to modify the output characteristics of lasers through parametric interactions of optical beams or second and third harmonic generation of laser radiation.

Here, we wish to report a theoretical and experimental investigation showing how the presence of a medium possessing a refractive index non-linearity inside the cavity of a Q switched solid state laser can modify its mode structure and give rise to the production of intense and ultra-short pulses of light.

The output electric field of a laser is equal to the sum of the electric fields of the individual modes of the cavity which are amplified by the laser medium, i.e. whose frequencies lie within the gain linewidth $\Delta\nu_G$ of the amplifying transition.

In the normal mode of oscillation of a laser (no perturbation inside the cavity), the phases of the modes are random and uncorrelated, and the output intensity is fluctuating randomly in time around its mean value equal to $N\bar{I}$ where N is the number of oscillating cavity modes and $\bar{I}$

is the average mode intensity.

It has been shown(1) that if the losses of the laser cavity are modulated at a frequency equal to the intermode spacing frequency $C/2L$ (L is the length of the cavity), then the output of the laser consists of a train of pulses which have the following properties:

(a) The pulsewidth is equal to the reciprocal of the gain linewidth $1/\Delta\nu_G$

(b) The pulses are separated in time by the double transit time of the light inside the cavity $2L/C$

(c) The peak power is equal to N times the average power of the laser where N is the number of coupled modes.

The introduction of a loss modulator inside a cavity couples the modes of this cavity in the following way. Suppose one mode at the frequency ν_0 is oscillating. When the electric field of this mode passes through the modulator operating at a frequency $\delta\nu$, sidebands are generated at frequencies $\nu_0 + \delta\nu$ and $\nu_0 - \delta\nu$. On the next pass through the modulator, sidebands of frequencies $\nu_0 + 2\delta\nu$ and $\nu_0 - 2\delta\nu$ will be generated and so on. If $\delta\nu$ is equal to $C/2L$ the intermode spacing frequency, then the sidebands correspond to resonance frequencies of the cavity modes. The modes are thus coupled together with a unique phase relationship. (The term mode locking is also applied to this phenomenon.) While in the non-mode locked case, the phases of the modes are random.

Using internal modulators, ultrashort pulses have been obtained in continuous wave gas lasers(2) (with a width of 2.5×10^{-9} sec) and solid state lasers(3) (8×10^{-11} sec) with a pulsewidth approaching the theoretical value $1/\Delta\nu_G$. Internal modulators have also been used to generate ultrashort pulses in pulsed solid state lasers where the duration of the pulsing ($\sim 1\ \mu$ sec) is larger than the modulating period[4,5]. The observed pulsewidths were 2×10^{-9} sec for Ruby and 0.5×10^{-9} for Nd: glass while the theoretical values are respectively 10^{-11} sec and 4×10^{-13} sec, indicating that the whole linewidth is not fully mode locked.

An increase in the output power of solid state lasers has been obtained by the technique of Q-switching[6]. The output of a non-mode locked Q-switched solid state laser consists typically of a pulse of 10 to 50×10^{-9} sec with a peak power of up to a few hundred megawatts. For these lasers, mode-locking has been obtained by inserting a saturable absorber inside the cavity.[7,8] A saturable absorber is an element whose optical transmission is an increasing function of the intensity of the incident beam. Pulses as short as a few 10^{-11} sec in Ruby lasers and a few 10^{-12} sec in Nd: glass lasers with peak intensities in excess of 10^9 watts have been observed by using this technique.

In this report, we present a new method for generating high intensity picosecond pulses in Q-switched solid state lasers.

We show theoretically that the introduction of a refractive index non-linearity inside a laser resonator gives rise to a mode-locked spectrum characteristic of the ultrashort pulse mode of oscillation[9]. The non-linearities are provided by anisotropic molecular liquids. The theoretical argument is presented in Part I.

In Part II we describe the experimental techniques and present the experimental results. Ultrashort pulses are observed in the output of a

Q-switched Ruby laser when a cell containing nitrobenzene or α-chloronaphthalene is placed inside the cavity[10]. The degree of mode locking is found to be a sensitive function of the orientational relaxation time of the molecules which can be controlled by changing the temperature of the liquid. Ultrashort pulses appear regularly only when the nitrobenzene is heated above $120^{o}C$. The pulsewidth ($\sim 10^{-11}$ sec) is measured accurately by the two photon fluorescence technique. Some observations of the stimulated Raman emission from nitrobenzene are presented.

A summary of the results is followed by a discussion where we give a physical argument to show how a pulse of light is shortened when traveling through a nonlinear index of refraction.

I. THEORETICAL INVESTIGATION

1-1. Statement of the problem.

In this report we consider the interaction of the longitudinal modes of a laser cavity with a medium which possesses a nonlinear index of refraction. In our analysis such a medium is a liquid with anisotropic molecules, i.e. molecules having only one axis of symmetry. These molecules have different polarizabilities along their axis of symmetry and along any other axis perpendicular to it. We call these polarizabilities $\alpha_{\parallel}$ and $\alpha_{\perp}$ respectively. A linearly polarized electric field applied to such liquid induces a nonlinear polarization in the medium which is proportional to the difference $(\alpha_{\parallel} - \alpha_{\perp})$ and to the cube of the electric field as we shall see later, and therefore produces a change in the dielectric constant of the medium proportional to the square of the electric field. When a liquid with anisotropic molecules is placed inside a laser resonator where the optical electric fields are large enough to produce an appreciable change of the dielectric constant, it couples the longitudinal modes of the laser cavity together in the following way. Let us assume that three modes of the cavity oscillate with frequencies ω_0, $\omega_0 + \Omega$, $\omega_0 - \Omega$. (Figure 1-1.) Ω is the radial intermode frequency $\Omega = \frac{\pi C}{L}$ where L is the optical length of the cavity.

The two modes (0) and (+1) for example, induce a change in the dielectric constant of the liquid $\Delta \varepsilon \propto E_0 E_1$ where E_0 and E_1 are the electric fields of the two modes. $\Delta \varepsilon$ has a component oscillating at the frequency $(\omega_0 + \Omega) - \omega_0 = \Omega$. The mode (-1) incident upon the liquid "sees" a modulation of the dielectric constant at frequency Ω

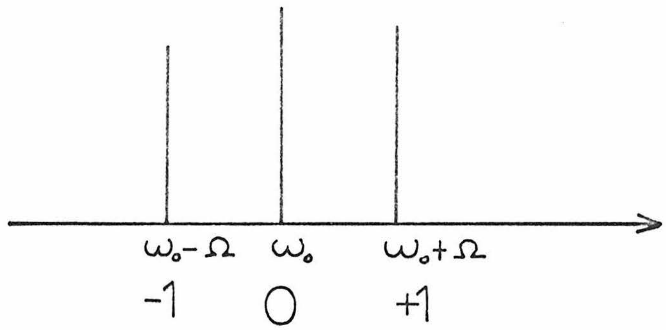

FIG 1.1 _ Three oscillating laser modes _

and a sideband at frequency $(\omega_0 - \Omega) + \Omega = \omega_0$ is generated, therefore coupling the three modes together, i.e. introducing relationships between the amplitudes and the phases of the three modes.

Solid state lasers such as the ruby laser and the neodymium laser with very large gain linewidth 2×10^{11} Hz for ruby, more than 10^{12} Hz for Nd^{3+}: glass can have a very large number of modes oscillating at the same time. For example, in the case of a ruby laser with a 1-meter-long cavity, as many as 600 modes of the cavity lie within the gain linewidth of the ruby and can oscillate simultaneously.

Although it is difficult to account for the interaction of such a large number of modes in a nonlinear medium, it can be seen by the above qualitative argument that a liquid with anisotropic molecules placed inside a laser resonator induces coupling and gives rise to power exchange between the equispaced laser modes.

Our problem is to find the amplitudes and the phases of the modes of a laser resonator containing a nonlinear liquid. For this purpose, we first calculate the dipole moment of an anisotropic molecule induced by an electric field. The electric field is then expressed as the sum over the cavity modes of the electric field of one cavity.

1-2. Average dipole moment of an anisotropic molecule induced by a linearly polarized optical electric field.

In this section, we consider the dipole moment induced on one anisotropic molecule by a linearly polarized optical electric field. The axis of symmetry of the molecule makes an angle θ with the direction of the electric field taken as the z-direction. See Figure 1.2.

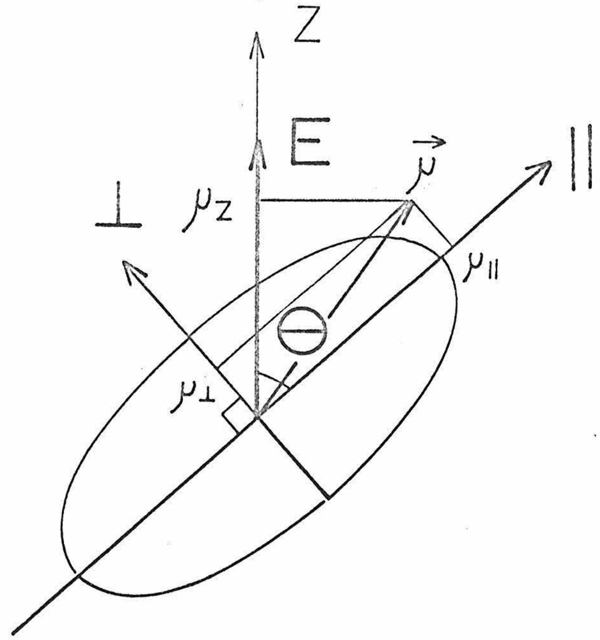

FIG 1-2 _ Orientation of an anisotropic molecule with an electric field _

Let us call $\alpha_{\parallel}$ and $\alpha_{\perp}$ the polarizability of the molecule along its axis of symmetry and an axis perpendicular to it and let us call μ_z, $\mu_{\parallel}$ and $\mu_{\perp}$ the components of the induced dipole moment along the direction of the field, the $\parallel$ and the $\perp$ axis.

The following relationships hold between μ_z, $\mu_{\parallel}$ and $\mu_{\perp}$.

$$\mu_{\parallel} = \alpha_{\parallel} E \cos\theta \quad , \quad \mu_{\perp} = \alpha_{\perp} E \sin\theta \tag{1.1}$$

and

$$\mu_z = \mu_{\parallel} \cos\theta + \mu_{\perp} \sin\theta \tag{1.2}$$

From the two above expressions, we find

$$\mu_z = E\,\alpha_{\parallel} \cos^2\theta + E_{\alpha_{\perp}} \sin^2\theta$$

or

$$\mu_z = E(\alpha_{\parallel} - \alpha_{\perp}) \cos^2\theta + \alpha_{\perp} E \tag{1.3}$$

The average induced dipole moment of one anisotropic molecule is found from (1.3) by replacing $\cos^2\theta$ by its statistical average $\overline{\cos^2\theta}$ taken over the ensemble of molecules. Then

$$\overline{\mu_z} = E(\alpha_{\parallel} - \alpha_{\perp}) \overline{\cos^2\theta} + \alpha_{\perp} E \tag{1.4}$$

with

$$\overline{\cos^2\theta} = \frac{\int_0^\pi \cos^2\theta \; f(\theta)\; d\Omega}{\int_0^\pi f(\theta)\; d\Omega}$$

where $f(\theta)\; d\Omega$ is the number of molecules whose axes of symmetry lie in the differential solid angle $d\Omega = 2\pi \sin\theta\, d\theta$.

When no electric field is applied, all orientations of the axis are equally probable and

$\frac{f(\theta)}{\int_0^\pi f(\theta)\; d\Omega} = \frac{1}{4\pi}$. In this case, $\overline{\cos^2\theta} = \frac{1}{2}\int_0^\pi \cos^2\theta \sin\theta\, d\theta = \frac{1}{3}$. When a strong electric field is applied to the liquid, it tends to align the molecules along its direction, and therefore $\overline{\cos^2\theta}$ will be different from $\frac{1}{3}$. We then write

$$\overline{\cos^2\theta} = s + \frac{1}{3} \tag{1.5}$$

The quantity s determines the average deviation of the orientation of anisotropic molecules in a liquid when an electric field is applied to it. s is the first diagonal element of the anisotropy tensor(11).

From (1.4) and (1.5) we find

$$\overline{\mu_z} = E(\alpha_{\parallel} - \alpha_{\perp})\, s + \frac{E}{3}(\alpha_{\parallel} + 2\alpha_{\perp}) \tag{1.6}$$

The anisotropy tensor element s obeys the following differential equation[11]

$$\frac{ds}{dt} + \frac{s}{\tau} = \frac{2\lambda}{3} E^2 \tag{1.7}$$

where E is the linearly polarized electric field.

τ is the time it takes for the molecules to regain their random orientation after the electric field has been turned off; it is often called the Debye relaxation time[12] or orientational relaxation time.

λ is a constant which can be found by the following argument.

The energy of a molecule whose axis makes an angle θ with the direction of the electric field is $W(\theta) = -\frac{1}{2}\mu_z E = -\frac{1}{2}(\alpha_{\parallel} - \alpha_{\perp}) \times E^2 \cos^2\theta - \frac{1}{2}\alpha_{\perp} E^2$ where the relationship (1.3) has been used. In thermal equilibrium, the average value of $\cos^2\theta$ is then:

$$\overline{(\cos^2\theta)}_{th} = \frac{\int_0^{\pi} \cos^2\theta \, e^{-\frac{W(\theta)}{kT}} \sin\theta \, d\theta}{\int_0^{\pi} e^{-\frac{W(\theta)}{kT}} \sin\theta \, d\theta} = \frac{\int_{-1}^{+1} e^{\gamma u^2} u^2 du}{\int_{-s}^{+s} e^{\gamma u^2} du}$$

where the change of variable $u = \cos\theta$ has been made and $\gamma = \frac{(\alpha_{\parallel} - \alpha_{\perp}) E^2}{2kT}$. For $\gamma \ll 1$, we find $\overline{(\cos^2\theta)}_{th} = \frac{1}{3} + \frac{4\gamma}{45}$ or from (1.5), $s_{th} = \frac{4\gamma}{45}$.

In equilibrium $\left(\frac{ds}{dt} = 0\right)$, the solution of Eq.(1.7) is $s_{th} = \frac{2\lambda}{3}\tau E^2$. From the last three relationships, it is found that

$$\lambda = \frac{3 s_{th}}{2} \frac{1}{\tau E^2} = \frac{3}{2} \times \frac{4}{45} \gamma \times \frac{1}{\tau E^2} = \frac{1}{15} \frac{(\alpha_{\parallel} - \alpha_{\perp})}{kT\tau} \tag{1.8}$$

The next step is to solve the differential equation (1.7) when the electric field is the sum of the electric fields of the modes of a laser resonator.

1-3. Orientation of anisotropic molecules in a laser resonator.

In order to find the average dipole moment of an anisotropic molecule in a laser cavity, we must solve the differential Eq.(1.7) for the anisotropy tensor element s. The electric field is now the sum of the electric fields of all the individual longitudinal modes of the cavity which lie within the gain linewidth of the laser medium.

1-3.1. Normal mode formalism.

In order to describe the mode spectrum of the laser resonator, we introduce a set of orthonormal electric and magnetic vector functions $\vec{E}_n(\vec{r})$ and $\vec{H}_n(\vec{r})$ as defined by Slater(13)(14). They are related by the following relationships

$$k_n \vec{E}_n(\vec{r}) = \nabla \times \vec{H}_n(\vec{r}) \quad , \quad k_n \vec{H}_n(\vec{r}) = \nabla \times \vec{E}_n(\vec{r}) \tag{1.9}$$

and

$$\nabla \cdot \vec{E}_n(\vec{r}) = \nabla \cdot \vec{H}_n(\vec{r}) = 0 \tag{1.10}$$

where k_n is a constant and n is the index mode number. According to (1.9) and (1.10), they satisfy the following differential equations

$$(\nabla^2 + k_n^2)\, \vec{E}_n(\vec{r}) = 0 \quad , \quad (\nabla^2 + k_n^2)\, \vec{H}_n(\vec{r}) = 0 \tag{1.11}$$

and they are defined to obey the normalization conditions.

$$\int_V \vec{E}_n(\vec{r})\cdot\vec{E}_m(\vec{r})d\vec{r} = \delta_{nm} \quad , \quad \int_V \vec{H}_n(\vec{r})\cdot\vec{H}_m(\vec{r})d\vec{r} = \delta_{nm} \tag{1.12}$$

The above integrations are performed over the total volume of the cavity. We shall assume that the electric fields inside the cavity are linearly polarized. This is the case in a solid-state laser where the solid-state rods are cut at Brewster's angle to minimize the reflection losses. In this case, the electric fields of all the longitudinal modes of the laser cavity have the same direction; the same is true for the magnetic fields. We will, therefore, consider the electric and magnetic fields as being scalar quantities and we will drop the arrows.

We express the total electric field $E(\vec{r},t)$ and the total magnetic field $H(\vec{r},t)$ inside the cavity as:

$$E(\vec{r},t) = -\sum_n \frac{1}{\sqrt{\varepsilon_0}} \; P_n(t)\; E_n(\vec{r}) \tag{1.13}$$

$$H(\vec{r},t) = \sum_n \frac{1}{\sqrt{\mu_0}} \; \omega_n\; q_n(t)\; H_n(\vec{r}) \tag{1.14}$$

where ω_n is defined by

$$k_n = \omega_n \sqrt{\mu_0 \varepsilon_0} \tag{1.15}$$

and where $E_n(\vec{r})$ and $H_n(\vec{r})$ are the electric and magnetic scalar functions defined above. The summation is performed over the total number of modes of the cavity. ε_0 and μ_0 are the dielectric constant and the permittivity of the medium filling the cavity.

$p_n(t)$ and $q_n(t)$ are unknown functions of time describing the amplitudes and phases of the individual longitudinal cavity modes. We shall solve for these functions by using expansions (1.13) and (1.14) into Maxwell's equation with an added polarization term proportional to the cube of the electric field.

1-3.2. Expression of the anisotropy tensor element s in a laser cavity.

With the help of the formalism presented in Section 1-3.1, we can rewrite the differential Eq.(1.7) as follows:

$$\frac{ds}{dt} + \frac{s}{\tau} = \frac{2\lambda}{3} \sum_a \sum_b \frac{1}{\varepsilon_0} p_a(t)\, p_b(t)\, E_a(\vec{r})\, E_b(\vec{r}) \quad . \qquad (1.17)$$

We assume a solution for $p_n(t)$ in the form

$$p_n(t) = i\sqrt{\frac{\omega_n}{2}} \left(D_n^*(t)\, e^{i\omega_n t} - D_n(t)\, e^{-i\omega_n t} \right) \qquad (1.18)$$

where ω_n is the optical frequency of the oscillating mode n and $D_n(t)$ and $D_n^*(t)$ are slowly varying functions of time compared to $e^{i\omega_n t}$

$$\text{i.e.} \quad \left|\frac{dD_n^*(t)}{dt}\right| \ll \omega_n |D_n^*(t)| \qquad (1.19)$$

$D_n^*(t)$ is the complex conjugate of $D_n(t)$. The phase and amplitude information of mode n is thus obtained from the solution for $D_n^*(t)$.

According to (1.18),

$$p_a(t)\, p_b(t) = -\frac{\sqrt{\omega_a\omega_b}}{2}\left(D_a^*(t)\, D_b^*(t)\, e^{i(\omega_a+\omega_b)t} - D_a^*(t)\, D_b(t) e^{i(\omega_a-\omega_b)t}\right) + C.C.$$

we notice that the product $p_a(t)\, p_b(t)$ is made up of two frequency components, one at frequency $\omega_a + \omega_b$ and the other one at frequency $\omega_a - \omega_b$. Both ω_a and ω_b are optical frequencies $\sim 10^{15}$ rd/sec ; then $\omega_a + \omega_b$ is an optical frequency. Since the orientational relaxation time of the liquid is of the order of 10^{-10} to 10^{-12} sec, the molecules cannot respond to fields at optical frequencies. The only term to which the molecules can respond is the term at frequency $\omega_a-\omega_b$ which ranges from $\omega_a - \omega_a = 0$ to $\omega_a - \omega_b = 2\pi\,\Delta\nu_G$ where $\Delta\nu_G$ is the gain linewidth of the laser medium. Then

$$\frac{ds}{dt} + \frac{s}{\tau} = \frac{2\lambda}{3}\sum_a\sum_b E_a(\vec{r})E_b(\vec{r})\,\frac{\sqrt{\omega_a\omega_b}}{2\,\varepsilon_0}\left(D_a^*(t)D_b(t)\, e^{i(\omega_a-\omega_b)t} + C.C.\right) \qquad (1.20)$$

We look for a solution for s in the following form.

$$s = \sum_a\sum_b s_{ab}^*(t)\, e^{i(\omega_a-\omega_b)t} + C.C \qquad (1.21)$$

where $s_{ab}^*(t)$ is a slowly varying function of time compared to $e^{i(\omega_a-\omega_b)t}$ when $a \neq b$. Therefore for $a \neq b$, we can neglect $\frac{ds_{ab}^*(t)}{dt}$ with respect to $(\omega_a-\omega_b)\; s_{ab}^*(t)$ and we write

$$\frac{ds}{dt} = \left(\sum_a \frac{ds^*_{aa}(t)}{dt} + \sum_a \sum_{b \neq a} i(\omega_a - \omega_b) s^*_{ab}(t)\, e^{i(\omega_a - \omega_b)t} \right) + \text{C.C.} \qquad (1.22)$$

With the help of expressions (1.20), (1.21) and (1.22), we find

$$\sum_a \frac{ds^*_{aa}(t)}{dt} + \sum_a \sum_{b \neq a} i(\omega_a - \omega_b)\, s^*_{ab}(t)\, e^{i(\omega_a - \omega_b)t} + \frac{1}{\tau} \sum_a \sum_b s^*_{ab}(t)\, e^{i(\omega_a - \omega_b)t}$$

$$= \frac{\lambda}{3\varepsilon_0} \sum_a \sum_b \sqrt{\omega_a \omega_b}\, D^*_a(t)\, D_b(t)\, e^{i(\omega_a - \omega_b)t}\, E_a(\vec{r})\, E_b(\vec{r}) \qquad (1.23)$$

From the above equation, we find the following expressions for $s^*_{ab}(t)$ and $s^*_{aa}(t)$

$$s^*_{ab}(t) = \frac{\lambda\tau}{3\varepsilon_0} E_a(\vec{r})\, E_b(\vec{r}) \sqrt{\omega_a \omega_b}\, \frac{D^*_a(t)\, D_b(t)}{1 + i(\omega_a - \omega_b)\tau} \quad \text{for} \quad a \neq b \qquad (1.24)$$

and

$$\frac{d}{dt}\left(\sum_a s^*_{aa}(t) \right) + \frac{1}{\tau}\left(\sum_a s^*_{aa}(t) \right) = \frac{\lambda}{3\varepsilon_0} \sum_a \omega_a\, D^*_a(t) D_a(t) E^2_a(\vec{r}) \qquad (1.25)$$

The last expression is a differential equation for the quantity $\sum_a s^*_{aa}(t)$. The summation is performed over all the oscillating modes of the cavity. The right-hand side of Eq.(1.25) is related to the total electromagnetic energy stored in the cavity $\mathcal{E}_T$ in the following way. $\mathcal{E}_T$ is defined as

$$\mathcal{E}_T = \frac{1}{2} \int_V \left(\varepsilon_0 E^2(\vec{r},t) + \mu_0 H^2(\vec{r},t) \right) d\vec{r} \tag{1.26}$$

where the integral is performed over the volume of the cavity. $E(\vec{r},t)$ and $H(\vec{r},t)$ are replaced by their expressions (1.13) and (1.14)

$$\mathcal{E}_T = \frac{1}{2} \sum_n \sum_m \left(p_n(t) p_m(t) \int_V E_n(\vec{r})\, E_m(\vec{r}) d\vec{r} + \omega_n \omega_m q_n(t) q_m(t) \int_V H_n(\vec{r}) H_m(\vec{r}) d\vec{r} \right)$$

and with the help of (1.12)

$$\mathcal{E}_T = \frac{1}{2} \sum_n \left(p_n^2(t) + \omega_n^2 q_n^2(t) \right) \tag{1.27}$$

or in terms of the functions $D_n^*(t)$ and $D_n(t)$ defined by (1.18)

$$\mathcal{E}_T = \sum_n \omega_n D_n^*(t)\, D_n(t) \tag{1.28}$$

<u>Note</u>: The derivation of expression (1.28) from expression (1.27) is given in the Appendix.

The quantity $\sum_a \omega_a D_a^*(t)\, D_a(t)\, E_a^2(\vec{r})$ is equal to the energy per unit volume inside the cavity since the integral of this quantity over the volume of the cavity is:

$$\int_V d\vec{r} \sum_a \omega_a D_a^*(t)\, D_a(t)\, E_a^2(\vec{r}) = \sum_a \omega_a D_a^*(t)\, D_a(t) = \mathcal{E}_T \;.$$

If the total electromagnetic energy stored inside the cavity is constant during the time the interaction between the modes takes place in the liquid, then the energy per unit volume can be considered a constant and the solution of the differential equation (1.25) is:

$$\sum_a s^*_{aa}(t) = \frac{\lambda\tau}{3\varepsilon_0}(1 - e^{-t/\tau}) \sum_a \omega_a D^*_a(t)\, D_a(t)\, E^2_a(\vec{r}) \qquad (1.29)$$

Expressions (1.24) and (1.29) are then used in (1.21) to find s.

$$s = \frac{\lambda\tau}{3\varepsilon_0}\Bigg((1 - e^{-t/\tau}) \sum_a \omega_a D^*_a(t)\, D_a(t)\, E^2_a(\vec{r})$$

$$+ \sum_a \sum_{b \neq a} \sqrt{\omega_a\omega_b}\, \frac{D^*_a(t)\, D_b(t)}{1+i(\omega_a-\omega_b)\,\tau}\, e^{i(\omega_a-\omega_b)t}\, E_a(\vec{r})\, E_b(\vec{r})\Bigg) + \text{C.C.} \qquad (1.30)$$

For times t long compared to the relaxation time τ of the liquid, we have

$$s = \frac{\lambda\tau}{3\varepsilon_0} \sum_a \sum_b \sqrt{\omega_a\omega_b}\, \frac{D^*_a(t)\, D_b(t)}{1+i(\omega_a-\omega_b)\tau}\, e^{i(\omega_a-\omega_b)t}\, E_a(\vec{r})\, E_b(\vec{r}) + \text{C.C.} \qquad (1.31)$$

We have expressed the anisotropy tensor element s in terms of the complex amplitudes $D_n(t)$ and $D^*_n(t)$ of the oscillating longitudinal modes of the cavity and of the space eigenvectors $E_n(\vec{r})$ of these modes. We are now in a position to calculate the total polarization induced in the nonlinear liquid placed inside a laser resonator.

1-4. Total nonlinear polarization induced in the anisotropic molecular liquid.

The average dipole moment of an anisotropic molecule is $\bar{\mu}_z$. It is related to the total average polarization P per unit volume by the relationship

$$P = N_0 \bar{\mu}_z \tag{1.32}$$

where N_0 is the number of molecules per unit volume.

$$P = N_0(\alpha_{\parallel} - \alpha_{\perp})\, s\, E + N_0(\alpha_{\parallel} + 2\alpha_{\perp})\frac{E}{3}$$

In M.K.S. units the displacement vector D is expressed as

$$D = \varepsilon_0 E + P = \left(\varepsilon_0 + N_0\left(\frac{\alpha_{\parallel} + 2\alpha_{\perp}}{3}\right)\right)E + N_0(\alpha_{\parallel} - \alpha_{\perp})\, s\, E$$

which can be written as:

$$D = \varepsilon_L E + P_{NL} \tag{1.33}$$

where ε_L is the dielectric constant of the liquid and P_{NL} is the total nonlinear polarization induced in the liquid.

$$P_{NL} = N_0(\alpha_{\parallel} - \alpha_{\perp})\, s\, E \tag{1.34}$$

We now express the nonlinear polarization P_{NL} in terms of the cavity mode wavefunctions $D_n(t)$ and $E_n(\vec{r})$. With the help of (1.13), (1.19) and (1.31), we find

$$P_{NL} = -\, i\, N_0(\alpha_{\parallel} - \alpha_{\perp}) \frac{\lambda\tau}{3\varepsilon_0\sqrt{2\varepsilon_0}} \sum_a \sum_b \sum_c \sqrt{\omega_a\omega_b\omega_c}\, E_a(\vec{r})E_b(\vec{r})E_c(\vec{r})$$

$$\times \left(\frac{D_a^*(t)\, D_b(t)}{1+i(\omega_a-\omega_b)\tau}\, e^{i(\omega_a-\omega_b)t} + C.C. \right) \left(D_c^*(t)\, e^{i\omega_c t} - D_c(t)\, e^{-i\omega_c t} \right) \quad (1.35)$$

The two products in expression (1.35) are written explicitly as

$$\frac{i\, D_a^*\, D_b\, D_c^*}{1+i(\omega_a-\omega_b)\tau}\, e^{i(\omega_a-\omega_b+\omega_c)t} + C.C.$$

$$+\, i\, \frac{D_a\, D_b^*\, D_c^*}{1+i(\omega_b-\omega_a)\tau}\, e^{i(\omega_b-\omega_a+\omega_c)\, t} + C.C. \quad (1.36)$$

The second part of expression (1.36) is obtained from the first part by interchanging the indices a and b. Since the double summation $\sum_a \sum_b$ will be performed, these two terms will give the same contribution to the nonlinear polarization and we can write

$$P_{NL} = -\, 2\, N_0(\alpha_{\parallel} - \alpha_{\perp}) \frac{2\lambda\tau}{3} \frac{1}{\sqrt{8\varepsilon_0^3}} \sum_a \sum_b \sum_c \sqrt{\omega_a\omega_b\omega_c}$$

$$\times\, E_a(\vec{r})\, E_b(\vec{r})\, E_c(\vec{r}) \left(\frac{i\, D_a^*(t)\, D_b(t)\, D_c^*(t)}{1 + i(\omega_a-\omega_b)\tau}\, e^{i(\omega_a-\omega_b+\omega_c)t} + C.C. \right) \quad (1.37)$$

At this point it is useful to define an important parameter of an anisotropic molecular liquid which is a measure of its anisotropy: the optical Kerr constant.

Definition of the Optical Kerr Constant ε_2.

Let us consider the case of only one oscillating mode, say, mode n, in the cavity. The nonlinear polarization P_{NL} can then be expressed according to (1.37) as:

$$P_{NL} = - 2 N_0(\alpha_{\parallel} - \alpha_{\perp}) \frac{2\lambda}{3} \sqrt{\frac{\omega_n^3}{8\varepsilon_0^3}} E_n^3(\vec{r}) \left(i D_n^* D_n D_n^* e^{i\omega_n t} + C.C. \right) \qquad (1.38)$$

The electric field of this mode is, according to (1.13) and (1.18),

$$E(\vec{r},t) = - \sqrt{\frac{\omega_n}{2\varepsilon_0}} \left(i D_n^* e^{i\omega_n t} + C.C. \right) E_n(\vec{r})$$

and the component of $E^3(\vec{r},t)$ oscillating at the frequency ω_n is

$$E^3(\vec{r},t) = - 3\sqrt{\frac{\omega_n^3}{8\varepsilon_0^3}} E_n^3(\vec{r}) \left(i D_n^* D_n D_n^* e^{i\omega_n t} + C.C. \right) \qquad (1.39)$$

With the help of (1.38) and (1.39) we can write

$$P_{NL} = 2 N_0(\alpha_{\parallel} - \alpha_{\perp}) \frac{2\lambda\tau}{3} \frac{1}{3} E^3(\vec{r},t) \qquad (1.40)$$

The nonlinear polarization induced in the anisotropic liquid by the applied electric field is proportional to the cube of the electric field. The coefficient of proportionality is called the optical Kerr constant of the liquid ε_2 and is expressed with the help of (1.8) as:

$$\varepsilon_2 = \frac{2}{3} N_0(\alpha_{\parallel} - \alpha_{\perp}) \frac{2\lambda\tau}{3} = \frac{4}{135} N_0 \frac{(\alpha_{\parallel} - \alpha_{\perp})^2}{kT} \tag{1.41}$$

This constant has been measured by various authors in various liquids in relation to the self-focusing of optical beams in anisotropic liquids(15). It is proportional to the square of the anisotropy $(\alpha_{\parallel} - \alpha_{\perp})$ of the molecules and inversely proportional to the temperature. It is an important parameter of our analysis; the larger the optical Kerr constant, the larger the coupling between the cavity modes. In the experimental part of our work, we will use liquids with a very large Kerr constant.

The nonlinear polarization P_{NL} is then expressed in terms of ε_2. From (1.37) and (1.41)

$$P_{NL} = -3\, i\, \varepsilon_2 \sum_a \sum_b \sum_c \sqrt{\frac{\omega_a \omega_b \omega_c}{8 \varepsilon_0^{\,3}}}\; E_a(\vec{r})\, E_b(\vec{r})\, E_c(\vec{r})$$

$$\times \frac{D_a^*(t) D_b(t) D_c^*(t)}{1 + i(\omega_a - \omega_b)\tau}\, e^{i(\omega_a - \omega_b + \omega_c)t} + \text{C.C.} \tag{1.42}$$

We have now expressed the nonlinear polarization induced in the anisotropic molecular liquid by the fields of the laser cavity modes in terms of the wave functions $D_n(t)\, E_n(\overline{r})$ of these modes. P_{NL} as expressed by (1.42) involves a triple summation over all the cavity modes of the product of the wave functions of three modes. Coupling of the cavity modes will occur through this nonlinear polarization only if the frequency components of P_{NL} are within the gain linewidth of the laser medium.

We are now in a position to find solutions for the complex amplitudes $D_n^*(t)$ and $D_n(t)$ of the oscillating cavity modes in the presence of a nonlinear dielectric. This is done in the next section by solving Maxwell's equations with the nonlinear polarization acting as a driving source.

1-5. Solution of Maxwell's equations with a third order nonlinear polarization.

In order to find an expression for the electric field in a laser resonator containing an anisotropic molecular liquid, we solve Maxwell's equations with a nonlinear polarization driving term. By this procedure we shall first find a differential equation obeyed by the mode amplitudes $D_n^*(t)$.

1-5.1. Differential equation for $D_n^*(t)$.

1-5.1.1. Maxwell's equations.

In M.K.S. units, Maxwell's equations for the electric field $E(\vec{r},t)$ and the magnetic field $H(\vec{r},t)$ are

$$\nabla \times H(\vec{r},t) = I + \frac{\partial D}{\partial t} \tag{1.43}$$

$$\nabla \times E(\vec{r},t) = - \frac{\partial B}{\partial t} \tag{1.44}$$

where all the fields are scalar fields.

The consitutive relations in the nonlinear medium are

$$I = \sigma E \tag{1.45}$$

$$D = \varepsilon_L E + P_{NL} \tag{1.46}$$

$$B = \mu_0 H \tag{1.47}$$

ε_L is the dielectric constant of the medium and σ is the electrical conductivity of the medium. We shall assume in the remainder of this analysis that the linear dielectric constant of the liquid is the same as the dielectric constant of the medium filling the cavity. This is not exactly true since the liquids which are used have an index of refraction equal to 1.5 and therefore a dielectric constant of 2.25 while air with a dielectric constant of 1 fills the cavity. A correct result is obtained, however, if we consider the equivalent path of the light inside the liquid, i.e. the physical length of the liquid cell multiplied by the index of refraction of the liquid. Using (1.45), (1.46) and (1.47), we can rewrite Maxwell's equations as

$$\nabla \times H = \sigma E + \frac{\partial}{\partial t} (\varepsilon_0 E + P_{NL}) \tag{1.48}$$

and

$$\nabla \times E = - \mu_0 \frac{\partial H}{\partial t} \tag{1.49}$$

Let us first replace the electric and magnetic fields by their expressions (1.13) and (1.14) into the second Maxwell equation (1.49)

$$-\sum_n \frac{1}{\sqrt{\varepsilon_0}} p_n(t) \; (\nabla \times E_n(\vec{r})) = -\sqrt{\mu_0} \sum_n \omega_n \frac{dq_n(t)}{dt} H_n(\vec{r}) \tag{1.59}$$

We then use relationship 1.9, multiply both sides by $H_m(\vec{r})$ and integrate over the volume of the cavity. With the help of (1.12), we find

$$\frac{1}{\sqrt{\varepsilon_0}} p_n(t)\, k_n = \sqrt{\mu_0}\, \omega_n \frac{dq_n(t)}{dt}$$

and from the definition (1.15) of ω_n

$$p_n(t) = q_n'(t) \qquad (1.60)$$

The second Maxwell equation (1.49) has provided a relationship between $p_n(t)$ and $q_n(t)$. To find a solution for $p_n(t)$ and $q_n(t)$ we use the first Maxwell equation (1.48).

According to (1.60) and (1.18) we can now express $q_n(t)$ and then the magnetic field $H(\vec{r},t)$ in terms of the complex amplitude $D_n(t)$ of the nth cavity mode in the following way:

$$q_n(t) = \frac{1}{\sqrt{2\omega_n}} \left(D_n^*(t)\, e^{i\omega_n t} + D_n(t)\, e^{-i\omega_n t} \right) \qquad (1.61)$$

since

$$\left|\frac{dD_n^*(t)}{dt}\right| \ll \omega_n \left|D_n^*(t)\right|$$

With the help of expressions (1.13), (1.14) and (1.42) we write the first Maxwell equation (1.48) as

$$\sum_n \frac{1}{\sqrt{\mu_0}}\,\omega_n q_n(t)(\nabla \times H_n(\vec{r})) = -\frac{\sigma}{\sqrt{\varepsilon_0}}\sum_n p_n(t)\,E_n(\vec{r}) - \sqrt{\varepsilon_0}\sum_n p_n'(t)E_n(\vec{r})$$

$$+\,3\,\varepsilon_2 \sum_a\sum_b\sum_c \sqrt{\frac{\omega_a\omega_b\omega_c}{8\varepsilon_0^{\,3}}}\;E_a(\vec{r})\,E_b(\vec{r})\,E_c(\vec{r})$$

$$\times\,(\omega_a-\omega_b+\omega_c)\left(\frac{D_a^*(t)\,D_b(t)\,D_c^*(t)}{1+i(\omega_a-\omega_b)\tau}\;e^{i(\omega_a-\omega_b+\omega_c)t} + \text{C.C.}\right) \qquad (1.62)$$

We recall that $\nabla \times H_n(\vec{r}) = k_n\,E_n(\vec{r})$ and that $k_n = \omega_n\sqrt{\mu_0\varepsilon_0}$ and we write the left-hand side of (1.62) as

$$\sum_n \frac{1}{\sqrt{\mu_0}}\,\omega_n q_n(t)\;k_n E_n(\vec{r}) = \sqrt{\varepsilon_0}\sum_n \omega_n^2 q_n(t)\;E_n(\vec{r})\;.$$

We then multiply both sides of Eq.(1.62) by $E_m(\vec{r})$ and integrate over the volume of the cavity. The last term of Eq.(1.62) represents the nonlinear polarization term. It will be non zero only inside the volume of the nonlinear medium, i.e. the volume of the liquid cell placed inside the optical resonator. Under these conditions and with the help of the normalization conditions (1.12), we find

$$\sqrt{\varepsilon_0}\,\omega_n^2 q_n(t) = -\frac{\sigma}{\sqrt{\varepsilon_0}}\;p_n(t) - \sqrt{\varepsilon_0}\;p_n'(t) + 3\;\varepsilon_2\sum_a\sum_b\sum_c\sqrt{\frac{\omega_a\omega_b\omega_c}{8\varepsilon_0^{\,3}}}\;S_{nabc}$$

$$\times\,(\omega_a-\omega_b+\omega_c)\left(\frac{D_a^*(t)\,D_b(t)\,D_c^*(t)}{1+i(\omega_a-\omega_b)\,\tau}\;e^{i(\omega_a-\omega_b+\omega_c)t} + \text{C.C.}\right) \qquad (1.63)$$

where S_{nabc} is defined by the relationship

$$S_{nabc} = \int_{\substack{\text{Volume}\\ \text{of the cell}}} E_n(\vec{r})\, E_a(\vec{r})\, E_b(\vec{r})\, E_c(\vec{r})\, dV \tag{1.64}$$

We shall calculate this parameter explicitly later.

We can also write expression (1.63) as follows by dividing through by $\sqrt{\varepsilon_0}$

$$\omega_n^2 q_n(t) + \frac{\sigma}{\varepsilon_0} p_n(t) + p_n'(t) = \frac{3\ \varepsilon_2}{2\sqrt{2}\ \varepsilon_0^2} \sum_a \sum_b \sum_c \sqrt{\omega_a \omega_b \omega_c}\, S_{nabc}$$

$$\times\ (\omega_a - \omega_b + \omega_c) \left\{ \frac{D_a^*(t)\ D_b(t)\ D_c^*(t)}{1 + i(\omega_a - \omega_b)\ \tau}\ e^{i(\omega_a - \omega_b + \omega_c)\ t} + \text{C.C.} \right\} \tag{1.65}$$

By using both Maxwell's equations, we have found two different relationships between $p_n(t)$ and $q_n(t)$. They are expressions (1.60) and (1.65). By eliminating $q_n(t)$, for example, between these two relationships, we can find a differential equation obeyed by $p_n(t)$. The configuration of the electric field inside the cavity will then be found by solving this equation. We shall follow this procedure with a slight difference however.

The equation $p_n(t) = q_n'(t)$ has allowed us to express $q_n(t)$ in terms of $D_n^*(t)$ and $D_n(t)$. We shall then replace $p_n(t)$ and $q_n(t)$ by their expressions (1.18) and (1.61) and we shall find a differential equation for $D_n^*(t)$, instead of $p_n(t)$.

$$\omega_n^2 \frac{1}{\sqrt{2\omega_n}} D_n^*(t)e^{i\omega_n t} + i\frac{\sigma}{\varepsilon_0}\sqrt{\frac{\omega_n}{2}} D_n^*(t)e^{i\omega_n t} - \sqrt{\frac{\omega_n}{2}}\,\omega_n D_n^*(t)e^{i\omega_n t} + i\sqrt{\frac{\omega_n}{2}}\,\frac{dD_n^*(t)}{dt}\,e^{i\omega_n t}$$

$$= \frac{3\,\varepsilon_2}{2\sqrt{2}\,\varepsilon_0^{\,2}} \sum_a\sum_b\sum_c \sqrt{\omega_a\omega_b\omega_c}\; S_{nabc}(\omega_a-\omega_b+\omega_c)\,\frac{D_a^*(t)D_b(t)D_c^*(t)}{1+i(\omega_a-\omega_b)\,\tau}\,e^{i(\omega_a-\omega_b+\omega_c)t} \tag{1.66}$$

In the equation (1.66), we look for combinations of the modes a, b, and c such that the nonlinear polarization induced by these modes oscillates at frequency ω_n, i.e. such that

$$\omega_a - \omega_b + \omega_c = \omega_n \tag{1.67}$$

since only this term can provide synchronous driving of the oscillation at ω_n. With this condition, the above expression is written as:

$$\frac{dD_n^*(t)}{dt} + \frac{\sigma}{\varepsilon_0} D_n^*(t) = -\frac{3i\varepsilon_2}{2\varepsilon_0^{\,2}}\sqrt{\omega_n} \sum_{a,b,c} \sqrt{\omega_a\omega_b\omega_c}\; S_{nabc}\,\frac{D_a^*(t)D_b(t)D_c^*(t)}{1+i(\omega_a-\omega_b)\,\tau} \tag{1.68}$$

This is our main working equation; it is a differential equation for the nth mode complex amplitude $D_n^*(t)$. A similar differential equation for $D_n(t)$ is obtained by taking the complex conjugate.

In a cavity with no nonlinear medium, the differential equation (1.68) reduces to $\frac{dD_n^*(t)}{dt} + \frac{\sigma}{\varepsilon_0} D_n^*(t) = 0$ the solution of which is $D_n^*(t) = A\,e^{-\frac{\sigma}{\varepsilon_0}t}$. The amplitudes of modes decrease with time if $\sigma > 0$

(passive cavity) or are amplified if $\sigma < 0$ (in a laser the gain or $\sigma < 0$ is provided by the active medium).

We shall assume that the gain provided by the laser medium is at each time equal to the losses of the cavity (reflection losses at the mirror and diffraction losses). In that case $\sigma = 0$ and the differential equation (1.68) is reduced to

$$\frac{dD_n^*(t)}{dt} = -\frac{3i\varepsilon_2}{2\varepsilon_0^2}\sqrt{\omega_n}\sum_{a,b,c}\sqrt{\omega_a\omega_b\omega_c}\,S_{nabc}\,\frac{D_a^*(t)D_b(t)D_c^*(t)}{1+i(\omega_a-\omega_b)\tau} \qquad (1.69)$$

We have obtained a differential equation for the wave function of one cavity mode $D_n^*(t)$ in terms of the wave functions of all the other modes. The mode n is coupled to the modes a, b, c, such that $\omega_n = \omega_a - \omega_b + \omega_c$, by a susceptibility which is proportional to $\frac{1}{1+i(\omega_a-\omega_b)\tau}$. When the number of oscillating modes is very large so that the effect of the end modes (at the low frequency and high frequency sides of the gain linewidth) can be neglected, the interaction between all the modes can be accounted for in a rather simple and accurate way.

We first reduce the triple summation $\sum_{a,b,c}\frac{D_a^*D_bD_c^*}{1+i(\omega_a-\omega_b)\tau}$ to a double summation and we calculate explicitly S_{nabc}.

<u>1-5.1.2. Expression of $\sum_{a,b,c}\frac{D_a^*D_bD_c^*}{1+i(\omega_a-\omega_b)\tau}$ as a double summation.</u>

The triple summation $\sum_{a,b,c}\frac{D_a^*D_bD_c^*}{1+i(\omega_a-\omega_b)\tau}$ is in fact a double summation since the mode indices a, b, c are such that

$$\omega_n = \omega_a - \omega_b + \omega_c \tag{1.67}$$

The mode spectrum of a laser resonator in the absence of the nonlinear liquid (and we will assume that the presence of the liquid does not change it appreciably) consists of a large number of modes equispaced in frequency. The (radian) frequency separation between two adjacent modes is Ω such that $\Omega = 2\pi \frac{C}{2L}$.

The frequency of the nth mode is defined as

$$\omega_n = n\Omega \tag{1.70}$$

where n is a very large number approximately equal to the number of half wavelengths contained in the length of the resonator $n \sim \frac{L}{\lambda/2}$. For $L = 1m$ and $\lambda = 1\mu$, we have $n \sim 2 \times 10^6$. With the definition (1.70), the condition (1.67) becomes $n = a - b + c$; this condition is satisfied if we take $a = n + m$, $b = n + m + p$, $c = n + p$. The summation over a, b and c becomes a summation over m and p and

$$\sum_{a,b,c} \frac{D_a^* D_b D_c^*}{1 + i(\omega_a - \omega_b)\tau} = \sum_{m,p} \frac{D_{n+m}^* D_{n+m+p} D_{n+p}^*}{1 - i p \Omega \tau} \tag{1.71}$$

We notice that the beating of modes $n + m$ and $n + m + p$ at frequency $p\Omega$ will modulate the dielectric constant seen by the mode $n + p$ and thus will generate a sideband at frequency $(n+p)\Omega - p\Omega = n\Omega = \omega_n$, which is the frequency of the nth mode. In this way, modes n, $n+m$, $n+m+p$, and $n+p$ are coupled together (see Figure 1.3). The nonlinear susceptibility is proportional to $\frac{1}{1-ip\Omega\tau}$.

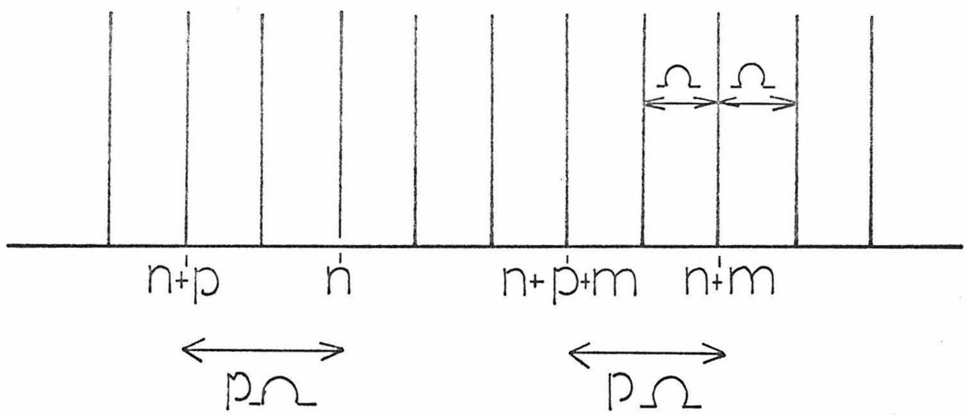

FIG 1-3 _ Coupling of the cavity modes n, n+p, n+m, n+p+m _ here m=5, p=-2

1-5.1.3. Explicit calculation of $S_{n,n+m,n+m+p,n+p} \equiv S_{n,m,p}$.

In the double summation over m and p, the term S_{nabc} is replaced by $S_{n,m,p}$ which is defined by (1.64).

$$S_{n,m,p} = \int_{\substack{\text{Volume}\\ \text{of the cell}}} E_n(\vec{r})\, E_{n+m}(\vec{r})\, E_{n+m+p}(\vec{r})\, E_{n+p}(r)\, dV \tag{1.72}$$

In order to evaluate explicitly this integral, we must specify the configuration of the liquid cell in the cavity.

A liquid cell of length 2ℓ is inserted in an optical cavity of length L at a distance L_0 from a mirror which is taken as the origin of the coordinates (see Figure 1.4).

The nth mode of the empty cavity has the following spatial dependence defined by (1.11) and (1.12).

$$E_n(\vec{r}) = \sqrt{\frac{2}{AL}}\ \sin\frac{n\pi x}{L} \tag{1.73}$$

where A is the cross section of the beam. We have, in (1.71), neglected the spatial dependence in the transverse direction. Then, according to (1.72),

$$S_{n,m,p} = \left(\frac{2}{AL}\right)^2 \int_{L_0-\ell}^{L_0+\ell} dx\ \sin\left(\frac{n\pi x}{L}\right)\ \sin\left[(n+m)\frac{\pi x}{L}\right] \sin\left[(n+m+p)\frac{\pi x}{L}\right] \sin\left[(n+p)\frac{\pi x}{L}\right] \int dA$$

The integration with respect to x yields

$$S_{n,m,p} = \frac{1}{V}\left(\frac{\ell}{L} + \frac{1}{2p\pi}\sin\frac{2\pi p\ell}{L}\cos\frac{2\pi pL_0}{L} + \frac{1}{2m\pi}\sin\frac{2m\pi\ell}{L}\cos\frac{2\pi mL_0}{L}\right) \tag{1.74}$$

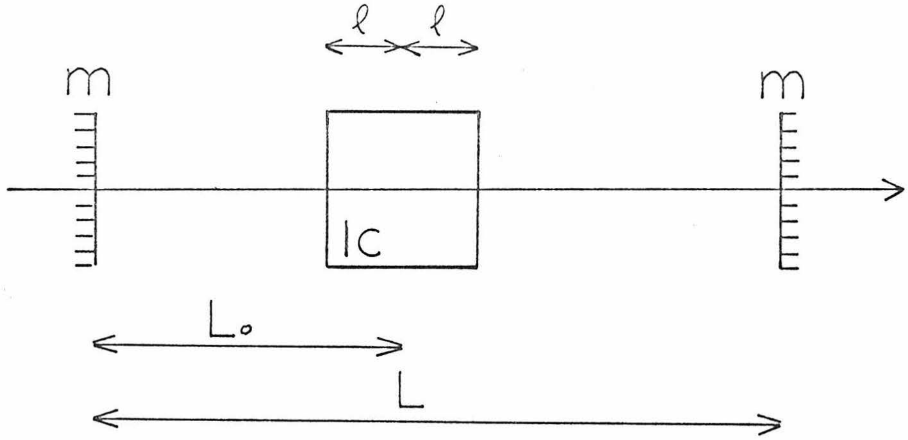

FIG 1-4 _The liquid cell (lc) is placed inside the cavity formed by the mirrors m _

where $V = AL$ is the volume occupied by the light in the cavity. We see that the quantity $S_{n,m,p}$ is roughly proportional to the ratio of the length of the liquid cell to the total length of the cavity.

All the frequencies ω_n, ω_{n+p}, ω_{n+m+p}, ω_{n+m} are optical frequencies which lie within the gain linewidth of the laser medium. Therefore we can write without any appreciable error.

$$\sqrt{\omega_n \omega_{n+p} \omega_{n+m+p} \omega_{n+m}} \sim \omega_n \, \omega_{n+m} \tag{1.75}$$

1-5.1.4. Final expression of the differential equation for $D_n^*(t)$.

With the help of (1.71), (1.74) and (1.75) we express the differential equation for $D_n^*(t)$ in the following way.

$$\frac{dD_n^*}{dt} = - \frac{3i \, \varepsilon_2 \, \omega_n}{2 \, \varepsilon_0^2 \, V} \sum_p \frac{D_{n+p}^*}{(1-ip\Omega\tau)} \sum_m \omega_{n+m} \, D_{n+m+p} \, D_{n+m}^*$$

$$\times \left(\frac{\ell}{L} + \frac{1}{2p\pi} \sin \frac{2\pi p \ell}{L} \cos \frac{2\pi p L_0}{L} + \frac{1}{2m\pi} \sin \frac{2m\pi\ell}{L} \cos \frac{2m\pi L_0}{L} \right) \tag{1.76}$$

The solution of this differential equation is very difficult in the general case where the liquid cell occupies an arbitrary position in the cavity. However, the physical situation will not be hindered if we solve the above equation for a special configuration of the cavity. This will be done in the next section.

1-5.2. Solution of the differential equation for $D_n^*(t)$.

We shall find a solution of the differential Eq.(1.76) for the following case: The liquid cell fills half of the laser cavity (see Figure 1.5).

In this case $L_0 = \ell = \frac{L}{4}$, $\sin \frac{2m\pi\ell}{L} \cos 2m \frac{\pi L_0}{L} = \sin \frac{m\pi}{2} \cos \frac{m\pi}{2} = \frac{1}{2} \sin m\pi$ and

$$\frac{1}{2m\pi} \sin \frac{2\pi m\ell}{L} \cos \frac{2\pi m L_0}{L} = \begin{cases} 0 & m \geq 1 \\ \frac{\ell}{L} & m = 0 \end{cases}$$

$$\frac{1}{2p\pi} \sin \frac{2\pi p\ell}{L} \cos \frac{2\pi p L_0}{L} = \begin{cases} 0 & p \geq 1 \\ \frac{\ell}{L} & p = 0 \end{cases}$$

Under these conditions, we write the differential equation (1.76) as:

$$\frac{dD_n^*}{dt} = - \frac{3i\ \varepsilon_2\ \omega_n}{2\ \varepsilon_0^2\ V} \frac{\ell}{L} \left(\sum_p \frac{D_{n+p}^*}{(1-ip\Omega\tau)} \sum_m \omega_{n+m}\, D_{n+m+p}\, D_{n+m}^* + D_n^* \sum_m \omega_{n+m}\, D_{n+m}\, D_{n+m}^* + D_n^* \sum_p \omega_{n+p} \frac{D_{n+p}\, D_{n+p}^*}{(1-ip\Omega\tau)} \right) \tag{1.77}$$

In order to specify the values of p and m over which the summations in (1.77) are performed, we must discuss in more detail the mechanism of power exchange between modes.

