A GENERAL SIMILARITY THEORY OF PARTIAL DIFFERENTIAL EQUATIONS AND ITS USE IN THE SOLUTION OF PROBLEMS IN AERONAUTICS

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ABSTRACT

A general similarity theory of systems of partial differential equations of any order in any number of independent variables is developed with the aid of the theory of continuous one-parameter groups of transformations. The theory is illustrated by means of several known examples of similarity equations, previously given without motivation, in Hydrodynamics. With the aid of the theory two new examples of similarity equations, one in Elasticity and one in Fluid Mechanics, have been found; these are discussed in the text.

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NOTATION

The following list of symbols refers to the notation used in Sections I - VII. The notation used in Sections VIII and IX is explicitly defined therein.

a	The numerical parameter in a one-parameter continuous group of transformations.
A	When used in functional notation denotes a similarity equation.
$C^{(n)}$	Usually used when referring to the differentiability properties of a function; that is: the function posses partial derivatives up to the n'th order with respect to all of its arguments and these are continuous functions of the arguments.
$G1, G2$	The facts given in the structure of a Theorem.
G_1	A continuous one-parameter group of transformations.
G_1^c	G_1 when expressed in terms of a canonical parameter t.
$G_1^{E_k}$	The k'th (k = 1,2 ...) enlargement of a G_1^c.
$G^1, \cdots, G^4$	The postulates for a group.
$I1, I2$	The hypothesis of a Theorem; that is: the antecedents of a logical statement.
$J\left(\frac{f^1, \ldots, f^p}{z^1, \cdots, z^p}\right)$	The Jacobian of the functions $f^1(z^1, \cdots, z^p)$, $\cdots, f^p(z^1, \cdots, z^p)$ with respect to the variables $z^1, \cdots, z^p$.

$R1, R2$	The conclusions or results of a Theorem; that is: the consequents of a logical statement.
S_{G_1}	A sub-group of transformations of G_1.
t	The canonical parameter of the group G_1.
U, V	Linear first order partial differential operators called the symbols of the group G_1^{c}.
$U^{(k)}$	Symbol of the group $G_1^{E_k}$ ($k = 1,2 \ldots$).
$x^1, \ldots, x^m$	The independent variables of a partial differential equation ($m \geq 2$).
$y_1, \cdots, y_n$	The dependent variables of a system of partial differential equations ($n \geq 1$).
$y_{j,\alpha_1 \ldots \alpha_k}$	$\dfrac{\partial^k y_j}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_k}} \quad \begin{bmatrix} j = 1, \cdots, n, \\ \alpha_1, \cdots, \alpha_k = 1, \cdots, m. \end{bmatrix}$.
$\bar{y}_{j,\alpha_1 \ldots \alpha_k}$	$\dfrac{\partial^k \bar{y}_j}{\partial \bar{x}^{\alpha_1} \cdots \partial \bar{x}^{\alpha_k}} \quad \begin{bmatrix} j = 1, \cdots, n, \\ \alpha_1, \cdots, \alpha_k = 1, \cdots, m. \end{bmatrix}$.
$\xi^{\alpha_1}, \xi_j, \xi_j^{\alpha_1}, \cdots \xi_j^{\alpha_1 \cdots \alpha_k}$	The vectors of the group $G_1^{E_k}$.
Φ	When used in functional notation denotes an arbitrary partial differential form.
$\in$	Contained in.
$\Longrightarrow$	Implies.
$\equiv$	Means or Identical to, as indicated by the context.

CONVENTIONS

Bar under repeated index	The summation convention is inoperative for the index.
Bar over variable of group	The transform of the variable of the group.

I. INTRODUCTION

Many of the exact solutions of the partial differential equations governing fluid flow phenomena, especially those of viscous flows, are obtained by the use of similarity equations. That is, by finding a transformation of dependent and independent variables which will yield a partial differential equation of the same order but in which the number of independent variables is reduced by one. One instance of this is given by the Blasius solution of the Prandtl boundary layer equations.

A review of the literature reveals that there exist no general methods for obtaining these so-called "similarity equations" from the given partial differential equations. It was then thought that the development of such a general method would be of importance to workers in Aeronautics since it would enable them to discover new solutions of the hydrodynamical equations by means of a rigorous procedure and in addition provide a motivation for all the presently known "similarity solutions" of fluid flow problems.

This research was initiated by a remark due to Michal (Ref. 1), that is: " ... one may suspect that continuous groups would be found helpful in attacking some of the more difficult problems of elasticity, hydrodynamics, aeronautics and meteorology." An attempt at filling this gap in the theory has been made by G. Birkhoff in his book "Hydrodynamics, A Study in Logic, Fact and Similitude." In this book it is noted, but not proved, that if a partial differential equation is invariant under a continuous one-parameter group of transformations than a similarity equation can be obtained from the given equation. This idea is one which reveals a basis for a theory of similarity equations and its further development is presented in this thesis.

Once it is known that a system of partial differential equations is invariant under a continuous one-parameter group of transformations the theory presented herein enables one to immediately predict the transformations of independent and dependent variables will give the similarity equations corresponding to the given system. The method is shown to hold for systems of partial differential equations of any order in any number of independent variables and it is immaterial whether these equations are linear or non-linear or of elliptic, hyperbolic or parabolic types.

The general theory developed in this thesis is illustrated by several known examples of similarity equations, previously given without motivation, in Aeronautics. Two new examples of similarity equations, one in Elasticity and one in boundary layer theory, have been obtained by use of the general theory and these are discussed herein. In Appendix II a considerable relaxation on the class of functions which can be considered in dimensional analysis is obtained by proving Buckingham's Pi-Theorem with the use of group-theoretic methods.

No attempt shall be made to give a detailed exposition of group theory in this thesis beyond that which is required to maintain the continuity of the text. Whenever group-theoretic results are used, we shall give reference to standard works on the subject.

II. PRELIMINARY REMARKS

2.10 Definition of Similarity Equations and Solutions

Before proceeding to review the available literature on the subject of similarity equations it is convenient to define our terminology at the outset:

Definition 2.1: Similarity Equations Those partial differential equations, $A=0$ (say), in $m-1$ independent variables which are obtained from a given partial differential equation, $\Phi=0$ (say), in m independent variables.

Definition 2.2: Similarity Solutions of Partial Differential Equations The solution (s) of the partial differential equation $\Phi=0$ which is (are) obtained by means of the solution (s) of the equation $A=0$.

