Aspects of Effective Field Theories from Scattering Amplitudes

Thesis by
Chia-Hsien Shen

In Partial Fulfillment of the Requirements for the
Degree of
Doctor of Philosophy

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2017
Defended May 15, 2017

Chia-Hsien Shen
ORCID: 0000-0002-5138-9971

ACKNOWLEDGEMENTS

The journey of graduate study is rewarding, but definitely a strenuous one. Along the way, many people helped and supported me to overcome all the challenges. All I can do is to use a tiny amount of paragraph to express my gratitude. The number of people who I should be thankful to are uncountable after six years. This acknowledgment only reflects a subset of people whose influence is so substantial that even a ignorant person like me cannot forget.

First of all, I would like to thank my advisor, Prof. Clifford Cheung. Cliff establishes himself as a great role model for me to follow. He pushes me forward with his endless passion for physics, demonstrates his fearless attitude in attacking a problem, and trains me to focus on the essence. I still remember asking stupid questions or writing terrible drafts back in the day. But Cliff still kept teaching me despite my ignorance. Once in a while, I got lost in the dark and did not know how to move forward. Nevertheless, Cliff did not give up on me and still patiently guided me through. It is his belief in me that motivates me to keep fighting. This PhD degree would not be possible without his unconditional support. I cannot emphasize more how grateful I am toward Cliff.

Second, I am extremely lucky to learn from another great physicist– Jaroslav Trnka. Although he started at Caltech as a postdoc, he treats me like his own student. Whenever we talk about physics, I am always amazed by his deep insight and breadth of knowledge on various aspects of amplitudes. Working with Jaroslav is a great pleasure, because he always challenges me with critical questions and provides a great vision for overall direction. I also thank him for providing a lot of opportunities to introduce me to the amplitude community. The only thing I regret is not having learned more from him while he was at Caltech. Under the supervision of Cliff and Jaroslav, it was a certainly privilege to learn from two young stars of their generation.

Many other senior faculties were very kind to advise a ignorant young student like me. I am indebted to Mark Wise for providing generous support which spanned from my postdoc application, building a great environment for the Caltech theory group, and serving on my committee. I also appreciate Sean Carroll and Frank Porter for serving on my committee as well.

My research would not be fruitful without my wonderful collaborators: Karol Kampf, Jiri Novotny, and Congkao Wen. I also thank Zvi Bern, Jacob Bourjaily, Lance Dixon, David Kosower, Song He, Yu-tin Huang, Oliver Schlotterer, John Joseph Carrasco and Henrik Johansson for advice and discussion.

Interaction with other fellow students is equally crucial to the growth of a PhD student. My PhD experience would be totally different without Enrico Herrmann. Although we have never written papers together, Enrico is always passionate and never stops asking me about my latest progress or discussing physics. Seeing his passion, working discipline, and breath of knowledge in physics motivates me to keep working harder. Both Cliff and Jaroslav are role models of a caliber that I probably could never achieve, but I hope I can be as good as you, Enrico. We also have been to so many places together, from TASI, Hong Kong, Montreal, Taiwan, Davis, Stockholm, KITP, and the coming trip in Edinburgh. I hope the list will keep growing, and we will have a chance to be actual collaborators in the future. I wish we all have a successful career as postdocs.

I had a fine time enjoying an office of my own, until the arrival of Alek Ridgway. He makes time in office not just fine, but quite happy. Working in the office during weekends is not lonely anymore. I will miss all the jokes and discussion in this office. It is impressive to see you grow so quickly. I am looking forward to your future papers written in Downs 418.

I am also grateful to other colleagues in the Caltech Theory Group for creating an lovely environment, including Tony Bartolotta, Charles Chun Jun Cao, Aidan Chatwin-Davies, Murat Kologlu, Petr Kravchuk, Ying-Hsuan, Tristan Mckinney, Du Pei, Jason Pollack, Grant Remmen, Ingmar Saberi, Ashmeet Singh, and Bogdan Stoica. It is my pleasure to be at Caltech with you all. I was also fortunate to know many excellent students in other schools, including Alex Edison, Michael Enciso, Sean Litsey, Andrew McLeod, Josh Nohle, JJ Stankowicz, and Julio Parra Martinez, whom I learned a lot from.

The friendship built during six years at Caltech will never be forgotten. I will miss forever the life shared with my dear VolleyBobo family, Ho-Hsuan and Chia-Wei, Lulu, Bobo, Sunny, Shun-Jia, Yi-Yin, Riva and Hanky. You guys are not just friends, but more like my family. I had a lot of ups and downs these years. Thanks to you, I could find a harbor to rest, to share the laughter and tears, and find courage for the future. The person always attached to our gathering is my roommate Guagua. He is probably not the best roommate by usual standard but he is kind and never hesitates to help. In addition, I would like thank Min-Feng, Linhan, Josiane, and Hsieh-Chen for friendship and support, especially in the early years.

Let me also thank Hui Chiu and Cicada Lin. It is amazing that we are such good friends even from just knowing each other very recently. I will never forget the support and life lessons you two gave me. There are other friends who played important parts in my life. If you did, you know you are in my heart even if your names are not listed here.

I cannot say how lucky I am to keep a close friendship with Maomao. I have always learned from you since my undergrad days. Thanks for your company all these years. Often I forget the importance of your support until the moment I need it. I hope we both can find some happiness in life in the future.

Lastly, being abroad limits my time spent with my family. I wish we could share more time together. I thank my family and in particular, my parents for their endless love.

ABSTRACT

On-shell methods offer an alternative definition of quantum field theory at tree-level. We first determine the space of constructible theories solely from dimensional analysis, Lorentz invariance, and locality. We show that all amplitudes in a renormalizable theory in four dimensions are constructible, but only a subset of amplitudes is constructible in non-renormalizable theories. The obstructions to effective field theories (EFTs) are then lifted for the non-linear sigma model, Dirac-Born-Infeld theory, and the Galileon, using the enhanced soft limits of their amplitudes.

We then systematically explore the space of scalar EFTs based on the soft limits and power counting of amplitudes. We prove that EFTs with arbitrarily soft behavior are forbidden by on-shell momentum shifts and recursion relations. The exceptional EFTs like the non-linear sigma model, Dirac-Born-Infeld theory, and the special Galileon lie precisely on the boundary of allowed theory space. Our results suggest that the exceptional theories are the natural EFT analogs of gauge theory and gravity because they are one-parameter theories whose interactions are strictly dictated by properties of the S-matrix.

Next, a new representation of the nonlinear sigma model is proposed to manifest the duality between flavor and kinematics. The action consists of only cubic interactions, which define the structure constants of an underlying kinematic algebra. The action is invariant under a combination of internal and spacetime symmetries whose conservation equations imply flavor-kinematics duality, ensuring that all Feynman diagrams satisfy kinematic Jacobi identities. Substituting flavor for kinematics, we derive a new cubic action for the special Galileon. The vanishing soft behavior of amplitudes is shown as a byproduct of the Weinberg soft theorem.

Finally, we derive a class of one-loop non-renormalization theorems that strongly constrain the running of higher dimension operators in four-dimensional quantum field theory. Our derivation combines unitarity and helicity selection rules at tree level. These results explain and generalize the surprising cancellations discovered in the renormalization of dimension six operators in the standard model.

PUBLISHED CONTENT AND CONTRIBUTIONS

This thesis is based on the following published materials, in the order of their appearances from Chap. 1 to Chap. 5. All the results are the outcomes of the active discussion and the tremendous efforts shared between me and my collaborators.

C. Cheung, C. H. Shen and J. Trnka, *Simple Recursion Relations for General Field Theories, Journal of High Energy Physics* **1506**, 118 (2015) [https://arxiv.org/abs/1502.05057]
CHS involved in the concepts of the project, carried out the calculations, and participate in the writing of the manuscript.

C. Cheung, K. Kampf, J. Novotny, C. H. Shen and J. Trnka, *Physics Review Letters* **116**, no. 4, 041601 (2016), [https://arxiv.org/abs/1509.03309]
CHS involved in the simplification of the calculations and participate in the writing of the manuscript.

C. Cheung, K. Kampf, J. Novotny, C. H. Shen and J. Trnka, *A Periodic Table of Effective Field Theories, Journal of High Energy Physics* **1702**, 020 (2017), [https://arxiv.org/abs/1611.03137]
CHS involved in the part of the proof, carried out the calculations, and participate in the writing of the manuscript.

C. Cheung and C. H. Shen, *Symmetry and Action for Flavor-Kinematics Duality, Physics Review Letters* **118**, no. 12, 121601 (2017), [https://arxiv.org/abs/1612.00868]
CHS shared the main discovery of the project, and participate in the writing of the manuscript.

C. Cheung and C. H. Shen, *Nonrenormalization Theorems without Supersymmetry, Physics Review Letters* **115**, no. 7, 071601 (2015), [https://arxiv.org/abs/1505.01844]
CHS resolved a loophole in the original proof and participate in the writing of the manuscript.

CONTENTS

Chapter 1

INTRODUCTION

Quantum field theory is a cornerstone of modern theoretical physics, whose conventional approach is to write down a Lagrangian and then derive all structures therein. However, there has been tremendous progress in the modern S-matrix program revealing many symmetries and dualities obscured by the traditional approach. These new structures show up in a wide range of theories, including Yang-Mills (YM), gravity, and effective field theories (EFTs).

The initial motivation of the modern S-matrix program is to reduce the complexities of the usual method of Feynman diagrams. Feynman diagram calculation introduces off-shell redundancies from gauge invariance and a choice of field basis which appear in intermediate processes but are absent in observables. The modern S-matrix program exploits physical criteria like Lorentz invariance and unitarity to construct scattering amplitudes directly and without the aid of a Lagrangian. The history can be traced back to the unitarity methods [1, 2] developed in the 90's for loop-level calculation. The second wave of revolution was led by the celebrated BCFW recursion relations which compute S-matrices in YM without using Feynman diagrams at all [3, 4]. On-shell recursion relations were soon extended to gravity theories [5, 6], supersymmetric theories [7], and, eventually, all renormalizable and some non-renormalizable theories [8]. In the context of planar $\mathcal{N} = 4$ super Yang-Mills theory, on-shell recursion is even generalized to all-loop order [9]. These developments made traditionally intractable calculations possible, and generated many surprisingly simple formulae of scattering amplitudes. Since the development of on-shell recursion relations, many other alternative formulations of S-matrix have been invented, *e.g.*, on-shell diagrams and positive Grassmannian [10, 11], Cachazeo-He-Yuan (CHY) formula [12–14], hexagon bootstrap [15, 16], flux tube S-matrix [17, 18], twistor methods [19–27], and amplituhedron [28, 29]. We refer readers interested in more detail to the pedagogical review [30].

Many surprising properties of field theory were discovered with the aid of the S-matrix program. These properties are usually very obscured or remained unexplained at the level of the Lagrangian, but manifest in the alternative formulations of amplitudes. For example, there is a remarkable squaring relation that connects gauge and gravity first discovered in string theory [31] and much later generalized as a color-kinematics duality by Bern, Carrasco, and Johansson (BCJ) [32–34], which applies at loop level. Later on, the structure was made manifest in the context of CHY formula [14] and generalized into a wider range of theories including EFTs.

Nevertheless, if we aim to find an on-shell reformulation of quantum field theory, we need to know "what is the space of on-shell reconstructible theories?" We systematically survey the landscapes of reconstructible theories in four dimensions in Chap. 2, simply based on locality, gauge invariance, and power counting. The space spans a wide range of theories, including *all* renormalizable theories. The recursion can even be simplified for specific cases such as the standard model and supersymmetric theories. For non-renormalizable theories, only a subset of amplitudes are constructible. That obstruction occurs in non-renormalizable theories is expected, since higher point vertices are independent of lower point ones.

However, there is a class of EFTs whose amplitudes are still surprisingly on-shell constructible. To achieve this, Chap. 3 uses soft limits as the defining properties of amplitudes. In cases of the non-linear sigma model (NLSM), Dirac Born-Infield, and the special Galileon, the enhanced soft limits dictate the infinite tower of interactions as gauge invariance in gravity, which enables the S-matrix to be constructible. The studies in Chap. 2 and 3 demonstrate both the potential and limitations of recursion relations as a self-contained formulation of quantum field theory.

In Chap. 4, we classify EFTs by the soft limits of their scattering amplitudes. This unifies seemingly different EFTs into a periodic table, calling for a deeper connection among these EFTs. Using only the factorization and soft limits of the S-matrix, we carve out whole swaths of EFT space, making the space of interesting theories very limited. This is analogous to the "four-particle test" in [35, 36], which rules out higher spin theories based on on-shell consistencies. Our proof relies heavily on the recursion relations for EFTs developed in Chap. 3, since they have soft limits built into the expressions. This no-go theorem establishes the NLSM, Dirac Born-Infield, and the special Galileon as the unique exceptional theories with enhanced soft limits, that are the scalar analogs of YM and gravity whose structures are fixed by gauge invariance.

In Chap. 5, we further investigate the connection between the previously mentioned EFTs and their resemblance to YM and gravity. The same squaring relation observed in gauge and gravity turns out also occur between the NLSM and the special Galileon. This serves as a novel example of double copy relations, given that EFTs do not appear in string theory as easily as in YM and gravity, suggesting a pure field theory origin. The full explanation of double copy structure remains an open question (cf. [37] and references therein).

Without the simultaneous complications of gauge invariance and field redefinition, we first propose an action for the NLSM to manifest the flavor-kinematics duality. The Feynman vertices in this action serve as the structure constant of the associated kinematic algebra. The action enables us to identify the symmetry origin for flavor-kinematics duality. As a byproduct, we show the Adler zero of pions is related to the Weinberg soft theorem in YM. These further strengthen the connection between these EFTs and YM/gravity.

In Chap. 6 we demonstrate a novel example of the power of the S-matrix program at loop level. Technical naturalness dictates that all operators not forbidden by symmetry are compulsory—and thus generated by renormalization. The vanishing of ultraviolet divergences are in turn a telltale sign of underlying symmetry. This is famously true in supersymmetry, where holomorphy enforces powerful non-renormalization theorems. Recent calculations in the standard model effective field theory [38] found miraculous non-renormalization among many of them which raises the question: "Is there any new symmetry in the standard model?"

We prove and generalize this non-renormalization to all higher dimensional operators in generic field content in four spacetime dimensions. However, this non-renormalization is not a consequence of new symmetry. The key is to map operator mixing into 1-loop amplitudes. The mixing occurs only if the associated 1-loop amplitude has non-vanishing cuts as products of tree amplitudes. However, there are certain helicity selection rules which forbid certain tree amplitudes and the associated 1-loop process. The results can be summarized into simple rules that constrain the renormalization of general EFT. Once again, we see the seemingly miraculous results in quantum field theory have neat derivations in terms of scattering amplitudes.

Chapter 2

RECURSION RELATIONS IN FOUR DIMENSIONS

2.1 Introduction

On-shell recursion relations are a powerful tool for calculating tree-level scattering amplitudes in quantum field theory. Practically, they are far more efficient than Feynman diagrams. Formally, they offer hints of an alternative boundary formulation of quantum field theory grounded solely in on-shell quantities. To date, there has been enormous progress in computing tree-level scattering amplitudes in various gauge and gravity theories with and without supersymmetry.

In this chapter we ask: *to what extent do on-shell recursion relations define quantum field theory?* Conversely, for a given quantum field theory, what is the minimal recursion relation, if any, that constructs all of its amplitudes? Here an amplitude is "constructible" if it can be recursed down to lower point amplitudes, while a theory is "constructible" if all of its amplitudes are either constructible or one of a finite set of seed amplitudes which initialize the recursion.

For our analysis we define a "covering space" of recursion relations, shown in Eq. (2.2), which includes natural generalizations of the BCFW [4] and Risager [39] recursion relations. These generalizations, defined in Eq. (2.12) and Eq. (2.13), intersect at a new "soft" recursion relation, defined in Eq. (2.15), that probes the infrared structure of the amplitude.

As usual, these recursion relations rely on a complex deformation of the external momenta parameterized by a complex number z. By applying Cauchy's theorem to the complexified amplitude, $\mathcal{M}(z)$, one relates the original amplitude to the residues of poles at complex factorization channels, plus a boundary term at $z = \infty$ which is in general incalculable. Consequently, an amplitude can be recursed down to lower point amplitudes if it vanishes at large z and no boundary term exists.

The central aim of this chapter is to determine the conditions for on-shell constructibility by determining when the boundary term vanishes for a given

Theory	YM	YM + ψ	YM + ϕ	YM + ψ + ϕ	Yukawa	Scalar	SUSY	SM
m	2	2	5 (3)	5 (3)	3	5 (3)	3	3

Table 2.1: Summary of the minimal m-line recursion relation needed to construct all scattering amplitudes in various renormalizable theories: Yang-Mills with matter of diverse spins and arbitrary representations, Yukawa theory, scalar theory, supersymmetric theories, and the standard model. The values in parentheses apply if every scalar has equal charge under a $U(1)$ symmetry. Here ϕ and ψ denote scalars and fermions, respectively.

amplitude. We define the large z behavior, γ, of an amplitude by

$$\mathcal{M}(z \to \infty) = z^{\gamma}, \tag{2.1}$$

for an n-point amplitude under a general m-line momentum shift, where $m \leq n$. Inspired by Ref. [40], we rely crucially on the fact that the large z limit describes the scattering of m hard particles against $n - m$ soft particles. Hence, the large z behavior of the n-point amplitude is equal to the large z behavior of an m-point amplitude computed in the presence of a soft background. Fortunately, explicit m-point amplitudes need not be computed, as γ can be stringently bounded simply from dimensional analysis, Lorentz invariance, and locality, yielding the simple formulas in Eq. (2.26), Eq. (2.27), Eq. (2.29), and Eq. (2.32). From these large z bounds, it is then possible to determine the minimal m-line recursion relation needed to construct an n-point amplitude for any given theory. If every amplitude, modulo the seeds, are constructible, then we define the theory to be m-line constructible.

Our results apply to a general quantum field theory of massless particles in four dimensions, which we now summarize as follows:

Renormalizable Theories

- Amplitudes with arbitrary external states are 5-line constructible.
- Amplitudes with any external vectors or fermions are 3-line constructible.
- Amplitudes with only external scalars are 3-line constructible if there is a $U(1)$ symmetry under which every scalar has equal charge.
- The above claims imply 5-line constructibility of all renormalizable quantum field theories and 3-line constructibility of all gauge theories with

fermions or complex scalars in arbitrary representations, all supersymmetric theories, and last but not least the standard model. The associated recursion relations are defined in Eq. (2.12) and Eq. (2.13).

Non-renormalizable Theories

- Amplitudes are m-line constructible for $(m-1)$-valent interactions without derivatives.
- Amplitudes are constructible for interactions with derivatives up to a certain order in the derivative expansion.
- The above claims imply m-line constructibility of all scalar and fermion $\phi^{m_1}\psi^{m_2}$ theories for $m_1+m_2 = m-1$, and of certain amplitudes in higher derivative gauge and gravity theories. The associated recursion relations are defined in Eq. (2.2).

Constructibility conditions for some familiar cases are presented in Tab. 2.1. These cases fully span the space of all renormalizable theories.

As we will see, our covering space of recursion relations naturally bifurcates according to the number of z poles in each factorization channel: one or two. For the former, the recursion relations take the form of standard shifts such as BCFW and Risager, which is the case for the 5-line and 3-line shifts employed for renormalizable theories. For the latter, the recursion relations take a more complicated form which is more cumbersome in practice, but necessary for some of the non-renormalizable theories.

The remainder of our chapter is as follows. In Sec. 2.2, we present a covering space of recursion relations for an m-line shift of an n-point amplitude, taking note of the generalizations of the BCFW and Risager momentum shifts. Next, we compute the large z behavior for these momentum shifts in Sec. 2.3. Afterwards, in Sec. 2.4, we present our main result, which is a classification of the minimal recursion relations needed to construct various renormalizable and non-renormalizable theories. Finally, we discuss examples in Sec. 2.5 and conclude in Sec. 2.6.

2.2 Covering Space of Recursion Relations

Definition

Let us now define a broad covering space of recursion relations subject to a loose set of criteria. In particular, we demand that the external momenta remain on-shell and conserve momenta for all values of z. In four dimensions, these conditions are automatically satisfied if the momentum deformation is a complex shift of the holomorphic and anti-holomorphic spinors of external legs[1],

$$\begin{aligned} \lambda_i &\rightarrow \lambda_i(z) = \lambda_i + z\eta_i, \quad i \in \mathcal{I} \\ \widetilde{\lambda}_i &\rightarrow \widetilde{\lambda}_i(z) = \widetilde{\lambda}_i + z\widetilde{\eta}_i, \quad i \in \widetilde{\mathcal{I}}, \end{aligned} \tag{2.2}$$

where η_i and $\widetilde{\eta}_i$ are reference spinors that may or may not be identified with those of external legs, and $\mathcal{I}$ and $\widetilde{\mathcal{I}}$ are disjoint subsets of the external legs. As shorthand, we will refer to the shift in Eq. (2.2) as an $[\widetilde{\mathcal{I}}, \mathcal{I}\rangle$-line shift. When the specific elements of $\mathcal{I}$ and $\widetilde{\mathcal{I}}$ are not very important, we will sometimes refer to this as an $[|\widetilde{\mathcal{I}}|, |\mathcal{I}|\rangle$-line shift, where the labels are the orders of $\mathcal{I}$ and $\widetilde{\mathcal{I}}$. For an m-line shift, $m = |\mathcal{I}| + |\widetilde{\mathcal{I}}|$. In this notation, the BCFW and Risager shifts are $[1,1\rangle$-line and $[3,0\rangle$-line shifts, respectively.

As we will see, the efficacy of recursion relations depends sensitively on the correlation between the helicity of a particle and whether its holomorphic or anti-holomorphic spinor is shifted. Throughout, we will define "good" and "bad" shifts according to the choices

$$(\mathcal{I}, \widetilde{\mathcal{I}}) = \begin{cases} (+,-), & \text{good shift} \\ (-,+), & \text{bad shift.} \end{cases} \tag{2.3}$$

For example, the bad shift for the case of BCFW yields a non-vanishing contribution at large z in non-supersymmetric gauge theories.

The resulting tree amplitude, $\mathcal{M}(z)$, is then complexified, but the original amplitude, $\mathcal{M}(0)$ is obtained by evaluating the contour integral $\oint dz\, \mathcal{M}(z)/z$ for a contour encircling $z = 0$. An on-shell recursion relation is then obtained by applying Cauchy's theorem to deform the contour out to $z = \infty$, in the process picking up all the residues of $\mathcal{M}(z)$ in the complex plane.

[1]There is a more general class of shifts in which both λ_i and $\widetilde{\lambda}_i$ are shifted for every particle. However, in the case momentum conservation imposes complicated non-linear relations among reference spinors which makes the study of large z behavior difficult.

As noted earlier, the momentum conservation must apply for arbitrary values of z, implying

$$\sum_{i\in\mathcal{I}} \eta_i \widetilde{\lambda}_i + \sum_{i\in\widetilde{\mathcal{I}}} \lambda_i \widetilde{\eta}_i \quad = \quad 0, \tag{2.4}$$

which should be considered as four constraints on η_i and $\widetilde{\eta}_i$, which are easily satisfied provided the number of reference spinors is sufficient.

Factorization

Next, consider a factorization channel of a subset of particles $\mathcal{F}$. The complex deformation of the momenta in Eq. (2.2) sends

$$P \to P(z) = P + zQ, \tag{2.5}$$

where P is the original momentum flowing through the factorization channel and Q is the net momentum shift, so

$$P = \sum_{i\in\mathcal{F}} \lambda_i \widetilde{\lambda}_i, \qquad Q = \sum_{i\in\mathcal{F}_\lambda} \eta_i \widetilde{\lambda}_i + \sum_{i\in\mathcal{F}_{\widetilde{\lambda}}} \lambda_i \widetilde{\eta}_i, \tag{2.6}$$

where $\mathcal{F}_\lambda$ and $\mathcal{F}_{\widetilde{\lambda}}$ are intersection of $\mathcal{F}$ with $\mathcal{I}$ and $\widetilde{\mathcal{I}}$.

As we will see, the physics depends crucially on whether Q^2 vanishes for all factorization channels or not. First of all, the large z behavior is affected because propagators in the complexified amplitude scale as

$$\frac{1}{(P+zQ)^2} \quad = \quad \begin{cases} z^{-1} \quad , \quad Q^2 = 0 \\ z^{-2} \quad , \quad Q^2 \neq 0 \end{cases}, \tag{2.7}$$

for a given factorization channel. Second, there is a very important difference in the structure of the recursion relation depending on whether Q^2 vanishes in all channels. If so, then each factorization channel has a simple pole at

$$z_* \quad = \quad -P^2/2P\cdot Q, \tag{2.8}$$

and the on-shell recursion relation takes the usual form,

$$\mathcal{M}(0) \quad = \quad \sum_{\mathcal{F}} \frac{1}{P^2} \mathcal{M}_{\mathcal{F}}(z_*) \mathcal{M}_{\bar{\mathcal{F}}}(z_*) + (\text{pole at } z = \infty), \tag{2.9}$$

where the sum is over all factorization channels and intermediate states, and $\mathcal{M}_{\mathcal{F}}$ and $\mathcal{M}_{\bar{\mathcal{F}}}$ are on-shell amplitudes corresponding to each side of the factorization channel. However, if Q^2 does not vanish, then each propagator is a quadratic in z and thus carries conjugate poles at

$$z_{\pm} \;=\; \frac{-P\cdot Q \pm \sqrt{(P\cdot Q)^2 - P^2Q^2}}{Q^2}. \tag{2.10}$$

Summing over both of these roots, we find a new recursion relation,

$$\begin{aligned}\mathcal{M}(0) \;=\; & \sum_{\mathcal{F}} \frac{1}{P^2}\left[\frac{z_+\mathcal{M}_{\mathcal{F}}(z_-)\mathcal{M}_{\bar{\mathcal{F}}}(z_-) - z_-\mathcal{M}_{\mathcal{F}}(z_+)\mathcal{M}_{\bar{\mathcal{F}}}(z_+)}{z_+ - z_-}\right] \\ & +(\text{pole at } z=\infty).\end{aligned} \tag{2.11}$$

Under conjugation of the roots, $z_+ \leftrightarrow z_-$, the summand is symmetric, so crucially, square roots always cancel in the final expression in the recursion relation. Of course, the intermediate steps in the recursion are nevertheless quite cumbersome in this case.

Recursion Relations

All known recursion relations can be constructed by imposing additional constraints on the momentum shift in Eq. (2.2) beyond the condition of momentum conservation in Eq. (2.4). In the absence of extra constraints, the reference spinors η_i and $\widetilde{\eta}_i$ are arbitrary so by Eq. (2.6), $Q^2 \neq 0$ generically. In this case the recursion relation will have square roots in intermediate steps.

On the other hand, if $Q^2 = 0$, then Q must be factorized into the product of two spinors. If Q is factorizable, then in the summand of Eq. (2.6) either the η_i and λ_i are proportional or the $\widetilde{\eta}_i$ and $\widetilde{\lambda}_i$ are proportional. For general external kinematics, *i.e.* the λ_i and $\widetilde{\lambda}_i$ are independent, these proportionality conditions can involve at most one external spinor. As we will see, this implies two distinct classes of recursion relation which can accommodate $Q^2 = 0$.

The first possibility is to shift only holomorphic spinors or only anti-holomorphic spinors subject to the constraint that the $\eta_i = c_i\eta$ and $\widetilde{\eta}_i = \widetilde{c}_i\widetilde{\eta}$ are all proportional to universal reference spinors η and $\widetilde{\eta}$. In each case, Eq. (2.6) factorizes into the form $Q = \eta(\ldots)$ and $Q = (\ldots)\widetilde{\eta}$, respectively. In mathematical terms, these scenarios correspond to the $[0, m\rangle$-line and $[m, 0\rangle$-line shifts,

$$\begin{aligned}
[0,m\rangle\text{-line:} \quad & \begin{cases} \lambda_i \to \lambda_i(z) = \lambda_i + z c_i \eta, \quad i \in \mathcal{I} \\ \sum_{i\in\mathcal{I}} c_i \widetilde{\lambda}_i = 0 \end{cases} \\
[m,0\rangle\text{-line:} \quad & \begin{cases} \widetilde{\lambda}_i \to \widetilde{\lambda}_i(z) = \widetilde{\lambda}_i + z \widetilde{c}_i \widetilde{\eta}, \quad i \in \widetilde{\mathcal{I}} \\ \sum_{i\in\widetilde{\mathcal{I}}} \widetilde{c}_i \lambda_i = 0 \end{cases}
\end{aligned} \tag{2.12}$$

where the constraints on c_i and $\widetilde{c}_i$ arise from momentum conservation. Of course, the $[0,m\rangle$-line and $[m,0\rangle$-line shifts are simply generalizations of the Risager shift with the only difference that here $m \leq n$ is arbitrary.

The second possibility is to shift only holomorphic spinors except for one or only anti-holomorphic spinors except for one. In this case the reference spinors must be proportional to a spinor of a specific external leg, which we denote here by λ_j or $\widetilde{\lambda}_j$. Thus, in each case, $\eta_i = c_i\lambda_j$ and $\widetilde{\eta}_i = \widetilde{c}_i\widetilde{\lambda}_j$, so we again have factorization, but of the form $Q = \lambda_j(\ldots)$ and $Q = (\ldots)\widetilde{\lambda}_j$. These correspond to $[1, m-1\rangle$-line and $[m-1, 1\rangle$-line shifts,

$$\begin{aligned}
&[1, m-1\rangle\text{-line:} \qquad \begin{cases} \lambda_i \to \lambda_i(z) = \lambda_i + zc_i\lambda_j, & i \in \mathcal{I} \\ \widetilde{\lambda}_j \to \widetilde{\lambda}_j(z) = \widetilde{\lambda}_j - z\sum_{i\in\mathcal{I}} c_i\widetilde{\lambda}_i, & j = \widetilde{\mathcal{I}} \end{cases} \\
&[m-1, 1\rangle\text{-line:} \qquad \begin{cases} \widetilde{\lambda}_i \to \widetilde{\lambda}_i(z) = \widetilde{\lambda}_i + z\widetilde{c}_i\widetilde{\lambda}_j, & i \in \widetilde{\mathcal{I}} \\ \lambda_j \to \lambda_j(z) = \lambda_j - z\sum_{i\in\widetilde{\mathcal{I}}} \widetilde{c}_i\lambda_i, & j = \mathcal{I} \end{cases}
\end{aligned} \tag{2.13}$$

where we have chosen a form such that momentum conservation is automatically satisfied. Note that the case $m = 2$ corresponds precisely to BCFW, so these shifts are a generalization of BCFW to arbitrary $m \leq n$.

Note that for $m \leq 3$, any momentum shift is necessarily of the form of the first or second possibility, so $Q^2 = 0$ automatically. Thus, $Q^2 \neq 0$ is only possible if $m > 3$.

Remarkably, while the recursion relations in Eq. (2.12) and Eq. (2.13) are naturally the generalizations of Risager and BCFW, they actually overlap for a specific choice of reference variables! In particular, consider the $[0,m\rangle$-line and $[m,0\rangle$-line shifts in Eq. (2.12) for the case of $\eta = \lambda_j$ and $\widetilde{\eta} = \widetilde{\lambda}_j$, and modifying the constraint from momentum conservation such that $\sum_{i\in\mathcal{I}} c_i\widetilde{\lambda}_i = \widetilde{\lambda}_j$ and $\sum_{i\in\widetilde{\mathcal{I}}} \widetilde{c}_i\lambda_i = \lambda_j$, respectively. In this case the recursion coincides with the form of the $[1, m-1\rangle$-line and $[m-1, 1\rangle$-line shifts in Eq. (2.13), with a curious feature that $\lambda_j(z) = \lambda_j(1-z)$ and $\widetilde{\lambda}_j(z) = \widetilde{\lambda}_j(1-z)$. We dub these "soft" shifts for the simple reason that when $z = 1$ the amplitude approaches a soft limit. For $m = 3$, the soft shift takes a particularly elegant form,

$$\text{3-line soft shift:} \quad \begin{cases} \lambda_1 \to \lambda_1(z) = \lambda_1 + z\frac{[23]}{[21]}\lambda_3 \\ \lambda_2 \to \lambda_2(z) = \lambda_2 + z\frac{[13]}{[12]}\lambda_3 \\ \widetilde{\lambda}_3 \to \widetilde{\lambda}_3(z) = \widetilde{\lambda}_3(1-z) \end{cases} , \text{ or} \tag{2.14}$$

$$\begin{cases} \widetilde{\lambda}_1 \to \widetilde{\lambda}_1(z) = \widetilde{\lambda}_1 + z\frac{\langle 23\rangle}{\langle 21\rangle}\widetilde{\lambda}_3 \\ \widetilde{\lambda}_2 \to \widetilde{\lambda}_2(z) = \widetilde{\lambda}_2 + z\frac{\langle 13\rangle}{\langle 12\rangle}\widetilde{\lambda}_3 \\ \lambda_3 \to \lambda_3(z) = \lambda_3(1-z) \end{cases} . \tag{2.15}$$

This shift offers an on-shell prescription for taking a soft limit. We will not make use of this shift in this chapter but leave a more thorough analysis of this soft shift for future work.

2.3 Large z Behavior of Amplitudes

The recursion relations in Eq. (2.9) and Eq. (2.11) apply when the amplitude does not have a pole at $z = \infty$. In this section we determine the conditions under which this boundary term vanishes. Although one could study the boundary term in BCFW or Risager shift instead, as in Ref. [41, 42], we will not proceed in this direction. Concretely, take the n-point amplitude, $\mathcal{M}$, deformed by an m-line shift where $m \leq n$. At large z, the shifted amplitude describes the physical scattering of m hard particles in a soft background parametrizing the remaining $n - m$ external legs. Thus, we can determine the large z behavior by applying a background field method: we expand the original Lagrangian in terms of soft backgrounds and hard propagating fluctuations, then compute the on-shell m-point "skeleton" amplitude, $\widetilde{M}$. If the skeleton amplitude vanishes at large z, then the boundary term is absent and the recursion relation applies. A similar approach was applied in Ref. [40] for BCFW for the case of a hard particle propagator, *i.e.* the skeleton amplitude for $m = 2$.

Crucially, it will not be necessary to explicitly compute the skeleton amplitude. Rather, from Lorentz invariance, dimensional analysis, and the assumption of local poles, we will derive general formulae for the large z behavior of m-line shifts of n-point amplitudes. Hence, our calculation of the large z scaling combines and generalizes two existing proofs in the literature relating to the BCFW [40] and all-line recursion relations[8].

Ansatz

The basis of our calculation is a general ansatz for the m-point skeleton amplitude for $m \leq n$,

$$\widetilde{\mathcal{M}} = \widetilde{g} \times \sum_{\text{diagrams}} \left(F \times \prod_{\text{vectors}} \varepsilon \times \prod_{\text{fermions}} u \right) \tag{2.16}$$

where the sum is over Feynman diagrams F, which are contracted into products over the polarization vectors ε and fermion wavefunctions u of the hard particles[2]. Here $\widetilde{g} = g \times B$ where g is a product of Lagrangian coupling constants and B is a product of soft field backgrounds and their derivatives. Note that $\widetilde{g}$ has free Lorentz indices since it contains insertions of the soft background fields and their derivatives. Crucially, since B is composed of backgrounds, it is always non-negative in dimension, so $[B] \geq 0$ and

$$[\widetilde{g}] = [g] + [B] \geq [g]. \tag{2.17}$$

For the special case of gravitational interactions, each insertion of the background graviton field is accompanied by an additional coupling suppression of by the Planck mass, so $[\widetilde{g}] = [g]$. This is reasonable because the background metric is naturally dimensionless so insertions of it do not change the dimensions of the overall coupling.

Note the skeleton amplitude receives dimensionful contributions from every term in Eq. (2.16) except the vector polarizations, so

$$[\widetilde{\mathcal{M}}] = 4 - m = [\widetilde{g}] + [F] + \sum_{\text{fermions}} 1/2, \tag{2.18}$$

via dimensional analysis. This fact will be crucial for our calculation of the large z scaling of the skeleton amplitude for various momentum shifts and theories.

Large z Behavior

We analyze the large z behavior of Eq. (2.16). The contribution from each Feynman diagram F can be expressed as a ratio of polynomials in momenta, so $F = N/D$. Here N arises from interactions while D arises from propagators. We define the large z behavior of the numerator and denominator as γ_N and γ_D where

$$N \sim z^{\gamma_N}, \qquad D \sim z^{\gamma_D}. \tag{2.19}$$

[2]Note that polarization vectors arise from any particle of spin greater than or equal to one.

We now compute the large z behavior of the external wavefunctions, followed by that of the Feynman diagram numerator and denominator, and finally the full amplitude.

External Wavefunctions. First, we study the contributions from external polarization vectors and fermion wavefunctions. For convenience, we define a "weighted" spin, $\widetilde{s}$, for each shifted leg of +/− helicity, which is simply the spin s multiplied by + if the angle/square bracket is shifted and − if the square/angle bracket is shifted. In mathematical terms,

$$\widetilde{s} \;=\; s \times \begin{cases} +, & \text{good shift} \\ -, & \text{bad shift} \end{cases}, \tag{2.20}$$

where good and bad shifts denote the correlation between helicity and the shift of spinor indicated in Eq. (2.3). As we will see, a multiplier of +/− tends to improve/worsen the large z behavior. In terms of the weighted spin, it is now straightforward to determine how the large z scaling of the polarization vectors and fermion wavefunctions,

$$\text{external wavefunction} \;\sim\; \begin{cases} z^{-\widetilde{s}}, & \text{boson} \\ z^{-(\widetilde{s}-1/2)}, & \text{fermion} \end{cases}. \tag{2.21}$$

So more positive values of $\widetilde{s}$, corresponding to good shifts, imply better large z convergence.

Numerator and Denominator. The numerator N of each Feynman diagram depends sensitively on the dynamics. However, for a generic shift, we can conservatively assume no cancellation in large z so the numerator scales at most as its own mass dimension,

$$\gamma_N \leq [N]. \tag{2.22}$$

The denominator D comes from propagators which are fully dictated by the topology of the diagram. Each propagator can scale as $1/z^2$ or $1/z$ at large z, depending on the details of shifts. Thus, the large z behavior of the denominator is constrained to be within

$$\frac{[D]}{2} \leq \gamma_D \leq [D]. \tag{2.23}$$

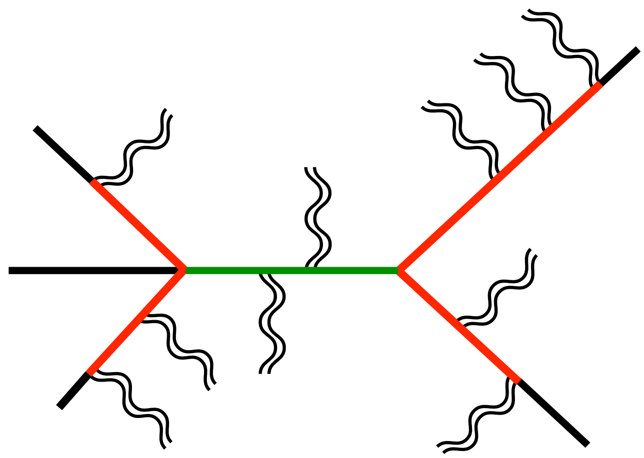


Figure 2.1: A skeleton diagram for a $Q^2 \neq 0$ shift. Here straight lines are hard particles and curved lines are soft backgrounds. Color segments are propagators, and red and green denotes those that scale as $1/z$ and $1/z^2$ at large z, respectively.

For the $Q^2 = 0$ shifts, every propagator scales as $1/z$ so $\gamma_D = [D]/2$. On the other hand, for the $Q^2 \neq 0$ shifts, we would naively expect that there is a $1/z^2$ from each propagator given that the reference spinors are arbitrary. However, this reasoning is flawed due to an important caveat. Since the theory contains soft backgrounds, the Feynman diagram can have 2-point interactions of the hard particle induced by an insertion of the soft background. If the 2-point interactions occur before the hard particle interacts with another hard particle, then Q is simply the momentum shift of a single external leg, so $Q^2 = 0$ accidentally, and the corresponding propagator scales as $1/z$ rather than $1/z^2$. It is simple to see that the number of such propagators is $[D] - \gamma_D$. See Fig. 2.1 for an illustration of this effect. Thus the large z behavior is constrained within the range of Eq. (2.23).