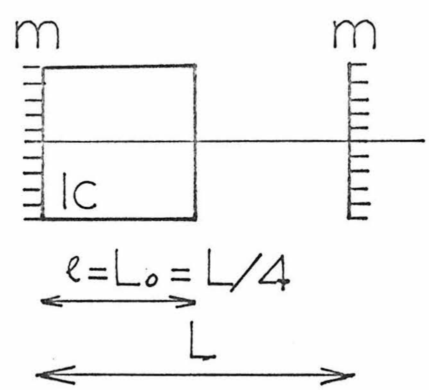

FIG 1-5 _ The liquid cell (lc) fills half of the laser cavity

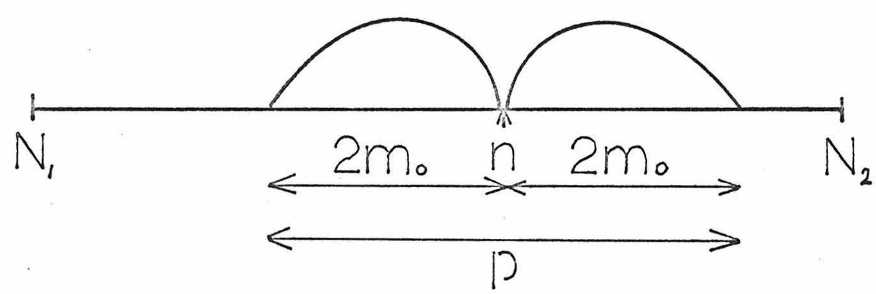

FIG 1-7 _ Description of the modes participating directly to the energy flow to and from mode n

In Section 1-5.1.2, we have described the mode coupling between four modes labeled n, $n+p$, $n+m+p$, $n+m$. See Figure 1.3. A nonlinear polarization is induced at frequency ω_m. The susceptibility for this process is $X = \frac{\varepsilon_2}{1-ip\Omega\tau}$. The dielectric constant seen by mode $n+p$ is modulated at frequency $p\Omega$ thereby creating a sideband at the frequency ω_m. We can write the susceptibility X as a real part X' plus an imaginary part $+iX''$ where X'' is a real function. The transfer of energy between the modes n and $n+p$ is proportional to X'' and in our case proportional to $X'' = \varepsilon_2 \frac{p\Omega\tau}{1+(p\Omega\tau)^2}$.

If p is positive, i.e. if the mode $n+p$ has a higher frequency than the mode n, energy will be transferred from mode $n+p$ into mode n at a rate proportional to $\frac{p\Omega\tau}{1+(p\Omega\tau)^2}$. If p is negative, i.e. if $\omega_n > \omega_{n+p}$, energy will flow from mode n into mode $n+p$ at the same rate. If we neglect the effects of the end modes, i.e. if we assume a large number of oscillating modes, the amplitude of each mode is likely to stay constant since as much power is fed into it from higher frequency modes as flows out of it to lower frequency modes.

The rate at which energy is transferred into one mode via another mode separated in frequency by $p\Omega$ is proportional to $\frac{p\Omega\tau}{1+(p\Omega\tau)^2}$. We call this function of p, $f(\frac{p}{m_0})$ where

$$m_0 = \frac{1}{\Omega\tau} \tag{1.78}$$

then

$$f(\frac{p}{m_0}) = \frac{p/m_0}{1 + (p/m_0)^2} \tag{1.79}$$

This function is depicted in Figure 1.6. $f(\frac{p}{m_0})$ increases for

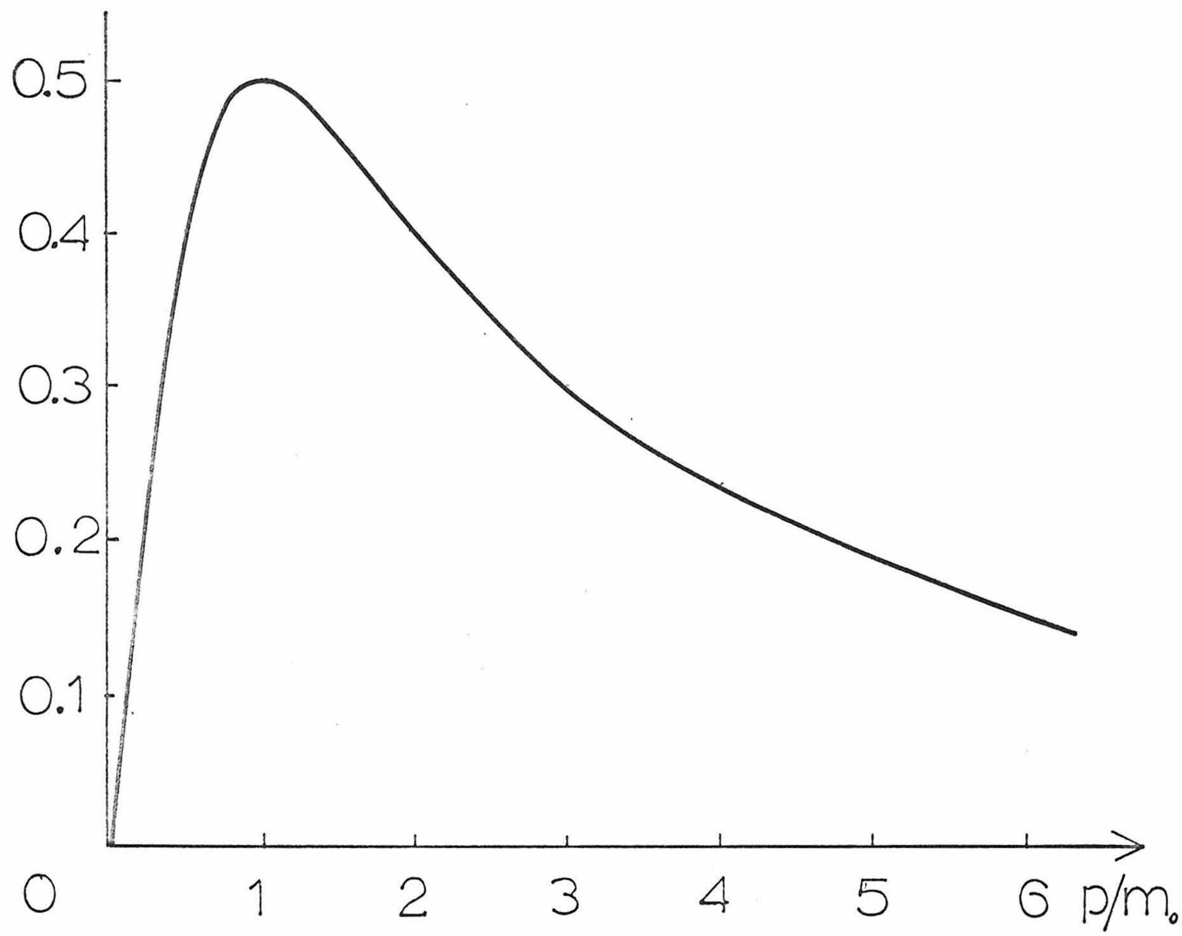

FIG 1-6 _Representation of the function $f(p/m_0) = \frac{p/m_0}{1+(p/m_0)^2}$

$p < m_0$ to its maximum value of $\frac{1}{2}$ for $p = m_0$ and decreases to zero for $p > m_0$. The energy flowing from or into mode n via mode $n + p$ decreases when $p > m_0$. In our analysis we shall neglect the interaction of the nth mode with the mode $n + p$ such that $p > 2m_0$; therefore we shall limit the summation over p to $-2m_0$ and $+2m_0$. See Figure 1.7.

We label N_1 and N_2 the lowest and highest frequency modes inside the gain linewidth. The total number of oscillating modes is $N = N_2 - N_1$.

We can now write the differential equation (1.77) as

$$\frac{dD_n^*}{dt} = - \frac{3i\ \varepsilon_2\ \omega_n}{2\ \varepsilon_0^2\ V} \frac{\ell}{L} \left(\sum_{p=-2m_0}^{2m_0} \frac{D_{n+p}^*}{(1-ip\Omega\tau)} \sum_{m=N_1-n}^{N_2-n} D_{n+m+p}\ D_{n+m}^* \right.$$

$$\left. + D_n^* \mathcal{E}_T + D_n^* \sum_{p=-2m_0}^{2m_0} \omega_{n+p} \frac{D_{n-p}\ D_{n+p}^*}{(1-ip\Omega\tau)} \right) \tag{1.80}$$

where $\mathcal{E}_T$ is the total energy stored in the cavity, according to (1.28),

$$\mathcal{E}_T = \sum_{m=N_1-n}^{N_2-n} \omega_{n+m}\ D_{n+m}^*\ D_{n+m}.$$

We shall look for a solution of (1.80) which has the following form

$$D_n^*(t) = \beta(t)\ e^{in\Phi(t)} \quad , \quad D_n(t) = \beta(t)\ e^{-in\Phi(t)} \tag{1.81}$$

where $\beta(t)$ and $\Phi(t)$ are real functions of time. In this assumed form the cavity modes have the same amplitude $\beta(t)$ and the phase difference between any two adjacent modes is the same and equal to $\Phi(t)$.

We now replace D_n^* and D_n in Eq.(1.80) by their expressions (1.81)

$$\left(\frac{d\beta(t)}{dt} + in\Phi'(t)\beta(t)\right)e^{in\Phi(t)} = -\frac{3i\ \varepsilon_2\ \omega_n}{2\ \varepsilon_0^2\ V}\ \frac{\ell}{L}\ e^{in\Phi(t)}$$

$$\left(\beta \sum_{p=-2m_0}^{2m_0} \left(\frac{1}{1-ip\Omega\tau}\right)\sum_m \omega_{n+m}\ \beta^2 + \beta\,\mathcal{E}_T + \beta \sum_{p=-2m_0}^{2m_0} \frac{\omega_{m+p}\ \beta^2}{(1-ip\Omega\tau)}\right) \tag{1.82}$$

We use the fact that the total energy $\mathcal{E}_T$ is

$$\mathcal{E}_T = \sum_m \omega_{n+m}\ D^*_{n+m}\ D_{n+m} = \sum_m \omega_{n+m}\ \beta^2$$

to write

$$\sum_{p=-2m_0}^{2m_0} \left(\frac{1}{1-ip\Omega\tau}\right)\sum_m \omega_{n+m}\ \beta^2 + \mathcal{E}_T = 2\,\mathcal{E}_T \sum_{p=0}^{2m_0}\left(\frac{1}{1+p^2\Omega^2\tau^2}\right)$$

The last term of Eq.(1.82) is roughly equal to $\frac{1}{N}$ the first term since $\dfrac{\omega_{n+p}\ \beta^2}{\sum_m \omega_{n+m}\ \beta^2} = \dfrac{\mathcal{E}_{n+p}}{\mathcal{E}_T} \sim \dfrac{1}{N}$ where N is the number of oscillating modes. Since N is very large, the last term of Eq.(1.82) can be neglected and we write

$$\frac{d\beta(t)}{dt} + in\Phi'(t)\beta(t) = - \frac{3i\ \varepsilon_2\ \omega_n}{\varepsilon_0^2\ V} \frac{\ell}{L} \mathcal{E}_T\ \beta \sum_{p=0}^{2m_0} \left(\frac{1}{1+p^2\Omega^2\tau^2}\right) \tag{1.83}$$

The right-hand side of Eq.(1.83) is purely imaginary; therefore

$$\frac{d\beta(t)}{dt} = 0 \qquad \text{and} \qquad \beta(t) \equiv \beta = \text{constant.} \tag{1.84}$$

The amplitudes of the cavity modes are constant in time and the solution for $\Phi(t)$ is

$$\Phi(t) = - \frac{3\ \varepsilon_2\ \Omega}{\varepsilon_0^2\ V} \frac{\ell}{L} \mathcal{E}_T \sum_{p=0}^{2m_0} \left(\frac{1}{1+p^2\Omega^2\tau^2}\right) t \tag{1.85}$$

Since $\Phi(t)$ is independent of the mode index n, the expression $D_n^*(t) = \beta(t)\, e^{in\Phi(t)}$ is indeed a solution of the differential equation (1.80). We write $D_n^*(t)$ explicitly with the help of (1.84) and (1.85)

$$D_n^*(t)\, e^{i\omega_n t} = \beta\, e^{i\omega_n \left(1 - \frac{3\varepsilon_2}{\varepsilon_0^2 V} \frac{\ell}{L} \mathcal{E}_T \sum_{p=0}^{2m_0} \left(\frac{1}{1+p^2\Omega^2\tau^2}\right)\right) t} \tag{1.86}$$

We notice that the resonant frequency of the nth mode has been slightly pulled from its initial value ω_n by an amount which is proportional to the stored energy $\mathcal{E}_T$. Therefore the presence of an anisotropic molecular liquid inside a laser resonator gives rise to a mode-locked spectrum of equal amplitudes and zero phases. The time envelope $E(t) \propto \left(\sum_n D_n^*(t)\, e^{i\omega_n t} + \text{C.C.}\right)$ where $D_n^*(t)$ is given by (1.86) consists of a train of ultrashort laser pulses of very high intensity. See Figure 1.8. These pulses are separated in time by the double transit

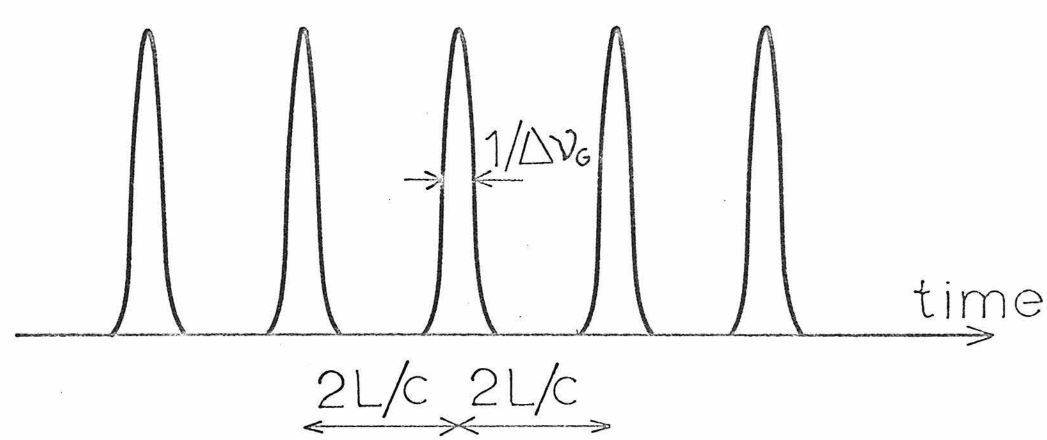

FIG 1-8_ Output of a mode-locked laser : a train of ultrashort pulses

time of the light inside the cavity $\frac{2L}{C}$ and approach a duration $T \sim (\Delta\nu_G)^{-1}$ where $\Delta\nu_G$ is the gain linewidth of the amplifying transition.

For a Ruby laser, $\Delta\nu \sim 10^{11}$ Hz. High intensity pulses of duration $T = 10^{-11}$ sec = 10 p sec should be obtained by placing an anisotropic molecular liquid inside the cavity of a Ruby laser.

For a Neodymium glass laser, $\Delta\nu \sim 2 \times 10^{12}$ Hz ; pulses as short as 0.5 p sec can be obtained from such a laser.

Since high intensity fields in the cavity are necessary to induce a sizeable nonlinearity of the index of refraction of the anisotropic molecular liquid, we expect to generate ultrashort pulses from Q-switched(6) solid state lasers. These lasers in their normal mode of operation (no anisotropic liquid, no saturable absorber dye inserted in the cavity) emit intense light pulses (~ 100 MW) with a duration of $\sim 10^{-8}$ sec.

In order to couple the longitudinal modes of the cavity in the way described above, sufficient energy exchange between the modes must take place during the time the laser is oscillating, i.e. $\sim 10^{-8}$ sec.

In the next section we define and calculate a time constant T_0 for the energy circulation between the modes to find out whether the above condition $T_0 < 10^{-8}$ sec can be satisfied under reasonable experimental conditions.

1-6. Energy circulation time constant.

1-6.1. Definition.

As a measure of the strength of the mode coupling due to the refractive index nonlinearity, we define a circulation time T_0 as the exponential time constant for the circulation of the energy in one mode due to its interactions with all others. If $\mathcal{E}_n$ is the energy of the nth mode,

$$\frac{1}{T_0} = \frac{1}{\mathcal{E}_n} \frac{d\mathcal{E}_n}{dt} \tag{1.87}$$

1-6.2. Calculation of T_0.

The energy of the nth mode is $\mathcal{E}_n = \omega_n D_n^* D_n = \omega_n \beta^2$ where D_n^* and D_n have been replaced by their expressions (1.81). Therefore, according to (1.87)

$$\frac{1}{T_0} = 2 \frac{\beta'}{\beta} \tag{1.88}$$

We have found in Section 1-5 that $\beta' = 0$ and β = constant. This is true because as much energy flows into the nth mode via the higher frequency modes as flows out of it via the lower frequency modes. We want to calculate the rate at which energy is flowing into the nth mode. Therefore, to find β', we shall keep in expression (1.82) only the terms where p is positive in the summation over p.

Only the real part of the right-hand side of (1.82) gives rise to power exchange between the modes. Equating $\beta'(t)$ to the real part of the right-hand side of (1.82) and using the definition (1.87) yields

$$\frac{1}{T_0} = \frac{3\ \varepsilon_2\ \omega_n}{\varepsilon_0^{\ 2}}\ \frac{\ell}{L}\ \frac{\mathcal{E}_T}{V} \sum_{p=1}^{2m_0} \frac{p\Omega\tau}{1+(p\Omega\tau)^2} \tag{1.89}$$

The rate at which energy is exchanged between modes is proportional to the electromagnetic energy stored per unit volume in the cavity $\frac{\mathcal{E}_T}{V}$, to the ratio of the length of the liquid cell to the length of the cavity $\frac{\ell}{L}$, to the optical Kerr constant ε_2 and to the term $\sum_{p=1}^{2m_0} \frac{p\Omega\tau}{1+(p\Omega\tau)^2}$ accounting for the number of modes interacting with any one mode. To give an estimation of this last quantity, we notice that we can write

$$\sum_{p=1}^{2m_0} \frac{p/m_0}{1+(p/m_0)^2} = \sum_{p=1}^{2m_0} f(\frac{p}{m_0}) \quad \text{with} \quad m_0 = \frac{1}{\Omega\tau}$$

The function $f(p/m_0)$ is depicted in Figure 1.6. In this figure, the sum $\sum_{p=1}^{2m_0} f(\frac{p}{m_0})$ is pictured by the area under the rectangles which have a height equal to $\frac{p/m_0}{1+(p/m_0)^2}$ and a base equal to 1. We see that we can approximate the area under the rectangles by the area under the curve, and therefore approximate the summation $\sum_{p=1}^{2m_0} \frac{p/m_0}{1+(p/m_0)^2}$ by the integral $\int_0^{2m_0} \frac{x/m_0}{1+(x/m_0)^2}\,dx$. We make the change of variable $y = x/m_0$ and the integral is equal to

$$m_0 \int_0^2 \frac{y\ dy}{1+y^2} = \frac{m_0}{2} \log(1+2^2) \sim \frac{m_0}{2} = \frac{1}{2\Omega\tau} \tag{1.90}$$

According to (1.89) and (1.90), we write

$$\frac{1}{T_0} = \frac{3}{2} \frac{\varepsilon_2 \, \omega_n}{\varepsilon_0^2} \frac{\ell}{L} \frac{\mathcal{E}_T}{V} \frac{1}{\Omega\tau} \tag{1.91}$$

We can now calculate a numerical value for T_0 in a typical experimental situation. However, in the literature, the measured values of the optical Kerr constant are given in electrostatic units. Since our analysis has been performed in MKS units, we must now relate ε_2 to the measured values in esu units.

For this purpose, we express the displacement vector and the change in the dielectric constant in both systems of units.

In electrostatic units, the displacement vector is expressed as $D_e = E_e + 4\pi P_e$ where E_e and P_e are the electric field and the polarization in electrostatic units. And in the liquid

$$P_e = X_L E_e + X_{NL} E_e^3$$

where X_L and X_{NL} are the linear and nonlinear susceptibilities. Therefore

$$D_e = E + 4\pi X_L E + 4\pi X_{NL} E^3$$

or

$$D_e = E\left((1+4\pi X_L) + 4\pi X_{NL} E^2\right)$$

$$D_e = (\varepsilon + \Delta\varepsilon) E_e$$

ε is the dielectric constant of the liquid in esu units. $\Delta\varepsilon$ is the change in dielectric constant due to the nonlinearities of the medium in esu units.

We now relate the change in dielectric constant $\Delta\varepsilon$ to the Kerr constant B_0 measured by M. Paillette[16]. This constant is measured by the following method.

The Kerr effect induced by a linearly polarized Q-switch ruby laser changes the index of refraction of a liquid for an Argon laser linearly polarized at 45° from the direction of polarization of the Ruby laser, therefore introducing a birefringence and a phase difference Φ between two perpendicular components of the Argon laser electric field, one perpendicular to the Ruby laser electric field, the other one perpendicular to it.

By this method, M. Paillette is able to measure a constant which is proportional to the difference between the index of refraction of the liquid parallel to the Ruby laser electric field $n_{\parallel}$ and perpendicular to it $n_{\perp}$. This change $n_{\parallel} - n_{\perp}$ is related to the change of the index of refraction Δn induced by a linearly polarized electric field upon itself by $2\Delta n = n_{\parallel} - n_{\perp}$.

According to M. Paillette, $\frac{\pi}{\lambda_A}(n_{\parallel} - n_{\perp}) = B_0 E_e^2$ where λ_A is the wavelength of an Argon laser in vacuum = 4880 Å. E_e is the electric field expressed in esu units, B_0 is the optical Kerr constant in esu units. Therefore

$$2\Delta n = \frac{B_0 E_e^2 \lambda_A}{\pi} \tag{1.92}$$

Since $\varepsilon = n^2$, the change of the dielectric constant $\Delta\varepsilon$ is related to the change of the index of refraction Δn by

$$\Delta\varepsilon = 2n\ \Delta n = \frac{n\ B_0\ E_e^{\ 2}\ \lambda_A}{\pi} \tag{1.93}$$

n is the index of refraction of the liquid. We have expressed the change of the dielectric constant in esu units. We now find an expression for the same quantity in MKS units. In MKS units the displacement vector is expressed as $D_M = \varepsilon_0 E_M + P_M$ where E_M and P_M are the electric field and the polarization in MKS units. In the nonlinear liquid, we have $P_M = X_L E_M + \varepsilon_2 E_M^3$, where X_L is the linear susceptibility and ε_2 is the Kerr constant in MKS units. Therefore,

$$D_M = \varepsilon_0\ E_M + X_L\ E_M + \varepsilon_2\ E_M^{\ 3}$$

or

$$D_M = \varepsilon_0\ E_M \left(\left(1 + \frac{X_L}{\varepsilon_0}\right) + \frac{\varepsilon_2}{\varepsilon_0}\ E_M^{\ 2} \right)$$

We can then write D_M in the following form

$$D_M = \varepsilon_0 (\varepsilon + \Delta\varepsilon)\ E_M$$

where ε is the dielectric constant of the medium in esu units (in MKS units, the dielectric constant is $\varepsilon_0\varepsilon$) and

$$\Delta\varepsilon = \frac{\varepsilon_2}{\varepsilon_0}\ E_M^{\ 2} \tag{1.94}$$

is the change in the dielectric constant in esu units when ε_2 and E_M are expressed in MKS units.

Therefore, from (1.93) and (1.94), we find a relationship between B_0 and ε_2.

$$\varepsilon_2 = n \frac{B_0 \lambda_A}{\pi} \varepsilon_0 \frac{E_e^2}{E_M^2} \tag{1.95}$$

The electric field in esu units E_e is related to the electric field in MKS units E_M: $E_e = \sqrt{4\pi\varepsilon_0}\, E_M$. Therefore

$$\varepsilon_2 = 4 \varepsilon_0^2 n B_0 \lambda_A \tag{1.96}$$

where ε_2 is expressed in MKS units and B_0 in esu units and $\varepsilon_0 = \frac{10^7}{4\pi C^2}$. We can now express the circulation time T_0 in terms of quantities in esu units. From (1.91) and (1.96),

$$\frac{1}{T_0} = \frac{3}{2} \frac{4\varepsilon_0^2 n B_0 \lambda_A}{\varepsilon_0^2} \frac{2\pi C}{\lambda_R} \frac{\mathcal{E}_T}{V} \frac{1}{\Omega\tau}$$

or

$$\frac{1}{T_0} = 12\pi C n \frac{\lambda_A}{\lambda_R} B_0 \frac{\ell}{L} \frac{\mathcal{E}_T}{V} \frac{1}{\Omega\tau} \tag{1.97}$$

where we have replaced ω_n by $2\pi \frac{C}{\lambda_R}$, λ_R is the ruby wavelength $= 6943$ Å. Numerical application: We calculate the circulation time T_0 for the following experimental situation. A 5 cm ($2\ell = 5$ cm) cell containing nitrobenzene is inserted into a laser cavity 1 meter long ($L = 100$ cm); the cross section of the beam is 1 cm^2; therefore,

$V = 100\ cm^3$ $B_0 = 2.9 \times 10^{-7}$ esu. The other parameters are $\mathcal{E}_T = 0.1$ Joule $= 10^6$ ergs and $\tau = 5 \times 10^{-11}$ sec. We find $T_0 \sim 10^{-9}$ sec.

For this experimental situation, the circulation time T_0 is of the order of 1 nanosecond and therefore sufficient energy exchange between cavity modes is expected to take place to produce effective mode coupling.

In our theoretical analysis, we have considered the effect of an anisotropic molecular liquid placed inside a laser resonator on the mode structure of the cavity. We have found that under reasonable experimental conditions, the nonlinear polarization induced in the liquid produces strong coupling between the cavity modes. A quasi equilibrium situation might be reached where all the modes have equal amplitudes and zero phase difference, thus giving rise to the production of ultrashort and intense pulses of light of duration equal to the inverse of the gain linewidth $\Delta\nu_G$ of the amplifying transition (in Ruby $\Delta\nu_G \sim 10^{11}$ Hz) and separated by the transit time of the light in the cavity $\frac{2L}{C}$.

An experiment has been performed to verify these predictions which is reported in the following pages.

II. EXPERIMENTAL INVESTIGATION

In Chapter II we report the experimental results obtained when a liquid cell filled with an anisotropic molecular liquid is placed inside the cavity of a Q-switched Ruby laser.

An evaluation of the important parameters of the experiment help us in selecting an anisotropic molecular liquid which induces strong mode coupling.

The experimental apparatus is described, the experimental techniques and results are then presented.

2-1. Parameters of the experiment.

In Part I we have shown by a theoretical argument that the presence of an anisotropic molecular liquid inside a laser resonator induces a mode-locked operation; the output of the laser then consists of a train of ultrashort pulses of light.

In order to perform an experiment and verify the above predictions, we consider the various parameters of the problem. The important parameters of the liquid system are:

- The Kerr constant ε_2
- The orientational relaxation time τ
- The length of the cell 2ℓ

The parameters for the laser system are:

- The length of the cavity L and then the frequency difference between adjacent modes $\Omega = \frac{\pi C}{L}$.
- The gain linewidth of the amplifying laser transition $\Delta\nu_G$ and therefore the number of oscillating modes $N = \frac{\Delta\nu_G}{C/2L}$.

A careful examination of the parameters of the liquid is needed to choose the liquid which induces a strong mode coupling. The parameters of the laser system are fixed by the equipment available in our laboratory. The experimental work is performed with a ruby laser Q-switched by a rotating prism. The wavelength is $\lambda = 6943$ Å; the gain linewidth measured with a Fabry-Perot etalan is $\Delta\nu_G = 1.8\ cm^{-1}$.

The laser cavity is 1 meter long, then $\Omega \sim 10^9$ rd/sec., and $N \sim \frac{1.8}{\frac{1}{200}} = 360$ modes are oscillating.

2-2. Selection of the anisotropic molecular liquid.

In this section we determine which specific liquid can be used to produce strong coupling between the modes of the laser resonator by examining the important parameters of the liquid with respect to the parameters of the laser systems described above.

The criterion we use to evaluate the parameters of the liquid is the following: We require the circulation time T_0 defined in Section 1-6 to be as short as possible so that sufficient energy exchange between the modes takes place during a Q-switch pulse to produce mode coupling.

According to 1.97, we shall use materials with a large Kerr constant B_0, i.e. molecules with a large anisotropy.

Another important parameter is the orientational relaxation time τ or rather the quantity $\Omega\tau$. According to (1.97), $\Omega\tau$ should be as small as possible, but we shall see that there is a lower limit to the possible values of $\Omega\tau$ by examining the physical significance of this parameter.

The rate at which energy is exchanged between the nth mode and the (n+p)th mode, for example, is proportional to the quantity $\frac{p\Omega\tau}{1+(p\Omega\tau)^2}$

which is equal to the imaginary part of the susceptibility of an anisotropic molecule in an electric field with radian frequency $p\Omega$. The maximum of this quantity is equal to $\frac{1}{2}$ and occurs at $p = 1/\Omega\tau$. See Figure 1.6.

If $\Omega\tau \gg 1$, the molecular orientation does not respond to optical envelope variations at frequency Ω or higher and the amount of refractive index nonlinearity is too small to couple the modes together.

If $\Omega\tau = 1$, the nth mode exchanges energy principally with adjacent modes. The rate at which it receives energy from higher frequency modes is maximum for the (n+1)th mode and decreases rapidly for the (n+2), (n+3) modes and so on. In order to couple more modes faster, the relaxation time τ of the molecule has to be made shorter, so that the molecules respond to more frequency components. But there is a limit as to how short $\Omega\tau$ should be. If $\Omega\tau \ll \frac{1}{N}$, where N is the total number of oscillating modes, the nth mode is coupled very weakly with all the other modes. In that case most of the energy is coupled outside the gain linewidth and lost without giving rise to any appreciable coupling between the oscillating modes. In Figure 2.1, the function $f(p\Omega\tau) = \frac{p\Omega\tau}{1+p^2\Omega^2\tau^2}$, which is a measure of the mode coupling strength, is presented for $\Omega\tau = 1$, $\frac{1}{N} < \Omega\tau < 1$ and $\Omega\tau < \frac{1}{N}$. To produce strong mode coupling, the relaxation time must be chosen so that

$$\frac{1}{N} < \Omega\tau < 1 \tag{2.1}$$

or according to the laser parameters.

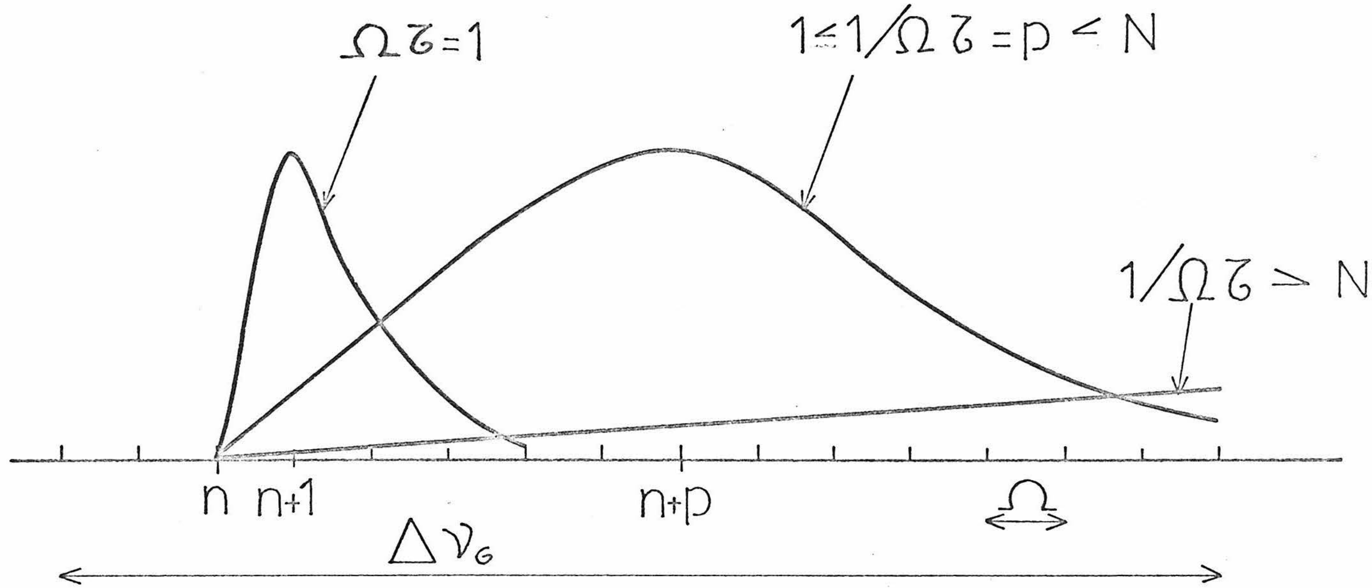

FIG 2-1 _ The strength of the mode coupling as represented by the function $p\Omega\tau/(1+(p\Omega\tau)^2)$ is a function of $\Omega\tau$

$$3 \times 10^{-12} \text{ sec} < \tau < 10^{-9} \text{ sec.}$$

The three criteria for choosing the nonlinear liquid are:

- Large Kerr constant B_0

- Shortest relaxation time τ as possible but no shorter than 3×10^{-12} sec

- The liquid must be transparent at the ruby wavelength.

In our experimental work, we used (a) Nitrobenzene and (b) α-chloronaphthalene as the anisotropic molecular liquids. In case (a), the Kerr constant is $B_0 = 2.9 \times 10^{-7}$ esu. At room temperature, the relaxation time is $\tau \sim 50 \times 10^{-12}$ sec. In case (b), $B_0 = 2.7 \times 10^{-7}$ esu and the room temperature relaxation time is $\tau = 53 \times 10^{-12}$ sec.

The relaxation time τ is given by:[12]

$$\tau = \frac{\eta V}{kT}$$

where η is the viscosity of the liquid

V is the volume of one molecule

T is the temperature of the liquid.

The viscosity η is a decreasing function of temperature[17]. It varies empirically as $A\, e^{\frac{B}{T}}$ where A and B are two empirical constants characteristic of the liquid. Therefore, the relaxation time τ is a decreasing function of the temperature. This dependence has been experimentally verified for nitrobenzene by Rank et al[18]. They measured the frequency shift of the stimulated Rayleigh line as a function of temperature from $T = 12^{o}C$ to $T = 117^{o}C$. Their results are plotted in Figure 2.2 and extrapolated to higher temperatures.

Therefore, by heating the liquid placed inside the laser resonator we expect to decrease the relaxation time τ (from its value of 50×10^{-12} sec at room temperature) and produce a faster mode coupling.

Note that carbon disulfide has a larger Kerr constant than nitrobenzene ($B_0 = 4.18 \times 10^{-7}$ esu)(16) but at room temperature its relaxation time is $\sim 10^{-12}$ sec which is not within the limits of the second criterion. Therefore CS_2 should be cooled down to increase its relaxation time; however, since it is less convenient to cool a liquid than to heat it, we did not use it in our experiment.

2-3. Description of the experimental apparatus.

2-3.1. The cell and temperature controller.

A cell has been specially designed and built for this experiment. A 5 cm long Kovar tube was cut at the Brewster angle of the Ruby wavelength; two glass windows were then cemented on the ends with a heat resistant epoxy. Kovar was chosen because its heat expansion coefficient is close to the one of glass and because of good heat conductivity.

A thin heating wire is glued around the Kovar tube. A thermistor (negative temperature resistivity coefficient) is inserted inside the cell. The temperature of the liquid in the cell is measured by measuring the resistance of the thermistor. This is done by placing it into one arm of a bridge of resistors. Two known and fixed resistors are placed in two other arms. In the fourth arm there is a variable

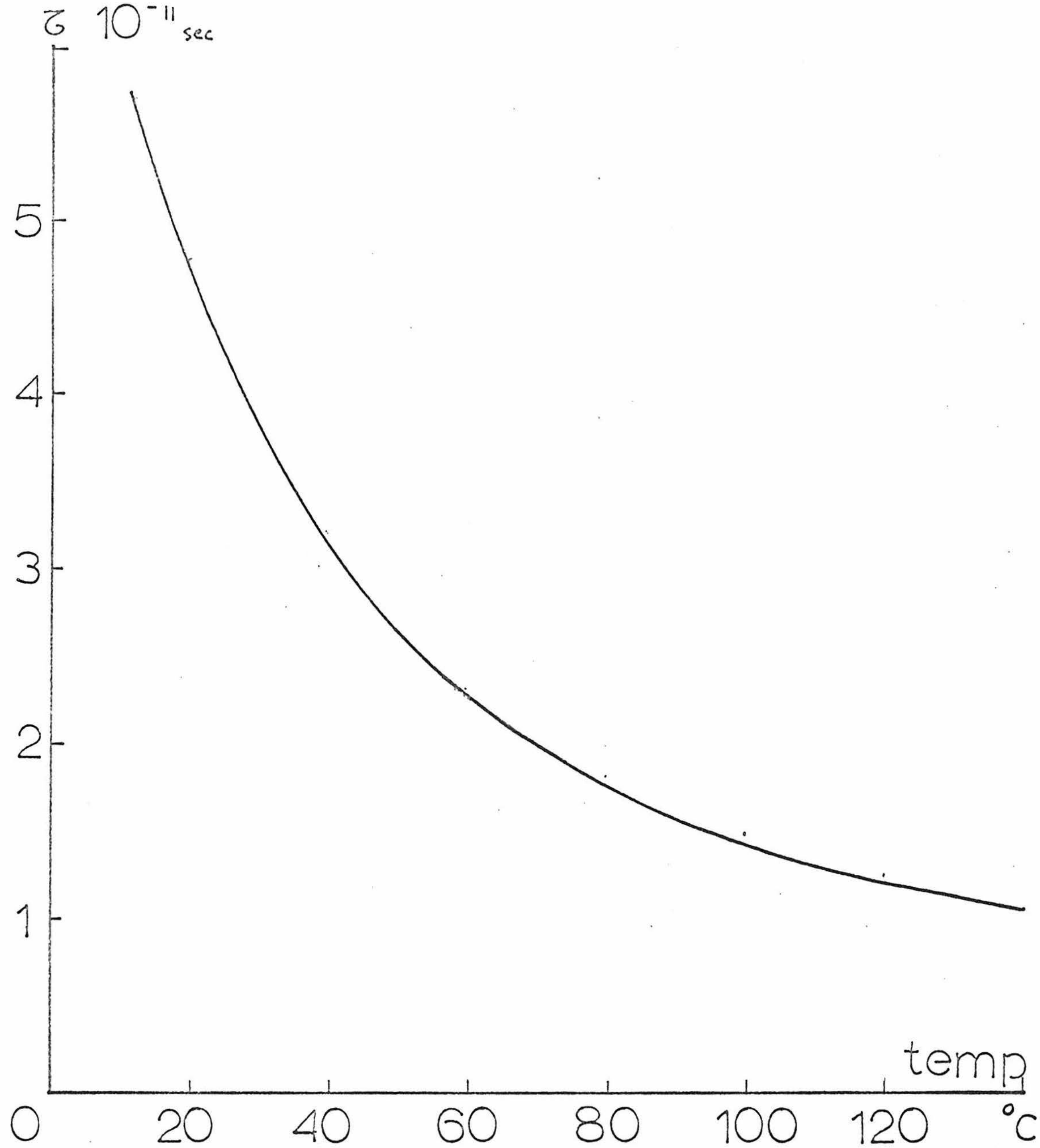

FIG 2-2 The orientational relaxation time τ as a function of temperature in nitrobenzene

resistor whose value determines the equilibrium condition of the bridge for a certain temperature. An on and off relay switch is in one diagonal of the arm and controls the power supply to the heating wire. This apparatus controls the temperature of the liquid to within one percent in the following way. First, the variable resistor is set to a fixed value; then the power supply of the heating wire is turned on. When the liquid is cold, the resistance of the thermistor is high, the bridge is not in equilibrium and current flows in a certain direction through the relay switch which remains closed. When the temperature of the liquid rises slightly above the present temperature, the resistance of the thermistor becomes smaller than the equilibrium resistance of the bridge and current flows in the opposite direction through the relay switch thus opening it and turning off the power supply to the heat resistor. During the experiments, we wait a few minutes so that the oscillations of the temperature around its preset equilibrium conditions are small. The range over which the temperature of the liquid can be controlled depends upon the sensitivity of the thermistor; in our case the range was from $-95^{\circ}c$ to $140^{\circ}c$.

2-3.2. The laser system.

The laser system and the liquid cell are described in Fig. 2.3. A Spacerays laser system with a maximum output of 2.84 Joules Q-switched is used. The Ruby rod 4-5/16" long and 1/2" in diameter is cut at the Brewster angle and placed along the common focus of a double elliptical cavity. It is pumped by two water-cooled linear Xenon flashtubes placed at the two other foci.

The laser is Q-switched by a glass prism, 1/2" x 1/2", rotating at

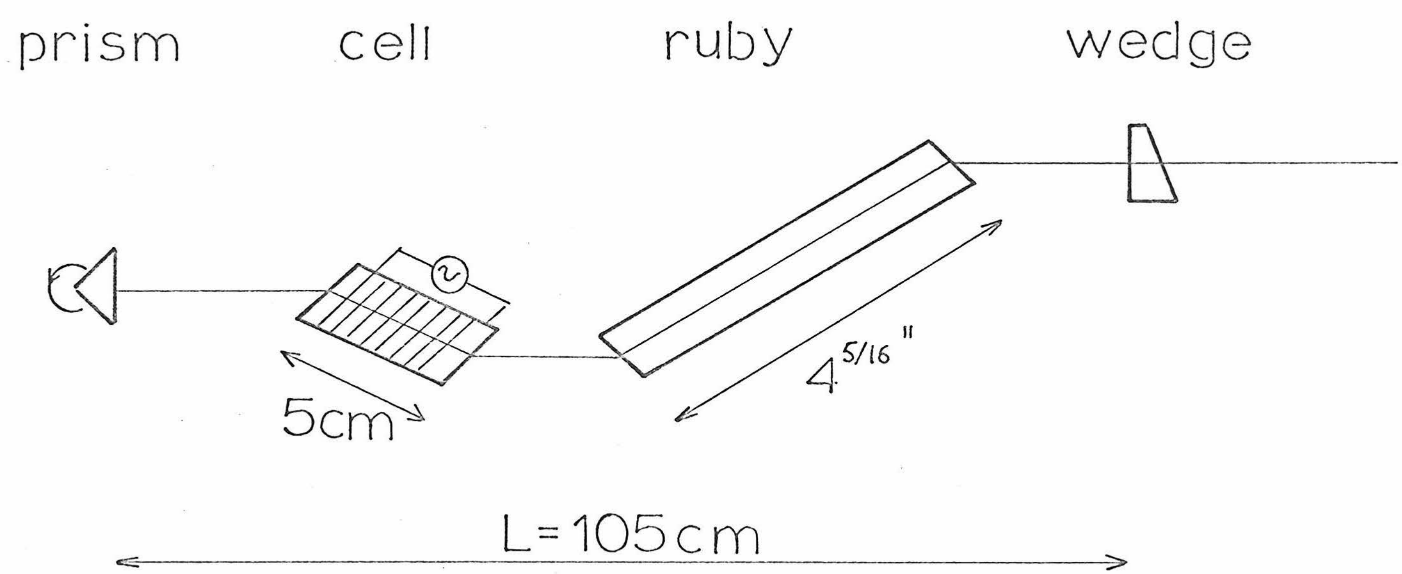

FIG 2-3 _ Description of the cell and the laser system a speed of 20.000 rpm.

The output mirror, a 55% reflecting dielectric mirror, is fixed. The back surface and the reflecting surface are at a wedge angle so that no light reflected from the back surface is coupled back into the cavity, and therefore the only oscillating cavity is formed by the reflecting surface of the mirror and the rotating prism.

The output energy of the laser is 1 Joule on the average in a giant pulse 50 nsec long.

The total cavity length is 1.05 m measured by observing the modulation period of the laser power with a fast pulse detector and a fast oscilloscope.

2-3.3. The measuring apparatus.

The measuring equipment which is used in the experiment is of two kinds.

-- The electric equipment which records the power of the laser + a Fabry-Perot etalon to measure the oscillating linewidth of the laser which can be easily modified to observe the stimulated Raman emission from the anisotropic molecular liquids.

-- An optical, two-photon fluorescence technique which allows us to measure accurately optical pulse lengths as short as 10^{-12} sec.

The measuring equipment is described in Figure 2.4.

2-3.3.1. The electronic measurement equipment.

30% of the laser output is intercepted by a beam splitter and passes through a diverging lens. 30% of the diverging beam is detected on a fast pulse detector (United Aircraft Model 1240) which has a S-1 photo emissive surface and a response time of less than 0.3

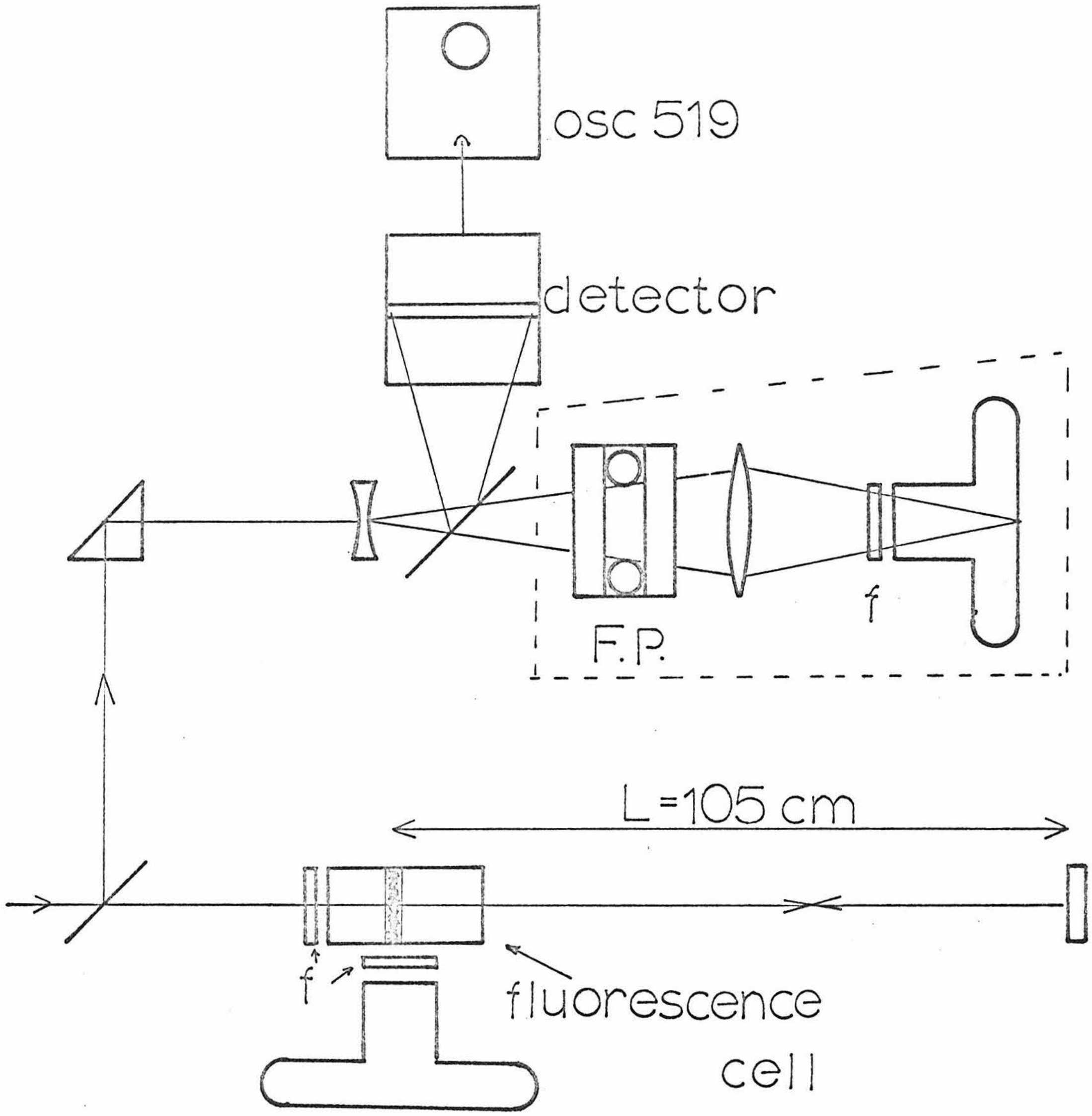

FIG 2-4 _ Description of the measuring apparatus.

f=filter F.P. = Fabry-Perot etalon

nanosecond. The output of the detector is fed into a Tektronix oscilloscope Type 519 with a measured risetime of 0.32 nsec and a vertical sensitivity of 8.9 v/cm.

The total response time of the detector and the oscilloscope is of the order of 0.6 nsec.

In order to measure the oscillating linewidth of the laser transition and to compare it with the duration of the ultrashort pulses, a Fabry-Perot etalon is set up behind the diverging lens. The concentric rings formed by the etalon are then focussed on the plate of a polaroid camera by a converging lens. To eliminate the background light, a ruby narrow band filter is placed in front of the camera.

The Fabry-Perot has been made by evaporating a silver layer on two flat glass surfaces. The separation between the two reflecting surfaces can be varied easily from 1/16" = 0.16 cm to 1/4" = 0.62 cm. The finesse of the etalon is equal to 5.

2-3.3.2. Modification of the apparatus for the observation of Stimulated Raman scattering.

The intense laser light incident upon the liquid induces transitions between the molecular vibration modes. A certain amount of light is emitted by a stimulated process from the liquid at a frequency equal to the frequency of the incident light ω_0 minus the vibrational frequency of the liquid ω_V; it is called the stimulated Raman scattered radiation.

The liquids we use in our experiment are known to exhibit this effect under intense optical electric fields. When stimulated Raman scattering takes place, a certain amount of energy is coupled out of the gain linewidth of the laser transition since the Raman frequency shift $\omega_V \sim 1300\ cm^{-1}$ is much larger than the gain linewidth $\Delta\nu_G \sim 2\ cm^{-1}$. This process represents a loss mechanism as far as the process of mode coupling within the gain linewidth is concerned.

It is therefore interesting to make some observations of the Stimulated Raman light. For this purpose, the electronic apparatus described in Section 2-3.3.1 is modified in the following way. See Figure 2.5.

The Fabry-Perot etalon is removed, and a second fast pulse photodetector (Applied Lasers Model FP 125) is set up. It has an S-1 photocathode and a risetime of 0.5 nsec. A 7.69 Corning filter is placed in front of the detector to absorb the laser light which is not transformed into Raman light. This filter has a transmission of less than 0.1% at the Ruby wavelength and a transmission of 70% at the Stokes wavelength of nitrobenzene (~ 7700 Å). A narrowband ruby filter with a bandwidth of 200 Å is placed in front of the United Aircraft (UA) detector to eliminate all the Raman light. Calibrated neutral density filters are also placed in front of the UA detector to attenuate the laser light. With all the filters, the UA detector (recording the laser light power) is 500 times less sensitive than the Applied Laser (AL) detector recording the Raman light. The output of both detectors is fed into the Tektronix 519 oscilloscope. The AL detector output is delayed by 100 nsec with respect to the output of the UA detector. The Raman and the laser intensities are thus recorded simultaneously on the same oscilloscope trace.

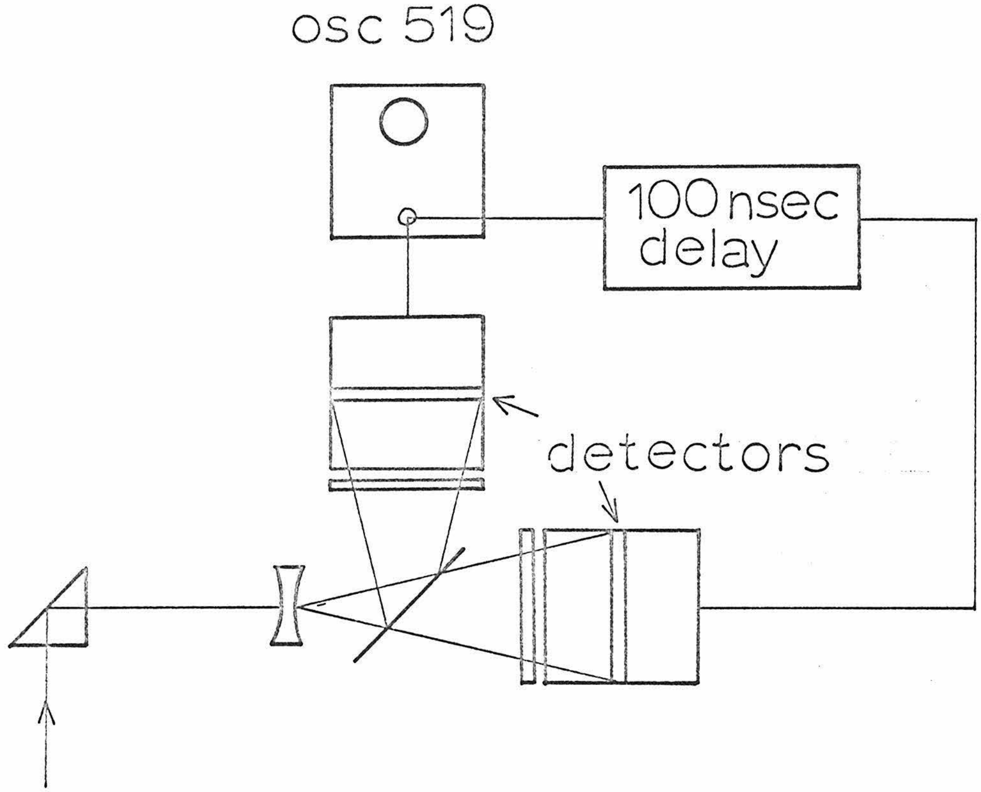

FIG 2-5 _ Electronic apparatus for observing stimulated Raman Scattering

2-3.3.3. The two-photon fluorescence technique.

In this experiment we want to detect and measure intense pulses of light which last no longer than 10^{-11} sec. No electronic equipment to date has a sufficiently fast response time to record these ultra-short pulses. An optical method has been devised recently[19][20] which gives a simple and accurate measurement of pulses as short as a few 10^{-12} sec. It is called the two-photon fluorescence technique.

This technique is based on the two-photon absorption and fluorescence in an organic dye. In our experiment, we use a nearly saturated solution of Rhodamine 6G in isopropyl alcohol. This dye has an absorption peak at a wavelength of 3500 Å close to the frequency of the second harmonic of the Ruby light (3470 Å).

The fluorescent intensity I_F is proportional to the square of the intensity of the laser light I_L since we assume a two-photon process $I_F = \alpha\, I_L^2$ (2.2) where α is the coefficient of proportionality.

The experimental setup used to record the ultrashort pulses is the following: A 5-cm cell containing Rhodamine 6G is placed on the axis of the laser cavity behind the beam splitter so that 70% of the laser intensity is incident upon it. See Figure 2.4.

A mirror (99% reflecting at 6943 Å) is placed behind the cell so that the laser beam passing through the cell is reflected back onto itself. The laser intensity $I_L(x,t)$ at some time t and some point in the cell at a distance x from the mirror is

$$I_L(x,T) = \frac{1}{T} \int_0^T \left(E(x,t) - E\left(x, t + \frac{2nx}{c}\right)\right)^2 dt \qquad (2.3)$$

$E(x,t)$ is the electric field of the optical traveling wave at its first passage and $E(x, t + \frac{2nx}{c})$ is the electric field of the reflected wave. T is a time equal to a few optical cycles. n is the index of refraction of the organic dye at the ruby wavelength. We assume that $E(x,t)$ has the following form

$$E(x,t) = V(t) \cos(\omega_0 t - kx) \tag{2.4}$$

ω_0 is the optical radian frequency, k is the wave number = $\frac{\omega_0 n}{c}$ and $V(t)$ is the time envelope of the wave which is a slowly varying function compared to $\cos \omega_0 t$. $V(t)$ is a real function.

We use expression (2.4) and perform the integration in (2.3), then

$$I_L(x) = \frac{1}{2} (V^2(t) + V^2(t+\tau)) - V(t)V(t+\tau) \cos(\omega_0 \tau + 2kx) \tag{2.5}$$

where $\quad \tau = \frac{2nx}{c}$.

The fluorescent intensity recorded on the film plate of a camera focused on the center of the cell is

$$I_R(x) = \beta \int_0^{T_0} I_F(x,t)\, dt \tag{2.6}$$

where T_0 is the duration of the laser burst and β is a constant coefficient. From relationships (2.2) and (2.5), we find

$$I_R(x) = \frac{\alpha\beta}{4}\left[\int_0^{T_0} V^4(t)\, dt + \int_0^{T_0} V^4(t+\tau)\, dt + \right.$$

$$- 4\cos(4kx) \int_0^{T_0} (V^3(t)\, V(t+\tau) + V(t)\, V^3(t+\tau))\, dt$$

$$\left. + 2\int_0^{T_0} V^2(t)\, V^2(t+\tau) dt + 4\cos^2(4kx) \int_0^{T_0} V^2(t)\, V^2(t+\tau)\, dt\right] \quad (2.7)$$

$I_R(x)$ as given by (2.7) would the recorded fluorescent intensity if the film had a spatial resolution smaller than a wavelength. However, usual films have a resolution of the order of 100μ. The film is not able to record optical spatial fluctuations (cos(4kx)). A spatial average is then performed over a few optical wavelengths. Then

$$I_R(x) = \frac{\alpha\beta}{4}\left[\int_0^{T_0} V^4(t) dt + \int_0^{T_0} V^4(t+\tau)\, dt + 4\int_0^{T_0} V^2(t)\, V^2(t+\tau)\, dt\right] \quad (2.8)$$

The first two terms in expression (2.8) are independent of position and represent the fluorescent tracks due to the incident and the reflected laser beams.