The above definitions will also apply when we are dealing with more than one dependent variable and hence with systems of simultaneous partial differential equations.

Definitions 2.1 and 2.2 are sufficiently general so that all known cases of similarity equations are subsumed under them. For the case $m=2$ the similarity equations are ordinary differential equations and this is the case which is usually thought of when the terminology "similarity equation" is used.

2.20 Review of the Literature

The two attempts to provide a background for a theory of similarity equations have been made by G. Birkhoff (Refs. 2 and 3). Birkhoff recognized that some similarity equations could be obtained from some given partial differential equations when they remained unchanged in form (invariant) under the transformations of a continuous one-parameter group of transformations. Birkhoff illustrated his contention by giving many examples; but, he did not succeed in formulating the problem from a general group-theoretic point of view or in proving that his contention holds in general.

In Section 8.30, after we have been introduced to some of the concepts of group theory and developed our results, we shall make a comparison between the method given by Birkhoff and the one which results from the theory presented herein.

There are many excellent books on the theory of continuous groups of transformations. An excellent introduction is contained in Cohen's book "An Introduction to the Lie Theory of One-Parameter Groups" (Ref. 4). A much more detailed and general exposition of the theory is given in Eisenhart's "Continuous Groups of Transformations" (Ref. 5). In this thesis considerable use has been made of the group theory contained in these two books.

III. CONTINUOUS ONE-PARAMETER GROUPS OF TRANSFORMATIONS

3.10 Definition of Continuous One-Parameter Groups of Transformations

In the succeeding work we shall be dealing exclusively with continuous one-parameter groups of transformations (also called Lie Groups in the literature) which we shall here define. The definition which we shall adopt is given in Ref. 5, p. 15, and is reproduced here for completeness.

Suppose that:

$$\bar{z}^i = f^i(z^1, \cdots, z^p; a) \; ; \; i=1, \cdots, p, \qquad (3.1)$$

denotes a set of transformations among the p variables z^i which in addition depends on a parameter a which can range over the real numbers and such that the functions f^i are in class $C^{(1)}$ with respect to their arguments.

<u>Definition 3.1</u>: <u>Continuous One-Parameter Group of Transformations</u>

The set of transformations (3.1) will be called a continuous one-parameter group of transformations if:

1. The functions f^i are such that: $f^i \varepsilon C^{(1)}$, or greater, in the z's and the a and, in particular:

$$J\left(\frac{f^1, \cdots, f^p}{z^1, \cdots, z^p}\right) \neq 0 .$$

2. The following group postulates are satisfied:

   G^1: <u>Existence of Identity</u>: The class of transformations (3.1) is such that there exists a value of the parameter a_o (say) for which

$$\bar{z}^i = f^i(z^1, \cdots, z^p; a_o) = z^i$$

G^2: Existence of Inverse: There exists a value of the parameter a_{-1} (say) such that:

$$z^i = f^i(\bar{z}^1, \cdots, \bar{z}^p; a_{-1}) .$$

G^3: Closure: There exists a value $a_3 = \varphi(a_1, a_2)$ of the parameter such that if:

$$\bar{\bar{z}}^i = f^i(\bar{z}^1, \cdots, \bar{z}^p; a_1)$$

and

$$\bar{z}^i = f^i(z^1, \cdots, z^p; a_2)$$

then:

$$\bar{\bar{z}}^i = f^i(z^1, \cdots, z^p; a_3) .$$

G^4: Associativity: The Associative Law holds for the set of transformations (3.1); that is, if we denote the transformations with parameter a_1 by T_{a_1}, etc. we must have:

$$T_{a_3}(T_{a_2} T_{a_1}) = (T_{a_3} T_{a_2}) T_{a_1} .$$

The transformations of (3.1) may be thought of as transforming a point $P(z)$ in a p-dimensional space V_p into a point $\bar{P}(\bar{z})$ in another p-dimensional space $\bar{V}_p$. If $\bar{P}(\bar{z})$ is the transform of a point $P(z)$ for a particular value of a then for small changes in the value of a we get points in the neighborhood of $\bar{P}(\bar{z})$.

Before proceeding to consider partial differential equations invariant under continuous one-parameter groups of transformations we shall need to develop some of the properties of such groups.

3.20 Some Properties of Continuous One-Parameter Groups of Transformations of a Special Type

In the treatment of one or more (systems) partial differential equations invariant under continuous one-parameter groups of transformations we shall need to know some of the properties of the groups with which we shall be dealing. These properties are listed or investigated in this section.

The groups which we shall deal with are of the special type:

$$\begin{aligned} &\bar{x}^i = f^i(x^1, \cdots, x^m; a); \; i=1,\cdots,m; \; m\geqslant 2; \\ &\bar{y} = f(y;a) \end{aligned} \tag{3.2}$$

when dealing with one partial differential equation and of the form:

$$G_1: \begin{aligned} &\bar{x}^i = f^i(x^1, \cdots, x^m; a); \; i=1,\cdots,m; \; m\geqslant 2; \\ &\bar{y}_j = f_j(y_j; a); \; j=1,\cdots,n; \; n\geqslant 1, \end{aligned} \tag{3.3}$$

when dealing with systems of partial differential equations in n unknowns y_j and m independent variables x^i. The sub-group of transformations of (3.3)

$$S_{G_1}: \bar{x}^i = f^i(x^1, \cdots, x^m; a); \; i=1,\cdots,m; \; m\geqslant 2,$$

shall be denoted by S_{G_1}.

Clearly, (3.2) is the special case of (3.3) for $n=1$, hence in our succeeding development we shall only be concerned with groups of the type (3.3).

From elementary group theory (Ref. 5, p. 34, Theorem 10.1) we have the following result.

Theorem 3.1

Any continuous one-parameter group of transformations is equivalent to a one-parameter group of translations.

There will then be no loss of generality if we consider the group (3.3) to be given in terms of a canonical parameter t such that

$$\frac{dx^i}{dt} = \xi^i(x^1, \ldots, x^m) \ , i=1, \cdots, m, \tag{3.4}$$

and

$$\frac{dy_j}{dt} = k_{\underline{j}}\, y_{\underline{j}} \quad , j=1, \ldots, n \, , \tag{3.5}$$

where $t=0$, $-t$, $t_1 + t_2$, are the values of the parameter which give the identity, inverse and the product of two transformations respectively.

