Immersing Essential Surfaces in Odd Dimensional Closed Hyperbolic Manifolds

Thesis by

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In Partial Fulfillment of the Requirements for the
degree of
Doctor of Philosophy

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2017
Defended October 7, 2016

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ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my advisor, Prof. Vladimir Makrovic, for his support, guidance and supervision throughout my five years of Ph.D. study. His continuous encouragement helped me all through my research and the writing of this thesis. Without his patient guidance, it would be impossible for me to finish this thesis.

I would like to thank the rest of my thesis committee Prof. Matilde Marcolli, Prof. Yi Ni, and Dr. Tengren Zhang, for their valuable comments and insightful questions. Their rigorous attitudes toward research are always great inspirations to me.

I am also grateful to Yi Liu for several helpful conversations and suggestions.

Finally, I would like to thank my parents for supporting me throughout this thesis and my life in general.

ABSTRACT

The surface subgroup theorem, proved by Kahn and Markovic, states that the fundamental group of every closed hyperbolic 3-manifold contains a closed hyperbolic surface subgroup. The criterion of incompressibility, a criterion to ensure that an immersing surface to be essential, has played an important role in their proof.

In this thesis, we generalize the criterion of incompressibility from dimension three to all higher dimensions. Then we use the mixing property of the geodesic flow to construct a closed immersed surface which satisfies the assumption of our criterion when the hyperbolic manifold is in an odd dimension. Together, we prove the surface subgroup theorem in all odd dimensions.

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INTRODUCTION

1.1 Background

Thurston's Geometrization conjecture stated that every closed 3-manifold can be decomposed in a canonical way (prime decomposition and JSJ decomposition) into pieces that each has one of eight types of geometric structure. In particular, Thurston had showed that the geometrization conjecture holds for all Haken manifolds. Haken manifolds, first introduced by Haken in 1961, are a special type of manifolds. They are oriented compact irreducible manifolds that contain a properly embedded two-sided incompressible surface other than S^2.

It has been proved that Haken-three manifolds admit a hierarchy, where they can be split up into three-balls along incompressible surfaces. And a manifold is virtually Haken if it is finitely covered by a Haken manifold. The famous Virtually Haken Conjecture (VHC) stated that every closed hyperbolic 3-manifold is virtually Haken. A relative question is whether a hyperbolic 3-manifold contains an immersed closed hyperbolic surface. Actually the resolution of this question is a crucial and foremost step in Agol's proof of VHC [Ago13]. Since for higher hyperbolic manifolds, their fundamental groups can completely determine the geometry according to the Mostow Rigidity Theorem. In this thesis, we ask the following question:

Question 1.1.1. *Let M^n be a closed hyperbolic manifold with $n \geq 3$. Does the fundamental group of M contain a surface subgroup?*

When $n = 3$, Kahn and Markovic [KM12] proved the Surface Subgroup Theorem, which states that, for any closed hyperbolic 3-manifold, the fundamental group always contains a surface subgroup. In this thesis, we answer this question confirmly when n is odd (cf. Theorem 1.1.2). We also point out that Cooper-Long-Reid [CLR97] gave a positive answer to this question in the case of cusped finite volume hyperbolic-3 manifolds. In his preprint, Liu [Liu16] proved that every closed hyperbolic 3-manifold contains an immersed quasi-Fuchsian closed subsurface of odd Euler characteristic.

We should also point out that Ursula has proved Theorem 1.1.2 in her paper [Ham15, Theorem 1]. However, we proved this result independently using a different approach. My advisor gave me this problem in 2012, and essentially I had done all the work by 2013. I chose to write the paper right now only because I am writing my thesis.

Lower dimensional closed hyperbolic manifolds have been well studied today. In dimension two, Teichmüller Theory tells us that the geometry of a closed hyperbolic surface is completely determined by its Fenchel-Nielsen coordinates. Kahn and Markovic [KM15] proved the Ehrenpreis conjecture, which essentially says that the geometry of any two closed hyperbolic surfaces can be arbitrarily close, up to taking finite covers. In dimension three, built on the results of Kahn-Markovic [KM12] and Wise [Wis09] , Agol [Ago13] proved the famous Virtually Haken Conjecture showing that any closed hyperbolic 3 manifold virtually contains a properly embedded two sided incompressible surface. The proof of the Virtually Haken Conjecture gives us a simple recipe for constructing all closed hyperbolic three-manifolds, the generic type of three-dimensional geometry that had not been fully explicated. The readers are referred to [AFW15] for a detailed survey on three dimensional manifolds.

However, the fundamental groups of higher dimensional closed manifolds are poorly understood, one reason being that for any given finitely presented group and any $n \geq 4$, there exists a closed n-manifold with the prescribed fundamental group. In this thesis, we will prove the following theorem which generalizes the Surface Subgroup Theorem to all odd dimensions.

Theorem 1.1.2. *Let M^n be a closed hyperbolic manifold in dimension n. If n is an odd number, then there exists a closed hyperbolic surface S_0 and a π_1-injective map $f : S_0 \to M$.*

We mainly follow Kahn and Markovic's construction and generalize their method to higher dimensions. The generalization works well, one reason being that the frame flow is exponential mixing (cf. Theorem 3.0.1) in a higher dimensional hyperbolic manifold. However, our proof is not valid for even n, and the reason is that, for even n, the numbers of pants on opposite sides are not necessarliy balanced. So in this case we cannot glue all the pants in a controlled way to form a closed surface. We believe that there is some homological obstruction in the even dimensional case. So we re-ask the following question about the homology of higher dimensional manifolds.

Question 1.1.3 ([LM15], Question 9.2)**.** *Is the (rational) good pants homology equivalent to the standard (rational) homology for higher dimensional closed hyperbolic manifold?*

1.2 Sketch of the Proof.

Our proof of Theorem 1.1.2 is based on the Kahn-Markovic construction. Using the exponential mixing property of the frame flow (cf. section 3.1), we can construct abundant incompressible pairs of pants from well connected tripods. By studying the geometry of these skew pants we show that they are all R-good. Next, a non-negative weight will be assigned to each R-good pair of pants according to the mixing rate of the tripods. By investigating the symmetry on each good curve, we show that it is possible to glue the good pants along their boundaries in a nearly unit sheering fashion. Together, we have constructed a closed immersed surface S, which is a candidate of the π_1-injective immersed surface. In the next chapter, we will prove the following incompressibility criterion.

Theorem 1.2.1. *For any closed hyperbolic manifold M^n, there exists $R_0 > 0$ such that the following holds. For any $R > R_0$, assume that S is an immersed closed surface in M with a pants decomposition C such that each pair of pants is R-good (see Definition 1.3.1) and the sheering twists along each gluing are $\frac{10}{R^2}$-close to 1. Then there exist a closed hyperbolic surface S_0, which has the same pants decomposition as S, and a π_1-injective map $f : S_0 \to M$.*

Remark 1.2.2. *The condition on sheering twists is to ensure that the thin parts of the pants won't accumulate. We note that the length of a seam is on the order of $e^{-\frac{R}{2}}$ when all lengths of three cuffs are close to R. If we disregard the condition on sheering, then it could happen that the thin parts of the surface accumulate together, and the surface might not be compressible .*

Proof strategy. Let α be a geodesic in S_0, where S_0 is a hyperbolic surface comprised by R-standard pairs of pants. Denote by $f(\alpha)$ the image of α in $S \subset M$. We can homotope $f(\alpha)$ to be a piecewise geodesic, where all the bending points are on the cuffs. Since the twists along each gluing are very close to 1, the thin parts of pants won't accumulate. We can show that every unit-length subarc of α

can cross at most $O(R)$ many cuffs. In Section 2.3, we will estimate the length and bending angles of $f(\alpha)$ and show that, after carefully choosing the bending points, all bending angles will be simultaneously small. In particular, it implies that the piecewise geodesic $f(\alpha)$ will have a small bending norm (see Section 2.2). In Corollary 2.2.4, we show that any piecewise geodesic in $\mathbb{H}^n$ with small bending norm is a quasi-geodesic with good constants. Hence, we've completed the proof that the loop $f(\alpha)$ is non null-homotopic.

The rest of this paper, Chapter 3, is to show that the closed immersed surface described in Theorem 1.2.1 exists. To construct such a surface, we first need to find a suitable finite collection of good pants such that the pants can be assembled to form a closed surface with twists close to 1. We can view a collection of good pants as a finite measure μ over the set of good pants. By some combinatorics argument, the gluing conditions can be translated into a linear system of inequalities on the measure μ. So the goal is to find positive integral solutions to some linear system. To this end, we use the mixing property of the frame flow to construct abundant good pairs of pants and assign a non-negative weight $\mu(\Pi)$ to each good pair of pants Π. We then show that measure μ constructed this way is a real solution to the linear system. Then by a standard rationalization procedure, an integral non-negative solution also exists. Hence,

1.3 Settings and Notations

Throughout this paper, M^n will be a closed hyperbolic n-manifold. The universal cover $\tilde{M}$ of M can be identified as a copy of the n-dimensional hyperbolic space $\mathbb{H}^n$. The group $\mathrm{SO}(n,1)$ is the orientation preserving isometry group of $\mathbb{H}^n$. We will regard the deck transformation group $\pi_1(M)$ as a torsion-free discrete cocompact subgroup of $\mathrm{SO}(n,1) = \mathrm{Isom}^+(\mathbb{H}^n)$. The topological space $\mathrm{SO}(n,1)$ can be identified as the orthonomal frame bundle of $\mathbb{H}^n$. The isometry group $\mathrm{SO}(n,1)$ acts on $\mathbb{H}^n$ transitively and the stabilizer group is $\mathrm{SO}(n)$, so the hyperbolic space $\mathbb{H}^n$ can be identified as $\mathrm{SO}(n,1)/\mathrm{SO}(n)$.

Every curve in M can be freely homotoped to a unique closed geodesic. Sometimes we abuse the notation of a curve and its geodesic representative. Let γ be a closed geodesic in M, and denote the unit normal vector bundle of γ as $N(\gamma)$. Then $N(\gamma)$ is an associated vector bundle of an $\mathrm{SO}(n-1)$ principal bundle. The parallel transport along γ acts on $N(\gamma)$ and fixes the fibers of $N(\gamma)$, and we call this action the monodromy action. Let $v \in N(\gamma)$ and denote $\gamma.v$ the monodromy action on v. The monodromy of γ is called ϵ close to the identity if for any $v \in N(\gamma)$ we have that $\gamma.v$ and v are ϵ close to each other. The shape of a closed geodesic is completely determined by its length and monodromy.

Let $\Sigma_{0,3}$ be a topological pair of pants (a three holed sphere with boundaries). A pair of pants in M is the homotopy class of a π_1-injective map $\Pi : \Sigma_{0,3} \to M$. While Π is not unique, we can always homotope it so that the boundary of $\Pi(\Sigma_{0,3})$ is a union of three closed geodesics in M, and we call the geodesic representatives the cuffs of the pants. The seams of $\Sigma_{0,3}$ are three properly embedded simple arcs connecting the three pairs of cuffs of $\Sigma_{0,3}$. The images of seams from $\Sigma_{0,3}$ can be homotopic to unique geodesics that are orthogonal to the cuffs of $\Pi(\Sigma_{0,3})$. We call these geodesic arcs the seams of Π. For every pair of cuffs C_1 and C_2, the seam η from C_1 to C_2 defines a unit normal vector $v \in N(C_1)$, pointing along η towards C_2. There are six feet for Π, exactly two on each cuff. We often abuse the notations of feet and their basepoints.

The collection of all closed curves in M is denoted by Γ and the collection of all pairs of pants in M is denoted by $\mathbf{\Pi}$. Let $R > 0$ be a fixed large number and $\epsilon > 0$ be a fixed small number, and use the notation $\Gamma_{R,\epsilon}$ to denote the collection of all closed geodesics such that the lengths are ϵ close to R and the monodromies are ϵ-close to the identity. We denote by $\Pi_{R,\epsilon}$ the collection of all pairs of pants in M such that all three cuffs (the geodesic representatives) are in $\Gamma_{R,\epsilon}$. Let $\gamma \in \Gamma$ and denote $N^2(\gamma) = N(\gamma) \times N(\gamma)$ to be the product manifold. We denote $N(\Gamma)$ to be the disjoint union of unit normal bundles of $N(\gamma)$ for all $\gamma \in \Gamma$. Similarly, we denote $N^2(\Gamma)$ to be the disjoint union of $N^2(\gamma)$ for all $\gamma \in \Gamma$. It is natural for us to study $N^2(\Gamma)$, as a pair of feet on a cuff is indeed an element of $N^2(\Gamma)$. When we glue two paris of pants along a common cuff, we want to match the two pairs of feet simultaneously.

Measures and boundary operators. Denote $\mathcal{M}(\mathbf{\Pi})$ to be the space of all finitely supported finite measures on the set of pair of pants in M, and denote by $\mathcal{M}(\Pi_{R,\epsilon})$ the space of all finitely supported finite measures on $\Pi_{R,\epsilon}$. For a measure space $\mathcal{M}$, we use the notation $\mathcal{M}^+$ to denote the subspace of all measures that are non-negative.

Then there is a natural boundary operator

$$\partial : \mathcal{M}(\mathbf{\Pi}) \to \mathcal{M}(N^2(\Gamma)),$$

defined by assigning $\partial\Pi$ to be the sum of the atomic measures supported at the three pairs of feet, where the mass of each atom is 1. To be precise, the boundary operation is defined as follows. For any $\Pi \in \mathbf{\Pi}$, let $v_{i,j}$ be the foot on the cuff C_i that points to the cuff C_j, where $i, j \in \mathbb{Z}_3$ and $i \neq j$. Denote by a_i^Π the atom measure on $\mathcal{M}(N^2(\Gamma))$ supported at the pair $(v_{i,i+1}, v_{i,i-1})$ with a total measure 1. We define

$$(\partial\mu)(\Pi) = \mu(\Pi) \cdot \sum_{i=1}^{3} a_i^\Pi$$

to be the measure on $\mathcal{M}(N^2(\Gamma))$ supported at the three pairs of feet. By definition, the restriction of the boundary operator on $\mathcal{M}(\Pi_{R,\epsilon})$ yields the following:

$$\partial : \mathcal{M}(\Pi_{R,\epsilon}) \to \mathcal{M}(N^2(\Gamma_{R,\epsilon})).$$

In the rest of this chapter we give some geometric definitions.

Distances and angles. Let p, q be two points in $\mathbb{H}^n$ and use $d(p, q)$ to denote the hyperbolic distance between them, and let $v(p, q)$ be the unit vector at p that is tangent to the geodesic segment from p to q. Let v, w be two unit tangent vectors based at a point $p \in \mathbb{H}^n$, and denote $\Theta(v, w)$ to be the unoriented angle between v and w. Let $p, q \in \mathbb{H}^n$ and v be a vector based at p, and denote $v@q$ to be the parallel transport of v along the geodesic segment p to q.

We can also define the complex distance between two oriented geodesics in $\mathbb{H}^n$. Let α and β be two oriented geodesics in $\mathbb{H}^n$; there exists a unique geodesic γ with an orientation perpendicular to both α and β. Let $p = \alpha \cap \gamma$ and $q = \beta \cap \gamma$, and let u be the tangent vector to α at p and let v be the tangent vector to β at q. The complex distance between α and β denoted as $\mathbf{d}(\alpha, \beta)$ is defined as follows: the real part of $\mathbf{d}$ is given by $d(p, q)$ and the imaginary part is $\Theta(u@q, v)$. Let p, q be two points on a oriented geodesic γ, then use $d^*_\gamma(p, q)$ to denote the real distance from p to q under the orientation of γ. If, in addition, u is a unit vector at p and v is a unit vector at q, we use $\mathbf{d}^*_\gamma(u, v)$ to denote the complex distance between u and v measured along γ.

Half lengths and twists. For a pair of pants Π in M, the three seams will cut Π into two singular regions whose boundaries are right angled hexagons. In dimension three, the two right angled hexagons are isometric (cf. [KM12]). However, in general we no longer have this property for higher dimensions. Namely, the two right angled hexagons could have different shapes as explained in [Tan94] when $n > 3$. Let C be a cuff of Π and let η_1, η_2 be the two seams on C. We can define the half length of C in two ways: the complex distance from η_1 to η_2 along C, or the complex distance from η_2 to η_1 along C. Denote by $\mathbf{hl}_1^{\Pi}(C)$ and $\mathbf{hl}_2^{\Pi}(C)$ the two half lengths on C. In dimension 3, these two definitions coincide. However, in higher dimensions, they may differ from each other. But we will show later that all pairs of pants we constructed will be R-good (cf. Definition 1.3.1), so the two half lengths will be very close to $\frac{R}{2}$.

The orientation of a pair of pants Π will induce the orientation on its boundary $\partial\Pi$. Two pants, Π_1 and Π_2, that share a boundary component C can be glued together if their induced orientations on C are the reverse of each other. Let (v_1, v_2) be the pair of feet on C from Π_1 and let (w_1, w_2) be the pair of feet on C from Π_2. We define the sheering twists $\{S_1^{\Pi_1,\Pi_2}(C),\ S_2^{\Pi_1,\Pi_2}(C)\}$ as follows:

$$S_1^{\Pi_1,\Pi_2}(C) = \mathbf{d}_C(v_1, -w_1) \ \text{ and } \ S_2^{\Pi_1,\Pi_2}(C) = \mathbf{d}_C(v_2, -w_2).$$

Recall that in dimension 3, we always have $S_1^{\Pi_1,\Pi_2}(C) = S_2^{\Pi_1,\Pi_2}(C)$. However, in higher dimensions, the two sheer twists can be different from each other. If both pairs of pants are good pants (see Definition 1.3.1), as all half lengths are close to $\frac{R}{2}$, the differences of the sheer twists will be bounded by $\frac{2}{R^2}$. So up to a small error, the sheer twists S_1 and S_2 are the same. Let $\delta > 0$, and say that the gluing is δ-close to a unit sheering if and only if both $\left|S_1^{\Pi_1,\Pi_2}(C) - 1\right|$ and $\left|S_2^{\Pi_1,\Pi_2}(C) - 1\right|$ are bounded by δ.

We give the following definition of good pants.