From our knowledge of Feynman diagrams, we can further relate the total number of propagators to the number of hard external legs, m, and the valency of the interactions, v, yielding

$$\frac{[D]}{2} \leq \left(\frac{m-v}{v-2}\right) + [B], \tag{2.24}$$

where $v \geq 3$ is the valency of the interaction vertices in the fundamental theory and the $[B]$ term arises because we have conservatively assumed that every single background field insertion contributes to a 2-point interaction to the amplitude.

Full Amplitude. Combing in the large z scaling of the external wavefunctions in Eq. (2.21) with that of the numerator and denominator of the the Feynman diagram in Eq. (2.19), we obtain

$$\begin{aligned}\gamma &= \gamma_N - \gamma_D - \sum_{\text{bosons}} \widetilde{s} - \sum_{\text{fermions}} (\widetilde{s} - 1/2) \\ &\leq 4 - m - [g] - \sum_{\text{all}} \widetilde{s} + [D] - \gamma_D - [B],\end{aligned} \tag{2.25}$$

where in the second line we have plugged in the inequality from Eq. (2.22), replaced $[N] = [F] + [D]$, and eliminated $[F]$ by solving Eq. (2.18). This is the master formula from which we will derive corresponding large z behaviors in $Q^2 \neq 0$ and $Q^2 = 0$ shifts. As expected, the above bound can be improved for $Q^2 = 0$ shifts because in this case the product of any two hard momenta only scales as z rather than z^2. We render the specific derivation in subsequent sections.

The general formula in Eq. (2.25) can be reduced to more illuminating forms by making the assumption of specific shifts. We consider the large z behavior for the $Q^2 \neq 0$ and $Q^2 = 0$ shifts in turn.

($Q^2 \neq 0$)

To start, we calculate the large z behavior for a general momentum shift defined in Eq. (2.2). As noted earlier, for arbitrary reference spinors, $Q^2 \neq 0$ as long as $m \geq 3$, which we assume here. The large z behavior is given by Eq. (2.25). The offset $[D] - \gamma_D$ is the number of propagators with $Q^2 = 0$ as discussed before. As shown for an example topology in Fig. 2.1, there is at least one soft background associated with each propagator for which $Q^2 = 0$. The canonical dimensions of fields leads to $[D] - \gamma_D - [B] \leq 0$. We conclude that

$$\boxed{\gamma \leq 4 - m - [g] - \sum_{\text{all}} \widetilde{s}.} \tag{2.26}$$

The large z convergence is best for the largest possible value for $\widetilde{s}$, which occurs if we only apply good shifts to external legs, so $\widetilde{s} = s$. As we will see, this particular choice has the best large z behavior of any shift. There is an inherent connection between $Q^2 \neq 0$ and improved z behavior of the amplitude, simply because in this case, propagators fall off with z^2 in diagrams.

($Q^2 = 0$)

Next, we compute the large z behavior of the momentum shift in Eq. (2.2) when $Q^2 = 0$. In these shifts, substituting $\gamma_D = [D]/2$ and Eq. (2.24) into Eq. (2.25) yields

$$\boxed{\gamma \leq 1 - \left(\frac{v-3}{v-2}\right)(m-2) - [g] - \sum_{\text{all}} \widetilde{s}.} \tag{2.27}$$

For trivalent interactions, $v = 3$, the bound is independent of m. For quadrivalent vertices, $v = 4$, the bound improves for larger numbers of shifted legs, m.

We showed previously that $Q^2 = 0$ can only occur for the $[0, m\rangle$-, $[m, 0\rangle$-, $[1, m-1\rangle$-, and $[m-1, 1\rangle$-line shifts defined in Eq. (2.12) and Eq. (2.13). Hence, we can learn more by considering the specific form of the large z shifts. In the subsequent sections we consider each of these cases in turn to derive additional bounds on the large z behavior.

$[0, m\rangle$-Line and $[m, 0\rangle$-Line Shifts. The $[0, m\rangle$-line and $[m, 0\rangle$-line shifts defined in Eq. (2.12) are a generalization of the Risager momentum shift, for which $Q^2 = 0$. To begin, let us consider the large z behavior of the $[0, m\rangle$-line shift; an identical argument will of course hold for the $[m, 0\rangle$-line shift. We only have to keep track of holomorphic spinors, since anti-holomorphic spinors are not shifted. To conservatively bound the large z behavior of the numerator of Eq. (2.16), we can simply sum the total number of holomorphic spinors and divide by two, since the reference spinors are proportional and thus vanish when dotted into each other. However, note that we must remember to count the holomorphic spinors coming from the numerator N as well as from the soft background B and external wavefunctions. Overall Eq. (2.21) gives the correct number of holomorphic spinors. Including all contributions yields

$$\gamma \;\leq\; \frac{1}{2}\left([N] + n_B - \sum_{\text{bosons}} \widetilde{s} - \sum_{\text{fermions}} (\widetilde{s} - 1/2) - [D]\right), \tag{2.28}$$

where n_B is the number of holomorphic spinor indices that come from soft background insertions. Again solving for $[F]$ with Eq. (2.18), and applying our arguments to both shifts, the large z behavior is

$$\boxed{\gamma \leq \begin{cases} \frac{1}{2}\left(4 - m - [g] - \sum\limits_{\text{all}} h + \Delta\right), & [0, m\rangle\text{-line} \\ \frac{1}{2}\left(4 - m - [g] + \sum\limits_{\text{all}} h + \Delta\right), & [m, 0\rangle\text{-line} \end{cases}} \tag{2.29}$$

where h denotes helicity and we have defined

$$\Delta = n_B - [B]. \tag{2.30}$$

In a theory with only spin $s \leq 1$ fields, soft background insertions contribute at most one holomorphic or anti-holomorphic spinor index to be contracted with. Thus, n_B is balanced by the dimension $[B]$, so $\Delta \leq 0$ in these theories. On the other hand, for a theory with spin $s \leq 2$ fields, *e.g.,* gravitons, then an insertion of a graviton background yields two spinor indices but only with one power of mass dimension. For these two cases we thus find

$$\Delta \quad \leq \quad \begin{cases} 0, & \text{theories with } s \leq 1 \\ n - m, & \text{theories with } s \leq 2 \end{cases} . \tag{2.31}$$

Eqs. (2.29) and (2.31) together give our final answer. For an all-line shift, $m = n$, so $\Delta = 0$ and this bound reduces to known result from Ref. [8]. Note that in some cases Eq. (2.27) is stronger than Eq. (2.29) so we have to consider both bounds at the same time.

$[1, m{-}1\rangle$-Line and $[m{-}1, 1\rangle$-Line Shifts. The $[1, m{-}1\rangle$-line and $[m{-}1, 1\rangle$-line shifts defined in Eq. (2.13) are a generalization of the BCFW momentum shift, for which $Q^2 = 0$. To start, consider a $[1, m{-}1\rangle$-line shift, where particle j has a shifted in anti-holomorphic spinor and all other shifts are on holomorphic spinors. To determine the large z behavior of the $[1, m-1\rangle$-line shift, we start with our earlier result on the $[0, m\rangle$-line shift. By switching the deformation on particle j from a shift of $|j]$ to a shift of $|j\rangle$, all the angle brackets associated with j change their scaling from 1 to z at large z for generic choice of $\widetilde{c}_i$ in Eq. (2.13). In the mean time, all square brackets involving particle j reduce from z to 1 because the reference spinor is $|j]$. The change in large z behavior from a $[0, m\rangle$-line shift to a $[1, m-1\rangle$-line shift is exactly the difference of the degrees between anti-holomorphic and holomorphic spinors of j, which is fixed by little group. Applying the reasoning to both shifts, we obtain

$$\boxed{\gamma \leq \begin{cases} \frac{1}{2}\left(4 - m - [g] - \sum\limits_{\text{all}} h + \Delta\right) + 2h_j, & [1, m-1\rangle\text{-line} \\ \frac{1}{2}\left(4 - m - [g] + \sum\limits_{\text{all}} h + \Delta\right) - 2h_j, & [m-1, 1\rangle\text{-line} \end{cases}} \tag{2.32}$$

where h_j is the helicity of particle j. We then see that the $[1, m-1\rangle$-line shift improves large z behavior of the $[0, m\rangle$-line shift if $h_j > 0$.

The above argument has a caveat in the special case of the $[1, 1\rangle$-line shift, *i.e.* the BCFW shift. Shifting the anti-holomorphic spinor of particle i and the holomorphic spinor of particle j, then the angle bracket $\langle ij\rangle$ does not scale as z at large z so Eq. (2.32) does not apply. Nevertheless, we can still use Eq. (2.27) which is valid for BCFW shift.

2.4 On-Shell Constructible Theories

In this section we at last address the question posed in the introduction: *what is the simplest recursion relation that constructs all on-shell tree amplitudes in a given theory?* To find an answer we consider the $Q^2 \neq 0$ momentum shift defined in Eq. (2.2) and the $Q^2 = 0$ momentum shifts defined in Eq. (2.12) and Eq. (2.13). We utilize our results for the large z behavior in Eq. (2.26), Eq. (2.27), Eq. (2.29), and Eq. (2.31). Throughout the rest of the chapter we restrict to the good momentum shifts defined in Eq. (2.3). Thus, we only shift the holomorphic spinors of plus helicity particles and the anti-holomorphic spinors of negative helicity particles, and the weighted spin of each leg is equal to its spin, $\tilde{s} = s$. Unless otherwise noted, we henceforth denote any scalar/fermion/gauge boson/graviton by $\phi/\psi/A/G$.

Renormalizable Theories

To begin we consider the generic momentum shift defined in Eq. (2.2), which has large z behavior derived in Eq. (2.26). Since a renormalizable theory only has marginal and relevant interactions, the mass dimension of the product of couplings in any scattering amplitude is $[g] \geq 0$. Plugging this into Eq. (2.26), we find that a 5-line shift suffices to construct any amplitude. This is also true for the 5-line shifts defined in Eq. (2.12) and Eq. (2.13), whose large z scaling is shown in Eq. (2.29) and Eq. (2.32) by conservatively plugging in $\Delta = 0$ for renormalizable theories. Consequently, 5-line recursion relations provide a purely on-shell, tree-level definition of any renormalizable quantum field theory. We must take as input the three and four point on-shell tree amplitudes, but this is quite reasonable, as a renormalizable Lagrangian is itself specified by interactions comprised of three or four fields.

Fortunately, simpler recursion relations are sufficient to construct a more restricted but still enormous class of renormalizable theories. To see this, consider a general 3-line momentum shift and its associated large z behavior shown in Eq. (2.27). The amplitude vanishes at large z provided the sum of the spins of the three shifted legs is greater than one. This is automatic if all three shifted particles are vectors or fermions. Such a shift can always be chosen unless the amplitude is composed of *i*) one vector and scalars, *ii*) two fermions and scalars, or *iii*) all scalars. In case *i*), we can apply a 3-line shift of the form $[\{\phi, \phi\}, \{A^+\}\rangle$ or $[\{A^-\}, \{\phi, \phi\}\rangle$, while in case *ii*), we can apply a 3-line shift of the form $[\{\phi, \phi\}, \{\psi^+\}\rangle$ or $[\{\psi^-\}, \{\phi, \phi\}\rangle$. In both cases the large z behavior is vanishing according to Eq. (2.32). Hence, any amplitude with an external vector or fermion is 3-line constructible.

This leaves case *iii*), which is the trickiest scenario: an amplitude with only external scalars. In general, such an amplitude is not 3-line constructible, but the story changes considerably if the scalars are covariant under a global or gauge $U(1)$ symmetry. Concretely, consider a 3-line shift of the form $[\{\phi, \phi, \phi\}, 0\rangle$ or $[0, \{\phi, \phi, \phi\}\rangle$. Moreover, let us assume that the shifted legs carry a net charge under the scalar $U(1)$ which is not equal to the charge of any other scalar in the spectrum. In this case, invariance under the scalar $U(1)$ requires that the amplitude has more than one additional external scalar with unshifted momenta. The charge cannot be accounted for by an external fermion with unshifted momenta, since the amplitude only has external scalars. From the perspective of the skeleton diagram describing the scattering of three hard particles in a soft background, the additional scalars correspond to more than one insertion of a soft scalar background, so as defined in Eq. (2.30), $\Delta < -1$. Thus, according to Eq. (2.29), the 3-line shift has vanishing large z behavior and the associated amplitudes are constructible. Note that the charge condition we have assumed is automatically satisfied if every scalar in the theory has equal charge under the scalar $U(1)$ and we shift three same-signed scalars.

It seems impossible for this 3-line recursion to construct all equal-charged $U(1)$ scalar amplitudes, especially with the presence of quartic potential. However, as three same-signed scalars are only available from six points, this 3-line recursion still takes three and four point amplitudes as seeds. The information of quartic potential still enters this special 3-line recursion. We will demonstrate with a simple ϕ^4 theory in the next section.

Putting everything together, we have shown that a 3-line shift can construct any amplitude with a vector or fermion, and any amplitude with only scalars if every scalar carries equal charge under a $U(1)$ symmetry. Immediately, this implies that any theory of solely vectors and fermions—*i*.e. any gauge theory with arbitrary matter content—is constructible[3]. Moreover, all amplitudes in Yukawa theory necessarily carry an external fermion, so these are likewise constructible. The standard model is also 3-line constructible simply because it has a single scalar—the Higgs boson—which carries hypercharge. Finally, we observe that all supersymmetric theories are constructible. The reason is that without loss of generality, the superpotential for such a theory takes the form $W = \lambda_{ijk}\phi_i\phi_j\phi_k$, where we have shifted away Polonyi terms and eliminated quadratic terms to ensure a massless spectrum. For such a potential there is a manifest R-symmetry under which every chiral superfield has charge 2/3. Consequently, all complex scalars in the theory have equal charge under the R-symmetry and all amplitudes are 3-line constructible. This then applies to theories with extended supersymmetry as well. The conditions for on-shell constructibility in some familiar theories is summarized in Tab. 2.1.

Non-renormalizable Theories

In what follows, we first discuss non-renormalizable theories which are constructible, *i.e.* for which all amplitudes can be constructed. As we will see, this is only feasible for a subset of non-renormalizable theories, so in general, the covering space of recursion relations does not provide an on-shell formulation of all possible theories. Second, we consider scenarios in which some but not all amplitudes are constructible within a given non-renormalizable theory. In many cases, amplitudes involving a finite number of higher dimension operator insertions can often be constructed by our methods.

Our analysis will depend sensitively on the dimensionality of coupling constants, which we saw earlier have a huge influence on the the large z behavior under momentum shifts. Table 2.2 summarizes the dimensions of coupling

[3]Note that such theories are constructible from BCFW, via a shift of any vector [43] or any same helicity fermions [44].

Theory	ϕ^v	ψ^v	F^v	R^v	Einstein (+ Maxwell)
$[g]$	$u(4-v)$	$u(4-3v/2)$	$u(4-2v)$	$2-n-2u(v-1)$	$2-n$

Table 2.2: The dimensionality of the coupling constant, $[g]$, for an n-point amplitude, where u denotes the number interaction vertices, which have minimal valency v.

constants in various theories[4]. Here v is the (minimal) valency of the vertex. F and R defined as vector field strength and Riemann tensor, respectively, and we have omitted indices and complex conjugations for simplicity. The superscript of an external state specifies its helicity. We keep the number of operator insertions, u, as a free parameter. At tree-level, it is constrained by the number of propagators, $u \leq [D]/2+1$, where $[D]/2$ is given in Eq. (2.24).

Constructible Theories. To start, consider a theory of scalars interacting via a ϕ^v operator. Following Eq. (2.24), and using that the dimensionality of backgrounds is positive, $[B] \geq 0$, we can bound the number of propagators by $[D]/2 \geq (m-v)/(v-2)$ for an m-point skeleton amplitude. The number of interaction vertices exceeds the number of propagators by one, so $u = [D]/2+1$. In an $[m,0\rangle$-line shift, substituting $[g] = u(4-v)$ from Table 2.2, and plugging into Eq. (2.29) with $\Delta = -[B] \leq 0$ for scalars, we have

$$\gamma \;\; \leq \;\; \frac{v-m}{v-2}. \tag{2.33}$$

Thus, we find that all amplitudes in ϕ^v theory are constructable for an $[m,0\rangle$-line shift where $m > v$ and the v point amplitude is taken as the input of the recursion relation[5]. Since the scalars have no spin, this large z also applies for the conjugate $[0,m\rangle$-line shift. Of course, this conclusion is completely obvious from the perspective of Feynman diagrams. In particular, since ϕ^v theory does not have any kinematic numerators, its amplitudes are constructible provided there is even one hard propagator, which happens as long as $m > v$.

Analogously, consider a theory of fermions interacting via ψ^v operators. Conservatively, we assume all soft fermions in the skeleton amplitude are emitted from $Q^2 = 0$ propagators

$$[D] - \gamma_D - [B] = \frac{n_f}{v-2} - \frac{3}{2}n_f, \tag{2.34}$$

[4]As pointed out in Ref. [8], we need to choose the highest dimension coupling if there are multiple of coupling constants.

[5]In fact, $m = v$ suffices to construct any amplitude with $v+1$ points or above. This can be derived if we treat soft background in $[D]/2$ more carefully.

where n_f is the number of soft fermion insertions. Substituting the above equation and the number of vertices $u = (m + n_f - 2)/(v - 2)$ into the large z behavior for a general m-line shift in Eq. (2.25), we find exactly the same expression for γ in Eq. (2.33). Thus, all amplitudes in ψ^v theory are constructible with generic m-line shift for $m > v$, and taking the v point amplitude as an input. Again, it is not surprising from Feynman diagrams. Note that we here required a general m-line shift with $Q^2 \neq 0$, such that the fermionic propagators $\not{P}/P^2$ scale as $1/z$ at large z. On the other hand, the recursion relation cannot work for a $Q^2 = 0$ momentum shift because the fermionic propagators do not fall off at large z.

It is straightforward to generalize the arguments above to a theory of scalars and fermions interacting via a $\phi^{v_1}\psi^{v_2}$. We find that this theory is fully constructible with a general m-line shift for $m > v_1 + v_2$.

Finally we consider perhaps the most famous constructible non-renormalizable theory: gravity. As is well-known, all tree-level graviton scattering amplitudes can be recursed via BCFW [40], taking the 3-point amplitudes as input. Still, let us see how each of our m-line shifts fare relative to BCFW. Throughout, we consider only good shifts, as defined in Eq. (2.3). Using Eq. (2.26) and Eq. (2.27), the large z behaviors of m-line shifts are

$$\gamma \;\leq\; \begin{cases} n + 2 - 3m, & Q^2 \neq 0 \text{ shift} \\ n - 1 - 2m, & Q^2 = 0 \text{ shift} \end{cases} . \tag{2.35}$$

With the $Q^2 \neq 0$ shifts, we can always construct an n-point amplitude with $m > (n + 2)/3$. Applying the above result to NMHV amplitudes for $m = 3$, we find $\mathcal{M} \lesssim z^{n-7}$ under a Risager 3-line shift, consistent with the known behavior z^{n-12} [45]. Generally, graviton amplitude can be constructed with $Q^2 = 0$ shifts if $m \geq n/2$. Ref. [8] shows amplitudes with total helicity $|h| \leq 2$ cannot be constructed from anti-holomorphic/holomorphic all-line shifts. We see this can be resolved if we choose to do "good" shift on only plus or negative helicity gravitons. Our large z analysis predicts the scaling grows linearly with n and this is indeed how the real amplitude behaves. From this point of view, the amplitude behaves surprisingly well under BCFW shift because the scaling doesn't grow as n increases.

An interesting comparison of our large z behavior is to use the KLT relations [31]. Consider the large z behavior of n point amplitudes under a

($m \geq 4$)-line $Q^2 \neq 0$ shift. A n point graviton amplitude $\mathcal{M}_{grav}$ can be schematically written as a "square" of gauge amplitudes $\mathcal{M}^2_{\text{gauge}}$ by the KLT relation

$$\begin{aligned} \mathcal{M}_{\text{grav}}|_{z\to\infty} &\sim s^{n-3}\mathcal{M}^2_{\text{gauge}}|_{z\to\infty} \\ z^{n+2-3m} &\geq z^{n-3}z^{8-4m} = z^{n+5-4m}, \end{aligned} \tag{2.36}$$

where we neglect all the permutation in particles and details of s-variables[6]. The KLT relation actually predicts a better large z behavior than our dimensional analysis.

Constructible Amplitudes. The above non-renormalizable theories are some limited examples which can be entirely defined by our on-shell recursions. Modifying these theories generally breaks the constructibility! For instrance, a theory of higher dimensional operator $\partial^2\phi^v$ cannot be constructed. This is clear from Feynman diagrams because the derivatives in vertices compensate the large z suppression from propagators. This implies the chiral Lagrangian is not constructible even with the best all-line shift[7]. In gauge theories, we cannot construct amplitudes where all vertices are higher dimensional F^v operators either.

Fortunately, we are usually interested in effective theories with some power counting on higher dimensional operators. If the number of operator insertions is fixed, then we can construct amplitudes with generic multiplicity. To illustrate this, consider amplitudes in a renormalizable theory (spin ≤ 1) with a single insertion of a d-dimensional operator. If we apply a general m-line $Q^2 \neq 0$ momentum shift, Eq. (2.26) gives

$$\gamma_{\text{gen}} \leq d - m - s. \tag{2.37}$$

In the worst case scenario, $s = 0$, we see an (d+1)-line shift suffices to construct any such amplitude. For $[0, m\rangle$- and $[m, 0\rangle$-line shifts, the sum of their large z scaling is

$$\gamma_{[0,m\rangle} + \gamma_{[m,0\rangle} \leq d - m, \tag{2.38}$$

where we use $\Delta = 0$ for theories with spin ≤ 1. The amplitude can always be constructed from one of them provided $m > d$. We see the input for recursion

[6] The inequality holds for $m \geq 4$ which is satisfied in any $Q^2 \neq 0$ shift.

[7] The chiral Lagrangian has the additional complication that there is an infinite tower of interactions generated at each order in the pion decay constant. To overcome this, it is important to use soft limits to relate them and construct the amplitudes [46].

relations are all amplitudes with d points and below. It is not surprising. After all, we need this input for a ϕ^v operator. If the amplitude has higher total spin/helicity, less deformation is needed to construct it. We will demonstrate this with the F^v operator in the next section. The result is similar to the conclusion of Ref. [8], but we can be more economical by choosing $(d+1)$-line or less rather than an all-line shift.

2.5 Examples

In this section, we illustrate the power of our recursion relations in various theories. The calculation is straightforward once the large z behavior is known.

YM + ψ + ϕ. Consider a gauge theory with fermion and scalar matter in the adjoint representation. In addition to the gauge interactions, there are Yukawa interactions of the form $\mathrm{Tr}(\phi\{\psi,\psi\})$. Here we construct the color-ordered amplitude $\mathcal{M}(\psi^-,\psi^-,\phi,\phi,\phi)$ via a 3-line shift $[\{2\},\{3,4\}\rangle$. The seed amplitudes for the recursion relation are

$$\begin{aligned}
\mathcal{M}(\psi^-,\psi^-,\phi) &= y\langle 12\rangle \\
\mathcal{M}(\psi^-,\psi^+,A^-) &= g\langle 31\rangle^2/\langle 12\rangle \\
\mathcal{M}(\phi,\phi,A^-) &= g\langle 31\rangle\langle 23\rangle/\langle 12\rangle \\
\mathcal{M}(\phi,\phi,\phi,\phi) &= g^2\left(1+\frac{[13]^2[24]^2}{[12][23][34][41]}\right) \\
\mathcal{M}(\psi^-,\psi^+,\phi,\phi) &= g^2\frac{[23][24]}{[12][34]}-y^2\frac{[24]}{[41]},
\end{aligned} \tag{2.39}$$

where y and g are the Yukawa and gauge coupling constants, respectively. There are only two non-vanishing factorization channels. Based on these seeds, it's straightforward to write down

$$\begin{aligned}
\mathcal{M}(\psi^-,\psi^-,\phi,\phi,\phi) =& yg^2\left(\frac{1}{[12]}+\frac{[14]^2[35]^2}{[13][12][34][45][51]}-\frac{[35][34]}{[13][23][45]}\right) \\
&+y^3\frac{[35]}{[23][51]}.
\end{aligned} \tag{2.40}$$

Note that the spurious pole [13] cancels between terms. From the final answer, we see that neither the BCFW shifts, like $[\{2\},\{3\}\rangle$ and $[\{1\},\{2\}\rangle$, nor the Risager shift on $[\{2,3,4\},0\rangle$ can construct the amplitude. Thus, a 3-line shift such as $[\{2\},\{3,4\}\rangle$ is necessary to construct theories with both gauge and Yukawa interactions.

$\mathcal{N}=1$ **SUSY.** We have shown all massless supersymmetric theories are 3-line constructible. Consider an $\mathcal{N}=1$ supersymmetric gauge theory with an $SU(3)$ flavor multiplet of adjoint chiral multiplets Φ_a. We assume a superpotential

$$W = i\lambda \mathrm{Tr}(\Phi_a[\Phi_b,\Phi_c]), \tag{2.41}$$

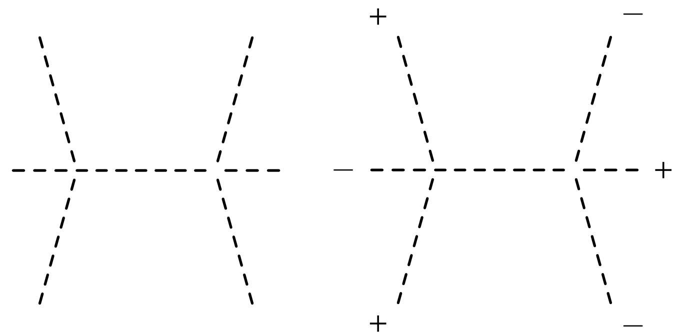
where a,b,c are fixed $SU(3)$ flavor indices, no summation implied. We apply our recursion relations on the (color-ordered) 6-point scalar amplitude $\mathcal{M}(\phi_a^-,\phi_b^-,\phi_c^-,\phi_c^+,\phi_b^+,\phi_a^+)$, where the superscripts and subscripts denote R-symmetry and flavor indices, respectively. In the massless limit, all scalars in the chiral multiplets carry equal R-charge. Therefore we can shift the three holomorphic scalars, namely, $[\{1,2,3\},0\rangle$. The relevant lower point amplitudes for recursion are

$$\begin{aligned}
\mathcal{M}(A^-,\phi_a^\pm,\phi_a^\mp) &= \frac{\langle 31\rangle\langle 12\rangle}{\langle 23\rangle}\\
\mathcal{M}(\phi_a^-,\phi_b^-,\phi_b^+,\phi_a^+) &= \frac{\langle 13\rangle\langle 42\rangle}{\langle 41\rangle\langle 23\rangle} + (1-\lambda^2)\\
\mathcal{M}(A^+,\phi_a^-,\phi_b^-,\phi_b^+,\phi_a^+) &= \frac{\langle 24\rangle\langle 53\rangle}{\langle 51\rangle\langle 12\rangle\langle 34\rangle} + (1-\lambda^2)\frac{\langle 52\rangle}{\langle 51\rangle\langle 12\rangle}.
\end{aligned} \tag{2.42}$$

Crucially, all of them are holomorphic in spinors. Under $[\{1,2,3\},0\rangle$ shift, it is straightforward to obtain the result by an MHV expansion from the above amplitudes [39, 47]

$$\begin{aligned}
&\mathcal{M}(\phi_a^-,\phi_b^-,\phi_c^-,\phi_c^+,\phi_b^+,\phi_a^+)\\
&= \frac{[6\eta][\eta 1]}{[61]\langle 5\not{P}_{61}\eta]\langle 2\not{P}_{61}\eta]}\left(\frac{\langle 24\rangle\langle 53\rangle}{\langle 34\rangle} + (1-\lambda^2)\langle 52\rangle\right)\\
&\quad + \frac{[3\eta][\eta 4]}{[34]\langle 2\not{P}_{34}\eta]\langle 5\not{P}_{34}\eta]}\left(\frac{\langle 51\rangle\langle 26\rangle}{\langle 61\rangle} + (1-\lambda^2)\langle 25\rangle\right)\\
&\quad + \frac{1}{P_{612}^2}\left(\frac{\langle 1\not{P}_{612}\eta]\langle 62\rangle}{\langle 2\not{P}_{612}\eta]\langle 61\rangle} + (1-\lambda^2)\right)\left(\frac{\langle 4\not{P}_{612}\eta]\langle 35\rangle}{\langle 5\not{P}_{612}\eta]\langle 34\rangle} + (1-\lambda^2)\right)\\
&\quad + \frac{1}{P_{561}^2}\left(\frac{\langle 3\not{P}_{561}\eta]\langle 24\rangle}{\langle 2\not{P}_{561}\eta]\langle 34\rangle} + (1-\lambda^2)\right)\left(\frac{\langle 6\not{P}_{561}\eta]\langle 51\rangle}{\langle 5\not{P}_{561}\eta]\langle 61\rangle} + (1-\lambda^2)\right),
\end{aligned} \tag{2.43}$$

where η is the reference spinor and $P_{\mathcal{F}}$ denotes the total momentum of the states in the factorization channel $\mathcal{F}$. We have verified numerically that the answer is, as expected, independent of reference η. Since the scalar amplitude is independent of the fermions, this result applies to any theory with the same bosonic sector. When $\lambda=1$, the $SU(3)$ flavor symmetry together with the $U(1)$ R-symmetry combine to form the $SU(4)$ R-symmetry of $\mathcal{N}=4$ SYM. Our expression agrees with known answer in this limit.



(a) general scalar. (b) $U(1)$ charged scalar.

Figure 2.2: Factorization channels in the 6-point scalar amplitude in ϕ^4 theory. The left and right diagrams show the factorization channels for the general case and the case of a $U(1)$ charged scalar, respectively.

ϕ^4 Theory. Next, consider amplitudes in a theory of interacting scalars. We have shown that a 5-line shift is sufficient to construct all amplitudes, while a 3-line shift suffices if every scalar has equal charge under a $U(1)$ symmetry. It is straightforward to see how these apply to the 6-point scalar amplitude in ϕ^4 theory. Applying a 5-line shift, the factorization channel is depicted in Fig. 2.2 where we sum over all non-trivial permutations of external particles. If the scalar is complex and carries $U(1)$ charge, namely $|\phi|^4$ theory, then only channels satisfying charge conservation can appear. Thus, three plus charged scalars never appear on one side of factorization. Consequently, shifting three plus charge scalars will construct the amplitude by exposing all physical poles.

ψ^4 Theory. From our previous discussion, we know four fermion theory can be constructed by a $Q^2 \neq 0$ 5-line shift. Consider a 6pt $\mathcal{M}(\psi^+, \psi^-, \psi^+, \psi^-, \psi^+, \psi^-)$ amplitude. Using a $[\{2,4\},\{1,3,5\}\rangle$ shift, we find

$$\begin{aligned}
&\mathcal{M}(\psi^+, \psi^-, \psi^+, \psi^-, \psi^+, \psi^-) \\
= &\sum_{\mathcal{P}(1,3,5),\mathcal{P}(2,4,6)} (-1)^\sigma \frac{[13]\langle 46\rangle}{4P_{456}^2} \left(\frac{z_{+,456}\langle 2|\hat{P}_{456}|5]|_{z_{-,456}} - (z_{+,456} \leftrightarrow z_{-,456})}{z_{+,456} - z_{-,456}} \right) \\
= &\sum_{\mathcal{P}(1,3,5),\mathcal{P}(4,5,6)} (-1)^\sigma \frac{[13]\langle 46\rangle\langle 2|P_{456}|5]}{4P_{456}^2},
\end{aligned} \tag{2.44}$$

where hatted variable is evaluated at factorization limit and $z_{\pm,456}$ are the two solutions of $\hat{P}_{456}^2 = 0$. The result is summed over permutation of $(1,3,5)$ and $(2,4,6)$ with σ being the number of total permutations. In the last line, we use the fact that $\langle 2|\hat{P}_{456}|5]$ is linear in z and only the non-deformed part survives after exchanging $z_{\pm,456}$. We see the final answer has no square root as claimed before.

Maxwell-Einstein Theory. We discuss the theory where a $U(1)$ photon minimally couples to gravity. The coupling constant has the same dimension as in GR (see Table 2.2). But as a photon has less spin than a graviton, the large z behavior is worse. We focus on the amplitudes with only external photons given that any amplitude with a graviton can be recursed by BCFW shift[43]. Using a m-line $Q^2 \neq 0$ shift, we find $\widetilde{\mathcal{M}} \lesssim z^{n+2-2m}$ at large z; thus, it's always possible to construct such an amplitude when $m > (n+2)/2$. Together with BCFW shift on gravitons, the theory is fully constructible! Using Eqs. (2.29) and (2.32), the results for $Q^2 = 0$ m-line shifts are

$$\gamma \leq \begin{cases} 1+n-3m/2, & \text{for} \quad [\{-,-,...\},0\rangle \\ n-3m/2, & \text{for} \quad [\{-,-,...\},\{+\}\rangle \end{cases} . \tag{2.45}$$

For the 4pt $\mathcal{M}(A^-,A^-,A^+,A^+)$ amplitude, we choose a $[\{1,2\};4\rangle$ shift so $\gamma < 0$. The inputs for recursions are 3pt functions obtained from consistency relation [35], $\mathcal{M}(A^-,A^+,G^-) = \langle 31\rangle^4/\langle 12\rangle^2$ and $\mathcal{M}(A^-,A^+,G^+) = [23]^4/[12]^2$. The amplitude then follows

$$\begin{aligned} \mathcal{M}(A^-,A^-,A^+,A^+) &= \frac{\langle 1\hat{P}_{24}|4]^4}{\langle 13\rangle^3[13][24]^2}\Big|_{z_{24}} + \frac{\langle 2\hat{P}_{14}|4]^4}{\langle 23\rangle^3[23][14]^2}\Big|_{z_{14}} \\ &= \langle 12\rangle^2[34]^2\left(\frac{1}{P_{24}^2}+\frac{1}{P_{14}^2}\right). \end{aligned} \tag{2.46}$$

F^v **Operators.** Consider amplitudes with a single insertion of a F^v operator. Applying an $[m,0\rangle$-line shift on minus helicity gluons and $[m-1,1\rangle$ m-line shift on all-but-one minus helicity gluons, Eq. (2.29) and Eq. (2.32) predict

$$\gamma \leq \begin{cases} v-m, & \text{for} \quad [\{-,-,...\},0\rangle \\ v-1-m, & \text{for} \quad [\{-,-,...\},\{+\}\rangle \end{cases} . \tag{2.47}$$

We conclude $[v+1,0\rangle$- and $[v-1,1\rangle$-line shifts suffice to construct the amplitude with the given helicity configuration.

The case of the F^3 operator has been studied extensively in Ref. [48]. Given the large z behavior above, the general MHV-like expression in Ref. [49] can be proven inductively by a $[\{-,-\},\{+\}\rangle$ shift. In addition, the vanishing of the boundary term in the $[\{-,-,...\},0\rangle$ shift directly proves the validity of CSW-expansion in Ref. [48]. We demonstrate it with the MHV-like amplitude $\mathcal{M}(A^-,A^-,A^-,A^+)$ where a single F^3 operator is inserted. Note that the all-minus amplitude $\mathcal{M}(1^-,2^-,3^-)=\langle 12\rangle\langle 23\rangle\langle 31\rangle$ is induced by an F^3 operator. Taking this as an input for the $[\{2,3\},4\rangle$ shift, we find

$$\begin{aligned}\mathcal{M}(A^-,A^-,A^-,A^+)=&\langle 12\rangle\langle 23\rangle\langle 31\rangle\left(\frac{\langle 23\rangle}{\langle 34\rangle\langle 2\hat{4}\rangle}|_{z_{12}}-\frac{\langle 12\rangle}{\langle 41\rangle\langle 2\hat{4}\rangle}|_{z_{23}}\right)\\=&\frac{\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}.\end{aligned} \tag{2.48}$$

This agrees with the result in Ref. [49, 50].

The case of the $\phi\,\mathrm{tr}(FF)$ operator, which is popular for the study of Higgs phenomenology, is very similar to F^3 operator. The MHV-like formula and CSW expansion in Ref. [49] can also be proved analogously.

R^v **Operators.** Such operators often arise in effective theories from string action. Consider amplitudes with a single insertion of an R^v operator. The amplitude scales as $z^{2v+n-3m}$ under an m-line $Q^2\neq 0$ shift. For a given R^r operator, any $(n>v)$-pt amplitude can be constructed under an all-line $Q^2\neq 0$ shift. If we use $Q^2=0$ shifts, Eq. (2.29) and Eq. (2.32) give

$$\gamma\leq\begin{cases}n+v-2m, & \text{for}\quad [\{-,-,...\},0\rangle\\ n+v-2-2m, & \text{for}\quad [\{-,-,...\},\{+\}\rangle\end{cases}. \tag{2.49}$$

So if the helicity configuration is available, the amplitude is constructible under the $[m,0\rangle$- and $[m-1,1\rangle$-line shifts for $m>(n+v)/2$ and $m>(n+v)/2-1$, respectively.

Consider the 4pt $\mathcal{M}(G^-,G^-,G^-,G^+)$ amplitude with one R^3 operator insertion. We adopt the $[\{2,3\};4\rangle$ shift to construct it. The amplitude factorizes into the anti-MHV amplitude in GR and $\mathcal{M}(G^-,G^-,G^-)=\langle 12\rangle\langle 23\rangle\langle 31\rangle$ induced by one insertion of R^3 operator. We find

$$\begin{aligned}&\mathcal{M}(G^-,G^-,G^-,G^+)\\&=(\langle 12\rangle\langle 23\rangle\langle 31\rangle)^2\times\left[\frac{\langle 12\rangle^2[41]}{\langle\hat{4}2\rangle^2\langle 41\rangle}\Big|_{z_{41}}+(\text{cyclic in }(1,2,3))\right],\\&=(\langle 12\rangle\langle 23\rangle\langle 31\rangle)^2\left[\frac{[41]\langle\xi 1\rangle^2}{\langle 41\rangle\langle\xi 4\rangle^2}+\frac{[42]\langle\xi 2\rangle^2}{\langle 42\rangle\langle\xi 4\rangle^2}+\frac{[43]\langle\xi 3\rangle^2}{\langle 43\rangle\langle\xi 4\rangle^2}\right]\\&=P_{12}^2\,\mathcal{M}(1_A^-,2_A^-,4_A^+,3_A^-)\mathcal{M}(1_A^-,2_A^-,3_A^-,4_A^+),\end{aligned} \tag{2.50}$$

where $|\xi\rangle$ is a reference spinor in 3-line shift. The result in the second line manifest the leading soft factor of particle 4. After canceling the reference spinor, the result in the last line is expressed in a KLT-relation form, where $\mathcal{M}(1_A^-, 2_A^-, 3_A^-, 4_A^+)$ is the corresponding amplitude in gauge theory with the F^3 operator given in Eq. (2.48). It agrees with Ref. [48]. It obvious from the answer that any $[m, 0\rangle$ shift cannot construct the amplitude.

2.6 Outlook

In this chapter we have determined the minimal set of recursion relations needed to construct renormalizable and non-renormalizable field theories of massless particles in four dimensions. We have shown that all renormalizable theories are constructible from a shift of five external momenta. Quite surprisingly, a shift of three external momenta suffices for a more restricted but still enormous class of theories: all renormalizable theories in which the scalars, if present, are charged equally under a $U(1)$ symmetry. Hence, we can construct all scattering amplitudes in any gauge theory with fermion and complex scalar matter, any supersymmetric theory, and the standard model.

Our results suggest several avenues for future work. Because our analysis hinges solely on dimensional analysis, Lorentz invariance, and locality, it should be possible to generalize our approach to a broader class of theories. In particular, there is the question of theories residing outside of four dimensions and involving massive particles. Moreover, one might study an expanded covering space of recursion relations that include multiple complex deformation parameters or simultaneous shifts of holomorphic and anti-holomorphic spinors of the same leg.

The recursion relations presented here might also offer new tools for studying the underlying properties of amplitudes. For example, the enhanced large z behavior of amplitudes at large momenta implies so-called “bonus relations” whose nature remains unclear. In addition, the soft shift defined in Eq. (2.15) gives a nicely on-shell regulator for the soft limit of the amplitude. Precise knowledge of the soft limit can uniquely fix effective theories [51], and will be useful next chapter. Finally, given a more complete understanding of on-shell constructibility at tree-level, we are better equipped to attack a much more difficult problem, which is developing a recursive construction for the loop integrands of general quantum field theories. This was accomplished for amplitudes in planar $\mathcal{N} = 4$ SYM [9], but with a procedure not obviously generalizable for less symmetric theories, where standard BCFW recursion induces ill-defined contributions in the forward limit. In principle, this somewhat technical obstruction might be eliminated by considering alternative momentum shifts.