From (3.4) and (3.5) we conclude that the finite transformations of the group, in terms of the canonical parameter t , are given by

$$G_1^c : \begin{array}{l} \bar{x}^i = f^i(x^1, \cdots, x^m; t) \, ; \ i=1, \cdots, m; \ m \geq 2; \\ \bar{y}_j = e^{k_{\underline{j}} t} y_{\underline{j}} \quad ; \ j = 1, \cdots, n \, ; \ n \geq 1 \, ; \end{array} \tag{3.6}$$

where the bar under the repeated indices indicates that the summation convention is inoperative and the notation G_1^c denotes the fact that the group G_1 is expressed in terms of the canonical parameter t.

3.30 The Enlargements of the Group

Assume that the y's of G_1^c are functions of $x^1, \cdots, x^m, m \geq 2$.

In analogy with the group theory of ordinary differential equations we wish to consider groups which are derived from G_1^c by adding to it the transformations among the partial derivatives of the $y_j(x^1, \cdots, x^m)$. In order to distinguish this way of adding transformations to the group G_1^c from that encountered in the theory of ordinary differential equations we shall call the derived groups <u>enlargements</u>, rather than <u>extensions</u>, of the group.

We shall then speak of the first, second, ..., k'th enlargements of the group G_1^c, denoting them by $G_1^{E_1}$, ..., $G_1^{E_k}$ respectively, according as the transformations of the first, second, ..., k'th partial derivatives of the $y_j(x^1,\cdots,x^m)$ are added successively to those of $G_1{}^c$, $G_1{}^{E_1}$, ..., $G_1{}^{E_{k-1}}$.

The first partial derivatives of the $y_j(x^1,\cdots,x^m)$, by (3.6), transform according to the relation:

$$\bar{y}_{j,\alpha} \overset{Df.}{=} \frac{\partial \bar{y}_j}{\partial \bar{x}^\alpha} = e^{k_{\underline{j}} t} \frac{\partial y_{\underline{j}}}{\partial \bar{x}^\alpha} = e^{k_{\underline{j}} t} \frac{\partial x^\beta}{\partial \bar{x}^\alpha} y_{\underline{j},\beta} = e^{k_{\underline{j}} t} \frac{\partial f^\beta(\bar{x}^1,\ldots,\bar{x}^m;-t)}{\partial \bar{x}^\alpha} y_{\underline{j},\beta}, \quad (3.7)^*$$

thus:

$$\bar{y}_{j,\alpha} = e^{k_{\underline{j}} t} A^\beta_\alpha \, y_{\underline{j},\beta}\,, \tag{3.8}$$

where:

$$A^\beta_\alpha = \frac{\partial f^\beta(\bar{x}^1,\ldots,\bar{x}^m;-t)}{\partial \bar{x}^\alpha} = \frac{\partial x^\beta}{\partial \bar{x}^\alpha}\,, \tag{3.9}$$

and:

$$x^\beta = f^\beta(\bar{x}^1,\cdots,\bar{x}^m;-t)$$

are the inverse transformations of the group S_{G_1}.

The second partial derivatives of the $y_j(x^1,\cdots,x^m)$ transform according to the relation

$$\begin{aligned} \bar{y}_{j,\alpha\beta} \overset{Df.}{=} \frac{\partial}{\partial \bar{x}^\beta}\left(\frac{\partial \bar{y}_j}{\partial \bar{x}^\alpha}\right) &= e^{k_{\underline{j}} t} \frac{\partial}{\partial \bar{x}^\beta}\left(A^\gamma_\alpha\, y_{\underline{j},\gamma}\right) \\ &= e^{k_{\underline{j}} t}\left[\frac{\partial A^\gamma_\alpha}{\partial x^\beta}\, y_{\underline{j},\gamma} + A^\gamma_\alpha A^\omega_\beta\, y_{\underline{j},\gamma\omega}\right], \end{aligned} \tag{3.10}$$

where the terms containing two repeated indices are summed over all combinations of the numbers 1, ..., m in such a way that all the partial derivatives of the y 's are taken into account without repetition.

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*For the remaining portion of this section Greek and Latin indices will range over the numbers 1, ..., m and 1, ..., n respectively.

The first and second enlargements of $G_1{}^c$ then consist of the transformations:

$$G_1^{E_1}: \quad \begin{aligned} \bar{x}^\alpha &= f^\alpha(x^1, \cdots, x^m; t); \\ \bar{y}_j &= e^{k_{\underline{j}} t} y_{\underline{j}} , \\ \bar{y}_{j,\alpha} &= e^{k_{\underline{j}} t} A^\gamma_\alpha y_{\underline{j},\gamma} \end{aligned} \tag{3.11}$$

and:

$$G_1^{E_2}: \quad \begin{aligned} \bar{x}^\alpha &= f^\alpha(x^1, \cdots, x^m; t); \\ \bar{y}_j &= e^{k_{\underline{j}} t} y_{\underline{j}} , \\ \bar{y}_{j,\alpha} &= e^{k_{\underline{j}} t} A^\gamma_\alpha y_{\underline{j},\gamma} \\ \bar{y}_{j,\alpha\beta} &= e^{k_{\underline{j}} t} \left[\frac{\partial A^\gamma_\alpha}{\partial \bar{x}^\beta} y_{\underline{j},\gamma} + A^\gamma_\alpha A^\omega_\beta y_{\underline{j},\gamma\omega} \right]. \end{aligned} \tag{3.12}$$

The procedure which must be followed for constructing further enlargements of $G_1{}^c$ is then clear from the above.

It remains to show that the transformations of $G_1{}^{E_1}, \ldots, G_1{}^{E_n}$ actually do form continuous one-parameter groups of transformations. To this effect we state the following result.

Lemma 3.1

If. 1. The transformations of $G_1{}^c$ form a continuous one-parameter group.	$\Rightarrow$	R1. The transformations of $G_1{}^{E1}, \ldots, G_1{}^{En}$ also form continuous one-parameter groups.	

Proof

To prove the above results it will be necessary to show that the postulates G^1-G^4 for a continuous one-parameter group are satisfied by the transformations of $G_1{}^{E_1}, \ldots, G_1{}^{E_n}$. The method of proof shall be given for $G_1{}^{E_1}$.