If the output of the laser consists of a train of ultrashort pulses of width $\frac{1}{\Delta\nu_G}$ separated by $\tau_0 = \frac{2L}{c}$ where L is the length of the laser cavity, then the last term will be different from zero only if $\tau = m\tau_0$ where m is an integer; i.e. the fluorescence will be brighter at the points of the cell where an incident pulse crosses a reflected pulse. The third term represents an enhancement of the fluorescence at the mirror due to the reflection of one pulse upon itself if $m = 0$, or at a distance L from the mirror due to the crossing of one pulse with the following one when $m = 1$. The width of these bright spots is equal to the duration of each individual pulse multiplied by the velocity of the light in the liquid. An ultra-short pulse of 10^{-11} sec with a Lorentzian shape gives a bright spot of 4.5 mm. The contrast ratio CR between the peak fluorescence intensity and the background intensity is defined as follows. According to (2.8)

$$CR = \frac{(I_R)_{peak}}{(I_R)_{back}} = \frac{\int_0^{T_0} V^4(t)\,dt + 2\int_0^{T_0} V^2(t)\,V^2(t+\tau)\,dt}{\int_0^{T_0} V^4(t)\,dt} \tag{2.9}$$

For perfectly coupled modes, $CR = 3$. For purely random phases, $CR = 1.5$.[21]

However, we wish to point out that a contrast ratio of 3 has never been experimentally observed for a ruby laser mode-locked by a saturable absorber which is known to produce ultrashort pulses. The reason for this discrepancy is not known. It may be due to deviation of the two-photon fluorescence intensity from an exact square law dependence on I_L.

The camera looks at the fluorescent track perpendicularly to it; therefore it records an average intensity over the spatial cross-section of the beam and the randomness of the spatial output from pulse to pulse may decrease the contrast ratio. In our experiment we shall compare the contrast ratios observed with the contrast ratios we obtained by mode locking the ruby laser with a saturable absorber (a solution of cryptocianine in methanol).

Our apparatus is set up so that we look for the enhancement of the fluorescence of the organic dye due to one ultrashort pulse with the next one. For this purpose the 99% reflecting mirror is placed at a distance from the center of the fluorescence cell equal to the optical length of the laser cavity, i.e. 1.05 m (this distance is physically shorter due to the index of refraction of the organic dye).

2-4. Presentation of the results.

In part I of this report, we have predicted theoretically the production of ultrashort laser pulses when an anisotropic molecular liquid is placed inside the laser resonator.

In Section 2-1, 2-2 and 2-3, we have described the experimental apparatus which allows us to detect the presence of these short light pulses.

In this section we present the results of our experimental investigation. The laser was first fired with no nonlinear refractive index medium inside the cavity. Figure 2.6 shows a typical picture of two-photon fluorescence dye cell. No ultrashort pulse was present in that case.

2-4.1. Observation of ultrashort pulses.

2-4.1.1. Nitrobenzene.

The liquid cell was filled with nitrobenzene. The temperature was varied from 6°C (its melting point) to 144°C (its boiling point is 211°

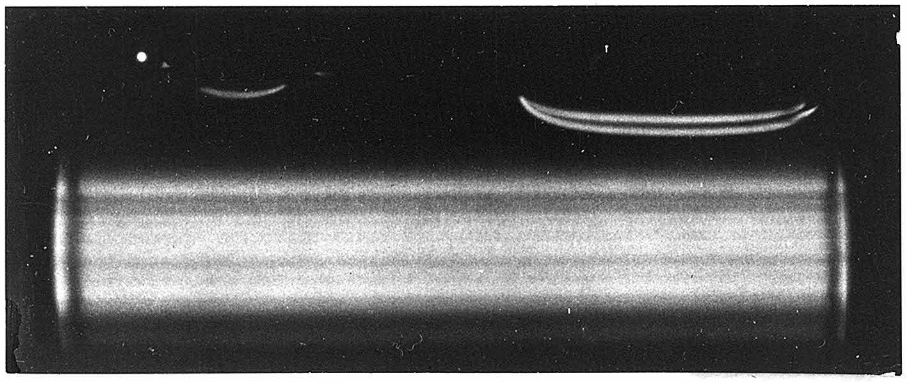

FIG 2-6 Two-photon fluorescence dye cell. No non linear medium in the cavity

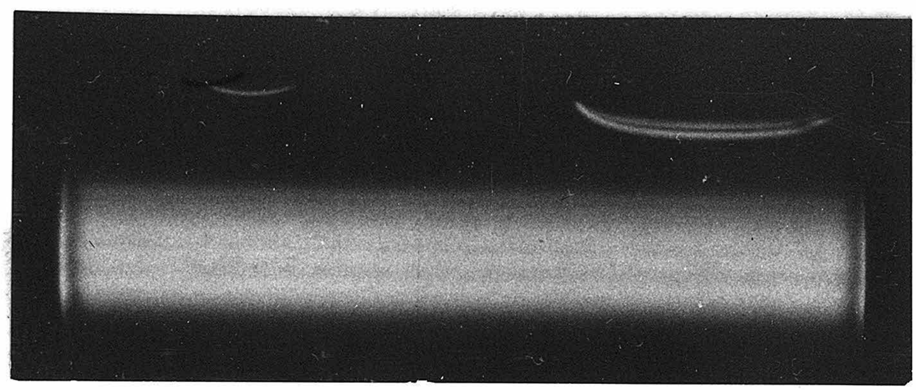

FIG 2-7 Two-photon fluorescence dye cell. Nitrobenzene at 25°C

C). No ultrashort short pulses were observed at room temperature and below. As the temperature was increased, a tendency for pulses to appear on the fluorescence track of the dye cell was noticed.

At 110°C, 126°C and 144°C, a brighter fluorescent spot appeared at the center of the dye cell. When the mirror behind the fluorescence cell was displaced by a certain amount, the bright fluorescent spot was displaced by the same amount divided by the index of refraction of the dye indicating that the bright fluorescent spot is due to the overlapping of one incident short pulse with the previous one reflected back by the mirror.

In Figure 2-7 we show the fluorescent dye cell when a 5 cm cell of Nitrobenzene at 25°C is inserted in the laser resonator. There is no indication of ultrashort pulses.

Figure 2.8 shows two consecutive pictures of the dye cell when the nitrobenzene has been heated to 126°C. At that temperature there is clear evidence of ultrashort pulses and therefore mode coupling. The contrast ratio estimated with a calibrated film is close to 2. It is not equal to 3, the theoretically predicted value for complete mode locking but it is the same as the one observed when a saturable absorber is used inside the cavity to generate ultrashort pulses.

At this temperature, the orientational relaxation time τ is estimated to be $\sim 10^{-11}$ sec from Figure 2.2; and therefore $\Omega\,\tau \sim 10^{-2}$.

The gain linewidth of the laser $\Delta\nu_G$ is measured with the Fabry-Perot etalon to be 1.8 cm^{-1}. The expected pulse width is $\frac{1}{\Delta\nu_G} \sim 1.8 \times 10^{-11}$ sec. The measured pulse width is $\Delta t = \frac{d}{G} \times \frac{n}{c}$ where d is the length of the fluorescent spot.

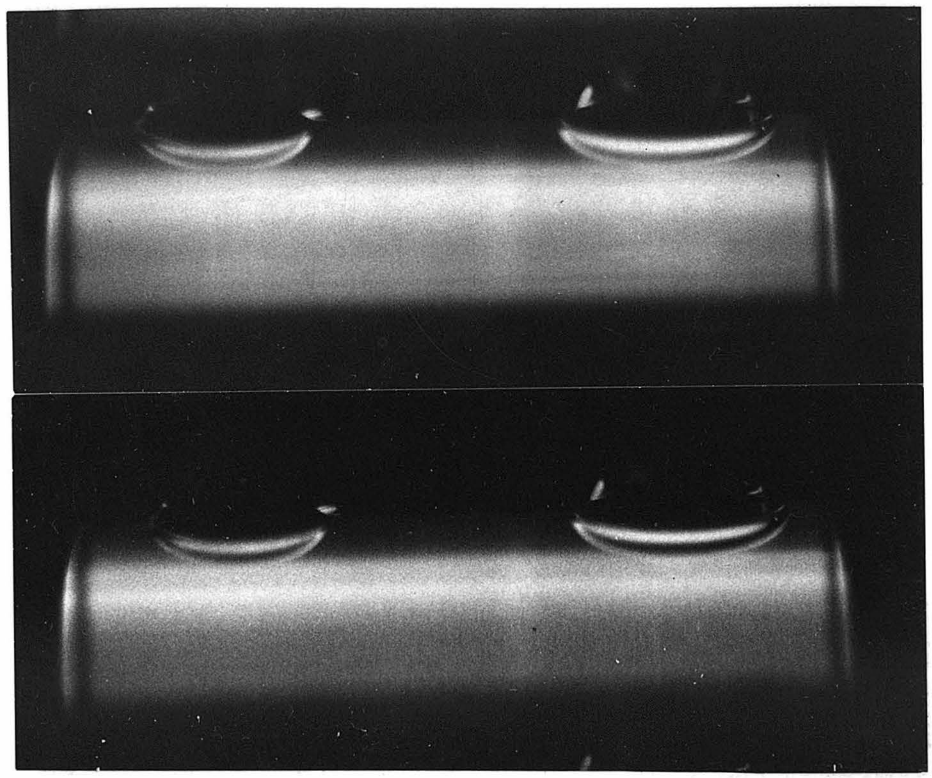

FIG 2-8 Two-photon fluorescence dye cell. Nitrobenzene at 126 °C.

n is the index of refraction of the dye solution ($n = 1.5$) and G is the magnification of the camera. The experimental value of Δt is 1.2×10^{-11} sec which, when compared to the expected value of 1.8×10^{-11} sec, indicates mode-locking across the full gain linewidth.

The amount of mode-locking is a sensitive function of the temperature of the anisotropic molecular liquid, i.e. of its orientational relaxation time. For nitrobenzene the pulsewidth (1.2×10^{-11} sec) is comparable to the relaxation time $\tau \sim 10^{-11}$ sec.

A noticeable variation of the pulsewidth as a function of temperature was not observed since the relaxation time of nitrobenzene varies slowly for temperatures over 100°C. See Figure 2.2.

2-4.1.2. α-Chloronaphthalene.

In this phase of the experiment the liquid cell within the optical resonator was filled with α-chloronaphthalene. This liquid has a Kerr constant $B_0 = 2.73 \times 10^{-7}$ esu[15] and a relaxation time $\tau = 5.3 \times 10^{-11}$ sec[22] at room temperature. These parameters are very close to the ones of nitrobenzene, $B_0 = 2.9 \times 10^{-7}$ esu and $\tau = 4.75 \times 10^{-11}$ sec at room temperature. Although no experimental data on the temperature dependence of the relaxation time is available for this material, τ is expected to decrease faster with temperature than nitrobenzene[21].

Ultrashort pulses were observed using α-chloronaphthalene heated to 62°C. At that temperature, τ is in the range of 2 to 4×10^{-11} sec. In Figure 2.9, two consecutive fluorescent tracks are shown and ultrashort pulses are apparent; the pulse width is estimated to be $\sim 1.2 \times 10^{-11}$ sec.

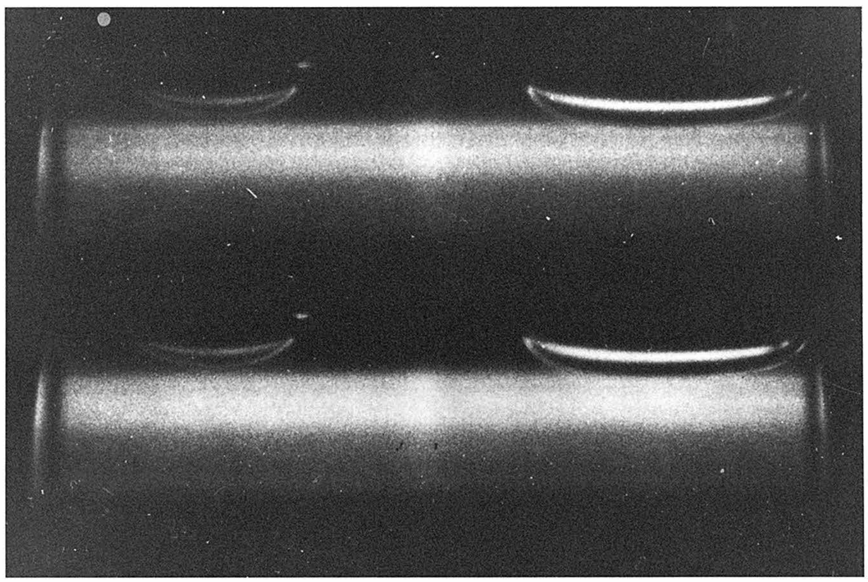

FIG 2-9 Two-photon fluorescence dye cell. α-chloronaphtalene at 62°C.

2-4.1.3. Carbon Disulfide.

CS_2 has a Kerr constant (4.18×10^{-7} esu)(15) larger than nitrobenzene and α-chloronaphthalene but at room temperature its relaxation time is $\sim 10^{-12}$ sec, too short a time to produce efficient mode locking as explained in Section 2-2.

To verify this assertion, CS_2 was placed in the Brewster angle cell inside the cavity. The photographs taken of the fluorescence cell showed a uniform light intensity along the cell indicating the absence of ultrashort pulses. Decreasing the temperature of Carbon Disulfide would increase its relaxation time and its Kerr constant and this should induce strong mode coupling. An attempt to observe this effect failed due to clouding of the CS_2 upon cooling. This can possibly be avoided by using extremely pure CS_2 which is distilled in Situo. This was not deemed practical in our experiment.

2-4.2. Observation of Stimulated Raman Scattering.

Stimulated Raman scattering was observed when the cell containing nitrobenzene was placed inside the optical cavity.

The Raman frequency shift for nitrobenzene is 1345 cm^{-1}. The powers of the Raman scattered light and of the ruby light are recorded simultaneously by the technique described in Section 2-3.3.2. A typical picture of the Tektronix 519 oscilloscope trace is shown in Figure 2.10. The first pulse represents the ruby light intensity, the second pulse is the Raman light intensity and the third pulse is the electronic reflection of the first pulse off the end of the 100 nsec delay cable. The detection of the Raman light is 500 times more sensitive than the detection of the ruby light.

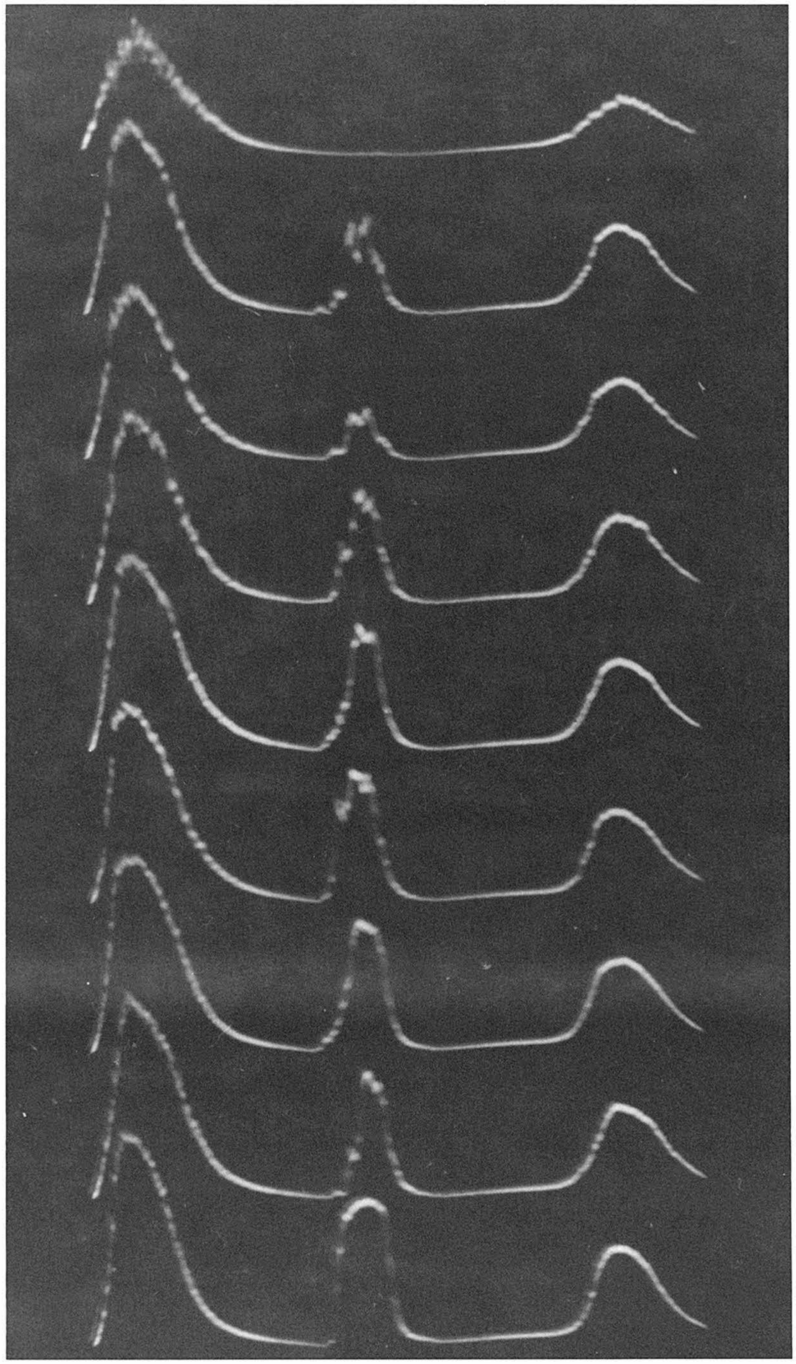

FIG 2-10 Detection of stimulated Raman emission from nitrobenzene at 25°C.

This picture was taken with nitrobenzene at 25°C, i.e. when no ultrashort pulse was present in the laser output. The Raman intensity is an increasing function of the laser intensity. Measurements performed at various liquid temperatures have shown a nearly exponential dependence of the Raman intensity on that of the laser. The gain coefficient is comparable to the theoretical value of 1.4×10^{-3} $cm^{-1}/\mu W/cm^2$ [23]. The Raman gain coefficient[24] is

$$g = \left(\frac{d\sigma}{d\Omega}\right) \frac{c^3}{\nu_S} \frac{1}{\Delta\nu} \left(1 - e^{-\frac{h\nu}{kT}}\right)\left(\frac{n_\ell}{V}\right)$$

where $\left(\frac{d\sigma}{d\Omega}\right)$ is the differential cross section of the Raman process per unit volume, per unit solid angle. ν_S and $\Delta\nu$ are the peak frequency and the width of the Raman scattered light. n_ℓ/V is the laser photon density.

As the temperature of the cell was increased, the Raman gain decreased. This is due to the fact that the Raman width $\Delta\nu$ is an increasing function of the temperature. We observed stimulated Raman emission both at low and high temperatures when ultrashort pulses do and do not occur regularly. The observed Raman intensities are at most 1/500 of the laser intensity.

No stimulated Raman scattering was observed when the cell inside the cavity was filled with α-chloronaphthalene. The frequency shift of the Raman light is 1368 cm^{-1} for α-chloronaphthalene and therefore our apparatus had the same sensitivity for detecting the Raman light as for nitrobenzene.

In liquids with a large Kerr constant (large molecular anisotropy), the observation of stimulated Raman emission is often accompanied by the phenomenon of self-focusing: when a beam with a gaussian intensity profile (large intensity at the center of the beam decreasing radially) travels through a liquid with a nonlinear index of refraction $n = n_0 + n_2 E^2$, the center of the beam will see a larger index of refraction than the outside part of beam and will travel slower through the liquid than the outside parts of the beam.

The liquid then has the effect of a converging lens and focuses the light into filaments. In these filaments, the intensity of the beam is very high and a large fraction of the ruby light is transformed into Raman light by stimulated Raman scattering.

In our experiment, we believe that stimulated Raman emission occurred in the absence of self-focusing for the two following reasons:

(1) No stimulated Raman emission was observed in α-chloronaphtalene which has nearly the same threshold for self-focusing and relaxation time as nitrobenzene.

(2) A nearly exponential dependence of the Raman intensity on that of the laser with a gain coefficient comparable to the theoretical value was measured.

The observation of the stimulated Raman emission has shown that only a small portion of the laser intensity is transformed into Raman light, and therefore only a small amount of energy is coupled outside the gain linewidth of the laser by this process. This energy loss is not sufficient to prevent the mode coupling and the production of ultrashort pulses by a refractive index nonlinearity as observed above.

DISCUSSION AND CONCLUSION

In part I of this report we have studied theoretically the behaviour of the longitudinal modes of a laser resonator when a cell containing an anisotropic molecular liquid was placed inside it. We have found that under certain conditions (large molecular anisotropy and an orientational relaxation time τ such that 3×10^{-12} sec $< \tau < 10^{-9}$ sec), the presence of a nonlinear refractive index inside a laser cavity can produce mode coupling and generate ultrashort pulses of ruby light.

In part II, an experiment was described which allowed us to observe the presence of pulses of ruby light as short as 10^{-11} sec when a cell containing nitrobenzene or α-chloronaphthalene was placed inside the cavity. The most important parameter of the experiment is the orientational relaxation time τ. Ultrashort pulses appeared only when τ was such that energy transfer between the cavity modes occurred in a time short compared the length of a Q-switched pulse.

The value of τ was varied by changing the temperature of the liquid.

Stimulated Raman scattering without self-focusing was observed when nitrobenzene was placed inside the cavity. Only 1/500 of the beam energy was shifted outside of the gain linewidth of the laser by this process. No stimulated Raman emission occurred when the nitrobenzene was replaced by α-chloronaphthalene.

The following simple and physical argument can be given to show how a pulse of light travelling back and forth in a laser cavity is shortened by the presence of a refractive index nonlinearity.

When a pulse of light (pulse (1) in Figure 2.11) travels through a medium with an index of refraction of the form $n = n_0 + n_2E^2$ where E^2 is the intensity of the pulse, the region of high intensity "sees" a larger index of refraction than the regions of low intensity and therefore travels slower through the medium. After a certain distance of propagation, the front part of the pulse has gained distance relative to the center and hence rises more slowly than the original pulse. Similarly the back portion drops off more steeply.[25] This case is depicted by pulse (2). Pulse (2) has been shortened with respect to pulse (1). Then, as the back end of the pulse drops off in a time comparable to the orientational relaxation time τ, the law $n = n_0 + n_2E^2$ is not obeyed any more because the molecules do not respond to the fast time change of the electric fields. Therefore the front end of the pulse rising more slowly is delayed more than the back end. The pulse is sharpened again; it is depicted by pulse (3).

This pulse-sharpening mechanism goes on as the number of passes through the liquid increases until the length of the pulse is of the order of τ, at which point the nonlinearities due to orientational Kerr effect have no effect on the pulse which is shown as pulse (4).

The pulse width could be made shorter only by decreasing the orientational relaxation time. This can be done by heating the liquid or by mixing it with a liquid of a lower viscosity. However, this would mean a smaller orientational Kerr constant and weaker mode coupling. The use of a temperature-controlled nonlinear dielectric constant inside the laser cavity may be, however, a practical way of generating ultrashort pulses of variable length.

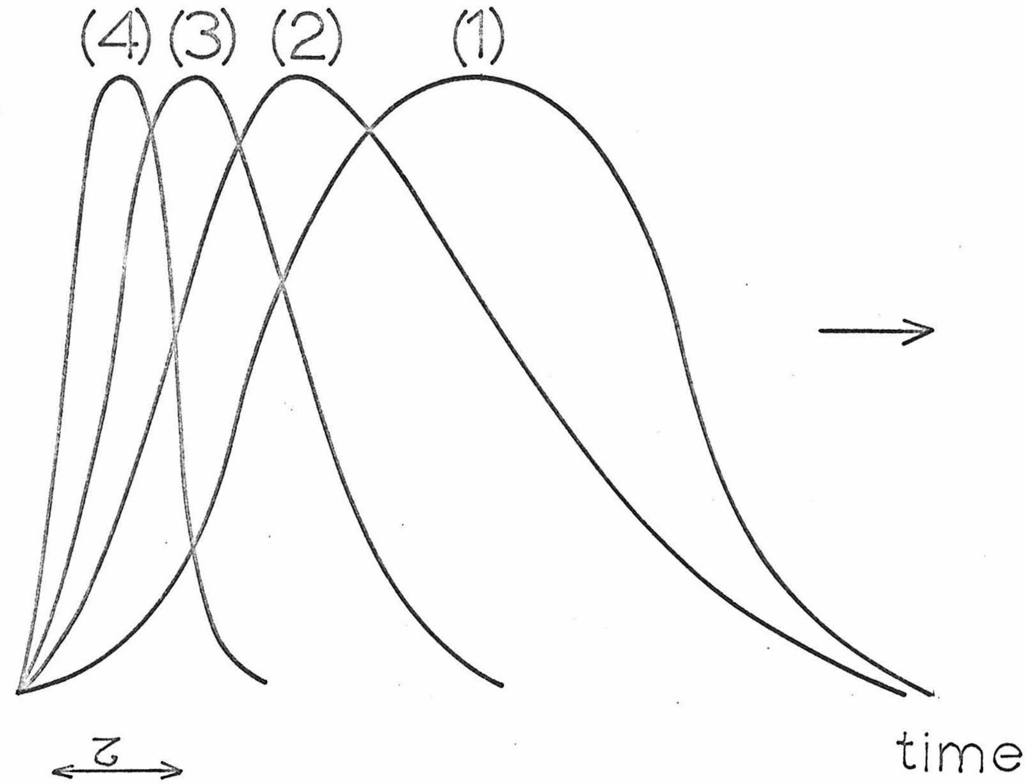

FIG 2-11 . Pulse sharpening in a non linear index of refraction

Another advantage of the use of this effect to generate ultrashort pulses is the following: Present techniques generally use a saturable absorber which tends to enhance the filamentary structure of the laser output, often causing damage to laser components. The liquids used here reduced the filamentary nature of the beam even when compared to that without any liquid. This can be seen from comparing the transverse structures in Figures 2.6 and 2.8, for example. No damage to the components was observed when ultrashort pulses were generated by the anisotropic molecular liquids, while it was a frequent problem with saturable absorbers.

APPENDIX

The total electromagnetic energy $\mathcal{E}_T$ stored inside the cavity is given by expression (1.27)

$$\mathcal{E}_T = \frac{1}{2}\sum_n \left(p_n^2(t) + \omega_n^2 q_n^2(t)\right) \tag{1.27}$$

where $p_n(t)$ and $q_n(t)$ are defined by (1.13).

We have expressed $p_n(t)$ in terms of the wave function $D_n(t)$ and its complex conjugate $D_n^*(t)$ in the following way

$$p_n(t) = i\sqrt{\frac{\omega_n}{2}}\left(D_n^*(t)e^{i\omega_n t} - D_n(t)e^{i\omega_n t}\right) \tag{1.18}$$

In this appendix we look for an expression for the electromagnetic energy $\mathcal{E}_T$ in terms of $D_n(t)$ and $D_n^*(t)$. For this purpose, we find a relationship between $q_n(t)$, $D_n(t)$ and $D_n^*(t)$ by solving the Maxwell's equation $\nabla \times E(\vec{r},t) = -\frac{\partial B}{\partial t}$.

This has been done in Section 1-5 and a relationship between $p_n(t)$ and $q_n(t)$ has been found

$$p_n(t) = q_n'(t) \tag{1.60}$$

With the help of (1.18) and (1.60), we can write

$$q_n(t) = \frac{1}{\sqrt{2\omega_n}}\left(D_n^*(t)e^{i\omega_n t} + D_n(t)e^{-i\omega_n t}\right) \tag{1.61}$$

and from (1.27), (1.18) and (1.61),

$$\mathcal{E}_T = \sum_n \omega_n D_n^*(t)\, D_n(t) \quad .$$

REFERENCES

1. A. Yariv, "Internal Modulation in Multimode Laser Oscillators", Journal of Applied Physics 36, No. 2, 388-391, February 1965.

2. L. E. Hargrove, R. L. Fork, and M. A. Pollack, "Locking of the He-Ne Laser Modes Induced by Synchronous Intracavity Modulation", Applied Physics Letters 5, No. 1, 4-5, 1 July 1964.

3. M. DiDomenico et al, "Generation of Ultrashort Optical Pulses by Mode Locking the YAG:Nd Laser", Applied Physics Letters 8, No. 7, 180-183, 1 April 1966.

4. T. Deutsch, "Mode-Locking Effects in an Internally Modulated Ruby Laser", Applied Physics Letters 7, No. 4, 15 August 1965.

5. A. J. DeMaria et al, "Mode Locking of a Nd^{3+} -Doped Glass Laser", Applied Physics Letters 8, No. 1, 22-24, 1 January 1966.

6. R. W. Hellwarth, Advances in Quantum Electronics, J. R. Singer, Ed. Columbia University Press, New York, pp. 334-441, 1961 .

7. H. W. Mocker and R. J. Collins, "Mode Competition and Self-Locking Effects in a Q-Switched Ruby Laser", Applied Physics Letters 7, No. 10, 270-273, 15 November 1965.

8. A. J. DeMaria, D. A. Stetser, and H. Heynau, "Self-Mode-Locking of Lasers with Saturable Absorbers", Applied Physics Letters 8, No.7, 174-176, 1 April 1966.

9. Jean-Pierre Laussade and Amnon Yariv, "Mode Locking and Ultrashort Laser Pulses by Anisotropic Molecular Liquids", Applied Physics Letters 13, No. 2, 65-66, 15 July 1968.

10. J. Comly, E. Garmire, J. P. Laussade and A. Yariv, "Observation of Mode Locking and Ultrashort Optical Pulses Induced by Anisotropic Molecular Liquids", Applied Physics Letters 13, 1 September 1968.

11. J. Frenkel, Kinetic Theory of Liquids, Dover Publications, Inc., New York, 1955, p. 283.

12. P. Debye, Polar Molecules, Dover Publications, Inc., Chapter V.

13. J. C. Slater, Microwave Electronics, Princeton, New Jersey, Van Nostrand, 1950, p. 57.

14. A. Yariv, "Parametric Interactions of Optical Modes", IEEE Journal of Quantum Electronics QE2, 30-37, February 1966.

15. C. C. Wang, "Nonlinear Susceptibility Constants and Self-Focusing of Optical Beams in Liquids", Physical Review 152, No. 1, 149-156, 2 December 1966.

16. M. Paillette, "Mesures de l'effet Kerr induit par une onde limineuse intense", Comptes Rendus de l'Academie des Sciences de Paris, t 262 (24 janvier 1966), pp. 264-267, Serie B.

17. See for example W. J. Moore, Physical Chemistry, Prentice Hall, New York, p. 421, 1950.

18. C. W. Cho, N. D. Foltz, D. H. Rank and T. A. Wiggins, "Stimulated Rayleigh Scattering", Physical Review Letters, Vol. 18, No. 4, pp. 107-109, 23 January 1967.

19. J. A. Giordmaine, P. M. Rentzepis, S. L. Shapiro and K. W. Wecht, "Two-photon Excitation of Fluorescence by Picosecond Light Pulses", Applied Physics Letters, Vol. 11, No. 7, pp. 216-218, 1 October 1967.

20. P. M. Rentzepis and M. A. Duguay, "Picosecond Light Pulse Display Using Two Different Optical Frequencies", Applied Physics Letters, Vol. 11, No. 7, pp. 218-220, 1 October 1967.

21. J. R. Klauder, M. A. Duguay, J. A. Giordmaine and S. L. Shapiro, "Correlation Effects in the Display of Picosecond Pulses by Two-Photon Techniques", Submitted to Applied Physics Letters.

22. N. D. Foltz, C. W. Cho, D. H. Rank, and T. A. Wiggins, "Stimulated Rayleigh Scattering in Liquids", The Physical Review, Vol. 165, No. 2, pp. 396-400, 10 January 1968.

23. F. J. McClung, "Study of Fundamental Requirements for High Brightness Raman Laser Sources", Semiannual Technical Summary Report, p. 6, Hughes Research Laboratory, Malibu, California, 1 October 1966 through 31 March 1967.

24. See for example A. Yariv, "Quantum Electronics", John Wiley, Inc., New York, 1967, eq. 23-2-5, p. 390.

25. F. DeMartini, C. H. Townes, T. K. Gustafson and P. L. Kelley, "Self Steepening of Light Pulses", The Physical Review, Vol. 164, pp. 312-323, December 1967.

FIGURE CAPTIONS

Figure 1.1. Three oscillating laser modes.

Figure 1.2. Orientation of an anisotropic molecule with an electric field.

Figure 1.3. Coupling of the cavity modes n, $n+p$, $n+m$, $n+p+m$.

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Figure 2.11. Pulse sharpening in a nonlinear index of refraction.

II. A THEORETICAL STUDY OF OPTICAL WAVE PROPAGATION THROUGH A RANDOM MEDIUM AND ITS APPLICATION TO OPTICAL COMMUNICATION

INTRODUCTION

The discovery and development of the laser has, in recent years, opened new perspectives in the area of optical communication. This is due mainly to the availability of a large bandwidth, theoretically up to 10^{12} Hz. Any optical communication system using a laser as an optical carrier transmits a modulated information through a layer of atmospheric turbulence. This is true when the communication link is from earth to a satellite, another planet, or from earth to earth. The atmospheric turbulence introduces random fluctuations in the amplitude and phase of the modulated optical signal and therefore distorts the information carried by the optical beam.

There are two principal detection schemes of modulated optical signals: (a) heterodyne detection, and (b) video detection. In scheme (a) the incoming modulated optical beam is mixed in a nonlinear detector with a local reference signal, while in scheme (b), the reference signal (i.e. the carrier) is transmitted together with the modulated signal along the communication path. The purpose of this report is to compare the theoretical performance of these two optical communication schemes through a randomly turbulent medium in terms of the parameters of this medium.

In order to achieve this goal, the propagation of an optical wave through a random medium has to be studied. There is a considerable and valuable amount of literature on this subject (1-5). The problem common to all these references is finding a solution for the wave equation $(\nabla^2 + k^2 n^2(\vec{x}))\, u(\vec{x}) = 0$ in a medium where the index of refraction $n(\vec{x})$ is a random process. No simple analytical expression for the wave function $u(\vec{x})$ of the propagating wave can be found under these circumstances. Tatarski in reference (1) uses an approximation (the Rytov approximation) in order to find an analytical expression for the wave function. The Rytov approximation has been contested theoretically (6) (7) and its limits of applicability have been determined experimentally (8). The results obtained from the use of the Rytov approximation are only valid for short propagation lengths in a weakly turbulent medium. The geometrical optics and the Born approximations have also been used to solve the wave equation, but their range of validity is severely limited.

Other methods make use of a correct power series expansion for the wave function $\overline{u(\vec{x})}$ for calculating its statistical properties such as the ensemble average $\overline{u(\vec{x})}$ and the correlation function $\overline{u(\vec{x}_1)\, u^*(\vec{x}_2)}$ (5). However, in the averaging process, approximations are made which seriously handicap the validity of the results obtained.

In this report we shall derive a power series expansion solution for the wave equation, in a form which is slightly different from that of reference (5). This enables us to find analytical expressions for $\overline{u(\vec{x})}$ and $\overline{u(\vec{x}_1)\, u^*(\vec{x}_2)}$ without any approximation for a turbulent medium in which the fluctuations of the index of refraction are a Gaussian random process.

In Chapter I we examine the wave equation satisfied by an optical wave propagating through a medium with a random index of refraction. A formal solution as a series expansion in powers of the variance of the refractive index fluctuations is derived. Various approximate methods are then reviewed and their results compared with the correct expression for the wave function. In Chapter II the statistical mean of the wave function $\overline{u(\vec{x})}$ is calculated and an exact analytical expression is obtained in terms of the correlation function of the refractive index fluctuations. In Chapter III the results derived in Chapter II are compared with expressions for $\overline{u(\vec{x})}$ obtained by using various approximations. In Chapter IV the correlation function $\overline{u(L,\vec{r}_1)\ u^*(L,\vec{r}_2)}$ of the wave function between two points in a plane perpendicular to the direction of propagation after a propagation distance L through the turbulent medium is calculated. An exact analytical expression is obtained in terms of the correlation function of the refractive index fluctuations. In Chapter V the results derived in Chapter IV are compared with results obtained for the same function $\overline{u(L,\vec{r}_1)\ u^*(L,\vec{r}_2)}$ using various approximations. In Chapter VI another useful statistical function of the propagating wave, the intensity correlation function $\overline{I(L,\vec{r}_1)\ I(L,\vec{r}_2)}$ is investigated by the same methods as in Chapters II and IV. Our exact theoretical analysis is incapable, in this case, of predicting the behavior of this function for any propagation length and turbulence strength. Chapter VII is devoted to the calculation of the intensity correlation function using various approximations. Both our results and the results of the approximation methods cannot adequately explain the experimental results of reference (8) and others.

In Chapter VIII we present and discuss recent experimental observations on the behavior of the intensity fluctuations in relation to our results. Several empirical formulas which "fit" the experimental data for the variance of the intensity fluctuations $\overline{I^2(L)}$ are introduced.

In Chapter IX the results of the previous chapters are applied to the comparison between two schemes of optical communication through the atmospheric turbulence (a) heterodyne detection, (b) video communication. In scheme (a) the detection of a phase modulated optical beam is performed by mixing it in a nonlinear detector with a local reference signal. In scheme (b) the reference beam is sent together with the signal beam through the same path in the turbulent medium. In both these cases the signal-to-noise ratio (S/N) is derived in terms of the statistical properties of the refractive index fluctuations. The Kolmogoroff model of turbulence is then used for obtaining a numerical comparison of the performances of the two methods of optical communication. The following performance criterion is introduced,

$$R = \frac{(S/N)_{\text{scheme(b)}}}{(S/N)_{\text{scheme(a)}}}$$

R is expressed explicitly in terms of the length of the communication link, the diameter of the receiving aperture, the strength of the turbulence and the wavelength. These results are then analyzed and discussed.

CHAPTER I - SOLUTION OF THE WAVE EQUATION IN A MEDIUM WITH A RANDOMLY HOMOGENEOUS INDEX OF REFRACTION

1.1 Statement of the Problem

The problem is to solve Maxwell's equations in a medium where the index of refraction is a homogeneous random field. An optical wave propagating through such a random medium satisfies the following scalar wave equation. (In deriving this equation we have neglected depolarization effects (9)).

$$[\nabla^2 + k^2 n^2(\vec{x})]\, u(\vec{x}) = 0 \tag{1.1}$$

where k is the wave number of the wave $= \frac{2\pi}{\lambda}$ in the absence of turbulence

λ is the wavelength of the wave

$n(\vec{x})$ is the index of refraction of the turbulent propagation medium.

In the absence of any turbulence we assume that $n(\vec{x}) = 1$. However, in the presence of turbulence, the index of refraction varies randomly in space. We shall assume that the index of refraction $n(\vec{x})$ takes the following form

$$n(\vec{x}) = 1 + \varepsilon\, n_1(\vec{x}) \tag{1.2}$$

The random field $n(\vec{x})$ is assumed to be homogeneous and isotropic. By homogeneous, it is meant that its mean value is a constant (independent of position) and its correlation function $B_n(\vec{x}_1,\vec{x}_2)$ between two points $\vec{x}_1$ and $\vec{x}_2$ depends only upon the difference $\vec{x}_1 - \vec{x}_2$. Here we choose the mean value of $n(\vec{x})$ to be equal to one, $\overline{n(\vec{x})} = 1$, therefore $\overline{n_1(\vec{x})} = 0$

$$B_n(\vec{x}_1,\vec{x}_2) = \overline{(n(\vec{x}_1) - \overline{n(\vec{x}_1)})\ ((n(\vec{x}_2) - \overline{n(\vec{x}_2)})}$$

$$= \varepsilon^2\ \overline{n_1(\vec{x}_1)\ n_1(\vec{x}_2)} = B_n(\vec{x}_1 - \vec{x}_2) \qquad (1.3)$$

For example, the mean and the correlation function of a random function $f(x,y,z)$ of the spaces coordinates x, y and z are defined as follows:

$$\overline{f(x,y,z)} = \lim_{L \to \infty} \left[\frac{1}{L^3} \int_{-1/2}^{1/2} dx \int_{-1/2}^{1/2} dy \int_{-L/2}^{L/2} dz \ \ f(x,y,z) \right]$$

$$B_f(\alpha,\beta,\gamma) = \overline{f(x+\alpha,\ y+\beta,\ z+\gamma)\ \ f(x,y,z)}$$

$$= \lim_{L \to \infty} \left(\frac{1}{L^3} \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \int_{-L/2}^{L/2} dz\ f(x+\alpha,y+\beta,z+\gamma)\ f(x,y,z) \right)$$

The bars denote an ensemble average over the total $\vec{x}$ space. By isotropic, it is meant that the correlation function $B_n(\vec{x}_1,\vec{x}_2)$ depends only upon the distance between the two points $\vec{x}_1$ and $\vec{x}_2$, i.e. $B_n(\vec{x}_1,\vec{x}_2) = B_n(|\vec{x}_1 - \vec{x}_2|)$ where $|\vec{x}_1 - \vec{x}_2|$ denotes the modulus of the vector $\vec{x}_1 - \vec{x}_2$.

The homogeneous random field $n_1(\vec{x})$ has the following two-dimensional Fourier-Stieljes spectral representation (10)

$$n_1(\vec{x}) = \int_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}\, dN(\vec{k},x) \tag{1.4}$$

where $dN(\vec{k},x)$ is another random field; its properties will be given later. The position of a point in space is determined by its coordinates x and $\vec{r}$. The x-axis is along the direction of propagation of the wave. $\vec{r}$ is a vector perpendicular to the x-axis. The problem is to find statistical information about the wave function $u(\vec{x})$, e.g. its mean $\overline{u(\vec{x})}$, its correlation function $\overline{u(\vec{x}_1)\; u^*(\vec{x}_2)}$ in terms of the statistical properties of the fluctuations of the index of refraction $n_1(\vec{x})$.

The wave equation 1.1 has been solved by various researchers [(1) - (5)] using various approximations: the geometrical optics approximation, the Born approximation, the Rytov approximation. The purpose of this report is not to review these approximations in detail, but during the development of our analysis we will mention how these approximations come about.

We shall first look for a solution of the wave equation 1.1 in the following manner. According to 1.2 we can write

$$n^2(\vec{x}) = [1 + \varepsilon\, n_1(\vec{x})]^2 \simeq 1 + 2\varepsilon\, n_1(\vec{x}) \tag{1.5}$$

Since ε is a very small quantity (of the order of 10^{-8} in the atmosphere), we can drop the term in ε^2 which gives a negligible contribution to the fluctuations of the square of the index of refraction. Therefore the wave equation takes the following form:

$$(\nabla^2 + k^2)\, u(\vec{x}) + 2\varepsilon\, k^2\, n_1(\vec{x})\, u(\vec{x}) = 0 \tag{1.6}$$

We shall look for a solution of $u(\vec{x})$ as a power series of ε

$$u(\vec{x}) = u_o(\vec{x}) + \varepsilon\, u_1(\vec{x}) + \varepsilon^2 u_2(\vec{x}) + \cdots + \varepsilon^n u_n(\vec{x}) + \cdots \tag{1.7}$$

We substitute the above expansion for $u(\vec{x})$ in the differential equation 1.6

$$(\nabla^2 + k^2) \sum_{p=0}^{\infty} \varepsilon^p u_p(\vec{x}) + 2k^2 n_1(\vec{x}) \sum_{p=0}^{\infty} \varepsilon^{p+1} u_p(\vec{x}) = 0 \tag{1.8}$$

We now look in expression 1.8 for the terms of a given power of ε and we find the following recursion relationship between $u_p(\vec{x})$ and $u_{p-1}(\vec{x})$

$$(\nabla^2 + k^2)\, u_p(\vec{x}) + 2k^2 n_1(\vec{x})\, u_{p-1}(\vec{x}) = 0 \tag{1.9}$$

The method for calculating the various terms in the expansion of $u(\vec{x})$ becomes now clearer. Knowing $u_{p-1}(\vec{x})$, we can calculate the following term $u_p(\vec{x})$ by means of a Green's function.

$$u_p(\vec{x}) = -2k^2 \int d\vec{x}'\, G(\vec{x},\vec{x}')\, n_1(\vec{x}')\, u_{p-1}(\vec{x}') \tag{1.10}$$

$G(\vec{x},\vec{x}')$ is, according to 1.9, the Green's function corresponding to the operator $(\nabla^2 + k^2)$ which is defined by the following equation:

$$(\nabla^2 + k^2)\, G(\vec{x},\vec{x}') = -\delta(\vec{x} - \vec{x}')$$

Therefore,

$$G(\vec{x},\vec{x}') = \frac{e^{ik|\vec{x} - \vec{x}'|}}{4\pi|\vec{x} - \vec{x}'|} \tag{1.11}$$

We shall calculate explicitly the first two terms $u_1(\vec{x})$ and $u_2(\vec{x})$ of the power series expansions 1.7, assuming that the solution of the wave equation is a plane wave in the absence of turbulence. We shall then assume a forumla for $u_{m-1}(\vec{x})$ which is valid for $m-1 = 1$ and 2, and show that $u_m(\vec{x})$ is obtained from $u_{m-1}(\vec{x})$ by replacing $m-1$ by m. In this manner we prove that the formula postulated for $u_{m-1}(\vec{x})$, valid for $m-1 = 1$ and 2 is valid for any $m-1$.

The zero order term $u_o(\vec{x})$ is the solution of the wave equation in the absence of turbulence ($\varepsilon = 0$). We assume that this solution is a plane wave propagating in the x direction with amplitude A_o

$$u_o(\vec{x}) = A_o\, e^{ikx} \tag{1.12}$$

For convenience we do not compute the terms $u_p(\vec{x})$ in the expansion of $u(\vec{x})$, but rather the quantities

$$\psi_p(\vec{x}) = \frac{u_p(\vec{x})}{u_o(\vec{x})} \tag{1.13}$$

and we call $\psi(\vec{x})$ the following series

$$\psi(\vec{x}) = \frac{u(\vec{x})}{u_o(\vec{x})} = 1 + \varepsilon\psi_1(\vec{x}) + \varepsilon^2\psi_2(\vec{x}) + \cdots + \varepsilon^p\psi_p(\vec{x}) + \cdots \tag{1.14}$$

$\psi(\vec{x})$ is the ratio of the wave function $u(\vec{x})$ in the presence of turbulence to the wave function $u_o(x)$. In the absence of turbulence $\psi(\vec{x}) = 1$.

We shall now proceed to compute $\psi_1(\vec{x})$ and $\psi_2(\vec{x})$.

1.2 Calculation of $\psi_1(\vec{x}) = u_1(\vec{x})/u_o(\vec{x})$

The first order term $\psi_1(\vec{x})$ in the expansion of $\psi(\vec{x})$ is defined by 1.13 as

$$\psi_1(\vec{x}) = \frac{u_1(\vec{x})}{u_o(\vec{x})} = \frac{u_1(\vec{x})}{A_o} e^{-ikx}$$

We then use relationship 1.10 to find

$$\psi_1(\vec{x}) = -2k^2 \int d\vec{x}' \; G(\vec{x},\vec{x}') \; e^{-ik(x-x')} \; n_1(\vec{x}') \qquad (1.15)$$

We then express the Green's function $G(\vec{x},\vec{x}')$ as follows:

$$G(\vec{x},\vec{x}') = G(\vec{x}-\vec{x}') = G(x-x' \;,\vec{r}-\vec{r}') = \int_{\vec{K}} G(\vec{K},x-x')\; e^{i\vec{K}\cdot(\vec{r}-\vec{r}')} d\vec{K} \qquad (1.16)$$

where $G(\vec{K},x-x')$ is the transverse two-dimensional Fourier transform of $G(\vec{x},\vec{x}')$ and is given by

$$(2\pi)^2 \; G(\vec{K},x-x') = \frac{-i}{2(k^2-K^2)^{1/2}} \exp[i(k^2-K^2)^{1/2}|x-x'|] \qquad (1.17)$$

We then replace $G(\vec{x},\vec{x}')$ expressed by 1.16 and 1.17 and $n_1(\vec{x})$ given by 1.4 into expression 1.15:

$$\psi_1(\vec{x}) = -\frac{2k^2}{(2\pi)^2}\int dx' \int d\vec{r}' \int_{\vec{K}} d\vec{K} \int_{\vec{K}'} \frac{-i}{2(k^2 - K^2)^{1/2}}$$

$$\times\, e^{i(k^2-K^2)^{1/2}|x-x'|}\, e^{-ik(x-x')}\, e^{i\vec{K}\cdot(\vec{r}-\vec{r}')}\, e^{i\vec{K}'\cdot\vec{r}'}\, dN(\vec{K}',x') \tag{1.18}$$

The integration with respect to $\vec{r}'$ is easily performed

$$\int d\vec{r}'\, e^{-i(\vec{K}-\vec{K}')\cdot\vec{r}'} = (2\pi)^2\delta(\vec{K}-\vec{K}')$$

Due to the presence of the Dirac delta function, we notice that the only values of $\vec{K}$ (the space Fourier transform variable of the Green's function) which will contribute to $\psi_1(\vec{x})$ will be the ones which are of the same order of magnitude as $\vec{K}'$, the space Fourier transform variable of the fluctuations of the index of refraction $n_1(\vec{x})$. At this point it is important to state which specific problem we are trying to solve. We are concerned with the propagation of an optical wave through atmospheric turbulence. By optical wave, we mean a wave with a wavelength of up to about $10\mu = 10^{-2}$mm.

We shall now examine qualitatively the structure of the atmospheric turbulence. The atmospheric turbulence is made up of inhomogeneities of different sizes called eddies (11). The energy in the largest eddies is obtained from large scale ordered motions in the atmosphere, like atmospheric winds, for example. Each eddy of size ℓ

is characterized by a parameter called the Reynolds number Re and is defined as follows

$$Re = \frac{v\ell}{\nu}$$

where v is the velocity of the turbulent

ν is the kinematic viscosity = μ/ρ

μ is the dynamic viscosity

ρ is the fluid density.

As the Reynolds number of an eddy of size ℓ is increased (by increasing the velocity) above a critical value Re_{CR}, the motion of the fluid becomes unstable and the eddy of size ℓ breaks up into eddies of smaller sizes ℓ' so that the Reynolds number of the eddies of size $\ell' < \ell$ becomes smaller than the critical value Re_{CR} and the motion of the fluid inside the eddies of size ℓ' becomes stable. As the Reynolds number of the eddy of size ℓ' increases further, it breaks up into eddies of smaller sizes. This process goes on until the size of the eddies reaches a minimum called the inner scale of turbulence ℓ_o . The eddies of size ℓ_o do not break into smaller eddies because their Reynolds number is always smaller than the critical value. Energy is transferred from the largest eddies to the smallest eddies of sizes ℓ_o at which point energy is directly transformed into heat via viscous motion of the fluid. In the atmosphere, wind shear provides the energy to maintain the turbulence. The size of the largest eddies is called the outer scale of turbulence L_o. Therefore the size ℓ of the inhomogeneities in the atmosphere is such that $\ell_o \leq \ell \leq L_o$. This range of energy containing eddies is called the inertial subrange by Kolmogoroff (12).

In the spectral representation of the random process $n_1(\vec{x}')$

$$n_1(\vec{x}') = \int_{\vec{K}'} e^{i\vec{K}'\cdot\vec{r}'} \, dN(\vec{K}',x')$$

the magnitude of $\vec{K}'$ is such that $\frac{2\pi}{L_o} \leq K' = \frac{2\pi}{\ell} \leq \frac{2\pi}{\ell_o}$. In the atmosphere ℓ_o is, typically, of the order of one millimeter and L_o of the order of a few meters. Therefore for an optical wave

$$K' \leq \frac{2\pi}{\ell_o} << \frac{2\pi}{\lambda} = k \tag{1.19}$$

since $\frac{K'}{k} < \frac{\lambda}{\ell_o} = \frac{1\mu}{1mm} = 10^{-3}$.

We can then express the two-dimensional Fourier transform $G(\vec{K},x-x')$ in 1.17

$$G(\vec{K},x-x') = -\frac{i}{2k(2\pi)^2} \exp[i(k - \frac{K^2}{2k}) \, |x-x'| \,] \tag{1.20}$$

We also notice that the propagation of a wave in the x-direction between two points of coordinates $(0,\vec{r})$ and $(x,\vec{r})$ will be affected by the random field $n_1(x',\vec{r}')$ or $dN(\vec{K}',x')$ only if the condition $0 \leq x' \leq x$ is fulfilled. Therefore in 1.18 we must perform the integration with respect to x' between the limits $x' = 0$ and $x' = x$.

Using expression 1.20, taking into account the above remark, performing the integrations with respect to $\vec{K}$ and $\vec{r}'$, we find

$$\psi_1(\vec{x}) = ik \int_0^x dx' \int_{\vec{K}'} e^{-\frac{iK'^2}{2k}(x-x')} e^{i\vec{K}'\cdot\vec{r}} dN(\vec{K}',x') \qquad (1.21)$$

We shall state again, because it is an important point, the approximation made above. We have assumed that the wavelength of the propagating wave is much smaller than the inner scale of the turbulence. This is especially true for the case of an optical wave propagating through the atmosphere and roughly exact for submillimeter waves. This assumption allowed us to approximate the two-dimensional Fourier transform of the Green's function by expression 1.20. This simplified greatly the calculation of $\psi_1(\vec{x})$ because of the cancellation of the terms $e^{ik(x-x')}$ and $e^{-ik(x-x')}$. We shall use the assumption

$$\lambda \ll \ell_o \qquad (1.22)$$

all along the development of our analysis. This inequality is an essential condition for its validity. We now proceed to calculate the second term $\psi_2(\vec{x})$ in the expansion for $\psi(\vec{x})$.