Definition 1.3.1 (Good pants)**.** *For a fixed large number $R > 0$, a pair of pants Π is called R-good if it satisfies the following:*

1. *For any cuff C of Π, the two half lengths satisfy:*
$$\left|\mathbf{hl}_1^{\Pi}(C) - \frac{R}{2}\right| \le \frac{1}{R^2}, \quad i = 1, 2;$$
2. *For any seam of Π, if we denote $\mathbf{d} = d + i\theta$ the complex length of the seam, then*
$$\frac{\theta}{d} \le \frac{1}{R^2} \ \text{ and } \ \left|\frac{d}{d_0} - 1\right| \le \frac{1}{R^2},$$
where d_0 is the length of a seam in an R-standard pair of pants.

Definition 1.3.2 (Well-glued)**.** *Let* Π_1 *and* Π_2 *be two pairs of R-good pants. Assuming that* Π_1 *and* Π_2 *share a common boundary cuff C, we say that* Π_1 *and* Π_2 *are well-glued along C if and only if the following conditions are satisfied.*

1. Π_1 *and* Π_2 *induce opposite orientations on* C*.*
2. *The gluing is* $\frac{10}{R^2}$*-close to the unit sheering; that is, the following inequalities hold:*
$$\left|S_1^{\Pi_1,\Pi_2}(C) - 1\right| \leq \frac{10}{R^2} \text{ and } \left|S_2^{\Pi_1,\Pi_2}(C) - 1\right| \leq \frac{10}{R^2}.$$

Remark 1.3.3. *For good pairs of pants, we have*

$$\left|S_1^{\Pi_1,\Pi_2} - S_2^{\Pi_1,\Pi_2}\right| \leq \frac{2}{R^2},$$

so, by the triangle inequality, equation $\left|S_1^{\Pi_1,\Pi_2}(C) - 1\right| \leq \frac{8}{R^2}$ *would imply that the two pants are well glued. So, to make sure that* Π_1 *and* Π_2 *are well glued along C, we only need to keep track of one foot from each side of C and have the stronger inequality with the bound* $\frac{8}{R^2}$*.*

Chapter 2

CRITERION OF INCOMPRESSIBILITY

2.1 Construction

The goal of this chapter is to prove Theorem 1.2.1. Let S be the immersed surface in the theorem, then S has a pants decomposition C, where each pair of pants is good (see Definition 1.3.1) and the gluing has a nearly unit sheering twist. We will start by constructing the closed surface S_0 and the map $f : S_0 \to M$.

Let S_0 be a hyperbolic closed surface that has the same pants decomposition as S. Then S is homeomorphic to S_0 as a topological surface. The geometry of the surface S_0 is determined by the (real) Fenchel-Nilsen coordinates associated to the pants decomposition. Let g be the genus of the surface S_0, then there are $6g - 6$ Fenchel-Nilsen coordinates, where $3g - 3$ of them are called lengths and the other $3g - 3$ are called twists. We let all the lengths be R and all the twists be 1. The surface S_0 can be viewed as the standard model for S.

We can choose f to be a representative in the homotopy class so that it maps the cuffs of S_0 to the cuffs of S. As the 1-complex made of cuffs and seams divides both surfaces into singular regions whose boundaries are right angled hexagons, so f can be extend to a map from S_0 to M. On each cuff, there are four feet which cut the cuff into four geodesic segments. We first let f map the feet of S_0 to the corresponding feet in S. Then we extend f to the whole cuff by similarities on each geodesic segments cut by those feet. Similarly, we extend f on each seam by similarity. We should just remember that the map f satisfies the following conditions: (1) f maps the boundary of a right angled hexagon to the boundary of some right angled hexagon. (2) When restricted to any side of a right angled hexagon, f is a $(1 + \frac{20}{R^2})$-bilipschitz map onto its image when R is large.

It is natural to lift f to the universal covers $f : \tilde{S}_0 \to \tilde{M}$. Here we abuse the notation of f and its lift. In surface S_0 the collection of cuffs is lifted to a geodesic lamination λ and the seams are lifted to geodesic segments connecting leaves of the lamination. The seams and the lamination divide the hyperbolic plane into right angled hexagons. Similarly, we can lift the cuffs in the universal cover of M and get a lamination λ' in $\mathbb{H}^n$.

Let $\gamma : [0, L] \to S_0$ be a closed geodesic in S_0 parametrized by its arc-length. We assume that $\gamma(0)$ is on a cuff of S_0. Denote by $\tilde{\gamma} : [0, L] \to \mathbb{H}^n$ a lift of γ in the universal cover of S_0. Let $C_0, C_1, \cdots, C_{k+1}$ be the leaves in the lamination λ that $\tilde{\gamma}$ intersects. We orient C_i so that the oriented angle from $\tilde{\gamma}$ to C_i is positive. Let P_i be the intersection point between $\tilde{\gamma}$ and C_i, for $i = 0, \cdots, k+1$. Let $C_i' = f(C_i)$ be the corresponding leaf in $\mathbb{H}^n$, for $i = 0, \cdots, k+1$, with the induced orientation. Let $D_i,\ i = 0, \cdots, k$ be the common orthogonal between C_i and C_{i+1}. Similarly, let $D_i',\ i = 0, \cdots, k$, be the common orthogonal between C_i' and C_{i+1}'.

We have the following lemma from [KM12, section2].

Lemma 2.1.1. *Assume that $d(P_j, P_{j+1}) < e^{-5}$ holds for $j = i, j+1, \cdots, i+m$, then for R large enough, we have $m < R$.*

The following result is a corollary of the previous lemma.

Lemma 2.1.2. *Let γ be a closed geodesic in S_0, then γ has length greater than e^{-5} for sufficiently large R.*

Proof. Let m be a positive integer. We consider the closed geodesic $m\gamma$, the m-times multiple of γ in S_0. Denote $\hat{\gamma}$ the lift of $m\gamma$ in the universal cover $\mathbb{H}^2$. As before, we denote by $P_0, \cdots, P_{k+1}$ the points at which $\hat{\gamma}$ intersects with the leaves of the lamination λ.

Assume that the length of γ is less than e^{-5}, then we have

$$d(P_i, P_{i+1}) < e^{-5},$$

for all i. By the previous lemma, we have $k+1 < R$. On the other hand, since $\hat{\gamma}$ is the lift of $m\gamma$, the intersection number $k+1 \geq m$. So we have $R > m$. However, m is an arbitrary positive integer. Contradiction! □

In the next section, we will define the bending norm of a piecewise geodesic and show that a piecewise geodesic is a quasi-geodesic, provided that the bending norm is small.

2.2 Piecewise Geodesic and Bending Norm

Let $\alpha : [0, \infty) \to \mathbb{H}^n$ be a piece-wise geodesic parametrized by its pathlength. We define the bending norm as follows.

Definition 2.2.1. *Let I be an open interval in $[0,\infty)$ such that the length of I is 1, and denote $\mu(I)$ to be the sum of all bending angles in the piece-wise geodesic subarc $\alpha(I)$. We define the bending norm of α as follows.*

$$||\alpha|| = \sup_I \mu(I),$$

where I varies over all open intervals of length 1. So the norm represents the maximal bending of a fixed length.

Let $\theta(t)$ be the oriented angle from $\alpha'(t)$ to the geodesic ray from $\alpha(0)$ through $\alpha(t)$. Let point P to be the starting point of α, namely, $P = \alpha(0)$. Denote $s(t)$ to be $\mathbf{d}(P, \alpha(t))$. Then $\theta(t)$ and $s(t)$ vary smoothly at any smooth point of α. In the following lemma we will discuss precisely how $\theta(t)$ and $s(t)$ change when t varies. Then we use these formulas to get a upper bound of $\theta(t)$.

Lemma 2.2.2. *There exists some universal constant $\epsilon_0 > 0$ such that the following holds. Let $\alpha : [0,\infty) \to \mathbb{H}^n$ be a piecewise geodesic, and denote the bending norm of α by ϵ. If $\epsilon \le \epsilon_0$, then, for all $t > 0$, $0 < \theta(t) \le 3\epsilon$.*

By reversing the orientation of α, we obtain the following corollary.

Corollary 2.2.3. *Under the assumption of Lemma 2.2.2, for any $t > 0$, the angle between the initial direction $\alpha'(0)$ and the geodesic ray from $\alpha(0)$ through $\alpha(t)$ is bounded by 3ϵ.*

Proof of Corollary 2.2.3. Let $t > 0$, we consider the piecewise geodesic segment $\beta : [0,t] \to \mathbb{H}^n$ defined by $\beta(s) = \alpha(t-s)$, for any $s \in [0,t]$. Then by definition, the bending norm of β is no more than ϵ, the bending norm of α. So the result of Lemma 2.2.2 applies to β. In particular, we have that the angle between $\beta'(t)$ and the geodesic ray from $\beta(0)$ through $\beta(t)$ is bounded by 3ϵ. Then the claim of this corollary follows. □

The purpose of Lemma 2.2.2 is to study the asymptotic behavior of a piecewise geodesic α. We show that, if the bending norm of α is small, then the piecewise geodesic will be a quasi-geodesic. Moreover, by Corollary 2.2.3, α will always be inside the cone of aperture 6ϵ with $\alpha(0)$ the vertex and $\alpha'(0)$ the direction of the axis. The following proof is based on [EMM04, Lemma 4.4].

Proof of the Lemma 2.2.2. Let us first examine the derivatives of $s(t)$ and $\theta(t)$ at smooth points. Let $\alpha(t)$ be a smooth point. We consider the triangle with three vertices P, $\alpha(t)$ and $\alpha(t+\Delta)$, where Δ is a small positive number such that $\alpha([t, t+\Delta])$ is a geodesic segment. The geometric measurements of this triangle are illustrated in the Figure 2.2. Then by the sine rule and cosine rule, we have the following inequalities.

$$\cosh s(t+\Delta) = \cosh s(t) \cosh \Delta + \sinh s(t) \sinh \Delta \cos \theta(t),$$

$$\sinh s(t+\Delta) \cdot \sin \theta(t+\Delta) = \sinh s(t) \cdot \sin \theta(t).$$

Notice that the above equations hold for all Δ small enough. Take the derivative with respect to Δ.

Then we get

$$s'(t) = \cos(\theta(t)) \text{ and } \theta'(t) = -\frac{\sin(\theta(t))}{\tanh(s(t))} \leq -\sin(\theta(t)).$$

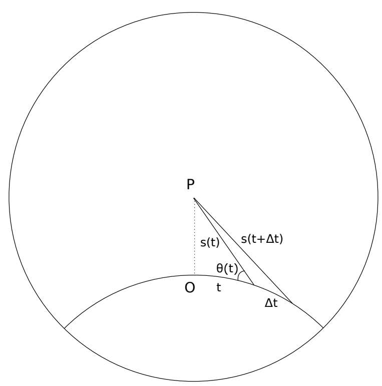

Figure 2.1: An illustration of the change rates of $s(t)$ and $\theta(t)$

So at a smooth point, $\theta(t)$ is always decreasing. In contrast, at a bending point, $\theta(t)$ may jump upwards up to ϵ. Let $\epsilon_0 = \frac{\pi}{6}$. We will prove by contradiction that

$$\forall t \in [0, \infty), \ t \leq \theta(t) \leq 3\epsilon < \frac{\pi}{2}.$$

Suppose this is false, then we can define t_2:

$$t_2 = \inf\{t : \theta(t) > 3\epsilon\}.$$

Since $\theta(0) = 0$, we must have $t_2 > 0$.

Similarly, we can define t_1 :

$$t_1 = \sup\{t : 0 < t < t_2 \text{ and } \theta(t) < 2\epsilon\}$$

and see that $t_1 > 0$. So for $t \in (t_1, t_2)$, we have

$$\theta(t_1) < 2\epsilon \le \theta(t) < 3\epsilon < \frac{\pi}{2}.$$

It follows that, at smooth points in (t_1, t_2),

$$\theta'(t) = -\sin(\theta(t)) < -\sin(2\epsilon) < -\epsilon.$$

We must have $t_2 > t_1 + 1$, because, by the definition of bending norm, θ can jump upwards at most ϵ in an open interval of length 1.

On the interval $[t_1, t_1 + 1]$, θ is decreasing more than $\sin(2\epsilon)$ over the smooth points and θ can increase at most ϵ over the bending points. Hence, we have

$$\theta(t_1 + 1) - \theta(t_1) < -\sin(2\epsilon) + \epsilon < 0.$$

Therefore $\theta(t_1+1) < \theta(t_1) < 2\epsilon$. This contradicts the definition of t_1 and completes the proof.

□

Under the condition of Lemma 2.2.2, we have $\theta(t) < 3\epsilon$ for all $t \ge 0$. Recall that

$$s'(t) = \cos(\theta(t)).$$

We can choose ϵ_0 small enough so that $\cos(3\epsilon_0) > 1 - \epsilon_0$. For instance, $\epsilon_0 = 0.23$ can be chosen. Then for any $0 < \epsilon < \epsilon_0$, the same inequality holds for ϵ. Then we have

$$s'(t) = \cos(\theta(t)) > \cos(3\epsilon) > 1 - \epsilon.$$

By integration, we obtain that

$$(1 - \epsilon)(t_2 - t_1) < \mathbf{d}(\alpha(t_1), \alpha(t_2)) \le t_2 - t_1,$$

for all $0 \le t_1 < t_2$. This shows that α is a bilipschitz map onto its image. So we have proved the following corollary.

Corollary 2.2.4. *In Lemma 2.2.2, we can choose ϵ_0 to be any positive number smaller than 0.23. Then, for any $0 < \epsilon < \epsilon_0$, we have*

$$(1 - \epsilon)|t_2 - t_1| \le \mathbf{d}(\alpha(t_2), \alpha(t_1)) \le |t_2 - t_1|.$$

In particular, α will be a $\frac{1}{1-\epsilon}$-bilipschitz map onto its image, where the image has the induced metric from $\mathbb{H}^n$.

The above bending norm technic will be used to prove the incompressibility theorem (Theorem 1.2.1) . In particular, we will prove the following claim in this chapter. We will use the setting from last section.

Claim 2.2.5. *On each geodesic C_i', there exists a point Q_i such that the following conditions hold when R large.*

(1). *$d(Q_0, f(P_0)) < \frac{20}{R}$ and $d(Q_{k+1}, f(P_{k+1})) < \frac{20}{R}$.*

(2). *If we denote by α the piecewise geodesic obtained by concatenating $Q_0, \cdots,$ Q_{k+1}, then the bending norm of α is bounded by $O\left(\frac{1}{R}\right)$.*

(3). *Denote by $l(\alpha)$ and L the path-lengths of α and $\tilde{\gamma}$ respectively, then*

$$\left|\frac{l(\alpha)}{L} - 1\right| = O(R^{-1}).$$

In the rest of this section, we show Claim 2.2.5 would imply Theorem 1.2.1. Assume that Claim 2.2.5 holds. Denote by ϵ_0 the constant in the Corollary 2.2.4. For any $0 < \epsilon < \epsilon_0$, we can choose R large enough such that the bending norm of α is smaller than ϵ and $\left|\frac{l(\alpha)}{L} - 1\right| < \epsilon$. Then by Corollary 2.2.4, we have

$$(1 - \epsilon)l(\alpha) \leq \mathbf{d}(Q_0, Q_{k+1}) \leq l(\alpha). \tag{2.1}$$

Hence, we obtain the following inequality:

$$(1 - 2\epsilon)L \leq \mathbf{d}(Q_0, Q_{k+1}) \leq (1 + \epsilon)L. \tag{2.2}$$

Since Q_0 is $\frac{20}{R}$-close to $f(P_0)$ and Q_{k+1} is $\frac{20}{R}$-close to $f(P_{k+1})$, we obtain

$$\mathbf{d}(Q_0, Q_{k+1}) - \frac{40}{R} \leq \mathbf{d}(f(P_0), f(P_{k+1})) \leq \mathbf{d}(Q_0, Q_{k+1}) + \frac{40}{R}. \tag{2.3}$$

By Lemma 2.1.2, L, the length of $\tilde{\gamma}$, is greater than e^{-5}, so we have $\frac{40}{R} < \epsilon L$ when R large. Hence, we obtain

$$(1 - 3\epsilon)L \leq \mathbf{d}(f(P_0), f(P_{k+1})) \leq (1 + 3\epsilon)L. \tag{2.4}$$

In particular, if $\epsilon < \frac{1}{3}$, the endpoints of $f(\tilde{\gamma})$ will be distinct, which means that $f(\gamma)$ is non null-homotopic. Therefore, the map $f : S_0 \to M$ is π_1-injective.

2.3 Preliminary Propositions

In this section, we will prove several preliminary results that will be used to prove Claim 2.2.5. The first proposition is a result from [Bow05, Section 14]. It answers the following questions. How would the length of P_iP_{i+1} change if the distance between C_i and C_{i+1} is dilated by a factor close to 1? And how would it effect the angle between P_iP_{i+1} and C_i (or C_{i+1}) ?

Proposition 2.3.1. *Let $ABCD$ and $A'B'C'D'$ be two quadrangles in a hyperbolic plane. Assume that $\angle B, \angle C, \angle B', \angle C'$ are right angles, and $|AB| = |A'B'|$, $|CD| = |C'D'|$. Let $0 < \epsilon < 1$, assume that*

$$\frac{1}{1+\epsilon} \leq \frac{|B'C'|}{|BC|} \leq 1+\epsilon, \tag{2.5}$$

then there exists a map $F : \mathbb{H}^2 \to \mathbb{H}^2$ such that:

- $F(A) = A',\ F(B) = B',\ F(C) = C',\ F(D) = D'$.
- *When restricted to side $AB,\ BC$ or CD, the map F is a similarity.*
- *The map F is a $(1+\epsilon)$-bilipschitz map.*

Moreover, we have $\sin|\angle DAB - D'A'B'| \leq \epsilon$.

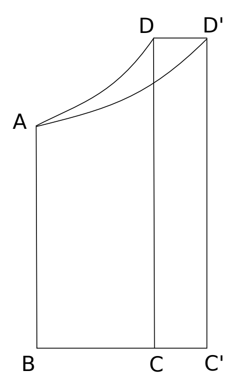

Figure 2.2: An illustration of the two quadrangles $ABCD$ and $A'B'C'D'$

Applying the above proposition twice, we get the following corollary to compare triangles.