Chapter 3

RECURSION RELATIONS FOR EFFECTIVE FIELD THEORIES

3.1 Introduction

As we have seen from the previous chapter, the on-shell recursion relations are typically inapplicable to EFTs. Such an limitation is unfortunate, as effective field theories provide a universal description of spontaneous symmetry breaking in all branches of physics, ranging from superconductivity to the strong interactions [52–54] to cosmology [55].

The aim of this chapter is to fill this gap. We derive a new class of recursion relations that fully construct the S-matrices of certain scalar effective field theories by harnessing an additional physical ingredient: the vanishing of amplitudes in the soft limit. This approach is logical because the soft behavior of the S-matrix actually encodes the interactions and symmetries of the corresponding effective field theory [51], thus giving a theory classification purely in terms of on-shell data. Our new recursion relations apply to any theory with enhanced soft limits, including the non-linear sigma model, Dirac-Born-Infeld theory, and the Galileon [56, 57].

3.2 Recursion and Factorization

On-shell recursion relations act on an initial seed of lower-point on-shell amplitudes to bootstrap to higher-point. Criteria like Lorentz invariance—which prescribes strict little group covariance properties of the amplitude [30]—are manifest provided the initial amplitudes and recursion relation maintain these properties at each step.

The property of factorization, on the other hand, enters less trivially. To access multiple factorization channels, the BCFW recursion relations [3, 4] employ a complex deformation of two external momenta,

$$p_1 \to p_1 + zq \quad \text{and} \quad p_2 \to p_2 - zq, \tag{3.1}$$

where q is fully fixed up to rescaling the on-shell conditions $q^2 = q \cdot p_1 = q \cdot p_2 = 0$. The original amplitude is extracted from the complexified amplitude $A_n(z)$

by contour integrating over an infinitesimal circle centered around $z = 0$. Cauchy's theorem then yields a new expression for the original amplitude,

$$A_n(0) = \frac{1}{2\pi i}\oint \frac{dz}{z} A_n(z) = -\sum_I \operatorname*{Res}_{z=z_I}\left(\frac{A_n(z)}{z}\right), \tag{3.2}$$

where I labels factorization channels at which the intermediate momentum $P_I(z)$ goes on-shell, so z_I is defined by $P_I(z_I)^2 = 0$. The residue at each pole is

$$-\operatorname*{Res}_{z=z_I}\left(\frac{A_n(z)}{z}\right) \;=\; A_{n_I}(z_I)\frac{1}{P_I^2}A_{\bar{n}_I}(z_I), \tag{3.3}$$

establishing a recursion relation in terms of the lower-point amplitudes A_{n_I} and $A_{\bar{n}_I}$ where $n_I + \bar{n}_I = n + 2$.

The above derivation fails when there is a non-zero residue at $z = \infty$. However, this boundary contribution is calculable in certain circumstances [41, 42, 58, 59] and moreover there exist any number of generalizations of BCFW recursion for which the amplitude vanishes at large z [8, 43]. Ultimately, this is not surprising because the boundary term literally encodes a class of factorization channels [40, 43]. Since BCFW recursion and its extensions apply to all renormalizable and some non-renormalizable theories [8, 43, 60], the corresponding S-matrices are completely fixed by Lorentz invariance and factorization.

3.3 Recursion and Soft Limits

In effective field theories, BCFW recursion and its generalizations are hindered by a non-zero boundary term at $z = \infty$.[1] Naively, this is attributable to the divergent behavior of non-renormalizable interactions at large momenta, but this is plainly false in gravity theories, which have terrible high energy behavior but are perfectly constructible via BCFW recursion. For effective field theories, the problem is simply more fundamental: amplitudes are *not* just fixed by factorization, and additional information is needed. In hindsight this is obvious since high-order contact operators in effective field theories are typically related to low-order contact operators not by factorization but by symmetries.

Since existing recursive technology already exploits amplitudes' singularities, a natural candidate for new physical information is amplitudes' zeros. The

[1]In previous work [46], we derived semi-off-shell recursion relations for the non-linear sigma model, though these methods do not generalize straightforwardly.

former are dictated by factorization while the latter require special kinematics at which the amplitude vanishes. Amplitudes in effective field theories typically vanish in the limit that $p \to 0$ for the momentum of an external particle, so there exists a classification of theories according to the degree of their soft behavior [51], σ, where

$$A_n \sim p^\sigma \quad \text{for} \quad p \to 0, \tag{3.4}$$

and $\sigma \geq 1$ is an integer. As shown in [51], higher values of σ correspond to more symmetry in the theory.

To exploit Eq. (3.4) we need a momentum shift that probes the soft limits of external legs. This is not accomplished by the BCFW shift in Eq. (3.1), which probes collinear but not soft behavior. For our purposes we define a “rescaling shift" on all external legs,

$$p_i \to p_i(1 - z a_i), \tag{3.5}$$

where the a_i are defined up to an overall rescaling and

$$\sum_{i=1}^{n} a_i p_i = 0, \tag{3.6}$$

to maintain momentum conservation. For $n < D+1$, a generic set of momenta p_i are linearly independent, so the only solution to Eq. (3.6) has all a_i equal, corresponding to total momentum conservation. Since this momentum shift simply rescales of all the momenta, it is not useful for recursion. For $n \geq D+1$, Eq. (3.6) is solved by

$$a_i = (-1)^i |p_1 \dots p_{i-1} p_{i+1} \dots p_{D+1}| \tag{3.7}$$

for $i = 1, ..., D+1$ with all other $a_i = 0$. When $n = D+1$, this solution again trivializes to a_i all equal, but for $n > D+1$ it is always possible to find distinct a_i provided p_i represent a general kinematic configuration.

The scaling shift in Eq. (3.5) is purposely chosen so that

$$A_n(z) \sim (1 - z a_i)^\sigma \quad \text{for} \quad z \to 1/a_i, \tag{3.8}$$

due to Eq. (3.4), thus recasting the soft behavior as a degree σ zero of the amplitude. To compute the amplitude we apply Cauchy’s theorem to a contour encircling all poles at finite z

$$\oint \frac{dz}{z} \frac{A_n(z)}{F_n(z)} = 0, \tag{3.9}$$

where the denominator factor is defined to be

$$F_n(z) = \prod_{i=1}^{n}(1 - a_i z)^{\sigma}. \tag{3.10}$$

The integrand of Eq. (3.9) is engineered to be non-singular at $z = 1/a_i$ since the poles introduced by $F_n(z)$ are cancelled by zeroes of the amplitude. Thus, the integrand of Eq. (3.9) has poles from factorization channels only, so in analogy with BCFW, the amplitude is

$$A_n(0) = -\sum_I \operatorname*{Res}_{z=z_{I\pm}} \left(\frac{A_n(z)}{zF_n(z)}\right), \tag{3.11}$$

where I again labels factorization channels. In contrast with BCFW, each factorization channel in $P_I(z)$ yields two poles $z_{I\pm}$ corresponding to the roots of

$$P_I^2 + 2P_I \cdot Q_I z + Q_I^2 z^2 = 0, \tag{3.12}$$

where $P_I(z) = P_I + zQ_I$ and where

$$P_I = \sum_{i\in I} p_i \quad \text{and} \quad Q_I = -\sum_{i\in I} a_i p_i. \tag{3.13}$$

Each residue is a product of lower-point amplitudes which can be rearranged into a new recursion relation,

$$A_n(0) = \sum_I \frac{1}{P_I^2} \frac{A_{n_I}(z_{I-})\, A_{\bar{n}_I}(z_{I-})}{(1 - z_{I-}/z_{I+})F_n(z_{I-})} + (z_{I+} \leftrightarrow z_{I-}). \tag{3.14}$$

Again, we assume a vanishing boundary term at $z = \infty$, which is achievable because $F_n(z)$ substantially improves the large z behavior of the integrand of Eq. (3.9). In the next section we determine the precise conditions under which the boundary term is zero.

3.4 Criteria for On-Shell Constructibility

Next, we determine the conditions under which the boundary term vanishes and the new recursion relation in Eq. (3.14) applies. Under the rescaling shift in Eq. (3.5), all momenta scale as z at large z. Consequently, if the n-point amplitude scales with m powers of momenta, then $A_n(z) \sim z^m$ and $F_n(z) \sim n\sigma$ so

$$\frac{A_n(z)}{F_n(z)} \sim z^{m-n\sigma}. \tag{3.15}$$

Demanding falloff at $z = \infty$ implies that

$$\text{on-shell constructible} \quad \leftrightarrow \quad m/n < \sigma. \tag{3.16}$$

At the level of the contact terms this is *exactly* the condition at which the soft limit of the amplitude is enhanced beyond the naive expectation given by the number of derivatives per field. So the set of amplitudes with special soft behavior are on-shell constructible.

To lift the criterion for on-shell constructibility from amplitudes to theories, we adopt the (ρ, σ) classification of scalar effective field theories presented in [51]. In particular, for operators of the form $\partial^m \phi^n$, we define a derivative power counting parameter

$$\rho = \frac{m-2}{n-2}, \tag{3.17}$$

so that an amplitude of a given ρ can factorize into two lower-point amplitudes of the same ρ. The simplest effective theories have a fixed value of ρ but mixed ρ theories also exist. The derivative power counting parameter ρ in Eq. (3.17), together with the soft limit degree σ defined in Eq. (3.4) define a two parameter classification of scalar effective field theories.

In terms of the (ρ, σ) classification, the criterion of on-shell constructibility in Eq. (3.16) becomes

$$(\rho - 1) < (\sigma - 1)\left(\frac{1}{1 - 2/n}\right). \tag{3.18}$$

For an effective field theory to be on-shell constructible requires that recursion relations apply for arbitrarily high n. In the large n limit, Eq. (3.18) yields a simple condition for on-shell constructibility,

$$\text{on-shell constructible} \quad \leftrightarrow \quad \rho \leq \sigma \text{ and } (\rho, \sigma) \neq (1,1), \tag{3.19}$$

which precisely coincides with the class of theories that exhibit enhanced soft behavior.

Examples of on-shell constructible theories are the non-linear sigma model $(\rho, \sigma) = (0, 1)$, Dirac-Born-Infeld theory $(\rho, \sigma) = (1, 2)$, and the general/special Galileon $(\rho, \sigma) = (2, 2)/(2, 3)$ [51][2]. Among these theories, we dub those with especially good soft behavior, $\rho = \sigma - 1$, “exceptional” theories. Exceptional

[2]Theories with higher shift symmetries [61, 62] violate this bound.

theories have a very interesting property: their soft behavior is not manifest term by term in the Feynman diagram expansion, and is only achieved after summing all terms into the amplitude. Note the close analogy with gauge invariance in Yang-Mills theory or diffeomorphism invariance in gravity, which similarly impose constraints among contact operators of different valency. The exceptional theories also play a prominent role in the scattering equations [14] and ambitwistor string theories [25], suggesting a deeper connection between these approaches and recursion.

For the exceptional theories, Eq. (3.19) is more than satisfied, yielding better large z falloff than is even needed for constructibility. Thus, our recursion relations generate so-called bonus relations defining identities among amplitudes. In principle this can be exploited, for example by introducing factors of $P_i(z)^2$ into the numerator of the recursion relation to eliminate certain factorization channels from the recursion relation. This is an interesting possibility we leave for future work.

Finally, let us address a slight caveat to the z scaling arguments discussed above. While all momenta scale as z at large z, it is *a priori* possible that cancellations modify the naive scaling of $A_n \sim z^m$ for an amplitude with m derivatives. This is conceivable because the a_i parameters in the momentum shift are implicitly related by the momentum conservation condition in Eq. (3.6). In particular, our recursions would fail if there were cancellations in propagator denominators such that they scaled less severely than z^2. That there is always a choice of a_i for which no such cancellations arise can be shown via proof by contradiction. In particular, assuming no such choice exists implies that cancellations occur for all values of a_i. But we can always perturb a given choice of a_i away from such a cancellation point by applying an additional infinitesimal momentum shift on a subset of $D+1$ external legs as defined in Eq. (3.7). Thus the starting assumption is false and there are generic values of a_i for which $A_n \sim z^m$ scales as expected.

3.5 Example Calculations

In this section we apply our recursion relations to scattering amplitudes in various effective field theories. We begin with amplitudes in exceptional theories. Curiously, the six-point amplitudes in the non-linear sigma model, Dirac-Born-Infeld, and the special Galileon, are, term by term, the “square" and “cube" of each other, reminiscent of the result of [14]. Afterwards, we consider the general Galileon, which is marginally constructible.

Non-Linear Sigma Model: $(\rho, \sigma) = (0, 1)$

As shown in [63], flavor-ordered scattering amplitudes in the non-linear sigma model vanish in the soft limit. We derive the flavor-ordered six-point amplitude A_6 by recursing the flavor-ordered four-point amplitude,

$$A_4 = s_{12} + s_{23}. \tag{3.20}$$

Since A_6 has three factorization channels, the recursion relation in Eq. (3.14) takes the form

$$A_6 = A_6^{(123)} + A_6^{(234)} + A_6^{(345)}, \tag{3.21}$$

corresponding to when P_{123}, P_{234}, and P_{345} go on-shell. Consider first the pole at $P_{123}^2(z) = 0$, whose roots are

$$z_\pm = \frac{-(P_{123} \cdot Q_{123}) \pm \sqrt{(P_{123} \cdot Q_{123})^2 - P_{123}^2 Q_{123}^2}}{Q_{123}^2}. \tag{3.22}$$

Plugging Eq. (3.13) into Eq. (3.14) we obtain

$$A_6^{(123)} = \frac{B}{P_{123}^2} \times \sum_{\substack{ij\in\{12,23\}\\ kl\in\{45,56\}}} C_{ijkl} + (z_+ \leftrightarrow z_-), \tag{3.23}$$

where for later convenience we have defined

$$C_{ijkl} = \frac{s_{ij} s_{kl}}{\prod\limits_{m\notin\{i,j,k,l\}} (1 - a_m z_-)}, \tag{3.24}$$

and $B = (1 - z_-/z_+)^{-1}$. We observe that $A_6^{(123)}$ is equal to the residue of a new function

$$\begin{aligned} A_6^{(123)} &= -\operatorname*{Res}_{z=z_\pm} \left[\frac{(s_{12}(z) + s_{23}(z))(s_{45}(z) + s_{56}(z))}{z P_{123}^2(z) F_6(z)} \right] \\ &= \frac{(s_{12} + s_{23})(s_{45} + s_{56})}{P_{123}^2} \\ &\quad + \sum_{i=1}^{6} \operatorname*{Res}_{z=z_i} \left[\frac{(s_{12}(z) + s_{23}(z))(s_{45}(z) + s_{56}(z))}{z P_{123}^2(z) F_6(z)} \right], \end{aligned} \tag{3.25}$$

which we have recast in terms of residues at $z = 0$ and $z_i = 1/a_i$ by Cauchy's theorem. Summing over factorization channels, we simplify the $z_i = 1/a_i$ residues to

$$\sum_{i=1}^{6} \operatorname*{Res}_{z=z_i} \frac{s_{12}(z) + ...}{zF_6(z)} = -(s_{12} + ...), \tag{3.26}$$

where ellipses denote cyclic permutations and we have again applied Cauchy's theorem. Our final answer is

$$A_6 = \left[\frac{(s_{12} + s_{23})(s_{45} + s_{56})}{P_{123}^2} + \dots \right] - (s_{12} + \dots), \tag{3.27}$$

which is the expression from the Feynman diagrams.

Dirac-Born-Infeld Theory: $(\rho, \sigma) = (1, 2)$

Amplitudes in Dirac-Born-Infeld theory are computed similarly with the notable exception that there is no flavor-ordering, so all expressions are permutation invariant. The four-point amplitude takes the form

$$A_4 = s_{12}^2 + s_{23}^2 + s_{13}^2, \tag{3.28}$$

which is the "square" of Eq. (3.20). The six-point scattering amplitude takes the form

$$A_6 = A_6^{(123)} + \dots, \tag{3.29}$$

where the ellipses denote permutations, totaling to the ten factorization channels of the six-point amplitude. As in Eq. (3.22), each factorization channel has two roots in z, so recursion yields

$$A_6^{(123)} = \frac{B}{P_{123}^2} \times \sum_{\substack{i,j \in \{1,2,3\} \\ k,l \in \{4,5,6\}}} C_{ijkl}^2 + (z_+ \leftrightarrow z_-), \tag{3.30}$$

which like before can be shown to be equal to the Feynman diagram expression. Interestingly, Eq. (3.30) is precisely the "square" form of Eq. (3.23).

Special Galileon: $(\rho, \sigma) = (2, 3)$

Next, consider the special Galileon [14, 51], whose existence was conjectured in [51] due to the existence of an S-matrix with the same derivative power counting as those in the Galileon but with even more enhanced soft behavior (at the same time the amplitudes in this theory were obtained using scattering equations [14]). Shortly after this work it was shown in [64] that this theory is a subset of the Galileon theories with a higher degree shift symmetry related by an S-matrix preserving duality [65–67].

Since the Galileon does not carry flavor, its amplitudes are permutation invariant. The four-point amplitude is

$$A_4 = s_{12}^3 + s_{23}^3 + s_{13}^3, \tag{3.31}$$

which is the “cube" of Eq. (3.20). Permutation symmetry implies that the amplitude is again of the form of Eq. (3.29), except here we find

$$A_6^{(123)} = \frac{B}{P_{123}^2} \times \sum_{\substack{i,j\in\{1,2,3\}\\ k,l\in\{4,5,6\}}} C_{ijkl}^3 + (z_+ \leftrightarrow z_-), \tag{3.32}$$

which is the “cube" of Eq. (3.23).

General Galileon: $(\rho, \sigma) = (2, 2)$

Finally, let us compute amplitudes in the general Galileon. As shown in [65], each n-point vertex of the D-dimensional Galileon is a Gram determinant,

$$V_n \quad = \quad G(\widehat{p}_1, p_2, \dots, p_n) = G(p_1, \widehat{p}_2, \dots, p_n) = \dots, \tag{3.33}$$

which is simply the determinant of the matrix s_{ij} with the row and column corresponding to the hatted momentum removed. The Gram determinant is by construction symmetric in its arguments. Furthermore,

$$G(\lambda p_1, p_2, \dots, p_n) \quad = \quad \lambda^2 G(p_1, p_2, \dots, p_n), \tag{3.34}$$

so crucially, the rescaling shift in Eq. (3.5) acts homogenously on the vertex. This allows for a major simplification of our recursion relation. Here we define the seed amplitudes for the recursion to be lower-point amplitudes for $n = 4, 5, \dots, D+1$.

For a concrete example, we now apply our new recursion relations to the eight-point amplitude A_8 for the Galileon with just a five-point vertex in $D = 4$. The amplitude factorizes into two five-point amplitudes which are simply vertices, *e.g.*, $A_5 = V_5 = G(p_1, p_2, p_3, p_4)$ and $A_{\bar{5}} = V_{\bar{5}} = G(p_5, p_6, p_7, p_8)$, with the intermediate leg corresponding to the missing column in the Gram determinant. We find that

$$\frac{A_5(z)A_{\bar{5}}(z)}{F_8(z)} = \frac{V_5(z)V_{\bar{5}}(z)}{F_8(z)} = V_5(0)V_{\bar{5}}(0), \tag{3.35}$$

applying the homogeneity property from Eq. (3.34) to cancel factors of $(1 - a_i z)^2$ in the numerator and denominator. Summing over factorization channels yields

$$\begin{aligned} A_8 &= \frac{B}{P_{1234}^2} \frac{V_5(0)\, V_{\bar{5}}(0)}{(1 - z_{I-}/z_{I+})} + (z_{I+} \leftrightarrow z_{I-}) + \ldots \\ &= G(p_1, p_2, p_3, p_4) \frac{1}{P_{1234}^2} G(p_5, p_6, p_7, p_8) + \ldots, \end{aligned} \tag{3.36}$$

where the ellipses denote permutations. This expression is manifestly equal to the Feynman diagram expression.

Note the similarity between the above manipulations and the derivation of the CSW rules for Yang-Mills amplitudes. While MHV amplitudes are invariant under square bracket shifts, the Galileon vertices literally rescale under the rescaling shift. Just as the CSW rules can be proven using the Risager three-line momentum shift [39], the Feynman diagram expansion of the general Galileon can be proven using our new recursion relations.

3.6 Outlook

We have derived a new class of recursion relations for effective field theories with enhanced soft limits, *i.e.*, the non-linear sigma model, Dirac-Born-Infeld theory, and the Galileon. Like gauge and diffeomorphism invariance, soft behavior dictates non-trivial relations among interactions of different valencies.

Our results open many avenues for future work. In particular, while we have considered fixed ρ theories here, it should be straightforward to generalize our results to mixed ρ theories such as the DBI-Galileon [68]. Also interesting would be to extend our results to theories with universal albeit non-vanishing soft behavior. For example, in the conformal Dirac-Born-Infeld model—corresponding to the motion of a brane in AdS—the soft limits of an n-point amplitude are not zero but related to the derivative of the $(n-1)$-point amplitude with respect to the AdS radius parameter. Last but not least, there is the question of how to utilize collinear or double-soft limits of amplitudes (for recent discussion see [69–72]).

Chapter 4

A PERIODIC TABLE OF EFFECTIVE FIELD THEORIES

4.1 Introduction

While much of the progress in S-matrix has centered on gauge theory and gravity, another important class of theories—effective field theories (EFTs)—have received substantially less attention, even though they play an important and ubiquitous role in many branches of physics. At the very minimum, the EFT approach provides a general parameterization of dynamics in a particular regime of validity, usually taken to be low energies. If the EFT has many free parameters then its predictive value is limited. However, in many examples the interactions of the EFT are dictated by symmetries, *e.g* as is the case for the Nambu-Goldstone bosons (NGBs) of spontaneous symmetry breaking. At the level of scattering amplitudes, these rigid constraints are manifested by special infrared properties. The archetype for this phenomenon is the Adler zero [73],

$$\lim_{p \to 0} A(p) = 0\,, \tag{4.1}$$

which dictates the vanishing of amplitudes when the momentum of an NGB is taken to be soft. This imprint of symmetry on the S-matrix is reminiscent of gravity, which is also an EFT with a limited regime of validity.

At the same time, the longstanding aim of the modern amplitudes program is to construct the S-matrix without the aid of a Lagrangian, thus relinquishing both the benefits and pitfalls of this standard approach. But without a Lagrangian, it is far from obvious how to incorporate the symmetries of an EFT directly into the S-matrix. However, recent progress in this direction [51] has shown that the symmetries of many EFTs can be understood as the *consequence* of a "generalized Adler zero" characterizing a non-trivial vanishing of scattering amplitudes in the soft limit. Here an amplitude is defined to have a "non-trivial" soft limit if it vanishes in the soft limit faster than one would naively expect given the number of derivatives per field.

By directly imposing a particular soft behavior at the level of the S-matrix, one can then *derive* EFTs and their symmetries from non-trivial soft behavior. From this "soft bootstrap" one can rediscover a subclass of so-called "excep-

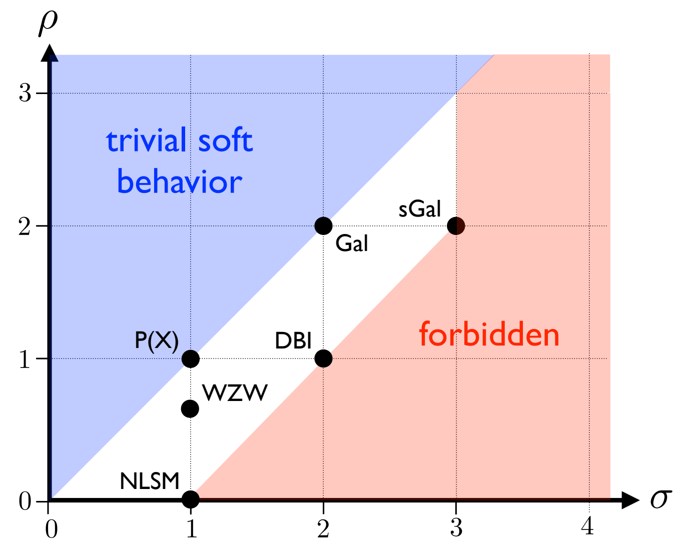


Figure 4.1: Plot summarizing the allowed parameter space of EFTs. The blue region denotes EFTs whose soft behavior is trivial due to the number of derivatives per interaction. The red region is forbidden by consistency of the S-matrix, as discussed in Sec. 4.5. The white region denotes EFTs with non-trivial soft behavior, with solid black circles representing known standalone theories. The d-dimensional WZW term theory corresponds to $(\rho, \sigma) = (\frac{d-2}{d-1}, 1)$. The exceptional EFTs all lie on the boundary of allowed theory space and $(\rho, \sigma) = (3, 3)$ is forbidden.

tional" EFTs [51] whose leading interactions are uniquely fixed by a single coupling constant. These exceptional theories include the non-linear sigma model (NLSM) [52–54], the Dirac-Born-Infeld (DBI) theory, and the so-called special Galileon [51, 64].

In [74], it was shown that the space of exceptional EFTs coincides precisely with the space of on-shell constructible theories via a new set of soft recursion relations. These very same EFTs also appeared in a completely different context from the CHY scattering equations [14], which are simple constructions for building the S-matrices for certain theories of massless particles. Altogether, these developments suggest that the exceptional theories are the EFT analogs of gauge theory and gravity. In particular, they are all simple one-parameter theories whose interactions are fully fixed by simple properties of the S-matrix.

In this chapter, we systematically carve out the theory space of all possible Lorentz invariant and local scalar EFTs by imposing physical consistency conditions on their on-shell scattering amplitudes. Our classification hinges on a set of physical parameters (ρ, σ, v, d) which label a given hypothetical EFT. Here ρ characterizes the number of derivatives per interaction, with a corresponding Lagrangian of the schematic form

$$\mathcal{L} = \partial^2 \phi^2 F(\partial^\rho \phi)\,, \tag{4.2}$$

for some function F. This power counting structure is required for destructive interference between tree diagrams of different topologies [51]. Meanwhile, the parameter σ is the soft degree characterizing the power at which amplitudes vanish in the soft limit,

$$\lim_{p\to 0} A(p) = \mathcal{O}(p^\sigma)\,. \tag{4.3}$$

Obviously, for sufficiently large ρ, a large of value σ is trivial because a theory with many derivatives per field will automatically have a higher degree soft limit. As shown in [51] the soft limit becomes non-trivial when

$$\begin{aligned} \sigma &\geq \rho \qquad \text{for} \quad \rho > 1\,, \\ \sigma &> \rho \qquad \text{for} \quad \rho \leq 1\,. \end{aligned} \tag{4.4}$$

The other parameters in our classification are v, the valency of the leading interaction, and d, the space-time dimension.

Taking a bottom up approach, we *assume* a set of values for (ρ, σ, v, d) to bootstrap scattering amplitudes which we then analyze for self-consistency. Remarkably, by fixing these parameters—without the aid of a specific Lagrangian or set of symmetries—it is possible to rule out whole swaths of EFT space using only properties of the S-matrix. Since our analysis sidesteps top down considerations coming from symmetries and Lagrangians, we obtain a robust system for classifying and excluding EFTs. This approach yields an overarching organizing principle for EFTs, depicted pictorially in Fig. 4.1 as a sort of "periodic table" for these structures. See Appendix A.3 for a brief summary of the EFTs discussed in this chapter. Our main results are as follows:

- The soft degree of all EFTs is bounded by the number of derivatives per interaction, so in particular, $\sigma \leq \rho + 1$. The exceptional EFTs—the NLSM, DBI, and the special Galileon—all saturate this bound.

- The soft degree of every non-trivial EFT is strictly bounded by $\sigma \leq 3$, so arbitrarily enhanced soft limits are forbidden.
- Non-trivial soft limits require the valency of the leading interaction be bounded by the spacetime dimension, so $v \leq d+1$. For $4 < v \leq d+1$, this is saturated by the Galileon [57, 75] and the Wess-Zumino-Witten (WZW) term for the NLSM [76, 77].
- The above constraints permit a theory space of single scalar EFTs and multiple scalar EFTs with flavor-ordering in general d populated by known theories: NLSM, DBI, the Galileon, and WZW. In principle this allows for new theories at the these same values of (ρ, σ, d, v) but we exclude this possibility in $d = 3, 4, 5$ by direct enumeration.

The core results of this chapter focus on the soft behavior of EFTs of a single scalar, or multiple scalars where there is a notion of flavor-ordering. However, we also briefly discuss the space of general EFTs with multiple scalars, as well as alternative kinematical regimes like the double soft or collinear limits.

This chapter is organized as follows. In Sec. 4.2, we define the parameters of the EFT theory space and outline our strategy for classification. We then derive soft theorems from general symmetry considerations in Sec. 4.3. The tools for classification—soft momentum shifts and recursion relations—are summarized in Sec. 4.4, and then applied to carve out the space of allowed EFTs in Sec. 4.5. In the permitted region, we search and enumerate EFTs numerically in Sec. 4.6. Other kinematics limits and more general classes of theories are considered in Sec. 4.7. Finally we conclude in Sec. 4.8.

4.2 Classification Scheme

As described in the introduction, scalar EFTs are naturally classified in terms of the set of parameters (ρ, σ, v, d). Here we review the definitions and motivations for these parameters, first in terms of the Lagrangian and then in terms of the S-matrix.

Lagrangians

The power counting parameter ρ is a measure of the number of powers of momentum associated with each interaction. As shown in [51], destructive interference among diagrams, *i.e.* cancellations, imposes a strict power counting condition relating the interactions of the EFT. In particular, suppose that the Lagrangian has a schematic form

$$\mathcal{L} = \sum_{m=0}^{\infty} \sum_{n=v}^{\infty} \lambda_{m,n}\, \partial^m \phi^n \,, \tag{4.5}$$

where $\lambda_{m,n}$ are coupling constants. Cancellations can occur between couplings of fixed

$$\rho = \frac{m-2}{n-2} \,, \tag{4.6}$$

where ρ is a fixed non-negative rational number. Here Eq. (4.5) is schematic, since we have suppressed Lorentz and internal indices so at a given order in m, n there are actually many coupling constants $\lambda_{m,n}$. This restriction still leaves a huge parameter space of viable EFTs.

In principle, one can combine interactions of different values of ρ into the same theory. However, cancellations among the interactions with either the smallest or the highest value of ρ are closed, so it is natural to focus first on fixed ρ theories.

In Eq. (4.5), v denotes the valency of the leading interaction. Naively, the minimum of possible valency is $v = 3$. However, the leading cubic vertex in a derivatively coupled theory of massless scalars can always be eliminated by equations of motion. This is obvious because the only possible non-zero 3pt amplitude of scalars is a constant, corresponding to a cubic scalar potential interaction. On the other hand, the on-shell 3pt amplitude for derivatively coupled scalars will vanish because there is no non-zero kinematic invariant built from three on-shell momenta. So without loss of generality we can take $v = 4$ as the minimum valency.

For concreteness, let us briefly enumerate a few simple examples of Lagrangians with fixed ρ. Consider first the very simplest case, $\rho = 0$, for a theory of a single scalar with only even interactions,

$$\mathcal{L}_{\rho=0} = \lambda_{2,4}(\partial^2\phi^4) + \lambda_{2,6}(\partial^2\phi^6) + \lambda_{2,8}(\partial^2\phi^8) + \ldots \tag{4.7}$$

Since each term only has two derivatives, the Lorentz structure of these terms is simple:

$$\phi^{n-2}(\partial^\mu\phi\, \partial_\mu\phi) \,. \tag{4.8}$$

It is straightforward to see that all on-shell tree-level scattering amplitudes in this theory are zero, corresponding to the fact that all the interactions are related by a field redefinition to the action for a free scalar. For a multiplet of scalars, this is no longer true, and the theory can have non-trivial scattering amplitudes.

For $\rho = 1$ the Lagrangian for a scalar with even interactions is

$$\mathcal{L}_{\rho=1} = \lambda_{4,4}(\partial^4\phi^4) + \lambda_{6,6}(\partial^6\phi^6) + \lambda_{8,8}(\partial^8\phi^8) + \dots \tag{4.9}$$

In this case, even for a single scalar field there are many possible ways to contract Lorentz indices. For example, the first term above could represent any of three different interactions,

$$\lambda^{(1)}_{4,4}(\partial^\mu\phi)(\partial_\mu\phi)(\partial^\nu\phi)(\partial_\nu\phi) + \lambda^{(2)}_{4,4}\phi^2(\partial^\mu\partial^\nu\phi)(\partial_\mu\partial_\nu\phi) + \lambda^{(3)}_{4,4}\phi(\partial^\mu\partial^\nu\phi)(\partial_\mu\phi)(\partial_\nu\phi)\,. \tag{4.10}$$

In fact, we can eliminate two of these terms via integration-by-parts identities and equations of motion. These relations are harder to track down for more complicated Lagrangians, but for our analysis we will thankfully not need to determine all of these identities.

Finally, let us stress that ρ need not be an integer, but is more generally an arbitrary rational number. As we will later see, a case of particular interest is $\rho = 2/3$, for which

$$\mathcal{L}_{\rho=\frac{2}{3}} = \lambda_{4,5}(\partial^4\phi^5) + \lambda_{6,8}(\partial^6\phi^8) + \lambda_{8,11}(\partial^8\phi^{11}) + \dots \tag{4.11}$$

A priori, quite extreme values of ρ are possible. For example, for $\rho = 13/11$ we have

$$\mathcal{L}_{\rho=\frac{13}{11}} = \lambda_{28,24}(\partial^{28}\phi^{24}) + \lambda_{54,46}(\partial^{54}\phi^{46}) + \dots \tag{4.12}$$

For such peculiar values of ρ, the leading valency v of the theory can be very high. Naively, this signals a serious obstruction to any program for explicit construction of all possible EFTs. In particular, any exhaustive search for EFTs at a fixed valency will always miss possible EFTs at higher valency. After all, the space of rational numbers ρ is dense. Remarkably, we will later on find general arguments bounding the allowed maximum valency of a consistent EFT, making an enumerative procedure feasible.

Although only theories with fixed ρ are considered in this thesis, we briefly comment on the scenario with multiple ρ interactions. This generally arises from loop induced interactions. For instance, the 1-loop correction of Eq. (4.9) yields

$$\mathcal{L}' = \lambda_{8,4}(\partial^8\phi^4) + \lambda_{10,6}(\partial^{10}\phi^6) + \lambda_{12,8}(\partial^{12}\phi^8) + \dots \tag{4.13}$$

The single insertion of the above operators corresponds to $\rho = 3, 2, 5/3$ for four, six, and eight points respectively. Given fixed loop order counting, we find the value of ρ decreases for higher point interactions. Suppose the associated amplitudes have soft limit $\sigma = 2$ (which we expect for the loop-correction of DBI theory). The amplitudes will have trivial soft limits at four points but become non-trivial starting at six points. We leave the study of multiple ρ theories to future work.

Scattering Amplitudes

Starting from a general Lagrangian of fixed power counting parameter ρ one can calculate the n-pt tree-level scattering amplitude using the corresponding Feynman rules. The resulting answer is a function of kinematical invariants together with the coupling constants $\lambda_{m,n}$. In turn, the $\lambda_{m,n}$ can be constrained by demanding that the amplitude conform to the enhanced soft limit of Eq. (4.3).

In principle, the soft degree σ can be any integer. However, $\sigma < 0$ corresponds to singular behavior in the soft limit, which is only possible if there are cubic interactions in the theory. As we argued previously, though, all such cubic interactions can be eliminated by equations of motion in a theory of derivatively coupled scalars. In contrast, such cubic interactions are physical in YM and gravity, where $\sigma = -1$. In any case, for scalar EFTs we have that $\sigma \geq 0$.

As the number of derivatives per field increases, so too will the soft degree. However, something interesting occurs when the soft degree exceeds the number of derivatives per field,

$$\sigma > \frac{m}{n}\,, \tag{4.14}$$

which is only possible if there is cancellation among diagrams. We define this to be an enhanced soft limit (see [51]). Rewriting this inequality in terms of ρ, we obtain

$$(\sigma - 1) > (\rho - 1) \times \left(1 - \frac{2}{n}\right)\,. \tag{4.15}$$

For a theory with enhanced soft behavior, this inequality should be true of all amplitudes. Thus we can take the large n limit, in which case the inequality approaches the inequalities in Eq. (4.4). This range defines a swath of EFT space that has enhanced soft behavior, which will be of our primary interest.

Ansatze

Fixing the power counting parameter ρ, the soft degree σ, the valency of the leading interaction v, and the spacetime dimension d, we can now place stringent constraints on the space of scalar EFTs. One way to compute the associated scattering amplitudes would be natural to enumerate all possible Lagrangian terms and calculate using Feynman diagrams. While this approach is straightforward, it is plagued with redundancies since integration-by-parts identities and field redefinitions induce an infinite set of Lagrangians corresponding to identical physics. Indeed, even a systematic enumeration of higher dimension operators in EFTs is a non-trivial task that remains an active area of research [78].

Here we bypass this complication by directly constructing the scattering amplitudes using ansatze. For a theory of scalars, the tree-level scattering amplitude A_n is a rational function of kinematic invariants $s_{ij} = (p_i + p_j)^2$, where A_n has poles only when $s_{i_1 i_2 \dots i_k} = (p_{i_1} + p_{i_2} + \cdots + p_{i_k})^2 = 0$. Note the absence of two particle poles, $s_{ij} = 0$, since the 3pt amplitude vanishes in a theory of derivatively coupled scalars. Schematically, the scattering amplitude ansatz is

$$A_{n,m}(s_{ij}) = \sum_{\text{topology}} \frac{N(s_{ij})}{D(s_{ij})} + A_{\text{contact}}(s_{ij}) \,, \tag{4.16}$$

where $m = \rho(n-2) + 2$ is the dimension of the amplitude, and counts the net power of momenta in the amplitude. Here the summation runs over all topologies involving internal exchanged scalars, allowing for all possible interactions consistent with ρ. These terms enter with propagator denominators collected into the function D, and the remaining numerator function is N. The second term A_{contact} corresponds to contributions that do not have propagator denominators, and is thus a local function of the kinematic invariants.

The amplitudes ansatz should satisfy several consistency conditions. First, it must factorize properly on poles, so

$$\lim_{P^2 \to 0} A_{n,m} = \sum \frac{A_L A_R}{P^2} \,, \tag{4.17}$$

where $P = (p_{i_1} + p_{i_2} + \cdots + p_{i_k})$ and the sum runs over internal states. Second, the amplitudes ansatz should respect all the permutation symmetries of a given diagram. For example, in a theory of a single scalar, all vertices should be permutation invariant under the exchange of external legs and all diagrams of the same topology should be related by permutations.

An ansatz consistent with the above conditions is a genuine scattering amplitude corresponding to the conjugacy class of physically equivalent Lagrangians that are identical up to off-shell redundancies like field redefinitions and identities from integration by parts. The immense advantage of these amplitudes ansatze is that these objects are free from such off-shell ambiguities and thus uniquely label distinct theories.

To be concrete, let us spell out this ansatz construction explicitly for the 4pt and 6pt amplitudes for a $\rho = 1$ theory. The unique 4pt amplitude for such a theory is

$$A_4 = \lambda_{4,1}(s_{12}^2 + s_{13}^2 + s_{23}^2)\,. \tag{4.18}$$

Since there is only one possible invariant, the corresponding Lagrangian must only describe one physical interaction parameterized by $\lambda_{4,1}$.

The 6pt amplitudes ansatz has a contact term and ten factorization terms,

$$A_6 = \left(\frac{\lambda_{4,1}^2(s_{12}^2 + s_{13}^2 + s_{23}^2)(s_{45}^2 + s_{46}^2 + s_{56}^2)}{s_{123}} + \text{permutations}\right) + A_{6,\text{contact}}\,, \tag{4.19}$$

where the permutations run through the other nine factorization channels. The factorization term is written so as to factorize properly into 4pt amplitudes while the contact term is

$$\begin{aligned} A_{6,\text{contact}} = {} & \alpha_1 s_{12}^3 + \alpha_2 s_{12}^2 s_{13} + \alpha_3 s_{12}^2 s_{34} + \alpha_4 s_{12}s_{13}s_{23} + \alpha_5 s_{12}s_{13}s_{14} + \alpha_6 s_{12}s_{23}s_{34} \\ & + \alpha_7 s_{12}s_{23}s_{45} + \alpha_8 s_{12}s_{34}s_{56} + \text{symmetrization in } (123456)\,. \end{aligned} \tag{4.20}$$

Not all these terms are independent, but kinematical identities eliminate all but two terms which can be chosen to be the terms proportional to α_1, α_2, α_4, α_5.