1.3 Calculation of $\psi_2(\vec{x})$

The second order term in ε^2 in the expansion 1.14 for $\psi(\vec{x})$ is $\psi_2(\vec{x})$ defined by 1.13. $\psi_2(\vec{x}) = u_2(\vec{x})/u_o(\vec{x})$ where $u_2(\vec{x})$ is expressed in terms of $u_1(\vec{x})$ by the recursion formula 1.10. First, we can write $\psi_2(\vec{x})$ as

$$\psi_2(\vec{x}) = \frac{u_2(\vec{x})}{u_o(\vec{x})} = \frac{u_2(\vec{x})}{A_o \exp(ikx)} = \frac{u_2(\vec{x}) \exp[-ik(x-x')]}{A_o \exp(ikx')}$$

$$= \frac{u_2(\vec{x}) \exp[-ik(x-x')]}{u_o(\vec{x}')} \tag{1.23}$$

We now make use of relationship 1.10 for $p = 2$

$$u_2(\vec{x}) = -2k^2 \int d\vec{x}' \, G(\vec{x},\vec{x}') \, n_1(\vec{x}') \, u_1(\vec{x}') \tag{1.24}$$

Combining 1.23 and 1.24, we find

$$\psi_2(\vec{x}) = -2k^2 \int d\vec{x}' \, G(\vec{x},\vec{x}') \, e^{-ik(x-x')} \, n_1(\vec{x}') \, \frac{u_1(\vec{x}')}{u_o(\vec{x}')} \tag{1.25}$$

We notice that $\dfrac{u_1(\vec{x}')}{u_o(\vec{x}')} = \psi_1(\vec{x}')$ and we replace $\psi_1(\vec{x}')$ in 1.25 by its expression 1.15.

$$\psi_2(\vec{x}) = (-2k^2)^2 \int d\vec{x}' \int d\vec{x}'' \, G(\vec{x},\vec{x}') e^{-ik(x-x')} \, G(\vec{x}',\vec{x}'')$$

$$\times \; e^{-ik(x'-x'')} \, n_1(\vec{x}') \, n_1(\vec{x}'')$$

In the following step the Green's functions and the fluctuations of the index of refraction n_1 are replaced by their two-dimensional Fourier transforms 1.16 and 1.4. Under the assumption $\lambda \ll \ell_o$, we can write, according to 1.20,

$$G(\vec{x},\vec{x}') \, e^{-ik(x-x')} = \int_{-\infty}^{\infty} d\vec{K}_1 \, \frac{(-i)}{2k(2\pi)^2} \exp[-i \, \frac{K_1^2}{2k}(x-x')] e^{i\vec{K}_1 \cdot (\vec{r}-\vec{r}')} \tag{1.26}$$

and then

$$\psi_2(\vec{x}) = \frac{(ik)^2}{(2\pi)^4} \int_0^x dx' \int_0^{x'} dx'' \int_{-\infty}^{\infty} d\vec{r}' \int_{-\infty}^{\infty} d\vec{r}'' \int d\vec{K}_1 \int d\vec{K}_2 \int_{\vec{K}'} \int_{\vec{K}''}$$

$$\times \left\{ \exp[-i \frac{K_1^2}{2k} (x-x')] \exp[-i \frac{K_2^2}{2k} (x'-x'')] \right.$$

$$\times \exp[i(\vec{K}_1\cdot(\vec{r}-\vec{r}') + \vec{K}_2\cdot(\vec{r}'-\vec{r}'') + \vec{K}'\cdot\vec{r}' + \vec{K}''\cdot\vec{r}'')]$$

$$\left. \times \; dN(\vec{K}',x')\, dN(\vec{K}'',x'') \right\} \qquad (1.27)$$

The integrations with respect to r' and $\vec{r}''$ yield

$$\int_{-\infty}^{\infty} d\vec{r}' \; e^{i(-\vec{K}_1 + \vec{K}_2 + \vec{K}')\cdot\vec{r}'} \int_{-\infty}^{\infty} d\vec{r}'' \; e^{i(-\vec{K}_2 + \vec{K}'')\cdot\vec{r}''}$$

$$= (2\pi)^4 \; \delta(\vec{K}_2 - \vec{K}_1 + \vec{K}')\; \delta(\vec{K}'' - \vec{K}_2)$$

The integrations with respect to $\vec{K}_1$ and $\vec{K}_2$ are then easily performed and

$$\psi_2(\vec{x}) = (ik)^2 \int_0^x dx' \int_0^{x'} dx'' \int_{\vec{K}'} \int_{\vec{K}''} \left\{ \exp\left[-i\frac{(\vec{K}'+\vec{K}'')^2}{2k}(x-x')\right] \right.$$

$$\left. \times \exp\left[-i\frac{K''^2}{2k}(x'-x'')\right] e^{i(\vec{K}'+\vec{K}'')\cdot\vec{r}} \; dN(\vec{K}',x')\, dN(\vec{K}'',x'') \right\} \qquad (1.28)$$

We can also express $\psi_2(\vec{x})$ in another form which will be useful later by writing

$$(\vec{K}' + \vec{K}'')^2 (x-x') + K''^2(x'-x'') = K'^2(x-x')$$

$$+ 2\vec{K}'\cdot\vec{K}''(x-x') + K''^2(x-x'')$$

then

$$\psi_2(\vec{x}) = (ik)^2 \int_0^x dx' \int_0^{x'} dx'' \int_{\vec{K}'} \int_{\vec{K}''} \left\{ \exp\left[-\frac{i}{2k} \left(K'^2(x-x') + K''^2(x-x'') \right.\right.\right.$$

$$\left.\left.\left. + 2\vec{K}' \cdot \vec{K}''(x-x') \right) \right] e^{i(\vec{K}' + \vec{K}'')\cdot\vec{r}} \; dN(\vec{K}', x') \; dN(\vec{K}'', x'') \right\} \qquad (1.29)$$

We notice that the calculation of the terms of various orders in the expansion of $\psi(\vec{x})$ is straightforward once the assumption $\lambda << \ell_o$ has been made. We shall now find the formula for the m^{th} term by the following method.

1.4 Calculation of $\psi_m(\vec{x})$

We shall first assume a formula for $\psi_{m-1}(\vec{x})$ which is valid for $m-1 = 1$ and 2, then we shall use the recursion formula to calculate $\psi_m(\vec{x})$ from $\psi_{m-1}(\vec{x})$ and find out that the expression for $\psi_m(\vec{x})$ is obtained by replacing $m-1$ by m in $\psi_{m-1}(\vec{x})$. Therefore if the formula proposed for $\psi_{m-1}(\vec{x})$ is valid for $m-1 = 1$ and 2, it will be valid for any m.

$\psi_m(\vec{x})$ is defined by 1.13 as $\psi_m(\vec{x}) = u_m(\vec{x})/u_o(\vec{x})$ which we can write

$$\psi_m(\vec{x}) = \frac{u_m(\vec{x})}{A_o \exp(ikx)} = \frac{u_m(\vec{x}) \exp[-ik(x-x')]}{A_o \exp(ikx')} = \frac{u_m(\vec{x}) \exp[-ik(x-x')]}{u_o(\vec{x}')} \qquad (1.30)$$

and $u_m(\vec{x})$ is given in terms of $u_{m-1}(\vec{x})$ by the recursion formula 1.10

$$u_m(\vec{x}) = -2k^2 \int d\vec{x}' \; G(\vec{x},\vec{x}') \; n_1(\vec{x}') \; u_{m-1}(\vec{x}') \qquad (1.31)$$

Therefore we use 1.30 and 1.31 to express $\psi_m(\vec{x})$ as follows

$$\psi_m(\vec{x}) = -2k^2 \int d\vec{x}'\ G(\vec{x},\vec{x}')\ e^{-ik(x-x')}\ n_1(\vec{x}')\ \frac{u_{m-1}(\vec{x}')}{u_o(\vec{x}')}$$

or

$$\psi_m(\vec{x}) = -2k^2 \int d\vec{x}'\ G(\vec{x},\vec{x}')\ e^{-ik(x-x')}\ n_1(\vec{x}')\ \psi_{m-1}(\vec{x}') \qquad (1.32)$$

We now postulate the following formula for $\psi_{m-1}(\vec{x})$ which is valid for $m-1 = 1$ and 2 (see expressions 1.21 and 1.29)

$$\psi_{m-1}(\vec{x}) = (ik)^{m-1} \int_0^x dx' \int_0^{x'} dx'' \cdots \int_0^{x^{(m-2)}} dx^{(m-1)}$$

$$\times \int_{\vec{K}'} \int_{\vec{K}''} \cdots \int_{\vec{K}^{(m-1)}} \left\{ \exp\left[i \left(\sum_{p=1}^{p=m-1} \vec{K}^{(p)} \right) \cdot \vec{r} \right] \right.$$

$$\times \exp\left[\left(-\frac{i}{2k}\right) \sum_{q=1}^{q=m-1} \left(\sum_{p=q}^{p=(m-1)} \vec{K}^{(p)} \right)^2 \left(x^{(q-1)} - x^{(q)} \right) \right]$$

$$\left. \times \left(\prod_{p=1}^{p=m-1} dN(\vec{K}^{(p)}, x^{(p)}) \right) \right\} \qquad (1.33)$$

where $0 \leq x^{(m-1)} \leq x^{(m-2)} \leq \cdots \leq x^{(p)} \leq x^{(p-1)} \leq \cdots \leq x'' \leq x' \leq x$

The symbol $\prod_{p=1}^{p=m-1} dN(\vec{K}^{(p)}, x^{(p)})$ stands for the product of the $m-1$ random functions $dN(\vec{K}^{(p)}, x^{(p)})$ evaluated for $\vec{K}^{(p)}$ at points $x^{(p)}$ where p runs from 1 to $m-1$, i.e.

$$\prod_{p=1}^{m-1} (\quad) = dN(\vec{K}', x')\ dN(\vec{K}'', x'') \cdots dN(\vec{K}^{(m-1)}, x^{(m-1)})$$

We now replace $\psi_{m-1}(\vec{x}')$, which is easily obtained from 1.33, into 1.32. The Green's function and the fluctuation of the index of refraction $n_1(\vec{x}')$ are expressed in terms of their two-dimensional Fourier transforms 1.20 and 1.4. $\psi_m(\vec{x})$ takes the following form

$$\psi_m(\vec{x}) = (-2k^2) \int_0^x dx' \int_{-\infty}^{\infty} d\vec{r}' \int d\vec{K}_1 \left(\frac{-i}{2k(2\pi)^2}\right) \exp\left[-i \frac{K_1^2}{2k} (x-x')\right]$$

$$\times\ e^{i\vec{K}_1\cdot(\vec{r}-\vec{r}')} \int_{\vec{K}'} e^{i\vec{K}'\cdot\vec{r}'} dN(\vec{K}',x')$$

$$\times\ (ik)^{m-1} \int_0^{x'} dx'' \int_0^{x''} dx''' \cdots \int_0^{x^{(m-1)}} dx^{(m)} \int_{\vec{K}''} \int_{\vec{K}'''} \cdots \int_{\vec{K}^{(m)}} \left\{ \exp\left[i\left(\sum_{p=2}^{p=m} \vec{K}^{(p)}\right)\cdot\vec{r}'\right]\right.$$

$$\left.\times\ \exp\left[\left(\frac{-i}{2k}\right) \sum_{q=2}^{q=m} \left(\sum_{p=q}^{p=m} \vec{K}^{(p)}\right)^2 (x^{(q-1)} - x^{(q)})\right] \left(\prod_{p=2}^{p=m} dN(\vec{K}^{(p)},x^{(p)})\right)\right\} \quad (1.34)$$

In this apparently complicated expression we can perform two straightforward integrations. The first is the integration with respect to $\vec{r}'$,

$$\int_{-\infty}^{\infty} d\vec{r}' \exp\left[i(-\vec{K}_1 + \vec{K}' + \sum_{p=2}^{p=m} \vec{K}^{(p)})\cdot\vec{r}'\right] = (2\pi)^2\, \delta(\vec{K}_1 - \sum_{p=1}^{p=m} \vec{K}^{(p)}) \quad (1.35)$$

The second is the integration with respect to $\vec{K}_1$ which is performed by replacing $\vec{K}_1$ by $\sum_{p=1}^{p=m} \vec{K}^{(p)}$ in expression 1.34. Carrying out these two integrations leads to the final expression of $\psi_m(\vec{x})$

$$\psi_m(\vec{x}) = (ik)^m \int_0^{x} dx' \int_0^{x'} dx'' \cdots \int_0^{x^{(m-1)}} dx^{(m)} \int_{\vec{K}'} \int_{\vec{K}''} \cdots \int_{\vec{K}^{(m)}}$$

$$\times \left\{ \exp\left[i \left(\sum_{p=1}^{p=m} \vec{K}^{(p)} \right) \cdot \vec{r} \right] \exp\left[\left(\frac{-i}{2k}\right) \sum_{q=1}^{q=m} \left(\sum_{p=1}^{p=m} \vec{K}^{(p)} \right)^2 \left(x^{(q-1)} - x^{(q)}\right) \right] \right.$$

$$\left. \times \left(\prod_{p=1}^{p=m} dN(\vec{K}^{(p)}, x^{(p)}) \right) \right\} \qquad (1.36)$$

We have proved that if $\psi_{m-1}(\vec{x})$ is given by expression 1.33, then the formula for $\psi_m(\vec{x})$ is obtained by replacing $m-1$ by m in 1.33. Therefore, since expression 1.33 is valid for $m-1=1$, which has been checked by the direct calculation of $\psi_1(\vec{x})$, it is valid for $m-1=2$, therefore for $m-1=3$ and so on; it is valid for any m.

We have calculated the general term $\psi_m(\vec{x})$ in the power series expansion of

$$\psi(\vec{x}) = \frac{u(\vec{x})}{u_o(\vec{x})} = \sum_{m=0}^{\infty} \varepsilon^m \psi_m(\vec{x})$$

This series is often called the Neumann series. Its convergence (5) is slow if the effect of multiple scattering is important.

At this point it is interesting to make reference to the various methods which have been used previously in the literature to solve the wave equation 1.1 .

1.5 Review of Approximate Methods

Many authors have previously tried to solve the wave equation $(\nabla^2 + k^2 n^2(\vec{x}))\, u(\vec{x}) = 0$, 1.1. All the methods which have been used are based on approximations which limit their range of validity. In Sections 1.1 to 1.4 we have derived an exact series solution of equation 1.1 for an optical wave propagating through the atmospheric turbulence. The series $\psi(\vec{x}) = \sum_{m=0}^{\infty} \varepsilon^m \psi_m(\vec{x})$ where $\psi_m(\vec{x})$ is given by 1.36 is not summable analytically. We shall now refer to approximations which allow us to find an analytical sum to the series $\psi(\vec{x})$ and then draw conclusions on the amplitude $A(\vec{x})$ and phase $\emptyset(\vec{x})$ of the wave function $u(\vec{x}) = A(\vec{x})\, e^{i\emptyset(\vec{x})}$ directly, rather than on statistical averages of these quantities.

<u>1.5.1 The Born approximation.</u> The Born or single scattering approximation $\psi_B(\vec{x})$ is obtained by keeping only the first term $\psi_1(\vec{x})$ given by 1.21 in the expression for $\psi(\vec{x})$, i.e.

$$\psi_B(\vec{x}) = \frac{u_B(\vec{x})}{u_o(\vec{x})} = 1 + i\varepsilon k \int_0^x dx' \int_{\vec{K}'} e^{-\frac{iK'^2}{2k}(x-x')}\, e^{i\vec{K}'\cdot\vec{r}}\, dN(\vec{K}',x') \tag{1.37}$$

The Born approximation is a good approximation for $\psi(\vec{x})$ only if $\varepsilon^2\overline{|\psi_1(\vec{x})|^2} \ll 1$ where the bar denotes an ensemble average defined in Section 1.1. We will show below that the above condition is satisfied for a length of propagation L through the atmospheric turbulence such that

$$\varepsilon^2 k^2 L_o L \ll 1 \tag{1.38}$$

where L_o is the outer scale of the turbulence. For an optical wave propagating through the atmosphere, the range of validity of the Born approximation is only a few meters.

<u>1.5.2 The geometrical optics approximation.</u> In this approximation we look for a solution to the wave equation in the near field of the smallest size scatterer of the random medium, i.e. at a distance L from the source such that $L \leq L_{CR}$ where L_{CR} is defined as follows:

$$L_{CR} = \frac{\ell_o^2}{\lambda} \tag{1.39}$$

where ℓ_o is the inner scale of the turbulence.

For a wavelength of 1μ, $L_{CR} \simeq 1m$. We make use of this approximation to evaluate the various terms $\psi_m(\vec{x})$ of $\psi(\vec{x})$. In the expression 1.21 of $\psi_1(L,\vec{r})$ we can neglect the term

$$\exp[i\,\frac{K'^2}{2k}(L-x')] \;,\; \text{since} \;\; \frac{K'^2}{2k}(L-x') \leq \frac{K'^2 L}{2k} \leq \left(\frac{2\pi}{\ell_o}\right)^2 \frac{L\lambda}{2\times 2\pi} = \pi\frac{\lambda L}{\ell_o^2} << 1$$

Therefore we can write, according to 1.4

$$\psi_1(L,\vec{r}) = ik\int_0^L dx' \int_{\vec{K}'} e^{i\vec{K}'\cdot\vec{r}}\, dN(\vec{K}',x') = ik\int_0^L dx'\, n_1(x',\vec{r}) \tag{1.40}$$

Similarly, we can neglect the term $\exp[-\frac{i}{2k}(K'^2(L-x') + K''^2(L-x'') + 2\vec{K}'\cdot\vec{K}''(L-x'))]$ in expression 1.29 for $\psi_2(\vec{x})$ and write

$$\psi_2(L,\vec{r}) = (ik)^2 \int_0^L dx' \int_0^{x'} dx'' \int_{\vec{K}'}\int_{\vec{K}''} e^{i(\vec{K}'+\vec{K}'')\cdot\vec{r}}\, dN(\vec{K}',x')\, dN(\vec{K}'',x'')$$

$$= (ik)^2 \int_0^L dx'\, n_1(x',\vec{r}) \int_0^{x'} dx''\, n_1(x'',\vec{r}) = \frac{1}{2}\left[ik\int_0^L dx'\, n_1(x',\vec{r})\right]^2 \tag{1.41}$$

Similarly, we can express $\psi_m(L,r)$ in the geometrical optics approximation as:

$$\psi_m(L,\vec{r}) = (ik)^m \int_0^L dx'\ n_1(x',\vec{r}) \int_0^{x'} dx''\ n_1(x'',\vec{r}) \cdots \int_0^{x^{(m-1)}} dx^{(m)} n_1(x^{(m)},\vec{r})$$

$$\psi_m(L,\vec{r}) = \frac{1}{m!}\left[\int_0^L dx'\ n_1(x',\vec{r})\right]^m \qquad (1.42)$$

The expressions 1.40, 1.41 and 1.42 for the terms $\psi_1(L,\vec{r})$, $\psi_2(L,\vec{r})$ and $\psi_m(L,\vec{r})$ lead to the following summation for $\psi_{(GO)}(L,\vec{r})$

$$\psi_{(GO)}(L,\vec{r}) = \sum_{m=0}^{\infty} \frac{(i\varepsilon k)^m}{m!}\left[\int_0^L dx'\ n_1(x',\vec{r})\right]^m$$

$$= \exp\left[i\varepsilon k \int_0^L dx'\ n_1(x',\vec{r})\right] \qquad (1.43)$$

under the condition $L \ll \frac{\ell_o^2}{\lambda}$. This expression is equivalent to the solution of reference (13) in the limits where the amplitude fluctuations are negligible.

1.5.3 The Rytov approximation (1). The Rytov approximation is used by many authors to solve the wave equation 1.1 . Its validity has been contested (6),(7). We shall discuss this approximation in more detail later in comparison to our results. At this point we notice that if a few terms are neglected in expression 1.36 for $\psi_m(\vec{x})$, the solution of the wave equation under the Rytov approximation is put in evidence.

In expression 1.36 we write

$$\sum_{q=1}^{q=m}\left(\sum_{p=1}^{p=m}\vec{K}^{(p)}\right)^2 \left(x^{(q-1)}-x^{(q)}\right)$$

$$=\sum_{p=1}^{m}(\vec{K}^{(p)})^2\,(x-x^{(p)})+2\sum_{p=1}^{m-1}\vec{K}^{(p)}\cdot\left(\sum_{q=1}^{m-p}\vec{K}^{(p+q)}\right)(x-x^{(p)}) \qquad (1.44)$$

For better comprehension we write down these summations explicitly for m=3

$$\sum_{q=1}^{3}\left(\sum_{p=1}^{3}\vec{K}^{(p)}\right)^2 \left(x^{(q-1)}-x^{(q)}\right) = (\vec{K}'+\vec{K}''+\vec{K}''')^2(x-x')+$$

$$+\ (\vec{K}''+\vec{K}''')^2\,(x'-x'')\ +\ \vec{K}'''^2(x''-x''')$$

$$= K'^2(x-x')+K''^2(x-x'')+K'''^2(x-x''')+2\vec{K}'\cdot(\vec{K}''+\vec{K}''')(x-x')$$

$$+\ 2\vec{K}''\cdot\vec{K}'''\,(x-x'')\ .$$

If, for some reason, the terms containing dot products of spatial wave vectors $\vec{K}^{(p)}\cdot\vec{K}^{(p+q)}$ are dropped in 1.44 and 1.36, $\psi_m(\vec{x})$ can be written as

$$\psi_m(\vec{x}) = (ik)^m\int_0^x dx'\int_0^{x'} dx''\cdots\int_0^{x^{(m-1)}} dx^{(m)}\int_{\vec{K}'}\int_{\vec{K}''}\cdots\int_{\vec{K}^{(m)}}\exp\left[i\sum_{p=1}^{p=m}\vec{K}^{(p)}\cdot\vec{r}\right]$$

$$\times\ \exp\left[\left(\frac{-i}{2k}\right)\sum_{p=1}^{m}(\vec{K}^{(p)})^2\,(x-x^{(p)})\right]\left(\prod_{p=1}^{m} dN(\vec{K}^{(p)},x^{(p)})\right)$$

$$= (ik)^m\int_0^x dx'\int_0^{x'} dx''\cdots\int_0^{x^{(m-1)}} dx^{(m)}\left(\prod_{p=1}^{m}\int_{\vec{K}^{(p)}}\exp\left[\frac{-i}{2k}(\vec{K}^{(p)})^2(x-x^{(p)})\right]\right.$$

$$\left.\times\ e^{i\vec{K}^{(p)}\cdot\vec{r}}\,dN(\vec{K}^{(p)},x^{(p)})\right)$$

Let us define $f(x^{(p)})$ by

$$f(x^{(p)}) = \int_{\vec{K}^{(p)}} \exp\left[\frac{-i}{2k}\,(\vec{K}^{(p)})^2 (x-x^{(p)})\right] e^{i\vec{K}^{(p)}\cdot\vec{r}}\, dN(\vec{K}^{(p)},x^{(p)}) \tag{1.45}$$

Then we can express $\psi_m(\vec{x})$ as

$$\psi_m(\vec{x}) = (ik)^m \int_0^x dx'\; f(x') \int_0^{x'} dx''\; f(x'')\cdots \int_0^{x^{(p-1)}} dx^{(p)}\, f(x^{(p)}) \times \cdots \int_0^{x^{(m-1)}} dx^{(m)}\, f(x^{(m)}) \tag{1.46}$$

It can be seen that the m-dimensional integral in 1.46 with the above limits is performed in a m-dimensional volume equal to $1/m!$, the volume of the m-dimensional "cube" of size x and yields

$$\psi_m(\vec{x}) = \frac{(ik)^m}{m!}\left[\int_0^x dx'\; f(x')\right]^m \tag{1.47}$$

$$\psi(\vec{x}) = \sum_{m=0}^{\infty} \varepsilon^m \psi_m(\vec{x}) = \exp\left[i\varepsilon k \int_0^x dx' \int_{\vec{K}'} e^{-i\frac{K'^2}{2k}(x-x')} \times e^{i\vec{K}'\cdot\vec{r}}\; dN(\vec{K}',x')\right] \tag{1.48}$$

$\psi(\vec{x})$ given by 1.48 is the solution of the wave equation under the Rytov approximation. We have obtained this solution by neglecting arbitrarily certain terms in the expression of $\psi_m(\vec{x})$. There is no justification for neglecting these terms. The Rytov approximation, as we shall see later, leads to erroneous information about some statistical averages of the wave function $u(\vec{x})$.

We have reviewed three types of approximations commonly made in order to solve the wave equation 1.1: The Born approximation, the geometrical optics approximation, and the Rytov approximation. The solutions corresponding to these approximations have been put in evidence by neglecting various factors in each term $\psi_m(\vec{x})$ of the series expansion $\psi(\vec{x}) = \sum_{m=0}^{\infty} \varepsilon^m \psi_m(\vec{x})$. (The exact solution to the wave equation is $u(\vec{x}) = A_o e^{ikx} \psi(\vec{x})$).

The motivation for the use of these approximations is to give a simple and analytical expression for $u(\vec{x})$ and therefore simple and analytical expressions for the amplitude $A(\vec{x})$ and the phase $\phi(\vec{x})$ by separating $u(\vec{x})$ into its real and imaginary parts. Conclusions about the statistics of the amplitude and phase of the wave separately can then be drawn.

A look at the formal exact solution $\psi(\vec{x}) = \sum_{m=0}^{\infty} \varepsilon^m \psi_m(\vec{x})$ where $\psi_m(\vec{x})$ is given by 1.36 shows us that it is hopeless to try and separate the wave function into its phase and amplitude without some approximation and therefore to calculate the statistical properties of the phase and the amplitude of the wave separately. However, we can hope to deduce statistical information about the wave function $u(\vec{x})$ itself. In the following parts of this report we shall evaluate the mean $\overline{u(\vec{x})}$, the correlation function of the wave function $\overline{u(L,\vec{r}_1)\, u^*(L,\vec{r}_2)}$ between two points $(L,\vec{r}_1)$ and $(L,\vec{r}_2)$ in a plane perpendicular to the direction of propagation of the wave at a distance L from the source, and we shall investigate the correlation function of the wave intensity $I(L,\vec{r})$ between these two points

$$\overline{I(L,\vec{r}_1)\, I(L,\vec{r}_2)} = \overline{u(L,\vec{r}_1)\, u^*(L,\vec{r}_1)\, u(L,\vec{r}_2)\, u^*(L,\vec{r}_2)} \; .$$

The asterisk denotes the complex conjugate.

CHAPTER II - CALCULATION OF $\overline{u(\vec{x})}$

We have, in Chapter I, found a formal solution of the wave equation $[\nabla^2 + k^2 + 2\varepsilon k^2 n_1(\vec{x})]\, u(\vec{x}) = 0$ as a power series expansion $u(\vec{x}) = A_o e^{ikx}\, \psi(\vec{x}) = A_o e^{ikx} \sum_{m=0}^{\infty} \varepsilon^m \psi_m(\vec{x})$ for an optical wave propagating through atmospheric turbulence. An expression for each term $\psi_m(\vec{x})$ of the series has been given, but it has not been possible to find a useful analytical sum to the series $\psi(\vec{x})$ without using some approximation. Therefore, no information on the phase and the amplitude of the wave is available directly from the series $\psi(\vec{x})$.

We now proceed to calculate the mean of the wave function $\overline{u(\vec{x})}$ or, rather, the statistical average $\overline{\psi(\vec{x})}$ of the sum $\psi(\vec{x})$, since

$$\overline{u(\vec{x})} = A_o\, e^{ikx}\, \overline{\psi(\vec{x})} \tag{2.1}$$

where

$$\overline{\psi(\vec{x})} = \sum_{m=0}^{\infty} \varepsilon^m\, \overline{\psi_m(\vec{x})} \tag{2.2}$$

In order to do so, some information on the nature of the statistics of the turbulent medium is needed. In the following paragraph the statistical properties of the random refractive index field are stated.

2.1 Statistical Properties of the Random Index of Refraction

The only random quantities appearing in $\psi_m(\vec{x})$ are the two-dimensional Fourier Stieljes transforms $dN(\vec{K}^{(p)}, x^{(p)})$ of the fluctuation of the index of refraction $n_1(x^{(p)}, \vec{r})$. The random process $dN(\vec{K}^{(p)}, x^{(p)})$ satisfies the following relations.

$$\overline{dN(\vec{K}^{(p)},x^{(p)})} = 0 \tag{2.3}$$

$$\overline{dN(\vec{K}^{(p)},x^{(p)})\,dN^*(\vec{K}^{(q)},x^{(q)})}$$

$$= \delta(\vec{K}^{(p)} - \vec{K}^{(q)})\; F_n(\vec{K}^{(p)},|x^{(p)} - x^{(q)}|)\,d\vec{K}^{(p)}d\vec{K}^{(q)} \tag{2.4}$$

where $F_n(\vec{K}^{(p)},|x^{(p)} - x^{(q)}|)$ is the transverse two-dimensional Fourier transform of the correlation function of the fluctuation $n_1(\vec{x})$ of the index of refraction, also called the two-dimensional spectral density function of $n_1(\vec{x})$. The correlation function $B_{n_1}(x_1,\vec{r}_1,x_2,\vec{r}_2)$ of $n_1(\vec{x})$ between two points $(x_1,\vec{r}_1)$ and $(x_2,\vec{r}_2)$ can be expressed as follows according to 1.4:

$$B_{n_1}(x_1,\vec{r}_1,x_2,\vec{r}_2) = \overline{n_1(x_1,\vec{r}_1)n_1(x_2,\vec{r}_2)}$$

$$= \int_{\vec{K}_1}\int_{\vec{K}_2} e^{i(\vec{K}_1\cdot\vec{r}_1 - \vec{K}_2\cdot\vec{r}_2)}\;\overline{dN(\vec{K}_1,x_1)\;dN^*(\vec{K}_2,x_2)}$$

and the transverse two-dimensional $F_n(\vec{K},|x_1 - x_2|)$ is defined by the following relation

$$B_{n_1}(x_1,\vec{r}_1,x_2,\vec{r}_2) = \int_{\vec{K}} e^{i\vec{K}\cdot(\vec{r}_1 - \vec{r}_2)}\; F_n(\vec{K}),|x_1 - x_2|)\;d\vec{K}$$

In order for the last two expressions of the correlation function to be compatible, the identity 2.4 must hold, i.e.

$$\overline{dN(\vec{K}_1,x_1)\;dN^*(\vec{K}_2,x_2)} = \delta(\vec{K}_1 - \vec{K}_2)\;F_n(\vec{K}_1|x_1 - x_2|)d\vec{K}_1\;d\vec{K}_2 \quad .$$

The two-dimensional Fourier transform $F_n(\vec{K},x)$ of $n_1(\vec{x})$ is related to the three-dimensional Fourier transform $\phi_n(\vec{K},K_x)$ by the relation

$$F_n(\vec{K},x) = \int_{-\infty}^{\infty} e^{iK_x x} \phi_n(\vec{K},K_x)\, dK_x$$

where

$$B_{n_1}(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dK_x dK_y dK_z\, e^{i(K_x x + K_y y + K_z z)} \phi_n(K_x,K_y,K_z)$$

It is assumed that the random process $dN(\vec{K},x)$ is a Gaussian random process and therefore possesses a multivariate Gaussian distribution. Its higher moments are then given by the following relations.

$$\overline{\prod_{p=1}^{m} dN(\vec{K}^{(p)},x^{(p)})} = 0 \qquad \text{if } m \text{ is odd}$$

$$\overline{\prod_{p=1}^{m} dN(\vec{K}^{(p)},x^{(p)})} = \text{sum of all the different permutations}$$

of m products $\overline{dN(\vec{K}^{(p)},x^{(p)})\, dN(\vec{K}^{(q)},x^{(q)})}$ if m is even (2.5)

There are $\dfrac{m!}{(m/2)!\; 2^{m/2}}$ such permutations. For better comprehension we write explicitly the above products for $m=3$ and $m=4$,

$$(m=3) \qquad \overline{dN_1\, dN_2\, dN_3} = 0$$

$$(m=4) \qquad \overline{dN_1\, dN_2\, dN_3\, dN_4} = \overline{dN_1\, dN_2} \cdot \overline{dN_3\, dN_4} + \overline{dN_1\, dN_3} \cdot \overline{dN_2\, dN_4} + \overline{dN_1\, dN_4} \cdot \overline{dN_2\, dN_3}$$

where the following notation has been used $dN(\vec{K}^{(p)},x^{(p)}) \equiv dN_p$.

According to relationships 2.5, the only terms which will contribute to $\overline{\psi(\vec{x})}$ will be the terms of 2.2 with m even, then

$$\overline{\psi(\vec{x})} = \sum_{p=0}^{\infty} \varepsilon^{2p} \overline{\psi_{2p}(\vec{x})} \tag{2.6}$$

with $\overline{\psi_o(\vec{x})} = 1$

We shall calculate a few terms in expansion 2.6 and show how these terms lead to an exact analytical sum for $\overline{\psi(\vec{x})}$. The first term in the series expansion for $\overline{\psi(\vec{x})}$ is $\overline{\psi_2(\vec{x})}$.

2.2 Calculation of $\overline{\psi_2(\vec{x})}$

$\psi_2(\vec{x})$ is given by expression 1.28. The random quantity appearing in $\psi_2(\vec{x})$ is $dN(\vec{K}',x')dN(\vec{K}'',x'')$; its statistical average is, according to 2.4,

$$\overline{dN(\vec{K}',x')\, dN(\vec{K}'',x'')} = \delta(\vec{K}'+\vec{K}'')\, F_n(\vec{K}',|x'-x''|)\, d\vec{K}'\, d\vec{K}'' .$$

We make use of the above relation to calculate $\overline{\psi_2(\vec{x})}$ and we perform the integration with respect to $\vec{K}''$, yielding

$$\overline{\psi_2(\vec{x})} = (ik)^2 \int_0^x dx' \int_0^{x'} dx'' \int_{-\infty}^{\infty} d\vec{K}' \exp[\frac{-iK'^2}{2k}(x'-x'')]F_n(\vec{K}',|x'-x''|) \tag{2.7}$$

$F_n(\vec{K}',|x'-x''|)$ is the two-dimensional spectral density function of the random index of refraction. It is a measure of the correlation of the index of refraction between points in the plane $X = x'$ and the plane $X = x''$, see Fig. 2.1. Clearly, this correlation is produced only by the inhomogeneities in the index of refraction which have a size ℓ larger than the distance between these two planes $\ell = |x' - x''|$.

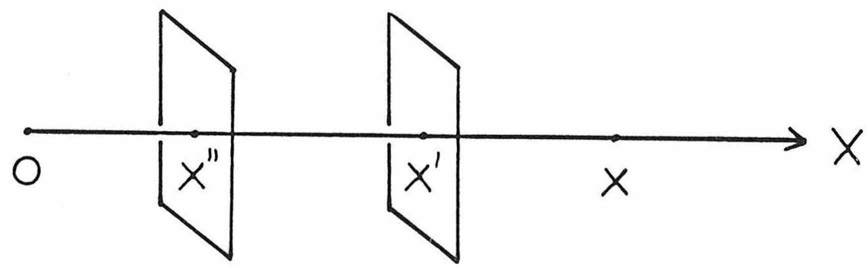


Fig. 2.1 The two-dimensional Fourier transform $F_n(\vec{K}, |x'-x''|)$ is a measure of the correlation of the index of refraction in two planes perpendicular to the direction of propagation of the wave

The wave number K' is related to the scale of the turbulence ℓ by $K' = 2\pi/\ell$. Therefore, $F_n(\vec{K}', |x'-x''|)$ will be appreciably different from zero only if the condition $K'|x' - x''| = 1$, equation 2.8, is satisfied. Therefore we can neglect the exponential term in the expression of $\overline{\psi_2(\vec{x})}$, since according to 2.8 and 1.22

$$\frac{K'^2}{2k}(x' - x'') \leq \frac{K'}{2k} \leq \frac{\lambda}{2\ell_o} \ll 1$$

This simplification is again due to the fact that the wavelength of the propagating wave is much smaller than the inner scale of the turbulence ℓ_o . We can then write

$$\overline{\psi_2(\vec{x})} = - k^2 \int_0^x dx' \int_0^{x'} dx'' \int_{-\infty}^{\infty} d\vec{K}' \; F_n(\vec{K}', |x' - x''|) \tag{2.9}$$

or, for later convenience

$$\overline{\psi_2(\vec{x})} = -k^2 \int_0^x dx' \int_0^{x'} dx'' \, f(|x'- x''|) = -\frac{k^2}{2} \int_0^x dx' \int_0^x dx'' \, f(|x'- x''|) \tag{2.10}$$

the function $f(|X|)$ is defined by the following relation

$$f(|X|) = \int_{-\infty}^{\infty} d\vec{K}' \, F_n(\vec{K}',|X|) \tag{2.11}$$

We shall now proceed to calculate the term of order ε^4 in the expansion of $\overline{\psi(\vec{x})}$, i.e. $\overline{\psi_4(\vec{x})}$.

2.3 Calculation of $\overline{\psi_4(\vec{x})}$

The calculation of the term of order ε^4 in the expansion of $\overline{\psi(\vec{x})}$ is a little more tedious than the calculation of $\overline{\psi_2(\vec{x})}$, but still straightforward. $\psi_4(\vec{x})$ is given by expression 1.36 with $n=4$, then

$$\overline{\psi_4(\vec{x})} = (ik)^4 \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^{x_3} dx_4 \int_{\vec{K}_1} \int_{\vec{K}_2} \int_{\vec{K}_3} \int_{\vec{K}_4} e^{(\vec{K}_1+\vec{K}_2+\vec{K}_3+\vec{K}_4)\cdot\vec{r}}$$

$$\times \exp\left[\frac{-i}{2k} \left\{(\vec{K}_1+\vec{K}_2+\vec{K}_3+\vec{K}_4)^2 \, (x-x_1) + (\vec{K}_2+\vec{K}_3+\vec{K}_4)^2 \, (x_1-x_2) + (\vec{K}_3+\vec{K}_4)^2 \, (x_2-x_3)\right.\right.$$

$$\left.\left. + K_4^2(x_3-x_4)\right\}\right] \overline{dN(\vec{K}_1,x_1) \, dN(\vec{K}_2,x_2) \, dN(\vec{K}_3,x_3) \, dN(\vec{K}_4,x_4)} \tag{2.12}$$

where, according to 2.5

$$\overline{dN_1 dN_2 dN_3 dN_4} = \overline{dN_1 dN_2} \cdot \overline{dN_3 dN_4} + \overline{dN_1 dN_3} \cdot \overline{dN_2 dN_4} + \overline{dN_1 dN_4} \cdot \overline{dN_2 dN_3} \tag{2.13}$$

The product $\overline{dN_1 dN_2} \cdot \overline{dN_3 dN_4}$ is proportional to $F_n(\vec{K}_1, |x_1 - x_2|)$ $F_n(\vec{K}_3, |x_3 - x_4|)$ i.e., proportional to the product of the correlation functions of the index of refraction between the point x_1 and x_2, x_3 and x_4.

$\overline{dN_1 dN_3} \cdot \overline{dN_2 dN_4} \propto F_n(\vec{K}_1, |x_1 - x_3|)\, F_n(\vec{K}_2, |x_2 - x_4|)$ and

$\overline{dN_1 dN_4} \cdot \overline{dN_2 dN_3} \propto F_n(\vec{K}_1, |x_1 - x_4|)\, F_n(\vec{K}_2, |x_2 - x_3|)$ where the following inequalities hold for x, x_1, x_2, x_3 and x_4

$$0 \leq x_4 \leq x_3 \leq x_2 \leq x_1 \leq x \tag{2.14}$$

Therefore we expect the term $\overline{12} \cdot \overline{34}$ to give a larger contribution to the integrals in 2.12 than the term $\overline{13} \cdot \overline{24}$ because the correlations of the index of refraction between the points x_1 and x_2 and the points x_3 and x_4 are larger than the correlations of the index of refraction between the points x_1 and x_3, and the points x_2 and x_4 since $|x_1 - x_2| \leq |x_1 - x_3|$ and $|x_3 - x_4| \leq |x_2 - x_4|$. But it is not clear that the term $\overline{12} \cdot \overline{34}$ will give a larger contribution than the term $\overline{14} \cdot \overline{23}$ since in the integrals of 2.12 all the limits x_1, x_2, x_3 will actually reach x.

Some authors (5),(14) have retained only the term $\overline{12} \cdot \overline{34}$ and neglected the terms $\overline{13} \cdot \overline{24}$ and $\overline{14} \cdot \overline{23}$ in the averaging procedure, i.e.

they have kept only the terms which represent the correlations between neighboring points. They have approximated

$$\overline{\prod_{p=1}^{m} dN(\vec{K}_p, x_p)}$$

as follows, when m is even

$$\overline{\prod_{p=1}^{m} dN(\vec{K}_p, x_p)} \equiv \overline{\prod_{p=1}^{m} dN_p} = \overline{dN_1 dN_2} \cdot \overline{dN_3 dN_4} \cdot --- \cdot \overline{dN_{m-1} dN_m} \qquad (2.15)$$

In this report we shall not make use of approximation 2.15 to calculate $\overline{\psi(\vec{x})}$. The correct expression of $\overline{\psi(\vec{x})}$ we shall obtain, will be compared in Appendix A with the expression obtained by using approximation 2.15.

We now calculate the contribution of each term of 2.13 to $\overline{\psi_4(\vec{x})}$.

2.3.1 Contribution of $\overline{dN_1 dN_2} \cdot \overline{dN_3 dN_4} \equiv \overline{12} \cdot \overline{34}$. We call this contribution $\psi_{4a}(\vec{x})$. According to 2.4,

$$\overline{dN_1 dN_2} \cdot \overline{dN_3 dN_4} = \delta(\vec{K}_1 + \vec{K}_2)\ \delta(\vec{K}_3 + \vec{K}_4)\ F_n(\vec{K}_1, |x_1 - x_2|)$$

$$F_n(\vec{K}_3, |x_3 - x_4|)\ d\vec{K}_1\ d\vec{K}_2\ d\vec{K}_3\ d\vec{K}_4$$

We use this relationship in the expression 2.12 for $\overline{\psi_4(\vec{x})}$, and perform the integrations with respect to $\vec{K}_2$ and $\vec{K}_4$ to find

$$\overline{\psi_{4a}(\vec{x})} = k^4 \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^{x_3} dx_4 \int_{-\infty}^{\infty} d\vec{K}_1 \int_{-\infty}^{\infty} d\vec{K}_3 \;\exp[\tfrac{-i}{2k}(K_1^2(x_1-x_2)$$

$$+ K_3^2(x_3-x_4))]\; F_n(\vec{K}_1,|x_1-x_2|)\; F_n(\vec{K}_3,|x_3-x_4|) \qquad (2.16)$$

We can neglect the exponential terms in 2.16 for the same reason which was already given in the calculation of $\overline{\psi_2(\vec{x})}$, i.e. $F_n(\vec{K}_1,|x_1-x_2|)$ and $F_n(\vec{K}_3,|x_3-x_4|)$ are appreciably different from zero only if $K_1|x_1-x_2| \lesssim 1$ and $K_3|x_3-x_4| \lesssim 1$. Therefore the coefficients of the exponentials in 2.16 are such that

$$\frac{K_1^2}{2k}(x_1-x_2) \lesssim \frac{K_1}{2k} \lesssim \frac{\lambda}{2\ell_o} << 1 \quad , \qquad \text{and}$$

$$\frac{K_3^2}{2k}(x_3-x_4) \lesssim \frac{K_3}{2k} \lesssim \frac{\lambda}{2\ell_o}, \quad << 1$$

We then obtain the following expression for $\overline{\psi_{4a}(\vec{x})}$

$$\overline{\psi_{4a}(\vec{x})} = k^4 \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^{x_3} dx_4 \int_{-\infty}^{\infty} d\vec{K}_1\; F_n(\vec{K}_1,|x_1-x_2|)$$

$$\times \int_{-\infty}^{\infty} d\vec{K}_3\; F_n(\vec{K}_3,|x_3-x_4|) \qquad (2.17)$$

or in terms of notation 2.11

$$\overline{\psi_{4a}(\vec{x})} = k^4 \int_0^x dx_1 \int_0^{x_1} dx_2\; f(|x_1-x_2|) \int_0^{x_2} dx_3 \int_0^{x_3} dx_4\; f(|x_3-x_4|) \qquad (2.18)$$

2.3.2 Contribution of $\overline{dN_1 dN_3} \cdot \overline{dN_2 dN_4} \equiv \overline{13} \cdot \overline{24}$ We call this contribution $\psi_{4b}(\vec{x})$. According to 2.4,

$$\overline{dN_1 dN_3} \cdot \overline{dN_2 dN_4} = \delta(\vec{K}_1 + \vec{K}_3)\ \delta(\vec{K}_2 + \vec{K}_4)\ F_n(\vec{K}_1, |x_1 - x_3|)\ F_n(\vec{K}_2, |x_2 - x_4|)$$

$$\times \quad d\vec{K}_1 d\vec{K}_2 d\vec{K}_3 d\vec{K}_4$$

After performing the integrations with respect to $\vec{K}_3$ and $\vec{K}_4$, we can write

$$\overline{\psi_{4b}(\vec{x})} = k^4 \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^{x_3} dx_4 \int_{-\infty}^{\infty} d\vec{K}_1 \int_{-\infty}^{\infty} d\vec{K}_2 \left\{ \exp\left[\frac{-i}{2k}(K_1^2(x_1 - x_3)\right.\right.$$

$$\left.\left. + K_2^2(x_2 - x_4))\right]\ F_n(\vec{K}_1, |x_1 - x_3|)\ F_n(\vec{K}_2, |x_2 - x_4|)\ d\vec{K}_1 d\vec{K}_2 \right\} \qquad (2.19)$$

We can neglect the exponential terms in 2.19 as in the calculation of $\overline{\psi_{4a}(\vec{x})}$ which yields

$$\overline{\psi_{4b}(\vec{x})} = k^4 \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^{x_3} dx_4\ f(|x_1 - x_3|)\ f(|x_2 - x_4|) \qquad (2.20)$$

where we have used notation 2.11 for $f(|X|)$.

2.3.3 Contribution of $\overline{dN_1 dN_4} \cdot \overline{dN_2 dN_3} \equiv \overline{14} \cdot \overline{23}$ We shall call it $\overline{\psi_{4c}(\vec{x})}$. According to 2.4,

$$\overline{dN_1 dN_4} \cdot \overline{dN_2 dN_3} = \delta(\vec{K}_1 + \vec{K}_4)\ \delta(\vec{K}_2 + \vec{K}_3)\ F_n(\vec{K}_1, |x_1 - x_4|)\ F_n(\vec{K}_2, |x_2 - x_3|)$$

$$\times \quad d\vec{K}_1 d\vec{K}_2 d\vec{K}_3 d\vec{K}_4$$

The integrations with respect to $\vec{K}_3$ and $\vec{K}_4$ are then performed in 2.12 to yield

$$\overline{\psi_{4c}(\vec{x})} = k^4 \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^{x_3} dx_4 \quad f(|x_1-x_4|)\, f(|x_2-x_3|) \qquad (2.21)$$

where again $f(|X|)$ is defined by 2.11.

2.3.4 Expression for $\overline{\psi_4(\vec{x})}$.

$\overline{\psi_4(\vec{x})} = \overline{\psi_{4a}(\vec{x})} + \overline{\psi_{4b}(\vec{x})} + \overline{\psi_{4c}(\vec{x})}$ or from 2.18, 2.20 and 2.21,

$$\overline{\psi_4(\vec{x})} = k^4 \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^{x_3} dx_4 \int_0^{x_4} f(|x_1-x_2|)\, f(|x_3-x_4|)$$

$$+ f(|x_1-x_3|)\, f(|x_2-x_4|) + f(|x_1-x_4|)\, f(|x_2-x_3|) \qquad (2.22)$$

The four-fold integration in 2.22 is performed in a four-dimensional volume equal to $1/4! = 1/24$, the volume of a four-dimensional "cube" with sides equal to x. We know that the following equality is true

$$\int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^{x_3} dx_4 \sum_{i,j,k,\ell} G(x_i, x_j, x_k, x_\ell)$$

$$= \int_0^x dx_1 \int_0^x dx_2 \int_0^x dx_3 \int_0^x dx_4 \; G(x_i, x_j, x_k, x_\ell) \qquad (2.23)$$

where $G(x_i, x_j, x_k, x_\ell)$ is any function defined in the four-dimensional "cube" with sides equal to x. The summation $\sum_{i,j,k,\ell} G(x_i, x_j, x_k, x_\ell)$

stands for all the possible G functions evaluated for all the permutations f,i,j,k and ℓ where the indices i,j,k and ℓ run from 1 to 4 and are all different in any one term of the sum. There are 4! such terms. In our case $G(x_i,x_j,x_k,x_\ell) = f(|x_i-x_j|)\, f(|x_k-x_\ell|)$ and

$$f(|x_1-x_2|)\, f(|x_3-x_4|) + f(|x_1-x_3|)\, f(|x_2-x_4|) + f(|x_1-x_4|)\, f(|x_2-x_3|)$$

$$= \frac{1}{8} \sum_{i,j,k,\ell} f(|x_i-x_j|)\, f(|x_k-x_\ell|) \tag{2.24}$$

where $\sum_{i,j,k,\ell} f(|x_i-x_j|)\, f(|x_k-x_\ell|)$ has been defined above. Therefore, with the help of 2.23 and 2.24 we can express $\overline{\psi_4(\vec{x})}$ as follows

$$\overline{\psi_4(\vec{x})} = \frac{k^4}{8} \int_0^x dx_1 \int_0^x dx_2 \int_0^x dx_3 \int_0^x dx_4\; f(|x_1-x_2|)\, f(|x_3-x_4|)$$

or

$$\overline{\psi_4(\vec{x})} = \frac{1}{2} \left[\frac{k^2}{2} \int_0^x dx_1 \int_0^x dx_2\; f(|x_1-x_2|)\right]^2 = \frac{1}{2}\, (\overline{\psi_2(\vec{x})})^2 \tag{2.25}$$

where $\overline{\psi_2(\vec{x})}$ is given by 2.10.

We have calculated the three first terms in the series expansion 2.6 for $\overline{\psi(\vec{x})}$, the statistical mean of the wave function in the presence of a turbulent propagating medium normalized to the wave function in the absence of a turbulence. We shall now calculate the general term of the series, i.e. $\overline{\psi_{2m}(\vec{x})}$.

2.4 Calculation of $\overline{\psi_{2m}(\vec{x})}$

$\psi_{2m}(\vec{x})$ is given for any m by expression 1.36. The calculation of $\overline{\psi_{2m}(\vec{x})}$ involves the following averaging

$$\overline{\prod_{p=1}^{p=2m} dN_p} = \text{sum of all the different permutations of } 2m \text{ products } \overline{dN_p dN_q}$$

where $\overline{dN_p dN_q} = \delta(\vec{K}_p + \vec{K}_q)\, F_n(\vec{K}_p, |x_p - x_q|)\, d\vec{K}_p d\vec{K}_q$. There are $(2m)!/m!2^m$ such products. After the integration with respect to $\vec{K}_q$ for example, any product $\overline{dN_p dN_q}$ gives the contribution $f(|x_p - x_q|)$ to the integral of $\overline{\psi_{2m}(\vec{x})}$ where $f(|X|) = \iint_{-\infty}^{\infty} d\vec{K}\, F_n(\vec{K}, |X|)$. The exponential terms in the expression of $\psi_{2m}(\vec{x})$ can be neglected for the reasons described above in the calculation of $\overline{\psi_2(\vec{x})}$, $(\lambda \ll \ell_0)$. $\overline{\psi_{2m}(\vec{x})}$ is then expressed as follows

$$\overline{\psi_{2m}(\vec{x})} = (-1)^m k^{2m} \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{2m-2}} dx_{2m-1} \int_0^{x_{2m-1}} dx_{2m}$$

$$\times\; P_1\Big\{ \prod_{p \neq q} f(|x_p - x_q|) \Big\} \qquad (2.26)$$

$\prod_{p \neq q} f(|x_p - x_q|)$ stands for the product of m terms $f(|x_p - x_q|)$ where the indices p and q are different and run from 1 to 2m. The symbol $P_1\{\prod_{p \neq q} f(|x_p - x_q|)\}$ stands for the sum of all permutations of the terms $\prod_{p \neq q} f(|x_p - x_q|)$ obtained by interchanging any two variables x_i and x_j in such a way as to give a different product. There are

$2m!/m!2^m$ such terms. An example of such a term is

$$f(|x_1-x_2|)\, f(|x_3-x_4|) \cdots f(|x_{2m-1} - x_{2m}|)$$

all the other terms of $P_1\{\ \}$ are obtained by interchanging the position of any two variables x_i and x_j in such a way as to give a different product. Interchanging x_1 and x_2, for example, would give the same product, since $f(|x_1-x_2|) = f(|x_2-x_1|)$. There are 2^m possibilities of interchanging any pair x_i and x_j in this way without changing the product. Interchanging x_1 and x_2 with x_3 and x_4 would not change the product, since $f(|x_1-x_2|)f(|x_3-x_4|) = f(|x_3-x_4|)\, f(|x_1-x_2|)$. There are $m!$ possibilities of interchanging any pair x_i and x_j without changing the product; therefore there are $2^m m!$ possibilities of interchanging any two variables x_i and x_j in any of the $(2m)!/2^m m!$ products without changing the product. To clarify the argument let us present explicitly the case $m=2$, then if we call $f(|x_i-x_j|) = \overline{ij}$, we can write the

$$\frac{(2m)!}{2^m m!} = \frac{4!}{2^2 \cdot 2!} = 3$$

terms of $P_1\{\ \}$ as $\overline{12}\cdot\overline{34} + \overline{13}\cdot\overline{24} + \overline{14}\cdot\overline{23}$. We can interchange the indices 1,2,3 and 4 in $2^m m! = 8$ ways in any of these products without changing the product, for example,

$$\overline{13}\cdot\overline{24} = \frac{1}{8}(\overline{13}\cdot\overline{24} + \overline{31}\cdot\overline{24} + \overline{31}\cdot\overline{42} + \overline{13}\cdot\overline{42} + \overline{24}\cdot\overline{13} + \overline{24}\cdot\overline{31} + \overline{42}\cdot\overline{31} + \overline{42}\cdot\overline{13})$$

therefore $\overline{12}\cdot\overline{34} + \overline{13}\cdot\overline{24} + \overline{14}\cdot\overline{23} = 1/8$ (sum of all the possible permutations of the indices i,j,k and ℓ in the product $\overline{ij}\cdot\overline{k\ell}$). There are 4! possible permutations.

In the same way for any m

$$P_1\{\prod_{p\neq q} f(|x_p-x_q|)\} = \frac{1}{m!\,2^m} P\{\prod_{p\neq q} f(|x_p-x_q|)\} \tag{2.27}$$

where $P\{\prod_{p\neq q} f(|x_p-x_q|)\}$ stands for the sum of all possible permutations of the indices in the product $\prod_{p\neq q} f(|x_p-x_q|)$, there are $(2m)!$ terms in this sum. According to 2.26 and 2.27

$$\overline{\Psi_{2m}(\vec{x})} = (-1)^m \frac{k^{2m}}{m!\;2^m} \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{2m-2}} dx_{2m-1} \int_0^{x_{2m-1}} dx_{2m}$$

$$\times\; P\{\prod_{p\neq q} f(|x_p-x_q|)\} \tag{2.28}$$

The above integration of the sum of all possible permutations of the coordinates $x_1, x_2, \cdots, x_{2m-1}, x_{2m}$ in a volume equal to $1/(2m)!$, the volume of a 2m-dimensional "cube" with sides equal to x, is equal to the integration of any one term of the permutation integrated over the 2m-dimensional cube, i.e.

$$\overline{\psi_{2m}(\vec{x})} = (-1)^m \frac{k^{2m}}{m!\;2^m} \int_0^x dx_1 \int_0^x dx_2 \int_0^x dx_3 \int_0^x dx_4 \cdots \int_0^{x_{2m-2}} dx_{2m-1} \int_0^{x_{2m-1}} dx_{2m}$$

$$\times\; (f(|x_1-x_2|)\; f(|x_3-x_4|) \cdots f(|x_{2m-1}-x_{2m}|))$$

or

$$\overline{\psi_{2m}(\vec{x})} = \frac{1}{m!}\left(-\frac{k^2}{2}\int_0^x dx_1 \int_0^x dx_2\ f(|x_1-x_2|)\right)^m = \frac{1}{m!}\left(\overline{\psi_2(\vec{x})}\right)^m \qquad (2.29)$$

where $\overline{\psi_2(\vec{x})}$ is given by 2.10

We have calculated the first three terms and the general term of the series expansion of $\overline{\psi(\vec{x})}$. These results are used in the next paragraph to find an analytical expression for $\overline{\psi(\vec{x})}$, therefore for $\overline{u(\vec{x})}$.