1. I1 $\Rightarrow$ that for the group $G_1{}^c$ the identity, the inverse and the

product of two transformations are given by values of the parameter $t=0$, $-t$ and $t_3=t_1+t_2$ respectively. We shall show that $t=0$, $-t$ and $t_3=t_1+t_2$ are also the values of the parameter t which give the identity, inverse and product transformations of G_1^{E1}.

2. (a) Existence of a Unit. $t=0$ is the value of the parameter which gives the identity transformation for G_1^{E1} for at $t=0$ we have

$$\bar{x}^\alpha = x^\alpha ,$$
$$\bar{y}_j = y_j ,$$
$$\bar{y}_{j,\alpha} = y_{j,\alpha} ,$$

which shows that postulate G^1 is satisfied.

(b) Existence of Inverse. $-t$ is the value of the parameter for the inverse transformations of G_1^{E1}. This is because:

I1 $\Rightarrow$

$$x^\alpha = f^\alpha(\bar{x}^1, \cdots, \bar{x}^m; -t) ,$$
$$y_j = e^{-k_{\underline{j}} t}\, \bar{y}_{\underline{j}} ,$$

then:

$$y_{j,\alpha} = e^{-k_{\underline{j}} t} \frac{\partial \bar{y}_{\underline{j}}}{\partial x^\alpha} = e^{-k_{\underline{j}} t} \frac{\partial \bar{x}^\beta}{\partial x^\alpha}\, \bar{y}_{\underline{j},\beta}$$
$$= e^{-k_{\underline{j}} t} \frac{\partial f^\beta(x^1, \cdots, x^m; t)}{\partial x^\alpha}\, \bar{y}_{\underline{j},\beta} ,$$

which shows that the conditions of postulate G^2 are satisfied.

(c) Closure.

I1 $\Rightarrow$ that if:

$$\bar{\bar{x}}^\alpha = f^\alpha(\bar{x}^1, \cdots, \bar{x}^m; t_1),$$
$$\bar{\bar{y}}_j = e^{k_{\underline{j}} t_1}\, \bar{y}_{\underline{j}} \tag{3.13}$$

and

$$\bar{x}^\beta = f^\beta(x^1, \cdots, x^m; t_2),$$
$$\bar{y}_i = e^{k_{\underline{i}} t_2} y_{\underline{i}} \tag{3.14}$$

then

$$\bar{\bar{x}}^\alpha = f^\alpha(x^1, \cdots, x^m; t_1 + t_2),$$
$$\bar{\bar{y}}_j = e^{k_{\underline{j}}(t_1+t_2)} y_{\underline{j}} \ . \tag{3.15}$$

From (3.13) we have that

$$\bar{\bar{y}}_{j,\alpha} = e^{k_{\underline{j}} t_1} \frac{\partial \bar{x}^\beta}{\partial \bar{\bar{x}}^\alpha} \bar{y}_{\underline{j},\beta} \tag{3.16}$$

and from (3.14)

$$\bar{y}_{j,\beta} = e^{k_{\underline{j}} t_2} \frac{\partial x^\gamma}{\partial \bar{x}^\beta} y_{\underline{j},\gamma} \ . \tag{3.17}$$

Combining (3.16) and (3.17) gives us the relation

$$\begin{aligned} \bar{\bar{y}}_{j,\alpha} &= e^{k_{\underline{j}}(t_1+t_2)} \frac{\partial x^\gamma}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^\beta}{\partial \bar{\bar{x}}^\alpha} y_{\underline{j},\gamma} \\ &= e^{k_{\underline{j}}(t_1+t_2)} \frac{\partial x^\gamma}{\partial \bar{\bar{x}}^\alpha} y_{\underline{j},\gamma} \\ &= e^{k_{\underline{j}}(t_1+t_2)} \frac{\partial f^\gamma(\bar{\bar{x}}^1, \ldots, \bar{\bar{x}}^m; -(t_1+t_2))}{\partial \bar{\bar{x}}^\alpha} y_{\underline{j},\gamma} \ . \end{aligned} \tag{3.18}$$

The relation (3.18) then verifies that postulate G^3 is satisfied for the transformations of ${G_1}^{E_1}$.

(d) <u>Associativity</u>. The elements of ${G_1}^{E_1}$ are defined by equations; hence postulate G^4 is satisfied since the process of substitution is associative.

3. The arguments of 2(a)-(d) show that the transformations of ${G_1}^{E_1}$ satisfy the postulates G^1-G^4 for a group. Hence the transformations of ${G_1}^{E_1}$ form a continuous one-parameter group of transformations. It is easy to see that a repetition of the arguments of 2(a)-(d) will

show that the transformations of $G_1^{E1}, \ldots, G_1^{En}$ also form continuous one-parameter groups of transformations. This proves R1.

In accordance with the above result we shall refer to the transformations of $G_1^{E1}, \ldots, G_1^{En}$ as groups without further qualifications.

3.40 The Vectors of the Enlargements of the Canonical Group

The vectors of the group G_1^c are defined to be the set of quantities: (Ref. 5, p. 33)

$$\xi^{\alpha_1} = \left.\frac{d\bar{x}^{\alpha_1}}{dt}\right)_{t=0}$$

and

$$\xi_j = \left.\frac{d\bar{y}_j}{dt}\right)_{t=0} .$$

The vectors of $G_1^{E1}, \ldots, G_1^{En}$ are obtained by successively adding to the vectors of G_1^c one each of the sets of quantities:

$$\xi_j^{\alpha_1} = \left.\frac{d\bar{y}_{j,\alpha_1}}{dt}\right)_{t=0} ,$$

$$\vdots \qquad \vdots$$

$$\xi_j^{\alpha_1\cdots\alpha_n} = \left.\frac{d\bar{y}_{j,\alpha_1\cdots\alpha_n}}{dt}\right)_{t=0} ,$$

where the indices $\alpha_1, \ldots, \alpha_n$ take values over their indicated range in such a way that all of the partial derivatives of the y_j are taken into account without repetition.