Corollary 2.3.2. *Let OAB be a right angled triangle in the hyperbolic plane, where $\angle AOB = \frac{\pi}{2}$. Let $0 < \epsilon < 1$, and choose a point A' on ray OA and a point B' on ray*

OB. Assume that we have

$$\frac{1}{1+\epsilon} \leq \frac{|OA'|}{|OA|} \leq 1+\epsilon \quad and \quad \frac{1}{1+\epsilon} \leq \frac{|OB'|}{|OB|} \leq 1+\epsilon.$$

Then we have the following length inequality on $|AB|$ and $|A'B'|$:

$$\frac{1}{1+3\epsilon} \leq \frac{|AB'|}{|AB|} \leq 1+3\epsilon.$$

We also have the following angle relations:

$$|\sin(\angle A'B'O - \angle ABO)|, |\sin(\angle B'A'O - \angle BAO)| \leq 2\epsilon.$$

Proof of Proposition 2.3.1. Let $\mathbb{R}^2_r$ be $\mathbb{R}^2$ with the metric

$$ds^2 = \cosh^2(y)dx^2 + dy^2.$$

Then $\mathbb{R}^2_r$ is isometric to the hyperbolic plane (cf. [Fen89, page 205]). Let $O = (0,0) \in \mathbb{R}^2_r$, then $(x,0)$ is a point of distance $|x|$ from O and (x,y) is a point at distance $|y|$ from $(x,0)$ such that $\overline{(x,y)(x,0)}$ is perpendicular to $\overline{(0,0)(x,0)}$.

We position the two quadrangles $ABCD$ and $A'B'C'D'$ as follows. As in Figure 2.2, we let $B = B' = O = (0,0)$, $A = A' = (0,|BA|)$, $C = (|BC|,0)$, $C' = (|BC'|,0)$, $D = (|BC|,|CD|)$ and $D' = (|BC'|,|C'D|)$. Under this identification, we set $F(x,y) = (ax,y)$, where $a = \frac{|B'C'|}{|BC|}$. Then the map F satisfies the first two conditions of the claim. Now we prove that F is a $(1+\epsilon)$-bilipschitz map.

Let u be a tangent vector in $\mathbb{R}^2_r$ based at (x,y). Assume that $u = x_1 \cdot \frac{\partial}{\partial x} + x_2 \cdot \frac{\partial}{\partial y}$, then we have

$$< u,u >= \cosh^2 y \cdot x_1^2 + x_2^2.$$

Denote $v = F_*(u)$, then v is based at $F(x,y) = (ax,y)$ and

$$v = ax_1 \cdot \frac{\partial}{\partial x} + x_2 \cdot \frac{\partial}{\partial y}.$$

Hence, we have

$$< v,v >= \cosh^2 y \cdot (ax_1)^2 + x_2^2.$$

Combined with equation (2.5), we obtain the following inequality

$$||v||^2 \leq (1+\epsilon)^2 ||u||^2.$$

Therefore, for any two points p and q,

$$d(F(p), F(q)) \leq (1+\epsilon)d(p,q),$$

Replacing F with its inverse F^{-1} in the above inequality, we obtain that

$$d(p,q) \leq (1+\epsilon)d(F(p), F(q)),$$

So F is $(1+\epsilon)$-bilipschitz.
Now we estimate $\angle DAD'$. Without loss of generality, we assume that $|BC'| \geq |BC|$. Denote M the middle point of the segment DD'. Then by hyperbolic geometry, in triangle ADD' we have

$$\sinh|DM| = \cosh|CD| \sinh\frac{|CC'|}{2},$$

Similarly, we have

$$\sinh|DA| \geq \sinh d(D, BA) = \cosh|CD| \sinh|BC|.$$

By comparing the above two equations,

$$\frac{\sinh|DM|}{\sinh|DA|} \leq \frac{\sinh\frac{|CC'|}{2}}{\sinh|BC|}.$$

By the sine rule of triangles in hyperbolic geometry, we have

$$\sin\angle DAM \leq \frac{\sinh|DM|}{\sinh|DA|} \leq \frac{\sinh\frac{|CC'|}{2}}{\sinh|BC|}.$$

When $\epsilon < 1$, the inequality

$$\sinh\epsilon x \leq \epsilon \sinh x,$$

holds for all $x \geq 0$. So we get

$$\sin\angle DAM \leq \frac{|CC'|}{2|BC|} \leq \frac{\epsilon}{2}.$$

Similarly, we have

$$\sin\angle DAM' \leq \frac{\epsilon}{2}.$$

So we get

$$\sin\angle DAD' \leq \epsilon.$$

When ϵ is small enough, we have $\angle DAD' \leq 2\epsilon$. □

A direct generalization of Proposition (2.3.1) is that we can dilate multiple distances at the same time. So we have the following corollary.

Corollary 2.3.3. *Assume that the distance between C_i and C_{i+1} is dilated by a factor of $1+\epsilon_i$, for $i = 0, \cdots, k$. If $|\epsilon_i|$ is universally bounded by a positive number ϵ, then the bending angle at P_i is at most 2ϵ and the dilation of $|P_iP_{i+1}|$ is ϵ-close to 1.*

Though the above corollary allows us to simultaneous adjust the distances between multiple pairs of C_i and C_{i+1}, the whole setting is still restricted in a hyperbolic plane. Recall that in $\mathbb{H}^n$, the distance between two geodesics is a complex number, where the real part denotes the real distance between the geodesics and the imaginary part denotes the angle difference. Next proposition will allow us to perturb C_{i+1} in $\mathbb{H}^n$ such that the complex distance between C_i and C_{i+1} can have a small imaginary part. To be precise, let α and β_1 be two oriented geodesics in $\mathbb{H}^n$. Denote by $\mathbf{d} = d + i\theta$ the complex distance between α and β_1. Let γ be their common orthogonal that is oriented from α to β_1. We parametrize α, β_1 by their arc-lengths and assume that $\alpha(0) = \alpha \cap \gamma$, $\beta_1(0) = \beta_1 \cap \gamma$. Let β be the geodesic passes through $\beta_1(0)$ along the direction $\alpha'(0)@\beta_1(0)$. Then α, β and γ are in a hyperbolic plane, and θ is the angle between β and β_1. We also parametrize β by path-length such that $\beta(0) = \beta_1(0)$. Choose $a, b \in \mathbb{R}$, denote by $A = \alpha(a)$, $B = \beta(b)$ and $B' = \beta_1(b)$. Then we will prove the following proposition.

Proposition 2.3.4. *For any $\epsilon > 0$, if $\frac{\theta}{d} < \epsilon$, the following inequalities*

$$1 \le \frac{|AB'|}{|AB|} \le 1 + 2\epsilon, \quad \sin \angle BAB' \le \epsilon, \tag{2.6}$$

hold for any $a, b \in \mathbb{R}$.

Proof. Choose M to be the middle point of BB'. By the cosine rule in the appendix, we have

$$\sinh |BM| = \cosh b \sin \frac{\theta}{2},$$

and

$$\sinh |BA| \ge \cosh b \sinh d.$$

Comparing the above two equations, we get

$$\frac{\sinh |BM|}{\sinh |BA|} \le \frac{\sin \frac{\theta}{2}}{\sinh d}. \tag{2.7}$$

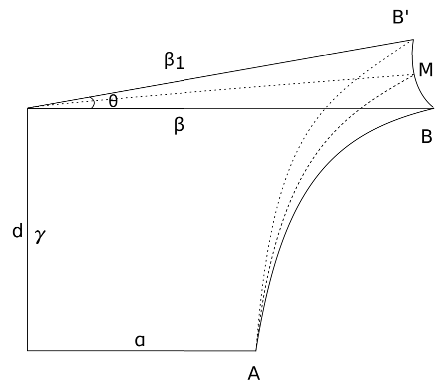

Figure 2.3: This figure illustrates the proof of Proposition 2.3.4

As for $x \geq 0$, we have $\sin x \leq x \leq \sinh x$, so we get

$$\frac{\sinh |BM|}{\sinh |BA|} \leq \frac{\theta}{2d},$$

Then by the sine rule in triangle, we have

$$\sin \angle MAB \leq \frac{\theta}{2d},$$

Similarly, we have

$$\sin \angle MAB' \leq \frac{\theta}{2d},$$

So we get

$$\sin BAB' \leq \frac{\theta}{d} \leq \epsilon.$$

From the cosine rule (see Appendix), we have the following formula:

$$\cosh |AB'| = \cosh a \cosh b \cosh d - \sinh a \sinh b \cos \theta \tag{2.8}$$

$$= \cosh a \cosh b(\cosh d - \cos \theta) + \cosh(a-b)\cos\theta, \tag{2.9}$$

Similarly, we have

$$\cosh |AB| = \cosh a \cosh b(\cosh d - 1) + \cosh(a-b). \tag{2.10}$$

If $|AB| \leq 1$, notice that the function $\frac{\sinh x}{x}$ is monotone increasing when x is positive, so we have

$$\frac{\sinh |AB|}{|AB|} \leq \sinh 1.$$

Combined with Equation (2.7), we have

$$\sinh |BM| \leq \sinh 1 \cdot \frac{\epsilon}{2}|AB|.$$

So we obtain

$$|BB'| = 2|BM| \leq \sinh 1 \cdot \epsilon |AB| < 2\epsilon |AB|.$$

Hence, $|AB'| \leq |AB|+|BB'| < (1+2\epsilon)|AB|$. It follows from (2.8) that $|AB'| \geq |AB|$. So it this case, the inequality (2.6) holds.

If $|AB| > 1$, by comparing (2.9) and (2.10), we get that

$$1 \leq \frac{\cosh |AB'|}{\cosh |AB|} \leq \frac{\cosh d - \cos\theta}{\cosh d - 1}.$$

Since the inequalities

$$1 - \cos x \leq \frac{x^2}{2} \leq \cosh x - 1,$$

hold for all $x \geq 0$, we get

$$\frac{\cosh d - \cos\theta}{\cosh d - 1} - 1 = \frac{1 - \cos\theta}{\cosh d - 1} \leq \frac{\theta^2}{d^2} \leq \epsilon^2.$$

It follows that $\frac{\cosh|AB'|}{\cosh|AB|} \leq 1 + \epsilon^2$. Since $\cosh(x + y) \geq \cosh x \cosh y$ for all x, y positive, we get

$$\cosh(|AB'| - |AB|) \leq 1 + \epsilon^2.$$

It follows that $|AB'| - |AB| \leq \sqrt{2}\epsilon$. Since $|AB| > 1$ as we assumed, the inequality (2.6) holds. □

2.4 Earthquake map

Recall that real Fenchel-Nilsen coordinates consist of lengths and twists. A real twist on C_i is the real distance between basepoints of the two feet on C_i. We want to adjust the real twists of C_i so that it matches the twists on C_i', and an earthquake map is such a transformation. The definition of earthquake map is given as follows.

Let l^* be an oriented geodesic in $\mathbb{H}^2$. For any $h \in R$, there is a unique isometry $E(l^*, h) \in \mathrm{PSL}(2, \mathbb{R})$ such that $E(l^*, h)$ fixes l^* and translates l^* a distance of h. For example, if l^* is the geodesic from 0 to ∞, then $E(l^*, h)$ is the transformation given by $z \mapsto e^h z$. We consider the lamination λ_0 comprised by leaves $C_0, \cdots, C_{k+1}$ and we associate a real number h_j to each leaf C_j, for $j = 0, \cdots, k + 1$. We view $\mu = (h_0, h_1, \cdots, h_{k+1})$ as a real measure on the lamination λ_0.

We are now ready to construct the earthquake map on the lamination λ_0. Denote the $k + 3$ components of $\mathbb{H}^2 \setminus \cup C_j$ from the left to the right by $\Delta_{-1}, \Delta_0, \cdots, \Delta_{k+1}$. We start with the extreme right line l_{k+1}. We apply the transformation $E(C_{k+1}, h_{k+1})$

on C_{k+1} and Δ_{k+1}. To the configuration $\Delta_k \cup E(C_{k+1}, h_{k+1})(\Delta_{k+1})$ apply $E(C_k, h_k)$. Continuing this process, we end up with the lines in $\mathbb{H}^2$,

$$C_0,\ E(C_0, h_0)(C_1), \cdots,\ E(C_0, h_0) \cdots E(C_k, h_k)(C_{k+1}).$$

We just denote by E the whole process of the earthquake associated to (λ_0, μ). Then the images of $C_0, \cdots, C_{k+1}$ under the earthquake E are $E(C_0), \cdots, E(C_{k+1})$ respectively. Each successive pair of lines bounds a sector of the plane isometric to one of the Δ_j. Geometrically, we are translating Δ_j a distance h_j from 0 to $k+1$ successively. Let C_i and C_j, $i < j$ be two leaves. We define the earthquake norm $|\mu|$ between C_i and C_j as follows:

$$|\mu|(C_i, C_j) = \sum_{i<p<j} |h_p|.$$

The readers are referred to [EM87, p.209-215] and [EMM04, Section 4] for more details on real earthquakes. The following proposition is a result from [EMM04, Theorem 4.12].

Proposition 2.4.1. *Let C_i and C_j be two leaves of the geodesic lamination (λ_0, μ). Let $x = |\mu|(C_i, C_j)$ be the earthquake norm between C_i and C_j. Let $E(C_i)$ and $E(C_j)$ be the images of C_i and C_j under the earthquake specified by $\mu = (h_0, \cdots, h_{k+1})$. Then*

$$e^{-\frac{x}{2}} d(C_i, C_j) \leq d(E(C_i), E(C_j)) \leq e^{\frac{x}{2}} d(C_i, C_j).$$

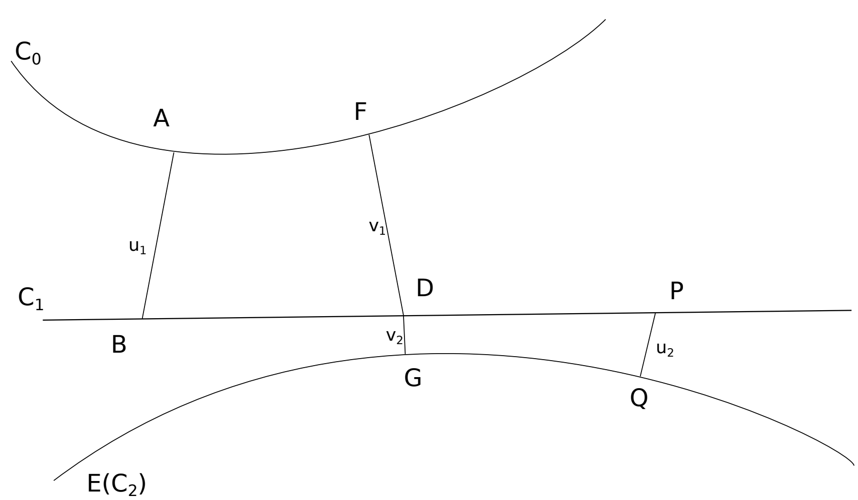

Figure 2.4: This figure illustrates the proof of Proposition 2.4.1

Proof. We only prove the case when there is only one leaf between C_i and C_j, and the general case follows from this special case. For simplicity, we just assume that $C_i = C_0$ and $C_j = C_2$. The earthquake $E = E(C_1, h_1)$ acts on the right half side of C_1. Let AH be the common orthogonal between C_0 and C_2, where A is on C_0 and H is on C_2. Let point B be the intersection between geodesic AH and C_1. Let points P and Q be the images of B and H under the earthquake map E respectively. Let D be a point on C_1, we denote by F the projection of D on C_0 and denote by G the projection of D on $E(C_2)$ (see Figure 2.4).

In this setting, we have $d(B, P) = |h_1| = x$. We claim that, if points F, D, G are on a geodesic, then point D must lie on the segment BP. Assume that F, D, G are in a line and point D is not on the segment BP, then, in Figure 2.4, D must be on the left side of B or on the right side of P. Without loss of generality, we assume that D is on the left side of B. By calculating the angles in quadrangle $ABDF$, we have

$$\angle FDB + \angle DBA < \pi.$$

Similarly, in quadrangle $DPQG$, we have

$$\angle GDP + \angle DPQ < \pi.$$

Summing these two equations, we obtain the inequality $\angle FDB + \angle GDP < \pi$, as $\angle DBA + \angle DPQ = \pi$. However, it contradicts the fact that B, D, G is on a geodesic. So we must have D on the segment BP. It follows that $|AF|, |GQ| < x$.

Now we choose D to be the middle point of BP, then $|BD| = |DP| = \frac{x}{2}$. Using the cosine rule in the appendix, we have

$$\sinh v_i \leq e^{\frac{x}{2}} \sinh u_i,$$

for $i = 1, 2$. Notice that

$$e^{\frac{x}{2}} \sinh u_i \leq \sinh(e^{\frac{x}{2}} u_i),$$

for $i = 1, 2$. It follows from the above observations that $v_i \leq e^{\frac{x}{2}} u_i$ for $i = 1, 2$. Since $v_1 + v_2$ is greater than or equal to the distance between $E(C_0)$ and $E(C_2)$, we have $d(E(C_0), E(C_2)) \leq e^{\frac{x}{2}} d(C_0, C_2)$.

The other direction follows immediately by considering the inverse process of the above earthquake. $\square$

Let O_iO_j be the common orthogonal between C_i and C_j such that O_i is on C_i and O_j is on C_j. Then $E(C_i)$ and $E(C_j)$ are the images of C_i and C_j under the earthquake respectively. We denote by $O_i'O_j'$ the common orthogonal between $E(C_i)$ and $E(C_j)$. Then the above proposition tells us that

$$e^{-\frac{x}{2}} \leq \frac{|O_i'O_j'|}{|O_iO_j|} \leq e^{\frac{x}{2}}. \tag{2.11}$$

Moreover, from the above proof we also obtain

$$d(E(O_i), O_i') \leq x, \quad d(E(O_j), O_j') \leq x. \tag{2.12}$$

So we have the following corollary.