In general, it is difficult to enumerate all of these kinematical identities analytically in order to reduce the ansatz to an independent basis of terms. Such a task is essentially equivalent to identifying an independent set of Lagrangian operators. However, by working with the ansatz directly, we can evaluate the ansatz numerically in order to remove the elements that generate numerically identical amplitudes.

Lastly, we note that in analogy with color-ordering in YM theory, it is sometimes possible to cleanly disaggregate the Lie algebraic and kinematic elements of the amplitude in an EFT of multiple scalars. For example, in the NLSM, a scattering amplitude A_n can be written as a sum over flavor-ordered amplitudes [46, 63]

$$A_n = \sum_{S/Z_n} \mathrm{T}r(T^{\sigma_{a_1}} T^{\sigma_{a_2}} \dots T^{\sigma_{a_n}})\, A_n^{(s)}(\sigma_{a_1}, \sigma_{a_2}, \dots \sigma_{a_n})\,. \tag{4.21}$$

After stripping off the Lie algebra structure, the flavor-ordered amplitudes are cyclically invariant with poles only in adjacent factorization channels like $s_{123} = 0$ and $s_{2345} = 0$. For these flavor-ordered amplitudes, the procedure for contracting ansatze is the same as before, only subject to the extra conditions of adjacent factorization and cyclic symmetry.

A priori, flavor ordering is not always possible in a general EFT of multiple scalars. In certain cases the flavor decomposition will involve multitrace terms, even in the tree-level scattering amplitude. While the bulk of this chapter is focused on the amplitudes for scalar field or the flavor-ordered amplitudes for multiple scalars, in Sec. 4.7 we also discuss some results for genuine multiple scalar field theories where the flavor-ordering is not assumed.

4.3 From Symmetries to Soft Limits

In this section we revisit the traditional field theory approach whereby the soft limit is derived from a byproduct of symmetry. From this perspective the vanishing of scattering amplitudes—*e.g.* the so-called Adler zero of NGBs—arises from spontaneous symmetry breaking in the EFT. Here the key observation is that the scattering amplitude of a soft NGB is closely related to the matrix element of the corresponding Noether current J^μ, in particular with a certain regular remainder function $R^\mu(p)$ obtained when the one-particle pole of the soft NGB is subtracted (cf. Eq. (4.28)) below[1]). Therefore, the soft behavior of amplitudes is dictated by the properties of the Noether currents of spontaneously broken symmetries.

The EFTs we consider here are derivatively coupled. Most of them are invariant with respect to the simple shift symmetry,

$$\phi(x) \to \phi(x) + a\,, \tag{4.22}$$

which is spontaneously broken, yielding a corresponding NGB field ϕ. Provided we have additional information on the Noether current of the shift sym-

[1]For further details see *e.g.* the textbook [79] and references therein.

metry at our disposal, we can further deduce soft limit properties of the scattering amplitudes beyond the leading Adler zero. This additional information is obtained from the enhanced symmetries of the theory.

While the technical steps of the subsequent analysis are somewhat complicated, our final conclusion is quite simple. In order to obtain an enhanced $\mathcal{O}\left(p^{n+1}\right)$ soft behavior of the amplitudes, it is sufficient that there is an additional non-linear (*i.e.* spontaneously broken) symmetry of the action of the form

$$\delta\phi\left(x\right) = \theta_{\alpha_1\ldots\alpha_n}\left[x^{\alpha_1}\ldots x^{\alpha_n} + \Delta^{\alpha_1\ldots\alpha_n}\left(x\right)\right], \tag{4.23}$$

where $\Delta^{\alpha_1\ldots\alpha_n}\left(x\right)$ is linear combination of local composite operators comprised of ϕ with coefficients that have polynomial dependence on x. More precisely, under some regularity assumptions (*e.g.*. absence of cubic vertices), and (almost) irrespective of the explicit form of $\Delta^{\alpha_1\ldots\alpha_n}\left(x\right)$, the very presence of the symmetry in Eq. (4.23) is sufficient condition for the $\mathcal{O}\left(p^{n+1}\right)$ behavior of the resulting scattering amplitudes corresponding to $\sigma = n+1$. Let us note that this result depends only on the $c-$number part of the general symmetry transformation Eq. (4.23). Therefore, theories invariant with respect to the transformation in Eq. (4.23) with the same polynomial $\alpha(x) = \theta_{\alpha_1\ldots\alpha_n}x^{\alpha_1}\ldots x^{\alpha_n}$ form a universality class of the same soft behavior.

We relegate the details of our proof to Appendix A.1, but here simply sketch the main steps of the argument. A crucial ingredient of the proof is an observation that the Noether currents of the shift symmetry and of the enhanced symmetry in Eq. (4.23) are in fact closely related (for more details see [80]). For single scalar EFTs this can be easily understood intuitively: there is only one NGB (which corresponds to the shift symmetry) but more than one non-linear (i.e. spontaneously broken) symmetry; thus the Noether currents cannot be independent. At the classical level there is another more precise argument. When we promote the global symmetries to local ones (i.e. when the parameters a and $\theta_{\alpha_1\ldots\alpha_n}$ become space-time dependent), the localized symmetry in Eq. (4.23) can be treated as a localized shift symmetry Eq. (4.22) with very special parameter

$$a \to \widehat{a}\left(x\right) = \theta_{\alpha_1\ldots\alpha_n}\left(x\right)\left[x^{\alpha_1}\ldots x^{\alpha_n} + \Delta^{\alpha_1\ldots\alpha_n}\left(x\right)\right]. \tag{4.24}$$

The above relations between currents express the Noether currents of the symmetry Eq. (4.23) in terms of the shift symmetry current J^{μ}, and more importantly put a constraint on the possible form of J^μ itself. At the quantum level[2] this constraint reads

$$\langle\alpha,\mathrm{out}|J^\mu(x)|\beta,\mathrm{in}\rangle\,\partial_\mu x^{\alpha_1}\dots x^{\alpha_n}=\partial_\mu\langle\alpha,\mathrm{out}|\Gamma^{\mu\alpha_1\dots\alpha_n}(x)|\beta,\mathrm{in}\rangle\;, \tag{4.25}$$

where $\Gamma^{\mu\alpha_1\dots\alpha_n}(x)=\gamma_A^{\mu\alpha_1\dots\alpha_n}(x)\,O^A(x)$ is some linear combination of local composite operators $O^A(x)$ with coefficients $\gamma_A^{\mu\alpha_1\dots\alpha_n}(x)$ with polynomial dependence on x. The explicit form of $\Gamma^{\mu\alpha_1\dots\alpha_n}(x)$, which depends on $\Delta^{\alpha_1\dots\alpha_n}(x)$, is irrelevant for the proof of the soft theorem.

Subtracting the one-particle pole in p (where $p=P_\beta-P_\alpha$ is a difference of momenta in the *in* and *out* state) on both sides of the relation in Eq. (4.25), we obtain a relation between the regular remainder function $R^\mu(p)$ of the matrix element of the shift current and the regular remainders $R^A(p)$ of the local operators $O^A(x)$. Such a relation reads

$$e^{-ip\cdot x}\partial_\mu x^{\alpha_1}\dots x^{\alpha_n}R^\mu(p)=\partial_\mu\left[\gamma_A^{\mu\alpha_1\dots\alpha_n}(x)\,e^{-ip\cdot x}\right]R^A(p)\,. \tag{4.26}$$

Assuming regularity[3] of the remainders for $p\to 0$, we can integrate over $\mathrm{d}^d x$ to obtain

$$p_\mu R^\mu(p)\partial^{\alpha_1}\dots\partial^{\alpha_n}\delta^{(4)}(p)=0\,. \tag{4.27}$$

The latter formula, together with

$$\langle\alpha+\phi(\mathbf{p}),\mathrm{out}|\beta,\mathrm{in}\rangle=\frac{1}{F}p_\mu R^\mu(p)\,, \tag{4.28}$$

which relates the remainder function to the NGB amplitude, is at the core of the soft theorems for theories with the enhanced symmetry in Eq. (4.23).

As an example let us consider theories which belong to the universality class of theories invariant with respect to Eq. (4.23) for which

$$\alpha(x)\propto\theta\cdot x\,. \tag{4.29}$$

Prominent members of this class are the general Galileon and DBI. While the former is invariant with respect to the linear shift

$$\delta_\theta\phi(x)=\theta\cdot x\,, \tag{4.30}$$

[2]Such a relation holds automatically at tree-level and we assume here that it is not spoiled by the quantum corrections.
[3]Regularity of R^μ for $p\to 0$ is guaranteed in the absence of the cubic vertices.

the latter has a nonlinearly realized $(d+1)$-dimensional Lorentz symmetry

$$\delta_\theta \phi\left(x\right) = \theta \cdot x - F^{-d} \theta \cdot \phi(x) \partial \phi\left(x\right). \tag{4.31}$$

Inserting the above $\alpha\left(x\right)$ into Eq. (4.27) we get

$$\begin{aligned} 0 &= p_\mu R^\mu(p) \partial^\beta \delta^{(d)}(p) \\ &= -\left[\partial^\beta \delta^{(d)}(p)\right] \left[\lim_{p\to 0} p_\mu R^\mu(p)\right] - \delta^{(d)}(p) \left[\lim_{p\to 0} \partial^\alpha \left(p_\mu R^\mu(p)\right)\right]. \end{aligned} \tag{4.32}$$

We recover thus not only the Adler zero condition

$$\lim_{p\to 0} p_\mu R^\mu(p) = 0\,, \tag{4.33}$$

but also an enhanced $\mathcal{O}\left(p^2\right)$ soft behavior corresponding to

$$\lim_{p\to 0} \partial^\alpha \left(p_\mu R^\mu(p)\right) = F \lim_{p\to 0} \partial^\alpha \left\langle \alpha + \phi(\mathbf{p}), \mathrm{out} \middle| \beta, \mathrm{in} \right\rangle = 0\,, \tag{4.34}$$

implying an Adler zero of the second degree. Further applications and generalizations can be found in Appendix A.1.

4.4 On-shell Reconstruction

As demonstrated in Chap. 3, enhanced soft behavior can be sufficiently constraining so as to fully dictate all tree amplitudes up to a single coupling constant. So for these exceptional EFTs, soft limits and factorization are enough information to fully determine the S-matrix. Since these EFTs are so special, they naturally reside near the boundary of the allowed regions of EFT space, which we verify explicitly in Sec. 4.5.

In the present section, we summarize the notion of on-shell constructibility in the previous two chapters. The concept of on-shell constructibility arose originally in YM theory and gravity, where tree-level amplitudes are fully fixed by two conditions: gauge invariance and factorization. The factorization condition, shown in Eq. (4.17), can then be imposed sequentially until all higher point amplitudes are reduced in terms of a set of input 3pt amplitudes. Said another way, the physical n-pt amplitude is the unique gauge invariant function which satisfies Eq. (4.17) in all channels.

Conveniently, the dual conditions of gauge invariance and factorization can be imposed automatically in YM and gravity using the celebrated BCFW recursion relations [3, 4]. These work by applying a complex shift of the momenta,

$$p_i \to p_i + zq\,, \qquad p_j \to p_j - zq\,, \tag{4.35}$$

where $q^2 = (p_i \cdot q) = (p_j \cdot q) = 0$ and the momentum conservation is preserved. Applying Cauchy's formula to the shifted amplitude $A_n(z)$, we can then reconstruct the original A_n using the products of shifted lower point amplitudes,

$$\int \frac{dz\, A_n(z)}{z} = 0 \quad \to \quad A_n = \sum_k \frac{A_L(z_k) A_R(z_k)}{P^2}\,, \tag{4.36}$$

where the sum is over all factorization channels for which $P^2(z_k) = 0$. Later on, the BCFW recursion relations were generalized to apply to a much broader class of theories [8, 43, 60].

An important requirement of Eq. (4.36) is that the shifted amplitude falls off at infinity, $A_n(z) \sim \frac{1}{z}$ for $z \to \infty$. If this is not true, then the recursion includes boundary terms which are difficult to calculate, though some progress has been recently made on that front [42, 58, 59]. For EFTs, amplitudes typically grow at large z as $A_n(z) \sim z^p$ where $p > 0$, so none of the standard recursion relations can be used.

This obstruction to recursion in EFTs is obvious from a physical perspective: typically there is an infinite tower of interactions in EFTs which produces contact terms in amplitudes. These contact terms cannot be constrained by factorization. So we need additional information to fix these unconstrained contact terms. In YM and gravity, gauge invariance dictates the appearance of contact terms and makes reconstruction feasible. In principle, it may be possible that these contact interactions can be fixed by leading and subleading soft theorems, and in particular recent work on conformal field structures for amplitudes suggest this may occur [81].

In scalar EFT, there is no gauge invariance to speak of, so it is natural to consider soft structure to relate cancellations between contact and pole terms. In particular, we call the amplitude A_n *soft limit constructible* if it is the unique function satisfying two conditions:

1. It has local poles and factorizes correctly on them according to Eq. (4.17).
2. It has required soft limit behavior $A_n = \mathcal{O}(p^\sigma)$.

Soft Shift	**Applicability**	**# of Soft Limits**	**Bound**
All-Line	$n > d+1$	n	$\frac{\rho-1}{\sigma-1} \geq \frac{v}{v-2}$
All-But-One-Line	$n > 4$	$n-1$	$\frac{\rho-2}{\sigma-2} \geq \frac{v-1}{v-2}$
All-But-Two-Line	$n \geq 3$ and $d \geq 4$	$n-2$	$\rho \geq \sigma - \frac{2}{v-2}$

Table 4.1: The first and second columns list soft momentum shifts and the conditions under which they can be applied to an amplitude with n legs to probe its soft limits. The third column lists the number of soft limits that are accessible by each soft shift when these criteria are satisfied. The fourth column lists the resulting constraints on EFTs with fixed (ρ, σ, d, v) proved in Section 4.5. As discussed in text, these constraints are derived by applying each soft shift to the leading non-trivial amplitude, which is an amplitude with $n = v$ legs.

Soft limit constructibility imposes non-trivial conditions on our classification parameters (ρ, σ) which we will review soon. In the subsequent sections we discuss how to probe soft limits while maintaining on-shell kinematics, as well as the construction of amplitudes from the above two criteria.

Soft Momentum Shifts

Our analysis makes heavy use of the soft momentum shift proposed in Chap. 3. This deformation maintains total momentum conservation and on-shell conditions while probing the soft limits of external particles. In [74] these momentum shifts were used to construct new recursion relations for scattering amplitudes in EFTs. However, here we need them as just a tool for probing the kinematics of scattering amplitudes.

The original soft momentum shift is applicable only when there are more than $d+1$ external legs in d spacetime dimensions. In order to probe the full EFT space, we develop a number of simple variations on the soft momentum shift. Although it seems to be a technical obstruction, we will see that the applicability has a one-to-one correspondence to the non-trivial soft limits in Sec. 4.5. We now discuss each momentum shift, whose properties are summarized in Table 4.1.

All-Line Soft Shift

We define the all-line soft shift by

$$p_i \;\; \to \;\; p_i(1 - z a_i)\,, \qquad 1 \leq i \leq n\,, \tag{4.37}$$

where the shifted momenta are automatically on-shell but momentum conservation requires

$$\sum_{i=1}^{n} a_i p_i = 0 \,. \tag{4.38}$$

Since this constraint is a relation among the momenta, it may or may not be satisfied depending on the number of momenta n relative to the space-time dimension d.

There are two configurations of a_i that are unphysical or not useful for probing the soft kinematic regimes of the amplitude. First, one can rescale all the a_i uniformly. This corresponds simply to a rescaling of the momentum deformation parameter z and therefore is not a new solution. Second, consider the case where the a_i all equal. This corresponds to a shift of the momentum of each leg by a constant times the momentum, which is also equivalent to a total rescaling of all the momenta. This class of momentum shifts does not probe any interesting kinematic regime of amplitudes provided the amplitude is a homogeneous function of momentum, which we assume here.

The above two configurations can be viewed as the "pure gauge" configurations of a_i. We can uniformly rescale or translate any solution of a_i and the result is still a solution by Eq. (4.38). When counting degrees of freedom, the two pure gauge directions need to be excluded. Subtracting these two configurations, only $n-2$ degrees of freedom among the a_i are of interest. The d constraints of Eq. (4.38) then reduce these to $n-d-2$ independent variables. Consequently, for $n \leq d+1$, the momenta are linearly independent so there are either no solutions to Eq. (4.38) or we have the trivial configuration where all a_i are equal.

Only for scattering amplitudes with sufficient numbers of external particles $n \geq d+2$ can we apply the soft shift in Eq. (4.37) with distinct a_i. In the marginal case $n = d+2$, the parameters a_i are completely fixed up to rescaling and translation. There are residual degrees of freedom when $n > d+2$. Note that the momentum conservation constraint in Eq. (4.38) implies that the a_i are implicitly dependent on the p_i, constrained so they actually represent $n-d-2$ independent parameters.

Moreover, $z \to 1/a_i$ corresponds to taking the soft limit of particle i. So for $n \geq d+2$ it is possible to apply an all-line soft shift that probes all the soft kinematic limits of the amplitude.

All-But-One-Line Soft Shift

Similarly, we can define an all-but-one-line shift by

$$p_i \rightarrow p_i(1 - za_i)\,, \qquad 1 \leq i \leq n-1 \tag{4.39}$$

$$p_n \rightarrow p_n + zq_n\,, \tag{4.40}$$

where momentum conservation and the on-shell conditions imply that

$$q_n = \sum_{i=1}^{n-1} a_i p_i\,, \qquad q_n^2 = q_n p_n = 0\,. \tag{4.41}$$

Here we are shifting all the external legs, but in such a way that all but one of the soft limits can be accessed by taking $z \to 1/a_i$.

The all-but-one-line shift is defined by $n-1$ parameters a_i. As before, the rescaling of a_i and the case where all a_i are equal correspond to a uniform rescaling of all the momenta, so only a subset of $n-3$ of these parameters are kinematically useful. Finally, the two on-shell conditions reduce these to $n-5$ independent variables, corresponding to distinct values of a_i.

In summary, the all-but-one-line shift acts non-trivially on any amplitude with $n \geq 5$ legs in all dimensions, and which can probe $n-1$ soft limits.

All-But-Two-Line Soft Shift

Lastly, we consider an all-but-two-line soft shift defined by

$$p_i \rightarrow p_i(1 - za_i)\,, \qquad 1 \leq i \leq n-2 \tag{4.42}$$

$$p_{n-1} \rightarrow p_{n-1} + zq_{n-1}\,, \tag{4.43}$$

$$p_n \rightarrow p_n + zq_n\,, \tag{4.44}$$

where momentum conservation and on-shell conditions imply

$$q_{n-1} + q_n = \sum_{i=1}^{n-2} a_i p_i\,, \qquad q_{n-1}^2 = q_n^2 = q_{n-1}p_{n-1} = q_n p_n = 0\,. \tag{4.45}$$

Here we treat the $n-2$ parameters a_i as free variables so that the two d-dimensional vectors q_{n-1} and q_n are constrained by the d constraints from momentum conservation. This corresponds to d degrees of freedom subject to four constraints, leaving $d-4$ degrees of freedom in q_{n-1} and q_n. Removing rescaling and translation as before, there are $n-4$ degrees of freedom in a_i. So the total number of independent variables are $(n-4)+(d-4)$.

In summary, for the general case $n \geq 5$, we find that the all-but-two-line soft shift acts nontrivially on any amplitude in $d \geq 3$ dimensions. For the special case of 4pt, the all-but-two-line soft shift only works for $d \geq 4$ but not $d = 3$.

Soft Recursion Relations

Next, we review the recursion relations for EFTs in [74] (see also the generalization in [82]) which is a crucial tool for bounding the space of consistent EFTs. To compute the n-pt amplitude, we first deform the momenta by any of the available soft shifts in Sec. 4.4. This promotes the amplitude A_n into a function of z,

$$A_n \to A_n(z)\,. \tag{4.46}$$

Then consider the contour integral

$$\oint \frac{dz}{z}\frac{A_n(z)}{F_n(z)} = 0\,, \tag{4.47}$$

where the denominator $F_n(z) = \prod_{i=1}^{n_s}(1 - a_i z)^\sigma$. The product in $F_n(z)$ runs from 1 to n_s, the number of external legs whose soft limits are accessible by the soft shift, given by the third column in Table 4.1. We can retrieve the original amplitude $A_n(0)$ by choosing the contour as an infinitesimal circle around $z = 0$. Cauchy theorem then relates the original amplitude as the (opposite) sum of all other residues. The possible poles correspond to factorization (poles in $A_n(z)$), soft limit ($F_n(z) = 0$), and the pole at infinity. However, the integrand is designed such that $A_n(z)/F_n(z)$ has no pole in the soft limit $z = 1/a_i$ since the amplitude vanishes as

$$A(z \to 1/a_i) \sim (1 - a_i z)^\sigma, \tag{4.48}$$

as we define in Eq. (4.3). If there is no pole at infinity, the original amplitude is equal to the sum of residues from factorization channel. For each factorization channel I, there are two poles $z_{I\pm}$ corresponding to the roots of

$$P_I^2(z) = P_I^2 + 2P_I \cdot Q_I z + Q_I^2 z^2 = 0\,, \tag{4.49}$$

where $P_I(z) = P_I + zQ_i$ and where

$$P_I = \sum_{i\in I} p_i \quad \text{and} \quad Q_I = -\sum_{i\in I} a_i p_i\,. \tag{4.50}$$

By locality, each residue is a product of lower-point amplitudes. Applying Cauchy theorem then yields the recursion relation

$$A_n(0) = \sum_I \frac{1}{P_I^2}\frac{A_L(z_{I-})\,A_R(z_{I-})}{(1 - z_{I-}/z_{I+})F(z_{I-})} + (z_{I+} \leftrightarrow z_{I-})\,. \tag{4.51}$$

The recursion relation above hinges on the absence of pole at infinity. The large z behaviors are $F_n(z) \sim z^{n_s\sigma}$ and $A_n(z) \sim z^m$ where A_n has m powers of momenta defined by Eq. (4.5). The function $A_n(z)/F_n(z)$ vanishes at infinity provided $m < n_s\sigma$ which can be written as

$$\sigma > \frac{2 + (n-2)\rho}{n_s} \tag{4.52}$$

in terms of ρ defined by Eq. (4.6). Remember that depending on n and d, each shift has its own applicability (see Table 4.1). For exceptional theories, $\sigma = \rho + 1$, we can use any of the three shifts in Sec. 4.4 to construct the amplitude starting from 5pt. This implies 4pt amplitudes dictate all other amplitudes. For theories on the non-trivial line $\sigma = \rho$, all-line and all-but-one-line soft shift can construct amplitudes with $\sigma > 1$ and $\sigma > 2$ respectively. Note that all-but-two-line is no longer applicable on this line. According to Table 4.1, theories with $\sigma = \rho = 2$ like the general Galileon need the 4pt to $(d+1)$pt scattering amplitudes as seeds for the recursion relation.

Example: Six point amplitude in NLSM

As an illustration of these recursion relations, consider the 6pt amplitude in NLSM. We use the all-but-one line soft shift so that our results apply in general dimensions. This momentum shift is applicable in all exceptional theories for amplitudes above 4pt. The flavor-ordered 4pt amplitude reads

$$A_4 = s_{12} + s_{23}\,. \tag{4.53}$$

The recursion relation in Eq. (4.51) can be rewritten as

$$A_6(0) = -\sum_I \mathrm{Res}_{z_{I\pm}} \left(\frac{A_L(z) A_R(z)}{z\, P_I^2(z)\, F(z)} \right) . \tag{4.54}$$

Note that we only probe soft limits of the first five legs, so $F(z) = \prod_{i=1}^5 f_i(z)$ where $f_i(z) = (1 - a_i z)$. For 6pt amplitude, the sub-amplitudes $A_L(z), A_R(z)$ are 4pt which have no poles. Thus, we can use the Cauchy theorem again term by term in the above equation

$$\begin{aligned} A_6 &= \sum_I \left\{ \frac{A_L A_R}{P_I^2} + \sum_i \underset{z=1/a_i}{\mathrm{Res}} \left(\frac{A_L(z) A_R(z)}{z P_I^2(z) F(z)} \right) \right\} \\ &= \left[\frac{(s_{12}+s_{23})(s_{45}+s_{56})}{P_{123}^2} + \dots \right] + \sum_{i,I} \underset{z=1/a_i}{\mathrm{Res}} \left(\frac{A_L(z) A_R(z)}{z P_I^2(z) F(z)} \right) , \end{aligned} \tag{4.55}$$

where the first term is the residue at $z = 0$, the second term sums over the residues from $F(z) = 0$ which corresponds to the soft limits, and ellipses denote cyclic permutations. We will identify the second term as the contact term in the amplitude.

$$A_{6,\text{contact}} = \sum_{i,I} \operatorname*{Res}_{z=1/a_i} \left(\frac{A_L(z)A_R(z)}{zP_I^2(z)F(z)} \right) . \tag{4.56}$$

For the flavor-ordered 6pt amplitude there are three factorization channels corresponding to when P_{123}, P_{234}, and P_{345} go on-shell. The above contact term can be decomposed into

$$A_{6,\text{contact}} = A^{(123)}_{6,\text{contact}} + A^{(234)}_{6,\text{contact}} + A^{(345)}_{6,\text{contact}} . \tag{4.57}$$

Considering the first term, we can plug Eq. (4.53) into Eq. (4.56), yielding

$$A_6^{(123)} = -\left(\frac{\hat{s}_{45} + \hat{s}_{56}}{f_2 f_3 f_4 f_5} \right) \Big|_{z=1/a_1} - \left(\frac{\hat{s}_{45} + \hat{s}_{56}}{f_1 f_2 f_4 f_5} \right) \Big|_{z=1/a_3} - \left(\frac{\hat{s}_{12} + \hat{s}_{23}}{f_1 f_2 f_3 f_5} \right) \Big|_{z=1/a_4} . \tag{4.58}$$

Here $\hat{s}_{ij}$ is the Mandelstam variable evaluated at shifted kinematics. Note that one of the sub-amplitudes cancels the propagator on the soft limit. For example, $P^2_{123}(1/a_1) = \hat{s}_{23} = A_L(1/a_1)$. The residue at $z = 1/a_1$ only shows up in $A_6^{(123)}$ and $A_6^{(234)}$. Combining the two yields

$$-\left(\frac{\hat{s}_{23} + \hat{s}_{34} + \hat{s}_{45} + \hat{s}_{56}}{f_2 f_3 f_4 f_5} \right) \Big|_{z=1/a_1} = \operatorname*{Res}_{z=1/a_1} \left(\frac{\hat{s}_{12} + \hat{s}_{23} + \hat{s}_{34} + \hat{s}_{45} + \hat{s}_{56} + \hat{s}_{61}}{z f_1 f_2 f_3 f_4 f_5} \right), \tag{4.59}$$

where we include $\hat{s}_{12} + \hat{s}_{61}$ in the numerator in the right-hand side since they vanish at $z = 1/a_1$. All residues at $z = 1/a_i$ can be combine into such form. Summing all of such gives

$$\begin{aligned} &\sum_{i=1}^{5} \operatorname*{Res}_{z=1/a_i} \left(\frac{\hat{s}_{12} + \hat{s}_{23} + \hat{s}_{34} + \hat{s}_{45} + \hat{s}_{56} + \hat{s}_{61}}{z f_1 f_2 f_3 f_4 f_5} \right) \\ &= -\left(s_{12} + s_{23} + s_{34} + s_{45} + s_{56} + s_{61} \right), \end{aligned} \tag{4.60}$$

where we use Cauchy theorem again to recast the sum into residue at the origin. Combining the non-contact terms, the final answer is

$$A_6 = \left[\frac{(s_{12} + s_{23})(s_{45} + s_{56})}{P^2_{123}} + \dots \right] - (s_{12} + \dots) , \tag{4.61}$$

where ellipses again denote cyclic permutations. The above expression is the same one obtained via Feynman diagrams.

4.5 Bounding Effective Field Theory Space

With an arsenal of momentum shifts and on-shell recursion relations, we are now ready to ascertain the allowed parameter space of EFTs. The aim of this section is to study the parameter space of EFTs as a function of (ρ, σ, d, v) and determine regions of theory space which are inconsistent with locality and Lorentz invariance. To exclude swaths of EFT parameter space, we will consider several consistency checks. The first will be a study of the soft limit of the leading interaction vertex of the EFT. The second will be a study of the locality properties of higher point amplitudes.

Soft Limit of the Leading Interaction

Consider an EFT with the fixed (ρ, σ, d, v). All amplitudes in this EFT have soft degree σ by assumption, including the leading non-vanishing amplitude A_v, where v is the valency of the lowest point interaction. Since A_v is comprised of a single vertex it has no factorization channels and is simply a polynomial function of the momenta. Given the definition of ρ in Eq. (4.6), this function contains $\rho(v-2)+2$ powers of momentum.

To begin, consider a soft momentum shift in Sec. 4.4 applied to A_v, lifting it to a complex function of z, so $A_v \to A_v(z)$. Since A_v is a contact amplitude, $A_v(z)$ is simply a polynomial in z. The degree of this polynomial is fixed by the mass dimension $\rho(v-2)+2$, since each momentum in the shift is linear in z.

At the same time, the vanishing soft limit corresponds to zeros of this polynomial. In particular, if v_s is the number of external legs whose soft limits can be probed by the soft momentum shift, then the total number of zeros are $v_s\sigma$ according to Eq. (4.3). Comparing the degree of polynomial with the number of zeros yields

$$\rho \geq \frac{v_s\sigma - 2}{v-2}\,. \tag{4.62}$$

Therefore, the most stringent bound on ρ requires the maximal v_s. Crucially, this depends on v and d as shown in Sec. 4.4 and so does the bound on ρ. These bounds are summarized in the fourth column of Table 4.1.

Altogether these bounds place a lower bound on ρ as a function of σ and v which excludes almost all possible EFTs with non-trivial soft limits. To explain these constraints, let us consider each of these bounds as a function of the leading interaction valency v relative to the space-time dimension d. Throughout, we assume space-time dimension $d \geq 4$.

The most general possible bounds arise from the all-but-two-line shift. As we are concerned with scalar theories, the lowest possible valency of the leading interaction is $v = 4$. From Table 4.1, the bound is weakest—that is, places the smallest lower bound on ρ—for $v = 4$ and becomes stronger as v grows. So conservatively, we can evaluate the all-but-two-line shift constraint from Table 4.1 for $v = 4$ to obtain a universal and remarkably stringent bound of

$$\rho \;\; \geq \;\; \sigma - 1\,. \tag{4.63}$$

Notably, this bound is exactly saturated by the exceptional theories discussed in [51], corresponding to the NLSM $(\rho, \sigma) = (0, 1)$, DBI theory $(\rho, \sigma) = (1, 2)$, and special Galileon $(\rho, \sigma) = (2, 3)$. Unsurprisingly, this result verifies that there are no theories with $\rho = 0, 1, 2$ with soft limits that are super-enhanced beyond these exceptional theories. This is expected because these exceptional theories each have a single coupling constant and are thus already so constrained by soft limits that they have no additional free parameters. Demanding a super-enhanced soft limit will over-constrain these theories, so no EFT exists with such properties. Less obvious is the statement that for general ρ—including rational but non-integer values—there are no theories with soft limits enhanced beyond the exceptional line defined by Eq. (4.63). Note that the proof here uses all-but-two-line shift which is valid only in $d \geq 4$. The same conclusion holds in $d = 3$, which we will revisit in the end.

For the all-line and all-but-one-line shifts we obtain more stringent constraints which are applicable only in specific ranges for v and d. First, consider the constraint in Table 4.1 from the all-line shift, which is applicable only when the valency v of the leading interaction is greater than $d + 1$. The resulting bound on ρ is a line that intersects the point at $(\rho, \sigma) = (1, 1)$, which describes a derivatively coupled theory of a single NGB, sometimes called $P(X)$ theory (see Appendix A.3). The slope of the boundary is $v/(v - 2) > 1$ so it is steeper than the $\rho = \sigma$ line that delineates the boundary between theories with trivial versus non-trivial soft limits. Since σ is a positive integer, we can exclude all EFTs with non-trivial soft limits for which $v > d + 1$. This result is consistent with the properties of known EFTs. In particular, the Galileon theory is known to have interaction vertices up to $v = d + 1$ valency but not higher.

Second, consider the constraint in Table 4.1 from the all-but-one-line shift, which is applicable only when $4 < v \le d+1$. Here the resulting bound intersects the Galileon theory at $(\rho, \sigma) = (2,2)$ with a slope of $(v-1)/(v-2) > 1$, which is again steeper than $\rho = \sigma$. Hence, this bound eliminates all EFTs with non-trivial soft limit $\sigma > 2$ and $v > 4$. The only allowed possibilities are then $(\rho, \sigma) = (2,2)$, which is consistent with the known Galileon theory, or $\sigma = 1$ with $\rho \ge (v-3)/(v-2)$, which is saturated by WZW theory. We will discuss the allowed region in depth in Sec. 4.6.

The above bounds significantly simplify the numerical search of possible theories. For a given dimension d, we only need to search leading amplitudes up to $v = d+1$. The inverse question is, given the leading valency v, what are the upper bounds on spacetime dimensions for which we do not expect to find new non-trivial amplitudes?

The answer is given by a simple statement in kinematics. For example, the 4pt kinematics in any $d \ge 3$, effectively lies in a three-dimensional subspace. This is easily seen in the center of mass frame, where the four spatial momenta lie in a plane. The generalization to high dimension is straightforward: the v-pt kinematics in the $d \ge v-1$ dimension only live in a $(v-1)$-dimensional subspacetime. If this is true, we can always take the soft limits within this $(v-1)$-dimensional subspacetime. It implies the enhanced soft limit at v-pt in $d \ge v-1$ dimensions must be present in $d = v-1$ already. The numerical search up to $d = v-1$ can saturate all non-trivial amplitudes at arbitrarily higher dimensions, which significantly reduces the space of possible theories that need to be checked.

The proof is analogous to 4pt. First consider the center of mass frame of the first two particles whose momenta are chosen as

$$\begin{aligned} p_1 &= \frac{E_{\mathrm{CM}}}{2}(1,1,0,\cdots,0)\,, \\ p_2 &= \frac{E_{\mathrm{CM}}}{2}(1,-1,0,\cdots,0)\,. \end{aligned} \tag{4.64}$$

Next, due to total momentum conservation, only $v-3$ momenta of the remaining $v-2$ particles are independent. Using spatial rotations (or the standard Gram-Schmidt decomposition), we can choose a basis where these $v-3$ momenta lie in a $(v-3)$-dimensional subspace. Together with the spatial part $p_{1,2}$, all spatial momenta can be chosen to reside in the first $v-2$ spatial components

$$p_i = (E_i, p_{i1}, \cdots, p_{i,v-2}, 0, \cdots, 0), \quad \forall\, i = \{3, \cdots, v\}\,. \tag{4.65}$$

Combining with the temporal component, we find the v-pt kinematics only live in a $(v-1)$-dimensional sub-spacetime as we claimed.

Let us come back to the case of $d = 3$. First, the same bound from the all-but-one-line and all-line shifts applies for $v \geq 5$ and $v \geq 6$ respectively. So we only need to consider the 4pt case in $d = 3$. Although we cannot use momentum shifts to prove Eq. (4.63), the 4pt kinematics always live in a three-dimensional subspace. Therefore, the 4pt kinematics should still satisfy $\rho = \sigma + 1$ as in higher dimensions, which can be verified explicitly. So all the bounds are the same for $d = 3$.

In summary, the leading valency v of EFTs with an enhanced soft limit must satisfy

$$v \leq d + 1\,, \tag{4.66}$$

while the enhanced soft limit should be present in

$$d = v - 1\,. \tag{4.67}$$

These imply that for the numerical search of the non-trivial leading amplitudes, we can focus on the line of $v = d + 1$. Moreover, if $v > 4$, then the soft degree and power counting parameters are bounded by

$$\sigma = 1 \text{ or } 2\,, \quad \text{ and } \quad \rho \geq (v-3)/(v-2)\,. \tag{4.68}$$

Locality of Higher Point Amplitudes

The bounds derived in the previous section imply that the soft degree of an EFT cannot exceed those of the exceptional EFTs. Nevertheless, these constraints still permit an infinite band in EFT space between the exceptional line $\rho = \sigma - 1$ and non-trivial line $\rho = \sigma$, as shown in Fig. 4.1. While we can constructively identify the known theories with $\sigma = 1, 2, 3$, there is *a priori* no restriction on EFTs of arbitrarily high soft degree beyond $\sigma > 3$, which we dub "super-enhanced" soft behavior. However, in this section we show how EFTs with such super-enhanced soft behavior are impossible.

As discussed in the previous section, an exceptional EFT must have a valency $v = 4$ for the leading interactions. Without loss of generality, the corresponding

4pt contact amplitude takes the form

$$A_4 = \sum_{b=0}^{\rho+1} \lambda_b\, s_{13}^{b}\, s_{12}^{\rho+1-b}, \tag{4.69}$$

where λ_b are coupling constants. From Eq. (4.69) we see that the soft degree is $\sigma = \rho + 1$ but can in principle be arbitrarily large. Hence, there is of yet no obvious obstruction to a theory with arbitrary high soft degrees.

To exclude such theories, we exploit the fact that exceptional theories are on-shell constructible [74]. Furthermore, in the previous section we showed that for $\sigma > 2$, the only contact amplitudes consistent with non-trivial soft behavior enter at 4pt. Altogether, this implies that all higher point amplitudes are fixed in terms of the 4pt amplitudes in Eq. (4.69) via on-shell recursion. Self-consistency then requires that the resulting higher point amplitude be independent of the precise way in which recursion is applied. Concretely, the recursion relation should produce scattering amplitudes which are independent of the specific momentum shift employed. For soft recursion relations, this means that the intermediate and unphysical momentum shift parameters a_i should cancel in the final expression, since the physical amplitude should only depend on Mandelstam variables. As shown in the example in Sec. 4.4, such a cancellation is highly non-trivial. In the following, we study this cancellation and use it to derive a no-go theorem for the existence of super-enhanced theories.

Our approach mirrors the so-called “four-particle test” of [35] (and see also [36]), where the consistency of higher spin theories was similarly studied via on-shell recursion. There it was shown that for theories of massless particles of spin greater than two, recursion relations yield different answers depending on the momentum shift used. This failure of recursion relations indicates an underlying tension between locality, factorization, and gauge invariance in the underlying theory. The same logic can be applied here: if soft recursion relations yield dependence on unphysical parameters in the final answer, then it is impossible to construct higher point amplitudes which are simultaneously local with the correct soft and factorization properties.

Since the details of the proof are rather technical, readers can skip the following and move to Sec. 4.6 if they are uninterested in the details. However, our final results from this analysis are that:

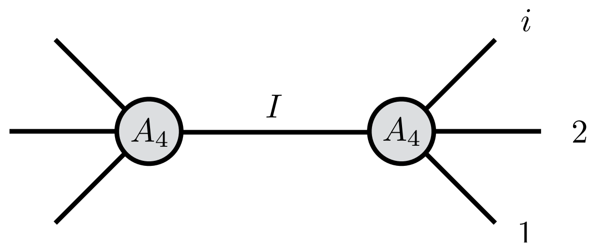


Figure 4.2: Factorization channel with spurious pole a_{12}.

- All EFTs with non-trivial soft behavior have $\rho < 3$. This claim is independent of flavor structure, and applies for single or multiple scalar EFTs.
- The NLSM is the unique EFT with flavor-ordered amplitudes that exhibit exceptional soft behavior, $\sigma = \rho + 1$.

We find the locality test imposes a stringent bound on the theory space of EFTs, as shown in Figure 4.1. Galileon theories live on the boundary of the allowed region.