The vectors $\xi_j^{\alpha_1}$ are then determined as follows:

$$\xi_j^{\alpha_1} = \left.\frac{d\bar{y}_{j,\alpha_1}}{dt}\right)_{t=0} = \left[k_{\underline{j}}\, e^{k_{\underline{j}}t} A_{\alpha_1}^{\gamma_1}\, y_{\underline{j},\gamma_1} + e^{k_{\underline{j}}t} \frac{\partial A_{\alpha_1}^{\gamma_1}}{\partial t}\, y_{\underline{j},\gamma_1} \right]_{t=0} . \tag{3.19}$$

By definition:

$$A_{\alpha_1}^{\gamma_1} = \frac{df^{\gamma_1}(\bar{x}^1, \ldots, \bar{x}^m; -t)}{\partial \bar{x}^{\alpha_1}} = \frac{\partial x^{\gamma_1}}{\partial \bar{x}^{\alpha_1}} ,$$

since $\bar{x}^{\alpha_1} = x^{\alpha_1}$ at $t=0$ we must have:

$$A_{\alpha_1}^{\gamma_1} = \delta_{\alpha_1}^{\gamma_1}, \tag{3.20}$$

where $\delta_{\alpha_1}^{\gamma_1}$ is the Kroneker delta.

Due to the continuity of the functions f^{γ_1} in the x's and t we have:

$$\frac{\partial A_{\alpha_1}^{\gamma_1}}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\partial x^{\gamma_1}}{\partial \bar{x}^{\alpha_1}}\right] = \frac{\partial}{\partial \bar{x}^{\alpha_1}}\left[\frac{\partial x^{\gamma_1}}{\partial t}\right];$$

therefore,

$$\left.\frac{\partial A_{\alpha_1}^{\gamma_1}}{\partial t}\right)_{t=0} = \left[\frac{\partial}{\partial \bar{x}^{\alpha_1}}\left(-\frac{\partial f^{\gamma_1}(x^1, \ldots, x^m; t)}{\partial t}\right)\right]_{t=0} = -\left[\frac{\partial \xi^{\gamma_1}(\bar{x}^1, \ldots, \bar{x}^m)}{\partial \bar{x}^{\alpha_1}}\right]_{t=0} = -\frac{\partial \xi^{\gamma_1}}{\partial x^{\alpha_1}}. \tag{3.21}$$

By (3.20) and (3.21) the expression (3.19) becomes:

$$\xi_j^{\alpha_1} = \left[k_{\underline{j}}\,\delta_{\alpha_1}^{\gamma_1} - \frac{\partial \xi^{\gamma_1}}{\partial x^{\alpha_1}}\right] y_{\underline{j},\gamma_1}. \tag{3.22}$$

The <u>vectors</u> <u>for</u> <u>$G_1^{E_1}$</u> <u>are</u>, by (3.3), (3.4) and (3.22), then

$$\xi^{\alpha_1} = \xi^{\alpha_1}(x^1, \cdots, x^m),$$

$$\xi_j = k_{\underline{j}}\, y_{\underline{j}},$$

$$\xi_j^{\alpha_1} = \left[k_{\underline{j}}\,\delta_{\alpha_1}^{\gamma_1} - \frac{\partial \xi^{\gamma_1}}{\partial x^{\alpha_1}}\right] y_{\underline{j},\gamma_1}.$$

The vectors $\xi_j^{\alpha_1\alpha_2}$ are given by:

$$\xi_j^{\alpha_1\alpha_2} = \left.\frac{d y_{j,\alpha_1\alpha_2}}{dt}\right)_{t=0} = \left\{k_{\underline{j}} e^{k_{\underline{j}} t}\left[\frac{\partial A_{\alpha_1}^{\gamma_1}}{\partial \bar{x}^{\alpha_2}}\, y_{\underline{j},\gamma_1} + A_{\alpha_1}^{\gamma_1} A_{\alpha_2}^{\gamma_2}\, y_{\underline{j},\gamma_1\gamma_2}\right]\right.$$

$$\left. + e^{k_{\underline{j}} t}\left[\frac{\partial^2 A_{\alpha_1}^{\gamma_1}}{\partial t\,\partial \bar{x}^{\alpha_2}}\, y_{\underline{j},\gamma_1} + \frac{\partial A_{\alpha_1}^{\gamma_1}}{\partial t} A_{\alpha_2}^{\gamma_2}\, y_{\underline{j},\gamma_1\gamma_2} + A_{\alpha_1}^{\gamma_1}\frac{\partial A_{\alpha_2}^{\gamma_2}}{\partial t}\, y_{\underline{j},\gamma_1\gamma_2}\right]\right\}_{t=0}.$$

By (3.20) and (3.21) the above expression becomes:

$$\xi_j^{\alpha_1\alpha_2} = k_{\underline{j}}\left[0+\delta_{\alpha_1}^{\gamma_1}\delta_{\alpha_2}^{\gamma_2}\, y_{\underline{j},\gamma_1\gamma_2}\right] - \frac{\partial^2\xi^{\gamma_1}}{\partial x^{\alpha_1}\partial x^{\alpha_2}}\, y_{j,\gamma_1} - \frac{\partial\xi^{\gamma_1}}{\partial x^{\alpha_1}}\delta_{\alpha_2}^{\gamma_2}\, y_{j,\gamma_1\gamma_2} - \delta_{\alpha_1}^{\gamma_1}\frac{\partial\xi^{\gamma_2}}{\partial x^{\alpha_2}}\, y_{j,\gamma_1\gamma_2}$$

$$= -\frac{\partial^2\xi^{\gamma_1}}{\partial x^{\alpha_1}\partial x^{\alpha_2}}\, y_{j,\gamma_1} + \left[k_{\underline{j}}\,\delta_{\alpha_1}^{\gamma_1}\delta_{\alpha_2}^{\gamma_2} - \delta_{\alpha_2}^{\gamma_2}\frac{\partial\xi^{\gamma_1}}{\partial x^{\alpha_1}} - \delta_{\alpha_1}^{\gamma_1}\frac{\partial\xi^{\gamma_2}}{\partial x^{\alpha_2}}\right] y_{\underline{j},\gamma_1\gamma_2}\,. \qquad (3.23)$$

Hence, the vectors of the second enlargement, $G_1^{E_2}$, of G_1^c are:

$$\xi^{\alpha_1} = \xi^{\alpha_1}(x^1,\cdots,x^m)\,,$$

$$\xi_j = k_{\underline{j}}\, y_{\underline{j}}\,,$$

$$\xi_j^{\alpha_1} = \left[k_{\underline{j}}\,\delta_{\alpha_1}^{\gamma_1} - \frac{\partial\xi^{\gamma_1}}{\partial x^{\alpha_1}}\right] y_{\underline{j},\gamma_1}\,,$$

$$\xi_j^{\alpha_1\alpha_2} = -\frac{\partial^2\xi^{\gamma_1}}{\partial x^{\alpha_1}\partial x^{\alpha_2}}\, y_{j,\gamma_1} + \left[k_{\underline{j}}\,\delta_{\alpha_1}^{\gamma_1}\delta_{\alpha_2}^{\gamma_2} - \delta_{\alpha_1}^{\gamma_1}\frac{\partial\xi^{\gamma_2}}{\partial x^{\alpha_2}} - \delta_{\alpha_2}^{\gamma_2}\frac{\partial\xi^{\gamma_1}}{\partial x^{\alpha_1}}\right] y_{\underline{j},\gamma_1\gamma_2}\,.$$

The procedure to be used in determining the vectors $\xi_j^{\alpha_1\alpha_2\alpha_3}, \ldots, \xi_j^{\alpha_1\cdots\alpha_n}$ is evident from the above description.