Corollary 2.4.2. *Let α be a geodesic that intersects both C_i and C_j. Denote $A = \alpha \cap C_i$ and $B = \alpha \cap C_j$. Assume that $x = |\mu|(C_i, C_j) < 1$. Then there exist a point A' on $E(C_i)$ and a point B' on $E(C_j)$ such that the following holds.*

(1) $d(A', E(A)) \leq x$ *and* $d(B', E(B)) \leq x$,

(2) $\frac{1}{1+x} d(A, B) \leq d(A', B') \leq (1+x) d(A, B)$,

(3) $|\angle A' - \angle A| < x$ *and* $|\angle B' - \angle B| < x$, *where $\angle A$ denotes the oriented angle between AB and C_i, and similarly we define $\angle B$, $\angle A'$, $\angle B'$.*

Proof. As above, we define O_iO_j to be the common orthogonal between C_i and C_j and define $O_i'O_j'$ to be the common orthogonal between $E(C_i)$ and $E(C_j)$. We choose A' on $E(C_i)$ such that $d^*_{E(C_i)}(O_i', A') = d^*_{C_i}(O_i, A)$, where d^* denotes the oriented distance. Similarly, we choose point B' on $E(C_j)$ such that $d^*_{E(C_j)}(O_j', B') = d^*_{C_j}(O_j, B)$. Since E is an isometry when restricted on each leaf, so by our constructions, we have

$$d^*_{E(C_i)}(O_i', A') = d^*_{E(C_i)}(E(O_i), E(A)),$$
$$d^*_{E(C_j)}(O_j', B') = d^*_{E(C_j)}(E(O_j), E(B)).$$

Then by (2.12), we have $d(A', E(A)) \leq x$ and $d(B', E(B)) \leq x$, so the claim (1) is satisfied. By Proposition 2.4.1, we have

$$e^{-\frac{x}{2}} \leq \frac{|O_i'O_j'|}{|O_iO_j|} \leq e^{\frac{x}{2}}.$$

When $x < 1$, we have inequalities $e^{\frac{x}{2}} \leq 1 + x$ and $\sin x \geq e^{\frac{x}{2}} - 1$. The rest of the statement just follows from the above observations and Proposition 2.3.1.

□

2.5 Short Segments and Long Segments

We call a segment P_iP_{i+1} short if its length is less than e^{-5}, and we call P_iP_{i+1} long if it's not short. The next lemma is a corollary of Lemma 3.2.1. It gives us the freedom to perturb the endpoints of a long segment.

Lemma 2.5.1. *Let α and β be any two geodesics in $\mathbb{H}^n$, choose $P, P' \in \alpha$ and $Q, Q' \in \beta$. Let $R > 0$. We assume that $d(P,Q) > \frac{1}{2}e^{-5}$, $d(P,P') = O(R^{-1})$ and $d(Q,Q') = O(R^{-1})$, then we have the following inequalities.*

$$\left|\frac{d(P',Q')}{d(P,Q)} - 1\right| = O(R^{-1}),$$

$$\Theta(v(P',Q'), v(P,Q)@P') = O(R^{-1}) \text{ and } \Theta(v(Q',P'), v(Q,P)@P') = O(R^{-1}).$$

In particular, the oriented angle between $v(P,Q)$ and α is $O(R^{-1})$-close to the oriented angle between $v(P',Q')$ and α. Similar results hold on the geodesic β.

For a long segment P_iP_{i+1}, we will prove the following result.

Lemma 2.5.2. *Assume that P_iP_{i+1} is a long segment. Then for any point Q_j on C'_j that is $\frac{10}{R}$-close to $f(P_j)$, where $j = i, i+1$, we have*

$$\left|\frac{d(Q_i,Q_{i+1})}{d(P_i,P_{i+1})} - 1\right| = O(R^{-1}). \tag{2.13}$$

Moreover, the angle between $v(Q_i,Q_{i+1})$ and C'_i is $O(R^{-1})$-close to the angle between $v(P_i,P_{i+1})$ and C_i. Similarly, the angle between $v(Q_{i+1},Q_i)$ and C'_{i+1} is $O(R^{-1}$-close to to the angle between $v(P_{i+1},P_i)$ and C_{i+1}.

Proof. We first consider the case when P_iP_{i+1} doesn't cross any seams. In this case, D_i is a seam in $\mathbb{H}^2$ and D'_i is a seam in $\mathbb{H}^n$. Denote $A = D_i \cap C_i$, $B = D_i \cap C_{i+1}$, $A' = D'_i \cap C'_i$ and $B = D'_i \cap C'_{i+1}$. We then choose point P'_i on ray $A'f(P_i)$ such that $|A'P'_i| = |AP_i|$. Similarly, we choose P'_{i+1} on ray $B'f(P_{i+1})$ such that $|B'P'_{i+1}| = |BP_{i+1}|$. Denote $\mathbf{d} = d + i\theta$ the complex distance between C'_i and C'_{i+1}. Denote d_0 the real distance between C_i and C_{i+1}. Then by definition of good pants, we have

$$1 - \frac{1}{R^2} \leq \frac{d}{d_0} \leq 1 + \frac{1}{R^2}, \quad \text{and} \quad \frac{\theta}{d} < \frac{1}{R^2}.$$

So we can apply Proposition 2.3.1 and Proposition 2.3.4 to compare quadrangle $B'A'P'_iP'_{i+1}$ with BAP_iP_{i+1}. For R sufficiently large, we have

$$1 - \frac{4}{R^2} \leq \frac{d(P'_i,P'_{i+1})}{d(P_i,P_{i+1})} \leq 1 + \frac{4}{R^2}, \tag{2.14}$$

$$|\angle A'P_i'P_{i+1}' - \angle AP_iP_{i+1}| \leq \frac{4}{R^2}. \tag{2.15}$$

Moreover, the direction $v(P_i', P_{i+1}')$ is almost in the plane that contains D_i' and C_i'. By our construction of f, we have

$$d(P_j', f(P_j)) = O(R^{-2}),$$

for $j = i, i+1$. So we get $d(Q_j, P_j') = O(\frac{1}{R})$, for $j = i$ or $i+1$. By Equation (2.14), $d(P_i', P_{i+1}') > \frac{1}{2}e^{-5}$ when R large. Then the statement of Lemma 2.5.2 just follows from Lemma 2.5.1.

Now we assume that P_iP_{i+1} crosses some seams. Denote $S_1, \cdots, S_m$ to be the intersection points between P_iP_{i+1} and seams. We denote $S_0 = P_i$ and $S_{m+1} = P_{i+1}$. Then S_jS_{j+1} lies in a standard right angled hexagon and cuts the hexagon into two regions, where one of the regions can only be a triangle, a quadrangle or a pentagon. Based on the shape of that region, we have three cases. In all cases, we will prove the following inequality

$$\left|\frac{|f(S_j)f(S_{j+1})|}{|S_jS_{j+1}|} - 1\right| = O(R^{-2}). \tag{2.16}$$

Moreover, we will show that the bending angle at each point $f(S_j)$ is $O(R^{-2})$. To study the angles, we denote by θ_j the entering angle of S_jS_{j+1} into the right angle hexagon and we denote by φ_j the exit angle of S_jS_{j+1} leaving the right angled hexagon (see Figure 2.5). Similarly, we define θ_j' and φ_j' for the segment $f(S_j)f(S_{j+1})$ in $\mathbb{H}^n$.

Case 1 *Triangle*. Denote by Δ the triangle that contains side S_jS_{j+1} and denote Δ' the triangle that is the image of Δ mapped by f. Recall that f is $(1+\frac{20}{R^2})$-bilipschitz when restricted on each side of a right angled hexagon, so Equation (2.16) follows from Corollary 2.3.2 by comparing triangle Δ' with Δ. Moreover, we have the following inequalities on angles:

$$\left|\theta_j' - \theta_j\right| = O(R^{-2}) \text{ and } \left|\varphi_j' - \varphi_j\right| = O(R^{-2}).$$

Case 2 *Quadrangle*. By the geometry of a right angled hexagon (cf. A.3), the length of a seam is on the order of $e^{-\frac{R}{2}}$. So the lengths of S_jS_{j+1} and $f(S_j)f(S_{j+1})$ are both $O(R^{-2})$-close to $\frac{R}{2}$. In this case, all angles $\theta_j, \theta_j', \varphi_j, \varphi_j'$ are exponentially close to $\frac{\pi}{2}$.

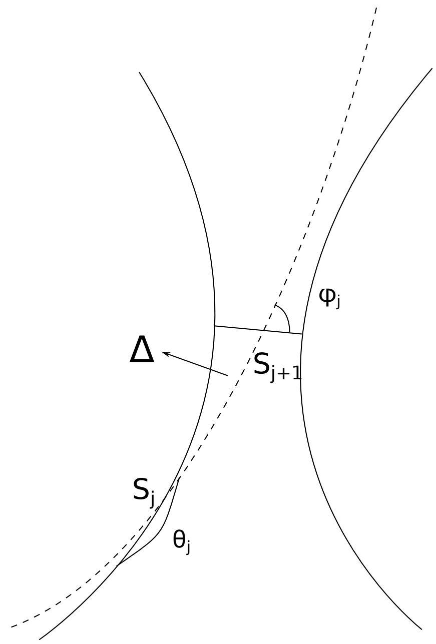

Figure 2.5: An illustration of the angles θ_j and φ_j in the case of a triangle

Case 3 *Pentagon*. Without loss of the generality, we assume that S_j is on a cuff and S_{j+1} is on a seam. Let point T be a foot of the seam that contains S_{j+1}, and denote $T' = f(T)$. Then T is exponentially close to S_{j+1} and T' is exponentially close to $f(S_{j+1})$. It is reduced to the quadrangle case, after replacing S_{j+1} by T and replacing $f(S_{j+1})$ by T'. Then Equation (2.16) follows from Lemma 3.2.1.

Since S_j, S_{j+1} and S_{j+2} are in a geodesic, so we have $\varphi_j = \theta_{j+1}$. By the above angle estimates, we have $|\varphi'_j - \theta'_{j+1}| = O(R^{-2})$. However, it is not enough to show that the bending angle at P'_{j+1} is $O(R^{-2})$ due to the freedom in dimension. Since any pair of pants in the surface S is R-good. So one can easily check that $v(S_{j+1},' S'_{j+2})$ is almost in the hyperbolic plane determined by S'_j, S'_{j+1} and the seam that contains S'_{j+1}. Therefore, the bending angle at S_j is $O(R^{-2})$. Now we consider the piecewise geodesic α by concatenating $f(S_0), \cdots, f(S_{m+1})$. Since the distance between two seams is roughly $\frac{R}{2}$, a unit length subarc of α can cross at most one bending point when R large. So the bending norm of α is bounded by $O(R^{-2})$. By Corollary 2.2.4, we have the following inequality:

$$\left| \frac{d(f(P_i), f(P_{i+1}))}{\sum_{i=0}^{m} \left| f(S_j) f(S_{j+1}) \right|} - 1 \right| = O(R^{-2}).$$

Combined with Equation (2.16), we proved

$$\left|\frac{d(f(P_i, f(P_{i+1}))}{d(P_i, P_{i+1})} - 1\right| = O(R^{-2}).$$

In particular, we have $d(f(P_i), f(P_{i+1})) \geq \frac{1}{2}e^{-5}$ when R large. So Inequality (2.13) follows from Lemma 2.5.1. One can check that the angle relations are also satisfied.

□

The above lemma shows that "long is flexible". In the above proof, It's important that P_iP_{i+1} is long. It gives us the freedom to perturb P'_i and P'_{i+1} such that Inequality (2.13) still hold. When P_iP_{i+1} is short, such a perturbation no longer works. We will use the earthquake technic when P_iP_{i+1} is short.

Now we consider the short segments. Any component of $\tilde{\gamma} \backslash \cup$ {long segments} is composed of short segments. Let P_iP_{i+m} be such a component. Then the twists on $C_{i+1}, \cdots, C_{i+m-1}$ are exactly 1, and the twists on $C'_{i+1}, \cdots, C'_{i+m-1}$ are all $\frac{10}{R^2}$-close to 1. We will perform a earthquake map E on the lamination λ. The measure $\mu = (h_0, \cdots, h_{k+1})$ associated to the earthquake is defined as follows: we set

$$h_j = \begin{cases} d(D'_{j-1}, D'_j) - 1, & \text{if } i < j < i+m \\ 0, & \text{otherwise.} \end{cases}$$

Here, $d(D'_{j-1}, D'_j)$ denotes the real distance between D'_{j-1} and D'_j. Denote by $E(C_i), \cdots, E(C_{i+m})$ the images of geodesics $C_i, \cdots, C_{i+m}$ under the earthquake E. By definition of a real earthquake, $E(D_j)$ is the common orthogonal between $E(C_j)$ and $E(C_{j+1})$. Then, by our construction of μ, the distance between $E(D_j)$ and $E(D_{j+1})$ is the same as the real distance between D'_j and D'_{j+1} for $j = i, \cdots, i+m-2$. By Lemma 2.1.1, we have $m < R$. Denote by x the earthquake norm, then

$$x = \sum_{j=i+1}^{i+m-1} |d(D'_{j-1}, D'_j) - 1| < \frac{10}{R}.$$

By Corollary 2.4.2, we can find a point R_i on $E(C_i)$ and a point R_{i+m} on $E(C_{i+m})$ such that the following hold.

1. $d(R_i, E(P_i)) < \frac{10}{R}$ and $d(R_{i+m}, E(P_{i+m})) < \frac{10}{R}$.
2. $\frac{1}{1+\frac{10}{R}} d(P_i, P_{i+m}) \leq d(R_i, R_{i+m}) \leq (1 + \frac{10}{R}) d(P_i, P_{i+m})$.

3. $\angle R_i$ is $\frac{20}{R}$-close to $\angle P_i$, and $\angle R_{i+m}$ is $\frac{20}{R}$-close to $\angle P_{i+m}$.

We connect R_i and R_{i+m} by a geodesic and denote by R_j the intersection between geodesic R_iR_{i+m} and $E(C_j)$, for $j = i+1, \cdots, i+m-1$. Now we can construct those points $Q_i, \cdots, Q_{i+m}$ in Claim 2.2.5. Since the distance between $E(D_j)$ and $E(D_{j+1})$ is the same as the real distance between D_j and D'_{j+1}, then there is a map from $\bigsqcup_{j=i}^{i+m} E(C_j)$ to $\bigsqcup_{j=i}^{i+m} C'_j$ such that all the feet are mapped to the corresponding feet and the map is an isometry when restricted on each connected component. Next, we choose Q_j to be the image of R_j under the map, for $j = i, \cdots, i+m$. Now we can apply Proposition 2.3.1 and Proposition 2.3.4 to estimate the length of Q_jQ_{j+1} and the bending angle at Q_j. Precisely, we get

$$\left|\frac{d(Q_j, Q_{j+1})}{d(R_j, R_{j+1})} - 1\right| \leq \frac{4}{R^2}, \tag{2.17}$$

for $j = i, \cdots, i+m-1$. Next, we consider the bending angle at Q_j. If the two feet on C'_j are in the exact opposite direction after parallel transporting, then by Proposition (2.3.1) and (2.4.1) the bending angle at Q_j is bounded by $\frac{4}{R^2}$. In general, the two feet on C_j, after parallel transporting at a same base point, can form an angle at most $\frac{10}{R^2}$. So the bending angle at Q_j is bounded by $\frac{14}{R^2}$.

By Equations (2.17), we obtain

$$(1 - \frac{4}{R^2})\left|R_iR_{i+m}\right| \leq \sum_{j=i}^{i+m-1} d(Q_j, Q_{j+1}) \leq (1 + \frac{4}{R^2})\left|R_iR_{i+m}\right|.$$

Combined with the second property of R_j above, we get

$$(1 - \frac{20}{R})\left|P_iP_{i+m}\right| \leq \sum_{j=i}^{i+m-1} d(Q_j, Q_{j+1}) \leq (1 + \frac{20}{R})\left|P_iP_{i+m}\right|, \tag{2.18}$$

when R large. It follows from $d(E(P_i), R_i) < \frac{10}{R}$ and our construction of Q_i that $d(Q_i, f(P_i)) < \frac{20}{R}$. By symmetry, $d(Q_{i+m}, f(P_{i+m})) < \frac{20}{R}$.

For each component of $\tilde{\gamma} \backslash \cup \{\text{long segments}\}$, we can define Q_i in the above way. If, for some $i \in \{0, \cdots, k+1\}$, Q_i is not defined yet, then either $P_i = P_{k+1}$ or P_iP_{i+1} is a long segment. For this i, we just let $Q_i = f(P_i)$. Thus, we have defined Q_i for all $i = 0, \cdots, k+1$. For a long segment P_iP_{i+1}, and from our construction Q_i (resp. Q_{i+1}) is $\frac{20}{R}$-close to $f(P_i)$ (resp. $f(P_{i+1})$), so the result of Lemma 2.5.2 holds. In the case of short segments, Inequality (2.18) holds and the bending angles at Q_i are bounded by $O(\frac{1}{R^2})$. So the bending norm is bounded by $O(\frac{1}{R})$. Therefore, the assumption of Claim 2.2.5 is satisfied and it completes the proof.

Chapter 3

MIXING OF THE FRAME FLOW

In this chapter, the main goal is to show that the surface in the assumption of Theorem 1.2.1 exists when R is large. We start with a classical theorem (cf. Theorem 3.0.1) in ergodic theory: the exponential mixing property of the frame flow in a hyperbolic manifold. Then we explain how to use this theorem to construct skew pants. Via studying the geometry of the pants, we show that the pants constructed are all good pants (cf. Definition 1.3.1). Finally, a weight will be assigned to each good pair of pants so that we can assemble them in a nice way (cf. Definition 1.3.2) to get the closed surface S. We basically follow the ideas of Kahn-Markovic in [KM12] and generalize the construction to higher dimensions.