Details of the Proof

We diagnose the self-consistency of super-enhanced soft behavior by analyzing the 6pt amplitude, by analogy with the 4pt test in higher spin gauge theory. Specifically, we consider the 6pt kinematics in $d = 3$ where we are allowed to apply all-line soft shift. For higher dimensional theories, we can always take a special 6pt kinematics restricted to $d = 3$. One might worry that the 6pt amplitude vanishes in this limit and thus trivializes the test. However, the non-trivial soft limits with $\rho > 2$ fix all amplitudes from 4pt amplitudes via the recursion relations. As we discussed in Section 4.5, the 4pt kinematics in $d = 3$ is already generic. We will see the spurious pole cancellation put constraints on the 4pt coupling constants. If the only consistent coupling constants are zero in the $d = 3$ special kinematics, then the 6pt amplitudes, which are given by the recursion, must be trivial even in generic kinematics. Therefore, the proof here applies to general $d \geq 3$.

Let us consider the 6pt amplitude obtained from recursion relations. As shown in Eq. (4.55), it can be decomposed into factorization terms (comprised of two 4pt vertices and a propagator) and the contact term (comprised of one 6pt vertex). The example presented in Eq. (4.55) is for the NLSM, but this decomposition is generally applicable.

First, we see that the factorization terms are manifestly independent of the shift parameters a_i. Hence, these cannot contain any spurious dependence on the momentum shift so we can ignore them. On the other hand, the contact term reads

$$A_{6,\mathrm{contact}} = \sum_{i,I} \operatorname*{Res}_{z=1/a_i} \left(\frac{A_L(z) A_R(z)}{z P_I^2(z) F(z)} \right) , \tag{4.70}$$

which can in principle depend on a_i, yielding an inconsistency. Conversely, consistency implies that Eq. (4.70) is a_i independent, so all spurious poles in these parameters must cancel. Here unphysical poles in a_i can only appear in the denominator of Eq. (4.70) because $A_{L,R}$ are 4pt amplitudes which are local functions of momenta, and thus local functions of a_i.

Let us determine what kind of spurious poles can arise from the above equation. Recall that $F(z) = \prod_{j=1}^{6} f_i^{\sigma}(z)$, where $f_j(z) = 1 - a_j z$ is the product of rescaling factors. Furthermore, observe that the rescaling factor of leg j evaluated at $z = 1/a_i$ is proportional to $(a_i - a_j)$, which induces a spurious pole. In general, the shifted propagator can also contain a similar form of spurious pole: for example, $P_{123}^2(1/a_2) = f_1 f_3 s_{13}$ is proportional to $(a_2 - a_1)(a_2 - a_3)$.

In what follows we analyze the unphysical pole at $a_1 \to a_2$ and show that the criterion that this singularity cancels in the final amplitude imposes a constraint on allowed EFTs. Here it is important that we can take the all-line soft shift in $d = 3$ at 6pt, so it is possible to send $a_1 \to a_2$ while keeping all other a_i distinct. Taking residue at $z = 1/a_2$ is then reminiscent of a double soft limit, where leg 2 is exactly soft, $p_2(1/a_2) = 0$, and leg 1 approaches soft $p_1(1/a_2) \sim (a_1 - a_2) p_1$ as $a_1 \to a_2$. As explained in the previous paragraph, the spurious pole in $a_1 - a_2$ only appears when taking the residue at $z = 1/a_1$ or $z = 1/a_2$. Legs 1 and 2 either appear on opposite sides of the factorization channel, or the same side, and we now consider these in turn.

If legs 1 and 2 are on different sides of factorization channel, we can always parametrize the 4pt amplitudes as

$$\begin{aligned} A_L(z) &= \sum_{b=0}^{\rho+1} \lambda_b\, \hat{s}_{1i}^{b}\, \hat{s}_{1j}^{\rho+1-b} \propto f_1^{\rho+1}(z)\,, \\ A_R(z) &= \sum_{b=0}^{\rho+1} \lambda_b\, \hat{s}_{2k}^{b}\, \hat{s}_{2l}^{\rho+1-b} \propto f_2^{\rho+1}(z)\,, \end{aligned} \tag{4.71}$$

where i, j, k, l label the on-shell legs in the amplitude other than legs 1 and 2. Recall that hatted Mandelstam variables are evaluated at shifted kinematics. Meanwhile, the internal propagator, $P_I^2(z)$, will never be singular as $a_1 \to a_2$, since the double soft limit does not yield a singularity from the propagator in this channel. Since $F(z) \propto f_1(z)^\sigma f_2(z)^\sigma$, Eq. (4.71) implies that the overall scaling of the contact factorization term is $(f_1 f_2)^\Delta$, where for later convenience we define

$$\Delta = \rho + 1 - \sigma\,. \tag{4.72}$$

Here $\Delta = 0$ for exceptional EFTs, while $\Delta = 1$ for EFTs with non-trivial behavior. Meanwhile, $\Delta > 1$ EFTs have trivial soft behavior that is guaranteed simply by large numbers of derivatives, and $\Delta < 0$ is forbidden by the arguments from the contact amplitude in the previous section. Putting this all together, since Δ is strictly non-negative, these terms can never produce a spurious pole as $a_1 \to a_2$.

Therefore, the spurious pole only appears when legs 1 and 2 are on the same side of the factorization channel. Namely, we only need to consider factorization channel $I = 123, 124, 125, 126$ as shown in Figure 4.2. In this case it is convenient to parametrize the 4pt amplitude

$$A_4(z) = \sum_{b=0}^{\rho+1} \lambda_b\, \hat{s}_{1i}^{b}\, \hat{s}_{12}^{\rho+1-b} \propto f_1^{\rho+1}(z) f_2^{\rho+1-b}(z)\,, \tag{4.73}$$

without loss of generality and where $i = 3, 4, 5, 6$. This is chosen so that the 4pt amplitude carries a factor of $f_1^{\rho+1}(z)$ that will overpower the $f_1^\sigma(z)$ factor in the denominator of the recursion. Thus, we find that spurious poles in $a_1 \to a_2$ are localized to the residue from f_2, *i.e.* the residue at $z = 1/a_2$ in four factorization channels $I = 123, 124, 125, 126$.

Consider the factorization $I = 12i$. We now combine the parameterization of the 4pt amplitude in Eq. (4.73), together with the recursion relation in

Eq. (4.70) to localize the spurious pole in $a_1 \to a_2$. We need to take the residue at $z = 1/a_2$ from

$$\frac{A_L(z)A_R(z)}{zP_I^2(z)F(z)} = \sum_{b=0}^{\rho+1} s_{12}^{\rho+1-b} \left(\frac{A_L(z)\, \lambda_{i,b} s_{1\hat{i}}^b f_1^\Delta(z)}{zP_I^2(z)F_{3456}(z) f_2^{b-\Delta}(z)} \right), \tag{4.74}$$

where $s_{1\hat{i}} = 2p_1 \cdot p_i(z)$, $F_{3456}(z) = (f_3 f_4 f_5 f_6)^\sigma$, and Δ is defined as in Eq. (4.72). Here we have kept the dependence of the coupling constant on i. The pole at $z = 1/a_2$ in the above equation is generally not a simple pole. The residue is then obtained through taking derivatives. However, the inverse propagator at $z = 1/a_2$ contains a spurious pole but not its derivative:

$$\begin{aligned} P_I^2(1/a_2) &= f_1(z) s_{1\hat{i}}\Big|_{z=1/a_2} \\ \frac{dP_I^2}{dz}(1/a_2) &\xrightarrow{a_1 \to a_2} -a_2(s_{1\hat{i}} + s_{2\hat{i}})\Big|_{z=1/a_2}. \end{aligned} \tag{4.75}$$

Therefore, the leading spurious pole in the residue occurs when all the derivatives act on $P_I^2(z)$ but not on the numerators. The worst spurious pole occurs for the maximal b with $\lambda_b \neq 0$ in Eq. (4.74).

Now we combine everything together. First take the residue from Eq. (4.74) and only keep the leading spurious term from b_{max}. Then, sum over factorization channels $I = 123, 124, 125, 126$. Finally we find

$$\frac{1}{(a_1 - a_2)^{b_{max} - 2\Delta}} \times \left[\sum_{i=3}^{6} A_L(1/a_2)\, \lambda_{i,b_{max}}\, s_{1\hat{i}}^\Delta (s_{1\hat{i}} + s_{2\hat{i}})^{b_{max} - \Delta - 1} \right]_{z=1/a_2}, \tag{4.76}$$

where we drop the irrelevant proportional constant. The spurious pole cancellation implies the numerator in the square bracket must vanish whenever the spurious pole forms, *i.e.*, $b_{max} > 2\Delta$.

In principle, there are several ways the above numerator can vanish. The most naive way is to forbid coupling constants whenever the spurious pole appears. The cancellation could also happen in the state sum in the multiple scalar case. The second possibility is to cancel the numerator in the summation of factorization channels. We only know the sufficient conditions for this to happen, which we will describe soon. But *a priori*, there could be accidental cancellations beyond our expectation and we have to check numerically for a given b_{max}. Strictly speaking, this is a loophole since we cannot check arbitrary high b_{max} numerically. However, we can localize the spurious pole to one single factorization using the so-called "bonus" relation. In such a case, the spurious pole cannot appear at all. We can close the loophole by combining numerical checks to sufficiently high b_{max} and after that using the proof via bonus relations. This proof using bonus relations is presented in Appendix A.2. Hence, we will assume no such accidental cancellation in what follows.

In the following, we will first discuss sufficient conditions for the spurious pole cancellation, which are satisfied by all known EFTs. These conditions are also necessary as supported by numerical checks and proofs from bonus relations. We will then show bounds in single and multiple scalars in turn.

Locality Test in Known EFTs

In the case of a single scalar, all the above constraints simplify dramatically since there is no state sum over flavors and the coupling constants $\lambda^{i,b}$ are universal. Moreover, when $a_1 \to a_2$ and z is evaluated at $1/a_2$, particle 1 and 2 are both soft. The left sub-amplitude A_L is then the universal 4pt amplitude of particle 3,4,5, and 6, which cannot be zero. We can furthermore factor out A_L in Eq. (4.76).

Consider exceptional theories in which $\Delta = 0$. Stripping off the universal A_L and coupling constants in Eq. (4.76) yields the numerator

$$\sum_{i=3}^{6} (s_{1\hat{i}} + s_{2\hat{i}})^{b_{max}-1}\,, \tag{4.77}$$

which is evaluated at $z = 1/a_2$. This has to vanish for $b_{max} > 0$ in generic kinematics. Recall that $z = 1/a_2$ corresponds to the double soft limit on the first two legs. The rest of momenta $\hat{p}_i$ form a 4pt kinematics, $\sum_{i=3}^{6} \hat{p}_i|_{z=1/a_2} = 0$. Therefore Eq. (4.77) is satisfied if

$$b_{max} = 2\,. \tag{4.78}$$

We check numerically up to $b_{max} = 10$, above which are ruled out by the bonus relations in Appendix A.2.

DBI straightforwardly satisfies the constraint because $\rho = 1$. On the other hand, the cancellation of spurious pole in special Galileon realizes in an interesting way

$$A_4 = s_{12}^3 + s_{13}^3 + s_{23}^3 = -3s_{12}^2 s_{13} - 3s_{12}s_{23}^2\,. \tag{4.79}$$

Although the amplitude has terms $\sim s^3$, on-shell kinematics cancels the leading term and satisfies the locality constraint.

For theories with flavor-ordered amplitudes, e.g., NLSM, the analysis is similar to the single scalar case except that we only sum over adjacent factorization channels and $\lambda_{b_{max}}$ depends on the ordering. We cannot cancel the spurious pole from $b_{max} = 2$ because global momentum conservation is no longer available when only adjacent factorization channels are summed. This can be checked numerically or be proven by the bonus relations in Appendix A.2. However, the spurious pole for $b_{max} = 1$ can be canceled if

$$\lambda_{3,1} + \lambda_{6,1} = 0\,. \tag{4.80}$$

We can check explicitly that the cyclic 4pt amplitudes in the NLSM are

$$\begin{aligned} A_4(1,2,3,I_{123}) &= -s_{13}\,, \\ A_4(6,1,2,I_{612}) &= s_{16} + s_{12}\,. \end{aligned} \tag{4.81}$$

So the coupling constants indeed have opposite sign and cancel the spurious pole.

For theories on non-trivial line, $\Delta = 1$. There is an extra factor of $s_{1\hat{i}}$ that ruins all the previous cancellation. Therefore, we do not know any sufficient condition to cancel spurious pole in the sum. This constrains

$$b_{max} \leq 2\,. \tag{4.82}$$

The bound is the same for exceptional theories. Again, we check numerically up to $b_{max} = 10$, beyond which is ruled out by the bonus relations.

We point out there is an intriguing similarity between exceptional EFTs and YM and gravity. Here we find the locality in DBI and special Galileon hinges on global momentum conservation, and locality in NLSM relies on cancellation between adjacent channels. This is completely analogous to the mechanism of how gauge invariance is realized in soft theorems in YM and gravity [83]. This could be a hint that these exceptional EFTs are closely related to YM and gravity.

Bounds on Single Scalar EFTs

As discussed before, we can factor out coupling constants and the sub-amplitude A_L in the case of single scalar. The locality test then demands $b_{max} \leq 2$. On the other hand, any pair of a_i, a_j could form a spurious pole. We can check the spurious pole in $a_1 - a_3$ from the parametrization of Eq. (4.73). The same bound applies if we replace b with $\rho+1-b$. Combining the two bounds on b, we find

$$2 \geq b \geq \rho+1-2\,, \tag{4.83}$$

which could be satisfied if $\rho \leq 3$. We find that ρ cannot be arbitrary.

Moreover, we can discuss the spurious pole in $a_2 - a_3$ and parametrize the 4pt amplitude using any two of the Mandelstam variable, s_{23}, s_{21}, s_{31}. The same bound $2 \geq b$ in Eq. (4.83) still applies to the power of any Mandelstam variable in any parametrization. From Eq. (4.83), the only permitted ansatz in $\rho = 3$ is $A_4 \propto s_{13}^2 s_{12}^2$. This is not allowed in the basis where we replace s_{13} with $-(s_{12}+s_{23})$. We conclude that for any non-trivial theories with a single scalar,

$$\rho \leq 2\,, \tag{4.84}$$

which is saturated by Galileon theories.

We can also bound theories with flavor-ordered amplitudes. They are very similar to single scalar theory except that only adjacent factorization channels are included. The spurious pole of a_1-a_2 only appears in channels $I = 123, 612$ and the spurious pole of $a_1 - a_3$ only appears in $I = 123$. As discussed in the locality test of the NLSM, the cancellation of a_1-a_2 only works with $b_{max} \leq 1$ because we lose momentum conservation. On the other hand, the spurious pole of $a_1 - a_3$ only appears in $I = 123$ and there is no cancellation. This demands $\rho + 1 - b_{max} = 0$ in the ansatz of Eq. (4.73). Combining both, we find

$$\rho = 0 \tag{4.85}$$

for exceptional theories with stripped amplitudes. The 4pt stripped amplitude with $\rho = 0$ is unique, which coincides with the NLSM one. As higher point amplitudes are uniquely specified by recursion, we conclude that NLSM is the *unique* exceptional theory with flavor ordering.

Bounds on Multiple Scalar EFTs

Next, let us consider the case of EFTs with multiple scalars. As noted earlier, some such theories admit flavor-ordered amplitudes, but this is not generic. We consider the generic multi-scalar case without assuming flavor-ordering here.

There are two complications in the case of multiple scalars. First, the coupling constant $\lambda_{i,b}$ now depends on the scalar species. Second, we need to sum over all possible intermediate states in Fig. 4.2. The sub-amplitude A_L is not universal and can no longer be factored out.

For example, consider the factorization channel $I = 123$. The subamplitude ansatze are

$$\begin{aligned} A_4(123I) &= \sum_{b=0}^{\rho+1} \lambda_{123I,b}\, s_{13}^{b}\, s_{12}^{\rho+1-b} \,, \\ A_4(456I) &= \sum_{b'=0}^{\rho+1} \lambda_{456I,b'}\, s_{45}^{b'}\, s_{46}^{\rho+1-b'} \,, \end{aligned} \tag{4.86}$$

where I labels an internal state. The key observation is that the internal state dependence only affects the coupling constants. So in the recursion, the coupling constants will only appear in a particular form

$$\lambda_{b,b'}^{123} \equiv \sum_I \lambda_{123I,b}\, \lambda_{456I,b'} \,, \tag{4.87}$$

where intermediate states I are summed over.

Even without knowing individual coupling constants, it is sufficient to constrain the $\lambda_{b,b'}^{123}$, which we dub "coupling constant square". If all of them are zero, then the 6pt amplitude must be trivial from recursion. This implies the 8pt amplitude is zero because it factorizes into the 4pt and 6pt ones. All the higher point amplitudes are then trivial by iterating this argument. We will focus on the constraints on these coupling constants squares in the following.

Plugging the ansatze in Eq. (4.86) into Eq. (4.76), the spurious pole cancellation requires

$$\sum_{i=3}^{6} \sum_{b'=0}^{\rho+1} \lambda^{12i}_{b_{\max},b'} \left(s_{\hat{4}\hat{5}}^{b'}\, s_{\hat{4}\hat{6}}^{\rho+1-b'} \right) s_{1\hat{i}}^{\Delta} (s_{1\hat{i}} + s_{2\hat{i}})^{b_{max}-\Delta-1} = 0 \tag{4.88}$$

for $b_{max} > 2\Delta$. We can check numerically if there is any choice of $\lambda^{12i}_{b_{max},b'}$ that can solve the above equation for generic kinematics for given b_{max} and ρ. We do not find a numerical solution for $b_{max} > 2$ for both exceptional theories and non-trivial theories, up to $\rho = 9$. The bonus relations in Appendix A.2 further rule out any such solution with $\rho > 9$. This constrains the coupling constant square $\lambda^{12i}_{b,b'}$ to have $b \leq 2$ for any b'.

Following the same the spurious pole analysis on any pair of $a_i - a_j$, both indices of the coupling constant square $\lambda^{123}_{b,b'}$ are restricted to be less than or equal to two, in any choice of Mandelstam variable basis. We find the same bound as in Eq. (4.83). As before, the ansatz of $\rho = 3$ is restricted to $s_{12}^2 s_{13}^2$ which is ruled out when switching to the basis of $s_{12}^2(s_{12} + s_{23})^2$. We conclude that the bounds on multi-scalar EFTs are identical to single scalar EFTs.

4.6 Classification of Scalar EFTs

In the previous sections we derived stringent exclusions on the (ρ, σ, v, d) parameter space of EFTs. However, these exclusions still allow for EFTs to exist in the range

$$d + 1 \geq v \quad \text{and} \quad 3 > \rho \geq \frac{\sigma(v-1) - 2}{v - 2}\,. \tag{4.89}$$

In what follows, we explicitly enumerate all scalar EFTs with non-trivial soft behavior, as defined by the window in Eq. (4.4). *A priori*, this would require scanning over values of (ρ, σ, v, d) and numerically determining whether there exists an amplitudes ansatz consistent with these assumptions. However, as shown earlier, for a given choice of (ρ, σ, v) it is always sufficient to check for the existence of EFTs in $d = v - 1$ dimensions, since no new theories can appear for $d > v - 1$. Thus for a given v we only have to check all possible (ρ, σ) regions in $d = v - 1$ dimensions.

In this section we enumerate and classify all possible EFTs for $v = 4, 5, 6$, which in turn exhausts all possible theories in $d = 3, 4, 5$. Our analysis begins with $v = 5$ and $v = 6$ theories, checking $n = v$ amplitudes. The $v = 4$ is special because the 4pt amplitude does not give any constraints since $\sigma = \rho + 1$ from 4pt kinematics. In this case we have to proceed further and consider 6pt amplitudes.

We distinguish between cases with permutation invariance among legs (corresponding to amplitudes of a single scalar) or cyclic invariance (corresponding to flavor-ordered amplitudes of multiple scalars). Note that for a single scalar with $\rho = 0$, the permutation invariant amplitudes ansatz vanishes identically because any Lagrangian of that form is just field redefinition of free scalar field theory. However, for multiple scalars with flavor-ordering, there is a non-trivial amplitudes ansatz.

Low Valency

In this subsection we enumerate scalar EFTs whose leading interactions are at low valency, corresponding to $v = 4, 5, 6$.

Case 1: $v = 5$

We begin with the case of leading valency $v = 5$. Here the corresponding critical dimension is $d = 4$, by which we mean that it is sufficient to scan for theories in $d = 4$ dimensions to enumerate all possible EFTs. Analyzing amplitudes in higher dimensionality is unnecessary simply because the kinematics of the $v = 5$ amplitude are constrained to $d = 4$ anyway.

We only consider EFTs which have non-trivial soft behavior and are thus on-shell constructible, so $\sigma \geq \rho$. Moreover, we restrict to the region defined in Eq. (4.89),

$$3 > \rho \geq \frac{4\sigma - 2}{3}\,, \tag{4.90}$$

which is in principle still permitted from our previous arguments. For $v = 5$, the only possible allowed pairs of (ρ, σ) compatible with (4.90) and non-triviality bound are $(\rho, \sigma) = (\frac{2}{3}, 1)$ and $(\rho, \sigma) = (2, 2)$.

In Figure 4.3, we use the symbol $\{a, b\}$ where a denote the number of solutions in the permutational invariant case and b the number of solutions in the cyclically invariant case. We also performed checks for cases satisfying $\sigma \geq \rho$ and $\rho < 3$ bounds but failing to meet Eq. (4.90). There is no solution and the previous proof is confirmed.

We see from the diagram that there is one interesting 5pt cyclically ordered amplitude for $(\rho, \sigma) = (\frac{2}{3}, 1)$,

$$A_5^{(\frac{2}{3},1)} = \epsilon_{\mu\nu\alpha\beta} p_1^\mu p_2^\nu p_3^\alpha p_4^\beta\,, \tag{4.91}$$

which arises precisely from the WZW term on the NLSM mentioned earlier. The presence of the Levi-Civita tensor implies that this solution exists only in $d = 4$ and not other dimensions.

Another interesting solution appears for $(\rho, \sigma) = (2, 2)$, and in $d = 4$ can be compactly represented by

$$A_5^{(2,2)} = \left(\epsilon_{\mu\nu\alpha\beta} p_1^\mu p_2^\nu p_3^\alpha p_4^\beta\right)^2. \tag{4.92}$$

In higher dimensions $d \geq 4$ this amplitude takes the form

$$A_5^{(2,2)} = \delta^{\mu_1 \dots \mu_4}_{\nu_1 \dots \nu_4} p_{1\mu_1} \dots p_{4\mu_4} p_1^{\nu_1} \dots p_4^{\nu_4} \,, \tag{4.93}$$

which is equal to the Gram determinant since $\delta^{\mu_1 \cdots \mu_n}_{\nu_1 \dots \nu_n} = \det(\delta^{\mu_i}_{\nu_j})^n_{i,j=1}$. Such amplitudes are both cyclic and permutational invariant in all legs. This amplitude arises from the 5pt interaction of the Galileon theory, for both a single and multiple scalar fields (cf. Appendix A.3), which exists in $d \geq 4$. This exhausts all interesting cases for leading valency $v = 5$.

Case 2: $v = 6$

For valency $v = 6$, it is sufficient to study EFTs restricting to the critical dimension $d = 5$ and the region in Eq. (4.89),

$$3 > \rho \geq \frac{5\sigma - 2}{4} \,. \tag{4.94}$$

For $v = 6$, the only non-trivial pairs (ρ, σ) satisfying (4.94) are $(\rho, \sigma) = (\frac{3}{4}, 1)$ and $(\rho, \sigma) = (2, 2)$. Indeed, there are two solutions for amplitudes, one for each point in the parametric space,

$$A_6^{(1,1)} = \epsilon_{\mu\nu\alpha\beta\kappa} p_1^\mu p_2^\nu p_3^\alpha p_4^\beta p_5^\kappa \,, \tag{4.95}$$

valid only in $d = 5$ which corresponds to the WZW model. The other solution is the 6pt Galileon, written in $d = 5$ as

$$A_6^{(2,2)} = \left(\epsilon_{\mu\nu\alpha\beta\kappa} p_1^\mu p_2^\nu p_3^\alpha p_4^\beta p_5^\kappa\right)^2 \,, \tag{4.96}$$

but in general $d > 4$ it takes the form (4.93) with five momenta involved.

Special Case: $v = 4$

As was discussed earlier the 4pt amplitudes are special due to 4pt kinematics. All kinematical invariants vanish if we set one of the momenta to zero. Therefore, for $(\partial^m \phi^4)$ we have $\rho = \frac{m-2}{2}$ and $\sigma = \frac{m}{2}$ which implies $\rho = \sigma - 1$. But we still have the inequality $\rho < 3$ and therefore, the only allowed cases are $(\rho, \sigma) = (0, 1), (1, 2), (2, 3)$. We can now directly explore all these cases with numerical methods and determine how many solutions are in each point of (ρ, σ) space. In order to check the existence of such theories we have to perform the test for 6pt amplitudes. The ansatz now contains the factorization terms with 4pt vertices as well as the 6pt contact term from the Lagrangian,

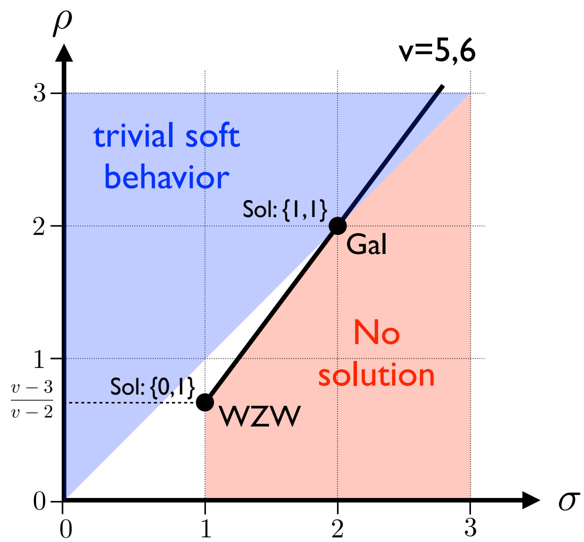


Figure 4.3: Plot summarizing the numerical search of EFTs with $v = 5, 6$. The blue region denotes the same trivial region as in Figure 4.1. The red region has no solution numerically. The only two points with solutions are the d-dimensional WZW theory, $(\rho, \sigma) = (\frac{v-3}{v-2}, 1)$, and the Galileon $(\rho, \sigma) = (2, 2)$. The label "Sol:{a,b}" denotes the number of solutions in permutation invariant and cyclic invariant amplitudes respectively.

$$\mathcal{L}_\rho = (\partial^{2\rho+2}\phi^4) + (\partial^{4\rho+2}\phi^6). \tag{4.97}$$

We perform the check in $d = 3, 4, 5$ as these are the only interesting cases. The results are summarized in Figure 4.4. The first solution for $(\rho, \sigma) = (0, 1)$ is with cyclic symmetry,

$$A_6^{(0,1)} = \frac{(s_{12}+s_{23})(s_{45}+s_{56})}{s_{123}} + \frac{(s_{23}+s_{34})(s_{56}+s_{61})}{s_{234}} + \frac{(s_{34}+s_{45})(s_{61}+s_{12})}{s_{345}} - (s_{12}+s_{23}+s_{34}+s_{45}+s_{56}+s_{61})\,, \tag{4.98}$$

which is the 6pt amplitude in the $SU(N)$ non-linear sigma model in any d. The solution for $(\rho, \sigma) = (1, 2)$ is with permutational symmetry,

$$A_6^{(1,2)} = \frac{(s_{12}s_{23}+s_{13}s_{23}+s_{12}s_{13})(s_{45}s_{46}+s_{46}s_{56}+s_{45}s_{56})}{s_{123}} - s_{12}s_{34}s_{56} + \text{permutations}\,, \tag{4.99}$$

which is the 6pt amplitude in the Dirac-Born-Infeld theory in any d. The last solution is a 4pt Galileon for $(\rho, \sigma) = (2, 2)$ which exists for both single and multiple scalar cases for $d > 2$. In the single scalar case there is an extra

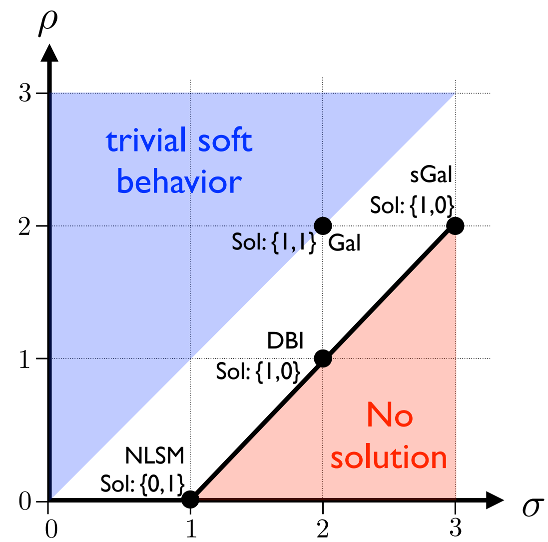


Figure 4.4: Plot summarizing the numerical search of EFTs with $v = 4$. The blue region denotes the same trivial region as in Figure 4.1. The red region has no solution numerically. The label "Sol:{a,b}" denotes the number of solutions in permutation invariant and cyclic invariant amplitudes respectively.

$\sigma = 3$ behavior giving us the special Galileon with $(\rho, \sigma) = (2, 3)$ while for the flavor-ordered case this enhanced soft limit is not present.

High Valency

The set of all possible values of ρ is $\rho = \frac{m-2}{v-2}$ where m is the number of derivatives in the interaction with constraint

$$3 > \frac{m-2}{v-2} \geq \frac{\sigma(v-1)-2}{(v-2)}, \tag{4.100}$$

and also $\sigma > \rho$ for $\sigma = 1$ and $\sigma \geq \rho$ for $\sigma > 1$. These inequalities can be easily solved and we can find all integers p which satisfy them, which would enumerate all possible solutions. For $\sigma = 1$ the constraint becomes

$$v > m \geq v - 1\,, \tag{4.101}$$

which only has a solution if $m = v - 1$. Therefore, the only possible allowed case is $(\rho, \sigma) = (\frac{v-3}{v-2}, 1)$. As this has $\rho < 1$ there can not be any permutational invariant amplitude with $\sigma = 1$ behavior – for a single scalar the theory must be derivatively coupled. However, we can have cyclically invariant v-pt amplitude,

$$A_v^{(\frac{v-3}{v-2},1)} = \epsilon_{\alpha_1\alpha_2\ldots\alpha_{v-1}} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{v-1}^{\alpha_{v-1}}. \tag{4.102}$$

This corresponds to the WZW term which exists only in $d = v-1$ dimensions. Of course, this is the only possible term if the number of derivatives $m = v-1$ is odd. For general v, we can not prove that the WZW term is the only solution for $v > 6$, but all theories have to sit at the point $(\rho,\sigma) = (\frac{v-3}{v-2}, 1)$.

For $\sigma = 2$ the inequality becomes $2(v-2) \geq m \geq 2(v-2)$ which forces $m = 2(v-2)$ and $\rho = 2$. So the only allowed case is $(\rho,\sigma) = (2,2)$. We know that this is exactly the powercounting of the $v$pt Galileon; in $d = v-1$ dimensions it is

$$A_v^{(2,2)} = (\epsilon_{\alpha_1\alpha_2\ldots\alpha_{v-1}} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{v-1}^{\alpha_{v-1}})^2, \tag{4.103}$$

but there is a general form analogous to (4.93) in any $d > v-2$. Note that this solution exists for both cyclic and permutational cases. What we cannot prove is that there are no solutions other than Galileon for $v > 6$, but they all have to sit at the point $(\rho,\sigma) = (2,2)$.

Exclusion Summary

To summarize, by direct evaluation we found all possible amplitudes with enhanced soft limit for $v = 4,5,6$ which gives all interesting theories for $d = 3,4,5$. We found that for $v = 4$ these theories are NLSM, DBI, Galileon and WZW theory. For $v = 5,6$ we have only Galileon and WZW. Both of these theories exist for $v > 6$, and in fact they both populate the only allowed points in the (ρ,σ) plane.

As a result, for $v = 4,5,6$ we enumerated all such theories and there cannot be any new ones. For $v > 6$, which is relevant only for $d > 5$, there is a possibility new theories can appear but they have to sit in the same (ρ,σ,v,d) spots degenerate with WZWs and Galileons.

4.7 More Directions

In this section we discuss several directions not included in the classification above. In particular, we first make some comments about the theories of multiple scalars that cannot be flavor-ordered. We solve this problem for the two flavor case and make some comments about three flavors. The landscape of theories for any number of flavors is still unknown.

We also explore more kinematical limits other than just the soft limit. In particular, we discuss the double soft limit when two momenta go to zero simultaneously, and the collinear limit when two of the momenta become proportional.

Multiple Scalars

This analysis exactly mirrors the strategy of [51], which constructed all single scalar effective theories consistent with factorization and a prescribed value of (ρ, σ). This procedure uniquely lands on well-known theories such as DBI and the Galileon, but also suggested evidence for a new effective theory known as the so-called special Galileon, whose enhanced shift symmetry is now fully understood [64].

Here, we apply the same procedure but allow for multiple species. As this constructive procedure is open ended, we restrict to the simplest case of $N = 2$ flavors throughout. We save $N = 3$ and higher to future work.

We start at 4pt, demanding that a general theory of the scalars ϕ_1 and ϕ_2 has an enhanced soft limit. However, we can see that this is automatic by the following argument. At a fixed value of the power counting parameter ρ, the 4pt amplitude A_4 should contain $2(\rho + 1)$ powers of momenta, so it is some polynomial in s, t, u with that degree. As we can always go to a basis that manifests a particular soft limit, *e.g.* the soft limit for leg 1 with $s = p_1 p_2$, $t = p_1 p_4$, $u = p_1 p_3$, then we have that

$$A_4 \overset{p \to 0}{\sim} p^{\rho+1}, \tag{4.104}$$

which means that $\sigma = \rho + 1$ generically, which corresponds to an enhanced soft limit at 4pt.

To move beyond the 4pt amplitude we must explicitly enumerate the vertices. First, it is easily seen that any cubic scalar interactions with derivatives can be eliminated via equations of motion, so, for example, the interaction $\lambda^{(3)}_{ijk} \partial_\mu \phi_i \partial^\mu \phi_j \phi_k$ can be removed by a field redefinition of the form

$$\phi_i \to \phi_i + \lambda^{(3)}_{ijk} \phi_j \phi_k. \tag{4.105}$$

Thus we can assume the absence of a 3pt vertex. With interactions that start at the 4pt vertex, the first amplitude of interest is a 6pt, which can receive contributions from the 4pt and 6pt vertex.

For the two derivative case, $\rho = 0$, the general action for $N = 2$ flavors is

$$\mathcal{L}_{\rho=0} = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_j (\delta_{ij} + \lambda^{(4)}_{ijkl} \phi_k \phi_l + \lambda^{(6)}_{ijklmn} \phi_k \phi_l \phi_m \phi_k + \ldots), \tag{4.106}$$

without loss of generality. For $\rho = 1$, the general action is

$$\mathcal{L}_{\rho=1} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_j(\delta_{ij} + \lambda^{(4)}_{ijkl}\partial_\nu\phi_k\partial^\nu\phi_l + \lambda^{(6)}_{ijklmn}\partial_\mu\phi_k\partial^\nu\phi_l\partial_\rho\phi_m\partial^\rho\phi_k + \ldots) \tag{4.107}$$

and there is a straightforward generalization to $\rho = 2$.

To construct the theory we then computed the 6pt scattering amplitude and demanded $\sigma = 1, 2, 3$ soft limits for the $\rho = 0, 1, 2$ cases. For $\rho = 0$ we find a single solution which corresponds to the $SO(3)/SO(2)$ NLSM, where the $N = 2$ flavors correspond to the two massless NGBs. For $\rho = 1$, we find two solutions. The first solution is simply two copies of the DBI theory for a 4D brane moving in 5D. The second solution is the DBI theory describing a 4D brane moving in 6D. Finally, for $\rho = 2$, the only possible theory in 4D corresponds to the single scalar special Galileon. In these cases the multi-flavor EFTs have the property that they can be rewritten as a sum of independent one-flavor Lagrangians after an orthogonal transformation. As a result, the Feynman rules for vertices are blind to the actual flavor combination of the legs.

Double Soft Limits

To begin, we consider the simultaneous soft limit of two particles, $p_j, p_k \to 0$. In the context of the NLSM, this limit is sensitive to the structure of the coset space [7, 63, 71, 84], and has been applied in the context of the scattering equations [85]. More recently, this kinematic regime was studied for gauge theory and gravity [70, 72].

Here we consider the double soft limit for a general scalar EFT. In this case the distinction between theories with trivial versus non-trivial behavior is different from that of the single soft limit since poles in the denominator can blow up. If $p_1, p_2 \to 0$ then all poles $s_{12a} \to 0$ where $a = 3, 4, \ldots, n$. For this reason factorization terms typically are singular, and will not have a smooth double soft limit.

For concreteness, let us consider two momenta, p_2, p_3, to be sent to zero,

$$p_2(t) = tp_2, \qquad p_3(t) = \alpha t p_3. \tag{4.108}$$

We also shift all other momenta in order to satisfy momentum conservation. The shifted amplitude is then inspected based on the degree of vanishing as

$t \to 0$,

$$A_n(t) = \mathcal{O}(t^\sigma). \tag{4.109}$$

It is simple to see that 5pt amplitudes are not interesting in this limit since $t \to 0$ yields an on-shell 3pt amplitude which is identically zero by our earlier kinematic arguments. Therefore, the first non-trivial case is the 6pt amplitude, which we now consider in detail. Furthermore, it is sufficient to fix to $d = 5$ for 6pt kinematics. No new solutions can exist for $d > 5$ but some of them can disappear when going to $d = 4$. For interesting cases in $d = 5$ we check if they are present in $d = 4$. While we do not have similar exclusion bounds as in the single soft limit case—presumably they do exist as well as double soft recursion relations—we can still fix $n = 6$ and increase the number of derivatives.

The first question is what is the meaning of “non-trivial" from the point of view of the double soft limit. Here it matters critically if we have $v = 4$ or $v = 6$. For $v = 6$ we have only a contact term and therefore $\sigma \geq 2\rho$ to get non-trivial soft limit behavior. If we have $v = 4$ then there are propagators in factorization terms which blow up for $p_2, p_3 \to 0$ and therefore, the behavior is not just the naive square of the single scalar soft limit. In particular, we get $\sigma \geq 2\rho - 1$. In Table 4.2 we summarize the number of solutions for $v = 4$ and $v = 6$. Note that $v = 4$ exists only for integer ρ. For $\rho = 1$ we have the straight inequalities for a non-trivial bound.

	$\rho = \frac{1}{2}$	$\rho = 1$	$\rho = \frac{3}{2}$	$\rho = 2$	$\rho = \frac{5}{2}$
$\sigma = 1$	0				
$\sigma = 2$		0			
$\sigma = 3$		0	0	1	
$\sigma = 4$				1	
$\sigma = 5$					0

Table 4.2: Number of solutions for double soft limit. We denote n the number of solution for $v = 4$ and n the number of solutions for $v = 6$.

We see that there are two interesting cases for $\rho = 2$, one for $v = 4$ and one for $v = 6$. We can easily identify both of them with Galileons. For $v = 4$ it is the 4pt Galileon (which also exists for $d = 4$) while for $v = 6$ it is the 6pt Galileon which is absent in $d = 4$ and lower. This can be easily shown from the representation of the Galileon vertex as a Gram determinant. As was shown before, each Gram determinant (for any number of points) scales like $\mathcal{O}(t^2)$ in the soft limit. For $v = 6$ we can obtain $\mathcal{O}(t^4)$ in the double soft limit. For $v = 4$ this is reduced by one power due to the propagators (when p_2 and p_3 are on the same side) which also scale like $\mathcal{O}(t)$, and in the end we get $\mathcal{O}(t^3)$. Note that the $\mathcal{O}(t^3)$ behavior of the special Galileon in the single soft limit is not propagated into the double soft limit case, and only the $\mathcal{O}(t^2)$ behavior is relevant. Here we performed the checks only for $\rho < 3$ but in principle, we should consider higher ρ or prove the same bound as in the single scalar field case.

Collinear Limits

The other natural limit to consider is the collinear limit where two of the momenta become proportional. This was recently studied from scattering equations [86]. We study it again in the context of single scalar EFT so we can choose $p_3 = \alpha p_2$ (for some parameter α) without a loss of generality. Unlike the single soft limit and double soft limit cases there are no theoretical expectations of how the amplitude should behave. In the Yang-Mills theory and gravity collinear limits are well understood and provide a pole and phase factor, respectively. In our case the situation is different as there are no 3pt vertices and the collinear limit never diverges. Therefore, we can pose the question in a similar way as in the soft limit case: when does the amplitude vanish at a given rate σ?