2.5 Calculation of $\overline{u(\vec{x})}$

According to 2.6, $\dfrac{\overline{u(\vec{x})}}{u_o(\vec{x})} = \overline{\psi(\vec{x})} = \sum\limits_{m=0}^{\infty} \varepsilon^{2m}\ \overline{\psi_{2m}(\vec{x})}$ or, with the help of 2.25 and 2.29,

$$\overline{\psi(\vec{x})} = 1 + \varepsilon^2\overline{\psi_2(\vec{x})} + \frac{\varepsilon^4}{2}\left(\overline{\psi_2(\vec{x})}\right)^2 + \cdots + \frac{1}{m!}\left(\varepsilon^2\overline{\psi_2(\vec{x})}\right)^m + \cdots$$

therefore,

$$\overline{\psi(\vec{x})} = \exp(\varepsilon^2\ \overline{\psi_2(\vec{x})}) \qquad (2.30)$$

with $$\overline{\psi_2(\vec{x})} = -\frac{k^2}{2}\int_0^x dx_1 \int_0^x dx_2\ f(|x_1-x_2|) = -\frac{k^2}{2}\int_0^x dx_1 \int_0^x dx_2 \int_{-\infty}^{\infty} d\vec{K} \times F_n(\vec{K},|x_1-x_2|)$$

The first statistical moment of an optical plane wave propagating a distance L through a randomly turbulent medium is given by the following expression in a plane perpendicular to the direction of propagation

$$\overline{u(L,\vec{r})} = A_o e^{ikL} \exp\left(- \frac{\varepsilon^2 k^2}{2} \int_0^L dx_1 \int_0^L dx_2 \int_{-\infty}^{\infty} d\vec{K}\, F_n(\vec{K},|x_1-x_2|)\right) \quad (2.31)$$

where $F_n(\vec{K},|x_1-x_2|)$ is the two-dimensional transverse Fourier transform of the correlation function $B_{n_1}(|x_1-x_2|,\vec{r})$ of the index of refraction fluctuations n_1 ,

$$B_{n_1}(x,\vec{r}) = \overline{n_1(\alpha,\vec{\beta})\, n_1(\alpha + x,\, \vec{\beta}+\vec{r})}$$

$$B_{n_1}(|x_1-x_2|,\vec{r}) = \int_{-\infty}^{\infty} d\vec{K}\, e^{i\vec{K}\cdot\vec{r}}\, F_n(\vec{K},|x_1-x_2|)$$

therefore,

$$\int_{-\infty}^{\infty} d\vec{K}\, F_n(\vec{K},|x_1-x_2|) = B_{n_1}(|x_1-x_2|,0) \quad (2.32)$$

and we can express $\overline{u(\vec{x})}$ as

$$\overline{u(L,\vec{r})} = A_o e^{ikL} \exp\left(- \frac{\varepsilon^2 k^2}{2} \int_0^L dx_1 \int_0^L dx_2\, B_{n_1}(|x_1-x_2|,0)\right) \quad (2.33)$$

We can also express $\overline{u(L,\vec{r})}$ in terms of the correlation function $B_n(x,\vec{r})$ of the index of refraction $n(\vec{x}) = 1 + \varepsilon n_1(x)$, which is defined as follows

$$B_n(x,\vec{r}) = \overline{(n(\alpha,\vec{\beta}) - 1)(n(\alpha + x,\, \vec{\beta} + \vec{r}) - 1)} = \varepsilon^2\, B_{n_1}(x,\vec{r}) \quad (2.34)$$

then

$$\overline{u(L,\vec{r})} = A_o e^{ikL} \exp\left(- \frac{k^2}{2} \int_0^L dx_1 \int_0^L dx_2\, B_n(|x_1-x_2|,0)\right) \quad (2.35)$$

Note that $\overline{u(L,\vec{r})}$ does not depend upon the transverse coordinate $\vec{r}$.

In Chapter I of this report we found a formal series expansion for the wave function $u(\vec{x})$ of an optical wave which satisfied the wave equation $(\nabla^2 + k^2 n^2(\vec{x}))\, u(\vec{x}) = 0$. This formal series solution does not lead to useful results unless some approximations are made to find an analytical sum of the series. These approximations limit the range of validity of all the results which are obtained by using them. However, an analytical expression for the first moment $\overline{u(L,\vec{r})}$ has been found without the use of any approximation; this result is therefore valid for any distance of propagation L and any strength of the turbulence. We shall later compare the result obtained in this section for $\overline{u(L,\vec{r})}$ with results which can be obtained by using the approximations described in Chapter I.

The correlation function $B_n(|x_1-x_2|)$ which appears in 2.35 depends only upon the difference $|x_1-x_2|$. In order to evaluate the integral $\int_0^L dx_1 \int_0^L dx_2\, B_n(|x_1-x_2|)$, we make the following change of variables: $x_2 - x_1 = \alpha$, $x_2 + x_1 = 2\beta$. The above integral is then replaced by the integral

$$2 \int_0^L d\alpha \int_{\alpha/2}^{L-\alpha/2} d\beta\, B_n(\alpha)$$

This can be seen by looking at Fig. 2.2. The integration with respect to β with α fixed is performed along a straight line at 45° from both axes. This integration is easily performed, since B_n depends only upon α , then

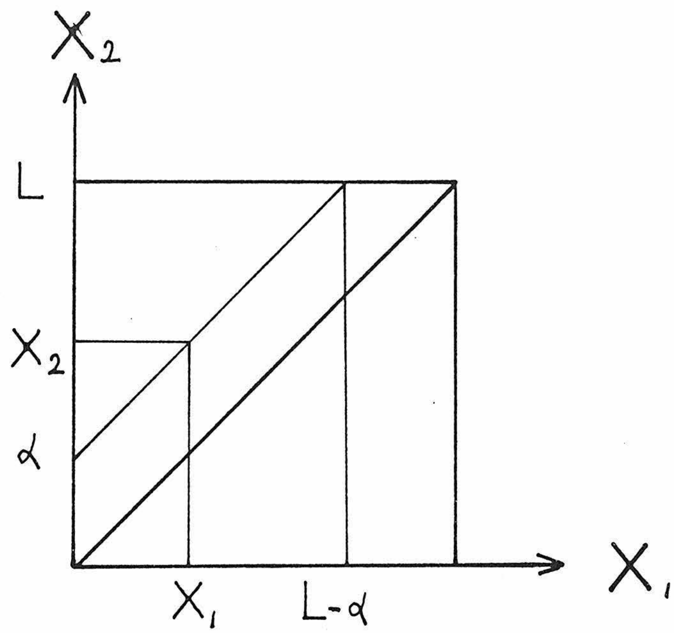


Fig. 2.2 Graphic illustration of the change of variables $x_2 - x_1 = \alpha$, $x_2 + x_1 = 2\beta$. The integration $\int_0^L dx_1 \int_0^L dx_2$ over the square of sides L is replaced by $2\int_0^L d\alpha \int_{\alpha/2}^{L-\alpha/2} d\beta$

$$\int_0^L dx_1 \int_0^L dx_2\, B_n(|x_1 - x_2|) = 2\int_0^L (L-\alpha)\, B_n(\alpha)\, d\alpha \qquad (2.36)$$

The correlation function $B_n(\alpha)$ between two points separated by a distance α is equal to zero if $\alpha \geqq L_o$ where L_o is the outer scale of the turbulence. L_o in the atmosphere is of the order of 1 to 10 meters. We are interested in propagation paths in the atmosphere much larger than a few meters. In this case $L \gg L_o$ and the integral in 2.36 is written

$$\int_0^L dx_1 \int_0^L dx_2 \, B_n(|x_1-x_2|) = 2L \int_0^{L_o} B_n(\alpha) \, dc \qquad (2.37)$$

$\overline{u(L,\vec{r})}$ is then expressed simply in terms of the correlation function B_n of the index of refraction.

$$\overline{u(L,\vec{r})} = A_o e^{ikL} \exp(-k^2 L \int_0^{L_o} B_n(\alpha) \, d\alpha) \qquad (2.38)$$

where $k = \frac{2\pi}{\lambda}$ is the wave number of the absence of turbulence ;

L_o is the outer scale of the turbulence.

The wave number k in the absence of turbulence is replaced by an effective wave number k_e in the presence of turbulence

$$\overline{u(L,\vec{r})} = A_o e^{ik_e L} \qquad (2.39)$$

with

$$\frac{k_e}{k} = 1 + ik \int_0^{L_o} B_n(\alpha) \, d\alpha \qquad (2.40)$$

In order to give an explicit expression for $\overline{u(L,\vec{r})}$ and k_e in terms of the parameters of the turbulent atmosphere, reference must be made to an existing model of the atmosphere. This is done in the next paragraph.

2.6 $\overline{u(\vec{x})}$ and the Kolmogoroff Model of Atmospheric Turbulence

The Kolmogoroff model of turbulence (12) has been described in Section 1.2. Using this model, Obukov (15) has found the following expression for the structure function $D_n(r)$ of the index of refraction on the basis of dimensional arguments.

$$D_n(r) = C_n^2 \, r^{2/3} \qquad \text{for} \quad \ell_o \leq r \leq L_o \tag{2.41}$$

$$D_n(r) = C_n^2 \, \ell_o^{2/3} \left(\frac{r}{\ell_o}\right)^2 \quad \text{for} \quad 0 \leq r \leq \ell_o$$

The structure function $D_n(r)$ of the index of refraction between two points separated by a distance r is defined as follows

$$D_n(r) = \overline{(n(\vec{x}_1) - n(\vec{x}_2))^2} \tag{2.42}$$

where $r = |\vec{x}_1 - \vec{x}_2|$ and the bar denotes the ensemble average over all points $\vec{x}_1$ and $\vec{x}_2$.

The parameter C_n is called the structure constant. It describes the strength of turbulence. C_n has been measured experimentally (16), (17); it is a decreasing function of the altitude (18). Along a horizontal path, C_n is of the order of $10^{-8}\ m^{-1/3}$ for a weak turbulence and of the order of $10^{-7}\ m^{-1/3}$ for a strong turbulence.

Our next goal is to relate the structure function $D_n(r)$ to the correlation function $B_n(r)$ to obtain an explicit expression for the correlation function which corresponds to the Kolmogoroff model. $B_n(r)$ is defined as follows

$$B_n(r) = \overline{(n(\vec{x}_1) - \overline{n(\vec{x}_1)})\,(n(\vec{x}_2) - \overline{n(\vec{x}_2)})}$$

and $D_n(r)$ is defined by 2.42. We can express $D_n(r)$ as

$$D_n(r) = \overline{n^2(\vec{x}_1)} + \overline{n^2(\vec{x}_2)} - \overline{2n(\vec{x}_1)\, n(\vec{x}_2)}$$

or

$$D_n(r) = 2 + 2\varepsilon^2 - 2\ \overline{n(\vec{x}_1)\ n(\vec{x}_2)} \qquad (2.43)$$

since

$$\overline{n^2(\vec{x})} = \overline{(1 + \varepsilon n_1(\vec{x}))^2} = 1 + \overline{2\varepsilon n_1(\vec{x})} + \varepsilon^2 \overline{n_1^2(\vec{x})} = 1 + \varepsilon^2$$

and

$$\overline{n_1(\vec{x})} = 0 \ , \quad \overline{n_1^2(\vec{x})} = 1 \ .$$

$B_n(r)$ can be expressed in the following way. Since $\overline{n(\vec{x})} = 1$,

$$B_n(r) = \overline{n(\vec{x}_1)\ n(\vec{x}_2)} - \overline{n(\vec{x}_1)} - \overline{n(\vec{x}_2)} + (\overline{n(\vec{x}_1)})\ (\overline{n(\vec{x}_2)}) = \overline{n(\vec{x}_1) n(\vec{x}_2)} - 1 \qquad (2.44)$$

From 2.43 and 2.44 it is found that

$$B_n(r) = \varepsilon^2 - \frac{1}{2} D_n(r)$$

Then for $r >> \ell_o$, $B_n(r) = \varepsilon^2 - \frac{1}{2} C_n^2\ r^{2/3}$. A simple relationship between ε the variance of the index of refraction fluctuations, C_n the structure constant, and L_o the outer scale of the turbulence is obtained by noticing that $B_n(L_o) = 0$, i.e. $\varepsilon^2 - \frac{1}{2} C_n^2\ L_o^{2/3} = 0$, then

$$B_n(r) = \frac{1}{2} C_n^2\ L_o^{2/3}\ (1 - (\frac{r}{L_o})^{2/3}) \qquad \ell_o \leq r \leq L_o$$

$$B_n(r) = \frac{1}{2} C_n^2\ L_o^{2/3}\ (1 - \frac{r^2}{L_o^{2/3}\ \ell_o^{4/3}}) \qquad 0 \leq r \leq \ell_o \qquad (2.45)$$

The integral $\int_0^{L_o} B_n(\alpha)\ d\alpha$ can then be evaluated as follows

$$\int_0^{L_o} B_n(\alpha)\, d\alpha = \frac{1}{2} C_n^2 L_o^{2/3} \left(\int_0^{\ell_o} \left(1 - \frac{\alpha^2}{L_o^{2/3}\, \ell_o^{4/3}} \right) d\alpha + \int_{\ell_o}^{L_o} \left(1 - (\frac{\alpha}{L_o})^{2/3} \right) d\alpha \right)$$

which leads to

$$\int_0^{L_o} B_n(\alpha)\, d\alpha = \frac{1}{5} C_n^2 L_o^{5/3} \left(1 + \frac{2}{3}(\frac{\ell_o}{L_o})^{5/3} \right) = \frac{1}{5} C_n^2 L_o^{5/3} \qquad (2.46)$$

since $\ell_o << L_o$ where ℓ_o is the inner scale of the turbulence. After propagating a distance L through a homogeneous and isotropic turbulent medium where the correlation function of the index of refraction is given by expression 2.45 (i.e. in the Kolmogoroff inertial subrange of turbulence), the statistical mean of an optical wave function is given by the following expression

$$\overline{u(L)} = A_o\, e^{ikL} \cdot e^{-\frac{1}{5} k^2 C_n^2 L_o^{5/3} L} \qquad (2.47)$$

The statistical mean $\overline{u(L)}$ is attenuated as $e^{-\alpha L}$ with $\alpha = \frac{1}{5} k^2 C_n^2 L_o^{5/3}$. The attenuation coefficient α is proportional to the strength of the turbulence described by C_n^2 and inversely proportional to the square of the wavelength.

<u>Numerical application.</u> For an optical wave with a wavelength of 1μ propagating in an intermediate turbulence $C_n = 3 \times 10^{-8}\ m^{-1/3}$, the attenuation coefficient is $\alpha = 0.33\ m^{-1}$ for $L_o = 10m$, and $\alpha = 7.1\ km^{-1}$ for $L_o = 1m$.

CHAPTER III - CALCULATION OF $\overline{u(\vec{x})}$ USING VARIOUS APPROXIMATIONS

In Chapter II an exact expression for the statistical mean of a wave function propagating in a turbulent medium has been obtained, the only assumption being that the wavelength of the wave is much smaller than the inner scale of the turbulence. This assumption is a very good one for the case of an optical wave propagating through the turbulent atmosphere. In Chapter III we shall compare the results obtained in Chapter II with the expressions for $\overline{u(\vec{x})}$ which are calculated from the various approximations described in paragraph 1.5. These are the Born, the geometrical optics, and the Rytov approximations.

3.1 $\overline{u(\vec{x})}$ in the Born Approximation: $\overline{u_B(\vec{x})}$

The Born approximation solution $u_B(x)$ of the wave equation 1.1 is given by expression 1.37. The statistical mean $\overline{u_B(\vec{x})}$ is then easily obtained with the help of condition 2.3.

$$\overline{u_B(\vec{x})} = A_o e^{ikx} \tag{3.1}$$

In the Born, or single scattering approximation, the coherent part of the wave $\overline{u(\vec{x})}$ does not decrease with distance. This is evident since the effects of multiple scattering are neglected. This is a good approximation when the condition

$$\frac{1}{5} k^2 C_n^2 L_o^{5/3} L << 1 \tag{3.2}$$

is satisfied. This condition represents the limit of validity of the Born approximation as stated in 1.38.

3.2 $\overline{u(\vec{x})}$ in the Geometrical Optics Approximation: $\overline{u_{(GO)}(L,\vec{r})}$

The solution of the wave equation 1.1 in the geometrical optics approximation $u_{(GO)}(\vec{x})$ is given by expression 1.43. Then

$$\overline{u_{(GO)}(L,\vec{r})} = A_o e^{ikL} \overline{\exp\left(i\varepsilon k \int_0^L dx' \; n_1(x',\vec{r}) \right)} \tag{3.3}$$

$n_1(x',\vec{r})$ and therefore $\int_0^L dx' \; n_1(x',r)$ are Gaussian random fields with zero mean. We can then apply to 3.3 the following property of a Gaussian random process β with zero mean,

$$\overline{e^{\beta}} = e^{\frac{1}{2}\overline{\beta^2}} \tag{3.4}$$

$$\overline{\exp\left(i\varepsilon k \int_0^L dx' \; n_1(x',\vec{r}) \right)} = \exp\left(- \frac{\varepsilon^2 k^2}{2} \int_0^L dx' \int_0^L dx'' \; \overline{n_1(x',\vec{r}) n_1(x'',\vec{r})} \right)$$

We notice that $\varepsilon^2 \; \overline{n_1(x',r) \; n_1(x'',r)} = B_n(x'-x'',0)$ where B_n is the correlation function of the index of refraction. We can then write

$$\overline{u_{(GO)}(L,\vec{r})} = A_o e^{ikL} \exp\left(- \frac{k^2}{2} \int_0^L dx' \int_0^L dx'' \; B_n(x'-x'',0) \right)$$

This expression for $\overline{u_{(GO)}(L,\vec{r})}$ is the same as the correct expression for $\overline{u(L,\vec{r})}$ obtained in Chapter II with no approximation

$$\overline{u_{(GO)}(L,\vec{r})} = \overline{u(L,\vec{r})} \tag{3.5}$$

Therefore, although the geometrical optics approximation gives results for the phase and the amplitude of an optical wave propagating in a turbulent medium which are valid only for propagation $\overline{\text{distances}\ L}$ such that $L \ll \ell_0^2/\lambda$, it leads to an expression for $\overline{u(L,\vec{r})}$ which is valid for all L .

3.3 $\overline{u(\vec{x})}$ in the Rytov Approximation: $\overline{u_{(RY)}(L,\vec{r})}$

The solution of the wave equation 1.1 in the Rytov approximation $u_{(RY)}(\vec{x})$ is given by expression 1.48 . We write it in the following form

$$u_{(RY)}(L,\vec{r}) = A_o e^{ikL} e^{i\varepsilon k\ \beta(L)} \tag{3.6}$$

where

$$\beta(L) = \int_0^L dx' \int_{\vec{K}'} e^{-i \frac{K'^2}{2k} (L-x')} e^{i\vec{K}'\cdot\vec{r}}\ dN(\vec{K}',x') \tag{3.7}$$

We use $\overline{\text{the relationship}}$ 3.4 for the Gaussian random process $\beta(L)$ to evaluate $\overline{u_{(RY)}(L,\vec{r})}$. From 3.6

$$\overline{u_{(RY)}(L,\vec{r})} = A_o\ e^{ikL}\ \overline{e^{i\varepsilon k\ \beta(L)}} = A_o e^{ikL}\ e^{-\frac{\varepsilon^2 k^2}{2}\ \overline{(\beta(L))^2}} \tag{3.8}$$

We now evaluate $\overline{(\beta(L))^2}$. From 3.7

$$\overline{(\beta(L))^2} = \int_0^L dx' \int_0^L dx'' \int_{\vec{K}'} \int_{\vec{K}''} \exp[\frac{-i}{2k}(K'^2(L-x') + K''^2(L-x''))]$$
$$\times\ e^{i(\vec{K}'+\vec{K}'')\cdot\vec{r}}\ \overline{dN(\vec{K}',x')\ dN(\vec{K}'',x'')} \tag{3.9}$$

The averaged quantity $\overline{dN(\vec{K}',x')\, dN(\vec{K}'',x'')}$ is replaced by its expression 2.4, and the integration with respect to $\vec{K}''$ is easily performed due to the presence of the delta function $\delta(\vec{K}'+\vec{K}'')$. It is then found that

$$\overline{(\beta(L))^2} = \int_0^L dx' \int_0^L dx'' \int_{-\infty}^{\infty} d\vec{K}\, \exp\left(\frac{-i}{2k} K^2(2L-x'-x'')\right) F_n(\vec{K},|x'-x''|) \quad (3.10)$$

or

$$\overline{(\beta(L))^2} = \int_0^L dx' \int_0^L dx'' \int_{-\infty}^{\infty} d\vec{K}\, \cos\left(\frac{K^2}{2k}(2L-x'-x'')\right) F_n(\vec{K},|x'-x''|)$$

since

$$\int_{-\infty}^{\infty} d\vec{K}\, \sin\left(\frac{K^2}{2k}(2L-x'-x'')\right) F_n(\vec{K},|x'-x''|) = 0 \quad \text{for symmetry reasons.}$$

In order to simplify expression 3.10, the same change of variables as in 2.5 is made, i.e. $x''- x' = \alpha$, $x''+ x' = 2\beta$. The integral $\int_0^L dx' \int_0^L dx''$ is then replaced by $2\int_0^L d\alpha \int_{\alpha/2}^{L-\alpha/2} d\beta$, and

$$\overline{(\beta(L))^2} = \int_{-\infty}^{\infty} d\vec{K}\; 2\int_0^L d\alpha \int_{\alpha/2}^{L-\alpha/2} d\beta\, \cos\frac{K^2\beta}{k}\; F_n(\vec{K},\alpha) \quad (3.11)$$

The integration with respect to β is then performed to yield

$$\overline{(\beta(L))^2} = 2 \int_{-\infty}^{\infty} d\vec{K} \int_0^L d\alpha\, F_n(\vec{K},\alpha) \frac{k}{K^2} \left(\sin \frac{K^2(L-\frac{\alpha}{2})}{k} - \sin \frac{K^2\alpha}{2k}\right) \qquad (3.12)$$

$F_n(\vec{K},\alpha)$ is different from zero only if $K\alpha < 1$ as in 2.8; therefore $\frac{K^2\alpha}{2k} < \frac{K}{2k} \leq \frac{\lambda}{2\ell_o} \ll 1$ and we can neglect the term $\sin\frac{K^2\alpha}{2k}$ in the integral 3.12. Also $\alpha \leq L_o$, therefore for distances of propagation L such that $L \gg L_o$ we can neglect $\alpha/2$ with respect to L and write with the help of 3.8 and 3.12 ,

$$\overline{u_{(RY)}(L,\vec{r})} = A_o e^{ikL} \exp\left(-\varepsilon^2 k^2 L \int_0^{L_o} d\alpha \int_{-\infty}^{\infty} d\vec{K} \frac{k}{K^2 L} \sin\left(\frac{K^2 L}{k}\right) F_n(\vec{K},\alpha)\right) \qquad (3.13)$$

This expression for the statistical mean of the wave function $u(\vec{x})$ obtained by using the Rytov approximation is the same as the correct expression 2.38 if $\frac{K^2L}{k} \ll 1$, i.e. if $L \ll \frac{\ell_o^2}{\lambda}$, the geometrical optics approximation limit, since in that case

$$\frac{k}{K^2L} \sin \frac{K^2L}{k} = 1 \qquad \text{and} \qquad \varepsilon^2 \int_{-\infty}^{\infty} d\vec{K}\, F_n(\vec{K},\alpha) = B_n(\alpha) \quad .$$

We write expression 3.13 in the following way

$$\overline{u_{(RY)}(L,\vec{r})} = A_o e^{ikL} \exp\left(-\varepsilon^2 k^2 L \int_0^{L_o} d\alpha \int_{-\infty}^{\infty} \left(1 - 1 + \frac{k}{K^2L} \sin \frac{K^2L}{k}\right) F_n(\vec{K},\alpha) d\vec{K}\right)$$

or

$$\overline{u_{(RY)}(L,\vec{r})} = \overline{u(L,\vec{r})} \exp\left(\varepsilon^2 k^2 L \int_0^{L_o} d\alpha \int_{-\infty}^{\infty} d\vec{K}\left(1 - \frac{k}{K^2L} \sin \frac{K^2L}{k}\right) F_n(\vec{K},\alpha)\right)$$

$$(3.14)$$

where $\overline{u(L,\vec{r})}$ is the correct expression given by 2.38. The Rytov approximation gives an expression for the statistical mean of the wave function which is valid for propagation distances and turbulence strengths such that the coefficient of the exponential in 3.14 is very small compared to one. A similar integral has been evaluated by Tatarski (19). It yields

$$\epsilon^2 k^2 L \int_0^{L_o} d\alpha \int_{-\infty}^{\infty} d\vec{K} \left(1 - \frac{k}{K^2 L} \sin \frac{K^2 L}{k}\right) F_n(\vec{K},\alpha) = 0.31\ C_n^2 k^{7/6} L^{11/6} \tag{3.15}$$

for $L >> \ell_o^2/\lambda$.

The validity of $\overline{u_{(RY)}(L,\vec{r})}$ is limited to propagation lengths L and turbulence strengths C_n such that

$$0.31\ C_n^2\ k^{7/6}\ L^{11/6} << 1 \tag{3.16}$$

<u>Numerical application.</u> For $\lambda = 6328\text{Å}$ the limit of validity is 7.4 km under intermediate turbulence ($C_n = 3 \times 10^{-8}\ m^{-1/3}$) and 2km under strong turbulence ($C_n = 10^{-7}\ m^{-1/3}$).

CHAPTER IV - CALCULATION OF THE CORRELATION FUNCTION

$B_u(L,\vec{r}_1,\vec{r}_2) = \overline{u(L,\vec{r}_1)\; u^*(L,\vec{r}_2)}$

In Chapter I a formal series solution of the wave equation $(\nabla^2 + k^2 n^2(\vec{x}))u(\vec{x})) = 0$ has been found. This solution has been used in Chapter II to calculate the statistical mean $\overline{u(\vec{x})}$ of the wave function. Although no useful information can be extracted without some approximation from the formal series solution for $u(\vec{x})$, an analytical expression for the statistical mean $\overline{u(\vec{x})}$ has been found without any approximation. In Chapter IV we calculate another important statistical quantity: the correlation function of the wave function between two points in a plane perpendicular to the direction of propagation of the wave after a propagation length L through random atmospheric turbulence. We call this correlation function $B_u(L,\vec{r}_1,\vec{r}_2) = \overline{u(L,\vec{r}_1)\,u^*(L,\vec{r}_2)}$ where the asterisk denotes the complex conjugate. The knowledge of the correlation function B_u is useful to calculate the signal-to-noise ratio in the output current of a detector in an optical heterodyne communication scheme. The method of calculating $B_u(L,\vec{r}_1,\vec{r}_2)$ is straightforward. In the absence of turbulence $B_u(L,\vec{r}_1,\vec{r}_2) = A_o^2$. The wave function $u(\vec{x})$ is expressed as a power series of ε , $u(\vec{x}) = \sum_{m=0}^{\infty} \varepsilon^m u_m(\vec{x})$ or, rather

$$\psi(\vec{x}) = \frac{u(\vec{x})}{u_o(\vec{x})} = \sum_{m=0}^{\infty} \varepsilon^m \psi_m(\vec{x})$$

where $\psi_m(\vec{x})$ is given by 1.36. Therefore,

$$B_u(L,\vec{r}_1,\vec{r}_2) = \overline{u(L,\vec{r}_1)\ u^*(L,\vec{r}_2)} = A_o^2 \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \varepsilon^{m+p}\ \overline{\psi_m(L,\vec{r}_1)\psi_p^*(L,\vec{r}_2)} \tag{4.1}$$

From condition 2.5 only the terms in B_u such that $m+p$ is an even number will be different from zero. We shall write

$$B_u(L,\vec{r}_1,\vec{r}_2) = A_o^2 \sum_{q=0}^{\infty} \varepsilon^{2q}\ B_{uq}(L,\vec{r}_1,\vec{r}_2) \tag{4.2}$$

where $B_{uq}(L,\vec{r}_1,\vec{r}_2)$ is defined by 4.1 and 4.2. We shall calculate B_{uq} for $q=0,1$ and 2 and show how these terms lead to a simple analytical expression for $B_u(L,\vec{r}_1,\vec{r}_2)$.

4.1 Calculation of B_{uo}

From 4.1 and 4.2 it is easily seen that

$$B_{uo} = \overline{\psi_o(L,\vec{r}_1)\ \psi_o^*(L,\vec{r}_2)} = 1 \tag{4.3}$$

4.2 Calculation of B_{u1}

$B_{u1}(L,\vec{r}_1,\vec{r}_2)$ is defined by relations 4.1 and 4.2

$$B_{u1} = \overline{\psi_2(L,\vec{r}_1)} + \overline{\psi_2^*(L,\vec{r}_2)} + \overline{\psi_1(L,\vec{r}_1)\ \psi_1^*(L,\vec{r}_2)} \tag{4.4}$$

The first two terms of B_{u1} have already been calculated. They are expressed by 2.10

$$\overline{\psi_2(L,\vec{r}_1)} + \overline{\psi_2^*(L,\vec{r}_2)} = -k^2 \int_0^L dx_1 \int_0^L dx_2\ f(|x_1-x_2|) \tag{4.5}$$

where $f(|X|)$ is defined by the relation 2.11

$$f(|X|) = \int_{-\infty}^{\infty} d\vec{K}\, F_n(\vec{K},|X|) \quad , \qquad \text{and} \qquad F_n(\vec{K},|X|)$$ is again the transverse two-dimensional Fourier transform of the correlation function of the index of refraction fluctuations. The last term of Bu_1 is calculated in the following way. From 1.21

$$\overline{\psi_1(L,\vec{r}_1)\psi_1^*(L,\vec{r}_2)} = k^2 \int_0^L dx_1 \int_0^L dx_2 \int_{\vec{K}_1}\int_{\vec{K}_2} \exp[\tfrac{-i}{2k}(K_1^2(L-x_1) - K_2^2(L-x_2))]$$

$$\times\ e^{i(\vec{K}_1\cdot\vec{r}_1 - \vec{K}_2\cdot\vec{r}_2)}\ \overline{dN(\vec{K}_1,x_1)\ dN^*(\vec{K}_2,x_2)} \qquad (4.6)$$

where according to 2.4

$$\overline{dN(\vec{K}_1,x_1)\ dN^*(\vec{K}_2,x_2)} = \delta(\vec{K}_1 - \vec{K}_2)\ F_n(\vec{K}_1,|x_1 - x_2|)\ d\vec{K}_1 d\vec{K}_2$$

The integration with respect to $\vec{K}_2$, for example, is easily performed in 4.6 due to the presence of the delta function $\delta(\vec{K}_1 - \vec{K}_2)$. The first exponential of 4.6 which becomes $\exp[(iK_1^2/2k)(x_1-x_2)]$ can be neglected when $\lambda \ll \ell_o$ for reasons which have already been stated (see paragraph 2.2). Then

$$\overline{\psi_1(L,\vec{r}_1)\psi_1^*(L,\vec{r}_2)} = k^2 \int_0^L dx_1 \int_0^L dx_2 \int_{-\infty}^{\infty} d\vec{K}\ e^{i\vec{K}\cdot(\vec{r}_1-\vec{r}_2)}\ F_n(\vec{K},|x_1-x_2|)$$

(4.7)

With the help of 4.4, 4.5, 2.11 and 4.7, it is found that

$$B_{u1}(L,\vec{r}_1,\vec{r}_2) = -k^2 \int_0^L dx_1 \int_0^L dx_2 \; F(|x_1-x_2|,\vec{r}_1-\vec{r}_2) \tag{4.8}$$

where $F(|X|,\vec{\rho})$ is defined by the following relation

$$F(|X|,\vec{\rho}) = \int_{-\infty}^{\infty} d\vec{K}(1 - e^{i\vec{K}\cdot\vec{\rho}}) \; F_n(\vec{K},|X|) \tag{4.9}$$

or

$$F(|X|,\vec{\rho}) = \frac{1}{\varepsilon^2} (B_n(|X|,0) - B_n(|X|,\vec{\rho})) \tag{4.10}$$

where B_n is the correlation function of the index of refraction.

4.3 Calculation of B_{u2}

$B_{u2}(L,\vec{r}_1,\vec{r}_2)$ is defined by relations 4.1 and 4.2; therefore,

$$B_{u2} = \overline{\psi_4(\vec{r}_1)} + \overline{\psi_4{}^*(\vec{r}_2)} + \overline{\psi_3(\vec{r}_1)\psi_1^*(\vec{r}_2)} + \overline{\psi_1(\vec{r}_1)\psi_3^*(\vec{r}_2)} + \overline{\psi_2(\vec{r}_1)\psi_2^*(\vec{r}_2)} \tag{4.11}$$

4.3.1 Calculation of $\overline{\psi_4(\vec{r}_1) + \psi_4^*(\vec{r}_2)}$

. The first two terms of B_{u2} have been calculated previously. From 2.25

$$\overline{\psi_4(L,\vec{r}_1)} + \overline{\psi_4^*(L,\vec{r}_2)} = \frac{1}{4}\left(k^2 \int_0^L dx_1 \int_0^L dx_2 \; f(|x_1-x_2|)\right)^2 \tag{4.12}$$

where $f(|X|)$ is defined by 2.11. We now calculate the third term of B_{u2}.

4.3.2 Calculation of $\overline{\psi_3(\vec{r}_1)\,\psi_1^*(\vec{r}_2)}$

. With the help of 1.21 and 1.36 for m=3, the following expression is obtained:

$$\overline{\psi_3(\vec{r}_1)\psi_1^*(\vec{r}_2)} = (ik)^3\,(-ik)\int_0^L dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^L dx_4 \int_{\vec{K}_1}\int_{\vec{K}_2}\int_{\vec{K}_3}\int_{\vec{K}_4}$$

$$\times\left\{e^{i(\vec{K}_1+\vec{K}_2+\vec{K}_3)\cdot\vec{r}_1}\left\{e^{-i\vec{K}_4\cdot\vec{r}_2}\exp\left(\frac{-i}{2k}\left((\vec{K}_1+\vec{K}_2+\vec{K}_3)^2(L-x_1)\right.\right.\right.\right.$$

$$\left.\left.+ (\vec{K}_2+\vec{K}_3)^2(x_1-x_2) + K_3^2(x_2-x_3) - K_4^2(L-x_4)\right)\right)$$

$$\left.\times\ \overline{dN(\vec{K}_1,x_1)\,dN(\vec{K}_2,x_2)\,dN(\vec{K}_3,x_3)\,dN^*(\vec{K}_4,x_4)}\right\} \qquad (4.13)$$

According to 2.5

$$\overline{1234^*} = \overline{12}\cdot\overline{34^*} + \overline{13}\cdot\overline{24^*} + \overline{14^*}\cdot\overline{23} \qquad (4.14)$$

where we have used the notation $dN(\vec{K}_m,x_m) \equiv m$. We shall calculate the contribution of the three terms of 4.14 to the average $\overline{\psi_3(\vec{r}_1)\psi_1^*(\vec{r}_2)}$.

4.3.2.1 Contribution of $\overline{12}\cdot\overline{34^*}$ The first term of 4.14 is $\overline{12}\cdot\overline{34^*}$ which can be written according to 2.4, $\overline{12}\cdot\overline{34^*} = \delta(\vec{K}_1+\vec{K}_2) \times \delta(\vec{K}_3-\vec{K}_4)\, F_n(\vec{K}_1,|x_1-x_2|)\, F_n(\vec{K}_3,|x_3-x_4|)\, d\vec{K}_1 d\vec{K}_2 d\vec{K}_3 d\vec{K}_4$. The integration with respect to $\vec{K}_2$ and $\vec{K}_4$ is easily performed in 4.13 because of the presence of the two delta functions $\delta(\vec{K}_1+\vec{K}_2)\,\delta(\vec{K}_3-\vec{K}_4)$. The coefficient of the exponential in 4.13 becomes

$$\frac{-i}{2k}\left(K_3^2(L-x_1) + (\vec{K}_3-\vec{K}_1)^2(x_1-x_2) + K_3^2(x_2-x_3-L+x_4)\right) = \frac{-i}{2k}\left(K_3^2(x_4-x_3)\right.$$

$$\left. + K_1^2(x_1-x_2) - 2\vec{K}_1\cdot\vec{K}_3(x_1-x_2)\right) .$$

This exponential can be neglected for the reasons described in Section 2.2, since $K_3|x_4-x_3| < 1$ and $K_1|x_1-x_2| < 1$. The contribution of

$\overline{12}\cdot\overline{34*}$ to the integrals in 4.13 is then

$$\int_{-\infty}^{\infty} d\vec{K}_1 \int_{-\infty}^{\infty} d\vec{K}_3 \; F_n(\vec{K}_1,|x_1-x_2|) \; e^{i\vec{K}_3\cdot(\vec{r}_1-\vec{r}_2)} \; F_n(\vec{K}_3,|x_3-x_4|) \qquad (4.15)$$

<u>4.3.2.2 Contribution of $\overline{13}\cdot\overline{24*}$</u> The second term of 4.14 is $\overline{13}\cdot\overline{24*}$ which can be written according to 2.4

$$\overline{13}\cdot\overline{24*} = \delta(\vec{K}_1+\vec{K}_3)\;\delta(\vec{K}_2-\vec{K}_4)\;F_n(\vec{K}_1,|x_1-x_3|)\;F_n(\vec{K}_2,|x_2-x_4|)$$
$$\times \; d\vec{K}_1\; d\vec{K}_2\; d\vec{K}_3\; d\vec{K}_4$$

The integration with respect to $\vec{K}_3$ and $\vec{K}_4$ is then easily performed. The coefficient of the exponential in 4.13 becomes

$$\frac{-i}{2k}\left(K_2^2(L-x_1) + (\vec{K}_2-\vec{K}_1)^2\,(x_1-x_2) + K_1^2(x_2-x_3) - K_2^2(L-x_4)\right)$$
$$= \frac{-i}{2k}\left(K_2^2(x_4-x_2) + K_1^2(x_1-x_3) - 2\vec{K}_1\cdot\vec{K}_2(x_1-x_2)\right)$$

This exponential can be neglected for an optical wave ($\lambda << \ell_o$) since

$$K_2|x_4-x_2| < 1\;,\quad K_1|x_1-x_3| < 1\;,\quad \text{and}$$

$$K_1(x_1-x_2) \leq K_1|x_1-x_3| < 1\;.$$

The contribution of $\overline{13}\cdot\overline{24*}$ to the integrals in 4.13 is then

$$\int_{-\infty}^{\infty} d\vec{K}_1 \int_{-\infty}^{\infty} d\vec{K}_2 \; F_n(\vec{K}_1|x_1-x_3|) \; e^{i\vec{K}_2\cdot(\vec{r}_1-\vec{r}_2)} \; F_n(\vec{K}_2,|x_2-x_4|) \qquad (4.16)$$

4.3.2.3 Contribution of $\overline{14^*}\cdot\overline{23}$ The last term of 4.14 is $\overline{14^*}\cdot\overline{23}$. From 2.4

$$\overline{14^*}\cdot\overline{23} = \delta(\vec{K}_1 - \vec{K}_4)\ \delta(\vec{K}_2 + \vec{K}_3)\ F_n(\vec{K}_1, |x_1 - x_4|)\ F_n(\vec{K}_2, |x_2 - x_3|) \times d\vec{K}_1\ d\vec{K}_2\ d\vec{K}_3\ d\vec{K}_4$$

The integration with respect to $\vec{K}_3$ and $\vec{K}_4$ is then performed and the coefficient of the exponential in 4.13 becomes

$$\frac{-i}{2k}\left(K_1^2(L - x_1) + K_2^2(x_2 - x_3) - K_1^2(L - x_4)\right) = -\frac{i}{2k}\left(K_1^2(x_4 - x_1) + K_2^2(x_2 - x_3)\right)$$

This exponential can be neglected, since $K_1|x_4 - x_1| < 1$ and $K_2|x_2 - x_3| < 1$. The contribution of $\overline{14^*}\cdot\overline{23}$ to the integrals in 4.13 is then

$$\int_{-\infty}^{\infty} d\vec{K}_1 \int_{-\infty}^{\infty} d\vec{K}_2\ F_n(\vec{K}_1, |x_1 - x_4|)\ e^{i\vec{K}_1\cdot(\vec{r}_1 - \vec{r}_2)}\ F_n(\vec{K}_2, |x_2 - x_3|) \qquad (4.17)$$

4.3.2.4 Expression of $\overline{\psi_3(\vec{r}_1)\ \psi_1^*(\vec{r}_2)}$ The contributions of the three terms in 4.14 have been calculated. We have already introduced the definition

$$f(|X|) = \int_{-\infty}^{\infty} d\vec{K}\ F_n(\vec{K}, |X|) \quad ; \qquad (2.11)$$

we now introduce the following definition:

$$h(|X|) = \int_{-\infty}^{\infty} d\vec{K}\ e^{i\vec{K}\cdot(\vec{r}_1 - \vec{r}_2)}\ F_n(\vec{K}, |X|) \qquad (4.18)$$

Then we can express $\overline{\psi_3(\vec{r}_1)\psi_1^*(\vec{r}_2)}$ in terms of the functions of f and h . Assembling together the results of 4.13, 4.14, 4.15, 4.16 and 4.17, it is found that

$$\overline{\psi_3(\vec{r}_1)\psi_1^*(\vec{r}_2)} = -k^4 \int_0^L dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^L dx_4 \; f(|x_1-x_2|)\, h(|x_3-x_4|)$$

$$+ f(|x_1-x_3|)\, h(|x_2-x_4|) + f(|x_2-x_3|)\, h(|x_1-x_4|) \quad (4.19)$$

Let us introduce the notation $G(x_i,x_j,x_k) = f(|x_i-x_j|) \times h(|x_k-x_4|)$. From this definition $G(x_i,x_j,x_k)=G(x_j,x_i,x_k)$ and we can write the integral of 4.19 as

$$\int_0^L dx_4 \int_0^L dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 (G(x_1,x_2,x_3) + G(x_1,x_3,x_2) + G(x_2,x_3,x_1))$$

$$= \frac{1}{2} \int_0^L dx_4 \int_0^L dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \; (G(x_1,x_2,x_3) + G(x_1,x_3,x_2) + G(x_2,x_3,x_1)$$

$$+ G(x_2,x_1,x_3) + G(x_3,x_1,x_2) + G(x_3,x_2,x_1))$$

$$= \frac{1}{2} \int_0^L dx_4 \int_0^L dx_1 \int_0^L dx_2 \int_0^L dx_3 \; G(x_1,x_2,x_3)$$

$$= \frac{1}{2} \int_0^L dx_1 \int_0^L dx_2 \; f(|x_1-x_2|) \int_0^L dx_3 \int_0^L dx_4 \; h(|x_3-x_4|)$$

and therefore

$$\overline{\psi_3(\vec{r}_1)\psi_1^*(\vec{r}_2)} = -\frac{k^4}{2}\int_0^L dx_1 \int_0^L dx_2\, f(|x_1-x_2|) \int_0^L dx_3 \int_0^L dx_4\, h(|x_3-x_4|) \tag{4.20}$$

where $f(|X|)$ and $h(|X|)$ are defined by 2.11 and 4.18.

4.3.3 Calculation of $\overline{\psi_1(\vec{r}_1)\psi_3^*(\vec{r}_2)}$ The expression for $\overline{\psi_1(\vec{r}_1)\psi_3^*(\vec{r}_2)}$ is obtained from the expression for $\overline{\psi_3(\vec{r}_1)\psi_1^*(\vec{r}_2)}$ by taking the complex conjugate and interchanging $\vec{r}_1$ and $\vec{r}_2$. It is easily seen that

$$\overline{\psi_1(\vec{r}_1)\psi_3^*(\vec{r}_2)} = \overline{\psi_3(\vec{r}_1)\psi_1^*(\vec{r}_2)} \tag{4.21}$$

4.3.4 Calculation of $\overline{\psi_2(\vec{r}_1)\psi_2^*(\vec{r}_2)}$ The last term to be calculated in the expression 4.11 of $B_{u2}(L,\vec{r}_1,\vec{r}_2)$ is $\overline{\psi_2(\vec{r}_1)\psi_2^*(\vec{r}_2)}$. This quantity is expressed as follows with the help of 1.36

$$\overline{\psi_2(\vec{r}_1)\psi_2^*(\vec{r}_2)} = (ik)^2\,(-ik)^2 \int_0^L dx_1 \int_0^{x_1} dx_2 \int_0^L dx_3 \int_0^{x_3} dx_4 \int_{\vec{K}_1}\int_{\vec{K}_2}\int_{\vec{K}_3}\int_{\vec{K}_4}$$

$$\times \Big\{ e^{i(\vec{K}_1+\vec{K}_2)\cdot\vec{r}_1}\, e^{-i(\vec{K}_3+\vec{K}_4)\cdot\vec{r}_2} \exp\big(\tfrac{-i}{2k}((\vec{K}_1+\vec{K}_2)^2\,(L-x_1) + K_2^2(x_1-x_2)$$

$$- (\vec{K}_3+\vec{K}_4)^2\,(L-x_3) - K_4^2(x_3-x_4))\big)$$

$$\times\, \overline{dN(\vec{K}_1,x_1)\; dN(\vec{K}_2,x_2)\; dN^*(\vec{K}_3,x_3)\; dN^*(\vec{K}_4,x_4)}\Big\} \tag{4.22}$$

Again, for a Gaussian random process $dN(\vec{K}_m,x_m) \equiv m$,

$\overline{123^*4^*} = \overline{12}\cdot\overline{3^*4^*} + \overline{13^*}\cdot\overline{24^*} + \overline{14^*}\cdot\overline{23^*}$ and again there will be three contributions to the integrals of 4.22. The method of calculation is the same as the one used in the detailed calculation of $\overline{\psi_3(\vec{r}_1)\psi_1^*(\vec{r}_2)}$. It is easily seen that the exponential terms in 4.22 can be neglected in every one of the three terms which can be expressed in the following way.

4.3.4.1 Contribution of $\overline{12}\cdot\overline{3^*4^*}$

$$\int_{-\infty}^{\infty} d\vec{K}_1 \int_{-\infty}^{\infty} d\vec{K}_3 \, F_n(\vec{K}_1, |x_1-x_2|) \, F_n(\vec{K}_3, |x_3-x_4|) \tag{4.23}$$

4.3.4.2 Contribution of $\overline{13^*}\cdot\overline{24^*}$

$$\int_{-\infty}^{\infty} d\vec{K}_1 \int_{-\infty}^{\infty} d\vec{K}_2 \, e^{i\vec{K}_1\cdot(\vec{r}_1-\vec{r}_2)} F_n(\vec{K}_1, |x_1-x_3|) \, e^{i\vec{K}_2\cdot(\vec{r}_1-\vec{r}_2)} F_n(\vec{K}_2, |x_2-x_4|) \tag{4.24}$$

4.3.4.3 Contribution of $\overline{14^*}\cdot\overline{23^*}$

$$\int_{-\infty}^{\infty} d\vec{K}_1 \int_{-\infty}^{\infty} d\vec{K}_2 \, e^{i\vec{K}_1\cdot(\vec{r}_1-\vec{r}_2)} F_n(\vec{K}_1, |x_1-x_4|) \, e^{i\vec{K}_2\cdot(\vec{r}_1-\vec{r}_2)} F_n(\vec{K}_2, |x_2-x_3|) \tag{4.25}$$

4.3.4.4 Expression of $\overline{\psi_2(\vec{r}_1)\psi_2^*(\vec{r}_2)}$. $\overline{\psi_2(\vec{r}_1)\psi_2^*(\vec{r}_2)}$ can be expressed in terms of the functions f and h defined by 2.11 and 4.18. Assembling together the results of 4.22, 4.23, 4.24 and 4.25, it is found that

$$\overline{\psi_2(\vec{r}_1)\psi_2^*(\vec{r}_2)} = k^4 \int_0^L dx_1 \int_0^{x_1} dx_2 \int_0^L dx_3 \int_0^{x_3} dx_4 \Big(f(|x_1-x_2|)\, f(|x_3-x_4|) + h(|x_1-x_4|)\, h(|x_2-x_3|) + h(|x_1-x_3|)\, h(|x_2-x_4|) \Big)$$

which can be written

$$\overline{\psi_2(\vec{r}_1)\psi_2^*(\vec{r}_2)} = \frac{k^4}{4}\left(\int_0^L dx_1 \int_0^L dx_2\ f(|x_1-x_2|)\right)^2$$

$$+ \frac{k^4}{2}\left(\int_0^L dx_1 \int_0^L dx_2\ h(|x_1-x_2|)\right)^2 \qquad (4.26)$$

4.3.5 Expression of $B_{u2}(L,\vec{r}_1,\vec{r}_2)$. We have calculated all the terms in the expression 4.11 of B_{u2} . The expressions for each term obtained in 4.12, 4.20, 4.21 and 4.26 are then collected together to yield

$$B_{u2} = \frac{k^4}{4}\left(\int_0^L dx_1 \int_0^L dx_2\ f(|x_1-x_2|)\right)^2 - k^4\left(\int_0^L dx_1 \int_0^L dx_2\ f(|x_1-x_2|)\right)$$

$$\times \left(\int_0^L dx_1 \int_0^L dx_2\ h(|x_1-x_2|)\right) + \frac{k^4}{4}\left(\int_0^L dx_1 \int_0^L dx_2\ f(|x_1-x_2|)\right)^2$$

$$+ \frac{k^4}{2}\left(\int_0^L dx_1 \int_0^L dx_2\ h(|x_1-x_2|)\right)^2$$

or

$$B_{u2} = \frac{k^4}{2}\left(\int_0^L dx_1 \int_0^L dx_2\ \big(f(|x_1-x_2|) - h(|x_1-x_2|)\big)\right)^2$$

The functions f, h and F defined respectively by 2.11, 4.18 and 4.9 obey the following relation $F(|X|,\vec{\rho}) = f(|X|) - h(|X|)$.

Then

$$B_{u2} = \frac{k^4}{2}\left(\int_0^L dx_1 \int_0^L dx_2 \; F(|x_1-x_2|, \vec{r}_1-\vec{r}_2)\right)^2$$

and according to 4.8,

$$B_{u2} = \frac{1}{2}(B_{u1})^2 \tag{4.27}$$

The calculation of the second order term B_{u2} in the power series expansion of the correlation function $B_u(L,\vec{r}_1,\vec{r}_2)$ was long but rather straightforward. The calculation of higher order terms becomes very tedious; however, the third order term B_{u3} has given the following result

$$B_{u3} = \frac{1}{6}(B_{u1})^3 \tag{4.28}$$

where B_{u1} is given by 4.8. In the next section we show how an analytical sum of the power series expansion for $B_u(L,\vec{r}_1,\vec{r}_2)$ can be found.

4.4 Expression of the Correlation Function $B_u(L,\vec{r}_1,\vec{r}_2)$

The correlation function $B_u(L,\vec{r}_1,\vec{r}_2)$ of the wave function between two points $(L,\vec{r}_1)$ and $(L,\vec{r}_2)$ in a plane perpendicular to the direction of propagation of the wave has been expressed in a power series expansion of ε^2, the variance of the index of refraction fluctuations.

$$B_u(L,\vec{r}_1,\vec{r}_2) = A_o^2 \sum_{q=0}^{\infty} \varepsilon^{2q} B_{uq} \tag{4.2}$$

An explicit calculation of the first terms in this expansion has shown that $$B_u = A_o^2\left(1 + \varepsilon^2 B_{u1} + \frac{(\varepsilon^2 B_{u1})^2}{2} + \frac{(\varepsilon^2 B_{u1})^3}{6} + \cdots\right)$$

which suggests that

$$B_u(L,\vec{r}_1,\vec{r}_2) = A_o^2 \exp(\varepsilon^2 B_{u1}) \tag{4.29}$$

Although it would be too tedious to calculate the general term B_{uq}, it can be seen that the averaging process allows us to neglect the exponential terms (other than $e^{i(\Sigma\vec{K}_p)\cdot\vec{r}}$) in all cases. This is again due to the assumption that the wavelength of the wave is much smaller than the inner scale of the turbulence. Therefore we would have obtained the correct expression for $B_u(L,\vec{r}_1,\vec{r}_2)$ if we had neglected the exponential terms in $u(L,\vec{r}_1)$ and $u^*(L,\vec{r}_2)$ (other than $e^{i\Sigma_p\vec{K}_p\cdot\vec{r}_1}$ and $e^{-i\Sigma_p\vec{K}_p\cdot\vec{r}_2}$) to start with, i.e. if we had used the expressions of the wave functions in the geometrical optics approximation 1.43.

We will now verify that by using the expression 1.43 for the wave functions $u(L,\vec{r}_1)$ and $u^*(L,\vec{r}_2)$ the correct result for the correlation function $B_u(L,\vec{r}_1,\vec{r}_2)$ given by 4.29 is obtained. From 1.43

$$B_u(L,\vec{r}_1,\vec{r}_2) = \overline{u(L,\vec{r}_1)u^*(L,\vec{r}_2)} = A_o^2\, \overline{\exp\left(i\varepsilon k \int_0^L dx_1 (n_1(x_1,\vec{r}_1) - n_1(x_1,\vec{r}_2))\right)} \tag{4.30}$$

which we can write

$$B_u = A_o^2\, \overline{e^{ik\beta}} \tag{4.31}$$

where β is defined by 4.30 and 4.31. Since $n_1(x,\vec{r})$ is a Gaussian random process with zero mean, B_u is expressed as follows:

$$B_u = A_o^2 e^{\frac{-k^2 \overline{\beta^2}}{2}} \tag{4.32}$$

and $\overline{\beta^2}$ is calculated in the following way

$$\overline{\beta^2} = \varepsilon^2 \int_0^L dx_1 \int_0^L dx_2 \; \overline{(n_1(x_1,\vec{r}_1) - n_1(x_1,\vec{r}_2))(n_1(x_2,\vec{r}_1) - n_1(x_2,\vec{r}_2))}$$

or

$$\overline{\beta^2} = \varepsilon^2 \int_0^L dx_1 \int_0^L dx_2 \left(\overline{n_1(x_1,\vec{r}_1)\, n_1(x_2,\vec{r}_1)} + \overline{n_1(x_1,\vec{r}_2)\, n_1(x_2,\vec{r}_2)} - \overline{n_1(x_1,\vec{r}_1)\, n_1(x_2,\vec{r}_2)} - \overline{n_1(x_1,\vec{r}_2)\, n_1(x_2,\vec{r}_1)} \right)$$

From the definition of the correlation function of the index of refraction

$B_n(|x_1-x_2|,|\vec{r}_1-\vec{r}_2|) = \varepsilon^2 \overline{n_1(x_1,\vec{r}_1)\, n_1(x_2,\vec{r}_2)}$, and with the help of 4.32, the correct expression for $B_u(L,\vec{r}_1,\vec{r}_2)$ is

$$B_u(L,|\vec{r}_1-\vec{r}_2|) = A_o^2 \exp\left(-k^2 \int_0^L dx_1 \int_0^L dx_2 \; (B_n(|x_1-x_2|,0) - B_n(|x_1-x_2|,|\vec{r}_1-\vec{r}_2|))\right) \tag{4.33}$$

According to 4.10, the above expression for the correlation function $B_u(L,|\vec{r}_1-\vec{r}_2|)$ is the same as the expression 4.29 suggested by the results for the first terms of the power series expansion 4.2 for $B_u(L,\vec{r}_1,\vec{r}_2)$.

Then the correlation function $B_u(L,\vec{r}_1,\vec{r}_2) = \overline{u(L,\vec{r}_1)u^*(L,\vec{r}_2)}$ of the wave function between two points $(L,\vec{r}_1)$ and $(L,\vec{r}_2)$ in a plane perpendicular to the direction of propagation of the wave after a propagation distance L through a randomly turbulent atmosphere is correctly given by expression 4.33. This expression is valid for any propagation distance L and any strength of the turbulence for an optical wave whose wavelength is much smaller than the inner scale of the turbulence. This expression is the same as the one which can be calculated from the geometrical optics approximation. The correlation function $B_u(L,\vec{r}_1,\vec{r}_2)$ is a function of the distance between the two points $(L,\vec{r}_1)$ and $(L,\vec{r}_2)$; we shall call it $\rho = |\vec{r}_1-\vec{r}_2|$. This is due to the assumption that the index of refraction $n(\vec{x})$ is an isotropic and homogeneous random field, i.e. the correlation function $B_n(\vec{x}_1,\vec{x}_2)$ of the index of refraction between two points $\vec{x}_1,\vec{x}_2$ is only a function of the distance between the two points $B_n(\vec{x}_1,\vec{x}_2) \equiv B_n(|\vec{x}_1-\vec{x}_2|)$. We can express $B_u(L,\rho)$ in a simpler way by making the same change of variables as in Section 2.5, $x_2-x_1 = \alpha$ and $x_1+x_2 = 2\beta$. The integral $\int_0^L dx_1 \int_0^L dx_2$ is then replaced by $2\int_0^L d\alpha \int_{\alpha/2}^{L-\alpha/2} d\beta$; the integration with respect to β is performed in 4.33. Then,

$$B_u(L,\rho) = A_o^2 \exp\left(-2k^2 L \int_0^L (B_n(\alpha) - B_n(\sqrt{\alpha^2+\rho^2}))d\alpha\right) \qquad (4.34)$$

Since the propagation distance L is much larger than any correlation length of the index of refraction, we have neglected α with respect to L to obtain the above expression for $B_u(L,\rho)$.

In order to obtain an explicit expression for $B_u(L,\rho)$ in terms of the parameters of the turbulent medium, we shall evaluate the integral in 4.34 for two different models of the atmospheric turbulence: the Kolmogoroff model and the model leading to the correlation function $B_n(\alpha) = \varepsilon^2 e^{-(\alpha/r_o)^2}$.