Denote $\mathcal{F}(\mathbb{H}^n)$ the set of $(n-1)$-frames $F_p = (p, u, E)$, where $p \in \mathbb{H}^n$, u is a unit vector at p and E is an $n-2$ orthonormal frame at p that is orthogonal to u. Let $\mathbf{g}_t, t \in \mathbb{H}$, be the frame flow that acts on $\mathcal{F}(\mathbb{H}^n)$ and let Λ be the Liouville measure on $\mathcal{F}(\mathbb{H}^n)$ that is invariant under the frame flow. Similarly, we can define the $(n-1)$-frame bundle $\mathcal{F}(M)$, and we have $\mathcal{F}(M) = \mathcal{F}(\mathbb{H}^n)/\pi_1(M)$. Let $F_p = (p, u, E_1)$ and $F_q = (q, v, E_2)$ be two $(n-1)$-frames in $\mathcal{F}(\mathbb{H}^n)$; write $E_1 = (u_1, \cdots, u_{n-2})$ and $E_2 = (v_1, \cdots, v_{n-2})$. We define the distance function $\mathbf{D}$ on $\mathbb{F}(\mathbb{H}^n)$ by

$$\mathbf{D}((p, u, E_1), (q, v, E_2)) = d(p, q) + \Theta(u', v) + \max_{1 \le i \le n-2} \Theta(u_i', v_i),$$

where $u', u_i' \in T_q^1(\mathbb{H}^n)$ are the parallel transports of u and u_i along the geodesic that connects p and q respectively. The distance function $\mathbf{D}$ is invariant under the action of $\mathrm{SO}(n, 1)$. We denote $B_\epsilon(\tilde{z}) = \{\tilde{w} \in \mathcal{F}(\mathbb{H}^n) : \mathbf{D}(\tilde{w}, \tilde{z}) < \epsilon\}$ to be the ϵ-ball around the frame $\tilde{z} \in \mathcal{F}(\mathbb{H}^n)$.

The following well-known result is called the exponential mixing of the frame flow. The theorem is originally proved by [Moo87]. The following precise formulation is cited from a paper of Luke Hartley.

Theorem 3.0.1 ([Moo87])**.** *Let M^n be a closed hyperbolic n-manifold, let $\mathcal{F}(M)$ be the frame bundle of M and let Λ be the Liouville measure on $\mathcal{F}(M)$ which is invariant under the frame flow $\mathbf{g}_t$. Then there exists a number $\mathbf{q} > 0$ that depends* *only on M such that the following holds: Let $\varphi, \psi : \mathcal{F}(M) \to \mathcal{R}$ be two C^1 functions; we have*

$$\left|\Lambda(\mathcal{F}(M)\int_{\mathcal{F}(M)}(\mathbf{g}_t^*\varphi)(x)\psi(x)d\Lambda(x) - \int_{\mathcal{F}(M)}\varphi(x)d\Lambda(x)\int_{\mathcal{F}(M)}\psi(x)d\Lambda(x)\right| \leq Ce^{-\mathbf{q}t},$$

where $C > 0$ only depends on the C^1-norm of φ and ψ.

For two functions $\varphi, \psi : \mathcal{F}(M) \to \mathbb{R}$ we set

$$(\varphi, \psi) = \int_{\mathcal{F}(M)}\varphi(x)\psi(x)d\Lambda(x).$$

Let $R > 0$ be a large positive number. And let $\epsilon > 0$ denote a positive number that is smaller than the injectivity radius of M. Then the projection map $\mathcal{F}(\mathbb{H}^n) \to \mathcal{F}(M)$ is injective on every ϵ-ball $B_\epsilon(z) \subset \mathcal{F}(\mathbb{H}^n),\ z \in \mathcal{F}(\mathbb{H}^n)$.

Fix a point $\tilde{z}_0 \in \mathcal{F}(\mathbb{H}^n)$. Let $F_{\tilde{z}_0} : \mathcal{F}(\mathbb{H}^n) \to [0,1)$ be a smooth function, supported in the ϵ-ball centered at $\tilde{z}_0$, such that

$$\int_{\mathcal{F}(\mathbb{H}^n)} F_{\tilde{z}_0}(X)d\Lambda(X) = 1.$$

For every $\tilde{z} \in \mathcal{F}(\mathbb{H}^n)$, there is a unique element A of $\mathrm{Isom}^+(\mathbb{H}^n)$ such that $A(\tilde{z}) = \tilde{z}_0$. We define function $F_{\tilde{z}} : \mathbb{H}^n \to [0,\infty]$ by pulling back $F_{\tilde{z}_0}$ via A. Hence the function $F_{\tilde{z}}$ is supported in the ϵ-ball centered at $\tilde{z}$. So the function $F_*(*)$ is $\mathrm{Isom}^+(\mathbb{H}^n)$-invariant in the following sense. For any $z_1, z_2 \in \mathcal{F}(\mathbb{H}^n)$, and any $g \in \mathrm{Isom}^+(\mathbb{H}^n)$,

$$F_{(g.z_1)}(g.z_2) = F_{z_1}(z_2).$$

Let $z = (p, u, E)$ be a frame; we define the opposite frame

$$-z = (p, -u, E).$$

The following function tells us how well two frames in $\mathcal{F}(\mathbb{H}^n)$ are connected.

Definition 3.0.2. *Let $z_j = (p_j, u_j, E_j) \in \mathcal{F}(\mathbb{H}^n),\ j = 1,2$ be two frames and set $\hat{z}_j = \mathbf{g}_{\frac{r}{4}}(z_j)$. Define*

$$\mathbf{a}_{\mathbb{H}^n}(z_1, z_2) = (\mathbf{g}_{\frac{r}{2}}^* F_{\hat{z}_1}, F_{-\hat{z}_2}).$$

We say that the frames z_1 and z_2 are (ϵ, r)-well connected if $\mathbf{a}_{\mathbb{H}^n}(z_1, z_2) > 0$.

Remark 3.0.3. *Let γ be the unique geodesic that connects p_1 and p_2. If the frames z_1 and z_2 are (ϵ, r)-well connected, then by "Chain Lemma" in section 3.2, the length of γ will be close to r and the parallel transport along γ will send z_1 to some frame that is close to $-z_2$, the opposite frame of z_2.*

Similarly, we can define how well two frames in $\mathcal{F}(M)$ are connected along a geodesic.

Definition 3.0.4. *Let $z_j = (p_j, u_j, E_j) \in \mathcal{F}(M)$, $j = 1, 2$ be two frames, and let γ be a geodesic segment in M that connects p_1 and p_2. Let $\tilde{z}_j \in \mathcal{F}(\mathbb{H}^n)$, $j = 1, 2$, be a lift of z_j such that the base points of $\tilde{z}_1$ and $\tilde{z}_2$ can be connected by a geodesic that lifts γ. We define*

$$\mathbf{a}_\gamma(z_1, z_2) = \mathbf{a}_{\mathbb{H}^n}(\tilde{z}_1, \tilde{z}_2).$$

We say that the frames z_1 and z_2 are (ϵ, r)-well connected along the geodesic γ if $\mathbf{a}_\gamma(z_1, z_2) > 0$.

Remark 3.0.5. *If the lift $\tilde{z}_1$ is chose, then there is a unique lift $\tilde{z}_2$ such that their basepoints can be connected by a geodesic that lifts γ. We should also point out that the definition doesn't depend on the choices of lifts as the function $F_*(*)$ is left-invariant under the action of $\pi_1(M)$.*

Choose two frames $z_1, z_2 \in \mathcal{F}(M)$, and let $\hat{z}_j = g_{\frac{r}{4}}(z_j)$, $j = 1, 2$. We define (with ϵ and r understood)

$$\mathbf{a}(z_1, z_2) = (\mathbf{g}^*_{\frac{r}{2}} F_{\hat{z}_1}, F_{-\hat{z}_2}).$$

Then

$$\mathbf{a}(z_1, z_2) = \sum_\gamma \mathbf{a}_\gamma(z_1, z_2),$$

where γ varies over all geodesic segments in M that connect p_1 and p_2. (Only finitely many numbers $\mathbf{a}_\gamma(z_1, z_2)$ are non-zero.)

The following result is a corollary of Theorem 3.0.1.

Corollary 3.0.6. *Fix any positive integer k, let $\epsilon = r^{-k}$, then for r large enough and any $z_1, z_2 \in \mathcal{F}(M)$, we have*

$$\mathbf{a}(z_1, z_2) = \frac{1}{\Lambda(\mathcal{F}(M))}(1 + O(e^{-\mathbf{q}\frac{r}{2}})),$$

where $\mathbf{q} > 0$ is a constant that depends only on M and k.

Remark 3.0.7. *The corollary tells us that we may choose ϵ to be a function dependent on r. In particular, one can choose $\epsilon(r)$ to be any polynomial in the variable r^{-1} for which the mixing property still holds when is r large.*

3.1 Tripods and θ-graph

There is a natural order three homeomorphism $\omega : \mathcal{F}(\mathbb{H}^n) :\rightarrow \mathcal{F}(\mathbb{H}^n)$ given by $\omega(p, u, E) = (p, \omega(u), E)$, where $\omega(u)$ is the the vector in $T_p^1(\mathbb{H}^n)$; that is, orthogonal to the $(n-2)$-frame E and such that the oriented angle between u and $\omega(u)$ is $\frac{2\pi}{3}$. (The plane orthogonal to E is oriented by the orientation of E and $T_p^1(\mathcal{F}(\mathbb{H}^n))$.) The homeomorhism ω commutes with the $SO(n, 1)$ action, hence it is well defined on $\mathcal{F}(M)$ by the projection.

It's easy to see that ω^3 is the identity and we let $\overline{\omega} = \omega^{-1}$. To any frame $z \in \mathcal{F}(\mathbb{H}^n)$ we can associate a tripod $T(z) = (z, \omega(z), \omega^2(z))$ and an "anti-tripod" $\overline{T}(z) = (z, \overline{\omega}(z), \overline{\omega}^2(z))$. We have the similar definitions for frames in $\mathcal{F}(M)$.

Let $z_p = (p, u, E_1), z_q = (q, v, E_2)$ be a pair of frames from $\mathcal{F}(M)$. Let γ_i, $i = 0, 1, 2$ be three geodesic segments that connect p and q in M. Then the three 1-cells $(\gamma_0, \gamma_1, \gamma_2)$ and two 0-cells (p, q) form a θ-graph. Then $\gamma_i\overline{\gamma}_{i+1}$, $i = 0, 1, 2$ are three closed curves based at p, where $\overline{\gamma_i}$ is the reverse of the path γ_i. In the non-generating case, we can homotope $\gamma_i\overline{\gamma}_{i+1}$, $i = 0, 1, 2$, to be three closed geodesics, which can bound a pair of pants Π.

Denote the pair $(T(z_p), \overline{T}(z_q))$, a pair of tripods with the first entry a tripod and the second entry an anti-tripod. Let $\gamma = (\gamma_0, \gamma_1, \gamma_2)$ be the triple of three geodesic segements that connect p and q. We can define how well the pair of tripods $(T(z_p), T(z_q))$ is connected along the triple of segments γ, and define

$$\mathbf{b}_\gamma(T(z_p), \overline{T}(z_q)) = \prod_{i=0}^{2} \mathbf{a}_{\gamma_i}(\omega^i(z_p), \overline{\omega}^i(z_q)).$$

The function $\mathbf{b}_\gamma(T(z_p), \overline{T}(z_q))$ quantifies how well the pair of tripods is connected. We say that $(T(z_p), \overline{T}(z_q))$ is well-connected along γ if $\mathbf{b}_\gamma(T(z_p), \overline{T}(z_q)) > 0$. If $(T(z_p), \overline{T}(z_q), \gamma)$ is well connected, we will show in Section 3.3 that the skew pair of pants Π associated to the θ-graph has length $O(\epsilon)$ close to $2(r - \log \frac{4}{3})$.

3.2 Chain Lemma

We are interested in the geometry of Π, the pair of pants associated to the θ-graph. In this section, we will prove some results in hyperbolic geometry that will be later used to study the geometry of pants. Readers are referred to Chapter 1 for the notations. The following lemma is a corollary of the Gauss-Bonnet Theorem and hyperbolic geometry.

Lemma 3.2.1 ([KM12], Section 4)**.** *Let $a, b, c \in \mathbb{H}^n$; v is a unit tangent vector based at a. Then the following inequalities hold:*

$$\Theta(v@b@c@a, v) \leq Area(abc) \leq |bc|,$$

$$\Theta(v(c, a).v(b, a)@c) \leq \Theta(v(a, b), v(a, c)) + Area(abc) \leq \Theta(v(a, b), v(a, c)) + |bc|,$$

where $Area(abc)$ denotes the hyperbolic area of the triangle abc.

Using the sine rule and the cosine rule in hyperbolic geometry, we obtain the following proposition.

Proposition 3.2.2. *Let ABC be a hyperbolic triangle. We denote $|CA| = b, |CB| = a, |AB| = c$ and denote $\eta = \pi - \angle ACB$.*

(1) Then there exists some constant $D > 0$ such that, for b large and a small, the inequality $\angle CAB \leq Dae^{-b}$ holds.

(2) Then there exists some constant $D > 0$ such that, for η small enough, the inequalities

$$\angle CAB \leq D\eta e^{-b} \text{ and } |c - (a + b)| \leq D\eta$$

hold.

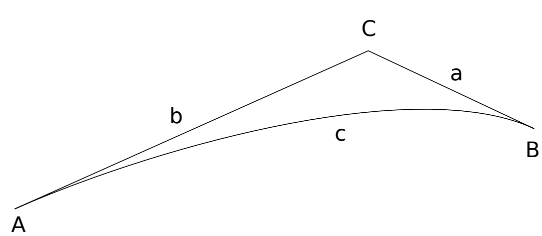

Figure 3.1: A hyperbolic triangle

Let v be a unit tangent vector at p; we denote $v@q$ to be the parallel transport of v along geodesic segment pq. Similarly, let $\tilde{z}$ be a frame at p; we denote $\tilde{z}@q$ to be the parallel transport of $\tilde{z}$ along geodesic pq.

The following theorem is called the higher dimensional "Chain Lemma."

Lemma 3.2.3. *Denote $\tilde{z}_i = (a_i, u_i, E_i), \tilde{w}_i = (b_i, v_i, F_i) \in \mathcal{F}(\mathbb{H}^n)$, $i = 1, \cdots, k$, and suppose that they satisfy the following.*

(1) Let $t_i = |a_i b_i|$, then $\tilde{w}_i = \mathbf{g}_{t_i}(\tilde{z}_i)$ and $t_i \geq Q$.

(2) $\mathbf{D}(\tilde{z}_{i+1}, \tilde{w}_i) \leq \epsilon$.

Then for ϵ small and Q large and some constant $D > 0$, the following inequalities hold.

$$\left| |a_1 b_k| - \sum_{i=1}^{k} |a_i b_i| \right| \leq kD\epsilon, \tag{3.1}$$

$$\Theta(v(a_1, b_1), v(a_1, b_k)) \leq kD\epsilon e^{-Q} \text{ and } \Theta(v(b_k, a_1), v(b_k, a_k) < kD\epsilon e^{-Q}, \tag{3.2}$$

$$\angle a_k a_1 b_1 < 2kD\epsilon e^{-Q}, \tag{3.3}$$

$$\mathbf{D}(\tilde{z}_1 @ a_k, \tilde{z}_k) \leq 14k\epsilon \quad \text{and} \quad \mathbf{D}(\tilde{z}_1 @ b_k, \tilde{w}_k) \leq 14k\epsilon. \tag{3.4}$$

Proof. We prove this lemma by induction. When $k = 1$, the statement is trivial. Now we suppose that the statement is true for some $k \geq 1$. We need to prove the statement for $k + 1$. We first show that

$$\pi - \angle a_1 a_{k+1} b_{k+1} = \Theta(-v(a_{k+1}, b_{k+1}), v(a_{k+1}, a_1)) \leq 4\epsilon. \tag{3.5}$$

By the assumption (2) of the lemma, we have $d(a_{k+1}, b_k) \leq \epsilon$. Then by Equation (3.1) and Proposition 3.2.2, we get

$$\angle a_{k+1} a_1 b_k \leq 2D_1 \epsilon e^{-Q} < \epsilon, \tag{3.6}$$

where D_1 is the constant from Proposition 3.2.2 for Q large enough. Then by Lemma 3.2.1, we have

$$\Theta(v(a_{k+1}, a_1), v(b_k, a_1) @ a_{k+1}) \leq \angle a_{k+1} a_1 b_k + |b_k a_{k+1}| \leq 2\epsilon.$$

Combining the assumption (2) of the theorem and the above inequality, and by the triangle inequality we obtain

$$\Theta(-v(a_{k+1}, b_{k+1}), v(a_{k+1}, a_1)) \leq \Theta(v(b_k, a_k), v(b_k, a_1)) + 3\epsilon.$$

Then, by Equation (3.2), we have $\Theta(-v(a_{k+1}, b_{k+1}), v(a_{k+1}, a_1)) \leq 4\epsilon$ when Q is large enough.

Next, we prove Inequalities (3.2). By the triangle inequality we have

$$\Theta(v(a_1, b_k), v(a_1, b_{k+1})) \leq \Theta(v(a_1, b_k), v(a_1, a_{k+1})) + \Theta(v(a_1, a_{k+1}), v(a_1, b_{k+1})).$$

By Equation (3.1) and Proposition 3.2.2, we have

$$\Theta(v(a_1, a_{k+1}), v(a_1, b_{k+1})) \leq 2D_1 \epsilon e^{-Q}.$$

Combining with Equation (3.6) shows

$$\Theta(v(a_1, b_k), v(a_1, b_{k+1})) \leq 4D_1 \epsilon e^{-Q} \leq D\epsilon e^{-Q}.$$

Together with the induction hypothesis, this proves the first inequality in (3.2). The second one just follows by symmetry. Inequality (3.3) follows from Equations (3.2) and (3.6).

It follows from Proposition 3.2.2 and Inequality (3.5) that $\Big||a_1 a_{k+1}| + |a_{k+1} b_{k+1}| - |a_1 b_{k+1}|\Big| \leq 4D_1 \epsilon$. By triangle inequality, we have $\Big||a_1 b_k| - |a_1 a_{k+1}|\Big| \leq \epsilon$. Thus, we obtain

$$\Big||a_1 b_k| + |a_{k+1} b_{k+1}| - |a_1 b_{k+1}|\Big| \leq D\epsilon.$$

Then by the induction hypothesis, we obtain Inequality (3.1).