To be more specific, we have to introduce a small parameter t which will control the distance from the collinear region. We shift momentum $p_3 \to p_3(t)$ where

$$p_3(t) = \alpha(1-t)p_2 - \alpha t(1-t)\frac{s_{23}}{\alpha(1-t)s_{12} + ts_{13}}p_1 + tp_3, \tag{4.110}$$

where s_{ab} are the invariants of unshifted momenta. In order to preserve the momentum conservation we have to shift also other momenta $p_4, \ldots, p_n$ but in a way which is regular for any value of t. The shift in Eq. (4.110) is more complicated in order to preserve the on-shell condition $p_3(t)^2 = 0$ and also control how we approach the collinear region. Note that for $t = 1$ we recover the original configuration, $p_3(t) = p_3$, and also other momenta become unshifted, while for $t = 0$ we get $p_3 = \alpha p_2$. Then the question is what is the rate at which the shifted amplitude $A_n(t)$ vanishes,

$$A_n(t) = \mathcal{O}(t^{\sigma}). \tag{4.111}$$

Unlike in the soft limit case there is no statement symmetry → collinear limit. Therefore, we have to rely just on the kinematical check. The only kinematical invariant which vanishes in this limit is s_{23}. Naively, in order to get the vanishing collinear limit in any pair of momenta each Feynman diagram would have to contain the product of all invariants s_{ij}, which pushes the derivative degree very high. We also do not have any argument about the leading valency of the Lagrangian.

We did the checks for 5pt amplitudes up to 18 derivatives, 6pt amplitudes up to 14 derivatives and 7pt amplitudes up to 10 derivatives, with no interesting results (no vanishing collinear limits) except one class of theories which are Galileons.

Galileons from collinear limits

For the Lagrangians of the type $(\partial^8\phi^5)$ there is one solution for the collinear limit vanishing for $d \geq 4$, and for $(\partial^{10}\phi^6)$ there is also one solution for $d \geq 5$. The solutions can be identified with the 5pt and 6pt Galileons which are then unique solutions to the problem of vanishing collinear limit. Moreover, the amplitudes in both cases vanish as $A(t) \sim \mathcal{O}(t^2)$. This can be understood from the definition of the Galileon vertex. The Gram determinant for $n = 5$ in $d = 4$ behaves by definition as

$$Gram_{d=4,n=5}\left[p_1, p_2, p_3(t), p_4(t), p_5(t)\right] \equiv (\epsilon_{\mu\nu\alpha\beta}p_{1\mu}p_{2\nu}p_{3\alpha}(t)p_{4\beta}(t))^2 = \mathcal{O}(t^2) \tag{4.112}$$

and similarly for $n = 6$ and $d = 5$. In higher dimensions some of indices are contracted together from both ϵ tensors but the scaling property is still valid. However, the collinear vanishing is the property of the contact term only, not the amplitude for higher n. The factorization terms spoil this property as they do not vanish in the collinear limit when both legs are on the opposite sides of the channel. In principle, there could be a cancellation between different Feynman diagrams, but this does not happen as the numerical checks show. We can also see it in the $(\partial^{10}\phi^6)$ case where there is no solution for the 6pt amplitude coming from the 4pt Galileon $(\partial^6\phi^4)$.

But still it is interesting to note that the collinear limit can be used to define the Galileons as unique theories based on the behavior in the collinear limit. It would be interesting to explore the kinematical space more exhaustively and also do it for multiple scalars.

4.8 Outlook

In this chapter we have mapped out the theory space of Lorentz invariant and local scalar effective field theories by studying the soft behavior of scattering amplitudes. The bulk of our discussion has focused on theories of a single scalar or multiple scalars which allow for flavor-ordering. We have derived bounds on the power counting and soft behavior of all possible consistent theories with enhanced soft limit and classified completely all the non-trivial cases in $d < 6$. Our final catalog of EFTs include NLSM, DBI, Galileon, and WZW term theory. A main takeaway of this chapter is that these theories are truly unique. We also commented on the theories with generic multiple scalars and different kinematical limits.

Remarkably, the exceptional theories discussed here coincide precisely with the EFTs constructed from the CHY representation [14] and which satisfy BCJ duality [87]. Moreover, there is evidence of new theories which are extensions of these exceptional theories [88, 89], suggesting a rich interplay between soft limits, BCJ duality, and CHY representation. Classifying theories based on various aspects can illuminate the relations among them. Insights into the soft structure of the S-matrix have also arisen in the program of asymptotic symmetries [69, 81, 83, 90–105].

There are many other directions viable for constructing theories from the properties of scattering amplitudes. The most natural direction is to consider other particle content (higher spins), other kinematical regimes (like the double soft limit or the collinear limit briefly mentioned in the chapter), loop-level correction [106], or curved backgrounds. More ambitiously, one might also consider non-relativistic theories [62], where amplitudes satisfy less symmetry, but must nevertheless exhibit locality and factorization. *A priori*, one would expect a far greater diversity in non-relativistic EFTs, so there is also the possibility that new theories might yet lie undiscovered.

This is the first step in the program of extending the developments in the study of scattering amplitudes in gauge theory and gravity to other quantum field theories, and EFTs are the furthest possible cousins. The recent progress on recursion relations and CHY representation in these theories show that there should be a completely new formulation for scattering in general QFTs.

Chapter 5

SYMMETRY FOR FLAVOR-KINEMATICS DUALITY FROM AN ACTION

5.1 Introduction

The study of scattering amplitudes has uncovered a beautiful duality between gauge theory and gravity concisely summarized by the mantra, gravity $\sim$ gauge^2. Bern, Carrasco, and Johansson (BCJ) [32] proposed a remarkable generalization of this squaring relation known as color-kinematics duality. BCJ showed that tree amplitudes in YM theory can be rearranged into the schematic form,

$$A \sim \sum_i \frac{C_i N_i}{D_i}, \tag{5.1}$$

where i sums over cubic topologies with propagator denominators D_i, color structures C_i, and kinematic numerators N_i. Here C_i and N_i satisfy Jacobi identities,

$$C_i + C_j + C_k = 0 \quad \text{and} \quad N_i + N_j + N_k = 0, \tag{5.2}$$

where i, j, k denote any triplet of cubic topologies which are the same except for a single propagator. That there exist N_i with the same algebraic relations as C_i is at the heart of color-kinematics duality. Since N_i and C_i are in this sense interchangeable, we can substitute the latter with the former, yielding the double copy,

$$M \sim \sum_i \frac{N_i N_i}{D_i}, \tag{5.3}$$

which is the graviton tree amplitude [33, 34]. The double copy has been generalized to include loops, supersymmetry, and matter fields (cf. [37] and references therein).

While color Jacobi identities are trivialized by an underlying Lie algebra, this is not so simple for kinematics. BCJ strongly suggests an underlying algebra for kinematic numerators, but this structure remains elusive except in limited contexts, *e.g.* for YM in the self-dual sector [107] and in the formalism of Cachazo, He, and Yuan (CHY) [12, 13, 108, 109].

In constrast, the nonlinear sigma model is a theory of Nambu-Goldstone bosons unburdened by gauge symmetry, thus offering a simpler path to the kinematic algebra. Flavor-kinematics duality in the NLSM has been explored at tree-level and on the worldsheet [87, 89, 110], though without mention of the double copy.

In this chapter, we present a new formulation of the NLSM in general spacetime dimension. This action is remarkably simple, comprised of just a handful of fields interacting via cubic Feynman vertices that play the role of structure constants in an underlying kinematic algebra. Flavor-kinematics duality then emerges as a symmetry: the kinematic Jacobi identities are literally the current conservation equations for a certain combination of internal and spacetime symmetries. In turn, all Feynman diagrams automatically satisfy Eq. (5.2). Applying the double copy construction, we then derive a new cubic action for the special Galileon theory [14, 51, 64], which describes a scalar coupled through a tower of higher derivative interactions. Lastly, we show how these formulations reproduce the vanishing soft behavior of amplitudes.

5.2 Warmup

As a preface to our main results, let us briefly review the theory of a biadjoint scalar. Though trivial in structure, this theory nicely illustrates how Jacobi identities arise from considerations of symmetry. The action is

$$S = \int \frac{1}{2}\phi^{a\bar{a}}\Box\phi^{a\bar{a}} + \frac{\lambda}{3} f^{abc} f^{\bar{a}\bar{b}\bar{c}} \phi^{a\bar{a}}\phi^{b\bar{b}}\phi^{c\bar{c}}, \tag{5.4}$$

where f^{abc} and $f^{\bar{a}\bar{b}\bar{c}}$ are the structure constants for a pair of global flavor symmetries. The equations of motion are

$$\frac{\delta S}{\delta \phi^{a\bar{a}}} = \Box\phi^{a\bar{a}} + \lambda f^{abc} f^{\bar{a}\bar{b}\bar{c}} \phi^{b\bar{b}}\phi^{c\bar{c}}. \tag{5.5}$$

The action is invariant under the global flavor rotations,

$$\delta\phi^{a\bar{a}} = f^{abc}\theta^b\phi^{c\bar{a}}, \tag{5.6}$$

whose associated Noether current is

$$J^a_\mu = -f^{abc}\phi^{b\bar{a}}\partial_\mu\phi^{c\bar{a}}. \tag{5.7}$$

Noether current conservation then implies that

$$\begin{aligned} \partial J^d &= -f^{dae}\phi^{a\bar{a}}\overleftrightarrow{\Box}\phi^{e\bar{a}} = 0 \\ &= \frac{\lambda}{3} f^{\bar{a}\bar{b}\bar{c}}\phi^{a\bar{a}}\phi^{b\bar{b}}\phi^{c\bar{c}}\left(f^{dae}f^{ebc} + f^{dbe}f^{eca} + f^{dce}f^{eab}\right), \end{aligned} \tag{5.8}$$

which is the Jacobi identity. Here $\overleftrightarrow{\Box} = \frac{1}{2}(\overrightarrow{\Box} - \overleftarrow{\Box})$ and we have used Eq. (5.5) together with the cyclicity of $f^{\bar{a}\bar{b}\bar{c}}$.

The above derivation is actually equivalent to the diagrammatic representation of the Jacobi identity that typically appears in the study of scattering amplitudes. In terms of Feynman diagrams, the d'Alembertian in Eq. (5.8) has the effect of canceling an internal propagator. The resulting triplet of objects—each equal to the Feynman diagrammatic numerator associated with a given cubic topology—satisfies the Jacobi identity.

5.3 Action

The strategy above can be applied directly to the NLSM, though doing so requires an alternative formulation of the theory. To begin, let us introduce the fields

$$X_\mu, \quad Y, \quad Z^\mu, \tag{5.9}$$

in the adjoint representation of a flavor symmetry. The XYZ fields interact via the remarkably simple action

$$S = \int Z^{a\mu} \boxdot X^a_\mu + \frac{1}{2} Y^a \boxdot Y^a, \tag{5.10}$$

where we have defined a modified d'Alembertian,

$$\boxdot()^a = \Box()^a + 2f^{abc} Z^{b\mu} \partial_\mu ()^c. \tag{5.11}$$

Expanded fully, the action becomes

$$\begin{aligned} S = \int \; & Z^{a\mu} \Box X^a_\mu + \frac{1}{2} Y^a \Box Y^a \\ & - f^{abc} \left(Z^{a\mu} Z^{b\nu} X^c_{\mu\nu} + Z^{a\mu} (Y^b \overleftrightarrow{\partial}_\mu Y^c) \right), \end{aligned} \tag{5.12}$$

where $X_{\mu\nu} = \partial_\mu X_\nu - \partial_\nu X_\mu$ and $\overleftrightarrow{\partial}_\mu = \frac{1}{2}(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)$. As we will soon see, the Nambu-Goldstone bosons of the NLSM are simultaneously described by Y and Z, so this formulation obscures Bose symmetry. Moreover, the cubic structure of the action hides the underlying parity of the NLSM interactions. These properties come at the cost of manifesting flavor-kinematics duality as a symmetry.

5.4 Scattering Amplitudes

The structure of the action in Eq. (5.10) is reminiscent of a "colored scalar", Y, coupled to the $(-)$ and $(+)$ components of a "YM field", X and Z. The interactions in Eq. (5.12) are then analogous to cubic $\overline{\text{MHV}}$ vertices. By general arguments in [111], all tree amplitudes trivially vanish except those with exactly two Y states or exactly one X state, with all other states given by Z.

Our claim is that the tree amplitudes of the NLSM are equal to the tree amplitudes

$$A(\ldots, Y_i, \ldots, Y_j, \ldots), \tag{5.13}$$

where i, j are arbitrary and the ellipses denote all other external particles, taken to be longitudinally polarized Z states for which $\epsilon_\mu = ik_\mu$ in units of the NLSM decay constant. Note that Bose symmetry is ultimately preserved since the final amplitude does not depend on which particles are chosen to be Y states.

As an illustration of this, let us turn to the four-particle amplitude. Using Feynman diagrams, we compute the kinematic numerators for the half-ladder topology for (Y_1, Z_2, Z_3, Y_4), yielding

$$N_s = s^2, \quad N_t = s^2 - u^2, \quad N_u = u^2, \tag{5.14}$$

where $N_t = N_s - N_u$ so the kinematic Jacobi identity is satisfied. Moreover, the resulting flavor-ordered amplitude precisely matches that of the NLSM,

$$A_4 = \frac{1}{2}(s + t), \tag{5.15}$$

in the convention that $[T^a, T^b] = i\sqrt{2} f^{abc} T^c$. Squaring the numerators via the double copy procedure, we obtain

$$M_4 = -stu, \tag{5.16}$$

which is the amplitude of the special Galileon.

Alternatively, we could have instead computed the kinematic numerators for the choice (Y_1, Y_2, Z_3, Z_4),

$$N_s = t^2 - u^2, \quad N_t = t^2, \quad N_u = -u^2, \tag{5.17}$$

or for the choice (Y_1, Z_2, Y_3, Z_4),

$$N_s = -s^2, \quad N_t = -t^2, \quad N_u = t^2 - s^2, \tag{5.18}$$

which give different numerators but the same amplitude.

We can generalize to n-particle scattering by computing all the kinematic numerators for half-ladder topologies, which form a complete basis for all tree amplitudes [32, 112]. For later convenience, we define

$$\tau_i = -2p_i \sum_{j<i} p_j \quad \text{and} \quad (\tau_i^{\pm})_\mu{}^\nu = \delta_\mu^\nu \tau_i \pm 2p_{i\mu} p_i^\nu, \tag{5.19}$$

as well as the kinematic variables

$$\begin{aligned}
\Sigma_{ij} &= \tau_i \tau_{i+1} \ldots \tau_{j-1} \tau_j \\
\Sigma_{ij}^{\pm} &= 2p_i \tau_{i+1}^{\pm} \tau_{i+2}^{\pm} \ldots \tau_{j-2}^{\pm} \tau_{j-1}^{\pm} p_j.
\end{aligned} \tag{5.20}$$

For each choice of Y states, it is a straightforward exercise to calculate the corresponding half-ladder numerators via Feyman diagrams, yielding

$$\begin{aligned}
N(Y_1, \ldots, Y_n) &= -\Sigma_{2,n-1} \\
N(Y_1, \ldots, Y_i, \ldots) &= \Sigma_{2,i-1} \Sigma_{i,n}^{-} \\
N(\ldots, Y_i, \ldots, Y_n) &= -\Sigma_{1,i}^{+} \Sigma_{i+1,n-1} \\
N(\ldots, Y_i, \ldots, Y_j, \ldots) &= \Sigma_{1,i}^{+} \Sigma_{i+1,j-1} \Sigma_{j,n}^{-},
\end{aligned} \tag{5.21}$$

where the ellipses denote external Z states. The first line of Eq. (5.21) is the simple numerator proposed in [89, 113]. We have checked that these expressions reproduce the tree amplitudes of the NLSM up to ten-particle scattering.

5.5 Equations of Motion

With the help of Feynman diagrams it is simple to check flavor-kinematics duality in specific examples. However, to derive more general principles, it will be convenient to study the classical field equations, which are a proxy to tree-level Feynman diagrams [114]. The Euler-Lagrange equations of motion for Eq. (5.10) are

$$\begin{aligned}
\frac{\delta S}{\delta X_\mu^a} &= \Box Z^{a\mu} + f^{abc}(2Z^{b\nu}\partial_\nu Z^{c\mu} + 2\partial_\nu Z^{b\nu} Z^{c\mu}) = 0 \\
\frac{\delta S}{\delta Y^a} &= \Box Y^a + f^{abc}(2Z^{b\nu}\partial_\nu Y^c + \partial_\nu Z^{b\nu} Y^c) = 0 \\
\frac{\delta S}{\delta Z^{a\mu}} &= \Box X_\mu^a - f^{abc}\left(2Z^{b\nu} X_{\mu\nu}^c + Y^b \partial_\mu Y^c\right) = 0,
\end{aligned} \tag{5.22}$$

where the divergence of the first equation yields $\Box\partial_\mu Z^\mu = 0$. This implies the classical field condition

$$\partial_\mu Z^\mu = 0, \tag{5.23}$$

whenever Z is an off-shell source. If Z is on-shell, then by the prescription of Eq. (5.13) it is longitudinally polarized, so Eq. (5.23) is still valid because of the on-shell condition. In any case, the bottom line is that Eq. (5.23) is generally true whenever the equations of motion are satisfied.

Since Eq. (5.23) is a constraint on classical fields, its implications for amplitudes are actually somewhat subtle. In particular, due to the nondiagonal kinetic term, an off-shell source Z propagates into X, which can then only interact via the field strength combination in Eq. (5.12). From this perspective Eq. (5.23) simply says that the longitudinal polarizations of X are projected out.

5.6 Symmetries

The action in Eq. (5.10) has a surprisingly rich set of local and global symmetries. We now present these symmetries and derive their associated Noether currents.

Local Transformation

To begin, consider the local transformation,

$$\delta X_\mu = \partial_\mu \theta, \tag{5.24}$$

for an adjoint-valued gauge parameter θ. Modulo boundary terms, the action shifts by

$$\delta S = -\int \partial_\mu Z^{a\mu} \Box \theta^a, \tag{5.25}$$

which is zero on the equations of motion by Eq. (5.23).

Global δ_X Transformation

The first global symmetry transformation is

$$\begin{aligned} \delta_X X_\mu &= \theta_{X\mu\nu} Z^\nu \\ \delta_X Y &= 0 \\ \delta_X Z^\mu &= 0, \end{aligned} \tag{5.26}$$

where $\theta_{X\mu\nu} = \partial_\mu \theta_{X\nu} - \partial_\nu \theta_{X\mu}$ is a constant antisymmetric matrix. While $\theta_{X\mu}$ is technically spacetime-dependent, it always enters with a derivative so the symmetry is still global. The action shifts by

$$\delta_X S = \theta_{X\mu\nu} \int f^{abc} \partial_\rho Z^{a\rho} Z^{b\mu} Z^{c\nu}, \tag{5.27}$$

which again vanishes on the equations of motion.

Global $\boldsymbol{\delta_Y}$ Transformation

The second global symmetry transformation is

$$\begin{aligned} \delta_Y X_\mu &= \theta_{Y\mu} Y \\ \delta_Y Y &= -\theta_{Y\mu} Z^\mu \\ \delta_Y Z^\mu &= 0, \end{aligned} \tag{5.28}$$

where $\theta_{Y\mu} = \partial_\mu \theta_Y$ is a constant vector. Again, θ_Y is spacetime-dependent but the symmetry is still global. The action transforms as

$$\delta_Y S = \theta_{Y\mu} \int f^{abc} \partial_\rho Z^{a\rho} Z^{b\mu} Y^c, \tag{5.29}$$

which is zero on the equations of motion.

Global $\boldsymbol{\delta_Z}$ Transformation

The third global transformation is

$$\begin{aligned} \delta_Z X_\mu &= \theta^\nu_Z \partial_\nu X_\mu + \partial_\mu \theta^\nu_Z X_\nu \\ \delta_Z Y &= \theta^\nu_Z \partial_\nu Y \\ \delta_Z Z^\mu &= \theta^\nu_Z \partial_\nu Z^\mu - \partial_\nu \theta^\mu_Z Z^\nu, \end{aligned} \tag{5.30}$$

for a transverse parameter, $\partial_\mu \theta^\mu_Z = 0$. This transformation is an infinitesmal diffeomorphism, where Y transforms as a scalar and X and Z as vectors. Here we will restrict to Poincare transformations, $\theta^\mu_Z = a^\mu + b^\mu{}_\nu x^\nu$, where a is a constant translation vector and b is a constant antisymmetric rotation matrix.

Noether Current Conservation

The global transformations δ_X, δ_Y, δ_Z are associated with a set of equations for Noether current conservation,

$$\begin{aligned} -\theta_{X\mu}\partial J^\mu_X &= \delta X^a_\mu \frac{\delta S}{\delta X^a_\mu} \\ -\theta_Y \partial J_Y &= \delta X^a_\mu \frac{\delta S}{\delta X^a_\mu} + \delta Y^a \frac{\delta S}{\delta Y^a} \\ -\theta^\mu_Z \partial J_{Z\mu} &= \delta X^a_\mu \frac{\delta S}{\delta X^a_\mu} + \delta Y^a \frac{\delta S}{\delta Y^a} + \delta Z^{a\mu} \frac{\delta S}{\delta Z^{a\mu}}, \end{aligned} \tag{5.31}$$

which with the transverse condition on θ_Z imply

$$\begin{aligned} \partial J^\mu_X &= -2Z^{a\nu}\overleftrightarrow{\Box}\partial_\nu Z^{a\mu} \\ \partial J_Y &= -2Z^{a\nu}\overleftrightarrow{\Box}\partial_\nu Y^a \\ \partial J_{Z\mu} &= 2Z^{a\nu}\overleftrightarrow{\Box}X^a_{\mu\nu} + Y^a\overleftrightarrow{\Box}\partial_\mu Y^a, \end{aligned} \tag{5.32}$$

which is the analog of Eq. (5.8) for the NLSM. Note that the cubic interactions also happen to be invariant under the local versions of the δ_X, δ_Y, δ_Z transformations.

5.7 Kinematic Algebra

To derive the kinematic algebra it is useful to introduce a unified description of the XYZ fields in terms of an adjoint-valued multiplet,

$$W_A = \begin{bmatrix} X_\mu \\ Y \\ Z^\mu \end{bmatrix}, \tag{5.33}$$

so the action in Eq. (5.12) becomes

$$S = \int \frac{1}{2} g^{AB} W^a_A \Box W^a_B + \frac{1}{3} f^{abc} F^{ABC} W^a_A W^b_B W^c_C. \tag{5.34}$$

Here capital Latin indices are raised and lowered by

$$g_{AB} = \begin{bmatrix} 0 & 0 & \delta^\mu_\nu \\ 0 & 1 & 0 \\ \delta^\nu_\mu & 0 & 0 \end{bmatrix} \quad \text{and} \quad g^{AB} = \begin{bmatrix} 0 & 0 & \delta^\nu_\mu \\ 0 & 1 & 0 \\ \delta^\mu_\nu & 0 & 0 \end{bmatrix}. \tag{5.35}$$

The kinematic structure constant F^{ABC} is a differential operator acting multilinearly on the fields, and it describes the cubic Feynman vertex. Contracted with a single field, $F^{ABC}W_B$ is represented by a matrix,

$$\begin{bmatrix} 0 & 0 & -\delta^\mu_\nu \overleftarrow{\partial} Z + \overleftarrow{\partial}_\nu Z^\mu \\ 0 & \frac{1}{2}(Z\overrightarrow{\partial} - \overleftarrow{\partial} Z) & \frac{1}{2}(\overleftarrow{\partial}_\nu Y - \partial_\nu Y) \\ \delta^\nu_\mu Z\overrightarrow{\partial} - Z^\nu \overrightarrow{\partial}_\mu & \frac{1}{2}(-Y\overrightarrow{\partial}_\mu + \partial_\mu Y) & \partial_\mu X_\nu - \partial_\nu X_\mu \end{bmatrix}, \tag{5.36}$$

which is manifestly antisymmetric as required. The commutative subgroup of the δ_Z transformations, *i.e.*, spacetime translations, form a natural Cartan subalgebra. In turn, the root vectors are literally momenta while δ_X and δ_Y are raising and lowering operators. We leave a full analysis of the kinematic algebra for future work.

The equations of motion in Eq. (5.22) then become

$$\frac{\delta S}{\delta W^a_A} \simeq \Box W^{aA} + f^{abc} F^{ABC} W^b_B W^c_C = 0, \tag{5.37}$$

where hereafter $\simeq$ will denote equality up to terms that vanish on the equations of motion either by Eq. (5.23) or by integration by parts, *e.g.* $\overleftarrow{\partial}_\mu Y + \partial_\mu Y + Y\overrightarrow{\partial}_\mu = 0$. The field variations in Eq. (5.26), Eq. (5.28), Eq. (5.30) become

$$\delta W^A = \begin{bmatrix} \delta Z^\mu \\ \delta Y \\ \delta X_\mu \end{bmatrix} \simeq F^{ABC}\theta_B W_C, \tag{5.38}$$

with the associated conservation equation,

$$\partial J^A = \begin{bmatrix} \partial J^\mu_X \\ \partial J_Y \\ \partial J_{Z\mu} \end{bmatrix} \simeq -F^{ABC} W^a_B \overleftrightarrow{\Box} W^a_C, \tag{5.39}$$

which is the kinematic analog of Eq. (5.8), proving

$$F^{DAE} F_E{}^{BC} + F^{DBE} F_E{}^{CA} + F^{DCE} F_E{}^{AB} \simeq 0, \tag{5.40}$$

which is precisely the kinematic Jacobi identity.

At the level of scattering amplitudes, these manipulations imply that all Feynman diagrams computed from Eq. (5.10) will automatically satisfy Jacobi identities up to terms that vanish on the transverse condition in Eq. (5.23).

5.8 Double Copy

The double copy procedure maps Eq. (5.1) to Eq. (5.3) via a simple substitution of flavor with kinematics. It is simple to show that the resulting double copy theory is the special Galileon, and in fact this is naturally anticipated by the CHY construction [14].

At the level of the action, the double copy is derived by mechanically dropping all flavor indices and doubling all kinematic structures in the interactions. Since the action in Eq. (5.10) is cubic, this is a trivial procedure. To see how this works, we introduce the fields

$$X_{\mu\bar{\mu}}, \quad Y, \quad Z^{\mu\bar{\mu}}, \tag{5.41}$$

which have doubled index structure relative to the NLSM. These new XYZ fields couple via the action

$$\begin{aligned} S = \int \; & Z^{\mu\bar{\mu}} \Box X_{\mu\bar{\mu}} + \frac{1}{2} Y \Box Y \\ & + 2\left(Z^{\mu\bar{\mu}} Z^{\nu\bar{\nu}} X_{\mu\nu\bar{\mu}\bar{\nu}} + Z^{\mu\bar{\mu}} (Y \overleftrightarrow{\partial}_{\mu} \overleftrightarrow{\partial}_{\bar{\mu}} Y) \right), \end{aligned} \tag{5.42}$$

where we have defined an analog of Riemann curvature, $X_{\mu\nu\bar{\mu}\bar{\nu}} = \partial_\mu \partial_{\bar{\mu}} X_{\nu\bar{\nu}} + \partial_\nu \partial_{\bar{\nu}} X_{\mu\bar{\mu}} - \partial_\mu \partial_{\bar{\nu}} X_{\nu\bar{\mu}} - \partial_\nu \partial_{\bar{\mu}} X_{\mu\bar{\nu}}$. Note that the barred and unbarred indices in Eq. (5.42) are separately contracted, exhibiting the expected twofold Lorentz invariance of the double copy.

Tree amplitudes of the special Galileon are then given by Eq. (5.13) except where the ellipses denote doubly longitudinal polarizations of the Z for which $\epsilon_{\mu\bar{\mu}} = i k_\mu k_{\bar{\mu}}$. It would be interesting to understand how this construction relates to the Galileon as the longitudinal mode of massive gravity [57].

5.9 Infrared Structure

Lastly, we turn to infrared properties. As the momentum p of a particle is taken to be soft, amplitudes in the NLSM and the special Galileon scale as $\mathcal{O}(p)$ [73] and $\mathcal{O}(p^3)$ [51, 64], respectively. Remarkably, these properties dictate virtually everything about these theories [51, 115], and can be leveraged to derive recursion relations for their amplitudes [74]. While this soft behavior is usually obscured at the level of the action, the $\mathcal{O}(p)$ scaling of the NLSM and $\mathcal{O}(p^2)$ scaling of the special Galileon have a simple explanation in our formulation.

In particular, consider the soft limit of a Nambu-Goldstone boson, taken here to be a longitudinal Z of the NLSM amplitude in Eq. (5.13). Since Z enters with a derivative, the corresponding kinematic numerator trivially scales as $\mathcal{O}(p)$. However, the hard leg from which Z is emitted enters with a nearly on-shell propagator with $\mathcal{O}(p^{-1})$, so the net scaling of the amplitude is $\mathcal{O}(1)$. Now observe from Eq. (5.12) that cubic interactions of Z take the form of gauge interactions modulo terms that vanish for longitudinal Z components. Although the action lacks the requisite quartic interactions needed for a genuine Z gauge symmetry, the soft Z limit is dictated solely by cubic interactions. In turn, the Z soft limit obeys the usual Weinberg soft theorems for gauge bosons [83], dropping contributions from lower point amplitudes with a longitudinal Z since they are odd and hence vanish by the underlying parity of the NLSM. Gauge invariance then implies that the amplitude for soft longitudinal Z emission is zero, eliminating the leading $\mathcal{O}(1)$ contribution but leaving the residual $\mathcal{O}(p)$ scaling of the NLSM. This cancellation can be verified via Feynman diagrams. Similarly, the $\mathcal{O}(p)$ contribution of the special Galileon vanishes by the Weinberg soft graviton theorem, however the further cancellation of $\mathcal{O}(p^2)$ terms is not obvious.

Remarkably, the leading nontrivial soft behavior of NLSM amplitudes is actually characterized by an underlying extended theory [88]. We can accommodate the structure by promoting Y to a biadjoint field with the additional cubic coupling, $f^{abc}f^{\bar{a}\bar{b}\bar{c}}Y^{a\bar{a}}Y^{b\bar{b}}Y^{c\bar{c}}$, which preserves all the Jacobi identities of the full action. We have verified that this modification reproduces the soft theorem in [88] up to ten-particle scattering.

5.10 Summary

In summary, we have reformulated the NLSM and special Galileon as theories of purely cubic interactions. At the expense of explicit Bose symmetry and parity of the Nambu-Goldstone bosons, these representations exhibit several elegant properties. In particular, they manifest flavor-kinematics duality as a symmetry, trivialize the double copy structure, and explain the vanishing soft behavior of amplitudes via the Weinberg soft theorem.

Chapter 6

NON-RENORMALIZATION THEOREM WITHOUT SUPERSYMMETRY

6.1 Introduction

In this chapter we derive a new class of non-renormalization theorems for non-supersymmetric theories. Our results apply to the one-loop running of the leading irrelevant deformations of a four-dimensional quantum field theory of marginal interactions,

$$\Delta\mathcal{L} = \sum_i c_i \mathcal{O}_i, \tag{6.1}$$

where $\mathcal{O}_i$ are higher dimension operators. At leading order in c_i, renormalization induces operator mixing via

$$(4\pi)^2 \frac{dc_i}{d\log\mu} = \sum_j \gamma_{ij} c_j, \tag{6.2}$$

where by dimensional analysis the anomalous dimension matrix γ_{ij} is a function of marginal couplings alone.

The logic of our approach is simple, making no reference to symmetry. Renormalization is induced by log divergent amplitudes, which by unitarity have kinematic cuts equal to products of on-shell tree amplitudes [1, 2]. If any of these tree amplitudes vanish, then so too will the divergence. Crucially, many tree amplitudes are zero due to helicity selection rules, which *e.g.* forbid the all minus helicity gluon amplitude in Yang-Mills theory.

For our analysis, we define the holomorphic and anti-holomorphic weight of an on-shell amplitude A by[1]

$$w(A) = n(A) - h(A), \qquad \overline{w}(A) = n(A) + h(A), \tag{6.3}$$

where $n(A)$ and $h(A)$ are the number and sum over helicities of the external states. Since A is physical, its weight is field reparameterization and gauge independent. The weights of an operator $\mathcal{O}$ are then invariantly defined by

[1]Holomorphic weight is a generalization of k-charge in super Yang-Mills theory, where the N^{k}MHV amplitude has $w = k + 4$.

minimizing over all amplitudes involving that operator: $w(\mathcal{O}) = \min\{w(A)\}$ and $\overline{w}(\mathcal{O}) = \min\{\overline{w}(A)\}$. In practice, operator weights are fixed by the leading non-zero contact amplitude[2] built from an insertion of $\mathcal{O}$,

$$w(\mathcal{O}) = n(\mathcal{O}) - h(\mathcal{O}), \qquad \overline{w}(\mathcal{O}) = n(\mathcal{O}) + h(\mathcal{O}), \tag{6.4}$$

where $n(\mathcal{O})$ is the number of particles created by $\mathcal{O}$ and $h(\mathcal{O})$ is their total helicity. For field operators we find:

$\mathcal{O}$	$F_{\alpha\beta}$	ψ_α	ϕ	$\bar{\psi}_{\dot{\alpha}}$	$\bar{F}_{\dot{\alpha}\dot{\beta}}$
h	$+1$	$+1/2$	0	$-1/2$	-1
$(w,\overline{w})$	$(0,2)$	$(1/2,3/2)$	$(1,1)$	$(3/2,1/2)$	$(2,0)$

where all Lorentz covariance is expressed in terms of four-dimensional spinor indices, so *e.g.* the gauge field strength is $F_{\alpha\dot{\alpha}\beta\dot{\beta}} = F_{\alpha\beta}\bar{\epsilon}_{\dot{\alpha}\dot{\beta}} + \bar{F}_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}$. The weights of all dimension five and six operators are shown in Fig. 6.1.

As we will prove, an operator $\mathcal{O}_i$ can only be renormalized by an operator $\mathcal{O}_j$ at one-loop if the corresponding weights $(w_i,\overline{w}_i)$ and $(w_j,\overline{w}_j)$ satisfy the inequalities

$$w_i \geq w_j \qquad \text{and} \qquad \overline{w}_i \geq \overline{w}_j, \tag{6.5}$$

and all Yukawa couplings are of a "holomorphic" form consistent with a superpotential. This implies a new class of non-renormalization theorems,

$$\gamma_{ij} = 0 \qquad \text{if} \qquad w_i < w_j \quad \text{or} \quad \overline{w}_i < \overline{w}_j, \tag{6.6}$$

which impose mostly zero entries in the matrix of anomalous dimensions. The resulting non-renormalization theorems for all dimension five and six operators are shown in Tab. 6.1 and Tab. 6.2.

Because our analysis hinges on unitarity and helicity rather than off-shell symmetry principles, the resulting non-renormalization theorems are general. Moreover, they explain the ubiquitous and surprising cancellations [38] in the one-loop renormalization of dimension six operators in the standard model [116–122]. Absent an explanation from power counting or spurions, the authors of [38] conjectured a hidden "holomorphy" enforcing non-renormalization among holomorphic and anti-holomorphic operators. We show here that this classification simply corresponds to $w < 4$ and $\overline{w} < 4$, so these cancellations follow immediately from Eq. (6.6), as shown in Tab. 6.2 .

[2]By definition, all covariant derivatives D are treated as partial derivatives ∂ when computing the leading contact amplitude.

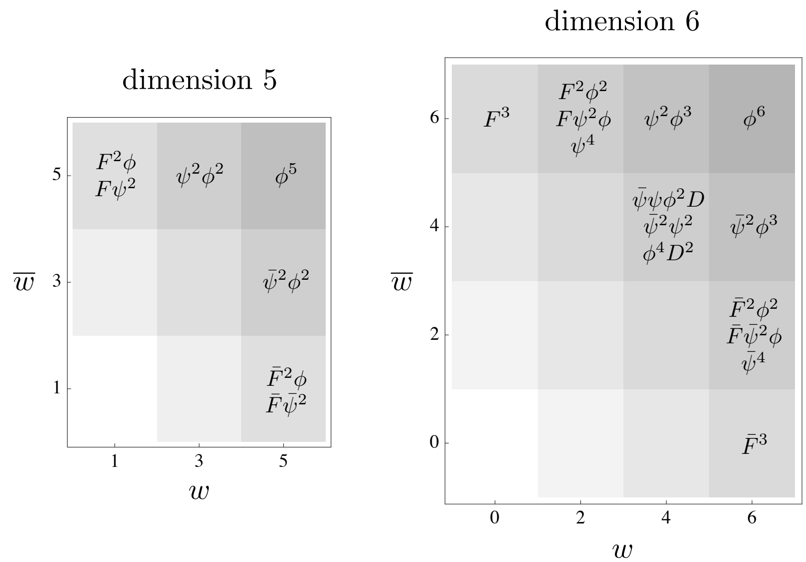


Figure 6.1: Weight lattice for dimension five and six operators, suppressing flavor and Lorentz structures, *e.g.* on which fields derivatives act. Our non-renormalization theorems permit mixing of operators into operators of equal or greater weight. Pictorially, this forbids transitions down or to the left.

6.2 Weighing Tree Amplitudes

To begin, we compute the holomorphic and anti-holomorphic weights $(w_n, \overline{w}_n)$ of a general n-point on-shell tree amplitude in a renormalizable theory of massless particles. We start at lower-point and apply induction to extend to higher-point.

The three-point amplitude is

$$A(1^{h_1}2^{h_2}3^{h_3}) = g \begin{cases} \langle 12\rangle^{r_3}\langle 23\rangle^{r_1}\langle 31\rangle^{r_2}, & \sum_i h_i \leq 0 \\ [12]^{\overline{r}_3}[23]^{\overline{r}_1}[31]^{\overline{r}_2}, & \sum_i h_i \geq 0 \end{cases} \tag{6.7}$$

where g is the coupling and each case corresponds to MHV and $\overline{\text{MHV}}$ kinematics, $|1] \propto |2] \propto |3]$ and $|1\rangle \propto |2\rangle \propto |3\rangle$. Lorentz invariance fixes the exponents to be $r_i = -\overline{r}_i = 2h_i - \sum_j h_j$ and $\sum_i r_i = \sum_i \overline{r}_i = 1 - [g]$ by dimensional analysis [35]. According to Eq. (6.7), the corresponding weights are

$$(w_3, \overline{w}_3) \quad = \quad \begin{cases} (4-[g], 2+[g]), & \sum_i h_i \leq 0 \\ (2+[g], 4-[g]), & \sum_i h_i \geq 0. \end{cases} \tag{6.8}$$

In a renormalizable theory, $[g] = 0$ or 1, so we obtain

$$w_3, \overline{w}_3 \geq 2, \tag{6.9}$$

for the three-point amplitude.

The majority of four-point tree amplitudes satisfy $w_4, \overline{w}_4 \geq 4$ because $w_4 < 4$ and $\overline{w}_4 < 4$ require a non-zero total helicity which is typically forbidden by helicity selection rules. To see why, we enumerate all possible candidate amplitudes with $w_4 < 4$. Analogous arguments will apply for $\overline{w}_4 < 4$.

Most four-point tree amplitudes with $w_4 = 1$ or 3 vanish since they have no Feynman diagrams, so

$$\begin{aligned} 0 &= A(F^+F^+F^\pm\phi) = A(F^+F^+\psi^\pm\psi^\pm) \\ &= A(F^+F^-\psi^+\psi^+) = A(F^+\psi^+\psi^-\phi) \\ &= A(\psi^+\psi^+\psi^+\psi^-). \end{aligned}$$

Furthermore, most amplitudes with $w_4 = 0$ or 2 vanish due to helicity selection rules, so

$$\begin{aligned} 0 &= A(F^+F^+F^+F^\pm) = A(F^+F^+\psi^+\psi^-) \\ &= A(F^+F^+\phi\,\phi) = A(F^+\psi^+\psi^+\phi). \end{aligned}$$

While Feynman diagrams exist, they vanish on-shell for the chosen helicities. This leaves a handful of candidate non-zero amplitudes,

$$0 \neq A(\psi^+\psi^+\psi^+\psi^+), A(F^+\phi\,\phi\,\phi), A(\psi^+\psi^+\phi\,\phi),$$

with $w_4 = 2, 3, 3$, respectively. These "exceptional amplitudes" are the only four-point tree amplitudes with $w_4 < 4$ that do not vanish identically.