3.50 The Symbols of the Canonical Group and Its Enlargements

Using the vectors of the groups S_{G_1}, G_1^c, $G_1^{E_1}$, ..., $G_1^{E_n}$ we define the following linear partial differential operators:

$$U^{(0)}f = \xi^{\alpha_1}\frac{\partial f}{\partial x^{\alpha_1}}\,, \qquad (3.24)$$

$$Uf = \xi^{\alpha_1}\frac{\partial f}{\partial x^{\alpha_1}} + k_j y_j \frac{\partial f}{\partial y_j}\,, \qquad (3.25)$$

$$U^{(1)}f = \xi^{\alpha_1}\frac{\partial f}{\partial x^{\alpha_1}} + k_j y_j \frac{\partial f}{\partial y_j} + \xi_j^{\alpha_1}\frac{\partial f}{\partial y_{j,\alpha_1}}\,, \qquad (3.26)$$

$$U^{(2)}f = \xi^{\alpha_1}\frac{\partial f}{\partial x^{\alpha_1}} + k_j y_j \frac{\partial f}{\partial y_j} + \xi_j^{\alpha_1}\frac{\partial f}{\partial y_{j,\alpha_1}} + \xi_j^{\alpha_1\alpha_2}\frac{\partial f}{\partial y_{j,\alpha_1\alpha_2}}\,, \qquad (3.27)$$

$$\vdots$$

$$U^{(n)}f = \xi^{\alpha_1}\frac{\partial f}{\partial x^{\alpha_1}} + k_j y_j \frac{\partial f}{\partial y_j} + \xi_j^{\alpha_1}\frac{\partial f}{\partial y_{j,\alpha_1}} + \cdots + \xi_j^{\alpha_1\cdots\alpha_n}\frac{\partial f}{\partial y_{j,\alpha_1\cdots\alpha_n}}\,, \qquad (3.28)$$

which we call the symbols of S_{G_1}, $G_1{}^c$, $G_1{}^{E_1}$, ..., $G_1{}^{E_n}$ respectively.

Elementary group theory provides us with the following result (Ref. 5, p. 62, Theorem 17.1).

Theorem 3.2

If. 1. $\mathcal{F}$ is an absolute invariant of a group G_1 $\Longleftrightarrow$ R1. $U\mathcal{F} = 0$.

By successive applications of Theorem 3.2 we conclude that the necessary and sufficient condition for any quantity to be an absolute invariant of the groups S_{G_1}, $G_1{}^c$, $G_1{}^{E_1}$, ..., $G_1{}^{E_n}$ respectively is that it satisfy one of the equations:

$$
\begin{aligned}
U^{(0)}f &= 0 , \\
Uf &= 0 , \\
U^{(1)}f &= 0 , \\
&\vdots \\
U^{(n)}f &= 0 .
\end{aligned}
$$

IV. PROPERTIES OF THE RELATIONS

$$z_\delta(x^1,\cdots,x^m) = g_\delta\big(y_1(x^1,\cdots,x^m),\cdots,y_n(x^1,\cdots,x^m),x^1,\cdots,x^m\big)$$

The quantities $g_\delta(y_1,\cdots,y_n,x^1,\cdots,x^m), \delta=1,\cdots,n,$ are defined in the statement of Lemma 4.1.

In succeeding sections it will be found necessary to consider relations of the type

$$z_\delta(x^1,\cdots,x^m)=g_\delta\big(y_1(x^1,\cdots,x^m),\cdots,y_n(x^1,\cdots,x^m),x^1,\cdots,x^m\big),\delta=1,\cdots,n, \quad (4.1)$$

since they will be useful in the investigation of partial differential equations which are invariant* under continuous one-parameter groups of transformations. Hence, in this section we determine some of their properties, the first of which is:

*This statement shall be made precise in Section V.

Lemma 4.1

Given:

G1. The set of equations

$$z_\delta(x^1,\cdots,x^m) = g_\delta(y_1,\cdots,y_n,x^1,\cdots,x^m),$$

where $\delta=1,\cdots,n$ and the $y_1,\cdots,y_n$ are considered as functions of $x^1,\cdots,x^m$.

2. The functions

$$g_\delta(y_1,\cdots,y_n,x^1,\cdots,x^m),\ \delta=1,\cdots,n,^*$$

when considered as functions of $m+n$ independent variables $y_1,\cdots,y_n$, $x^1,\cdots,x^m$, are functionally independent absolute invariants of the group

$$G_1:\ \begin{array}{l} \bar{x}^i = f^i(x^1,\cdots,x^m;a),\ i=1,\cdots,m,\ m\geq 2, \\ \bar{y}_j = f_j(y_j;a),\ j=1,\cdots,n,\ n\geq 1, \end{array}$$

where: a is a numerical parameter and $a=a_0$ (say) gives the identity transformation.

If. 1. Under the transformations of G_1:

$$\bar{y}_\delta(\bar{x}^1,\cdots,\bar{x}^m) = y_\delta(\bar{x}^1,\cdots,\bar{x}^m)$$

where: $\delta=1,\cdots,n$.

$\rightleftarrows$

R1. Under the transformations of G_1:

$$z_\delta(\bar{x}^1,\cdots,\bar{x}^m) = z_\delta(x^1,\cdots,x^m).$$

Proof ($\Rightarrow$)

1. G1 $\Rightarrow$ that under the transformations of the variables

*That these absolute invariants exist and are independent can be shown by elementary group theory. It is possible to choose them so that the Jacobian $J\left(\frac{g_1,\cdots,g_n}{y_1,\cdots,y_n}\right)\neq 0$.