It remains to prove (3.4). Let $E(i)$ be the i-th vector in the $(n-2)$-frame E. Then by the definition of $\mathbf{D}$, we have

$$\mathbf{D}(\tilde{z}_1 @ a_{k+1}, \tilde{z}_{k+1}) = \pi - \angle a_1 a_{k+1} b_{k+1} + \max_{1 \leq i \leq n-2} \Theta\Big(E_1(i) @ a_{k+1}, E_{k+1}(i)\Big), \quad (3.7)$$

$$\mathbf{D}(\tilde{z}_1 @ b_{k+1}, \tilde{w}_{k+1}) = \angle a_1 b_{k+1} a_{k+1} + \max_{1 \leq i \leq n-2} \Theta(E_1(i) @ b_{k+1}, F_{k+1}(i)). \quad (3.8)$$

By Inequality (3.5), $\pi - \angle a_1 a_{k+1} b_{k+1} \leq 4\epsilon$. We also have $\angle a_1 b_{k+1} a_{k+1} \leq \pi - \angle a_1 a_{k+1} b_{k+1} \leq 4\epsilon$. By the induction hypothesis, we have

$$\Theta(E_1(i) @ a_k, E_k(i)) \leq 14k\epsilon,$$

and

$$\Theta(E_1(i) @ b_k, F_k(i)) \leq 14k\epsilon,$$

for all $i \in \{1, \cdots, n-2\}$. We have the following inequalities:

$$\begin{aligned}\Theta(E_1(i)@a_{k+1}, E_{k+1}(i)) &= \Theta(E_1(i)@a_{k+1}@b_k, E_{k+1}(i)@b_k)\\ &\le \epsilon + \Theta(E_1(i)@a_{k+1}@b_k, F_k(i)) \quad (\text{as } \mathbf{D}(\tilde{w}_k, \tilde{z}_{k+1}) \le \epsilon)\\ &\le \epsilon + \Theta(E_1(i)@a_{k+1}@b_k, E_1(i)@a_k@b_k) + \Theta(E_1(i)@a_k@b_k, F_k(i)).\end{aligned}$$

We first estimate $\Theta(E_1(i)@a_k@b_k, F_k(i))$ and have

$$\begin{aligned}\Theta(E_1(i)@a_k@b_k, F_k(i)) &= \Theta(E_1(i)@a_k@b_k@a_k, F_k(i)@a_k)\\ &= \Theta(E_1(i)@a_k, F_k(i)@a_k)\\ &= \Theta(E_1(i)@a_k, E_k(i)) \le 14k\epsilon.\end{aligned}$$

By the triangle inequality,

$$\Theta(E_1(i)@a_{k+1}@b_k, E_1(i)@a_k@b_k)$$

$$\le \Theta(E_1(i)@a_{k+1}@b_k, E_1(i)@b_k) + \Theta(E_1(i)@b_k, E_1(i)@a_k@b_k).$$

In triangle $a_1a_{k+1}b_k$, we apply Lemma 3.2.1 and obtain

$$\Theta(E_1(i)@a_{k+1}@b_k, E_1(i)@b_k) \le |b_{k+1}a_k| \le \epsilon.$$

In triangle $a_1a_kb_k$, we apply Lemma 3.2.1 and obtain

$$\Theta(E_1(i)@b_k, E_1(i)@a_k@b_k) \le \text{Area}(a_1a_kb_k) \le \pi - \angle a_1a_kb_k \le 4\epsilon.$$

Combining these estimates together, we have

$$\Theta(E_1(i)@a_{k+1}, E_{k+1}(i)) \le (14k+6)\epsilon. \tag{3.9}$$

Now we can estimate $\Theta(E_1(i)@b_{k+1}, F_{k+1}(i))$ as follows:

$$\begin{aligned}\Theta(E_1(i)@b_{k+1}, F_{k+1}(i)) &= \Theta(E_1(i)@b_{k+1}@a_{k+1}, F_{k+1}(i)@a_{k+1})\\ &= \Theta(E_1(i)@b_{k+1}@a_{k+1}, E_{k+1}(i))\\ &\le \Theta(E_1(i)@b_{k+1}@a_{k+1}, E_1(i)@a_{k+1}) + \Theta(E_1(i)@a_{k+1}, E_{k+1}(i))\\ &\le 4\epsilon + (14k+6)\epsilon \quad (\text{By Lemma 3.2.1 and (3.9) })\\ &= (14k+10)\epsilon.\end{aligned}$$

Therefore, combining Equations (3.7) and (3.8), we get

$$\mathbf{D}(\tilde{z}_1@a_{k+1}, \tilde{z}_{k+1}) \le 14(k+1)\epsilon$$

and

$$\mathbf{D}(\tilde{z}_1@b_{k+1}, \tilde{w}_{k+1}) \le 14(k+1)\epsilon.$$

This completes the induction step. □

Let $\tilde{z}_i = (a_i, u_i, E_i) \in \mathcal{F}(\mathbb{H}^n)$ and $\tilde{w}_i = (b_i, v_i, F_i) \in \mathcal{F}(\mathbb{H}^n)$ be "chain" frames, $i = 1, \cdots, k$, under the condition of Theorem 3.2.3. Suppose that $A \in \mathrm{Isom}^+(\mathbb{H}^n)$ maps $\tilde{z}_1$ to $\tilde{z}_k$. Then we want to know $l(A)$ the length of A and the position of the axis of A. The following proposition will answer these questions.

Proposition 3.2.4. *Let $p, q \in \mathbb{H}^n$ and $A \in Isom^+(\mathbb{H}^n)$ be such that $A(p) = q$. Suppose that for every unit tangent vector u based at p we have $\Theta(A(u), u@q) \leq \epsilon$. Then for ϵ small enough and $d(p,q)$ large enough, and for some constant $D > 0$, we have*

(1) the transformation A is loxodromic;

(2) $|l(A) - d(p,q)| \leq D\epsilon$;

(3) if $axis(A)$ denotes the axis of A, then $d(p, axis(A), d(q, axis(A)) \leq D\epsilon$;

(4) The monodromy of A is $D\epsilon$-close to the identity.

Remark. *Here the monodromy measures the difference between parallel transport along $axis(A)$ and the transformation of A.*

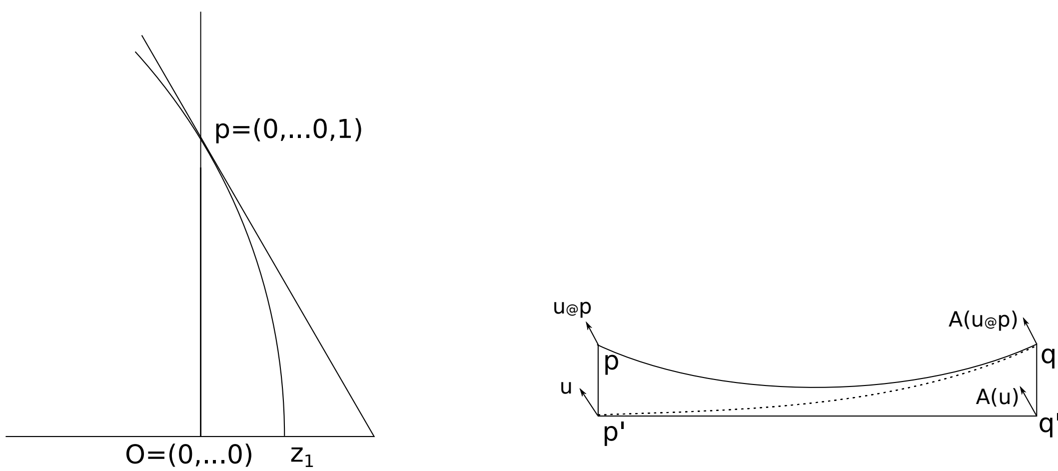

Figure 3.2: The figure on the left illustrates the position of z_1, and the figure on the right illustrates the action of A on vectors.

Proof. We may assume that the points p and q lie on the geodesic that connects 0 and ∞, such that $p = (0, \cdots, 0, 1) \in \mathbb{H}^n$ and $q = (0, \cdots, 0, x)$ for some $x > 1$. Now we consider the sequences $A^n(p)$, $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, we connect $A^n(p)$ and $A^{n+1}(p)$ by a geodesic segment. Then we obtain a piecewise geodesic γ such that

$A^n(p)$, $n \in \mathbb{Z}$, are the bending points. Notice that $\Theta(A(u), u@q) \leq \epsilon$ holds for any unit tangent vector u based at p. In particular, we can choose $u = v(p,q)$, then we get that $\Theta\Big(-v(q,p), v(q, A(q))\Big) \leq \epsilon$. In other words, the bending angle at $q = A(p)$ is bounded by ϵ. Since the isometry A^{n-1} maps q to $A^n(p)$, so the bending angle at $A^n(p)$ is also bounded by ϵ. By Lemma 2.2.2, we get that $\angle pqA^{-n}(p) \leq 3\epsilon$ for all $n \in \mathbb{Z}^+$ when Q large and ϵ small. By symmetry we get that $qpA^n(p) \leq 3\epsilon$ for all $n \in \mathbb{Z}^+$. By Corollary 2.2.4, we get that

$$\lim_{n\to\infty} d(p, A^n(p)) = \infty \text{ and } \lim_{n\to\infty} d(p, A^{-n}(p)) = \infty.$$

So the limit points $z_1 = \lim_{n\to-\infty} A^n(p)$ and $z_2 = \lim_{n\to\infty} A^n(p)$ exist. Hence, the isometry A is loxodromic. By the above angle estimates, we get that the angle between geodesic z_1p and the x_n-axis is bounded by 3ϵ, and the angle between geodesic qz_2 and the x_n-axis is also bounded by 3ϵ. We denote $||\cdot||$ the Euclidean norm in R^{n-1}. Then we have $||z_1|| \leq \tan 3\epsilon \leq 4\epsilon$ when ϵ is small enough. Similarly, we get

$$||z_2|| \geq 2x \cot 3\epsilon \geq \frac{2x}{\tan 3\epsilon} \geq \frac{x}{2\epsilon}$$

when ϵ is small enough. This shows that $d(p, axis(A)) \leq D_1\epsilon$, for some constant $D_1 > 0$. The inequality $d(q, aixs(A)) \leq D_1\epsilon$ follows by symmetry.

Now we show that $|l(A) - d(p,q)| \leq 2D_1\epsilon$. Denote p' the projection of p on the axis of A, and denote q' the projection of q on the axis of A. Then we have $d(p,p') \leq D_1\epsilon$ and $d(q,q') \leq D_1\epsilon$. Since $A(p) = q$, we get $A(p') = q'$. Then $l(A) = d(p', q')$, so we get

$$|l(A) - d(p,q)| \leq d(p,p') + d(q,q') \leq 2D_1\epsilon.$$

It remains to prove that the monodromy of A is $D\epsilon$-close to the identity. Let u be any unit tangent vector based at p'. By Lemma 3.2.1, we get

$$\Theta(u@q, u@p@q) \leq |pp'| \leq D_1\epsilon.$$

By our assumption, we have $\Theta(A(u@p), u@p@q) \leq \epsilon$. So we get

$$\Theta(A(u@p), u@q) \leq (D_1 + 1)\epsilon.$$

On the other hand, in triangle $p'qq'$, we apply Lemma 3.2.1 and get

$$\Theta(u@q, u@q'@q) \leq |qq'| \leq D_1\epsilon.$$

Combining the above inequalities, we obtain

$$\Theta(A(u@p), u@q'@q) \leq (2D_1 + 1)\epsilon.$$

Notice that A is an isometry, so it preserves parallel transport. In particular, the parallel transport of $A(u)$ along geodesic segment $q'q$ is exact $A(u@p)$, as A maps geodesic segment $p'p$ to geodesic segment $q'q$. In other words, we have $A(u@p)@q' = A(u)$. So we get $\Theta(A(u@p)@q', u@q'@q@q') \leq (2D_1+1)\epsilon$. And that is,

$$\Theta(A(u), u@q') \leq (2D_1 + 1)\epsilon \leq D\epsilon.$$

□

The following Lemma will provide a bridge between Lemma 3.2.3 and Proposition 3.2.4.

Lemma 3.2.5. *Let $\tilde{z}_i = (a_i, u_i, E_i), \tilde{w}_i = (b_i, v_i, F_i)$, $i = 1, \cdots, k$ be $(n-1)$-frames that satisfy the condition of Lemma 3.2.3. Let $A \in Isom^+(\mathbb{H}^n)$ be the unique isometry that maps $\tilde{z}_1$ to $\tilde{z}_k$. Assume that $\tilde{w}_k = A(\tilde{w}_1)$, then for ϵ small and Q large, A is a loxodromic transformation and*

$$\left| l(A) - \sum_{i=1}^{k-1} |a_i b_i| \right| \leq kD\epsilon$$

for some constant $D > 0$. Moreover, a_i, b_i are in the $kD\epsilon$-neighborhood of axis(A), and the monodromy of A is $(Dk\epsilon)$-close to the identity.

Proof. Denote $v_1 = v(a_1, b_1)$. We first prove that $\Theta(Av_1, v_1@a_k) \leq 5\epsilon$. Recall that for Q large enough, Inequality (3.5) holds:

$$\pi - \angle a_1 a_k b_k \leq 4\epsilon.$$

Since $A(\tilde{z}_1) = \tilde{z}_k$ and $A(\tilde{w}_1) = \tilde{w}_k$, so $A(v_1) = v(a_k, b_k)$. Since parallel transport preserves angles, we have

$$\Theta(v_1, v(a_1, a_k)) = \Theta(v_1@a_k, -v(a_k, a_1)).$$

By Equation (3.3), we can get $\Theta(v_1, v(a_1, a_k)) \leq \epsilon$ when Q is large. So we get the following string of inequalities:

$$\begin{aligned}
\Theta(A(v_1), v_1@a_k) &= \Theta(v(a_k, b_k), v_1@a_k) \\
&\leq \Theta(v(a_k, b_k), -v(a_k, a_1)) + \Theta(-v(a_k, a_1), v_1@a_k) \\
&= (\pi - \angle a_1 a_k b_k) + \Theta(v(a_1, a_k), v_1) \\
&\leq 4\epsilon + \epsilon = 5\epsilon.
\end{aligned}$$

Next, we write $E_1 = (v_2, \cdots, v_{k-1})$. Then $v_1, \cdots, v_{k-1}$ is an $(n-1)$-frame based at a_1. Since we have $\mathbf{D}(\tilde{z}_1@a_k, \tilde{z}_k) \leq 14k\epsilon$, so the inequality $\Theta(v_i@a_k, A(v_i)) \leq 14k\epsilon$ for all $i = 1, \cdots, n-1$. Therefore, there exist some small $\epsilon > 0$ and some constant $D > 0$ such that

$$\Theta(u@a_k, A(u)) \leq kD\epsilon$$

holds for every vector u based at a_1. Hence, the isometry A satisfies the condition of Proposition 3.2.4 with ϵ replaced by $kD\epsilon$. The lemma now follows from Proposition 3.2.4. □

3.3 Good Pants

In this section, we will show that skew pants from well connected pairs of tripods are always good pants (see Definition 1.3.1). Being a good pair of pants requires some conditions on the lengths of cuffs and complex lengths of seams. The following lemma gives us some estimations on the half lengths of cuffs.

Lemma 3.3.1. *Let $z = (p, u, E)$ and $w = (q, v, F)$ be two frames in $\mathcal{F}(M)$. Assume that the tripods $T(z)$ and $\bar{T}(w)$ are well connected by a triple $\gamma = (\gamma_0, \gamma_1, \gamma_2)$. Denote δ the closed geodesic that is homotopic to $\gamma_0 \cup \gamma_1$. Then, for r large and ϵ small, there exists some constant $D > 0$ such that*

$$\left| l(\delta) - 2r + 2\log\frac{4}{3} \right| \leq D\epsilon.$$

Moreover, the monodromy of δ is $D\epsilon$-close to the identity.

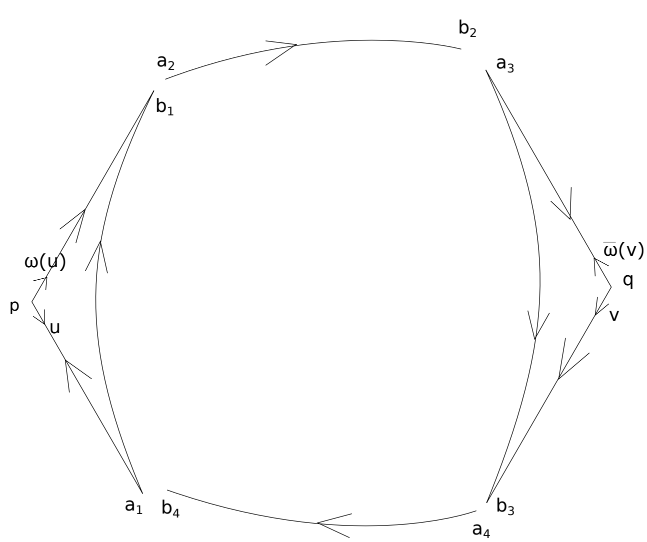

Figure 3.3: Chains of long geodesic segments

Proof. We define $z_1 = -\mathbf{g}_{\frac{r}{4}}(z)$ and $w_1 = \mathbf{g}_{\frac{r}{4}}(\omega(z))$. Similarly, we define $z_3 = -\mathbf{g}_{\frac{r}{4}}(\bar{\omega}(w))$ and $w_3 = \mathbf{g}_{\frac{r}{4}}(w)$. Since $T(z)$ and $\bar{T}(w)$ are well connected by $\gamma = (\gamma_0, \gamma_1, \gamma_2)$, there exist frames z_2, w_2, z_4, w_4 such that the following hold: (1) The frame z_2 is ϵ-close to w_1 and the frame z_4 is ϵ-close to w_3; (2) $w_2 = \mathbf{g}_{\frac{r}{2}}(z_2)$ and $w_4 = \mathbf{g}_{\frac{r}{2}}(z_4)$. Denote by a_i the basepoint of the frame z_i and denote by b_i the basepoint of the frame w_i, for $i = 1, 2, 3, 4$.