The exceptional amplitudes all require internal or external scalars, so they are absent in theories with only gauge bosons and fermions, *e.g.* QCD. The second and third amplitudes involve super-renormalizable cubic scalar interactions, which we do not consider here. The first amplitude arises from Yukawa couplings of non-holomorphic form: that is, $\phi\psi^2$ together with $\bar{\phi}\psi^2$, which in a supersymmetric theory would violate holomorphy of the superpotential. In the standard model, Higgs doublet exchange generates an exceptional amplitude proportional to the product up-type and down-type Yukawa couplings. This diagram will be important later when we consider the standard model. In summary,

$$w_4, \overline{w}_4 \geq 4, \tag{6.10}$$

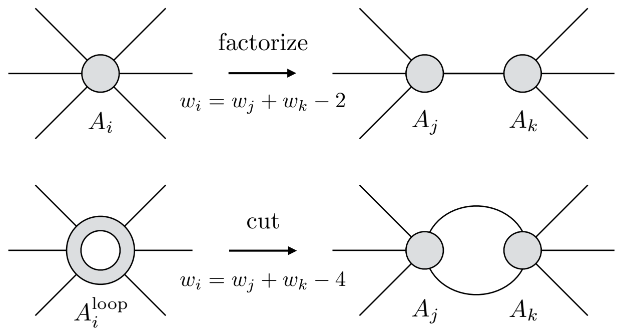


Figure 6.2: Diagrams of tree factorization and one-loop unitarity, with the weight selection rules from Eqs. (6.12) and (6.18).

for the four-point amplitude, modulo exceptional amplitudes.

Finally, consider a general higher-point tree amplitude, A_i, which on a factorization channel equals a product of amplitudes, A_j and A_k,

$$\text{fact}[A_i] \quad = \quad \frac{i}{\ell^2} \sum_h A_j(\ell^h) A_k(-\ell^{-h}), \tag{6.11}$$

depicted in Fig. 6.2. If the total numbers and helicities of A_i, A_j, and A_k, are (n_i, h_i), (n_j, h_j), and (n_k, h_k), then $n_i = n_j + n_k - 2$ and $h_i = h_j + h_k$, since either side of the factorization channel carries equal and opposite helicity. Thus, the corresponding weights, $(w_i, \overline{w}_i)$, $(w_j, \overline{w}_j)$, and $(w_k, \overline{w}_k)$, satisfy the following tree selection rule,

$$\text{tree rule:} \qquad \begin{aligned} w_i &= w_j + w_k - 2 \\ \overline{w}_i &= \overline{w}_j + \overline{w}_k - 2. \end{aligned} \tag{6.12}$$

We have already shown that $w_3, \overline{w}_3 \geq 2$ and $w_4, \overline{w}_4 \geq 4$ modulo the exceptional diagrams. Since all five-point amplitudes factorize into three and four-point amplitudes, Eq. (6.12) implies that $w_5, \overline{w}_5 \geq 4$. Induction to higher-point then yields the main result of this section,

$$w_n, \overline{w}_n \quad \geq \quad \begin{cases} 2, & n = 3 \\ 4, & n > 3, \end{cases} \tag{6.13}$$

which, modulo exceptional amplitudes, is a lower bound on the weights of n-point tree amplitudes in a theory of massless particles with marginal interactions. Note that even when exceptional amplitudes exist, $w_n, \overline{w}_n \geq 2$.

An important consequence of Eq. (6.12) is that attaching renormalizable interactions to an arbitrary amplitude A_j—perhaps involving irrelevant interactions—can only produce an amplitude A_i of greater or equal weight. To see why, note that A_i factorizes into A_j and an amplitude A_k composed of only renormalizable interactions, where $w_k, \overline{w}_k \geq 2$ by Eq. (6.13). Eq. (6.12) then implies that $w_i \geq w_j$ and $\overline{w}_i \geq \overline{w}_j$, so the minimum weight amplitude involving a higher dimension operator is the contact amplitude built from a single insertion of that operator.

6.3 Weighing One-Loop Amplitudes

The weights of one-loop amplitudes are obtained from generalized unitarity and the tree-level results of the previous section. The leading order renormalization of higher dimension operators is encoded in the anomalous dimension matrix γ_{ij} describing how $\mathcal{O}_i$ is radiatively generated by $\mathcal{O}_j$ and loops of marginal interactions. In practice, γ_{ij} is extracted from the one-loop amplitude A_i^{loop} built around an insertion of $\mathcal{O}_j$ with the same external states as the tree amplitude A_i built around an insertion of $\mathcal{O}_i$. Any divergence in A_i^{loop} must then be absorbed by the counterterm A_i, which implies non-zero γ_{ij}. By dimensional analysis, a necessary condition for renormalization is that $\mathcal{O}_i$ and $\mathcal{O}_j$ have equal mass dimension, but as we will see, this is not a sufficient condition because of our non-renormalization theorems.

The Passarino-Veltman (PV) reduction [123] of the one-loop amplitude A_i^{loop} is

$$A_i^{\text{loop}} = \sum_{\text{box}} d_4 I_4 + \sum_{\text{triangle}} d_3 I_3 + \sum_{\text{bubble}} d_2 I_2 + \text{rational},$$

which sums over topologies of scalar box, triangle, and bubble integrals, I_4, I_3, and I_2. Tadpole integrals vanish for massless particles. The integral coefficients d_4, d_3, and d_2 are rational functions of external kinematic data. Ultraviolet log divergences arise from the scalar bubble integrals in the PV reduction, where in dimensional regularization, $I_2 \to 1/(4\pi)^2\epsilon$. Separating ultraviolet divergent and finite terms, we find

$$A_i^{\text{loop}} = \frac{1}{(4\pi)^2\epsilon} \sum_{\text{bubble}} d_2 + \text{finite}, \tag{6.14}$$

which implies a counterterm tree amplitude,

$$A_i = -\frac{1}{(4\pi)^2\epsilon} \sum_{\text{bubble}} d_2, \tag{6.15}$$

so $A_i^{\text{loop}} + A_i$ is finite.

Generalized unitarity [1, 2] fixes integral coefficients by relating kinematic singularities of the one-loop amplitude to products of tree amplitudes. The two-particle cut in a particular channel is

$$\text{cut}[A_i^{\text{loop}}] = \sum_{h_1,h_2} A_j(\ell_1^{h_1}, \ell_2^{h_2}) A_k(-\ell_1^{-h_1}, -\ell_2^{-h_2}), \tag{6.16}$$

where ℓ_1, ℓ_2 and h_1, h_2 are the momenta and helicities of the cut lines and A_j and A_k are on-shell tree amplitudes corresponding to the cut channel, as depicted in Fig. 6.2.

Applying this cut to the PV reduction, we find

$$\text{cut}[A_i^{\text{loop}}] = d_2 + \text{terms depending on } \ell_1, \ell_2, \tag{6.17}$$

where the ℓ_1, ℓ_2 dependent terms correspond to two-particle cuts of triangle and box integrals. Famously, the divergence of the one-loop amplitude is related to the two-particle cut [7, 124, 125]. However, a kinematic singularity is present only if A_j and A_k are four-point amplitudes or higher, corresponding to "massive" bubble integrals. When A_j or A_k are three-point amplitudes, the associated "massless" bubble integrals are scaleless and vanish in dimensional regularization. We ignore these subtle contributions for now but revisit them later.

Eqs. (6.15), (6.16), and (6.17) imply that the total numbers and helicities (n_i, h_i), (n_j, h_j), (n_k, h_k) of A_i, A_j and A_k satisfy $n_i = n_j + n_k - 4$ and $h_i = h_j + h_k$, and thus the one-loop selection rule,

$$\text{one-loop rule:} \qquad \begin{aligned} w_i &= w_j + w_k - 4 \\ \overline{w}_i &= \overline{w}_j + \overline{w}_k - 4 \end{aligned} \tag{6.18}$$

where $(w_i, \overline{w}_i)$, $(w_j, \overline{w}_j)$, and $(w_k, \overline{w}_k)$ are the corresponding amplitude weights. For each γ_{ij} we identify A_i and A_j with tree amplitudes built around insertions of $\mathcal{O}_i$ and $\mathcal{O}_j$, and A_k with a tree amplitude of the renormalizable theory. As noted earlier, the amplitudes on both sides of the cut must be four-point or higher for a non-trivial unitarity cut, so Eq. (6.13) implies that $w_k, \overline{w}_k \geq 4$, absent exceptional amplitudes. Eq. (6.18) then implies that $w_i \geq w_j$ and $\overline{w}_i \geq \overline{w}_j$, which is the non-renormalization theorem of Eq. (6.5). If exceptional amplitudes with $w_k, \overline{w}_k = 2$ are present from non-holomorphic Yukawas, then Eq. (6.5) is violated by exactly two units.

		$F^2\phi$	$F\psi^2$	$\psi^2\phi^2$	$\bar{F}\bar{\psi}^2$	$\bar{F}^2\phi$	$\bar{\psi}^2\phi^2$	ϕ^5
	$(w,\bar{w})$	$(1,5)$	$(1,5)$	$(3,5)$	$(5,1)$	$(5,1)$	$(5,3)$	$(5,5)$
$F^2\phi$	$(1,5)$							
$F\psi^2$	$(1,5)$							
$\psi^2\phi^2$	$(3,5)$							
$\bar{F}^2\phi$	$(5,1)$							
$\bar{F}\bar{\psi}^2$	$(5,1)$							
$\bar{\psi}^2\phi^2$	$(5,3)$							
ϕ^5	$(5,5)$							

Table 6.1: Anomalous dimension matrix for dimension five operators in a general quantum field theory. The shaded entries vanish by our non-renormalization theorems.

Fig. 6.1 shows the weight lattice for all dimension five and six operators in a general quantum field theory. We employ the operator basis of [126], so redundant operators, *e.g.* those involving $\Box\phi$, are eliminated by equations of motion. Our non-renormalization theorems imply that operators can only renormalize operators of equal or greater weight, which in Fig. 6.1 forbids transitions that move down or to the left. The form of the anomalous dimension matrix for all dimension five and six operators is shown in Tab. 6.1 and Tab. 6.2.

6.4 Infrared Divergences

We now return to the issue of massless bubble integrals. While these contributions formally vanish in dimensional regularization, this is potentially misleading because ultraviolet and infrared divergences enter with opposite sign $1/\epsilon$ poles. Thus, an ultraviolet divergence may be present if there is an equal and opposite virtual infrared divergence [7, 124, 125]. Crucially, the Kinoshita-Lee-Nauenberg theorem [128, 129] maintains that all virtual infrared divergences are canceled by an inclusive final state sum incorporating tree-level real emission of an unresolved soft or collinear particle. Inverting the logic, if real emission is infrared finite, then there can be no virtual infrared divergence and thus no ultraviolet divergence. As we will see, this is true of the massless bubble contributions which were discarded but could *a priori* violate Eq. (6.5).

To diagnose potential infrared divergences in A_i^{loop}, we analyze the associated amplitude for real emission, $A_{i'}^{\text{real}}$. In the infrared regime, the singular part of this amplitude factorizes: $A_{i'}^{\text{real}} \to A_i S_{i\to i'} + A_j S_{j\to i'}$, where A_i and

		F^3	$F^2\phi^2$	$F\psi^2\phi$	ψ^4	$\psi^2\phi^3$	$\bar{F}^3$	$\bar{F}^2\phi^2$	$\bar{F}\bar{\psi}^2\phi$	$\bar{\psi}^4$	$\bar{\psi}^2\phi^3$	$\bar{\psi}^2\psi^2$	$\bar{\psi}\psi\phi^2 D$	$\phi^4 D^2$	ϕ^6
	$(w,\bar{w})$	$(0,6)$	$(2,6)$	$(2,6)$	$(2,6)$	$(4,6)$	$(6,0)$	$(6,2)$	$(6,2)$	$(6,2)$	$(6,4)$	$(4,4)$	$(4,4)$	$(4,4)$	$(6,6)$
F^3	$(0,6)$			$\times$	$\times$	$\times$			$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$F^2\phi^2$	$(2,6)$				$\times$	$\times$				$\times$	$\times$	$\times$			$\times$
$F\psi^2\phi$	$(2,6)$									$\times$				$\times$	$\times$
ψ^4	$(2,6)$	$\times$	$\times$			$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	y^2		$\times$	$\times$
$\psi^2\phi^3$	$(4,6)$	$\times^*$									y^2				$\times$
$\bar{F}^3$	$(6,0)$			$\times$	$\times$	$\times$			$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$\bar{F}^2\phi^2$	$(6,2)$				$\times$	$\times$				$\times$	$\times$	$\times$			$\times$
$\bar{F}\bar{\psi}^2\phi$	$(6,2)$				$\times$									$\times$	$\times$
$\bar{\psi}^4$	$(6,2)$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$			$\times$	$\bar{y}^2$		$\times$	$\times$
$\bar{\psi}^2\phi^3$	$(6,4)$					$\bar{y}^2$	$\times^*$								$\times$
$\bar{\psi}^2\psi^2$	$(4,4)$		$\times$		$\bar{y}^2$	$\times$		$\times$		y^2	$\times$			$\times$	$\times$
$\bar{\psi}\psi\phi^2 D$	$(4,4)$														$\times$
$\phi^4 D^2$	$(4,4)$				$\times$					$\times$		$\times$			$\times$
ϕ^6	$(6,6)$	$\times^*$		$\times$	$\times$		$\times^*$		$\times$	$\times$		$\times$			

Table 6.2: Anomalous dimension matrix for dimension six operators in a general quantum field theory. The shaded entries vanish by our non-renormalization theorems, in full agreement with [38]. Here y^2 and $\bar{y}^2$ label entries that are non-zero due to non-holomorphic Yukawa couplings, $\times$ labels entries that vanish because there are no diagrams [127], and $\times^*$ labels entries that vanish by a combination of counterterm analysis and our non-renormalization theorems.

A_j are tree amplitudes built around insertions of $\mathcal{O}_i$ and $\mathcal{O}_j$, and $S_{i\to i'}$ and $S_{j\to i'}$ are soft-collinear functions describing the emission of an unresolved particle. The soft-collinear functions from marginal interactions diverge as $1/\omega$ and $1/\sqrt{1-\cos\theta}$ in the soft and collinear limits, respectively, where ω and θ are the energy and splitting angle characterizing the emitted particle. By dimensional analysis, irrelevant interactions have additional powers of soft or collinear momentum rendering them infrared finite—a fact we have verified explicitly for all dimension five and six operators. Since the phase-space measure is $\int d\omega\,\omega \int d\cos\theta$, infrared divergences require that $S_{i\to i'}$ and $S_{j\to i'}$ both arise from soft and/or collinear marginal interactions.

For soft emission, the hard process is unchanged [83]. Since $A_i S_{i\to i'}$ and $A_j S_{j\to i'}$ contribute to the same process, A_i and A_j must have the same external states and thus equal weight, $w_i = w_j$. While massless bubbles do contribute infrared and ultraviolet divergences not previously accounted for, this is perfectly consistent with the non-renormalization theorem in Eq. (6.5), which allows for operator mixing when $w_i = w_j$. Violation of Eq. (6.5) instead requires from infrared divergences when $w_i < w_j$. However, the corresponding soft emission would induce a hard particle helicity flip and thus be subleading in the soft limit and finite upon $\int d\omega$ integration.

Similarly, collinear emission is divergent for $w_i = w_j$ but finite for $w_i < w_j$. Since $A_i S_{i\to i'}$ and $A_j S_{j\to i'}$ have the same external states and weight, restricting to $w_i < w_j$ means that $w(S_{i\to i'}) > w(S_{j\to i'})$. Eq. (6.8) then implies that $S_{i\to i'}$ and $S_{j\to i'}$ are collinear splitting functions generated by on-shell $\overline{\text{MHV}}$ and MHV amplitudes. As a result, the interference term $S^*_{j\to i'} S_{i\to i'}$ carries net little group weight with respect to the mother particle initiating the collinear emission. Rotations of angle ϕ around the mother particle axis act as a little group transformation on $S^*_{j\to i'} S_{i\to i'}$, yielding a net phase $e^{2i\phi}$ in the differential cross-section. Integrating over this angle yields $\int_0^{2\pi} d\phi\, e^{2i\phi} = 0$, so the collinear singularity vanishes upon phase-space integration.

In summary, since real emission is infrared finite for $w_i < w_j$, there are no corresponding ultraviolet divergences from massless bubbles. The non-renormalization theorems in Eq. (6.5) apply despite infrared subtleties.

6.5 Application to the Standard Model

Our results apply to the standard model and its extension to higher dimension operators [38, 116–122]. A tour de force calculation of the full one-loop anomalous dimension matrix of dimension six operators [117–119] unearthed a string of miraculous cancellations not enforced by a manifest symmetry and visible only after the meticulous application of equations of motion [38]. Lacking an explicit Lagrangian symmetry, the authors of [38] conjectured an underlying "holomorphy" of the standard model effective theory.

The cancellations in [38] are a direct consequence of the non-renormalization theorems in Eq. (6.5) and Eq. (6.6), based on a classification of holomorphic ($w < 4$), anti-holomorphic ($\overline{w} < 4$), and non-holomorphic operators ($w, \overline{w} \geq 4$), and violated only by exceptional amplitudes ($w, \overline{w} = 2$) generated by non-holomorphic Yukawas. The shaded entries in Tab. 6.2 denote zeroes enforced by our non-renormalization theorems. Entries marked with $\times$ trivially vanish because there are no associated Feynman diagrams, while entries marked with $\times^*$ vanish because the expected divergences in $\psi^2\phi^3$ and ϕ^6 are accompanied by a counterterm of the form $\phi^4 D^2$ [119] which is forbidden by our non-renormalization theorems.

The superfield formalism offers an enlightening albeit partial explanation of these cancellations [130] and analogous effects in chiral perturbation theory [131]. These results are clearly connected to our own via the "effective" supersymmetry of tree-level QCD [132–135], and merits further study.

6.6 Outlook

We have derived a new class of one-loop non-renormalization theorems for higher dimension operators in a general four-dimensional quantum field theory. Since our arguments follow from unitarity and helicity, they are broadly applicable and explain the peculiar cancellations observed in the dimension six renormalization of the standard model.

Non-renormalization at higher loop orders remains an open question. However, Eq. (6.5) will likely fail at two-loop since helicity selection rules are violated by finite one-loop corrections [136–139]. Another avenue for future study is higher dimensions, where helicity is naturally extended [140] and dimensional reduction offers a bridge to massive theories. Finally, it would be interesting to link our results to conventional symmetry arguments like those of [130]. Indeed, our definition of weight is reminiscent of both R-symmetry and twist, which relate to existing non-renormalization theorems.

Appendix A

A.1 Proof of the Soft Theorem

In this Appendix we give detailed proof of the soft theorem mentioned in the Section 3. While the bulk of this paper focuses on tree-level scattering amplitudes, we present here a non-perturbative proof which to our knowledge does not exists in the literature. For simplicity we restrict ourselves to theory with single NGB, while the generalization to multiple flavors is straightforward.

Review of the Adler Zero

For our analysis it will be helpful to briefly review the derivation of the Adler zero for the amplitudes of NGBs (see *e.g.* the textbook [79] and references therein). To begin, consider a theory of a single NGB corresponding to the spontaneous breaking of a one-parameter continuous symmetry. In most cases such a symmetry acts non-linearly on the NGB field according to

$$\phi\left(x\right) \to \phi\left(x\right) + a, \tag{A.1}$$

which has an associated Noether current $J^\mu(x)$. The NGB couples to the current with a strength parameterized by the decay constant, F, so

$$\langle 0|J^\mu(x)|\phi(\mathbf{p})\rangle = ip^\mu F e^{-ip\cdot x}. \tag{A.2}$$

The matrix elements of the current $J^\mu(x)$ has a pole as $p^2 \to 0$ whose residue is related to the amplitude for the NGB emission,[1]

$$\begin{aligned}\langle \alpha, \mathrm{out}|J^\mu(0)|\beta, \mathrm{in}\rangle &= \frac{i}{p^2}\langle 0|J^\mu(0)|\phi(\mathbf{p})\rangle \langle \alpha + \phi(\mathbf{p}), \mathrm{out}|\beta, \mathrm{in}\rangle + R^\mu(p) \\ &= -\frac{p^\mu}{p^2}F\langle \alpha + \phi(\mathbf{p}), \mathrm{out}|\beta, \mathrm{in}\rangle + R^\mu(p)\end{aligned} \tag{A.3}$$

where $p^\mu = P^\mu_{\beta,\mathrm{in}} - P^\mu_{\alpha,\mathrm{out}}$ is the difference between the in and out momenta, and $R^\mu(p)$ denotes a remainder function which is regular as $p^2 \to 0$. Due to conservation of J^μ we can dot Eq. (A.3) into p^μ to obtain the equation

$$\langle \alpha + \phi(\mathbf{p}), \mathrm{out}|\beta, \mathrm{in}\rangle = \frac{1}{F}p_\mu R^\mu(p), \tag{A.4}$$

[1] Here and in what follows we tacitly assume that all the momentum conservation $\delta-$functions are removed from the matrix elements. I.e., R^μ does not contain momentum conservation $\delta-$functions.

so $p_\mu R^\mu(p)/F$ can be thought of as an off-shell extension of the amplitude. The behavior of the amplitude in the soft NGB limit $p \to 0$ can be therefore inferred from the properties of the remainder function $R^\mu(p)$. Provided the theory does not have a cubic vertex, then $R^\mu(p)$ is regular for $p \to 0$, which implies that

$$\lim_{p\to 0} \langle \alpha + \phi(\mathbf{p}), \mathrm{out}|\beta, \mathrm{in}\rangle = \frac{1}{F}\lim_{p\to 0} p_\mu R^\mu(p) = 0. \tag{A.5}$$

This condition is precisely the Adler zero for NGB soft emission.

Classical Current Relations

It is straightforward to extend our results to the case of a generalized shift symmetry,

$$\phi \to \phi + \delta_\theta \phi\,(x) \tag{A.6}$$

where the variation takes the form

$$\delta_\theta \phi\,(x) = \theta_j \alpha_A^j(x) O^A\,[\phi]\,(x)\,. \tag{A.7}$$

Here θ_j are infinitesimal parameters, $\alpha_A^j\,(x)$ are fixed polynomial functions, and $O^A\,[\phi]\,(x)$ are local but generally composite operators constructed from $\phi\,(x)$ and its derivatives.

Classically, we can consider the local shift transformation, $\phi(x) \to \phi(x) + a(x)$, with a shift parameter with special value of $a\,(x) = \widehat{a}\,(x)$, namely with

$$\widehat{a}\,(x) = \theta_j\,(x)\,\alpha_A^j\,(x)\,O^A\,[\phi]\,(x)\,,$$

which coincides with the localized version of the transformation Eq. (A.7) with parameters $\theta_j \to \theta_j\,(x)$. This induces a relation between the Noether current of the shift symmetry $J^\mu\,(x)$ and the Noether current $J^{(j)\mu}\,(x)$ corresponding to the transformation Eq. (A.7) (see [80] for general discussion and further details)

$$\int d^d x\ \partial\widehat{a} \cdot J = \int d^d x\ \partial\theta_j \cdot J^{(j)}\,(x)\,. \tag{A.8}$$

Explicitly, we obtain

$$\int d^d x \left[\partial\theta_j \alpha_A^j O^A\,[\phi] + \theta_j \partial\alpha_A^j O^A\,[\phi] + \theta_j \alpha_A^j \partial O^A\,[\phi]\right] \cdot J = \int d^d x\ \partial\theta_j \cdot J^{(j)}\,(x)\,. \tag{A.9}$$

Invariance of the action with respect to the global form of the transformation Eq. (A.7) means that for constant θ_j, the integrand on the left-hand side of the previous equation is a total derivative

$$\left(\partial\alpha_A^j O^A[\phi] + \alpha_A^j \partial O^A[\phi]\right)\cdot J = \partial_\alpha\left(\beta_I^{\alpha j}\mathcal{O}^I[\phi]\right),$$

where β_I^j are known functions and $\mathcal{O}^I$are local composite operators. Inserting the latter into Eq. (A.9) we get

$$\int d^dx\ \partial\theta_j\cdot J^{(j)}(x) = \int d^dx\left[J\cdot\partial\theta_j\alpha_A^j O^A[\phi] + \theta_j\partial_\alpha\left(\beta_I^{\alpha j}\mathcal{O}^I[\phi]\right)\right]$$

and thus

$$J^{(j)\mu} = \alpha_A^j O^A[\phi]\,J^\mu - \beta_I^{\mu j}\mathcal{O}^I[\phi].$$

To summarize, we get two algebraic off-shell identities

$$\begin{aligned}\left(\partial\alpha_A^j O^A[\phi] + \alpha_A^j\partial O^A[\phi]\right)\cdot J &= \partial\cdot\beta_I^j\mathcal{O}^I[\phi] + \beta_I^j\cdot\partial\mathcal{O}^I[\phi]\\ J^{(j)} &= \alpha_A^j O^A[\phi]\,J - \beta_I^j\mathcal{O}^I[\phi],\end{aligned}\qquad\text{(A.10)}$$

which reveal the underlying dependence between the currents: conservation of $J^{(j)}$ is a consequence of conservation of J.

Let us now apply these relations to the case when $O^1[\phi]=1$, i.e. when we can rewrite Eq. (A.7) in the form

$$\delta_\theta\phi(x) = \theta_j\left[\alpha^j(x) + \alpha_B^j(x)\,O^B[\phi](x)\right].\qquad\text{(A.11)}$$

Such a transformation can be understood as a generalization of the simple shift symmetry Eq. (A.1) or more generally of the polynomial shift symmetry discussed in [62] and [61]. Note again that $\alpha^j(x)$ and $\alpha_B^j(x)$ are polynomials. Then the first of the relations, Eq. (A.10), reads

$$\partial\alpha^j\cdot J = -\partial\cdot\left(\alpha_B^j O^B[\phi]\,J - \beta_I^j\mathcal{O}^I[\phi]\right) + \alpha_B^j O^B[\phi]\,\partial\cdot J.\qquad\text{(A.12)}$$

From now we will assume just this special form of the relation between currents.

Quantum Current Relations

Another important assumption is that the above mentioned relations survive quantization, so for the renormalized quantum operators we have the current conservation equation,

$$\partial\cdot\left\langle\alpha,\mathrm{out}|J^{(j)}(x)|\beta,\mathrm{in}\right\rangle = \partial\cdot\langle\alpha,\mathrm{out}|J(x)|\beta,\mathrm{in}\rangle = 0,\qquad\text{(A.13)}$$

as well as the relation

$$\begin{aligned}
& \partial\alpha^{j}(x)\cdot\langle\alpha,\mathrm{out}|J(x)|\beta,\mathrm{in}\rangle \\
= \; & -\partial\cdot\left\langle\alpha,\mathrm{out}|\alpha_{B}^{j}(x)\,O^{B}[\phi](x)\,J(x)-\beta_{I}^{j}(x)\,\mathcal{O}^{I}[\phi](x)\,|\beta,\mathrm{in}\right\rangle \\
& +\alpha_{B}^{j}(x)\left\langle\alpha,\mathrm{out}|O^{B}[\phi](x)\,\partial\cdot J(x)\,|\beta,\mathrm{in}\right\rangle .
\end{aligned} \tag{A.14}$$

Evaluated between on-shell in and out states, we obtain

$$\left\langle\alpha,\mathrm{out}|O^{B}[\phi](x)\,\partial\cdot J(x)\,|\beta,\mathrm{in}\right\rangle = 0 \tag{A.15}$$

as a consequence of the Ward identities for the current J. Therefore,

$$\partial\alpha^{j}(x)\cdot\langle\alpha,\mathrm{out}|J(x)|\beta,\mathrm{in}\rangle = \partial\cdot\left\langle\alpha,\mathrm{out}|\gamma_{C}^{j}(x)\,\mathfrak{O}^{C}[\phi](x)\,|\beta,\mathrm{in}\right\rangle ,$$

where we denoted collectively all the $c-$number functions $\alpha_{B}^{j}(x)$ and $\beta_{I}^{j}(x)$ as $\gamma_{C}^{j}(x)$ and the local operators $O^{B}[\phi](x)\,J(x)$ and $\mathcal{O}^{I}[\phi](x)$ as $\mathfrak{O}^{C}[\phi](x)$. We then obtain

$$e^{-ip\cdot x}\partial\alpha^{j}(x)\cdot\langle\alpha,\mathrm{out}|J(0)|\beta,\mathrm{in}\rangle = \partial\cdot\left[\gamma_{C}^{j}(x)\,e^{-ip\cdot x}\right]\left\langle\alpha,\mathrm{out}|\mathfrak{O}^{C}[\phi](0)\,|\beta,\mathrm{in}\right\rangle , \tag{A.16}$$

with $p = P(\beta_{\mathrm{in}}) - P(\alpha_{\mathrm{out}})$ for any in and out states. For special choice $\langle\alpha,\mathrm{out}| = \langle 0|$ and $|\beta,\mathrm{in}\rangle = |\phi^{i}(\mathbf{p})\rangle$ we get

$$\partial\alpha^{j}(x)\cdot\langle 0|J(0)|\phi(\mathbf{p})\rangle = \partial\cdot\left[\gamma_{C}^{j}(x)\,e^{-ip\cdot x}\right]\left\langle 0|\mathfrak{O}^{C}[\phi](0)\,|\phi(\mathbf{p})\right\rangle . \tag{A.17}$$

Since the left-hand side of Eq. (A.16) has a NGB pole for $p^2 \to 0$, this must be reproduced on the right-hand side. Therefore at least one matrix element $\left\langle\alpha,\mathrm{out}|\mathfrak{O}^{C}[\phi](0)\,|\beta,\mathrm{in}\right\rangle$ develops a pole. In general we can write

$$\left\langle\alpha,\mathrm{out}|\mathfrak{O}^{C}[\phi](0)\,|\beta,\mathrm{in}\right\rangle = \frac{i}{p^2}\left\langle 0|\mathfrak{O}^{C}[\phi](0)\,|\phi(\mathbf{p})\right\rangle\langle\alpha+\phi(\mathbf{p}),\mathrm{out}|\beta,\mathrm{in}\rangle + R^{C}(p) \tag{A.18}$$

where $R^{C}(p)$ is a remnant regular for $p^2 \to 0$ and therefore at least one matrix element $\left\langle 0|\mathfrak{O}^{C}[\phi](0)\,|\phi^{i}(\mathbf{p})\right\rangle$ must be nonzero.

Inserting Eq. (A.3) and Eq. (A.18) into Eq. (A.16), together with Eq. (A.17), we obtain the following relation between the remainder functions

$$e^{-ip\cdot x}\partial\alpha^{j}(x)\cdot R(p) = \partial\cdot\left[\gamma_{C}^{j}(x)\,e^{-ip\cdot x}\right]R^{C}(p). \tag{A.19}$$

In what follows we will assume that all the remnants are regular also for $p \to 0$, i.e. there are no problems with cubic vertices.

Integrating this over $\mathrm{d}^d x$ we get

$$\widetilde{\partial \alpha^j}(p) \cdot R(p) = i\widetilde{\alpha^j}(p) p \cdot R(p) = 0, \tag{A.20}$$

which should be view as distributions and the tildes here denote Fourier transform. Because $p \cdot R(p)$ is related to the amplitude via Eq. (A.4), we can infer additional information on the soft behavior of the amplitude on top of Eq. (A.5). As we will see in the next subsection, Eq. (A.20) is the key formula for deriving the soft theorems for NGBs. Let us note that it depends only on the $c-$number part of the general symmetry transformation Eq. (A.11). Therefore, theories invariant with respect to the transformation Eq. (A.11) with the same $\alpha^j(x)$ form universality classes with the same soft behavior. In the next subsection we will illustrate application of this formula in more detail.

Derivation of Soft Theorems

As shown above, the existence of a non-linearly realized shift symmetry in Eq. (A.1) together with the absence of cubic vertices implies the presence of the Adler zero, *i.e.* that the amplitude with one soft emission behaves at least as $\mathcal{O}(p)$ for $p \to 0$.

This result and the case when for $\alpha(x) = \theta \cdot x$ mentioned in the main text can be easily generalized for the class of theories invariant with respect to the generalized polynomial shift symmetries

$$\delta_\theta \phi(x) = \theta_{\alpha_1 \ldots \alpha_n} \left[x^{\alpha_1} \ldots x^{\alpha_n} + \alpha_B^{\alpha_1 \ldots \alpha_n}(x) O^B[\phi](x) \right], \tag{A.21}$$

which corresponds to $\alpha^j(x) \to \alpha^{\alpha_1 \ldots \alpha_n}(x) \equiv x^{\alpha_1} \ldots x^{\alpha_n}$. Instead of Eq. (4.32) we get in this case

$$\begin{aligned} 0 &= p_\mu R^\mu(p) \partial^{\alpha_1} \ldots \partial^{\alpha_n} \delta^{(4)}(p) \qquad \text{(A.22)} \\ &= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^k \left[\lim_{p \to 0} \partial^{\alpha_1} \ldots \partial^{\alpha_k} p_\mu R^\mu(p) \right] \partial^{\alpha_{k+1}} \ldots \partial^{\alpha_n} \delta^{(4)}(p) \end{aligned}$$

and thus for $k = 0, \ldots, n$

$$\lim_{p \to 0} \partial^{\alpha_1} \ldots \partial^{\alpha_k} p_\mu R^\mu(p) = 0. \tag{A.23}$$

Using the correspondence in Eq. (A.4) we conclude that the amplitude has $\mathcal{O}\left(p^{n+1}\right)$ soft behavior, *i.e.* an Adler zero of the $(n+1)$th order.

It is also straightforward to generalize the above result to the case of symmetries in Eq. (A.21) with traceless tensor $\theta_{\alpha_1\cdots\alpha_n}$. The special Galileon is a member of this class, and is symmetric with respect to the "hidden Galileon symmetry" [64] (see also Appendix A.3)

$$\delta_s\phi\left(x\right)=\theta_{\alpha\beta}\alpha^2x^\alpha x^\beta-\theta_{\alpha\beta}\partial^\alpha\phi\left(x\right)\partial^\beta\phi\left(x\right),$$

where $\theta_{\alpha\beta}=\theta_{\beta\alpha}$ satisfies $\theta^\alpha_\alpha=0$. Instead of rewriting the general formula Eq. (A.23) for traceless tensor $\theta_{\alpha_1\ldots\alpha_n}$, we will illustrate it just on this concrete example. In this case we have from Eq. (A.11)

$$\alpha^j\left(x\right)\to\alpha^{\mu\nu}\left(x\right)=x^\mu x^\nu-\frac{1}{d}x^2\eta^{\mu\nu}.$$

Taking the Fourier transform, we obtain

$$\widetilde{\alpha^{\mu\nu}}\left(p\right)=-(2\pi)^d\Pi^{\mu\alpha\nu\beta}\partial_\alpha\partial_\beta\delta^{(d)}(p) \tag{A.24}$$

$$\Pi^{\mu\alpha\nu\beta}=\eta^{\mu\alpha}\eta^{\nu\beta}-\tfrac{1}{d}\eta^{\mu\nu}\eta^{\alpha\beta}, \tag{A.25}$$

which with Eq. (A.20) implies that

$$\begin{aligned}0&=-p_\sigma R^\sigma(p)\Pi^{\mu\alpha\nu\beta}\partial_\alpha\partial_\beta\delta^{(d)}(p)\\&=-\Pi^{\mu\alpha\nu\beta}\left\{\left[\partial_\alpha\partial_\beta\delta^{(d)}(p)\right]\left[\lim_{p\to0}p_\sigma R^\sigma(p)\right]-\left[\partial_\alpha\delta^{(d)}(p)\right]\left[\lim_{p\to0}\partial_\beta p_\sigma R^\sigma(p)\right]\right.\\&\quad\left.-\left[\partial_\beta\delta^{(d)}(p)\right]\left[\lim_{p\to0}\partial_\alpha p_\sigma R^\sigma(p)\right]+\delta^{(d)}(p)\left[\lim_{p\to0}\partial_\alpha\partial_\beta p_\sigma R^\sigma(p)\right]\right\}.\end{aligned}$$

We have thus soft theorems in the form[2]

$$\lim_{p\to0}\left(\eta^{\mu\alpha}\eta^{\nu\beta}-\frac{1}{d}\eta^{\mu\nu}\eta^{\alpha\beta}\right)\partial_\alpha\partial_\beta\left\langle\alpha+\phi(\mathbf{p}),\mathrm{out}|\beta,\mathrm{in}\right\rangle=0. \tag{A.26}$$

Taking the soft NGB momentum to be on-shell, we see that the soft limit vanishes with two powers of momenta, leaving $\mathcal{O}\left(p^3\right)$ behavior for the amplitude.

To summarize, the soft theorems above hold for an EFT that is invariant with respect to the generalized polynomial shift symmetry in Eq. (A.6). On the quantum level this means that the relations in Eq. (A.14) and Eq. (A.15) apply. Note that at tree-level, the relations Eq. (A.14) and (A.15) are satisfied automatically and therefore the symmetry (and the absence of the cubic vertices) provides us with a sufficient condition for enhanced soft limit of the tree-level amplitudes.

[2]This equation also appears in [64].

A.2 Bounds on ρ from Bonus Relations

This appendix shows how to obtain rigorous bounds on the power counting parameter ρ in non-trivial theories from bonus relations. We first introduce bonus relations in recursion and then apply them to the spurious pole cancellation.

In normal recursion relations, inputs from all factorization channels are needed. However, for sufficiently high σ, it is possible to eliminate certain factorization channels from the recursion relation by introducing factors like $B(z) = P^2(z)/P^2(0)$ directly into the recursion relation

$$\oint \frac{dz}{z}\frac{A_6(z)}{F(z)}B(z). \tag{A.27}$$

These terms evaluate to unity at $z = 0$, and do not spoil large z behavior, provided the soft behavior is sufficiently enhanced. To isolate the spurious pole cancellation, we choose $B(z) = P^2_{124}(z)P^2_{125}(z)P^2_{126}(z)/P^2_{124}P^2_{125}P^2_{126}$ such that the spurious pole of $a_1 - a_2$ only appears in the channel $P^2_{123}(z) = 0$. It relies on the fact that $A_6(z)/F(z)$ vanishes faster then $1/z^6$,

$$\text{Bonus relation:} \begin{cases} \text{Exceptional theory:} & \rho \geq 4 \\ \text{Non-trivial theory:} & \rho \geq 5 \end{cases} \tag{A.28}$$

which must be satisfied in order to eliminate these factorization channels from the recursion.

We can identify the spurious pole using "bonus" recursion relations as the derivation for Eq. (4.76). The only difference is the extra factor of $B(z)$ which is proportional to $f_1^3(z)$ when taking the residue at $z = 1/a_2$. Since there is only one single term, we can drop all overall kinematic invariants and the spurious pole becomes

$$\frac{\lambda_{3,b_{\max}}A_L(z)}{(a_1 - a_2)^{b_{\max}-2\Delta-3}}, \tag{A.29}$$

where the spurious pole power is shifted by 3 from $B(z)$. This has to vanish identically when $b_{max} - 2\Delta - 3$. We discuss the single and multiple scalars in turn.

For single scalar, there is no state sum and $A_L(z)$ can be dropped. As in Eq. (4.83), we find $2\Delta + 3 \geq b_{\max} \geq \rho + 1 - (2\Delta + 3)$ which can be satisfied for

$$\begin{cases} \text{Exceptional theory:} & \rho \leq 5 \\ \text{Non-trivial theory:} & \rho \leq 9. \end{cases} \tag{A.30}$$

These rigorous bounds truncate the range of numerical checks on the spurious pole cancellation in Eq. (4.76). In the case of stripped amplitudes, we only need to eliminate two factorization channels which is viable for $\rho > 0$. Specifically, choosing $B(z) = P_{234}^2(z)P_{612}^2(z)/P_{234}^2P_{345}^2$ yields a spurious pole in $a_1 - a_2$ unless $b_{max} \leq 1$. This rigorous derivation matches the previous numerical evidence. So we still conclude that the NLSM is the unique exceptional theory with stripped amplitudes.