$x^1, \cdots, x^m$ of G_1 we have:

$$z_\delta(\bar{x}^1, \cdots, \bar{x}^m) = g_\delta(y_1(\bar{x}^1, \cdots, \bar{x}^m), \cdots, y_n(\bar{x}^1, \cdots, \bar{x}^m), \bar{x}^1, \cdots, \bar{x}^m). \tag{4.1}$$

2. G2 $\Rightarrow$ that under the transformations of G_1

$$g_\delta(\bar{y}_1(\bar{x}^1, \cdots, \bar{x}^m), \cdots, \bar{y}_n(\bar{x}^1, \cdots, \bar{x}^m), \bar{x}^1, \cdots, \bar{x}^m) = g_\delta(y_1(x^1, \cdots, x^m), \cdots, y_n(x^1, \cdots, x^m), x^1, \cdots, x^m). \tag{4.2}$$

3. I1 $\Rightarrow$, by (4.1) and (4.2), that:

$$\begin{aligned} z_\delta(\bar{x}^1, \cdots, \bar{x}^m) &= g_\delta(y_1(\bar{x}^1, \cdots, \bar{x}^m), \cdots, y_n(\bar{x}^1, \cdots, \bar{x}^m), \bar{x}^1, \cdots, \bar{x}^m) \\ &= g_\delta(\bar{y}_1(\bar{x}^1, \cdots, \bar{x}^m), \cdots, \bar{y}_n(\bar{x}^1, \cdots, \bar{x}^m), \bar{x}^1, \cdots, \bar{x}^m) \\ &= g_\delta(y_1(x^1, \cdots, x^m), \cdots, y_n(x^1, \cdots, x^m), x^1, \cdots, x^m) \\ &= z_\delta(x^1, \cdots, x^m). \end{aligned}$$

This completes the proof of the sufficiency.

<u>Proof</u> ($\Leftarrow$)

1. R1, G1 $\Rightarrow$

$$g_\delta(y_1(\bar{x}^1, \cdots, \bar{x}^m), \cdots, y_n(\bar{x}^1, \cdots, \bar{x}^m), \bar{x}^1, \cdots, \bar{x}^m) = g_\delta(y_1(x^1, \cdots, x^m), \cdots, y_n(x^1, \cdots, x^m), x^1, \cdots, x^m). \tag{4.3}$$

2. G2 $\Rightarrow$ (4.2), hence on combining (4.2) and (4.3) we must have the following relation

$$g_\delta(y_1(\bar{x}^1, \cdots, \bar{x}^m), \cdots, y_n(\bar{x}^1, \cdots, \bar{x}^m), \bar{x}^1, \cdots, \bar{x}^m) = g_\delta(\bar{y}_1(\bar{x}^1, \cdots, \bar{x}^m), \cdots, \bar{y}_n(\bar{x}^1, \cdots, \bar{x}^m), \bar{x}^1, \cdots, \bar{x}^m). \tag{4.4}$$

For the purposes of the following argument it is important to note that the notation of (4.4) indicates that the arguments on either side of the equality are connected by the same functional relation. Since there are n such relations (4.4) and the arguments in the (n + 1)'th to the (n + m)'th places of the relations are the same it is evident, since $J\left(\frac{g_1, \cdots, g_n}{y_1, \cdots, y_n}\right) \neq 0$, that the only way in which the equality can be maintained is for:

$$\bar{y}_\delta(\bar{x}^1, \cdots, \bar{x}^m) = y_\delta(\bar{x}^1, \cdots, \bar{x}^m).$$

This proves the necessity.

Lemma 4.2

Given:

G1,2 of Lemma 4.1

G3. $\eta_1(x^1,\cdots,x^m),\cdots,\eta_{m-1}(x^1,\cdots,x^m)$ are functionally independent absolute invariants of the sub-group:*

S_{G_1}: $\bar{x}^i=f^i(x^1,\cdots,x^m;a)$; $i=1,\cdots,m$, $m\geq 2$,

of the Group G_1.

$$\left.\begin{array}{l}\text{If. 1. Under the transformations of } G_1\text{:}\\ \bar{y}_\delta(\bar{x}^1,\cdots,\bar{x}^m)=y_\delta(\bar{x}^1,\cdots,\bar{x}^m),\\ \text{where: } \delta=1,\cdots,n.\end{array}\right\}\begin{array}{c}\Longrightarrow\\ \Longleftarrow\end{array}\left\{\begin{array}{l}\text{R1.}\\ z_\delta(x^1,\cdots,x^m)=F_\delta(\eta_1,\cdots,\eta_{m-1}).\end{array}\right.$$

Proof

1. R1 of Lemma 4.1 $\Longrightarrow$ that the functions $z_\delta(x^1,\cdots,x^m)$ are absolute invariants of the sub-group S_{G_1} of G_1.

By elementary group theory every absolute invariant of S_{G_1} must be a function of $\eta_1,\cdots,\eta_{m-1}$, hence:

$$z_\delta(x^1,\cdots,x^m)=F_\delta(\eta_1,\cdots,\eta_{m-1}).$$

This completes the proof of Lemma 4.2.

Definition 4.1: Invariant Solutions of Partial Differential Equations

That class of solutions of a partial differential equation or systems of partial differential equations which satisfy the relation:

*That these $m-1$ absolute invariants exist and are independent can be shown by elementary group theory.

$$\overline{y}_\delta(\overline{x}^1,\cdots,\overline{x}^m) = y_\delta(\overline{x}^1,\cdots,\overline{x}^m) \quad \begin{bmatrix} m \geqslant 2, \\ \delta = 1,\cdots,n,\ n \geqslant 1. \end{bmatrix}$$

under the transformations of G_1; that is, the y_δ's are the same functions of the $x^1,\cdots,x^m$ that the $\overline{y}_\delta$'s are of the corresponding $\overline{x}^1,\cdots,\overline{x}^m$.