Write $z_i = (a_i, u_i, E_i)$ and write $w_i = (b_i, v_i, F_i)$. We denote by a_1b_1 the geodesic segment from a_1 to b_1 that is homotopic rel endpoints to

$$\mathbf{g}_{[0,\frac{r}{4}]}(a_1, u_1) \cdot \mathbf{g}_{[0,\frac{r}{4}]}(p, \omega(u))$$

Similarly, we define a_3b_3. Let $z_i' = (a_i, v(a_i, b_i), E_i)$ and $w_i' = (b_j, -v(b_i, a_i), F_i)$ for $i = 1, 3$. In triangle pa_1b_1, we have $d(p, a_1) = d(p, b_1) = \frac{r}{4}$ and $\angle a_1pb_1 = \frac{2\pi}{3}$. So the following inequalities hold:

$$\angle pa_1b_1 \leq D_1 e^{-\frac{r}{4}},$$

$$\left||a_1b_1| - \frac{r}{2} + \log\frac{4}{3}\right| \leq D_1 e^{-\frac{r}{4}},$$

for some constant $D_1 > 0$. We can choose r large enough so that $\angle pa_1b_1 \leq \epsilon$ and $\left||a_1b_1| - \frac{r}{2} + \log\frac{4}{3}\right| \leq \epsilon$. Notice that $\mathbf{D}(z_1', z_1) = \angle pa_1b_1 \leq \epsilon$. Similarly, $\mathbf{D}(z_3', z_3)$, $\mathbf{D}(w_1', w_1)$, $\mathbf{D}(w_3', w_3)$ are all bounded by ϵ. It's easy to see that $w_1' = \mathbf{g}_{|a_1b_1|}(z_1')$ and $w_3' = \mathbf{g}_{|a_3b_3|}(z_3')$. So the frames

$$(z_1', w_1', z_2, w_2, z_3', w_3', z_4, w_4)$$

satisfy the condition of Lemma 3.2.5. The result just follows from Lemma 3.2.5.

□

Next, we consider the positions of feet (with direction) on δ. Denote Π the skew pants associated to the pair of tripods $T(z), \bar{T}(w), \gamma$. Then there are two feet on δ in the pants Π. The next lemma shows that we can predict the position of the feet very well even if we have no information about the third connection γ_2. We first define the "predicted feet" $\mathbf{f}_\delta$ on the cuff δ from the data of $T(z), \bar{T}(w), \gamma_0, \gamma_1$. Then we show that no matter how we choose the third connection γ_2, as long as it is well connected, the actual feet on δ are exponentially close to our predictions.

The "predicted feet" map $\mathbf{f}_\delta$. We first define the geodesic ray $\alpha_p : [0,\infty) \to \mathbb{H}^n$ by $\alpha_p(0) = p$, $\alpha'_p(0) = \bar{\omega}(u)$. Then, for any good connection γ_2, the angle between geodesic $\alpha_p(0)$ and γ_2 is exponentially small. Similarly, we define the geodesic ray $\alpha_q : [0,\infty) \to M$ by $\alpha_q(0) = q, \alpha'_q(0) = \omega(v)$. Then for $t \in [0,\infty)$ and $i = 0, 1$, we let β_i^t be the geodesic segment homotopic relative endpoints to the piecewise geodesic arc

$$\overline{\alpha_p[0,t]} \cdot \gamma_i \cdot \alpha_q[0,t].$$

Denote β_i^∞ the limiting geodesic of $\beta_i(t)$, when $t \to \infty$. We let $f_i^t \in N(\delta)$ be the feet of the common orthogonal between β_i^t and δ. And we let $f_i = f_i^\infty$ be the limit of f_i^t. Then we define $\mathbf{f}_\delta = \{f_1, f_2\}$ to be the pair of the feet. We denote $\mathrm{feet}_\delta(T_p, \bar{T}_q, \gamma)$ to be the actual pair of the feet of the skew pants Π. Then we have the following result.

Lemma 3.3.2 ([KM12], Proposition 4.9)**.** *Assume that tripods T_p and $\bar{T}_q$ are well connected by $\gamma = (\gamma_0, \gamma_1, \gamma_2)$, then for r large and ϵ small, we have*

$$\mathbf{d}(\mathrm{feet}_\delta(T_p, T_q, \gamma), \mathbf{f}_\delta(T_p, \bar{T}_q, \gamma)) \leq De^{-\frac{r}{4}}$$

for some constant $D > 0$.

Remark. *Notice that* $\mathbf{d}$ *is the complex distance function between two pairs of vectors. We define* $\mathbf{d}$ *as follows:*

$$\mathbf{d}\left((f_1, f_2), (h_1, h_2)\right) = \max\{\mathbf{d}_0(f_1, h_1), \mathbf{d}_0(f_2, h_2)\},$$

where $\mathbf{d}_0$ *measures the usual (complex) distance between two vectors in $N(\delta)$.*

Since the monodromy of δ is very close to the identity, one can check that the complex distance between f_1 and f_2 on $N(\delta)$, measured in either direction of δ, is $D\epsilon$ close to $\frac{R}{2} = r - \log\frac{4}{3}$ for some constant $D > 0$. Combining the result of Lemma 3.3.2, we get that both half lengths $\mathbf{hl}_1^\Pi(\delta)$ and $\mathbf{hl}_2^\Pi(\delta)$ are $D\epsilon$-close to $\frac{R}{2}$ for some constant $D > 0$. So we have proved the following proposition.

Proposition 3.3.3. *For any pair of pants Π constructed from well connected tripods and for any cuff δ of Π, we have the following inequality:*

$$\left|\mathbf{hl}_i^\Pi(\delta) - \frac{R}{2}\right| \leq D\epsilon, \ , i = 1, 2$$

for some constant $D > 0$.

The following lemma is a corollary of Proposition 3.3.3. It shows that Π is a good pair of pants (see Definition 1.3.1).

Lemma 3.3.4. *Denote* $\mathbf{d} = d + i\theta$ *to be the complex distance between two cuffs of* Π*, then there exists some constant* $D > 0$ *such that*

$$\frac{\theta^2}{d^2} \leq D\epsilon,$$

when r *is sufficiently large and* ϵ *is sufficiently small. Moreover, if we denote* d_0 *the length of the seam in an* R*-standard pair of pants, then we have the following inequality*

$$\left|\frac{d^2}{d_0^2} - 1\right| \leq D\epsilon.$$

Proof. The three seams cut Π into two right angled hexagons H_1 and H_2. We can lift H_1 to be an embedded right angled hexagon in $\mathbb{H}^n$. Let $ABCDEF$ be the vertices of H_1 (see Figure 3.4). By Lemma 3.3.2, for some constant $K > 0$ we get the following estimates:$|AB|$, $|CD|$, $|EF|$ are all $K\epsilon$ close to $\frac{R}{2}$. Moreover, we have

$$\Theta(v(A, F)@B, v(B, C)) \leq K\epsilon, \tag{3.10}$$

$$\Theta(v(B, C)@D, v(D, E)) \leq K\epsilon, \tag{3.11}$$

$$\Theta(v(E, D)@F, v(F, A)) \leq K\epsilon. \tag{3.12}$$

From the proof of Lemma 3.3.1, one can check that the real distance between AB and EF is bounded by $2K\epsilon$. So $|AF|, |BC|, |DE|$ are all bounded by $2K\epsilon$.

We may assume that the complex distance between FE and AB is $\mathbf{d} = d + i\theta$. In the quadrangle $EFAB$, we use the cosine rule (see Appendix) and obtain

$$\begin{aligned}\cosh|EB| &= \cosh|FE|\cosh|AB|\cosh d - \sinh|FE|\sinh|AB|\cos\theta \\ &= \cosh|FE|\cosh|AB|(\cosh d - \cos\theta) + \cosh(|AB| - |FE|)\cos\theta.\end{aligned}$$

Since the lengths $|EB|, |FE|, |AB|$ are all $5K\epsilon$-close to $\frac{R}{2}$, we obtain that

$$\left|\frac{\cosh d - \cos\theta}{2e^{-\frac{R}{2}}} - 1\right| \leq K_1\epsilon$$

for some constant $K_1 > 0$ when R large and ϵ small. When d and θ are both small, we have that $(\cosh d - \cos\theta) \sim \frac{1}{2}(d^2 + \theta^2)$. So we obtain that

$$\left|\frac{d^2 + \theta^2}{4e^{-\frac{R}{2}}} - 1\right| \leq K_2\epsilon \tag{3.13}$$

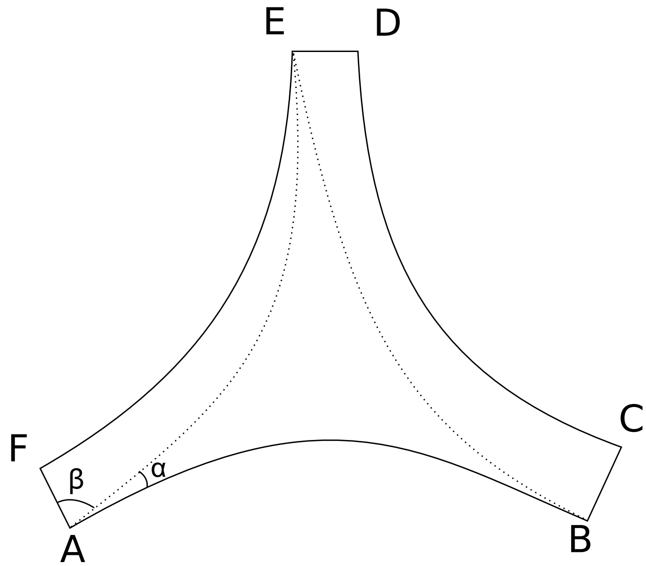

Figure 3.4: A right angled hexagon in $\mathbb{H}^n$

for some constant $K_2 > 0$. Hence, the lengths $|AF|, |BC|, |ED|$ are bounded by $O(e^{-\frac{R}{4}})$. We claim that $\Theta(v(A, F)@E, -v(E, D)) \leq (K + 1)\epsilon$. To prove the claim, consider the triangle AFE. By Lemma 3.2.1 we get

$$\Theta(v(A, F)@F@E, v(A, F)@E) \leq |AF| \leq \epsilon$$

when R is sufficiently large. Notice that $v(A, F)@F = -v(F, A)$ and $v(F, A)@E$ is $K\epsilon$-close to $v(E, D)$. Hence by triangle inequality, we get that

$$\Theta(v(A, F)@E, -v(E, D)) \leq (K + 1)\epsilon, \tag{3.14}$$

which proves the claim.

Similarly, using Lemma 3.2.1 in triangle BCD, we can get $\Theta(v(B, C)@D, -v(D, E)) \leq (K + 1)\epsilon$. Using Lemma 3.2.1 in triangle EDB, we get

$$\Theta(v(B, C)@D@E, v(B, C)@E) \leq |ED| \leq \epsilon$$

when R is large and ϵ is small. Since $-v(D, E)@E = v(E, D)$, combining the equations above, we obtain that

$$\Theta(v(B, C)@E, v(E, D)) \leq (K + 2)\epsilon.$$

In other words,

$$\Theta(v(E, D)@B, v(B, C)) \leq (K + 2)\epsilon. \tag{3.15}$$

Now we consider the triangle AEB and the vector $v(A, F)@E@B@A$. By equations (3.14), (3.15) and (3.10), we get

$$\Theta(v(A, F)@E@B@A, -v(A, F)) \leq 3(K + 1)\epsilon. \tag{3.16}$$

On the other hand, let u be any unit vector at A that is orthogonal to the plane EAB. We always have $u@E@B@A = u$. Also, by Gauss-Bonnet Theorem, for any unit tangent vector u at A that is tangent tn the plane EBA, we have

$$\Theta(u@E@B@A, u) = Area(EBA) < \pi.$$

So Equation (3.16) essentially tells us that $v(A, F)$ is very close to the plane that contains AEB. In particular, one can check that the angle between $v(A, F)$ and the plane that contains AEB is less than $2(K + 1)\epsilon$ when ϵ is small enough.

Denote F' to be the projection of F onto the plane that contains EAB. From the Euclidean geometry, we have

$$\cos \angle FAE = \cos \angle FAF' \cos \angle F'AE.$$

Since $\angle FAF' \leq 2(K + 1)\epsilon$, we have

$$\cos \angle FAF' \geq 1 - 2(K + 1)^2\epsilon^2.$$

Denote $\alpha = \angle EAB$, $\beta = \angle FAE$. Then $\alpha = \frac{\pi}{2} - \angle F'AE$, so we get

$$\cos \beta \geq (1 - 2(K + 1)^2\epsilon^2) \sin \alpha.$$

Now we look at the triangle AEF. By the sine rule, we have

$$\sin \beta = \frac{\sinh |EF|}{\sinh |AE|}.$$

Denote $a = |EF|$, then $|a - \frac{R}{2}| \leq K\epsilon$. By the cosine rule, $\cosh |AE| = \cosh a \cosh d$. So we get the following inequality:

$$\sin^2 \beta = \frac{\sinh^2 a}{\cosh^2 a \cosh^2 d - 1}.$$

We rewrite it and obtain

$$\begin{aligned}
\cosh^2 d - 1 &= \frac{1}{\cosh^2 a} \cdot \left(\frac{\sinh^2 a}{\sin^2 \beta} + 1\right) - 1 \\
&= \frac{1}{\cosh^2 a} \cdot \left(\frac{\sinh^2 a}{\sin^2 \beta} - \sinh^2 a\right) \\
&= \tanh^2 a \cdot \cot^2 \beta \\
&\geq \tanh^2 a \cdot \cos^2 \beta \\
&\geq \tanh^2 a \cdot [1 - 2(K + 1)^2\epsilon^2]^2 \cdot \sin^2 \alpha.
\end{aligned}$$

In triangle AEB, the lengths of three sides, $|AE|$, $|EB|$ and $|AB|$, are all $K\epsilon$-close to $\frac{R}{2}$. By the cosine rule in a triangle, we obtain

$$\left|\frac{1-\cos\alpha}{2e^{-\frac{R}{2}}}-1\right| \leq K_3\epsilon \tag{3.17}$$

for some constant $K_3 > 0$. So we have $\alpha \sim 2e^{-\frac{R}{4}}$. Moreover, the inequality $\sin^2\alpha \geq 4(1-K_4\epsilon)e^{-\frac{R}{2}}$ holds for some constant $K_4 > 0$.

Since a is $K\epsilon$-close to $\frac{R}{2}$, we get

$$\left|\tanh^2 a - 1\right| = \frac{1}{\cosh^2(a)} \leq 4e^{-2a} = O(e^{-R}).$$

Together, we get

$$\sinh^2 d = \cosh^2 d - 1 \geq 4(1-K_5\epsilon)e^{-\frac{R}{2}}$$

for some constant $K_5 > 0$. When d is small, we have $\sinh^2 d \sim d^2$. Combining with Equation (3.13), we obtain that

$$1 - K_6\epsilon \leq \frac{d^2}{4e^{-\frac{R}{2}}} \leq 1 + K_2\epsilon \tag{3.18}$$

for some constant $K_6 > 0$.

Using Equation (3.13) again, we get $\frac{\theta^2}{4e^{-\frac{R}{2}}} \leq (K_6 + K_2)\epsilon$. So there exists some constant $K_7 > 0$ such that

$$\frac{\theta^2}{d^2} \leq K_7\epsilon.$$

By the cosine rule in a flat right angled hexagon, we can get $d_0 = 2e^{-\frac{R}{4}} + O(e^{-\frac{3R}{4}})$. So the last result in the Lemma follows from Equation (3.18).

□

By Remark 3.0.7, $\epsilon = \epsilon(r)$ can be chosen to be any polynomial in r^{-1}. In particular, we can choose $\epsilon = r^{-6}$. Notice that $R = 2(r - \log\frac{4}{3})$. So we get that

$$\frac{\theta}{d} = O(R^{-3})$$

$$\left|\frac{d}{d_0} - 1\right| = O(R^{-3}).$$

Hence, we have proved that any pair of pants constructed from well connected tripods is indeed good in the sense of Definition 1.3.1.

3.4 Measures on Good Pants

In this section, we will show that, when n is an odd number, there exists an immersed surface S that satisfies the assumption of Theorem 1.2.1 when R is sufficiently large. Denote $\Pi(R)$ to be the collection of all $R-$good pants. The goal is to find a suitable finite collection of good pants such that they can be well glued along their boundaries to form a closed surface.

Let Π_1 and Π_2 be two good pants such that they share a common boundary C. By Definition 1.3.2, Π_1 and Π_2 are well-glued if the inequality

$$\left|S_i^{\Pi_1,\Pi_2}(C) - 1\right| \le \frac{10}{R^2} \tag{3.19}$$

holds for $i = 1, 2$. To satisfy the assumption of Theorem 1.2.1, all glued pants must be well-glued. For each good curve C, we will first study the distribution of feet on C. As in Section 1.3, we denote $N(C)$ to be the unit normal bundle of C, and we denote

$$N^2(C) = \Big\{\{f_1, f_2\} : f_1 \in N(C), f_2 \in N(C)\Big\}$$

to be the set of all pairs of feet in $N(C)$.

Let $A_t : N(C) \to N(C)$, $t \in \mathbb{R}$, to be the transformation that transports a distant t along C, and denote $O : N(C) \to N(C)$ to be the antipodal map, where $O(v) = -v$ for any $v \in N(C)$. Then A_t and O act on $N^2(C)$ as well. It's easy to see that $A_t \cdot A_s = A_s \cdot A_t$ and $A_t \cdot O = O \cdot A_t$ for any $s, t \in \mathbb{R}$. Denote $\text{feet}_C(\Pi)$ to be the pair of feet on C in Π. Let $\mathcal{A} = A_1 \cdot O$. Then Equation (3.19) is equivalent to the following:

$$\mathbf{d}(\mathcal{A}(\text{feet}_C(\Pi_1)), \text{feet}_C(\Pi_2)) \le \frac{10}{R^2}, \tag{3.20}$$

where $\mathbf{d}$ measures the complex distance between two pairs of feet. Since we can view a finite collection of good pants as a finite integral positive measure on good pants, so we have the following result.

Claim 3.4.1. *For R sufficiently large, if there exists a finite integral positive measure μ on R-good pants such that $\mathcal{A}_*(\partial\mu)$ and $\partial\mu$ are $\frac{10}{R^2}$-equivalent (see A.2), then there exists an immersed surface S that satisfies the assumption of Theorem 1.2.1.*

The above claim is a corollary of Proposition A.2.3. So it remains to show that such a measure μ exists.