4.5 The Correlation Function $B_u(L,\rho)$ and the Kolmogoroff Model of Atmospheric Turbulence

In the Kolmogoroff model of atmospheric turbulence the correlation function $B_n(\alpha)$ of the index of refraction is given by 2.45

$$B_n(\alpha) = \begin{cases} B_{n'}(\alpha) = \frac{1}{2} C_n^2 L_o^{2/3}\left(1 - \frac{\alpha^2}{L_o^{2/3}\, \ell_o^{4/3}}\right) & \text{for } 0 \leq \alpha \leq \ell_o \\ B_{n''}(\alpha) = \frac{1}{2} C_n^2 L_o^{2/3}\left(1 - \frac{\alpha^{2/3}}{L_o^{2/3}}\right) & \text{for } \ell_o \leq \alpha \leq L_o \\ 0 \text{ for } \alpha \geq L_o \end{cases}$$

We introduce the following definition

$$G(\rho,L_o,\ell_o) = \int_0^{L_o} B_n(\alpha)\, d\alpha - \int_0^{\sqrt{L_o^2 - \rho^2}} B_n(\sqrt{\alpha^2+\rho^2})\, d\alpha \qquad (4.35)$$

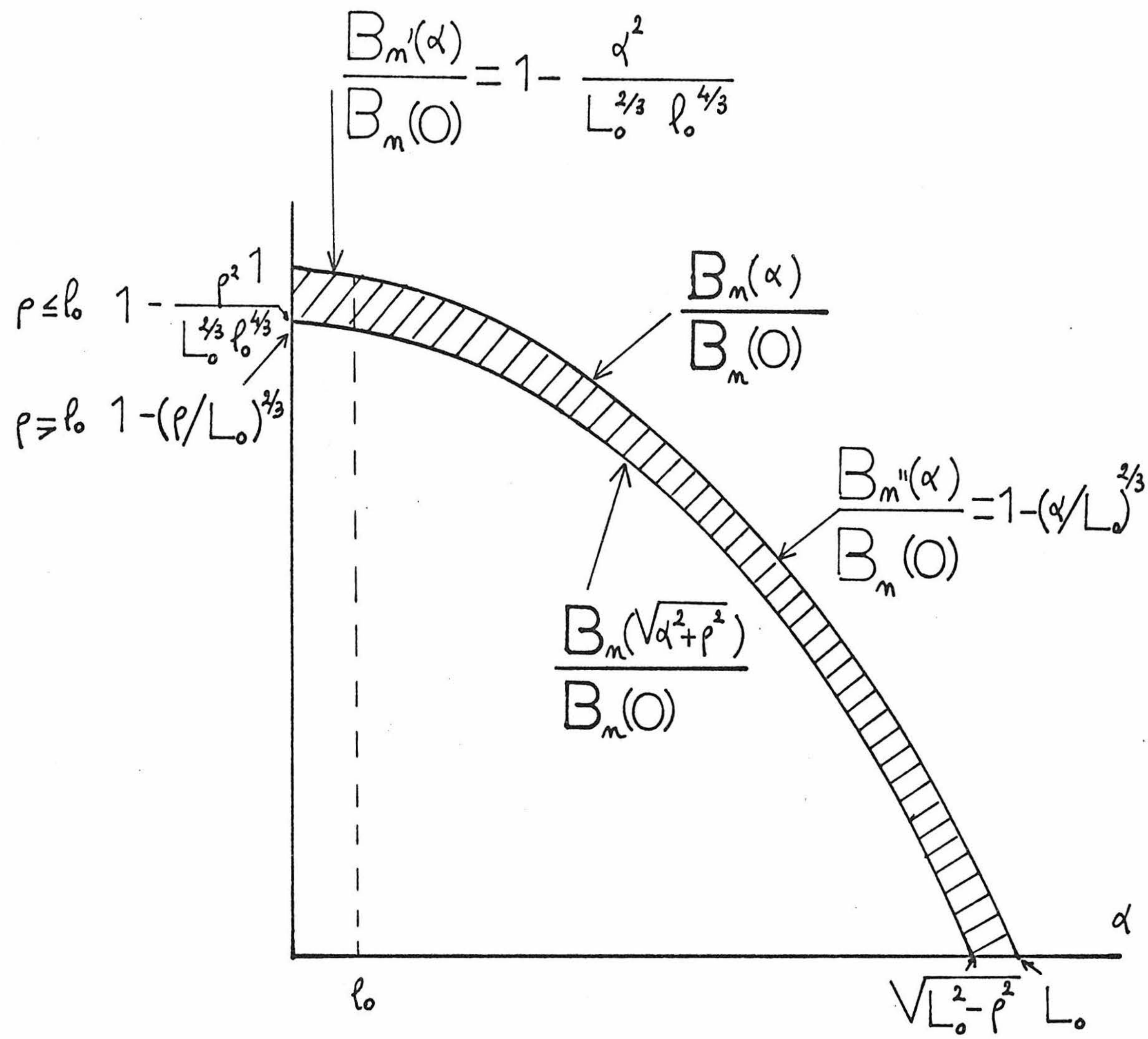


FIG 4-1 _ The area in between the two curves is

$$G(\rho,\ell_0,L_0) = \int_0^{L_0} B_n(\alpha)\,d\alpha - \int_0^{\sqrt{L_0^2-\rho^2}} B_n(\sqrt{\alpha^2+\rho^2})\,d\alpha$$

$G(\rho,L_o,\ell_o)$ is equal to the integral appearing in expression 4.34, since $B_n(\alpha) = 0$ for $\alpha \geq L_o$ and $B_n(\sqrt{\alpha^2+\rho^2}) = 0$ for $\alpha^2 \geq L_o^2 - \rho^2$. $G(\rho,L_o,\ell_o)$ is equal to the area in between the two curves of Fig. 4.1. We shall evaluate $G(\rho,L_o,\ell_o)$ explicitly in the three following cases: $0 \leq \rho \leq \ell_o$, $\ell_o \leq \rho \leq L_o$ and $\rho \geq L_o$.

<u>4.5.1 $B_u(L,\rho)$ for $\rho \leq \ell_o$</u> In the case where ρ is smaller than the inner scale of turbulence, $G(\rho,L_o,\ell_o)$ is given by the following sum of integrals

$$G(\rho,L_o,\ell_o) = \int_0^{\ell_o} B_{n'}(\alpha)\, d\alpha + \int_0^{L_o} B_{n''}(\alpha)\, d\alpha - \int_0^{\sqrt{\ell_o^2-\rho^2}} B_{n'}(\sqrt{\rho^2+\alpha^2})\, d\alpha$$

$$- \int_{\sqrt{\ell_o^2-\rho^2}}^{\sqrt{L_o^2-\rho^2}} B_{n''}(\sqrt{\rho^2+\alpha^2})\, d\alpha$$

where $B_{n'}$ and $B_{n''}$ are given by 2.45. The explicit calculation of the above integrals is carried out and it is found, with the help of 4.35, that

$$B_u(L,\rho) = A_o^2\left(\exp\left(-k^2L\, C_n^2\left\{\frac{4}{15}\ell_o^{5/3} + \frac{1}{3}\frac{(\ell_o^2-\rho^2)^{1/2}(\ell_o^2+2\rho^2)}{\ell_o^{4/3}} + \int_{\sqrt{\ell_o^2-\rho^2}}^{\sqrt{L_o^2-\rho^2}} (\alpha^2+\rho^2)^{1/3}\, d\alpha + L_o^{5/3}\left(\frac{2}{5} - \left(1 - \frac{\rho^2}{L_o^2}\right)^{\frac{1}{2}}\right)\right\}\right)\right) \quad (4.36)$$

for $\rho \leq \ell_o$.

<u>4.5.2 $B_u(L,\rho)$ for $\ell_o \leq \rho \leq L_o$</u> In the case $\ell_o \leq \rho \leq L_o$, $G(\rho,L_o,\ell_o)$ is given by the following expression

$$G(\rho,L_o,\ell_o) = \int_0^{\ell_o} B_{n'}(\alpha)\, d\alpha + \int_{\ell_o}^{L_o} B_{n''}(\alpha)\, d\alpha - \int_0^{\sqrt{L_o^2 - \rho^2}} B_{n''}(\sqrt{\rho^2 + \alpha^2})\, d\alpha$$

where $B_{n'}$ and $B_{n''}$ are given by 2.45. With the help of 4.34 and 4.35, the expression for $B_u(L,\rho)$ is obtained

$$B_u(L,\rho) = A_o^2 \exp\left(-k^2 L\, C_n^2 \left\{\frac{4}{15}\ell_o^{5/3} + \int_0^{\sqrt{L_o^2-\rho^2}} (\alpha^2 + \rho^2)^{1/3}\, d\alpha + L_o^{5/3}\left(\frac{2}{5} - \left(1 - \frac{\rho^2}{L_o^2}\right)^{1/2}\right)\right\}\right) \quad (4.37)$$

for $\ell_o \leq \rho \leq L_o$.

<u>4.5.3 $B_u(L,\rho)$ for $\rho \geq L_o$</u> . In the case where ρ is larger than the outer scale of the turbulence,

$$G(\rho,L_o,\ell_o) = \int_0^{L_o} B_n(\alpha)\, d\alpha = \frac{1}{5} C_n^2 L_o^{5/3}$$

Then the correlation function $B_u(L,\rho)$ is given by the following expression

$$B_u(L,\rho) = A_o^2\, e^{-\frac{2}{5} k^2 C_n^2 L_o^{5/3} L} \qquad \text{for } \rho \geq L_o \qquad (4.38)$$

We notice, with the help of 1.47, that for $\rho \geq L_o$, $B_u(L,\rho) = (\overline{u(L)})(\overline{u^*(L)})$. The correlation $B_u(L,\rho)$ for $\rho \geq L_o$ is equal to the product of the two independent averages $\overline{u(L,\vec{r}_1)}$ and $\overline{u(L,\vec{r}_2)}$.

In this section we have calculated the correlation function $B_u(L,\rho)$ explicitly in terms of the parameters of the turbulent atmosphere, the inner scale of the turbulence ℓ_o, the outer scale of the turbulence L_o, and the structure constant C_n . Expressions have been found for various ranges of ρ : $\rho \leq \ell_o$, $\ell_o \leq \rho \leq L_o$ and $\rho \geq L_o$. In all cases the correlation function $B_u(L,\rho)$ is a decreasing function of the propagation length L, of the strength of the turbulence C_n and of the distance between points ρ and an increasing function of the wavelength of the wave λ .

4.6 The Correlation Function $B_u(L,\rho)$ with a Refractive Index Correlation Function $B_n(\alpha) = \varepsilon^2 e^{-\alpha^2/r_o^2}$

The correlation function

$$B_n(\alpha) = \varepsilon^2 e^{-\alpha^2/r_o^2} \tag{4.39}$$

has often been used in calculations relating to the problem of wave propagation in turbulent media. Although it gives an incorrect description of the correlation of the index of refraction in the atmospheric turbulence, the calculations are usually easier and lead to simple analytical expressions where the only parameters describing the turbulence are the variance of index of refraction fluctuations ε^2 and the correlation length r_o . Reasonable values of r_o vary from a few centimeters to a few meters. For propagation lengths L much larger than r_o , the limits of the integrals in 4.34 can be extended to infinity and we can write according to 4.39

$$\int_0^L (B_n(\alpha) - B_n(\sqrt{\alpha^2+\rho^2}))\, d\alpha = \varepsilon^2 \int_0^\infty (e^{-\alpha^2/r_o^2} - e^{-(\alpha^2+\rho^2)/r_o^2}\, d\alpha$$

$$= \varepsilon^2 (1 - e^{-\rho^2/r_o^2}) \int_0^\infty e^{-\alpha^2/r_o^2}\, d\alpha = \varepsilon^2 \frac{\sqrt{\pi}}{2} r_o (1 - e^{-\rho^2/r_o^2})$$

Then the following expression is found for the correlation function $B_u(L,\rho)$

$$B_u(L,\rho) = A_o^2 \exp\left(-\sqrt{\pi}\, \varepsilon^2 k^2 r_o L(1 - e^{-\rho^2/r_o^2})\right)$$

4.7 Conclusion of Chapter IV

In Chapter IV the two-point correlation function $\overline{u(L,\vec{r}_1)u^*(L,\vec{r}_2)}$ of the wave function of an optical wave propagating in a turbulent medium has been calculated. The only approximations made are based on the assumption that the wavelength of the wave is much smaller than the inner scale of the turbulence. The starting point is the formal power series expansion for $u(\vec{x})$ which was found in Chapter I. Although no information about the phase and the amplitude of the wave could be obtained from the power series expansion for $u(\vec{x})$ without some approximations, we were able to obtain a correct and analytical expression for the correlation function $\overline{u(L,\vec{r}_1)u^*(L,\vec{r}_2)} = B_u(L,\rho)$. This correlation function is expressed only in terms of the wave number k , the distance of propagation L , and the correlation function of the index of refraction $B_n(\alpha)$,

$$B_u(L,\rho) = A_o^2 \exp\left(-2k^2 L \int_0^L (B_n(\alpha) - B_n(\sqrt{\alpha^2+\rho^2}))\, d\alpha\right) \tag{4.34}$$

This is a function of the distance $\rho = |\vec{r}_1 - \vec{r}_2|$ between the two points. $B_u(L,\rho)$ has been calculated explicitly in terms of the parameters of the turbulent medium for two different models of the turbulence:

(a) The Kolmogoroff model with the index of refraction correlation function given by 2.45

(b) A model for which the correlation function of the index of refraction is given by $B_n(\alpha) = \varepsilon^2 e^{-\alpha^2/r_o^2}$.

The correct expression 4.34 for $B_u(L,\rho)$ which is valid for all propagation lengths and all turbulence strengths is the same as the expression for $B_u(L,\rho)$ which is obtained with the help of the geometrical optics approximation. This is only true for a propagating wave with a wavelength much smaller than the inner scale of the turbulence.

In the next chapter, we shall compare the results obtained in Chapter IV with the expressions for $B_u(L,\rho)$ which are obtained by using the various approximations described in Section 1.5.

CHAPTER V - THE CORRELATION FUNCTION $B_u(L,\rho) = \overline{u(L,\vec{r}_1)u^*(L,\vec{r}_2)}$ IN VARIOUS APPROXIMATIONS

The correlation function $\overline{u(L,\vec{r}_1)u^*(L,\vec{r}_2)}$ was calculated in Chapter IV. In order to calculate this function, the formal power series expansion for $u(L,\vec{r}_1)$ and $u^*(L,\vec{r}_2)$ obtained in Chapter I was used and an analytical expression for $B_u(L,\rho)$ was obtained. In this chapter we compare the results of Chapter IV with the expressions for the correlation function $B_u(L,\rho)$ obtained by using various approximations: the Born approximation, the geometrical optics approximation, and the Rytov approximation.

5.1 $B_u(L,\rho)$ in the Born Approximation: $B_{u(B)}(L,\rho)$

The solution of the wave equation 1.1 in the Born or single scattering approximation is given by 1.37. We can express the correlation function $B_{u(B)}(L,\rho)$ as

$$B_{u(B)}(L,\rho) = \overline{A_o^2\left(1 + i\epsilon k \int_0^L dx_1 \int_{\vec{K}_1} e^{-i\frac{K_1^2}{2k}(L-x_1)} e^{i\vec{K}_1\cdot\vec{r}_1} dN(\vec{K}_1,x_1)\right)}$$

$$\times \overline{\left(1 - i\epsilon k \int_0^L dx_2 \int_{\vec{K}_2} e^{i\frac{K_2^2}{2k}(L-x_2)} e^{-i\vec{K}_2\cdot\vec{r}_2} dN^*(\vec{K}_2,x_2)\right)}$$

With the help of relations 2.3 and 2.4, one obtains

$$B_{u(B)}(L,\rho) = A_o^2\left(1 + \varepsilon^2 k^2 \int_0^L dx_1 \int_0^L dx_2 \int_{\vec{K}} e^{i\vec{K}\cdot(\vec{r}_1-\vec{r}_2)} F_n(\vec{K},|x_1-x_2|)\right)$$

and it is easily seen that the above expression can be transformed to yield

$$B_{u(B)}(L,\rho) = A_o^2\left(1 + 2K^2 L \int_0^L B_n(\sqrt{\alpha^2+\rho^2})\, d\alpha\right) \qquad (5.1)$$

where B_n is the correlation function of the index of refraction. This expression is only valid when the following condition is fulfilled

$$2k^2 L \int_0^L B_n(\alpha)\, d\alpha \ll 1$$

or, for a Kolmogoroff spectrum, $\frac{2}{5} k^2 C_n^2 L_o^{5/3} L \ll 1$.

5.2 $B_u(L,\rho)$ in the Geometrical Optics Approximation: $B_{u(GO)}(L,\rho)$

In the geometrical optics approximation $L \ll \ell_o^2/\lambda$, the solution of the wave equation 1.1 is given by 1.43. It has been shown in Section 4.4 that the correlation function $\overline{u(L,\vec{r}_1)u^*(L,\vec{r}_2)}$ obtained from the geometrical optics approximation is the same as the correct expression obtained without any approximation.

Although the geometrical optics approximation is valid for propagation distances L such that $L \ll \ell_o^2/\lambda$, the expressions for the statistical mean of an optical wave function $\overline{u(L,\vec{r})}$ and for the correlation function $\overline{u(L,\vec{r}_1)u^*(L,\vec{r}_2)}$ obtained by using this approximation are valid for any length of propagation and any turbulence strength. This is true provided that the wavelength of the wave is much smaller than the smallest scale of the turbulence ℓ_o.

5.3 $B_u(L,\rho)$ in the Rytov Approximation: $B_{u(RY)}(L,\rho)$

According to 1.48, the correlation function $B_{u(RY)}(L,\rho)$ in the Rytov approximation is

$$B_{u(RY)} = A_o^2 \, \overline{\exp\left(i\varepsilon k \int_0^L dx_1 \int_{\vec{K}_1} \left(e^{-i\frac{K_1^2}{2k}(L-x_1)} e^{i\vec{K}_1\cdot\vec{r}_1} dN(\vec{K}_1,\vec{r}_1) - e^{i\frac{K_1^2}{2k}(L-x_1)} e^{-i\vec{K}_1\cdot\vec{r}_2} dN^*(\vec{K}_1,\vec{r}_2)\right)\right)} \tag{5.2}$$

which can be written

$$B_{u(RY)} = A_o^2 \, \overline{e^{i\varepsilon k\beta}} \tag{5.3}$$

where β is defined by 5.2 and 5.3. β is a Gaussian random process with zero mean, since $dN(\vec{K}_1,\vec{r}_1)$ and $dN^*(\vec{K}_1,\vec{r}_2)$ are Gaussian random processes with zero mean. Then, according to 3.4,

$$B_{u(RY)} = A_o^2 \, e^{-\frac{\varepsilon^2 k^2}{2}\overline{\beta^2}} \tag{5.4}$$

$\overline{\beta^2}$ is given by the following expression:

$$\overline{\beta^2} = \int_0^L dx_1 \int_0^L dx_2 \int_{\vec{K}_1} \int_{\vec{K}_2}$$

$$\times \overline{\left(e^{-i\frac{K_1^2}{2k}(L-x_1)} e^{i\vec{K}_1\cdot\vec{r}_1} dN(\vec{K}_1,\vec{r}_1) - e^{i\frac{K_1^2}{2k}(L-x_1)} e^{-i\vec{K}_1\cdot\vec{r}_2} dN^*(\vec{K}_1,\vec{r}_2)\right)}$$

$$\times \overline{\left(e^{-\frac{iK_2^2}{2k}(L-x_2)} e^{i\vec{K}_2\cdot\vec{r}_1} dN(\vec{K}_2,\vec{r}_1) - e^{i\frac{K_2^2}{2k}(L-x_2)} e^{-i\vec{K}_2\cdot\vec{r}_2} dN^*(\vec{K}_2,\vec{r}_2)\right)} \quad (5.5)$$

or

$$\overline{\beta^2} = \int_0^L dx_1 \int_0^L dx_2 \int_{\vec{K}_1} \int_{\vec{K}_2} \Bigg(e^{-\frac{i}{2k}(K_1^2(L-x_1)+K_2^2(L-x_2))} e^{i(\vec{K}_1+\vec{K}_2)\cdot\vec{r}_1}$$

$$\times \overline{dN(\vec{K}_1,\vec{r}_1)\, dN(\vec{K}_2,\vec{r}_1)} + e^{\frac{i}{2k}(K_1^2(L-x_1)+K_2^2(L-x_2))} e^{-i(\vec{K}_1+\vec{K}_2)\cdot\vec{r}_2}$$

$$\times \overline{dN^*(\vec{K}_1,\vec{r}_2) dN^*(\vec{K}_2,\vec{r}_2)} - e^{\frac{i}{2k}(K_1^2(L-x_1)-K_2^2(L-x_2))} e^{i(\vec{K}_2\cdot\vec{r}_1-\vec{K}_1\cdot\vec{r}_2)}$$

$$\times \overline{dN^*(\vec{K}_1,\vec{r}_2) dN(\vec{K}_2,\vec{r}_1)}$$

$$- e^{\frac{i}{2k}(K_2^2(L-x_2)-K_1^2(L-x_1))} e^{i(\vec{K}_1\cdot\vec{r}_1-\vec{K}_2\cdot\vec{r}_2)} \overline{dN(\vec{K}_1,\vec{r}_1) dN^*(\vec{K}_2,\vec{r}_2)} \Bigg)$$

The statistical averaging is performed with the help of relation 2.4. It introduces delta functions $\delta(\vec{K}_1-\vec{K}_2)$ or $\delta(\vec{K}_1+\vec{K}_2)$; the integration with respect to $\vec{K}_2$ is then performed to yield

$$\overline{\beta^2} = 2 \int_0^L dx_1 \int_0^L dx_2 \int_{-\infty}^{\infty} d\vec{K} \left(\cos(\frac{K^2}{2k}(2L-x_1-x_2)) - \cos \vec{K}\cdot\vec{\rho} \right) \times F_n(\vec{K},|x_1-x_2|) \quad (5.6)$$

where $\vec{\rho} = \vec{r}_1-\vec{r}_2$. We then follow the same procedure which was used in Section 3.3 for the caculation of $\overline{u(L,\vec{r})}$ in the Rytov approximation.

In order to simplify expression 5.6, the following change of variables is introduced: $x_2-x_1 = \alpha$, $x_2+x_1 = 2\beta$. The integration with respect to β is then performed. For propagation distances L much larger than the outer scale of the turbulence and for a wavelength much smaller than the inner scale of the turbulence, $\overline{\beta^2}$ can be expressed as follows (see 2.3)

$$\overline{\beta^2} = 4L \int_0^L d\alpha \int_{-\infty}^{\infty} d\vec{K} \left(\frac{k}{K^2L} \sin(\frac{K^2L}{k}) - \cos K\cdot\vec{\rho}\right) F_n(\vec{K},\alpha)$$

or

$$\overline{\beta^2} = 4L \int_0^L d\alpha \int_{-\infty}^{\infty} d\vec{K}\left(\left(\frac{k}{K^2L} \sin \frac{K^2L}{k} - 1\right) + (1- \cos \vec{K}\cdot\vec{\rho})\right) F_n(\vec{K},\alpha) \quad (5.7)$$

$F_n(\vec{K},\alpha)$ is the transverse two-dimensional Fourier transform of the correlation function of the index of refraction; therefore,

$$\varepsilon^2 \int_{-\infty}^{\infty} d\vec{K}\ (1- \cos(\vec{K}\cdot\vec{\rho}))\ F_n(\vec{K},\alpha) = B_n(\alpha,\vec{\rho}=0) - B_n(\alpha,\vec{\rho})$$

$$= B_n(\alpha) - B_n(\sqrt{\alpha^2+ \rho^2})$$

for a homogeneous and isotropic index of refraction.

With the help of this last result, of expressions 5.7 and 5.4, and of the correct expression 4.34 for the correlation function $B_u(L,\rho)$, the correlation function $B_{u(RY)}(L,\rho)$ in the Rytov approximation is expressed in terms of $B_u(L,\rho)$ and of a correction factor:

$$B_{u(RY)}(L,\rho) = B_u(L,\rho)\ \exp\left|2\varepsilon^2 k^2 L \int_0^L d\alpha \int_{-\infty}^{\infty} d\vec{K}\ \left(1 - \frac{k}{K^2 L} \sin \frac{K^2 L}{k}\right) F_n(\vec{K},\alpha)\right| \tag{5.8}$$

The correction factor is the square of the correction factor which appeared in the expression 3.14 for $\overline{u_{(RY)}(L,\rho)}$. Since the correction factor in 5.8 does not depend upon ρ , $B_{u(RY)}(L,\rho)$ has the correct dependence upon ρ , the transverse coordinate. But the validity of $B_{u(RY)}(L,\rho)$ is limited to propagation distances and turbulence strength such that the correction factor in 5.8 remains much smaller than one, i.e. from 3.15, $0.62\ C_n^2 k^{7/6} L^{11/6} \ll 1$ in the Kolmogoroff spectrum.

CHAPTER VI - INVESTIGATION OF THE INTENSITY CORRELATION FUNCTION

$B_I(L,\vec{r}_1,\vec{r}_2) = \overline{I(L,\vec{r}_1)\ I(L,\vec{r}_2)}$

6.1 Introduction

Analytical expressions for the statistical mean $\overline{u(L,\vec{r})}$, the correlation function $\overline{u(L,\vec{r}_1)u^*(L,\vec{r}_2)}$ of an optical wave propagating through a randomly turbulent medium, have been obtained in Chapters II and IV under the condition $\lambda \ll \ell_o$, where λ is the wavelength of the wave and ℓ_o is the inner scale of the turbulence. In this section the investigation of another statistical quantity will be performed: the correlation function of the wave intensity $I(\vec{x}) = u(\vec{x})u^*(\vec{x})$ between two points $(L,\vec{r}_1)$ and $(L,\vec{r}_2)$ in a plane perpendicular to the direction of propagation of the wave after a propagation length L through a randomly turbulent atmosphere. It is defined as follows:

$$B_I(L,\vec{r}_1,\vec{r}_2) = \overline{I(L,\vec{r}_1)I(L,\vec{r}_2)} = \overline{u(L,\vec{r}_1)u^*(L,\vec{r}_1)u(L,\vec{r}_2)u^*(L,\vec{r}_2)} \tag{6.1}$$

The method of calculation is straightforward. The correlation function B_I is expressed as a power series expansion of ε, following 1.14

$$B_I = A_o^4 \sum_{m,p,s,t} \varepsilon^{(m+p+s+t)} \overline{\psi_m(1)\psi_p^*(1)\psi_s(2)\psi_t^*(2)} \tag{6.2}$$

where the indices m,p,s and t run from zero to infinity. The notation $\psi_m(1) \equiv \psi_m(L,\vec{r}_1)$ has been used and the ψ's of various orders are given by 1.36. The only terms contributing to B_I are the terms for which $m+p+s+t$ is an even number and we can write

$$B_I = A_o^4 \sum_{q=0}^{\infty} \varepsilon^{2q} B_{Iq} \quad \text{where} \quad B_{Iq} = \sum_{m+p+s+t=2q} \overline{\psi_m(1)\psi_p^*(1)\psi_s(2)\psi_t^*(2)} \tag{6.3}$$

The explicit calculation of $B_{Iq}(L,\vec{r}_1,\vec{r}_2)$ for $q=0$ and $q=1$ will be carried out in the remainder of this section. The complexity of the calculations for $q > 1$ will become apparent. However, the correct expression for $B_{I2}(L,\vec{r}_1,\vec{r}_2 = \vec{r}_1)$ will be given. It will then be shown that it is not possible to find a useful analytical expression for $B_I(L,\vec{r}_1,\vec{r}_2)$ and even $B_I(L,\vec{r},\vec{r}) = \overline{I^2(L,\vec{r})}$ without some approximations.

6.2 The Term of Order Zero $B_{I0}(L,\vec{r}_1,\vec{r}_2)$

From 6.3 and the fact that $\psi_o(L,\vec{r}) = 1$, we can readily write

$$B_{I0}(L,\vec{r}_1,\vec{r}_2) = 1 \tag{6.4}$$

6.3 The Term of Order One $B_{I1}(L,\vec{r}_1,\vec{r}_2)$

The term of order one in the power series expansion for $B_I(L,\vec{r}_1,\vec{r}_2)$ is given in 6.3 by the combinations of the indices m,p,s and t for which $m+p+s+t = 2$, i.e. the following $1+3+2\times3 = 10$ combinations

$$B_{I1} = \overline{(\psi_2(1)+\psi_2^*(1)+\psi_1(1)\psi_1^*(1))} + \overline{(\psi_2(2)+\psi_2^*(2)+\psi_1(2)\psi_1^*(2))} + \overline{(\psi_1(1)+\psi_1^*(1))(\psi_1(2)+\psi_1^*(2))} \tag{6.4a}$$

According to expression 4.4, the first two terms of expression 6.4a are respectively equal to $B_{u1}(L,\vec{r}_1,\vec{r}_1)$ and $B_{u1}(L,\vec{r}_2,\vec{r}_2)$ where $B_{u1}(L,\vec{r}_1,\vec{r}_2)$ is the first order term in the expansion 4.2 of $B_u(L,\vec{r}_1,\vec{r}_2)$, the wave function correlation function. The result for $B_{u1}(L,\vec{r}_1,\vec{r}_2)$ is given by 4.8. It is then found that

$$B_{u1}(L,\vec{r}_1,\vec{r}_1) = B_{u1}(L,\vec{r}_2,\vec{r}_2) = -k^2 \int_0^L dx_1 \int_0^L dx_2 \; F(|x_1-x_2|,0) = 0$$

since $F(|X|,\vec{\rho}=0) = 0$ according to the definition 4.9. The only term different from zero in 6.4a is the last one; then

$$B_{I1} = \overline{(\psi_1(1) + \psi_1^*(1))(\psi_1(2) + \psi_1^*(2))} \qquad (6.4b)$$

$\psi_1(\vec{x})$ is given by 1.21; therefore B_{I1} can be expressed as

$$B_{I1}(L,\vec{r}_1,\vec{r}_2 = (ik)(+ik) \int_0^L dx_1 \int_0^L dx_2 \int_{\vec{K}_1} \int_{\vec{K}_2}$$

$$\overline{\left(e^{-i\frac{K_1^2}{2k}(L-x_1)} e^{i\vec{K}_1\cdot\vec{r}_1} dN(\vec{K}_1,\vec{r}_1) - e^{i\frac{K_1^2}{2k}(L-x_1)} e^{-i\vec{K}_1\cdot\vec{r}_1} dN^*(\vec{K}_1,\vec{r}_1) \right)}$$

$$\overline{\left(e^{-i\frac{K_2^2}{2k}(L-x_2)} e^{i\vec{K}_2\cdot\vec{r}_2} dN(\vec{K}_2,\vec{r}_2) - e^{i\frac{K_2^2}{2k}(L-x_2)} e^{i\vec{K}_2\cdot\vec{r}_2} dN^*(\vec{K}_2,\vec{r}_2) \right)}$$

A similar calculation has been carried out in Section 5.3 (see equation 5.5). The statistical averaging is performed with the help of relation 2.4 which introduces the delta functions $\delta(\vec{K}_1 + \vec{K}_2)$ and $\delta(\vec{K}_1 - \vec{K}_2)$. The integration with respect to $\vec{K}_2$ is then carried out

$$B_{I1}(L,\vec{\rho}) = 2k^2 \int_0^L dx_1 \int_0^L dx_2 \int_{-\infty}^{\infty} d\vec{K} \left(\cos \vec{K}\cdot\vec{\rho} - \cos(\frac{K^2}{2k}(2L-x_1-x_2) - \vec{K}\cdot\vec{\rho})\right) \times F_n(\vec{K},|x_1-x_2|) \quad (6.5)$$

In order to simplify this integral, the following change of variables is made. $x_2-x_1 = \alpha$, $x_2+x_1 = 2\beta$. The integration with respect to β is easily performed; we make use of the assumption $\lambda << \ell_o$ to find

$$B_{I1}(L,\vec{\rho}) = 4k^2L \int_0^L d\alpha \int_{-\infty}^{\infty} d\vec{K} \left(1 - \frac{k}{K^2L} \sin \frac{K^2L}{k}\right) F_n(\vec{K},\alpha)\cos \vec{K}\cdot\vec{\rho} \quad (6.6)$$

A similar expression is considered by Tatarski (20) (the reason for the similarity will become apparent in Section 7.3.) The same arguments given by Tatarski will be used here to transform B_{I1} in the following way. The limit in the integral $\int_0^L d\alpha\, F_n(\vec{K},\alpha)$ can be extended to infinity since the function $F_n(\vec{K},\alpha)$ is zero for $\alpha > L_o$, the outer scale of the turbulence. The two-dimensional Fourier transform $F_n(K_y,K_z,\alpha)$ of the correlation function of the index of refraction fluctuations is related to the three-dimensional Fourier transform $\phi_n(K_x,K_y,K_z)$ in the following way:

$$\phi_n(K_x,K_y,K_z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(K_x\alpha)\, F_n(K_y,K_z,\alpha)\, d\alpha$$

therefore $\displaystyle\int_0^{\infty} F_n(K_y,K_z,\alpha)d\alpha = \pi\phi_n(0,K_y,K_z)$.

For a homogeneous and isotropic random index of refraction, the three-dimensional Fourier transform $\phi_n(K_x,K_y,K_z)$ is only a function of the magnitude of the vector (K_x,K_y,K_z), i.e. $\phi_n(K_x,K_y,K_z) = \phi_n(\sqrt{K_x^2+K_y^2+K_z^2})$. Recalling that $K^2= K_y^2 + K_z^2$, we have $\phi_n(0,K) = \phi_n(K)$. Then with the help of all the above considerations $B_{I1}(L,\vec{\rho})$ is expressed as

$$B_{I1}(L,\vec{\rho}) = 4\pi\, k^2 L \int_{-\infty}^{\infty} d\vec{K}\left(1 - \frac{k}{K^2 L}\sin\frac{K^2 L}{k}\right) \phi_n(K)\cos(\vec{K}\cdot\vec{\rho}) \qquad (6.7)$$

Polar coordinates are then used to simplify this expression, the integral $\int_{-\infty}^{\infty} d\vec{K}$ is replaced by $\int_0^{2\pi} d\phi \int_0^{\infty} K dK$. The integration with respect to ϕ is then easily performed, since

$$\int_0^{2\pi} d\phi\, \cos(K\rho\cos\phi) = 2\pi\, J_o(K\rho)$$

where J_o is the Bessel function of order zero. Then finally

$$B_{I1}(L,\rho) = 8\pi^2 k^2 L \int_0^{\infty} J_o(K\rho)\left(1 - \frac{k}{K^2 L}\sin\frac{K^2 L}{k}\right) \phi_n(K)\, K dK \qquad (6.8)$$

The first order term (in ε^2) $B_{I1}(L,\rho)$ in the power series expansion for the intensity correlation function $B_I(L,\rho)$ is given by expression 6.8 in terms of the three-dimensional Fourier transform $\phi_n(K)$ of the correlation function of the index of refraction fluctuations.

The next step would be to calculate the second order term (in ε^4) $B_{I2}(L,\rho)$; this calculation, although very tedious, has been attempted for the simpler case $\rho = 0$; the results are given in the following section.

6.4 The Term of Order Two $B_{I2}(L,\vec{r},\vec{r})$

The calculation of the second order term (in ε^4) $B_{I2}(L,\vec{r}_1,\vec{r}_2)$ involves the computation of all the terms in 6.3 for which the indices $m+p+s+t = 4$. There are 35 such combinations. B_{I2} can be expressed as

$$
\begin{aligned}
B_{I2}(L,\vec{r}_1,\vec{r}_2) &= B_{u2}(L,\vec{r}_1,\vec{r}_1) + B_{u2}(L,\vec{r}_2,\vec{r}_2) \\
&\quad + \overline{(\psi_1(1) + \psi_1^*(1))(\psi_3(2) + \psi_3^*(2) + \psi_2(2)\psi_1^*(2) + \psi_2^*(2)\psi_1(2))} \\
&\quad + \text{same term as above, where } \vec{r}_1 \text{ and } \vec{r}_2 \text{ are interchanged} \\
&\quad + \overline{(\psi_2(1) + \psi_2^*(1) + \psi_1(1)\psi_1^*(1))(\psi_2(2) + \psi_2^*(2) + \psi_1(2)\psi_1^*(2))}
\end{aligned}
\qquad (6.9)
$$

The first two terms are equal to zero according to 4.27, since $B_{u2}(L,\vec{r},\vec{r}) = 0$. The computation of all the other terms in 6.9 has been carried out. The result for $B_{I2}(L,\vec{r},\vec{r})$ is given by expression 6.10. This expression is rather formidable; many attempts to greatly simplify it have failed. This is mainly due to the presence of the terms involving the dot product $\vec{K}_1 \cdot \vec{K}_2$ in the integrals. If there were no dot products $\vec{K}_1 \cdot \vec{K}_2$, expression 6.10 would reduce to its first term equal to $\frac{1}{2}(B_{I1})^2$ and we could expect $B_I(L,\vec{r},\vec{r})$ to be equal to $\exp(\varepsilon^2 B_{I1}(L,\vec{r},\vec{r}))$ where $B_{I1}(L,\vec{r},\vec{r})$ is given by 6.8 with $\rho = 0$, but there is no apparent reason to drop the dot products $\vec{K}_1 \cdot \vec{K}_2$. We emphasize this point in view of the upcoming comparison of our results with similar results obtained by using various approximations.

$$
\begin{aligned}
B_{I2}(L,0) = \frac{1}{2}\Bigg(2k^2 \int_0^L dx_1 \int_0^L dx_2 \int_{-\infty}^{\infty} d\vec{K}\ \Big(1 - \cos \frac{K^2}{2k}(2L-x_1-x_2)\Big)\ F_n(\vec{K},|x_1-x_2|)\Bigg)^2 \\
+ k^4 \int_0^L dx_1 \int_0^L dx_2 \int_0^L dx_3 \int_0^L dx_4 \int_{-\infty}^{\infty} d\vec{K}_1 \int_{-\infty}^{\infty} d\vec{K}_2\ F_n(\vec{K}_1,|x_1-x_2|)\ F_n(\vec{K}_2,|x_3-x_4|) \\
\times \Bigg[-8 + 12\cos\Big(\frac{K_1^2}{2k}(2L-x_1-x_2)\Big) - \cos\Big(\frac{K_1^2}{2k}(2L-x_1-x_2)\Big)\cos\Big(\frac{K_2^2}{2k}(2L-x_3-x_4)\Big) \\
- 12\cos\left(\frac{1}{2k}\Big(K_1^2(2L-x_1-x_2) + 2\vec{K}_1\cdot\vec{K}_2(L-x_1)\Big)\right) + 8\cos\Big(\frac{\vec{K}_1\cdot\vec{K}_2}{k}(L-x_1)\Big)\Bigg] \\
+ 2k^4 \int_0^L dx_1 \int_0^{x_1} dx_2 \int_0^L dx_3 \int_0^{x_3} dx_4 \int_{-\infty}^{\infty} d\vec{K}_1 \int_{-\infty}^{\infty} d\vec{K}_2 \Bigg[\cos\Big(\frac{K_1^2}{2k}(2L-x_1-x_3) + \frac{\vec{K}_1\cdot\vec{K}_2}{k}(2L-x_1-x_3)\Big)\cos\Big(\frac{K_2^2}{2k}(2L-x_2-x_4)\Big) \\
\times\ F_n(\vec{K}_1,|x_1-x_3|)\ F_n(\vec{K}_2,|x_2-x_4|) \\
+ \cos\Big(\frac{K_1^2}{2k}(2L-x_1-x_4) + \frac{\vec{K}_1\cdot\vec{K}_2}{k}(2L-x_1-x_3)\Big)\cos\Big(\frac{K_2^2}{2k}(2L-x_2-x_3)\Big)\ F_n(K_1,|x_1-x_4|)\ F_n(\vec{K}_2,|x_2-x_4|)\Bigg]
\end{aligned}
\tag{6.10}
$$

The calculation of B_{I3} and higher order terms becomes impractical and worthless since no analytical expression for the correlation function $B_I(L,\vec{r}_1,\vec{r}_2)$ can be anticipated from the look of the first order (in ε^2) and second order (in ε^4) terms. In the next section, the results obtained for the correlation function $B_I(L,\vec{r}_1,\vec{r}_2)$ and $B_I(L,\vec{r},\vec{r}) = B_I(L,0)$ are summed up.

6.5 Considerations on the Variance of the Intensity Fluctuations $B_I(L,0) = \overline{(I(L,\vec{r}))^2}$ and on the Intensity Correlation Function $B_I(L,\vec{r}_1-\vec{r}_2) = \overline{I(L,\vec{r}_1)I(L,\vec{r}_2)}$

The variance of the intensity fluctuations $B_I(L,0) = \overline{(I(L,\vec{r}))^2}$ of an optical wave propagating a distance L through a randomly turbulent atmosphere has been expressed in a power series expansion of ε^2, the variance of the index of refraction fluctuations

$$B_I(L,0) = A_o^4 \sum_{q=0}^{\infty} \varepsilon^{2q} B_{Iq}(L,0) \tag{6.3}$$

The terms q=0,1 and 2 have been explicitly calculated. The results are given by expressions 6.4, 6.8 with $\rho = 0$, and 6.10. Since there is no apparent recursion relationship relating $B_{I1}(L,0)$ and $B_{I2}(L,0)$, no guess can be formulated for an analytical expression for the sum $B_I(L,0) = A_o^4(1 + \varepsilon^2 B_{I1}(L,0) + \varepsilon^4 B_{I2}(L,0) + \cdots)$. In the view of a later discussion, we notice that $B_{I2}(L,0)$ given by 6.10 is not equal to $\frac{1}{2}(B_{I1}(L,0))^2$ or $-\frac{1}{2}(B_{I1}(L,0))^2$. It is then concluded that there exists no exact simple theoretical expression for the variance of the intensity fluctuations $B_I(L,0)$ while such exact expressions could be found for $u(L,\vec{r})$ and the correlation function $u(L,\vec{r}_1)\ u^*(L,\vec{r}_2)$. Analytical expressions chosen to fit the experimental data have been proposed by some authors (21),(22); these expressions, which are not the results of a theoretical investigation, are useful, however, in practical calculations. We shall extend this discussion in Chapter VIII.

The first order term $B_{I1}(L,\vec{r}_1,\vec{r}_2)$ in the power series expansion 6.3 for the intensity correlation function $B_I(L,\vec{r}_1,\vec{r}_2)$ has been explicitly calculated. The second order term $B_{I2}(L,\vec{r}_1,\vec{r}_2)$ has also been calculated, although the result has not been explicitly written down because of its complexity. No simple recursion relationship between $B_{I1}(L,\vec{r}_1,\vec{r}_2)$ and $B_{I2}(L,\vec{r}_1,\vec{r}_2)$ can be found without some approximations; therefore there exists no simple analytical expression for the correlation function $B_I(L,\vec{r}_1-\vec{r}_2)$ which is valid for any propagation distance L and any strength of turbulence. In the next section we shall calculate this function using various approximations.

CHAPTER VII - THE INTENSITY CORRELATION FUNCTION $B_I(L,\rho)$ UNDER VARIOUS APPROXIMATIONS

7.1 $B_I(L,\rho)$ in the Born Approximation: $B_{I(B)}(L,\rho)$

7.1.1 Formal calculation of $B_{I(B)}(L,\rho)$ In the Born approximation, the solution of the wave equation 1.1 is given by expression 1.37

$$u_{(B)}(L,\vec{r}) = A_o e^{ikL}(1+\varepsilon\psi_1(L,\vec{r}))\ ,\ \text{where}\ \ \psi_1(L,\vec{r})\ \text{is given by 1.21}$$

Therefore, the correlation function $B_{I(B)}$ can be expressed as follows:

$$B_{I(B)} = A_o^4\left(1+\varepsilon^2\left[\overline{\psi_1(1)\psi_1^*(1)} + \overline{\psi_1(2)\psi_1^*(2)} + \overline{(\psi_1(1)+\psi_1^*(1))(\psi_1(2)+\psi_1^*(2))}\right] + \varepsilon^4\ \overline{\psi_1(1)\psi_1^*(1)\ \psi_1(2)\psi_1^*(2)}\right) \qquad (7.1)$$

The notations $\psi_1(L,\vec{r}_1) \equiv \psi_1(1)$ and $\psi_1(L,\vec{r}_2) \equiv \psi_1(2)$ have been used. Let us first look at the terms in ε^2 which have been calculated previously. The first two terms are given by expression 4.7, where $\vec{r}_1 = \vec{r}_2$, therefore $\overline{\psi_1(1)\psi_1^*(1)} + \overline{\psi_1(2)\psi_1^*(2)} =$

$$2k^2\int_0^L dx_1\int_0^L dx_2\int_{-\infty}^{\infty} d\vec{K}\ F_n(\vec{K},|x_1-x_2|)$$

which can be expressed as

$$\frac{4}{\varepsilon^2}\,k^2L\int_0^L B_n(\alpha)\ d\alpha \qquad (7.2)$$

where $B_n(\alpha)$ is the correlation function of the index of refraction. The third term in ε^2 in 7.1 is equal to $B_{I1}(L,\vec{r}_1,\vec{r}_2)$ which is given by 6.5 and 6.8. We call $B_{I1(B)}$ the term in ε^2 and write

$$B_{I1(B)}(L,\rho) = 4k^2L\left(\int_0^{L_o} B_n(\alpha)\,d\alpha + 2\pi^2\int_0^{\infty} J_o(K\rho)\left(1-\frac{k}{K^2L}\sin\frac{K^2L}{k}\right) \times \emptyset_n(K)\,KdK\right) \tag{7.3}$$

The term in ε^4 has already been computed in the calculation of $B_{I2}(L,\vec{\rho})$, (see 6.9). We call this term $B_{I2(B)}$

$$\begin{aligned} B_{I2(B)} = {} & \varepsilon^4k^4\left(\int_0^L dx_1\int_0^L dx_2\int_{-\infty}^{\infty} d\vec{K}\, F_n(\vec{K},|x_1-x_2|)\right)^2 \\ & + \varepsilon^4k^4\left(\int_0^L dx_1\int_0^L dx_2\int_{-\infty}^{\infty} d\vec{K}\cos(\vec{K}\cdot\vec{\rho})\, F_n(\vec{K},|x_1-x_2|)\right)^2 \\ & + \varepsilon^4k^4\left(\int_0^L dx_1\int_0^L dx_2\int_{-\infty}^{\infty} d\vec{K}\cos\left(\frac{K^2}{2k}(x_1+x_2)+\vec{K}\cdot\vec{\rho}\right) F_n(\vec{K},|x_1-x_2|)\right)^2 \end{aligned} \tag{7.4}$$

Each of the three terms appearing in $B_{I2(B)}$ is equal to $\frac{1}{4}$ of the square of one term in $B_{I1(B)}$, i.e. if $B_{I1(B)} = a+b+c$, then

$$B_{I2(B)} = \frac{1}{4}(a^2+b^2+c^2) \quad , \text{ where } \quad a = 4k^2L\int_0^{L_o} B_n(\alpha)\,d\alpha \tag{7.5}$$

$$b = 4k^2L\; 2\pi^2\int_0^{\infty} J_o(K\rho)\,\emptyset_n(K)\,K\,dK = 4k^2L\int_0^{L_o} B_n\left(\sqrt{\alpha^2+\rho^2}\right)d\alpha \tag{7.6}$$

and $$c = -4k^2L\,2\pi^2 \int_0^\infty J_o(K\rho)\,\frac{k}{K^2L}\,\sin\left(\frac{K^2L}{k}\right)\,\phi_n(K)\,K\,dK$$

The intensity correlation function $B_{I(B)}(L,\rho)$ in the Born approximation is given by the following expression

$$B_{IB}(L,\rho) = A_o^4(1 + B_{I1(B)} + B_{I2(B)}) \tag{7.7}$$

where $B_{I1(B)}$ and $B_{I2}(B)$ are given by 7.3 and 7.4.

We now calculate these quantities explicitly in the Kolmogoroff spectrum for $\rho = 0$.

7.1.2 Explicit expression of $B_{I(B)}(L,0)$ in the Kolmogoroff spectrum. In this paragraph the variance of the intensity fluctuations $B_{I(B)}(L,0) = \overline{(I_{(B)}(L,\vec{r}))^2}$ is explicitly expressed in terms of the parameters of the turbulent medium in the Kolmogoroff spectrum, i.e. the model of atmospheric turbulence in which the correlation function of the index of refraction is given by 2.45. According to 7.7,

$$B_{IB}(L,0) = A_o^4(1 + B_{I1(B)}(L,0) + B_{I2(B)}(L,0))$$

and $$B_{I1(B)}(L,0) = a + b + c\ ,$$

where
$$a = 4k^2L \int_0^{L_o} B_n(\alpha)\,d\alpha = \frac{4}{5}\,k^2C_n^2L_o^{5/3}L$$

The last equality has been written with the help of 2.46. For $\rho = 0$, $b = a$ according to 7.5 and 7.6. The expression

$$b + c = 8\,{}^{2}k^{2}L \int_{0} \left(1 - \frac{k}{K^{2}L} \sin \frac{K^{2}L}{k}\right) \phi_{n}(K)\, K\, dK$$

has been calculated by Tatarski (19). The result is

$$b + c = 1.23\, C_n^2\, k^{7/6}\, L^{11/6} = \varepsilon^2 B_{I1}(L,0) \tag{7.8}$$

The quantity $b + c$ is also equal to the first order term $B_{I1}(L,0)$ in the power series expansion 6.3 of the variance of the intensity fluctuations $B_I(L,0)$ (see equation 6.6 with $\rho = 0$).

Let us look at the ratio

$$\frac{a}{b+c} = \frac{0.8k^2\, C_n^2\, L_o^{5/3}\, L}{1.23k^{7/6}\, C_n^2\, L^{11/6}} = 0.6\, \frac{k^{5/6}\, L_o^{5/3}}{L^{5/6}}$$

or $\frac{a}{b+c} = 0.6\, (\frac{2\pi L_o^2}{\lambda\, L})^{5/6}$. This ratio is much larger than one for distances L such that $L < L_o^2/\lambda \sim 10^6 m$ for $L_o = 1$ and $\lambda = 1\mu$. Therefore, for short distances where we expect the Born approximation to hold, we can neglect $b + c$ with respect to a and also write $a^2 + b^2 + c^2 \sim 3a^2$. The variance of the intensity fluctuations in the Born approximation is then expressed as follows in the Kolmogoroff spectrum:

$$B_{I(B)}(L,0) = A_o^4 \left(1 + \frac{4}{5}\, k^2 C_n^2 L_o^{5/3} L + \frac{3}{4}\left(\frac{4}{5}\, k^2 C_n^2 L_o^{5/3} L\right)^2\right)$$

This expression is only valid for $\frac{4}{5}\, k^2 C_n^2 L_o^{5/3} L << 1$, in which case

$$B_{IB}(L,0) = A_o^4 (1 + 0.8k^2 C_n^2 L_o^{5/3} L) \tag{7.9}$$

7.2 $B_I(L,\rho)$ in the Geometrical Optics Approximation: $B_{I(GO)}(L,\rho)$

The solution of the wave equation 1.1 in the geometrical optics approximation is given by expression 1.43. The correlation function of the intensity fluctuations in the geometrical optics approximation: $B_{I(GO)}(L,\rho)$ is then given by

$$B_{I(GO)}(L,\rho) = \overline{u(L,\vec{r}_1)u^*(L,\vec{r}_1)u(L,\vec{r}_2)u^*(L,\vec{r}_2)}$$

$$= \overline{A_o^4 \exp\left(i\varepsilon k \int_0^L dx(n_1(x,\vec{r}_1) - n_1(x,\vec{r}_1) + n_1(x,\vec{r}_2) - n_1(x,\vec{r}_2))\right)}$$

therefore

$$B_{I(GO)}(L,\rho) = A_o^4 \tag{7.10}$$

The geometrical optics approximation does not account in a satisfactory way for the intensity fluctuations of the wave, although it gave correct expressions for $\overline{u(L,\vec{r})}$ and the correlation function $\overline{u(L,\vec{r}_1)u^*(L,\vec{r}_2)}$.

7.3 $B_I(L,\rho)$ in the Rytov Approximation: $B_{I(RY)}(L,\rho)$

The solution of the wave equation 1.1 in the Rytov approximation is given by expression 1.48 or, in terms of $\psi_1(L,\vec{r})$, the first order term 1.21 in the expansion 1.14 for $\psi(L,\vec{r}) = u(L,\vec{r})/u_o(L)$

$$u_{(RY)}(L,\vec{r}) = A_o e^{ikL} e^{\varepsilon\psi_1(L,\vec{r})}$$

The intensity correlation function is then expressed as follows:

$$B_{I(RY)}(L,\vec{\rho}) = A_o^4 \, \overline{\exp\, \varepsilon(\psi_1(L,\vec{r}_1)+ \psi_1^*(L,\vec{r}_1)+ \psi_1(L,\vec{r}_2)+ \psi_1^*(L,\vec{r}_2))} \quad (7.11)$$

or we can write

$$B_{I(RY)} = A_o^4 \, \overline{e^{\varepsilon\beta(L,\rho)}} \quad (7.12)$$

where $\beta(L,\rho)$ is defined by 7.11 and 7.12. $\beta(L,\rho)$ is a Gaussian random variable with zero mean; therefore

$$B_{I(RY)} = A_o^4 \, e^{\varepsilon^2\overline{\beta^2}/2} \quad (7.13)$$

Then $\overline{\beta^2} = \overline{(\psi_1(1)+ \psi_1^*(1) + \psi_1(2) + \psi_1^*(2))^2}$ where the notation $\psi_1(L,\vec{r}_1) \equiv \psi_1(1)$ has been used,

$$\overline{\beta^2} = 2\overline{(\psi_1(1)+ \psi_1^*(1))(\psi_1(2)+ \psi_1^*(2))} + \overline{(\psi_1(1)+ \psi_1^*(1))^2} + \overline{(\psi_1(2) + \psi_1^*(2))^2}$$

The first term in $\overline{\beta^2}$ is equal to $2B_{\underline{I1}}(L,\rho)$ (see expression 6.4b). We notice that the last two terms in $\overline{\beta^2}$ can be written as

$$\overline{(\psi_1(1)+ \psi_1^*(1))^2} + \overline{(\psi_1(2)+ \psi_1^*(2))^2} = B_{I1}(L,\vec{r}_1,\vec{r}_1)+ B_{I1}(L,\vec{r}_2,\vec{r}_2)$$
$$= 2B_{I1}(L,0)$$

Therefore $\overline{\beta^2} = 2B_{I1}(L,\rho) + 2B_{I1}(L,0)$, and from 7.13,

$$B_{I(RY)}(L,\rho) = A_o^4 \exp\, \varepsilon^2(B_{I1}(L,\rho) + B_{I1}(L,0)) \quad (7.14)$$

or

$$B_{I(RY)}(L,\rho) = A_o^4 \exp\left(8\pi^2 k^2 L \int_0^\infty (1+J_o(K\rho))(1-\frac{k}{K^2 L}\sin\frac{K^2 L}{k}) \times \phi_n(K)\, K\, dK\right) \qquad (7.15)$$

This expression represents the correlation function in the Rytov approximation of the intensity fluctuations of an optical wave propagating a distance L through random atmospheric turbulence. $\phi_n(K)$ is the three-dimensional Fourier transform of the correlation function of the index of refraction. This expression is in agreement with Tatarski since in its terms the correlation function $B_{I(RY)}(L,\rho)$ can be expressed as follows: (see Appendix B),

$$B_{I(RY)}(L,\rho) = \overline{I(L,\vec{r}_1)I(L,\vec{r}_2)} = A_o^4 \exp(4\overline{X^2} + 4B_A(\rho)) \qquad (7.16)$$

where $B_A(\rho)$ is the correlation function of the logarithm of the amplitude of the wave and $\overline{X^2} = B_A(0)$. $B_A(\rho)$ and $\overline{X^2}$ are given by equations 7.54 and 7.50 in Tatarski (23); their expressions used in 7.16 lead to expression 7.15.

In order to compare the expression $B_{I(RY)}(L,\rho)$ and the correct expression for $B_I(L,\rho)$ calculated in Chapter VI, we can only look at the first order term in ε^2 since only this term has been explicitly found in the expansion of $B_I(L,\rho)$. The first order term in the correct expression for $B_I(L,\rho)$ is $B_{I1}(L,\rho)$ given by 6.8. The first order term in the Rytov approximation is, according to 7.14,

$B_{I1}(L,\rho) + B_{I1}(L,0)$. Even in the first order in ε^2 the correct expression for $B_I(L,\rho)$ and its counterpart in the Rytov approximation differ; therefore the expression for $B_{I(RY)}(L,\rho)$ is expected to give a valid result when $B_{I1}(L,0) << 1$ or, according to 7.8, $1.23\ C_n^2\ k^{7/16}\ L^{11/6} << 1$ in the Kolmogoroff model of turbulence. This result has been verified experimentally by measuring the variance of the intensity fluctuations $\overline{I^2}$. In the next section we examine the experimental results available in the literature for the variance of the intensity fluctuations $\overline{I^2}$ and relate them to the results obtained in Chapter VI and other results obtained by various authors.