For multiple scalars, plugging ansatze in Eq. (4.86) into Eq. (A.29) gives

$$\frac{\sum_{b'} \lambda^{123}_{b_{\max},b'} s_{\hat{4}\hat{5}}^{b'} \, s_{\hat{4}\hat{6}}^{\rho+1-b'}}{(a_1 - a_2)^{b_{\max}-2\Delta-3}}. \tag{A.31}$$

Note that the 4pt kinematics $\hat{p}_{3,4,5,6}$ is generic. Since the momenta $p_{3,4,5,6}$ are only constrained by 6pt kinematics with $p_{1,2}$, they are sufficient to construct generic 4pt kinematics under the shift. The two Mandelstam variables $s_{\hat{4}\hat{5}}, s_{\hat{4}\hat{6}}$ are therefore independent. The vanishing of the spurious pole then requires $\lambda^{123}_{b,b'} = 0$ unless $b \leq 2\Delta + 3$ for any b'. The bounds are the same as in the single scalar case, Eq. (A.30).

In sum, bonus relations rigorously constrain the upper limits of ρ. This is supplementary to the numerical checks of Eq. (4.76), which applies to lower ρ, then Eq. (A.28). Combining the two establishes the proof of $\rho < 3$ for all non-trivial theories, independent of the flavor structure.

A.3 Catalog of Scalar Effective Field Theories

Here we list known scalar EFTs and their Lagrangians. These theories typically have generalized shift symmetries, and most have non-trivial soft behavior in scattering amplitudes.

Non-linear Sigma Model and WZW Term

The $SU(N)$ non-linear sigma model can be defined by the following Lagrangian:

$$\mathcal{L} = \frac{F^2}{4}\mathrm{Tr}\,(\partial^\mu U \partial_\mu U^\dagger), \quad \text{where} \quad U = \exp\left(\frac{i}{F}\phi\right), \tag{A.32}$$

where $\phi = \phi^a T^a$ is the (N^2-1)-plet (octet for $N = 3$) of pseudoscalar mesons. The Lagrangian is invariant under the chiral symmetry $U(x) \to V_R U(x) V_L^\dagger$ with unitary matrices $V_{R,L}$. The axial part of this symmetry is realized non-linearly as $\phi \to \phi + a + \ldots$ where the ellipses stand for terms that are at least quadratic in field ϕ and this implies that the axial symmetry is spontaneously broken. Following the theorem in Sec..., the soft limits of scattering amplitudes vanish, $A = \mathcal{O}(p)$. This theory for $N = 2, 3$ is famously used for the description of low energy degrees of freedom of QCD.

The other theory of this kind involving the same multiple of particles is the following Lagrangian

$$\mathcal{L} = \frac{1}{4}\mathrm{T}r(\partial_\mu\phi\partial^\mu\phi) + \lambda\epsilon_{\mu\nu\alpha\beta}\,\mathrm{T}r(\phi\,\partial^\mu\phi\,\partial^\nu\phi\,\partial^\alpha\phi\,\partial^\beta\phi). \tag{A.33}$$

It possesses the shift symmetry $\phi \to \phi + a$ and has thus the $O(p)$ behavior. This Lagrangian can be obtained as $\phi \to 0$ limit of the famous Wess-Zumino-Witten term

$$S_{WZW} = i\lambda\epsilon^{ABCDE}\int d^5x\mathrm{T}r(U^\dagger\partial_A UU^\dagger\partial_B UU^\dagger\partial_C UU^\dagger\partial_D UU^\dagger\partial_E U)\,, \tag{A.34}$$

which corresponds to the chiral anomaly. Generalization of (A.33) beyond $d = 4$ is obvious:

$$\mathcal{L} = \frac{1}{4}\mathrm{T}r(\partial_\mu\phi\partial^\mu\phi) + \lambda\epsilon_{\mu_1\ldots\mu_d}\,\mathrm{T}r(\phi\,\partial^{\mu_1}\phi\ldots\partial^{\mu_d}\phi)\,. \tag{A.35}$$

Such a theory correspons to $v = d + 1$, $\sigma = 1$ and $\rho = (d-2)/(d-1)$.

Dirac-Born-Infeld Theory

The so-called DBI Lagrangian for the single scalar field in d-dimensions reads

$$\mathcal{L} = -F^d\sqrt{1 - \frac{\partial\phi\cdot\partial\phi}{F^d}} + F^d. \tag{A.36}$$

The action can be obtained by description of a d-brane fluctuating in the $(d+1)$-dimensional spacetime with a flat metric diag$(\eta_{\alpha\beta}, -1)$. As a consequence this theory must be invariant under the shift symmetry and $(d+1)$-dimensional Lorentz symmetry

$$\phi \to \phi + a + \theta\cdot x - F^{-d}\theta\cdot\phi(x)\partial\phi(x)\,. \tag{A.37}$$

DBI corresponds to the theory with $\sigma = 2$ and $\rho = 1$.

$P(X)$ Theory

The DBI discussed above can be considered a special case of a general class of theories,

$$\mathcal{L} = F^d P\Big(\frac{\partial\phi\cdot\partial\phi}{F^d}\Big)\,, \tag{A.38}$$

occasionally referred to in the context of inflaton cosmology as $P(X)$ theories. Here P is a Taylor expansion of the form $P(x) = \frac{1}{2}x + \mathcal{O}(x^2)$. This theory is manifestly invariant under the shift symmetry $\phi \to \phi + a$ and thus exhibits $\sigma = 1$ and $\rho = 1$. This soft behavior is trivial, since the soft degree matches the number of derivatives per field.

Galileon

Lagrangian of the so-called Galileon in d-dimension consists of $d+1$ terms

$$\mathcal{L} = \sum_{n=1}^{d+1} d_n \phi \mathcal{L}^{\text{der}}_{n-1}\,, \tag{A.39}$$

with the total derivative term at valency n explicitly given by

$$\mathcal{L}^{\text{der}}_n = \varepsilon^{\mu_1\ldots\mu_d}\varepsilon^{\nu_1\ldots\nu_d}\prod_{i=1}^{n}\partial_{\mu_i}\partial_{\nu_i}\phi\prod_{j=n+1}^{d}\eta_{\mu_j\nu_j} = (-1)^{d-1}(d-n)!\det\left\{\partial^{\nu_i}\partial_{\nu_j}\phi\right\}^{n}_{i,j=1}. \tag{A.40}$$

For example in $d = 4$ we have

$$\begin{aligned}
\mathcal{L}^{\text{der}}_0 &= -4! \\
\mathcal{L}^{\text{der}}_1 &= -6\Box\phi \\
\mathcal{L}^{\text{der}}_2 &= -2\left[(\Box\phi)^2 - \partial\partial\phi : \partial\partial\phi\right] \\
\mathcal{L}^{\text{der}}_3 &= -\left[(\Box\phi)^3 + 2\partial\partial\phi\cdot\partial\partial\phi : \partial\partial\phi - 3\Box\phi\partial\partial\phi : \partial\partial\phi\right] \\
\mathcal{L}^{\text{der}}_4 &= -\left[(\Box\phi)^4 - 6\,(\Box\phi)^2\,\partial\partial\phi : \partial\partial\phi + 8\Box\phi\partial\partial\phi\cdot\partial\partial\phi : \partial\partial\phi\right. \\
&\quad \left. -6\partial\partial\phi\cdot\partial\partial\phi\cdot\partial\partial\phi : \partial\partial\phi + 3\,(\partial\partial\phi : \partial\partial\phi)^2\right].
\end{aligned} \tag{A.41}$$

This Lagrangian has a lowest interaction term with valency 3, but as shown in [65] we can always remove it using a duality transformation, which doesn't change the structure of other vertices. The Galileon Lagrangian represents the most general theory for a single scalar whose equation of motion involves just the second derivatives of the field and is invariant under the Galilean symmetry

$$\phi \to \phi + a + b\cdot x\,. \tag{A.42}$$

According to the soft theorem this theory has $\sigma = 2$ and $\rho = 2$.

Special Galileon

In [51] it was found that the Galileon with the 4pt interaction term in $d = 4$ has even stronger soft limit behavior than naively predicted by the symmetry argument. In fact, $A_n \sim \mathcal{O}(p^3)$ rather than just $A_n \sim \mathcal{O}(p^2)$. This was a signal for a hidden symmetry which was indeed discovered shortly after in [64]. The special Galileon can be obtained from (A.39) with

$$d_{2n} = \frac{(-1)^d}{(2n)!(d-2n+1)!}\frac{1}{\alpha^{2(n-1)}}\,, \qquad d_{2n+1} = 0\,. \tag{A.43}$$

In the case of four dimensions there is only one interaction term

$$\mathcal{L}_{int} = \frac{1}{4!}\frac{1}{\alpha^2}\phi\mathcal{L}_3^{\mathrm{d}er}\,. \tag{A.44}$$

The hidden symmetry is given by

$$\phi \to \phi + \theta^{\mu\nu}(\alpha^2 x_\mu x_\nu - \partial_\mu\phi\partial_\nu\phi)\,. \tag{A.45}$$

According our definition this means that $\sigma = 3$ and $\rho = 2$.

Multi-field Galileon

There are at least two possibilities for how to generalize the Galileon Lagrangian for scalar multiplet. The first one is a straightforward $U(N)$ symmetric generalization of the $n-$point interaction term

$$\mathcal{L}_n = \varepsilon^{\mu_1\ldots\mu_d}\varepsilon^{\nu_1\ldots\nu_d}\mathrm{Tr}\,(\phi\partial_{\mu_1}\partial_{\nu_1}\phi\ldots\partial_{\mu_n}\partial_{\nu_n}\phi)\prod_{j=n+1}^{d}\eta_{\mu_j\nu_j}$$

where $\phi = \phi^a T^a$ and T^a are the generators of $U(N)$. The corresponding action is invariant with respect to the linear shift symmetry and the $U(N)$ symmetry

$$\begin{aligned}\phi^a &\to \phi^a + b^a + c^a\cdot x\\ \phi &\to U\phi U^+, \quad U\in SU(N)\end{aligned}$$

which is responsible for the $\mathcal{O}(p^2)$ soft behavior of the scattering amplitudes. Moreover, because of the single trace structure of the interaction terms, the full amplitudes can be flavor-ordered and cyclically ordered Feynman rules can be formulated. Of course we could also include interaction terms with multiple traces without spoiling the symmetry and soft limit properties, e.g.

$$\begin{aligned}\mathcal{L}_{n,k_1,\ldots,k_m=d} &= \varepsilon^{\mu_1\ldots\mu_d}\varepsilon^{\nu_1\ldots\nu_d}\prod_{j=n+1}^{d}\eta_{\mu_j\nu_j}\mathrm{Tr}\left(\phi\partial_{\mu_1}\partial_{\nu_1}\phi\ldots\partial_{\mu_{k_1}}\partial_{\nu_{k_1}}\phi\right)\\ &\quad\times\prod_{r=2}^{m}\mathrm{Tr}\left(\partial_{\mu_{k_{r-1}+1}}\partial_{\nu_{k_{r-1}+1}}\phi\ldots\partial_{\mu_{k_r}}\partial_{\nu_{k_r}}\phi\right),\end{aligned}$$

however then the usual stripping of the amplitudes is not possible.

Another generalization follows the brane construction described in [64]. Such generalization has naturally a $O(N)$ symmetry as a remnant of the Lorentz symmetry of the $d+N$ dimensional target space in which the $d-$dimensional brane propagates. As shown in [64], there are only even-point vertices allowed by symmetry and the $2n-$point Lagrangian has the general form

$$\begin{aligned}\mathcal{L}_{2n} &= \varepsilon^{\mu_1\ldots\mu_d}\varepsilon^{\nu_1\ldots\nu_d}\prod_{j=2n+1}^{d}\eta_{\mu_j\nu_j} \\ &\quad\times\sum_{a_i=1}^{N}\phi^{a_1}\partial_{\mu_1}\partial_{\nu_1}\phi^{a_1}\partial_{\mu_2}\partial_{\nu_2}\phi^{a_2}\partial_{\mu_3}\partial_{\nu_3}\phi^{a_2}\ldots\partial_{\mu_{2n-1}}\partial_{\nu_{2n-1}}\phi^{a_n}\partial_{\mu_{2n}}\partial_{\nu_{2n}}\phi^{\alpha_n}.\end{aligned}$$

The action is invariant with respect to the linear shift symmetry and the $O(N)$ symmetry

$$\begin{aligned}\phi^a &\to \phi^a + b^a + c^a\cdot x \\ \phi^a &= R^a_b\phi^b,\quad R\in O(N)\end{aligned}$$

and thus the $\mathcal{O}(p^2)$ soft limit is guaranteed. This generalization does not allow for the usual stripping of the amplitudes.

Multi-field DBI

The natural generalization of the single scalar DBI Lagrangian can be obtained as the lowest order action of the $d-$ dimensional brane propagating in $d+N$ dimensional flat space. The embeding of the brane is described by

$$X^A = Y^A(\xi),$$

where $A = 0,1,\ldots,d+N-1$, and the parameters are $\xi\equiv\xi^\mu$ where $\mu = 0,\ldots,d-1$. The induced metric on the brane is

$$\mathrm{d}s^2 = \eta_{AB}\partial_\mu Y^A\partial_\nu Y^B\mathrm{d}\xi^\mu\mathrm{d}\xi^\nu \equiv g_{\mu\nu}\mathrm{d}\xi^\mu\mathrm{d}\xi^\nu$$

and the leading order reparameterization invariant action reads

$$S = -F^d\int\mathrm{d}^d\xi\sqrt{(-1)^{d-1}\det\left(g_{\mu\nu}\right)} = -F^d\int\mathrm{d}^d\xi\sqrt{(-1)^{d-1}\det\left(\eta_{AB}\partial_\mu Y^A\partial_\nu Y^B\right)}$$

where F is a constant with $\dim F = 1$. Let us fix a new parameterization in terms of parameters x^μ where

$$x^\mu = Y^\mu(\xi)$$

and denote

$$Y^{d-1+j}\left(\xi\left(x\right)\right)=\frac{\phi^{j}\left(x\right)}{F^{d/2}},\quad j=1,\ldots,N.$$

Then

$$g_{\mu\nu}=\eta_{\mu\nu}-\frac{1}{F^{d}}\sum_{j}\partial_{\mu}\phi^{j}\partial_{\nu}\phi^{j}=\eta_{\mu\alpha}\left(\delta_{\nu}^{\alpha}-\frac{1}{F^{d}}\eta^{\alpha\beta}\sum_{j}\partial_{\beta}\phi^{j}\partial_{\nu}\phi^{j}\right)$$

and after some algebra we get

$$\sqrt{(-1)^{d-1}\det g}=1+\sum_{N=1}^{\infty}\sum_{n=1}^{N}\frac{(-1)^{n}}{2^{n}n!}\sum_{\sum_{j=1}^{n}m_{j}=N,\ m_{j}\geq1}^{\infty}\prod_{j=1}^{n}\frac{1}{m_{j}}\mathrm{Tr}\alpha^{m_{j}}$$

where the $N\times N$ matrix α is defined as

$$\alpha^{ij}=\frac{1}{F^{d}}\partial\phi^{i}\cdot\partial\phi^{j}.$$

We get then for the first three terms of the Lagrangian

$$\begin{aligned}
\mathcal{L}_{N=1} &= \frac{1}{2}\mathrm{Tr}\alpha=\sum_{i}\frac{1}{2}\partial\phi^{i}\cdot\partial\phi^{i}\\
\mathcal{L}_{N=2} &= \frac{1}{2^{1}}\frac{\mathrm{Tr}\alpha^{2}}{2}-\frac{1}{2^{2}2!}\left(\mathrm{Tr}\alpha\right)^{2}=\frac{1}{4F^{d}}\sum_{j,i}\left(\partial\phi^{i}\cdot\partial\phi^{j}\partial\phi^{j}\cdot\partial\phi^{i}-\frac{1}{2}\partial\phi^{i}\cdot\partial\phi^{i}\partial\phi^{j}\cdot\partial\phi^{j}\right)\\
\mathcal{L}_{N=3} &= \frac{1}{2}\frac{\mathrm{Tr}\alpha^{3}}{3}-\frac{1}{4}\mathrm{Tr}\alpha\frac{\mathrm{Tr}\alpha^{2}}{2}+\frac{1}{48}\left(\mathrm{Tr}\alpha\right)^{3}\\
&= \frac{1}{F^{2d}}\sum_{j,i,k}\left(\frac{1}{6}\partial\phi^{i}\cdot\partial\phi^{j}\partial\phi^{j}\cdot\partial\phi^{k}\partial\phi^{k}\cdot\partial\phi^{i}-\frac{1}{8}\partial\phi^{k}\cdot\partial\phi^{k}\partial\phi^{i}\cdot\partial\phi^{j}\partial\phi^{j}\cdot\partial\phi^{i}\right.\\
&\left.\quad+\frac{1}{48}\partial\phi^{k}\cdot\partial\phi^{k}\partial\phi^{i}\cdot\partial\phi^{i}\partial\phi^{j}\cdot\partial\phi^{j}\right).
\end{aligned}$$

The action is invariant with respect to the linearly realized $O(N)$ flavour rotations (ϕ^{i} being in the defining representation)

$$\delta^{(ij)}\phi^{k}=\delta^{jk}\phi^{i}-\delta^{ik}\phi^{j},$$

and non-linearly realized Minkowski rotations and boosts in the $d+N$ dimensional space

$$\begin{aligned}
\delta^{(\alpha j)}x^{\mu} &= \eta^{\mu\alpha}\frac{\phi^{j}}{F^{d/2}}\\
\delta^{(\alpha j)}\phi^{k} &= F^{d/2}x^{\alpha}\delta^{jk}.
\end{aligned}$$

The latter symmetry is responsible for the $\mathcal{O}(p^{2})$ soft limit of the scattering amplitudes. However, the structure of the Lagrangian does not allow for introduction of flavor-ordered amplitudes.

BIBLIOGRAPHY

[1] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys.* **B435** (1995) 59–101, [hep-ph/9409265].

[2] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B425** (1994) 217–260, [hep-ph/9403226].

[3] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, *Nucl. Phys.* **B715** (2005) 499–522, [hep-th/0412308].

[4] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602, [hep-th/0501052].

[5] F. Cachazo and P. Svrcek, *Tree level recursion relations in general relativity*, hep-th/0502160.

[6] J. Bedford, A. Brandhuber, B. J. Spence and G. Travaglini, *A Recursion relation for gravity amplitudes*, *Nucl. Phys.* **B721** (2005) 98–110, [hep-th/0502146].

[7] N. Arkani-Hamed, F. Cachazo and J. Kaplan, *What is the Simplest Quantum Field Theory?*, *JHEP* **09** (2010) 016, [0808.1446].

[8] T. Cohen, H. Elvang and M. Kiermaier, *On-shell constructibility of tree amplitudes in general field theories*, *JHEP* **04** (2011) 053, [1010.0257].

[9] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, S. Caron-Huot and J. Trnka, *The All-Loop Integrand For Scattering Amplitudes in Planar N=4 SYM*, *JHEP* **01** (2011) 041, [1008.2958].

[10] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, *A Duality For The S Matrix*, *JHEP* **03** (2010) 020, [0907.5418].

[11] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. B. Goncharov, A. Postnikov and J. Trnka, *Scattering Amplitudes and the Positive Grassmannian.* Cambridge University Press, 2012.

[12] F. Cachazo, S. He and E. Y. Yuan, *Scattering of Massless Particles in Arbitrary Dimensions*, *Phys. Rev. Lett.* **113** (2014) 171601, [1307.2199].

[13] F. Cachazo, S. He and E. Y. Yuan, *Scattering of Massless Particles: Scalars, Gluons and Gravitons*, *JHEP* **07** (2014) 033, [1309.0885].

[14] F. Cachazo, S. He and E. Y. Yuan, *Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM*, *JHEP* **07** (2015) 149, [1412.3479].

[15] L. J. Dixon, J. M. Drummond, M. von Hippel and J. Pennington, *Hexagon functions and the three-loop remainder function*, *JHEP* **12** (2013) 049, [1308.2276].

[16] S. Caron-Huot, L. J. Dixon, A. McLeod and M. von Hippel, *Bootstrapping a Five-Loop Amplitude from Steinmann Relations*, 1609.00669.

[17] B. Basso, A. Sever and P. Vieira, *Spacetime and Flux Tube S-Matrices at Finite Coupling for N=4 Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **111** (2013) 091602, [1303.1396].

[18] B. Basso, J. Caetano, L. Cordova, A. Sever and P. Vieira, *OPE for all Helicity Amplitudes*, *JHEP* **08** (2015) 018, [1412.1132].

[19] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189–258, [hep-th/0312171].

[20] R. Roiban, M. Spradlin and A. Volovich, *On the tree level S matrix of Yang-Mills theory*, *Phys. Rev.* **D70** (2004) 026009, [hep-th/0403190].

[21] F. Cachazo and D. Skinner, *Gravity from Rational Curves in Twistor Space*, *Phys. Rev. Lett.* **110** (2013) 161301, [1207.0741].

[22] F. Cachazo, L. Mason and D. Skinner, *Gravity in Twistor Space and its Grassmannian Formulation*, *SIGMA* **10** (2014) 051, [1207.4712].

[23] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, *The S-Matrix in Twistor Space*, *JHEP* **03** (2010) 110, [0903.2110].

[24] L. Mason and D. Skinner, *Ambitwistor strings and the scattering equations*, *JHEP* **07** (2014) 048, [1311.2564].

[25] E. Casali, Y. Geyer, L. Mason, R. Monteiro and K. A. Roehrig, *New Ambitwistor String Theories*, *JHEP* **11** (2015) 038, [1506.08771].

[26] Y. Geyer, L. Mason, R. Monteiro and P. Tourkine, *Loop Integrands for Scattering Amplitudes from the Riemann Sphere*, *Phys. Rev. Lett.* **115** (2015) 121603, [1507.00321].

[27] Y. Geyer, L. Mason, R. Monteiro and P. Tourkine, *One-loop amplitudes on the Riemann sphere*, *JHEP* **03** (2016) 114, [1511.06315].

[28] N. Arkani-Hamed and J. Trnka, *The Amplituhedron*, *JHEP* **10** (2014) 030, [1312.2007].

[29] N. Arkani-Hamed and J. Trnka, *Into the Amplituhedron*, *JHEP* **12** (2014) 182, [1312.7878].

[30] H. Elvang and Y.-t. Huang, *Scattering Amplitudes*, 1308.1697.

[31] H. Kawai, D. C. Lewellen and S. H. H. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, *Nucl. Phys.* **B269** (1986) 1–23.

[32] Z. Bern, J. J. M. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev.* **D78** (2008) 085011, [0805.3993].

[33] Z. Bern, J. J. M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602, [1004.0476].

[34] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, *Gravity as the Square of Gauge Theory*, *Phys. Rev.* **D82** (2010) 065003, [1004.0693].

[35] P. Benincasa and F. Cachazo, *Consistency Conditions on the S-Matrix of Massless Particles*, 0705.4305.

[36] D. A. McGady and L. Rodina, *Higher-spin massless S-matrices in four-dimensions*, *Phys. Rev.* **D90** (2014) 084048, [1311.2938].

[37] J. J. M. Carrasco, *Gauge and Gravity Amplitude Relations*, 1506.00974.

[38] R. Alonso, E. E. Jenkins and A. V. Manohar, *Holomorphy without Supersymmetry in the Standard Model Effective Field Theory*, *Phys.Lett.* **B739** (2014) 95–98, [1409.0868].

[39] K. Risager, *A Direct proof of the CSW rules*, *JHEP* **12** (2005) 003, [hep-th/0508206].

[40] N. Arkani-Hamed and J. Kaplan, *On Tree Amplitudes in Gauge Theory and Gravity*, *JHEP* **04** (2008) 076, [0801.2385].

[41] B. Feng, J. Wang, Y. Wang and Z. Zhang, *BCFW Recursion Relation with Nonzero Boundary Contribution*, *JHEP* **01** (2010) 019, [0911.0301].

[42] Q. Jin and B. Feng, *Recursion Relation for Boundary Contribution*, *JHEP* **06** (2015) 018, [1412.8170].

[43] C. Cheung, *On-Shell Recursion Relations for Generic Theories*, *JHEP* **03** (2010) 098, [0808.0504].

[44] C. Schwinn and S. Weinzierl, *On-shell recursion relations for all Born QCD amplitudes*, *JHEP* **04** (2007) 072, [hep-ph/0703021].

[45] M. Bianchi, H. Elvang and D. Z. Freedman, *Generating Tree Amplitudes in N=4 SYM and N = 8 SG*, *JHEP* **09** (2008) 063, [0805.0757].

[46] K. Kampf, J. Novotny and J. Trnka, *Recursion relations for tree-level amplitudes in the $SU(N)$ nonlinear sigma model*, *Phys. Rev.* **D87** (2013) 081701, [1212.5224].

[47] F. Cachazo, P. Svrcek and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, *JHEP* **09** (2004) 006, [hep-th/0403047].

[48] J. Broedel and L. J. Dixon, *Color-kinematics duality and double-copy construction for amplitudes from higher-dimension operators*, *JHEP* **10** (2012) 091, [1208.0876].

[49] L. J. Dixon, E. W. N. Glover and V. V. Khoze, *MHV rules for Higgs plus multi-gluon amplitudes*, *JHEP* **12** (2004) 015, [hep-th/0411092].

[50] L. J. Dixon and Y. Shadmi, *Testing gluon selfinteractions in three jet events at hadron colliders*, *Nucl. Phys.* **B423** (1994) 3–32, [hep-ph/9312363].

[51] C. Cheung, K. Kampf, J. Novotny and J. Trnka, *Effective Field Theories from Soft Limits of Scattering Amplitudes*, *Phys. Rev. Lett.* **114** (2015) 221602, [1412.4095].

[52] J. A. Cronin, *Phenomenological model of strong and weak interactions in chiral U(3) x U(3)*, *Phys. Rev.* **161** (1967) 1483–1494.

[53] S. Weinberg, *Dynamical approach to current algebra*, *Phys. Rev. Lett.* **18** (1967) 188–191.

[54] S. Weinberg, *Nonlinear realizations of chiral symmetry*, *Phys. Rev.* **166** (1968) 1568–1577.

[55] C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan and L. Senatore, *The Effective Field Theory of Inflation*, *JHEP* **03** (2008) 014, [0709.0293].

[56] G. R. Dvali, G. Gabadadze and M. Porrati, *4-D gravity on a brane in 5-D Minkowski space*, *Phys. Lett.* **B485** (2000) 208–214, [hep-th/0005016].

[57] A. Nicolis, R. Rattazzi and E. Trincherini, *The Galileon as a local modification of gravity*, *Phys. Rev.* **D79** (2009) 064036, [0811.2197].

[58] B. Feng, K. Zhou, C. Qiao and J. Rao, *Determination of Boundary Contributions in Recursion Relation*, *JHEP* **03** (2015) 023, [1411.0452].

[59] Q. Jin and B. Feng, *Boundary Operators of BCFW Recursion Relation*, *JHEP* **04** (2016) 123, [1507.00463].

[60] C. Cheung, C.-H. Shen and J. Trnka, *Simple Recursion Relations for General Field Theories*, *JHEP* **06** (2015) 118, [1502.05057].

[61] K. Hinterbichler and A. Joyce, *Goldstones with Extended Shift Symmetries*, *Int. J. Mod. Phys.* **D23** (2014) 1443001, [1404.4047].

[62] T. Griffin, K. T. Grosvenor, P. Horava and Z. Yan, *Scalar Field Theories with Polynomial Shift Symmetries*, *Commun. Math. Phys.* **340** (2015) 985–1048, [1412.1046].

[63] K. Kampf, J. Novotny and J. Trnka, *Tree-level Amplitudes in the Nonlinear Sigma Model*, *JHEP* **05** (2013) 032, [1304.3048].

[64] K. Hinterbichler and A. Joyce, *Hidden symmetry of the Galileon*, *Phys. Rev.* **D92** (2015) 023503, [1501.07600].

[65] K. Kampf and J. Novotny, *Unification of Galileon Dualities*, *JHEP* **10** (2014) 006, [1403.6813].

[66] P. Creminelli, M. Serone, G. Trevisan and E. Trincherini, *Inequivalence of Coset Constructions for Spacetime Symmetries*, *JHEP* **02** (2015) 037, [1403.3095].

[67] C. De Rham, L. Keltner and A. J. Tolley, *Generalized galileon duality*, *Phys. Rev.* **D90** (2014) 024050, [1403.3690].

[68] C. de Rham and A. J. Tolley, *DBI and the Galileon reunited*, *JCAP* **1005** (2010) 015, [1003.5917].

[69] F. Cachazo and A. Strominger, *Evidence for a New Soft Graviton Theorem*, 1404.4091.

[70] T. Klose, T. McLoughlin, D. Nandan, J. Plefka and G. Travaglini, *Double-Soft Limits of Gluons and Gravitons*, *JHEP* **07** (2015) 135, [1504.05558].

[71] Y.-J. Du and H. Luo, *On single and double soft behaviors in NLSM*, *JHEP* **08** (2015) 058, [1505.04411].

[72] P. Di Vecchia, R. Marotta and M. Mojaza, *Double-soft behavior for scalars and gluons from string theory*, *JHEP* **12** (2015) 150, [1507.00938].

[73] S. L. Adler, *Consistency conditions on the strong interactions implied by a partially conserved axial vector current*, *Phys. Rev.* **137** (1965) B1022–B1033.

[74] C. Cheung, K. Kampf, J. Novotny, C.-H. Shen and J. Trnka, *On-Shell Recursion Relations for Effective Field Theories*, *Phys. Rev. Lett.* **116** (2016) 041601, [1509.03309].

[75] G. W. Horndeski, *Second-order scalar-tensor field equations in a four-dimensional space*, *Int. J. Theor. Phys.* **10** (1974) 363–384.

[76] J. Wess and B. Zumino, *Consequences of anomalous ward identities*, *Physics Letters B* **37** (Nov., 1971) 95–97.

[77] E. Witten, *Global aspects of current algebra*, *Nuclear Physics B* **223** (Aug., 1983) 422–432.

[78] B. Henning, X. Lu, T. Melia and H. Murayama, *2, 84, 30, 993, 560, 15456, 11962, 261485, ...: Higher dimension operators in the SM EFT*, 1512.03433.

[79] S. Weinberg, *The quantum theory of fields. Vol. 2: Modern applications.* Cambridge University Press, 2013.

[80] T. Brauner and H. Watanabe, *Spontaneous breaking of spacetime symmetries and the inverse Higgs effect*, *Phys. Rev.* **D89** (2014) 085004, [1401.5596].

[81] C. Cheung, A. de la Fuente and R. Sundrum, *4D Scattering Amplitudes and Asymptotic Symmetries from 2D CFT*, 1609.00732.

[82] H. Luo and C. Wen, *Recursion relations from soft theorems*, *JHEP* **03** (2016) 088, [1512.06801].

[83] S. Weinberg, *Infrared photons and gravitons*, *Phys. Rev.* **140** (1965) B516–B524.

[84] I. Low, *Double Soft Theorems and Shift Symmetry in Nonlinear Sigma Models*, *Phys. Rev.* **D93** (2016) 045032, [1512.01232].

[85] F. Cachazo, S. He and E. Y. Yuan, *New Double Soft Emission Theorems*, *Phys. Rev.* **D92** (2015) 065030, [1503.04816].

[86] D. Nandan, J. Plefka and W. Wormsbecher, *Collinear limits beyond the leading order from the scattering equations*, 1608.04730.

[87] G. Chen and Y.-J. Du, *Amplitude Relations in Non-linear Sigma Model*, *JHEP* **01** (2014) 061, [1311.1133].

[88] F. Cachazo, P. Cha and S. Mizera, *Extensions of Theories from Soft Limits*, *JHEP* **06** (2016) 170, [1604.03893].

[89] J. J. M. Carrasco, C. R. Mafra and O. Schlotterer, *Abelian Z-theory: NLSM amplitudes and alpha'-corrections from the open string*, 1608.02569.

[90] A. Strominger, *Asymptotic Symmetries of Yang-Mills Theory*, *JHEP* **07** (2014) 151, [1308.0589].

[91] T. He, P. Mitra, A. P. Porfyriadis and A. Strominger, *New Symmetries of Massless QED*, *JHEP* **10** (2014) 112, [1407.3789].

[92] T. He, P. Mitra and A. Strominger, *2D Kac-Moody Symmetry of 4D Yang-Mills Theory*, *JHEP* **10** (2016) 137, [1503.02663].

[93] D. Kapec, M. Pate and A. Strominger, *New Symmetries of QED*, 1506.02906.

[94] A. Strominger, *Magnetic Corrections to the Soft Photon Theorem*, *Phys. Rev. Lett.* **116** (2016) 031602, [1509.00543].

[95] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner, *Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems*, *Proc. Roy. Soc. Lond.* **A269** (1962) 21–52.

[96] R. K. Sachs, *Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times*, *Proc. Roy. Soc. Lond.* **A270** (1962) 103–126.

[97] G. Barnich and C. Troessaert, *Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited*, *Phys. Rev. Lett.* **105** (2010) 111103, [0909.2617].

[98] A. Strominger, *On BMS Invariance of Gravitational Scattering*, *JHEP* **07** (2014) 152, [1312.2229].

[99] T. He, V. Lysov, P. Mitra and A. Strominger, *BMS supertranslations and Weinberg's soft graviton theorem*, *JHEP* **05** (2015) 151, [1401.7026].

[100] D. Kapec, V. Lysov, S. Pasterski and A. Strominger, *Semiclassical Virasoro symmetry of the quantum gravity $\mathcal{S}$-matrix*, *JHEP* **08** (2014) 058, [1406.3312].

[101] V. Lysov, S. Pasterski and A. Strominger, *'s Subleading Soft Theorem as a Symmetry of QED*, *Phys. Rev. Lett.* **113** (2014) 111601, [1407.3814].

[102] T. T. Dumitrescu, T. He, P. Mitra and A. Strominger, *Infinite-Dimensional Fermionic Symmetry in Supersymmetric Gauge Theories*, 1511.07429.

[103] F. E. Low, *Bremsstrahlung of very low-energy quanta in elementary particle collisions*, *Phys. Rev.* **110** (1958) 974–977.

[104] T. H. Burnett and N. M. Kroll, *Extension of the low soft photon theorem*, *Phys. Rev. Lett.* **20** (1968) 86.

[105] C. D. White, *Factorization Properties of Soft Graviton Amplitudes*, *JHEP* **05** (2011) 060, [1103.2981].

[106] G. Goon, K. Hinterbichler, A. Joyce and M. Trodden, *Aspects of Galileon Non-Renormalization*, 1606.02295.

[107] R. Monteiro and D. O'Connell, *The Kinematic Algebra From the Self-Dual Sector*, *JHEP* **07** (2011) 007, [1105.2565].

[108] F. Cachazo, S. He and E. Y. Yuan, *Scattering equations and Kawai-Lewellen-Tye orthogonality*, *Phys. Rev.* **D90** (2014) 065001, [1306.6575].

[109] R. Monteiro and D. O'Connell, *The Kinematic Algebras from the Scattering Equations*, *JHEP* **03** (2014) 110, [1311.1151].

[110] Y.-J. Du and C.-H. Fu, *Explicit BCJ numerators of nonlinear simga model*, *JHEP* **09** (2016) 174, [1606.05846].

[111] C. Cheung and C.-H. Shen, *Symmetry and Action for Flavor-Kinematics Duality*, *Phys. Rev. Lett.* **118** (2017) 121601, [1612.00868].

[112] V. Del Duca, L. J. Dixon and F. Maltoni, *New color decompositions for gauge amplitudes at tree and loop level*, *Nucl. Phys.* **B571** (2000) 51–70, [hep-ph/9910563].

[113] J. J. M. Carrasco, C. R. Mafra and O. Schlotterer, *Semi-abelian Z-theory: NLSM+phi^3 from the open string*, 1612.06446.

[114] M. D. Schwartz, *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2014.

[115] C. Cheung, K. Kampf, J. Novotny, C.-H. Shen and J. Trnka, *A Periodic Table of Effective Field Theories*, 1611.03137.

[116] C. Grojean, E. E. Jenkins, A. V. Manohar and M. Trott, *Renormalization Group Scaling of Higgs Operators and* $\Gamma(h \to \gamma\gamma)$, *JHEP* **1304** (2013) 016, [1301.2588].

[117] E. E. Jenkins, A. V. Manohar and M. Trott, *Renormalization Group Evolution of the Standard Model Dimension Six Operators I: Formalism and lambda Dependence*, *JHEP* **1310** (2013) 087, [1308.2627].

[118] E. E. Jenkins, A. V. Manohar and M. Trott, *Renormalization Group Evolution of the Standard Model Dimension Six Operators II: Yukawa Dependence*, *JHEP* **1401** (2014) 035, [1310.4838].

[119] R. Alonso, E. E. Jenkins, A. V. Manohar and M. Trott, *Renormalization Group Evolution of the Standard Model Dimension Six Operators III: Gauge Coupling Dependence and Phenomenology*, *JHEP* **1404** (2014) 159, [1312.2014].

[120] J. Elias-Miró, J. Espinosa, E. Masso and A. Pomarol, *Renormalization of dimension-six operators relevant for the Higgs decays $h \to \gamma\gamma, \gamma Z$*, *JHEP* **1308** (2013) 033, [1302.5661].

[121] J. Elias-Miró, J. Espinosa, E. Masso and A. Pomarol, *Higgs windows to new physics through d=6 operators: constraints and one-loop anomalous dimensions*, *JHEP* **1311** (2013) 066, [1308.1879].

[122] J. Elias-Miró, C. Grojean, R. S. Gupta and D. Marzocca, *Scaling and tuning of EW and Higgs observables*, *JHEP* **1405** (2014) 019, [1312.2928].

[123] G. Passarino and M. Veltman, *One Loop Corrections for e+ e- Annihilation Into mu+ mu- in the Weinberg Model*, *Nucl.Phys.* **B160** (1979) 151.

[124] Y.-t. Huang, D. A. McGady and C. Peng, *One-loop renormalization and the S-matrix*, *Phys.Rev.* **D87** (2013) 085028, [1205.5606].

[125] L. Dixon, *Notes on the one-loop QCD β-function without ghosts*, *private communication* (2002) .

[126] B. Grzadkowski, M. Iskrzynski, M. Misiak and J. Rosiek, *Dimension-Six Terms in the Standard Model Lagrangian*, *JHEP* **1010** (2010) 085, [1008.4884].

[127] E. E. Jenkins, A. V. Manohar and M. Trott, *Naive Dimensional Analysis Counting of Gauge Theory Amplitudes and Anomalous Dimensions*, *Phys.Lett.* **B726** (2013) 697–702, [1309.0819].

[128] T. Kinoshita, *Mass singularities of Feynman amplitudes*, *J.Math.Phys.* **3** (1962) 650–677.

[129] T. Lee and M. Nauenberg, *Degenerate Systems and Mass Singularities*, *Phys.Rev.* **133** (1964) B1549–B1562.

[130] J. Elias-Miró, J. Espinosa and A. Pomarol, *One-loop non-renormalization results in EFTs*, 1412.7151.

[131] J. Gasser and H. Leutwyler, *Chiral Perturbation Theory to One Loop*, *Annals Phys.* **158** (1984) 142.

[132] S. J. Parke and T. Taylor, *Perturbative QCD Utilizing Extended Supersymmetry*, *Phys.Lett.* **B157** (1985) 81.

[133] Z. Kunszt, *Combined Use of the Calkul Method and N=1 Supersymmetry to Calculate QCD Six Parton Processes*, *Nucl.Phys.* **B271** (1986) 333.

[134] L. J. Dixon, *Calculating scattering amplitudes efficiently*, hep-ph/9601359.

[135] L. J. Dixon, J. M. Henn, J. Plefka and T. Schuster, *All tree-level amplitudes in massless QCD*, *JHEP* **1101** (2011) 035, [1010.3991].

[136] Z. Bern, L. J. Dixon and D. A. Kosower, *One loop corrections to five gluon amplitudes*, *Phys.Rev.Lett.* **70** (1993) 2677–2680, [hep-ph/9302280].

[137] Z. Bern, G. Chalmers, L. J. Dixon and D. A. Kosower, *One loop N gluon amplitudes with maximal helicity violation via collinear limits*, *Phys.Rev.Lett.* **72** (1994) 2134–2137, [hep-ph/9312333].

[138] G. Mahlon, *Multi - gluon helicity amplitudes involving a quark loop*, *Phys.Rev.* **D49** (1994) 4438–4453, [hep-ph/9312276].

[139] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *One loop gauge theory amplitudes with an arbitrary number of external legs*, hep-ph/9405248.

[140] C. Cheung and D. O'Connell, *Amplitudes and Spinor-Helicity in Six Dimensions*, *JHEP* **0907** (2009) 075, [0902.0981].