Let $\tilde{\mu}$ be the measure on well-connected pairs of tripods given by

$$d\tilde{\mu}(T(z), \bar{T}(w), \gamma) = \mathbf{b}_\gamma(T(z), \bar{T}(w)) d\lambda_T(z, w, \gamma),$$

where $\lambda_T(z, w, \gamma)$ is the product of the Liouville measure Λ on the first two terms and the counting measure on γ. Since $\mathbf{b}_\gamma(T(z), \bar{T}(w))$ has compact support, so $\tilde{\mu}$ is finite. To each pair of (ϵ, r) well-connected tripods, we can associate a pair of pants Π. We denote $\Pi = \pi(T(z), \bar{T}(w), \gamma)$. In the previous section, we proved that π maps well-connected pairs of tripods to good pairs of pants. So the projection induces a measure on good pants by $\mu = \pi_* \tilde{\mu}$. We will show that this measure μ is a real (positive) solution to the system of inequalities in Claim 3.4.1.

To any frame z we can associate a bipod $B(z) = (z, \omega(z))$, and likewise to any frame z we associate the "anti-bipod" $\bar{B}(z) = (z, \bar{\omega}(z))$. We say that $\left(B(z), \bar{B}(w), \gamma_0, \gamma_1\right)$ is a well connected pair of bipods along the pair of segment γ_0 and γ_1 if

$$\mathbf{a}_{\gamma_0}(z, w)\mathbf{a}_{\gamma_1}(\omega(z), \bar{\omega}(w)) > 0.$$

Let δ be the closed geodesic that is homotopic to $\gamma_0 \cup \gamma_1$. For any third connection γ_2 such that $\mathbf{a}_{\gamma_2}(\omega(z), \omega(w)) > 0$, we can get a pair of well-connected tripods $(T(z), \bar{T}(w), \gamma_0, \gamma_1, \gamma_2)$ from the bipods. Denote by $\mathbf{f}_\delta(T(z), \bar{T}(w), \gamma_0, \gamma_1)$ the predicted feet. Then, by Lemma 3.3.2, no matter what the third connection γ_2 is, we have

$$\mathbf{d}(\mathbf{f}_\delta(T(z), \bar{T}(w), \gamma_0, \gamma_1), \text{feet}_\delta(T(z), \bar{T}(w), \gamma)) \le De^{-\frac{r}{4}}$$

for some constant $D > 0$. Since $\mathbf{f}_\delta$ doesn't rely on the third connection γ_2, so we can define $\mathbf{f}_\delta$ on well connected pairs of bipods by:

$$\mathbf{f}_\delta(B(z), \bar{B}(w), \gamma_0, \gamma_1) = \mathbf{f}_\delta(T(z), \bar{T}(w), \gamma_0, \gamma_1).$$

Let S_δ be the set of well connected bipods $(B(z), \bar{B}(w), \gamma_0, \gamma_1)$ such that $\gamma_0 \cup \gamma_1$ is homotopic to δ. The set of S_δ carries the natural measure λ_B which is the product of the Liouville measures on the first two terms and the counting measure on the last two terms. Let C_δ be the set of well-connected tripods $(T(z), \bar{T}(w), \gamma)$ for which $\gamma_0 \cup \gamma_1$ is homotopic to δ, and let $\chi : C_\delta \to S_\delta$ be the forgetting map, so $\chi(T(z), \bar{T}(w), \gamma_0, \gamma_1, \gamma_2) = (B(z), \bar{B}(w), \gamma_0, \gamma_1)$. Then by definition, we have

$$\partial\mu\Big|_{N^2(\delta)} = (\text{feet}_\delta)_*(\tilde{\mu}\Big|_{C_\delta}). \tag{3.21}$$

As $\mathbf{f}_\delta$ is $De^{-\frac{r}{4}}$-close to feet_δ, so it suffices to prove that $(\mathbf{f}_\delta)_*(\tilde{\mu}|_{C_\delta})$ and $\mathcal{A}_*((\mathbf{f}_\delta)_*(\tilde{\mu}|_{C_\delta}))$ are $\frac{9}{R^2}$-equivalent when R is large. Let $\mathbf{f}_\delta = (f^1_\delta, f^2_\delta)$, where f^1 is the first projection and f^2 is the second projection. By Remark 1.3.3, it is enough to show that $(f^1_\delta)_*((\tilde{\mu}|_{C_\delta})$ and $\mathcal{A}_*((f^1_\delta)_*(\tilde{\mu}|_{C_\delta}))$ are $\frac{6}{R^2}$-equivalent. Since $\mathbf{f}_\delta$ is well defined on well connected bipods, we have $(f^1_\delta)_*(\tilde{\mu}|_{C_\delta}) = (f^1_\delta)_*(\chi_*(\tilde{\mu}|_{C_\delta}))$.

We will consider two natural measures on S_δ. The first one is $\chi_*(\tilde{\mu}|_{C_\delta})$. The other is ν_δ, defined on S_δ by

$$d_{\nu_\delta}(B(z), \bar{B}(w), \gamma_0, \gamma_1) = \mathbf{a}_{\gamma_0}(z, w)\mathbf{a}_{\gamma_1}(\omega(z), \bar{\omega}(w))d\lambda_B(z, w, \gamma_0, \gamma_1),$$

where we recall that $\lambda_B(z, w, \gamma_0, \gamma_1)$ is the product of the Liouville measure on the first two terms and the counting measure on the last two. Then by Corollary 3.0.6, the two measures satisfy the fundamental inequality

$$\left| \frac{d\chi_*(\tilde{\mu}|_{C_\delta})}{d\nu_\delta(B(z), \bar{B}(w), \gamma_0, \gamma_1)} - \frac{1}{\Lambda(\mathcal{F}(M))} \right| < De^{-\mathbf{q}\frac{r}{2}}. \tag{3.22}$$

Now we study the symmetry on S_δ. Choose a point x_0 on δ, and denote F the fiber of x_0 in $N(\delta)$. Then F is isometric to S^{n-2}. Let A be the monodromy of δ. Then A acts on F, the $n-2$ dimensional sphere. We can choose an $(n-1)$-frame in F such that under this frame the monodromy A acts on F as the following block matrix

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & \cdots & 0 & 0 \\ -\sin\theta_1 & \cos\theta_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cos\theta_k & \sin\theta_k \\ 0 & 0 & \cdots & -\sin\theta_k & \cos\theta_k \end{pmatrix}$$

for some $\theta_1, \cdots, \theta_k$, where $k = \frac{n-1}{2}$. Let $C(A)$ be the isometry group of $N(\delta)$ such that each element preserves F and it acts on F as the following matrix

$$\begin{pmatrix} \cos\varphi_1 & \sin\varphi_1 & \cdots & 0 & 0 \\ -\sin\varphi_1 & \cos\varphi_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cos\varphi_k & \sin\varphi_k \\ 0 & 0 & \cdots & -\sin\varphi_k & \cos\varphi_k \end{pmatrix}$$

for some $0 \le \varphi_1, \cdots, \varphi_k < 2\pi$. Then A commutes with all elements of $C(A)$. As before, we denote by A_t the parallel transport of a distance t along δ. Then, for any

$t \in \mathbb{R}$, A_t commutes with all elements of $C(A)$. Let $I(A) = C(A) \times \{A_t : 0 \leq t < l(\delta)\}$, then $I(A)$ is an isometry group on $N(\delta)$. We have the following inequality

$$\mathbf{a}_{\gamma_0}(z, w)\mathbf{a}_{\gamma_1}(\omega(z), \bar{\omega}(w)) = \mathbf{a}_{g(\gamma_0)}\left(g(z), g(w)\right)\mathbf{a}_{g(\gamma_1)}\left(g(\omega(z)), g(\bar{\omega}(w))\right),$$

for all $g \in I(A)$, as the affinity function $\mathbf{a}$ is left invariant. Also, $I(A)$ naturally acts on S_δ as isometry, so the group action leaves invariant the measure $\lambda(B)$. It follows that $I(A)$ will leave the measure ν_B invariant.

On the other hand, the predicted feet $\mathbf{f}_\delta$ is equivariant under the group action of $I(A)$; that is, for each $g \in I(A)$, we have

$$\mathbf{f}_\delta\left(B(g(z)), \bar{B}(g(w)), g(\gamma_0), g(\gamma_1)\right) = g\left(\mathbf{f}_\delta(B(z), \bar{B}(w), \gamma_0, \gamma_1)\right).$$

It follows from the above two observations that the measure $(\mathbf{f}_\delta)_*\nu_B$ is invariant under the $I(A)$ action. By projection, we get that the measure $(f_\delta^1)_*\nu_B$ on $N(\delta)$ is invariant under the $I(A)$ action. Since $I(\delta)$ may not be transitive on $N(\delta)$, it is not necessary that $(f_\delta^1)_*\nu_B$ is a multiple of the Euclidean measure on $N(\delta)$. However, for any $v \in N(\delta)$, the group $I(A)$ acts transitively on $I(A)\cdot v$, so, when restricted on the orbit $I(A)\cdot v$, the measure $(f_\delta^1)_*\nu_B$ is a multiple of the Euclidean measure. By the definition of $C(A)$ and $I(A)$, the orbit $I(A)\cdot v$ is a torus T^m for some $2 \leq m \leq k$. Notice that the antipodal map O is in $C(A)$. So $\mathcal{A} = A_1 \cdot O$ is also in $I(A)$. Hence, when restricted on a orbit $I(A)\cdot v$, the measure $\mathcal{A}_*((f_\delta^1)_*\nu_B)$ is a multiple of the Euclidean measure. Denote $\beta_\delta = (f_\delta^1)_*(\chi_*(\tilde{\mu}|_{C_\delta}))$, and denote λ to be the Euclidean measure on the orbit $I(A)\cdot v$. Then by Equation (3.22), when restricted on $I(A)\cdot v$, there exists some constant K_δ such that

$$K_\delta \leq \left|\frac{d\beta_\delta}{d\lambda}\right| \leq K_\delta(1 + D_2 e^{-\mathbf{q}\frac{R}{2}}),$$

where $D_2 > 0$ is some constant.

The following lemma shows that any C^0 measure on the torus T^m that is close to the Euclidean measure is obtained by pull-back the Euclidean measure by a diffeomorphism that is C^0-close to the identity.

Lemma 3.4.2. *Let $g : \mathbb{R}^m \to \mathbb{R}$ be a C^0 function on $\mathbb{R}^m$ that is well defined on the quotient $T^m = \mathbb{R}^m/\mathbb{Z}^m$, and such that:*

1. *For some $0 < \delta \leq \frac{1}{4}$, we have*

$$1 - \delta \leq g(\bar{x}) \leq 1 + \delta$$

for all $\bar{x} \in \mathbb{R}^m$.

2. *The following equality holds:*

$$\int_{[0,1]^m} g(\bar{x})d\bar{x} = 1,$$

where $d\bar{x} = dx_1 \cdots dx_m$ is the volume form.

Then we can find a C^1 diffeomorphism $h : T^m \to T^m$ such that:

1. *$g(\bar{x})d\bar{x} = h^*(d\bar{x})$, where $h^*(d\bar{x})$ is the pull-back of the volume form by the diffeomorphism h.*

2. *There exists some constant $D > 0$ such that the inequality*

$$||h(\bar{x}) - \bar{x}|| \leq D\delta$$

holds for every $\bar{x} \in \mathbb{R}^m$.

Since $K_\delta \leq \left|\frac{d\beta_\delta}{d\lambda}\right| \leq K_\delta(1+D_2e^{-\mathbf{q}\frac{R}{2}})$ and the Euclidean measure λ is invariant under $\mathcal{A}$, we have

$$K_\delta \leq \left|\frac{d\mathcal{A}^*(\beta_\delta)}{d\lambda}\right| \leq K_\delta(1 + D_2e^{-\mathbf{q}\frac{R}{2}}).$$

By Lemma 3.4.2, when restricted on a orbit $I(A) \cdot v$, the measure β_δ is $De^{-\mathbf{q}\frac{R}{4}}$-equivalent with the measure $K_1\lambda$ for some $K_1 > 0$ and some constant $D > 0$. Similarly, the measure $\mathcal{A}^*(\beta_\delta)$ is $De^{-\mathbf{q}\frac{R}{4}}$-equivalent with the measure $K_2\lambda$ for some $K_2 > 0$. As $\mathcal{A}^*(\beta_\delta)$ and β_δ have the same total measure on the orbit $I(A) \cdot v$, so we can choose $K_2 = K_1$. Then by Proposition A.2.2, β_δ and $\mathcal{A}^*(\beta_\delta)$ are $2De^{-\mathbf{q}\frac{R}{4}}$-equivalent measures on the orbit $I(A) \cdot v$. Notice that $N(\delta)$ is the union of all orbits $\bigcup I(A) \cdot v$. As $2De^{-\mathbf{q}\frac{R}{4}}$ will be universally less than $\frac{1}{R^2}$ when R is large enough, the two measures β_δ and $\mathcal{A}^*(\beta_\delta)$ are $\frac{1}{R^2}$-equivalent measures on $N(\delta)$. Recall that $\beta_\delta = (f^1_\delta)_*(\chi_*(\tilde{\mu}|_{C_\delta}))$. So we have shown that

$$(\mathbf{f}_\delta)_*(\tilde{\mu}|_{C_\delta}) \text{ and } \mathcal{A}_*((\mathbf{f}_\delta)_*(\tilde{\mu}|_{C_\delta})$$

are $\frac{1}{R^2}$-equivalent measures on $N(\delta)$. From equation 3.21, we have

$$\mathcal{A}_*(\partial\mu) \text{ and } \partial\mu$$

are $\frac{1}{R^2}$-equivalent.

This just shows that μ is a real positive solution to the linear system. Since both measures μ and $\partial\mu$ are atomic, the linear system has only finitely many inequalities. Then the standard rationalization procedure implies that this system has a positive integral solution. The existence of the surface S just follows from Claim 3.4.1.

Appendix A

A.1 Cosine Rule

Let H be a right angled hexagon in $\mathbb{H}^3$ with sides L_k, $k \in \mathbb{Z}_6$. Set $\sigma_k = \mathbf{d}_{L_k}(L_{k-1}, L_{k+1})$ to be the complex distance between side L_{k-1} and side L_{k+1}. Then we have the following cosine rules. (see [Fen89, Section 6])

$$\frac{\sinh \sigma_1}{\sinh \sigma_4} = \frac{\sinh \sigma_2}{\sinh \sigma_5} = \frac{\sinh \sigma_3}{\sinh \sigma_6}, \tag{A.1}$$

$$\cosh \sigma_k = \frac{\cosh(\sigma_{k+3}) - \cosh(\sigma_{k+1})\cosh(\sigma_{k-1})}{\sinh(\sigma_{k+1})\sinh(\sigma_{k-1})}. \tag{A.2}$$

Assume that H is an R-standard right angled hexagon in $\mathbb{H}^2$. Namely, we have $\sigma_1 = \sigma_3 = \sigma_5 = R + i\pi$. Then the shape of H is complete determined, and we have $\sigma_2 = \sigma_4 = \sigma_6$. Moreover,

$$\sigma_{2j} = 2e^{-\frac{1}{4}R} + i\pi + O(e^{-\frac{3R}{4}}), \tag{A.3}$$

for $j = 0, 2, 4$. From the pentagon formula, the hyperbolic distance between opposite sides of the hexagon can be estimated as

$$\mathbf{d}(L_k, L_{k+3}) = \frac{R}{4} + \ln 2 + o(\frac{1}{R^{100}}). \tag{A.4}$$

The constant $\ln 2$ in the above formula is the inefficiency of angle $\frac{\pi}{2}$. The readers are referred to [KM15, Chapter 4] for more details on the theory of inefficiency.

Let α and β be two geodesics in $\mathbb{H}^n$. We can always find a hyperbolic space $\mathbb{H}^3$ that contains both α and β. So we have the following cosine rules for quadrilaterals in $\mathbb{H}^n$ (cf. [Fen89, section 5]).

Lemma A.1.1. *Let α and β be two oriented geodesics in $\mathbb{H}^n$, and let γ be the common orthogonal of α and β. We parametrize α and β by their arc-lengths, and let $\alpha(0) = \alpha \cap \gamma$, $\beta(0) = \beta \cap \gamma$. Let $\mathbf{d}(\alpha, \beta) = d + i\theta$ be the complex distance between α and β. Then the distance between $\alpha(s)$ and $\beta(t)$ satisfies the following equation:*

$$\cosh d(\alpha(s), \beta(t)) = \cosh s \cosh t \cosh d - \sinh s \sinh t \cos \theta. \tag{A.5}$$

Moreover, if $\theta = 0$, then the real distance between $\alpha(s)$ and the geodesic β satisfies the following equation,

$$\sinh(d(\alpha(s), \beta)) = \sinh d \cosh s. \tag{A.6}$$

The following result is a corollary of the previous lemma.

Corollary A.1.2. *In Lemma A.1.1, if $s = t > 0$ and $\theta = 0$, then the distance $\alpha(s)$ and $\beta(t)$ satisfies the following equation.*

$$\sinh \frac{|\alpha(s)\beta(t)|}{2} = \cosh s \sinh \frac{d}{2}.$$

A.2 δ-equivalent Measures

In this section, we will give the definitions of δ-equivalent measures and show some properties of them.

Let (X, d) be a metric space. For $A \subset X$ and $\delta > 0$, let

$$N_\delta(A) = \{x \in X : \text{ there exists } a \in A \text{ such that } d(x, a) \leq \delta\}$$

be the δ-neighborhood of A.

Definition A.2.1. *Let μ, ν be two compacted supported boreal measures on X such that $\mu(X) = \nu(X) < \infty$. Letting $\delta > 0$, we say that μ and ν are δ-equivalent measures if and only if, for every Borel subset A, we have $\mu(A) \leq \nu(N_\delta(A))$.*

One can check that the definition is symmetric in μ and ν. The following propositions follow from the above definition.

Proposition A.2.2. *Suppose μ and ν are δ_1-equivalent, and ν and η are δ_2-equivalent, then μ and η are $(\delta_1 + \delta_2)$-equivalent.*

Proposition A.2.3 ([KM12], Theorem 3.2)**.** *Suppose that A and B are finite sets with same number of elements. Denote by Λ_A and Λ_B the standard counting measures on A and B respectively. Suppose that there are maps $f : A \to X$ and $g : B \to X$ such that the measures $f_*\Lambda_A$ and $g_*\Lambda_B$ are δ-equivalent for some $\delta > 0$. Then there exists a bijection $h : A \to B$ such that $d(g(h(a), f(a)) \leq \delta$ for every $a \in A$.*

The above proposition is a corollary of the Hall's Marriage Theorem.

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