CHAPTER VIII - PROBLEMS OF CURRENT INTEREST IN THE FIELD OF OPTICAL WAVE PROPAGATION THROUGH A RANDOM MEDIUM

In Chapter VI we have tried to find an analytical expression for the intensity correlation function $B_I(L,\vec{r}) = \overline{I(L,\vec{r}_1)\ I(L,\vec{r}_1+\vec{r})}$ and the variance of the intensity fluctuations $B_I(L,0) = \overline{I^2(L,\vec{r})}$. We have shown that it is not possible to find such expression without some approximations. We have calculated explicitly the first two terms $B_{I1}(L,0)$ and $B_{I2}(L,0)$ in the power series expansion 6.3 for $B_I(L,0)$,

$$B_I(L,0) = \overline{I^2(L,\vec{r})} = A_o^4(1+ \varepsilon^2 B_{I1}(L,0) + \varepsilon^4 B_{I2}(L,0) + \cdots) \tag{8.1}$$

Using the Kolmogoroff spectrum, we have derived the following expression for the first term $\varepsilon^2\ B_{I1}(L,0)$ (we call it σ_1^2) in terms of the parameters of the turbulence and of the wave. According to 7.8,

$$\varepsilon^2 B_{I1}(L,0) = \sigma_1^2 = 1.23\ C_n^2\ k^{7/6}\ L^{11/6} \tag{8.2}$$

L is the distance of propagation of the wave through the atmospheric turbulence, k is the wave number and C_n is the structure constant.

In this chapter we compare our results with recent experimental results. We point out their salient features and we discuss how the work in the field of optical propagation through random media can be extended to explain the results.

Some experimental work has been performed recently to determine the dependence of the intensity fluctuations upon the turbulence conditions (8),(24),(16),(25). In these references the function

$$\sigma_I^2 = \overline{(\log I - \overline{\log I})^2}$$

is investigated. σ_I^2 is the variance of the fluctuations in the logarithm of the wave intensity. If we assume that $\log I$ is normally distributed, we can write

$$\sigma_I^2 = \overline{(\log I - \overline{\log I})^2} = \log\left(\frac{\overline{I^2}}{(\overline{I})^2}\right) = \log\left(\frac{B_I(L,0)}{(B_u(L,0))^2}\right) \tag{8.3}$$

where $B_I(L,0)$ is the variance of the intensity fluctuations and $B_u(L,0)$ is the average intensity.

According to the results of Chapter IV, we have $B_u(L,0) = A_o^2$. With the help of 8.1, 8.2 and 8.3, we write

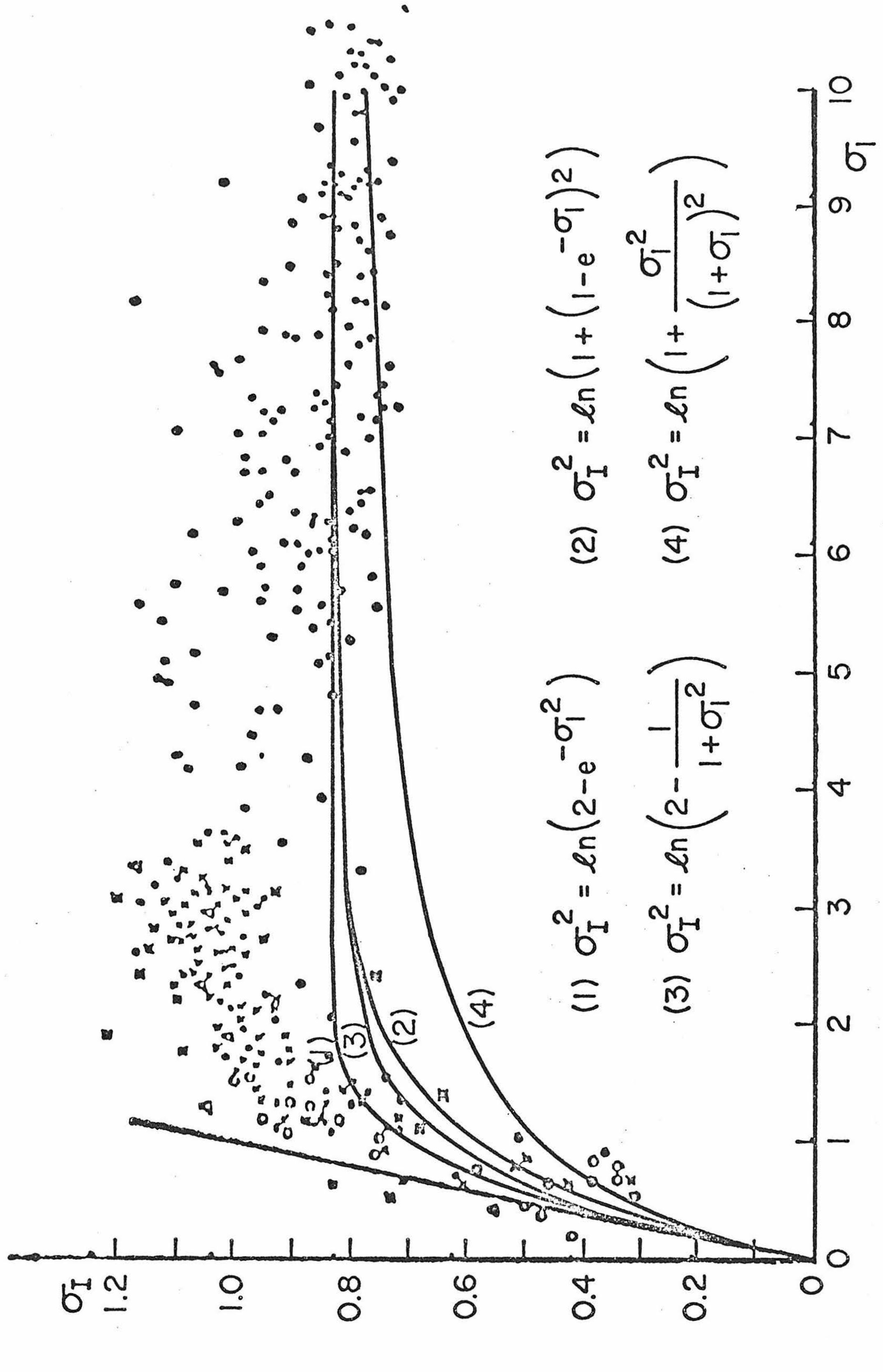
$$\sigma_I^2 = \log(1 + \sigma_1^2 + \varepsilon^4 B_{I2}(L,0) + \cdots) \tag{8.4}$$

Figure 8.1 shows the results of Gracheva (24). σ_I^2 is plotted as a function of σ_1^2 . σ_I^2 increases as σ_1^2 for small σ_1^2 .

From 8.4 we see that if $\sigma_1^2 << 1$, then

$$\sigma_I^2 \sim \log(1 + \sigma_1^2) \sim \sigma_1^2 \quad .$$

Then as σ_1^2 approaches the value of one, σ_I^2 increases more slowly, passes through a maximum, and eventually saturates at the value of 0.8 when $\sigma_1^2 >> 1$. Other workers have observed the saturation of the variance of the logarithm of the intensity for strong turbulence and long propagation distances ($\sigma_1^2 >> 1$) but failed to notice a pronounced maximum of this function.



σ_I
0
0.2
0.4
0.6
0.8
1.0
1.2
0
1
2
3
4
5
6
7
8
9
10
σ_l
(1)
(2)
(3)
(4)
(1) $\sigma_I^2 = \ln\left(2 - e^{-\sigma_l^2}\right)$
(2) $\sigma_I^2 = \ln\left(1 + \left(1 - e^{-\sigma_l}\right)^2\right)$
(3) $\sigma_I^2 = \ln\left(2 - \frac{1}{1+\sigma_l^2}\right)$
(4) $\sigma_I^2 = \ln\left(1 + \frac{\sigma_l^2}{(1+\sigma_l)^2}\right)$

Note that in the Rytov approximation $\sigma_I^2 = \sigma_1^2$; therefore this approximation is valid only when $\sigma_1^2 << 1$.

No satisfactory theoretical explanation has been given for the observed behavior of σ_I^2 when σ_1^2 is larger than one. However, it is interesting to point out the work of Salpeter (26) in a similar area. This author theoretically investigated the behavior of the variance $\overline{I^2}$ of the intensity of a wave after a passage through a phase screen. A phase screen is a medium which modifies the phase of a wave passing through but not its amplitude. He found that $\overline{I^2}$ is an increasing function of the parameter $\beta = \sqrt{Z\phi_o/ka^2}$ where Z is the distance from the phase screen; k is the wave number. ϕ_o and a are two parameters which characterize the phase screen, ϕ_o is the rms phase disturbance and a is the correlation length of the phase fluctuations. $\overline{I^2}$ increases as β^2 for $\beta < 1$ and has a maximum for $\beta = 1$. The value $\beta = 1$ corresponds to a distance $Z = ka^2/\phi_o$ which is the average focal length of the phase screen. $\overline{I^2}$ then reaches a constant value for $\beta > 1$. Although the similarity between a thin phase screen and a turbulent atmosphere is not evident, the analysis of Salpeter could be used to explain the experimental results obtained for σ_I^2 .

Some authors have proposed empirical formulas for σ_I^2 as a function of σ_1^2 designed to fit the experimental data. These formulas which are derived from intuitive arguments can be useful for practical calculations. We give four examples:

(1) $\sigma_I^2 = \ell n(2 - e^{-\sigma_1^2})$ Reference (21)

(2) $\sigma_I^2 = \ell n(1 + (1 - e^{-\sigma_1})^2)$ Reference (22)

(3) $\sigma_I^2 = \ln(2 - \frac{1}{1 + \sigma_1^2})$

(4) $\sigma_I^2 = \ln(1 + \frac{\sigma_1^2}{(1 + \sigma_1)^2})$

These four functions are plotted in Figure 8.1, together with the experimental data of Gracheva. The fit is good for $\sigma_1 << 1$ and $\sigma_1 >> 1$ but, since the four functions are monotonous functions, the empirical formulas are not in good agreement with the experimental results when σ_1 has a value close to one.

More experimental results on the behavior of the intensity fluctuations are needed in order to obtain a precise evaluation of any optical system operating through the atmospheric turbulence.

Another important question partially unanswered, concerning the propagation of an optical wave through a randomly turbulent atmosphere, is: What is the probability distribution of its intensity fluctuation? Some authors (1) have suggested that it is a log-normal distribution; i.e. $\ln I$ is normally distributed. Others (5)(21) have predicted a Rayleigh distribution. By a simple physical argument we show how these two distributions can arise.

(1) Log-normal distribution. Assume that we have an optical link through the atmospheric turbulence where the receiver is inside the turbulent area. Let us suppose also that the turbulent medium is composed of a large number of slabs. A_o is the amplitude of the optical electric field at the source. After having passed through the first slab, the amplitude of the field is equal to A_o multiplied by

a random factor R_1 . R_1 is a random function of the turbulence in the first slab. The amplitude of the field is $A_1 = A_o R_1$ at the output of the first slab. At the output of the second slab the amplitude is $A_2 = A_1 R_2 = A_o R_1 R_2$ where R_2 is a random function of the turbulence in the second slab. At the receiver, the wave has passed through N slabs and the field is

$$A = A_o R_1 R_2 \cdots R_N$$

Its logarithm is

$$\ell n\, A = \ell n\, A_o + \sum_{n=1}^{N} \ell n\, R_n \;.$$

If the lengths of the slabs are larger than the outer scale of the turbulence L_o , we can consider that the random functions $R_1, R_2 \cdots R_N$ are independent of each other. According to the central limit theorem (30) $\ell n\, A$ tends to a Gaussian distribution, i.e., the amplitude and the intensity have a log-normal distribution.

(2) Rayleigh distribution. The optical field at the receiver can in some cases be equal to the sum of a large number of random fields which have been scattered independently by different portions of the turbulence. This situation occurs when the effect of multiple scsttering is important, i.e. when the unscattered part of the wave function $\overline{u(L)}$ calculated in Chapter II is such that $|\overline{u(L)}| << A_o$ or, according to 2.47, $\frac{1}{5} k^2 C_n^2 L_o^{5/3} L >> 1$. We have seen that under intermediate turbulence with $L_o = 1\text{m}$, this condition is fulfilled for $L > 7.1$ km.

The application of the central limit theorem to a large number of independently scattered fields leads to the conclusion that the field at the receiver has a Gaussian distribution. The phase of such fields has a uniform probability distribution and its amplitude has a Rayleigh probability distribution (31).

Experimental results to date (8)(24)(16)(25) show evidence of a log-normal distribution. However, the accuracy of these results has been disputed (21) for the following reason. The experiments have tested the distribution around its mean value where the log-normal and the Rayleigh distributions are very similar. A valuable test would be to examine carefully the tails of the probability distribution function where the two distributions differ. An X^2 test performed on the experimental data would provide useful information on the shape of the probability distribution of the intensity fluctuations.

In this chapter we have pointed out the following problems of current interest in the field of optical wave propagation through a random medium: (a) A theoretical explanation of the saturation of the variance of the intensity fluctuations with turbulence strength and propagation distance. (b) What is the probability distribution function of the intensity fluctuations?

The answers to these problems would be useful in evaluating the performances of various optical communication schemes through the atmospheric turbulence as we shall see in the following chapter.

CHAPTER IX - OPTICAL COMMUNICATION THROUGH RANDOM ATMOSPHERIC TURBULENCE

9.1 Introduction

In the previous chapters we have considered the propagation of an optical wave through a random atmospheric turbulence, and we have derived statistical information for the wave function in terms of the statistical properties of the index of refraction of the turbulent medium. Analytical expressions for the mean wave function $\overline{u(\vec{x})}$ and the correlation function $\overline{u(L,\vec{r}_1)u^*(L,\vec{r}_2)}$ have been obtained. The intensity correlation function $\overline{I(L,\vec{r}_1)I(L,\vec{r}_2)}$ has also been investigated. In this chapter these results are applied to the comparative evaluation of two schemes of optical communication:

Scheme (1). Heterodyne detection of a phase-modulated optical beam

Scheme (2). Video communication

In both schemes the information to be transmitted is impressed upon an optical carrier at frequency ω_o by means of a phase modulator operating at a frequency ω_m . Sidebands at frequencies $\omega_o - \omega_m$ and $\omega_o + \omega_m$ are generated. In Scheme (1) the local oscillator electric field at a fixed frequency $\omega_o + \Delta\omega$ is mixed in the plane of the nonlinear detector with the modulated optical electric field. The output current of the detector has an oscillating component at a frequency equal to the difference frequency between the local oscillator frequency and the modulated optical signal frequency $\Delta\omega \pm \omega_m$. The output current of the detector is then fed into an amplifier tuned around the frequency $\Delta\omega$. Apart from the fixed frequency shift $\Delta\omega$, the output current of the amplifier is a replica of the modulating signal, i.e. the information to be transmitted. It is very important that the frequency shift $\Delta\omega$ be constant in time. Therefore, in a practical heterodyne optical communication system, using two lasers--one as the optical carrier and the other as the local oscillator--sophisticated electronic equipment is needed to stabilize the frequency of each laser and to lock their frequencies together.

An alternative method is video communication. In Scheme (2) the reference signal is transmitted with the phase modulated optical beam along the communication path. In practice, the linearly polarized electric vector of a laser is divided into two linearly polarized components at right angles to each other. One component is phase-modulated; the other component is used as the transmitted reference. Since the carrier frequency and the reference frequency are the same, the output current of the nonlinear detector is proportional to the modulating voltage. The detection of the phase-modulated information in Scheme (2) does not require a local oscillator and therefore the complexity of the receiving apparatus is reduced, while the complexity of the transmitting apparatus is not increased. In this chapter we shall calculate the signal-to-noise ratios (S/N) in the output current of the detector for both schemes of communication in terms of the statistical functions of the wave function which have been evaluated in the previous chapters. The signal-to-noise ratios for heterodyne and video communication schemes will then be compared and their comparative performances will be numerically expressed in terms of the length of the communication link and the strength of the turbulence and diameter of the detector.

9.2 Signal-to-Noise Ratio in Optical Mixing Detection

The detection of a phase-modulated optical signal is accomplished by mixing it with a reference signal on a square law detector such as a photomultiplier. In the most general case, we assume that both the reference and the information signal have random amplitudes and phases due to random fluctuations of the refractive index along the propagation path in the atmosphere. The amplitudes and phases are not constant over the area of the detector; at each point M on the detector defined by r and θ they are functions of r and θ, see Fig. 9.1.

The modulated signal in the plane of the detector is then

$$E_S(r,\theta) = A_S(r,\theta)\, e^{i[\omega_o t + \phi_m(t) + \phi_S(r,\theta)]} \tag{9.1}$$

$A_S(r,\theta)$ and $\phi_S(r,\theta)$ are the random amplitude and phase due to the turbulent nature of the transmission medium. ω_o is the frequency of the optical wave. $\phi_m(t)$ is the modulation phase which contains the information.

We shall consider in the remainder $\phi_m(t) = \delta \sin \omega_m t$,

δ = modulation index

ω_m = modulation frequency

The reference signal is written as

$$E_R(r,\theta) = A_R(r,\theta)\, e^{i((\omega_o-\Delta\omega)t + \phi + \phi_R(r,\theta))} \tag{9.2}$$

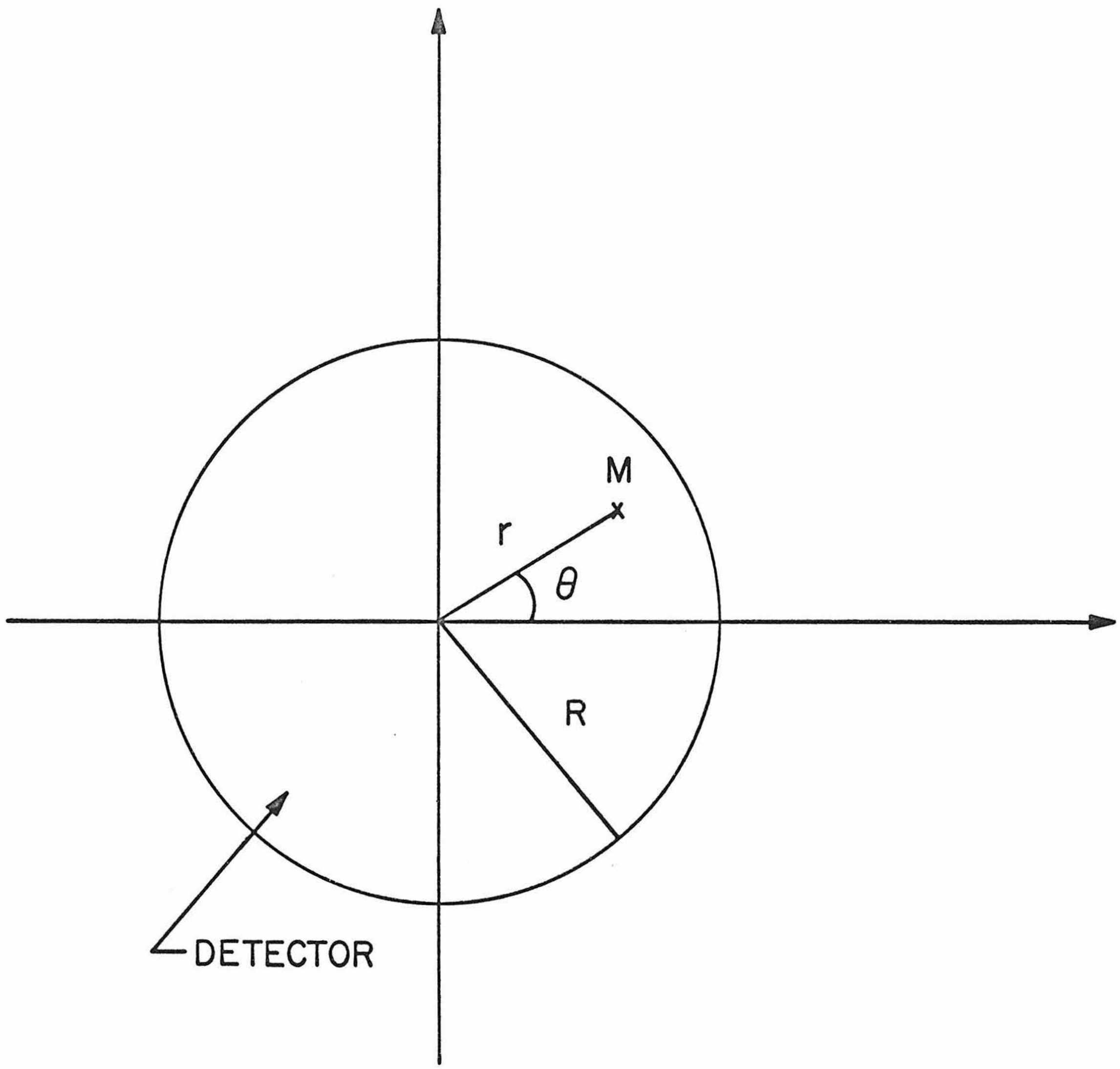


Figure 9.1 NOTATIONS IN THE PLANE OF THE DETECTOR

R is the radius of the detector

where $A_R(r,\theta)$ and $\phi_R(r,\theta)$ are the random amplitude and phase due to the turbulent transmission medium.

ϕ is a constant phase factor

A_S^2 and A_R^2 are the intensities of the light information signal and reference signal in watts/m^2.

In the general case, the optical carrier frequency and the frequency of the reference beam are not equal; we call the difference frequency $\Delta\omega$. The total field in the plane of the detector is

$$E_T(r,\theta) = E_S(r,\theta) + E_R(r,\theta) = A_S(r,\theta)\, e^{i[\omega_o t + \phi_m(t) + \phi_S(r,\theta)]} + A_R(r,\theta)\, e^{i((\omega_o - \Delta\omega)t + \phi + \phi_R(r,\theta))} \qquad (9.3)$$

The output current of the nonlinear detector is

$$i(t) = \frac{\eta q}{2h\nu} \int_0^{2\pi} d\theta \int_0^R r\, dr\, E_T(r,\theta)\, E_T^*(r,\theta) \qquad (9.4)$$

R is the radius of the detector

η is the quantum efficiency of the detector

ν is the frequency of the optical wave $(\nu = \omega_o/2\pi)$

In the following expressions $\int_0^{2\pi} d\theta \int_0^R r\, dr$ is replaced by $\int_\Sigma d\sigma$.

Σ is the area of the detector.

From 9.3 we find

$$E_T(r,\theta)\, E_T^*(r,\theta) = [E_S(r,\theta) + E_R(r,\theta)][E_S(r,\theta) + E_R(r,\theta)]^*$$
$$= A_S^2(r,\theta) + A_R^2(r,\theta) + 2A_R(r,\theta)\, A_S(r,\theta) \times \mathrm{Re}\, e^{i[\Delta\omega t + \delta \sin \omega_m t - \phi + \phi_S(r,\theta) - \phi_R(r,\theta)]}$$

and the output current is

$$i(t) = \frac{\eta q}{2h\nu} \text{ Re} \int_{\Sigma} d\sigma \left[A_S^2(r,\theta) + A_R^2(r,\theta) + 2A_S(r,\theta)\, A_R(r,\theta) \times e^{i[\Delta\omega t + \delta \sin \omega_m t - \emptyset + \emptyset_S(r,\theta) - \emptyset_R(r,\theta)]}\right]$$

The current $i(t)$ has a d.c. component

$$i_{DC} = \frac{\eta q}{2h\nu} \int_{\Sigma} d\sigma \, [A_S^2(r,\theta) + A_R^2(r,\theta)] \tag{9.5}$$

and an oscillating part

$$i_S(t) = \frac{\eta q}{h\nu} \text{ Re} \int_{\Sigma} d\sigma \, A_R(r,\theta) A_S(r,\theta) e^{i[\Delta\omega t + \delta \sin \omega_m t - \emptyset + \emptyset_S(r,\theta) - \emptyset_R(r,\theta)]} \tag{9.6}$$

Putting $\emptyset = \pi/2$, we write

$$\text{Re } \{e^{i[\Delta\omega t + \delta \sin \omega_m t - \frac{\pi}{2} + \emptyset_S(r,\theta) - \emptyset_R(r,\theta)]}\}$$

$$= \sin(\delta \sin \omega_m t) \cos(\Delta\omega t + \Delta\emptyset(r,\theta)) + \cos(\delta \sin \omega_m t) \times \sin(\Delta\omega t + \Delta\emptyset(r,\theta)) \tag{9.7}$$

where $\Delta\emptyset(r,\theta) = \emptyset_S(r,\theta) - \emptyset_R(r,\theta)$ (9.8)

If we consider only the terms oscillating at frequencies $\Delta\omega$ and $\Delta\omega \pm \omega_m$, then

$$\text{Re } \{e^{i[\ ]}\} = 2J_1(\delta) \sin \delta_m t \cos(\Delta\omega t + \Delta\emptyset(r,\theta)) + J_o(\delta) \sin(\Delta\omega t + \Delta\emptyset(r,\theta)) \tag{9.9}$$

In the case of video communication, the frequencies of the optical carrier beam and of the reference beam are equal; then $\Delta\omega = 0$ and

$J_o(\delta) \sin \Delta (r,\theta)$ represents an additional d.c. part of the current which is introduced by the random phase fluctuations. The d.c. current i_{DC} of 9.5 is thus more accurately given by

$$i_{DC} = \frac{\eta q}{2h\nu} \int_{\Sigma} d\sigma \, [A_S^2(r,\theta) + A_R^2(r,\theta) + 2J_o(\delta) \, A_S(r,\theta) A_R(r,\theta) \sin \Delta\phi(r,\theta)] \tag{9.10}$$

Using 9.6 and 9.9 we find the signal part of the current

$$i_S(t) = \frac{\eta q}{h\nu} 2J_1(\delta) \sin \omega_m t \int_{\Sigma} d\sigma \, A_R(r,\theta) \, A_S(r,\theta) \cos(\Delta\omega t + \Delta\phi(r,\theta)) \tag{9.11}$$

Since A_R, A_S and $\Delta\phi$ are random fields, the output currents i_{DC} and $i_S(t)$ are also random fields. A statistical averaging will have to be performed later. With the help of expressions 9.10 and 9.11, the signal-to-noise ratios for both local reference and transmitted reference can be calculated.

9.3 Signal-to-Noise Ratio for Heterodyne Detection

In a heterodyne detection scheme, a local reference is mixed with the incoming optical signal at the detector. The amplitude of the reference signal is constant over the plane of the detector; we call it $A_R(r,\theta) \equiv A_R$.

The reference signal does not contain any phase fluctuations $\phi_R(r,\theta) = 0$. We shall drop the index S and call $\Delta\phi(r,\theta) = \phi_{(1)}(r,\theta)$. We call $A_{(1)}(r,\theta)$ the amplitude of the modulated signal. From 9.11 the signal part of the current is

$$i_S(t) = \left(\frac{\eta q}{h\nu}\right) 2J_1(\delta)\, A_R \sin \omega_m t \int_\Sigma d\sigma\, A_{(1)}(r,\theta)\cos(\Delta\omega t + \emptyset_{(1)}(r,\theta)) \qquad (9.12)$$

Expression 9.12 is the correct form of the output current of the nonlinear detector at frequency $\Delta\omega \pm \omega_m$ in heterodyne detection. The optical signal is phase modulated at a frequency ω_m, therefore it contains sidebands at the frequencies $\omega_o + \omega_m$ and $\omega_o - \omega_m$. The current $i_S(t)$ as in 9.12 is obtained by the beating of both the sidebands of the signal beam with the reference beam at frequency $\omega_o - \Delta\omega$ at the nonlinear detector. The term $\sin \omega_m t \cos(\Delta\omega t + \emptyset_{(1)})$ in 9.12 is seen to be the sum of two terms with frequencies $\Delta\omega \pm \omega_m$,

$$\sin \omega_m t \cos(\Delta\omega t + \emptyset_{(1)}) = \frac{1}{2}\left|\sin((\Delta\omega + \omega_m)t + \emptyset_{(1)}) - \sin((\Delta\omega - \omega_m)t + \emptyset_{(1)})\right|.$$

Each term corresponds to the mixing of one sideband of the optical signal with the reference at frequency $\omega_o - \Delta\omega$. The term $\sin \omega_m t \cos(\Delta\omega t + \emptyset_{(1)})$ has widely been incorrectly replaced by $\sin((\omega_m + \Delta\omega)t + \emptyset_{(1)})$. The term $\sin((\omega_m + \Delta\omega)t + \emptyset_{(1)})$ represents twice the contribution of one sideband to the output current of the detector at frequency $\Delta\omega + \omega_m$, while the term $\sin \omega_m t \cos(\Delta\omega t + \emptyset_{(1)})$ which describes correctly the output current, represents the sum of the contributions of the two sidebands at frequencies $\omega_o \pm \omega_m$.

We shall, however, in the remainder of this paper, make use of the formally incorrect expression 9.13 for the output current of the detector because it is simpler to relate it to known functions describing the turbulence:

$$i_S(t) = \frac{\eta q}{h\nu} 2J_1(\delta)\, A_R \int_\Sigma d\sigma\, A_{(1)}(r,\theta) \sin((\Delta\omega + \omega_m)t + \emptyset_{(1)}(r,\theta)) \qquad (9.13)$$

In Appendix C we shall make use of expression 9.12 to calculate the signal power in the output current of the detector. We will show that the correct result which is then obtained is analytically more complex but quantitatively is at least within a factor 1/2 of the incorrect result obtained by using 9.13.

The signal power in the output current of the detector in the case of heterodyne detection $S_{(1)}(t)$ is defined as $S_{(1)}(t) = i_S^2(t)$ where $i_S(t)$ is given by expression 9.13

$$S_{(1)}(t) = i_S^2(t) = \left(\frac{\eta q}{h\nu}\right)^2 4J_1^2(\delta)\, A_R^2 \iint_{\Sigma\,\Sigma} d\sigma_1 d\sigma_2\, A_{(1)}(r_1,\theta_1)$$

$$\times\; A_{(1)}(r_2,\theta_2)\,\sin((\Delta\omega+\omega_m)t + \phi_{(1)}(r_1,\theta_1))\,\sin((\Delta\omega+\omega_m)t+\phi_{(1)}(r_2,\theta_2))$$

We call $\phi_{(1)}(r_1,\theta_1) \equiv \phi_{(1)1}$ $\qquad A_{(1)}(r_1,\theta_1) \equiv A_{(1)1}$

$\phi_{(1)}(r_2,\theta_2) \equiv \phi_{(1)2}$ $\qquad A_{(1)}(r_2,\theta_2) \equiv A_{(1)2}$

In order to find the time average of $S_{(1)}(t)$ we make use of the relationship:

$$\sin((\Delta\omega + \omega_m)t + \phi_{(1)1})\,\sin((\Delta\omega + \omega_m)t + \phi_{(1)2})$$

$$= \frac{1}{2}\left[\cos(\phi_{(1)1} - \phi_{(1)2}) - \cos(2(\Delta\omega + \omega_m)t + \phi_{(1)1} + \phi_{(1)2})\right]$$

The time average of $\cos(2(\Delta\omega + \omega_m)t + \phi_{(1)1} + \phi_{(1)2})$ is zero over an averaging time much larger than the period $2\pi/(\Delta\omega + \omega_m)$ of the oscillating current. Therefore the time average of the total signal power in the output current of the detector will be

$$S_{(1)} = \left(\frac{\eta q}{h\nu}\right)^2 2J_1^2(\delta)\, A_R^2 \int_\Sigma\int_\Sigma d\sigma_1\, d\sigma_2\, A_{(1)1}\, A_{(1)2} \times \cos(\phi_{(1)1} - \phi_{(1)2})$$

We can also write $S_{(1)}$ as

$$S_{(1)} = \left(\frac{\eta q}{h\nu}\right)^2 2J_1^2(\delta)\, A_R^2 \int_\Sigma\int_\Sigma d\sigma_1\, d\sigma_2\, A_{(1)1}\, A_{(1)2}\, e^{i(\phi_{(1)1} - \phi_{(1)2})}$$

since $\int_\Sigma\int_\Sigma d\sigma_1\, d\sigma_2\, A_{(1)1}\, A_{(1)2} \sin(\phi_{(1)1} - \phi_{(1)2}) = 0$

due to the symmetry of the double integration with respect to the coordinates r_1, θ_1 and r_2, θ_2.

At this point we notice that the wave function $u(\vec{x})$ which is the solution of the wave equation 1.1 is expressed in terms of its amplitude $A(\vec{x})$ and phase $\phi(\vec{x})$ as $u(\vec{x}) = A(\vec{x})\, e^{i\phi(\vec{x})}$ and

$$A_{(1)1}A_{(1)2}e^{i(\phi_{(1)1} - \phi_{(1)2})} = A_{(1)}(L,\vec{r}_1)\, e^{i\phi_{(1)}(L,\vec{r}_1)} \times A_{(1)}(L,\vec{r}_2)\, e^{-i\phi_{(1)}(L,\vec{r}_2)}$$

$$= u_{(1)}(L,\vec{r}_1)\, u^*_{(1)}(L,\vec{r}_2)$$

according to our notations. L is the distance of propagation through the random medium. The total statistically averaged power is then:

$$\overline{S_{(1)}(L)} = \left(\frac{\eta q}{h\nu}\right)^2 2J_1^2(\delta)\, A_R^2 \int_\Sigma\int_\Sigma d\sigma_1\, d\sigma_2\, \overline{u_{(1)}(L,\vec{r}_1)\, u^*_{(1)}(L,\vec{r}_2)}$$

This expression for the power $\overline{S_{(1)}}$ is directly related to the correlation function of the wave function between two points in a plane perpendicular to the direction of propagation of the wave $B_u(L,\rho)$, since $B_{u(1)}(L,\rho) = \overline{u_{(1)}(L,\vec{r}_1)u^*_{(1)}(L,\vec{r}_2)}$ where $\rho = |\vec{r}_1 - \vec{r}_2|$. This function has been evaluated in Chapter IV. Then

$$\overline{S_{(1)}(L)} = \left(\frac{\eta q}{h\nu}\right)^2 2J_1^2(\delta)\, A_R^2 \int_\Sigma \int_\Sigma d\sigma_1\, d\sigma_2\, B_{u(1)}(L,\rho) \tag{9.14}$$

We have kept the subscript (1) in $B_{u(1)}(L,\rho)$. It means that $B_{u(1)}(L,\rho)$ is the correlation function of the wave function $u_{(1)}(L,r)$ of the optical beam carrying the information.

From 9.10 the d-c part of the output current of the detector is

$$i_{DC} = \frac{\eta q}{2h\nu} A_R^2\, \pi R^2 \tag{9.15}$$

where the optimal conditions for heterodyne detection have been assumed, i.e. the amplitude of the reference signal is much larger than the amplitude of the information signal $A_R >> \overline{A(r,\theta)}$.

The shot noise power $N_{(1)}$ associated with i_{DC} is

$$N_{(1)} = 2q\, B\, i_{DC} = \frac{\eta q^2}{h\nu} B\, A_R^2\, \pi R^2 \tag{9.16}$$

where q = electronic charge = 1.6×10^{-19} coulomb

B = bandwidth of the circuit following the detector

R = radius of the detector.

The signal-to-noise ratio $S_{(1)}/N_{(1)}$ for heterodyne detection of a phase-modulated signal transmitted over a path L in the atmospheric turbulence is then given by combining 9.14 and 9.16. The result is

$$\overline{(S_{(1)}(L)/N_{(1)})} = (S/N)_{(1)} = \frac{\eta}{h\nu B}\, 2J_1^2(\delta)\, \frac{1}{\pi R^2} \int_\Sigma \int_\Sigma d\sigma_1\, d\sigma_2\, B_{u(1)}(L,\rho) \tag{9.17}$$

The signal-to-noise $(S/N)_{(1)}$ in the output current of the detector for the heterodyne detection of a phase modulated optical signal is given by expression 9.17 in terms of the correlation function $B_{u(1)}(L,\rho)$ of the optical signal wave function. It is sensitive to both amplitude and phase fluctuations introduced by the turbulent propagating medium.

9.4 Signal-to-Noise Ratio for Video Communication

In the video communication scheme the modulated optical signal is mixed on a nonlinear detector with a transmitted reference signal.

The amplitude and phase fluctuations of both the information signal and the reference signal are the same because they follow the same atmospheric path; then

$$\phi_R(r,\theta) = \phi_S(r,\theta)$$

$$\Delta\phi(r,\theta) = 0 \tag{9.18}$$

If the laser electric vector has equal components in the directions of polarization of the reference and information signals, then

$$A_R(r,\theta) = A_S(r,\theta) \tag{9.19}$$

We shall call $A_{(2)}(r,\theta) = A_R(r,\theta) = A_S(r,\theta)$ the common amplitude. Note that the subscript (2) refers to either the part of the optical beam which carries the information signal or the part which is used as the transmitted reference signal.

The signal part of the output current at ω_m of the detector is given with the help of 9.11, 9.18 and 9.19

$$i_S = \frac{\eta q}{h\nu} \sqrt{2} \; J_1(\delta) \int_{\Sigma} d\sigma \; A^2_{(2)}(r,\theta) \tag{9.20}$$

The total signal power $S_{(2)}$ in the output current of the detector is

$$S_{(2)} = i_S i_S^* = \left(\frac{\eta q}{h\nu}\right)^2 2J_1^2(\delta) \int_{\Sigma}\int_{\Sigma} d\sigma_1 \; d\sigma_2 \; A^2_{(2)}(r_1,\theta_1) A^2_{(2)}(r_2,\theta_2) \tag{9.21}$$

It is then noticed that

$$A^2_{(2)}(r_1,\theta_1)\; A^2_{(2)}(r_2,\theta_2)$$

$$= u_{(2)}(L,\vec{r}_1)u^*_{(2)}(L,\vec{r}_1)u_{(2)}(L,\vec{r}_2)u^*_{(2)}(L,\vec{r}_2)$$

$$= I_{(2)}(L,\vec{r}_1)\; I_{(2)}(L,\vec{r}_2)$$

The total statistically averaged power $\overline{S_{(2)}}$ involves the statistical average $\overline{I_{(2)}(L,\vec{r}_1)\; I_{(2)}(L,\vec{r}_2)}$ which is equal to the intensity correlation function $B_{I(2)}(L,\rho)$ which has been examined in Chapter VI. We can then write from 9.21

$$\overline{S_{(2)}(L)} = \left(\frac{\eta q}{h\nu}\right)^2 2J_1^2(\delta) \int_{\Sigma}\int_{\Sigma} d\sigma_1 \; d\sigma_2 \; B_{I(2)}(L,\rho) \tag{9.22}$$

The d-c part of the output current of the detector due to optical signal power is, from expressions 9.10, 9.18 and 9.19

$$i_{DC} = \frac{\eta q}{h\nu} \int_{\Sigma} d\sigma \, A^2_{(2)}(r,\theta) = \frac{\eta q}{h\nu} \int_{\Sigma} d\sigma \, I_{(2)}(L,\vec{r}) \qquad (9.23)$$

where $I_{(2)}(L,\vec{r})$ is the intensity of either the reference beam or the information signal beam at a point $(L,\vec{r})$ in a plane perpendicular to the direction of propagation of the wave at a distance L from the source.

We shall assume in the derivation of the noise power that the shot noise due to the incoming optical signal overcomes every other noise term in the detector output current. This assumption will be discussed in Appendix D and proved to be correct for large enough optical powers in the plane of the detector.

The noise power $N_{(2)}$ in the output current of the detector is

$$N_{(2)} = 2q\, B\, i_{DC} = 2qB \frac{\eta q}{h\nu} \int_{\Sigma} d\sigma \, I_{(2)}(L,\vec{r})$$

B is the bandwidth of the circuits following the detector. The total average noise power is

$$\overline{N_{(2)}(L)} = 2 \frac{\eta q^2}{h\nu} B \int_{\Sigma} d\sigma \, \overline{I_{(2)}(L,\vec{r})} = 2 \frac{\eta q^2}{h\nu} B \int_{\Sigma} d\sigma \, B_{u(2)}(L,0) \qquad (9.24)$$

since

$$B_{u(2)}(L,0) = \overline{u_{(2)}(L,\vec{r}) \, u^*_{(2)}(L,\vec{r})} = \overline{I_{(2)}(L,\vec{r})}$$

The signal-to-noise ratio

$$\overline{S_{(2)}}/\overline{N_{(2)}} \equiv (S/N)_{(2)}$$

for a video communication link which uses a transmitted reference to demodulate a phase modulated information signal is found by combining 9.22 and 9.24

$$(S/N)_{(2)} = \frac{\eta}{h\nu B} \; J_1^2(\delta) \; \frac{\int_\Sigma \int_\Sigma d\sigma_1 \, d\sigma_2 \, B_{I(2)}(L,\rho)}{\int_\Sigma d\sigma \, B_{u(2)}(L,0)} \tag{9.25}$$

The signal-to-noise ratio $(S/N)_{(2)}$ in the output of the detector for video communication is given by expression 9.25 in terms of the intensity correlation function of the optical signal wave function $B_{I(2)}(L,\rho)$ and the average $B_{u(2)}(L,0)$. It is only sensitive to amplitude fluctuations and not to phase fluctuations.

9.5 Comparison of Performances

9.5.1 Definition and expression of a performance criterion. We have calculated the signal-to-noise ratios for both heterodyne and video communication links through a randomly turbulent medium. In the first scheme the S/N is influenced by the amplitude and phase fluctuations introduced by the turbulent propagation medium, while in the second scheme only amplitude fluctuations affect the S/N. The purpose of this section is to examine and compare the two communication schemes. We define the quantity R

$$R = \frac{(S/N)_{(2)}}{(S/N)_{(1)}}$$ which is a measure of the relative performance of a video communication system over a heterodyne communication system. From 9.17 and 9.25

$$R = \frac{1}{2}\frac{1}{B_{u(2)}(L,0)} \times \frac{\int\limits_{\Sigma}\int\limits_{\Sigma} d\sigma_1 \, d\sigma_2 \, B_{I(2)}(L,\rho)}{\int\limits_{\Sigma}\int\limits_{\Sigma} d\sigma_1 \, d\sigma_2 \, B_{u(1)}(L,\rho)} \tag{9.26}$$

since $B_{u(2)}(L,0)$ is independent of r and θ and therefore

$$\int\limits_{\Sigma} d\sigma \, B_{u(2)}(L,0) = B_{u(2)}(L,0) \int\limits_{\Sigma} d\sigma = \pi R^2 B_{u(2)}(L,0)$$

We now evaluate R in the absence of any turbulence; we call it R_o. In that case, $B_{u(1)}(L,\rho) = A^2_{o(1)}$, $B_{u(2)}(L,0) = A^2_{o(2)}$ and $B_{I(2)}(L,\rho) = A^4_{o(2)}$ where $A_{o(1)}$ and $A_{o(2)}$ are the amplitudes of the phase modulated optical beam for heterodyne and video communcation schemes respectively. Then

$$R_o = \frac{1}{2} \frac{A^2_{o(2)}}{A^2_{o(1)}} \tag{9.27}$$

We suppose that the same laser optical beam with amplitude A_o at the plane of the detector is used as a carrier for both communication schemes.

In the case of heterodyne detection, the phase of the total laser electric vector is modulated; then $A_{o(1)} = A_o$. In the case of video communication the laser polarization is chosen to have equal components along two perpendicular directions. One polarization component is used as the information signal, the other as the transmitted reference signal. Then only the amplitude $A_o/\sqrt{2}$ is modulated and the amplitude of the reference signal is also $A_o/\sqrt{2}$, Fig. 9.2a.

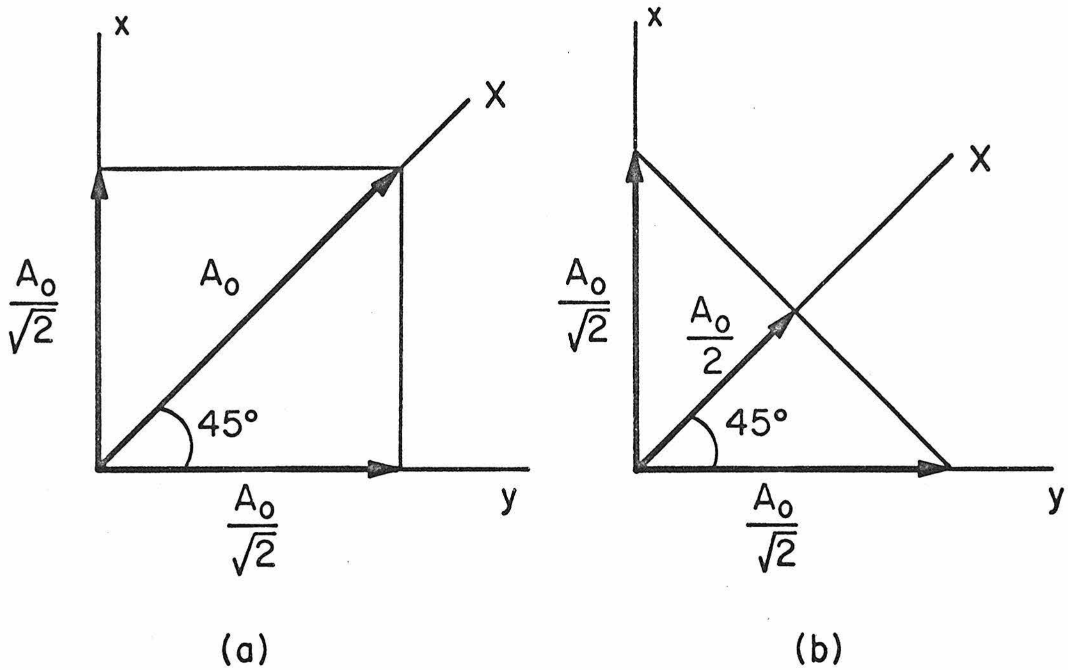


Figure 9.2 CONFIGURATION OF THE POLARIZATIONS IN THE CASE OF A COMMUNICATION SCHEME WITH A TRANSMITTED REFERENCE

The direction X is the direction of the laser polarization and of the parallel polarizer

A polarizer is used at the detector to mix the information with the reference signal, Fig. 9.2b. If the polarizer is parallel to the laser polarization, then the amplitude of the modulated optical signal along this direction is

$$\frac{A_o}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{A_o}{2}$$

The amplitude of the reference signal is also $A_o/2$. Therefore $A_{o(2)} = A_o/2$ and $R_o = 1/8$. This minimum value for R is obtained in the absence of any turbulence; under these circumstances the heterodyne detection scheme has a signal-to-noise ratio 8 times larger than the shot-noise-limited video communication scheme. However, as both communication links go through a turbulent atmosphere, the figure of merit R increases from its minimal value of 1/8; the longer the communication link and the stronger the turbulence, i.e. the more adverse the communication conditions, the larger R becomes. When R is larger than one, the video communication scheme is definitely preferable to the heterodyne detection scheme.

In the next section we determine numerically R for various communication conditions where the important parameters are the length of the communication link, the strength of the turbulence and the diameter of the detector.

9.5.2 Quantitative comparison. In order to find a numerical estimate for R , we replace the functions $B_{u(1)}(L,\rho)$ and $B_{I(2)}(L,\rho)$ by expressions found in Chapters IV and VI. According to 4.34,

$$B_{u(1)}(L,\rho) = A^2_{o(1)} \exp(-G(L,\rho)) \tag{9.28a}$$

where

$$G(L,\rho) = 2k^2L\left(\int_0^{L_o} B_n(\alpha)d\alpha - \int_0^{\sqrt{L_o^2-\rho^2}} B_n(\sqrt{\alpha^2+\rho^2})\,d\alpha\right) \tag{9.28b}$$

$B_n(\alpha)$ is the correlation function of the refractive index fluctuations. The following numerical calculations are performed for the Kolmogoroff model of turbulence. In that case the correlation function $B_n(\alpha)$ is given by 2.45.

Our theory has proven insufficient to find an exact analytical expression for the intensity correlation function $B_{I(2)}(L,\rho)$. For this numerical application we shall choose an expression for $B_{I(2)}(L,\rho)$ which "fits" the experimental data described in Chapter VIII

$$B_{I(2)}(L,\rho) = A^4_{o(2)}\left(1 + \frac{\sigma_1^2(L,\rho)}{1+\sigma_1^2(L,\rho)}\right) = A^4_{o(2)}\left(2 - \frac{1}{1+\sigma_1^2(L,\rho)}\right) \tag{9.29}$$

where $\sigma_1^2(L,\rho)$ is equal to the first order term in ε^2 in the power series expansion 6.3 of $B_I(L,\rho)$. In terms of the notations of Chapter VI, and according to 6.8

$$\sigma_1^2(L,\rho) = \varepsilon^2 B_{I1}(L,\rho) = 8\pi^2\varepsilon^2 k^2 L \int_0^\infty J_o(K\rho)$$

$$\times \left(1 - \frac{k}{K^2 L} \sin \frac{K^2 L}{k}\right) \phi_n(K) K \, dK \qquad (9.30)$$

$\varepsilon^2 \phi_n(K)$ is the three-dimensional Fourier transform of the correlation function of the refractive index fluctuations in the Kolmogoroff spectrum (27)

$$\varepsilon^2 \phi_n(K) = \begin{cases} 0.033\ C_n^2\ K^{-11/3} & \text{for} \quad K < K_m = 5.48/\ell_o \\ 0 & \text{for} \quad K > K_m \end{cases} \qquad (9.31)$$

For $\rho = 0$ with the help of 7.8, we have

$$\sigma_1^2(L,0) = \varepsilon^2 B_{I1}(L,0) = 1.23\ C_n^2\ k^{7/6}\ L^{11/6} \qquad (9.32)$$

$\sigma_1^2(L,0)$ is exactly equal to the variance of the fluctuations in the logarithm of the wave intensity when it is smaller than one, i.e.

$$\overline{(\log I(L) - \overline{\log I(L)})^2} = \sigma_1^2(L,0) \qquad \text{for} \quad \sigma_1^2(L,0) << 1$$

and also

$$B_I(L) = \overline{(I(L))^2} = 1 + \sigma_1^2(L,0) \qquad \text{for} \quad \sigma_1^2(L,0) << 1$$

In Tatarski's notations $\sigma_1^2(L,0)$ is called $\overline{4X^2}$ (23). It is called σ_1^2 in the experimental papers of Gracheva et al (8),(24) and also in DeWolf (21). $\sigma_1^2(L,0)$ is equivalent to $C_\ell^S(0)$ in Ochs et al

(16),(25).

With the help of 9.28a, 9.29 and 9.26 we can write

$$R = \frac{1}{8}\,\frac{\displaystyle\iint_{\Sigma\Sigma} d\sigma_1\, d\sigma_2\, \left(2 - \frac{1}{1+\sigma_1^2(L,\rho)}\right)}{\displaystyle\iint_{\Sigma\Sigma} d\sigma_1\, d\sigma_2\, \exp(-G(L,\rho))} \qquad (9.33)$$

The double integration $\int_\Sigma\int_\Sigma d\sigma_1\, d\sigma_2$ is performed over the area of the detector; it is equal to

$$\int_0^{D/2} r_1 dr_1 \int_0^{2\pi} d\theta_1 \int_0^{D/2} r_2 dr_2 \int_0^{2\pi} d\theta_2$$

where D is the diameter of the detector. This four-fold integral can be replaced by the following integral (28),(29)

$$\pi\int_0^D \left(D^2 \operatorname{Arc\,cos}(\rho/D) - \rho\sqrt{D^2-\rho^2}\right) \rho\, d\rho$$

since the functions to be integrated, $\left(2 - \frac{1}{1+\sigma_1^2(L,\rho)}\right)$ and $\exp(-G(L,\rho))$ depend only upon ρ , the distance between the two points of coordinates (r_1,θ_1) and (r_2,θ_2) in the plane of the detector. We will therefore use the following formula for a numerical estimation of R

$$R = \frac{1}{8}\,\frac{\displaystyle\int_0^1 \left(\cos^{-1}x - x\sqrt{1-x^2}\right)\left(2 - \frac{1}{1+\sigma_1^2(L,Dx)}\right)x\, dx}{\displaystyle\int_0^1 \left(\cos^{-1}x - x\sqrt{1-x^2}\right) \exp(-G(L,Dx))x\, dx} \qquad (9.34)$$

where the change of variable $x = \rho/D$ has been used and $G(L,Dx)$ is given by expression 9.28b and from 9.30 and 9.31

$$\sigma_1^2(L,Dx) = 8\pi^2(0.033)\, C_n^2 k^2 L \int_0^{K_m} J_o(K\,Dx)\left(1 - \frac{k}{K^2 L} \sin \frac{K^2 L}{k}\right) \times K^{-8/3}\, dK \qquad (9.35)$$

$K_m = 5.48/\ell_o$; ℓ_o is the inner scale of the turbulence.

Numerical calculations of R as a function of D, the diameter of the detector for various propagation distances L in various turbulence strengths have been performed. The results are shown in Figures 9.3 to 9.5 for a wavelength of 1μ and in Figures 9.6 and 9.7 for a wavelength of 10μ. The values of R vs D for the case $\lambda = 1\mu$ are presented for three turbulent strengths:

Fig. 9.3 Weak turbulence: $C_n = 10^{-8} m^{-1/3}$
Fig. 9.4 Intermediate turbulence: $C_n = 3 \times 10^{-8} m^{-1/3}$
Fig. 9.5 Strong turbulence: $C_n = 10^{-7} m^{-1/3}$.

In a weak turbulence R is smaller than one for a propagation distance $L = 10^3$m for any reasonable receiver diameter. For $L = 10^4$m R becomes larger than one for D = 46 cm, while when $L = 10^5$m, R is larger than one for D > 7 cm.

In the case of an intermediate turbulence, the values of the receiver diameter for which the ratio R is larger than one are smaller than in the case of a weak turbulence. It is even more true for a strong turbulence. As an example, let us examine a communication link of 10 km at a wavelength of 1μ through the atmospheric

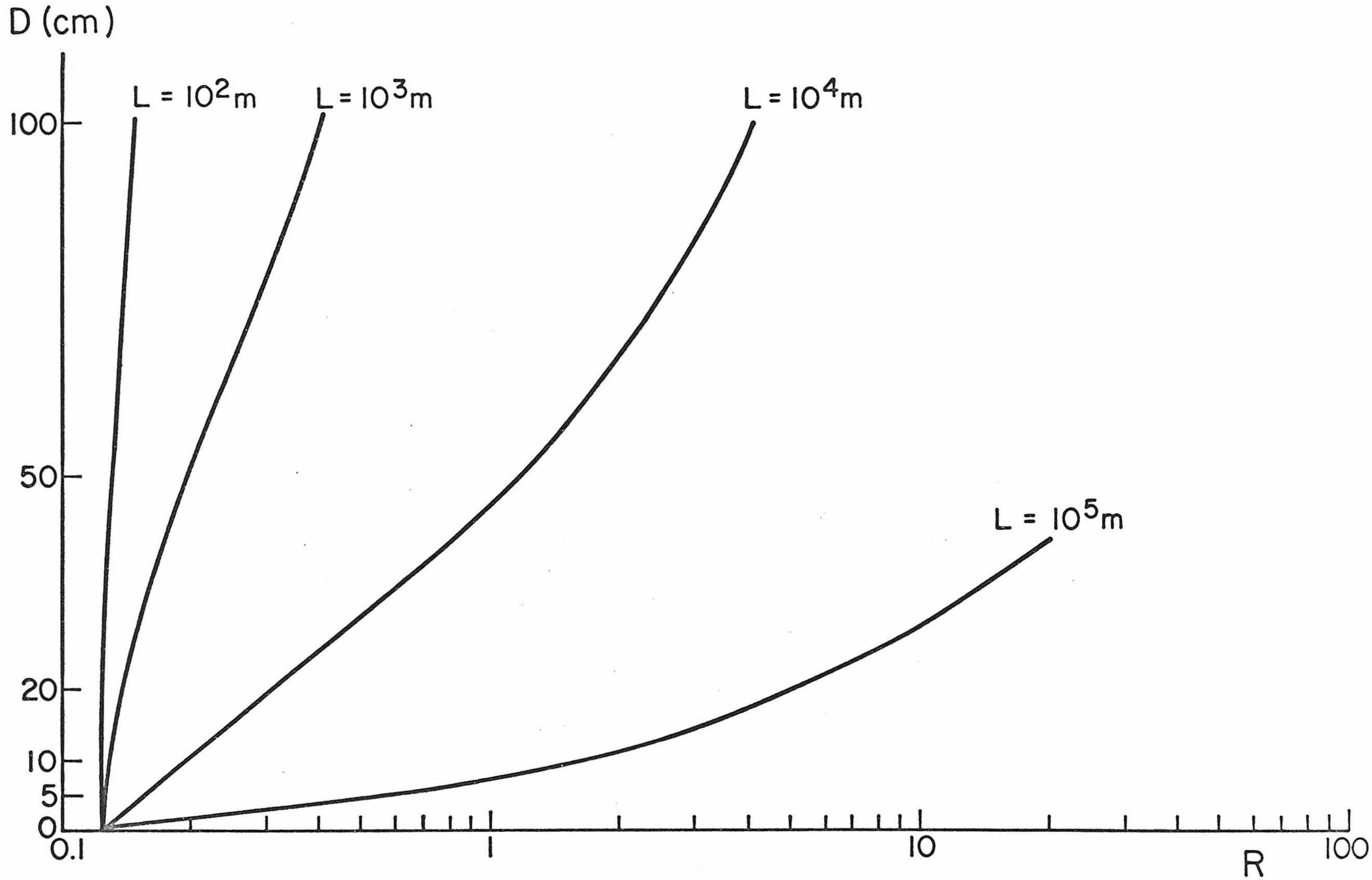


Figure 9.3 "R" vs "D" FOR A WEAK TURBULENCE $C_n = 10^{-8} m^{-1/3}$ AT $\lambda = 1\mu$

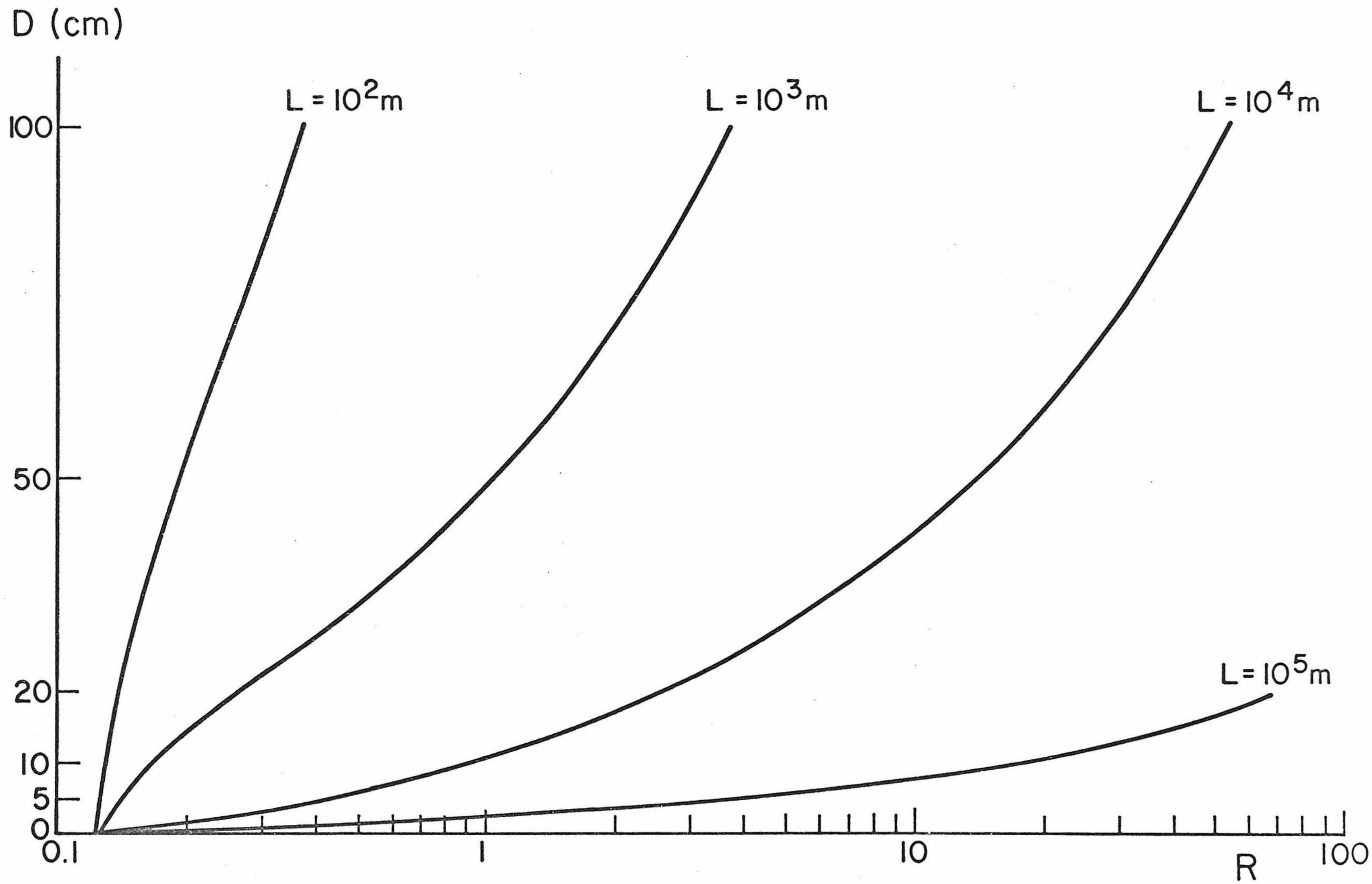


Figure 9.4 "R" vs "D" FOR AN INTERMEDIATE TURBULENCE $C_n = 3 \times 10^{-8} m^{-1/3}$ AT $\lambda = 1\mu$

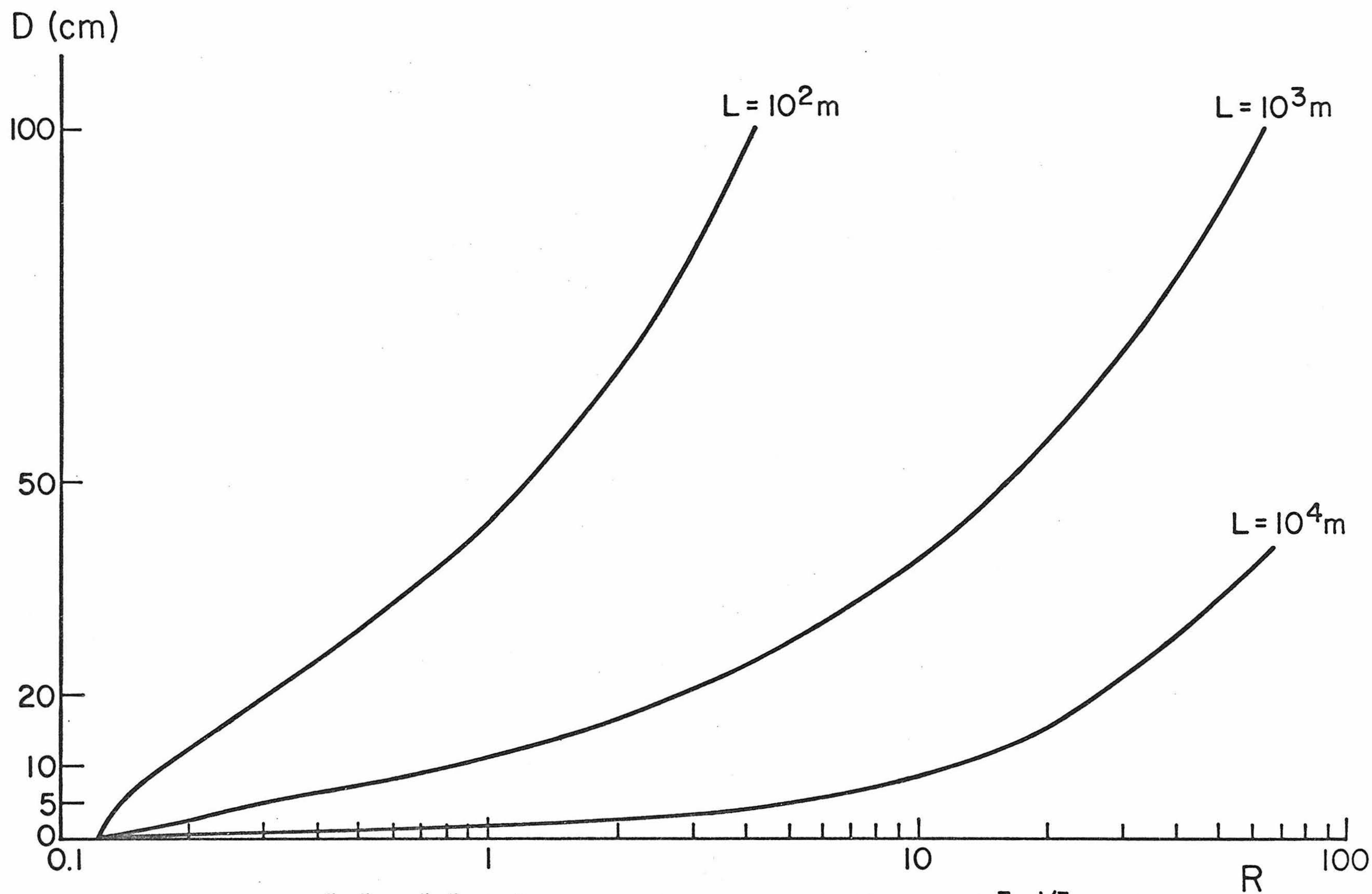


Figure 9.5 "R" vs "D" FOR A STRONG TURBULENCE $C_n = 10^{-7} m^{-1/3}$ AT $\lambda = 1\mu$

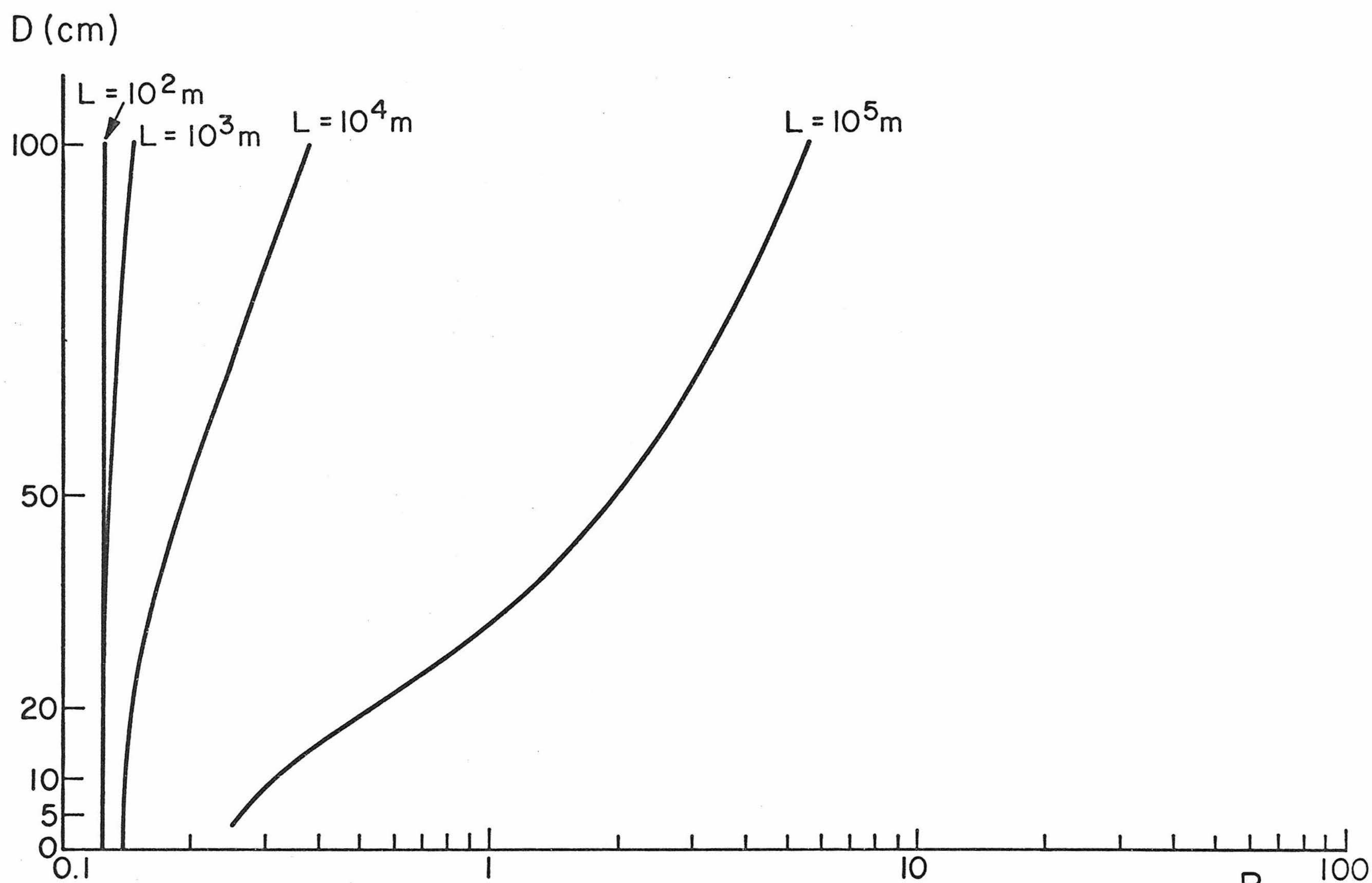


Figure 9.6 "R" vs "D" FOR AN INTERMEDIATE TURBULENCE $C_n = 3 \times 10^{-8} m^{-1/3}$ AT $\lambda = 10\mu$

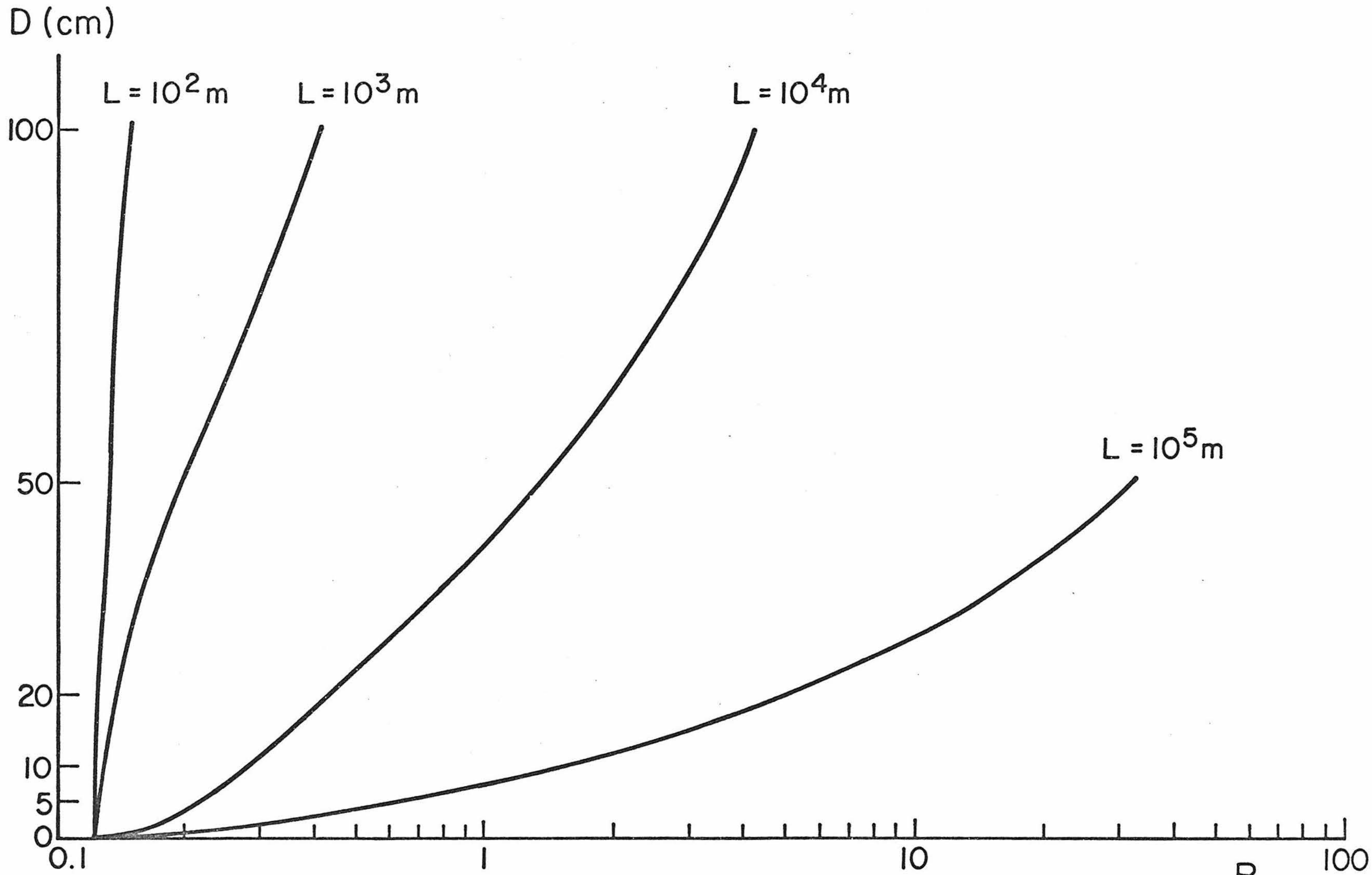


Figure 9.7 "R" vs "D" FOR A STRONG TURBULENCE $C_n = 10^{-7} m^{-1/3}$ AT $\lambda = 10\mu$

turbulence. Under a weak turbulence the video communication scheme has a larger signal-to-noise ratio than the heterodyne detection scheme ($R > 1$) for a receiver diameter larger than 46 cm. This value drops to 11 cm for an intermediate turbulence and to 2 cm for a strong turbulence. It is then concluded that for long propagation through the atmospheric turbulence where large receiver diameters are needed to detect a sizable amount of power, the video communication scheme is definitely more favorable than the heterodyne detection scheme. This is even more so, the stronger the turbulence and the longer the propagation path. This is due to the fact that the heterodyne S/N is very sensitive to the phase fluctuations impressed upon the signal beam by the atmospheric turbulence, while in the video communication scheme there is a cancellation of the phase fluctuations between the "reference" and the "signal" part of the beam.

The heterodyne S/N is a rapidly decreasing function of the detector diameter. In Figures 9.8 and 9.9 we show the S/N normalized to its value in the absence of turbulence as a function of the receiver diameter D for various propagation lengths under intermediate and strong turbulences at a wavelength of 1μ . These curves follow closely the curves "R" vs "D".

By comparing Figures 9.4 and 9.6, for example, we notice that for two similar communication links (same L and C_n), R is always smaller for a wavelength of 10μ than for a wavelength of 1μ. This is due to the fact that the correlation function $B_u(L,\rho)$ as expressed by 4.36 and 4.38 varies as e^{-1/λ^2} with wavelength and therefore decreases more slowly with a longer wavelength.

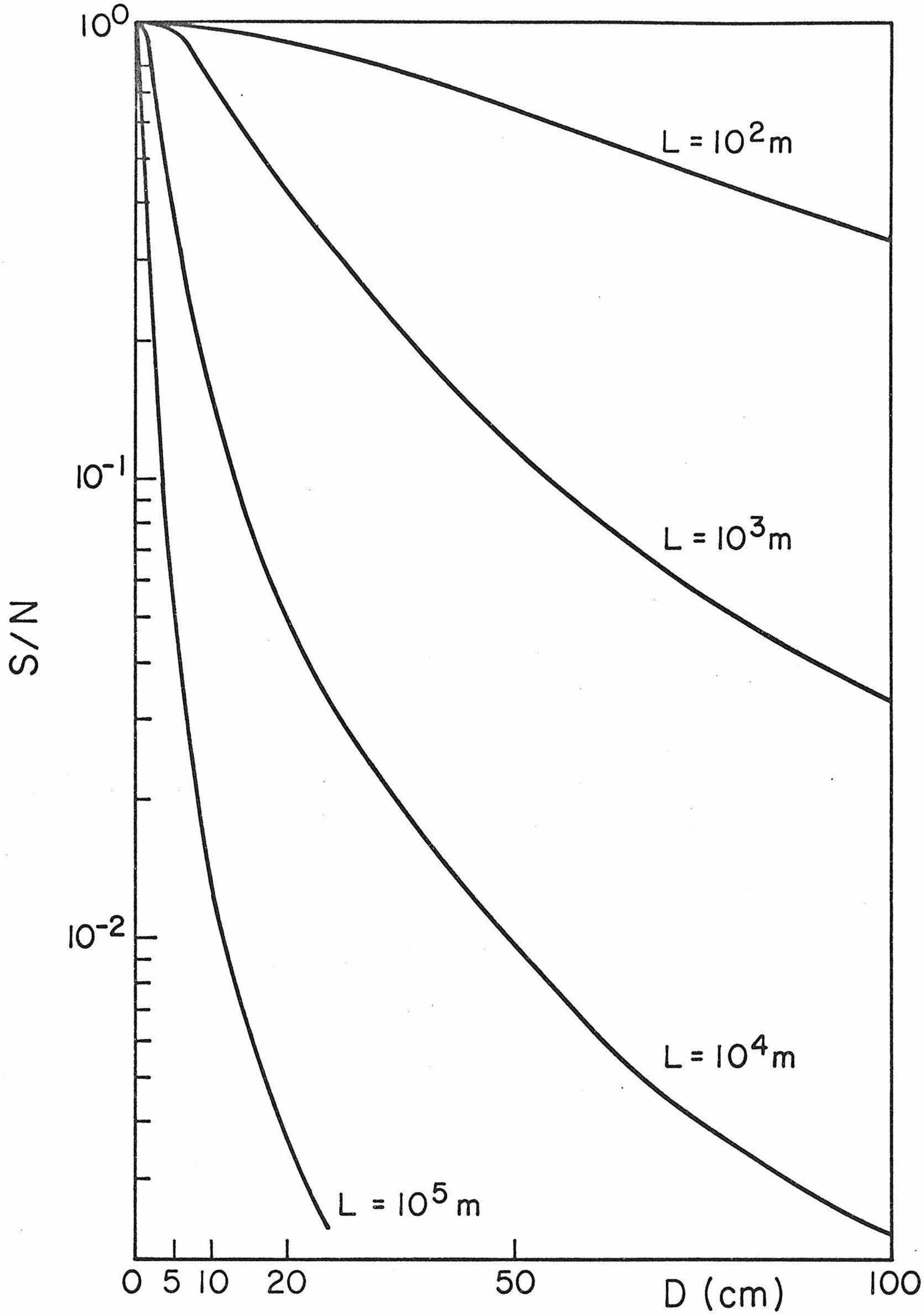


Figure 9.8 NORMALIZED "S/N" vs "D" FOR HETERODYNE DETECTION AT 1μ IN AN INTERMEDIATE TURBULENCE $C_n = 3 \times 10^{-8} m^{-1/3}$

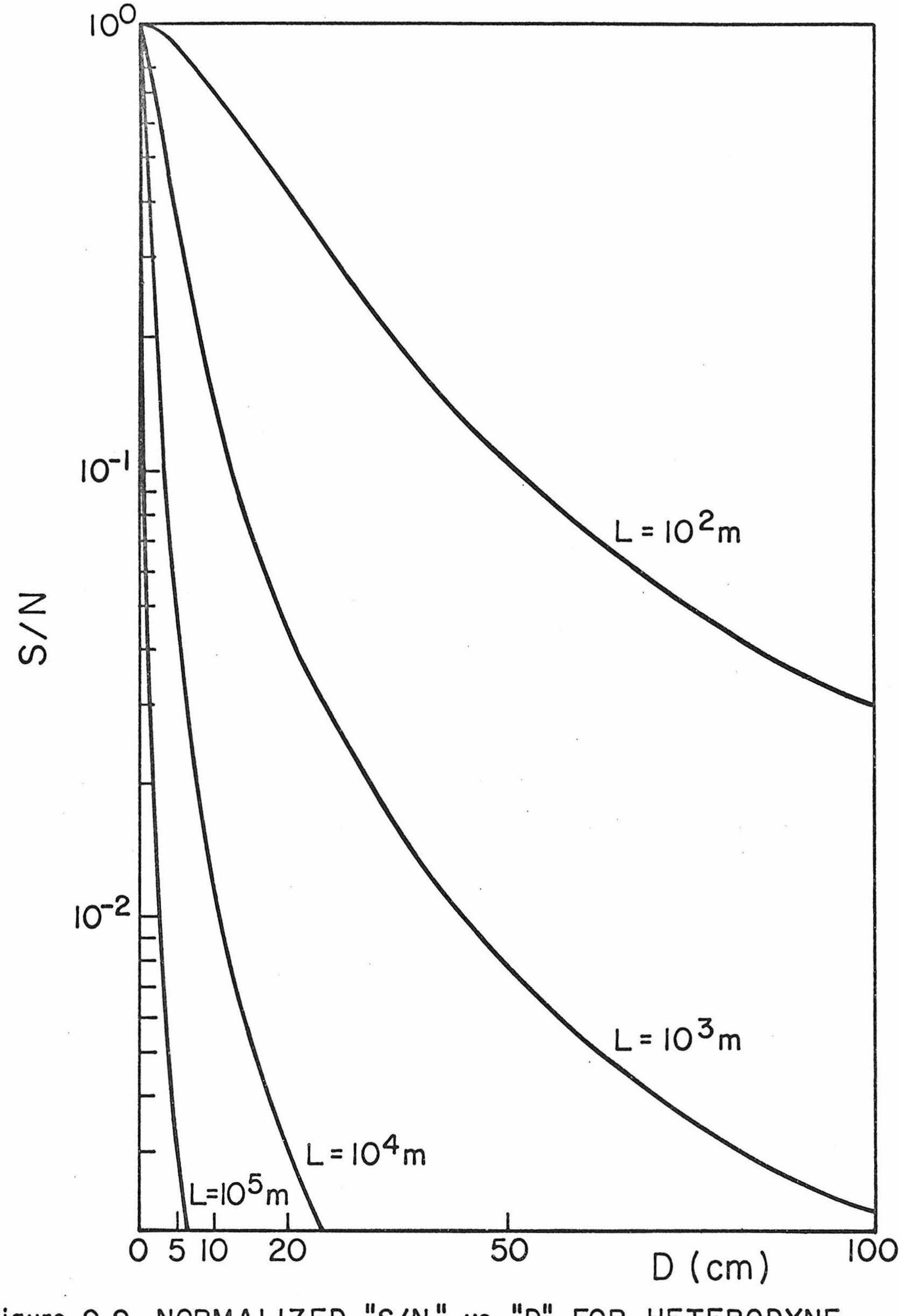


Figure 9.9 NORMALIZED "S/N" vs "D" FOR HETERODYNE DETECTION AT 1μ IN A STRONG TURBULENCE $C_n = 10^{-7} m^{-1/3}$

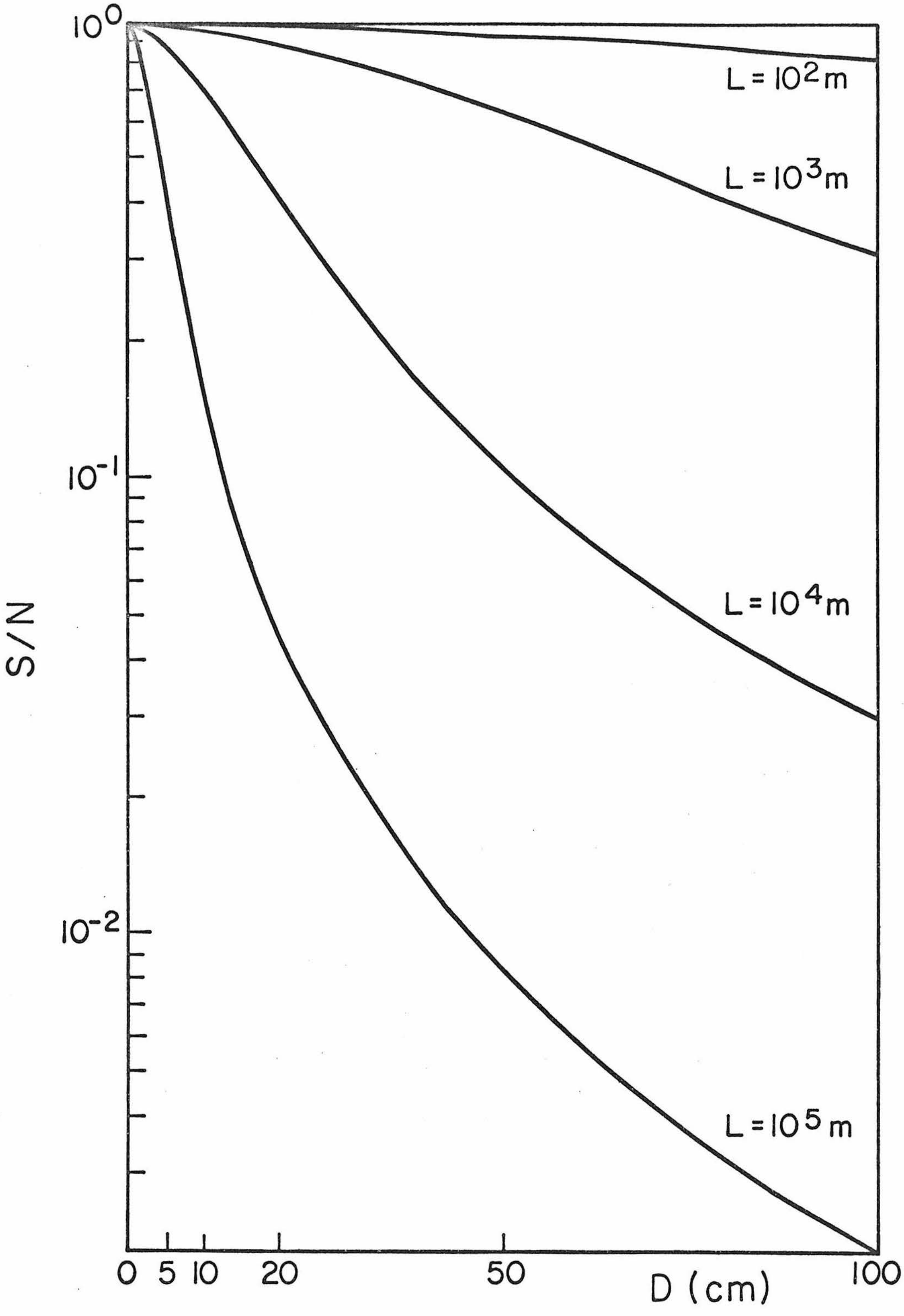


Figure 9.10 NORMALIZED "S/N" vs "D" FOR HETERODYNE DETECTION AT 10μ IN A STRONG TURBULENCE $C_n = 10^{-7} m^{-1/3}$

This effect can be seen by comparing Figures 9.9 and 9.10. In Figure 9.10 we have plotted the normalized signal-to-noise ratio for heterodyne detection at 10μ vs the diameter of the detector in a strong turbulence. Under similar conditions the S/N for heterodyne detection decreases faster for 1μ than 10μ.

9.6 Conclusion

In Chapter IX we have calculated the signal-to-noise ratios for two schemes of optical communication through the atmospheric turbulence: (a) heterodyne detection, (b) video communication. A numerical estimate of their comparative performance has been calculated for various propagation distances, turbulent conditions and detector diameters.

In the absence of turbulence, the signal-to-noise ratio of scheme (a), $(S/N)_a$ is 8 times larger than $(S/N)_b$.

In atmospheric turbulence under realistic communications conditions, $(S/N)_b$ becomes larger than $(S/N)_a$. (For a communication link at a wavelength of 1μ over a distance of 10 km through intermediate turbulence, $(S/N)_b = 10(S/N)_a$ when the diameter of the detector is 42 cm.)

The performances of the video communication scheme are even better for longer communication lengths and stronger turbulent conditions. This is true because this scheme is not sensitive to the random phase fluctuations introduced by the turbulent atmosphere. In scheme (b) the phase fronts of the signal beam and the reference beam are always in coincidence because they experience the same phase distortion. In scheme (a) only the phase of the signal beam is distorted by the atmospheric turbulence, and in the plane of the detector there is destructive interference between the distorted signal phase front and the plane reference phase front. Scheme (b) is more sensitive to amplitude fluctuations than scheme (a), but they play a lesser part than phase fluctuations.

CONCLUSION

We have studied theoretically the propagation of an optical wave through a random medium. We have derived a power series expansion for the wave function $u(\vec{x})$ which satisfies the wave equation $(\nabla^2 + k^2 n^2(\vec{x}))\, u(\vec{x}) = 0$ where $n(\vec{x})$ is the index of refraction of the turbulent medium. Exact analytical expressions have been obtained for the average wave function $\overline{u(\vec{x})}$ and the two-point correlation function $\overline{u(\vec{x}_1) u^*(\vec{x}_2)}$ in terms of the two-point correlation function of the index of refraction $\overline{n(\vec{x}_1)\, n(\vec{x}_2)}$. The intensity correlation function $\overline{I(\vec{x}_1)\, I(\vec{x}_2)} = \overline{u(\vec{x}_1)\, u^*(\vec{x}_1)\, u(\vec{x}_2)\, u^*(\vec{x}_2)}$ has also been investigated, but no simple analytical expression for it can be found without some approximations which limit its validity. Our results are compared with similar results which have been obtained by using approximate methods: The Born, the geometrical optics, and the Rytov approximations.

The Kolmogoroff model of turbulence is then used to find explicit expressions for the statistical moments of the wave function $u(\vec{x})$ in terms of the turbulence strength, the length of propagation and the wavelength of the wave.

In Chapter VII some problems of current interest are pointed out. These concern mainly the behavior of the wave intensity fluctuations for long propagation paths and strong turbulent conditions and the probability distribution function of the intensity.

The signal-to-noise ratios in the output current of a nonlinear detector are then calculated for two schemes of optical communication through atmospheric turbulence:

Scheme (a) - Heterodyne detection,

Scheme (b) - Video Communication.

$(S/N)_a$ is a function of the correlation function of the wave function $\overline{u(\vec{x}_1)\, u^*(\vec{x}_2)}$ which has been calculated in Chapter IV. $(S/N)_b$ is a function of the intensity correlation function $\overline{I(\vec{x}_1) I(\vec{x}_2)}$ which has been investigated in Chapter VI.

A numerical comparison of these two schemes of communication through a turbulent atmosphere is performed by computing the ratio $R = (S/N)_b/(S/N)_a$ in terms of the propagation distance, the turbulence strength, the diameter of the detector, and the wavelength.

It is found that for long propagation distances in weak turbulence, or for a propagation distance of a few kilometers in intermediate turbulence, the video communication scheme has a larger signal-to-noise ratio than the heterodyne detection scheme. Scheme (b) performs even better for longer distances or stronger turbulences. This is due to the fact that $(S/N)_a$ is very sensitive to the random phase fluctuations introduced by the turbulent medium, while in scheme (b) the phase fluctuations are cancelled between the reference and the signal portion of the beam since both go through the same atmospheric path.

It is found that a communication link at a wavelength of 10μ is less sensitive to phase fluctuations than at a wavelength of 1μ. Our analysis of the signal-to-noise ratios has been based on the following assumption: In the case of video communication the largest source of noise in the detector is the shot noise due to the optical signal. A careful evaluation of all the communication parameters (areas of the transmitter and the receiver, length of propagation, wavelength, transmitted power, noise equivalent power of the detector) is needed in a particular communication system to determine whether this condition is satisfied.

REFERENCES

1. V. I. Tatarski, Wave Propagation in a Turbulent Medium, McGraw-Hill Book Company, New York, N.Y., 1961.

2. L. A. Chernov, Wave Propagation in a Random Medium, McGraw-Hill Book Company, New York, N.Y., 1960.

3. J. B. Keller, "Wave propagation in random media", Proc. 13th Symposium on Applied Mathematics, New York, American Math. Soc. 1962, pp. 227-246.

4. F. C. Karal and J. B. Keller, "Elastic, electromagnetic and other waves in a random medium", J. Math. Phys. 5, 537-547 (1964).

5. W. P. Brown, Jr., "Propagation in random media--cumulative effect of weak inhomogeneities", IEEE Trans. on Antennas and Propagation AP-15, 81 (1967).

6. W. P. Brown, Jr., "Validity of the Rytov approximation in optical propagation calculations", J. Opt. Soc. Am. 56, 1045 (1966).

7. W. P. Brown, Jr., "Validity of the Rytov approximation", J. Opt. Soc. Am. 57, 1539 (1967).

8. M. E. Gracheva and A. S. Gurvich, "Strong fluctuations in the intensity of light propagated through the atmosphere close to the earth", Izvestya VUZ Radiofizika 8, 717-724 (1965).

9. Tatarski, Ibid., p.93.

10. A. M. Yaglom, An Introduction to the Theory of Stationary Random Functions, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962.

11. Tatarski, Ibid., p.27.

12. A. Kolmogoroff, "The local structure of turbulence in incompressible viscous fluid for very large Reynolds' numbers", Comptes Rendus (Doklady) de l'Academie des Sciences de l'URSS XXX, 301-305 (1941).

13. Tatarski, Ibid., pp. 93-97.

14. D. A. DeWolf, "Multiple scattering in a random continuum", Radio Science 2 (New Series), 1379-1392 (1967).

15. A. M. Obukhov, "On the distribution of energy in the spectrum of turbulent flow", Doklady Akad.Nauk SSSR 32, 19 (1941).

16. G. R. Ochs, R. R. Bergman and J. R. Snyder, "Laser beam scintillations over horizontal paths from 5.5 to 145 kilometers", to be published.

17. I. Goldstein, P. A. Miles and A. Chabot,"Heterodyne measurements of light propagation through atmospheric turbulence", Proc. of the IEEE 53, 1172 (1965).

18. R. E. Hufnagel and N. R. Stanley,"Modulation transfer function associated with image transmission through turbulent media", J. Opt. Soc. Am. 54, 52 (1964).

19. Tatarski, Ibid., p. 153.

20. Tatarski, Ibid., eq. (7.46), p. 135.

21. D. A. DeWolf, "Saturation of irradiance fluctuations due to turbulent atmosphere", J. Opt. Soc. Am. 58, 461-466 (1968).

22. H. T. Yura, "Electromagnetic Field and Intensity Fluctuations in a Weakly Inhomogeneous Medium", Memorandum RM-5697-PR, The RAND Corporation, July 1968.

23. Tatarski, Ibid.,pp.136,137.

24. M. E. Gracheva, Izv. Vysshikh Uchebn. Zavedenii, Radiofiz. 10, 775 (1967).

25. G. R. Ochs and R. S. Lawrence, "Saturation of laser beam scintillation under conditions of strong atmospheric turbulence", to be published.

26. E. E. Salpeter, "Interplanetary scintillations", Astrophys. J. 147, 433-448 (1967).

27. Tatarski, Ibid., p. 151.

28. Tatarski, Ibid., p. 233.

29. D. L. Fried, Optical heterodyne detection of an atmospherically distorted signal wave front", Proc. IEEE 55, 57 (1967).

30. H. Cramer, Mathematical Methods of Statistics, Princeton, N. J., Princeton University Press, 1954, p. 213.

31. W. B. Davenport and W. L. Root, Random Signals and Noise, McGraw-Hill Book Company, New York, 1958, pp. 158-165.

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Figure 2.1 The two-dimensional Fourier transform $F_n(\vec{K}, |x'-x''|)$ is a measure of the correlation of the index of refraction in two planes perpendicular to the direction of propagation of the wave.

Figure 2.2 Graphic illustration of the change of variables $x_2 - x_1 = \alpha$, $x_2 + x_1 = 2\beta$. The integration $\int_0^L dx_1 \int_0^L dx_2$ over the square of sides L is replaced by

$$2 \int_0^L d\alpha \int_{\alpha/2}^{L-\alpha/2} d\beta$$

Figure 4.1 The area in between the two curves is

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Figure 9.4 "R" vs "D" for an intermediate turbulence $C_n = 3 \times 10^{-8} m^{-1/3}$ at $\lambda = 1\mu$

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APPENDIX A

CALCULATION OF $\overline{u(\vec{x})}$ USING APPROXIMATION 2.15: $\overline{u_{(A)}(\vec{x})}$

The approximation 2.15 is the following

$$\overline{\prod_{p=1}^{2m} dN_p} = \overline{dN_1 dN_2} \times \overline{dN_3 dN_4} \times \cdots \times \overline{dN_{2m-1} dN_{2m}}$$

where the notation $dN_p \equiv dN(\vec{K}_p, x_p)$ has been used. The correct expression for $\overline{\prod_{p=1}^{2m} dN_p}$ is given by 2.5. In the approximation 2.15 only one of the $(2m)!/m!2^m$ permutations of m products $\overline{dN_p dN_q}$ is kept. In this case, let us call

$$\overline{\psi_{(A)}(\vec{x})} = \frac{\overline{u_{(A)}(\vec{x})}}{u_o(x)} = \sum_{m=0}^{\infty} \varepsilon^{2m} \, \overline{\psi_{(A)2m}(\vec{x})} \qquad \text{(A-1)}$$

where $\overline{\psi_{(A)2m}(\vec{x})}$ are the statistical averages of the functions $\psi_{2m}(\vec{x})$ (given by 1.36) which are calculated with approximation 2.15. It is easily seen that $\overline{\psi_{(A)0}(\vec{x})} = 1$ and

$$\overline{\psi_{(A)2}(L)} = \overline{\psi_2(L)} = -\,k^2 \int_0^L dx_1 \int_0^{x_1} dx_2 \, f(|x_1 - x_2|) \qquad \text{(A-2)}$$

where the function $f(|X|)$ is defined by relation 2.11. In the calculation of $\overline{\psi_{(A)4}(L)}$ only the contribution of the term $\overline{dN_1 dN_2} \cdot \overline{dN_3 dN_4}$ to the integrals of 2.12 is considered.

Therefore, according to 2.18

$$\overline{\psi_{(A)4}(L)} = \overline{\psi_{4a}(L)} = k^4 \int_0^L dx_1 \int_0^{x_1} dx_2\ f(|x_1-x_2|) \int_0^{x_2} dx_3 \int_0^{x_3} dx_4\ f(|x_3-x_4|) \tag{A.3}$$

and it is easy to see that under approximation 2.15 the general term $\overline{\psi_{(A)2m}(L)}$ can be expressed as follows:

$$\overline{\psi_{(A)2m}(L)} = (-1)^m k^{2m} \int_0^L dx_1 \int_0^{x_1} dx_2\ f(|x_1-x_2|) \int_0^{x_2} dx_3 \int_0^{x_3} dx_4\ f(|x_3-x_4|)$$

$$\cdots\cdot \int_0^{x_{2p}} dx_{2p+1} \int_0^{x_{2p+1}} dx_{2p+2}\ f(|x_{2p+1} - x_{2p+2}|) \cdots$$

$$\int_0^{x_{2m-2}} dx_{2m-1} \int_0^{x_{2m-1}} dx_{2m}\ f(|x_{2m-1} - x_{2m}|) \tag{A.4}$$

The expressions A.2, A.3 and A.4 are then collected into the series expansion A.1:

$$\overline{\psi_{(A)}(L)} = 1 - \varepsilon^2 k^2 \int_0^L dx_1 \int_0^{x_1} dx_2\ f(|x_1-x_2|) \Bigg(1 - \varepsilon^2 k^2 \int_0^{x_2} dx_3 \int_0^{x_3} dx_4$$

$$\times\ f(|x_3-x_4|) + \cdots$$

$$+ (-1)^{m-1} (\varepsilon^2 k^2)^{m-1} \int_0^{x_2} dx_3 \int_0^{x_3} dx_4\ f(|x_3-x_4|) \cdots$$

$$\int_0^{x_{2m-2}} dx_{2m-1} \int_0^{x_{2m-1}} dx_{2m}\ f(|x_{2m-1} - x_{2m}|) + \cdots \Bigg)$$

or

$$\overline{\psi_{(A)}(L)} = 1 - \varepsilon^2 k^2 \int_0^L dx_1 \int_0^{x_1} dx_2 \; f(|x_1 - x_2|) \Big(1 + \varepsilon^2 \overline{\psi_{(A)}(x_2)} + \cdots$$

$$+ \varepsilon^{2m-2} \; \overline{\psi_{(A)2m-2}(x_2)} + \cdots \Big)$$

and with the help of A.1 we obtain the following integral equation for $\overline{\psi_{(A)}(L)}$

$$\overline{\psi_{(A)}(L)} = 1 - \varepsilon^2 k^2 \int_0^L dx_1 \int_0^{x_1} dx_2 \; f(|x_1 - x_2|) \; \overline{\psi_{(A)}(x_2)} \qquad \text{(A.5)}$$

Using approximation 2.15 and the same method, DeWolf found the correct result

$$\overline{\psi(L)} = \exp -\varepsilon^2 k^2 \int_0^L dx_1 \int_0^{x_1} dx_2 \; f(|x_1 - x_2|)$$

(which is not a solution of equation A.5) due to some mysterious intuition.

Brown (5) used approximation 2.15 and, with a slightly different method, derived the integral equation A.5. (It is the equivalent of equation 16 in Reference (5)).

APPENDIX B

CALCULATION OF THE INTENSITY CORRELATION FUNCTION $B_{I(RY)}(L,\rho)$ IN THE RYTOV APPROXIMATION USING TATARSKI'S NOTATIONS (1)

In order to compare our results for the intensity correlation function B_I in the Rytov approximation and Tatarski's results, we now calculate this function using Tatarski's notations.

The wave function $u(\vec{x})$ can be expressed as

$$u(\vec{x}) = A(\vec{x})\, e^{i\phi(\vec{x})} = e^{\log A(\vec{x}) + i\phi(\vec{x})} = A_o e^{\chi(\vec{x}) + i\phi(\vec{x})} \tag{B.1}$$

where the notation $\chi(\vec{x}) = \log(\frac{A(\vec{x})}{A_o})$ has been used.

A_o is defined by the relationship

$$\log A_o = \overline{\log A(\vec{x})} \tag{B.2}$$

so that

$$\overline{\chi(\vec{x})} = \overline{\log A(\vec{x})} - \log A_o = 0 \tag{B.3}$$

A_o is also equal to the amplitude of the wave in the absence of turbulence. The intensity correlation function is expressed as follows

$$\begin{aligned} B_I(L,\rho) &= \overline{u(L,\vec{r}_1)u^*(L,\vec{r}_1)u(L,\vec{r}_2)u^*(L,\vec{r}_2)} \\ &= \overline{e^{2(\chi(L,\vec{r}_1)+\chi(L,\vec{r}_2))}} \end{aligned} \tag{B.4}$$

In the Rytov approximation, the amplitude of the wave has a log-normal distribution; therefore $\chi(\vec{x}) = \log A(\vec{x})$ has a Gaussian distribution and zero mean according to B.3. Then (see 3.4),

$$B_{I(RY)}(L,\rho) = e^{\overline{2(\chi(L,\vec{r}_1) + \chi(L,\vec{r}_2))^2}} \qquad (B.5)$$

$$\overline{(\chi(L,\vec{r}_1)+\chi(L,\vec{r}_2))^2} = \overline{(\chi(L,\vec{r}_1))^2} + \overline{(\chi(L,\vec{r}_2))^2} + \overline{2\chi(L,\vec{r}_1)\chi(L,\vec{r}_2)}$$

$\overline{\chi(L,\vec{r}_1)\chi(L,\vec{r}_2)}$ is the correlation function of the logarithm of the amplitude fluctuations. It is called $B_A(\rho)$ by Tatarski, with $\rho = |\vec{r}_1-\vec{r}_2|$ and $\overline{(\chi(L,r_1))^2} = \overline{(\chi(L,r_2))^2} = B_A(0) = \overline{\chi^2}$.

$\overline{\chi^2}$ is the variance of the logarithm of the amplitude fluctuations.

With the help of these notations we can rewrite expression B.5 as

$$B_{I(RY)}(L,\rho) = e^{4\overline{\chi^2} + 4B_A(\rho)} \qquad (B.6)$$

APPENDIX C

CORRECT EXPRESSION FOR THE SIGNAL POWER IN THE OUTPUT CURRENT OF THE DETECTOR IN AN OPTICAL HETERODYNING DETECTION SCHEME

In this appendix we calculate the signal power in the output current of a detector in the heterodyne detection of a phase-modulated optical beam. In order to do so, we make use of the correct expression 9.12 for the output current of the detector.

$$i_S(t) = \left(\frac{\eta q}{h\nu}\right) 2J_1(\delta)\, A_R \sin \omega_m t \int_\Sigma d\sigma\, A(r,\theta) \cos \phi(r,\theta) \qquad (9.12)$$

We dropped the subscript (1) with the understanding that $A(r,\theta)$ and $\phi(r,\theta)$ are the amplitude and the phase of the optical beam carrying the information signal. The total signal power at the frequency ω_m is

$$S(t) = i_S^2(t) = \left(\frac{\eta q}{h\nu}\right)^2 4J_1^2(\delta)\, A_R^2 \sin^2\omega_m t \int_\Sigma \int_\Sigma d\sigma_1\, d\sigma_2\, A_1 A_2 \cos \phi_1 \cos \phi_2$$

where $A_1 = A(r_1,\theta_1)$ $\qquad A_2 = A(r_2,\theta_2)$

$\phi_1 = \phi(r_1,\theta_1)$ $\qquad \phi_2 = \phi(r_2,\theta_2)$

The time averaged signal power is then

$$S = \left(\frac{\eta q}{h\nu}\right)^2 2J_1^2(\delta)\, A_R^2 \int_\Sigma \int_\Sigma d\sigma_1\, d\sigma_2\, A_1 A_2 \cos \phi_1 \cos \phi_2 \qquad (C.1)$$

In Tatarski's notation we can write

$$A_1 A_2 \cos\phi_1 \cos\phi_2 = A_o^2 \, e^{(\chi_1+\chi_2)} \cos\phi_1 \cos\phi_2$$

$$= \frac{A_o^2}{4} e^{(\chi_1+\chi_2)} (e^{i\phi_1} + e^{-i\phi_1})(e^{i\phi_2} + e^{-i\phi_2})$$

and

$$\overline{A_1 A_2 \cos\phi_1 \cos\phi_2} = \frac{A_o^2}{2} \overline{e^{(\chi_1+\chi_2)} (e^{i(\phi_1-\phi_2)} + e^{i(\phi_1+\phi_2)})}$$

We write

$$\overline{A_1 A_2 \cos\phi_1 \cos\phi_2} = \frac{A_o^2}{2} (\overline{e^C} + \overline{e^{C'}})$$

where $C = \chi_1 + \chi_2 + i(\phi_1 - \phi_2)$ and $C' = \chi_1 + \chi_2 + i(\phi_1 + \phi_2)$

We shall assume in this calculation that χ_1, χ_2, ϕ_1, ϕ_2 are normally distributed functions with zero mean; then C and C' are normally distributed functions with zero mean $\overline{C} = \overline{C'} = 0$.

Therefore

$$\overline{e^C} = e^{\frac{1}{2}\overline{C^2}} \qquad \text{and} \qquad \overline{e^{C'}} = e^{\frac{1}{2}\overline{C'^2}} \tag{C.2}$$

$\overline{C^2}$ is calculated by assuming that the fluctuations of the amplitudes and of the phases are uncorrelated.

$$\overline{C^2} = \overline{(\chi_1 + \chi_2 + i(\phi_1 - \phi_2))^2} = \overline{2\chi^2} + 2\overline{\chi_1\chi_2} - \overline{(\phi_1 - \phi_2)^2}$$

and

$$\overline{C'^2} = \overline{(\chi_1 + \chi_2 + i(\phi_1 + \phi_2))^2} = \overline{2\chi^2} + 2\overline{\chi_1\chi_2} - \overline{(\phi_1 + \phi_2)^2}$$

$$= \overline{C^2} - 4\overline{\phi_1\phi_2} \tag{C.3}$$

$\overline{\phi_1 \phi_2}$ is the correlation function of the phase fluctuations; we call it $B_\phi(\vec{r}_1, \vec{r}_2)$.

With the help of C.2 and C.3 we obtain

$$\overline{A_1 A_2 \cos \phi_1 \cos \phi_2} = A_o^2 \, e^{\frac{1}{2}\overline{C^2}} \, \frac{1}{2}(1 + e^{-2B_\phi(\vec{r}_1, \vec{r}_2)}) \qquad (C.4)$$

The incorrect expression 9.13 for output current of a detector in a heterodyne communication scheme led to the following averaging

$$\overline{A_1 A_2 e^{i(\phi_1 - \phi_2)}} = A_o^2 \, \overline{e^{X_1 + X_2 + i(\phi_1 - \phi_2)}} = A_o^2 \, \overline{e^{C}} = A_o^2 \, e^{\frac{1}{2}\overline{C^2}} \qquad (C.5)$$

The discrepancy between expressions C.4 and C.5 comes from the factor

$$\frac{1}{2}(1 + e^{-2B_\phi(\vec{r}_1, \vec{r}_2)}) \qquad (C.6)$$

The correlation function of the phase fluctuations has never been calculated yet, and a detailed knowledge of the term C.6 is not available. However, we know that the correlation function $B_\phi(\vec{r}_1, \vec{r}_2)$ is a positive function; therefore

$$0 < e^{-2B_\phi(\vec{r}_1, \vec{r}_2)} < 1$$

and

$$\frac{1}{2} < \frac{1}{2}\,(1 + e^{-2B_\phi(\vec{r}_1, \vec{r}_2)}) < 1$$

From C.6 we can write

$$\overline{S} = \left(\frac{nq}{h\nu}\right)^2 2J_1^2(\delta)\, A_R^2 \int_\Sigma \int_\Sigma d\sigma_1\, d\sigma_2\, B_u(L,\rho)\, \frac{1}{2}\left(1 + e^{-2B_\phi(\vec{r}_1,\vec{r}_2)}\right) \qquad \text{(C.7)}$$

$\overline{S}$ as given by C.7 is the correct statistical average signal power in the output current of the detector. The largest discrepancy between C.7 and expression 9.14 occurs when $e^{-2B_\phi(\vec{r}_1,\vec{r}_2)}$ is very small, i.e. when $B_\phi(\vec{r}_1,\vec{r}_2)$ is the largest. This occurs when $\vec{r}_1 = \vec{r}_2$ in the plane of the detector. However, $\overline{S}$ given by 9.14 is at most twice $\overline{S}$ given by C.7. Therefore, expression 9.14 is a good approximation to the correct expression C.7; it is handier to use in practical calculations at least until a knowledge of the correlation function of the phase fluctuations is available.

APPENDIX D

CONDITION FOR A SHOT-NOISE LIMITED DETECTION IN A VIDEO OPTICAL COMMUNICATION SCHEME

Our analysis of the S/N for the video communication scheme has been based on the assumption that the detection is shot-noise (quantum noise) limited, i.e. the shot noise inherent to the d.c. current due to the signal overcomes all the other possible noise sources. This is only true for sufficiently large optical signal power.

The noise power in a photomultiplier is

$$N = 2q\ B\ g^2 R(i_{DC} + I_D) + 4kTB$$

where g = amplification factor of the photomultiplier

B = i.f. bandwidth

q = electron charge = 1.6×10^{-19} coulomb

i_{DC} = D.C. photocurrent due to the optical signal

I_D = dark current

kTB = thermal noise

R = resistance of the output load

The noise power includes optical signal shot noise, dark current shot noise and thermal noise.

If the detector is exposed to a large amount of background light, then a corresponding shot noise term must be added to the noise power. However, we shall assume that the background light, if any, has been made much smaller than the signal light by optical filtering techniques.

For large values of the gain factor g and of the output load resistance R, the contribution of the thermal noise to the total noise power is negligible compared to the shot noise part. Then the detection of the video communication scheme will be shot-noise limited if the cathode photo current I is made larger than the dark current I_D,

$$\overline{i_{DC}} > I_D \qquad \text{(D.1)}$$

With the help of 9.23 we obtain

$$\overline{i_{DC}} = \frac{\eta q}{h\nu} \int_{\Sigma} d\sigma \, \overline{I_{(2)}(L,\vec{r})}$$

or

$$\overline{i_{DC}} = \frac{\eta q}{h\nu} \frac{P}{4} \qquad \text{(D.2)}$$

where P is the laser power incident on the receiving aperture area in the absence of turbulence. The condition D.1 is written

$$P > 4 I_D \frac{h\nu}{\eta q}$$

Taking $\eta = 10^{-2}$ and a dark current $I_D = 10^{-14}A$, the laser power should be greater than 10^{-11} watts at the plane of the detector.