Lemma 4.3

Given:

G1. A continuous one-parameter group of transformations of the form:

$$G_1: \quad \begin{aligned} \overline{x}^i &= f^i(x^1,\cdots,x^m;a);\ i=1,\cdots,m,\ m\geqslant 2, \\ \overline{y}_j &= f_j(y_j;a)\ ;\ j=1,\cdots,n,\ n\geqslant 1. \end{aligned}$$

If. 1. $\eta_1(x^1,\cdots,x^m),\cdots,\eta_{m-1}(x^1,\cdots,x^m)$, $g_1(y_1,\cdots,y_n,x^1,\cdots,x^m)$, $\cdot\ \cdot\ \cdot$, $g_n(y_1,\cdots,y_n,x^1,\cdots,x^m)$ are $m+n-1$ functionally independent absolute invariants of G_1 in Class $C^{(1)}$.

$\Longrightarrow$

R1. $\mathcal{R}\left[\mathcal{J}\left(\dfrac{\eta_1,\cdots,\eta_{m-1}}{x^1,\cdots,x^m}\right)\right] = m-1$.

2. $\dfrac{\partial g_j}{\partial y_k} \neq 0$ $(j,k=1,\cdots,n)$ for fixed j and all k.

3. Also $\mathcal{J}\left(\dfrac{g_1,\cdots,g_n}{y_1,\cdots,y_n}\right) \neq 0$.

Proof

Proof of R1

1. First we note that elementary group theory (Ref. 5, p. 62) gives us the following facts:

(a) G1 $\Longrightarrow$ that the absolute invariants of I1 exist since they are characterized by the solutions of a homogeneous first order partial differential equation whose independent variables are the variables of the group.

(b) The $\eta_1(x^1,\cdots,x^m),\cdots,\eta_{m-1}(x^1,\cdots,x^m)$ are absolute invariants of the sub-group:

$$S_{G_1}:\ \bar{x}^i = f^i(x^1, \cdots, x^m; a)\ ;\ i=1, \cdots, m,\ m \geq 2\ .$$

(c) Any other absolute invariant of S_{G_1} is a function of the η's.

2. Since the η's are functionally independent we immediately have the result (Ref. 6, p. 9) that

$$R\left[J\left(\frac{\eta_1, \cdots, \eta_{m-1}}{x^1, \cdots, x^m}\right)\right] = m-1\ ,$$

where: R denotes the rank of the Jacobian matrix

This proves R1.

<u>Proof of R2</u>

1. R2 can be proved by contradiction.

Suppose that:

$$\frac{\partial g_j}{\partial y_k} = 0 \text{ for any fixed } j \text{ and all } k\ ;$$

then we conclude that the functions g_j must be independent of the y's and only functions of the x's absolutely invariant under the transformations of G_1. But, if such is the case the g's must then be absolute invariants of S_{G_1} and hence functions of the $\eta_1, \cdots, \eta_{m-1}$. This contradicts the hypothesis I1 and completes the proof of R2.

<u>Proof of R3</u>

This fact is evident from elementary group theory as stated in the footnote to Lemma 4.1.

Lemma 4.4

Given:

G1 of Lemma 4.2

If. 1. $\eta_1(x^1,\cdots,x^m),\cdots,\eta_{m-1}(x^1,\cdots,x^m)$, $g_1(y_1,\cdots,y_n,x^1,\cdots,x^m),\cdots,$ $g_n(y_1,\cdots,y_n,x^1,\cdots,x^m)$ are $m+n-1$ functionally independent absolute invariants of G_1.

2. $F_\delta(\eta_1,\cdots,\eta_{m-1}), \delta=1,\cdots,n,$ is a set of functions in Class $C^{(1)}$.

$\Longrightarrow$

R1. The relations

$$F_\delta(\eta_1,\cdots,\eta_{m-1})= g_\delta(y_1,\cdots,y_n,x^1,\cdots,x^m),$$

where $\delta=1,\cdots,n$, which implicitly define the y's as functions of the x's, can be inverted for the y's in some finite domain of the $(x^1,\cdots,x^m)$-space.

Proof

To show that the above result is true we have to demonstrate that the conditions of the implicit function theorem are satisfied (see, for instance Ref. 7, p. 132, Theorem 23.1).

1. By elementary group theory the following facts are at our disposal:

(a) The $g_\delta(y_1,\cdots,y_n,x^1,\cdots,x^m), \delta=1,\cdots,n,$ are at least $C^{(1)}$-continuous functions of $y_1,\cdots,y_n,x^1,\cdots,x^m$ defined in some finite domain of the $(y_1,\cdots,y_n,x^1,\cdots,x^m)$-space.

(b) The $\eta_\omega(x^1,\cdots,x^m)$, $\omega=1,\cdots,m-1$, are at least $C^{(1)}$-continuous functions of $x^1,\cdots,x^m$ defined in that part of the $(x^1,\cdots,x^m)$-space which is included in the domain of definition of the g's.

2. G1, I1,2 $\Longrightarrow$, by 1(a,b), that the set of functions

$$f_\delta(y_1,\cdots,y_n,x^1,\cdots,x^m)= g_\delta(y_1,\cdots,y_n,x^1,\cdots,x^m)-F_\delta(\eta_1,\cdots,\eta_{m-1})=0 \qquad (4.5)$$

are defined and continuous, at least in Class $C^{(1)}$, throughout the whole of some finite domain $\mathcal{N}_0$ of the $(y_1, \cdots, y_n, x^1, \cdots, x^m)$-space.

By Lemma 4.3 we then know that:

$$J\left(\frac{f_1, \cdots, f_n}{y_1, \cdots, y_n}\right) \neq 0$$

at any point $(y_{1_0}, \cdots, y_{n_0}, x_0^1, \cdots, x_0^m)$ of the domain $\mathcal{N}_0$.

3. The arguments of Steps 1, 2 and R2 of Lemma 4.3 show that the conditions of the implicit function theorem are satisfied at every point of the domain $\mathcal{N}_0$. Hence, equations (4.5) define $y_1, \cdots, y_n$ as single valued functions, at least in Class $C^{(1)}$, of $x^1, \cdots, x^m$ over the whole of some finite domain in the $(x^1, \cdots, x^m)$-space. This then completes the proof of Lemma 4.4.

V. CONFORMALLY INVARIANT DIFFERENTIAL FORMS

5.10 Definition of Conformally, Constant Conformally and Absolutely Invariant Differential Forms

Before considering the invariance of differential forms it is necessary to define them, that is:

Definition 5.1: Differential Form of the n'th Order in m Independent Variables $\equiv$ A function, usually in Class $C^{(1)}$ or greater, of the form:

$$\Phi\left(x^1,\cdots,x^m, y(x^1,\cdots,x^m), \frac{\partial y}{\partial x^1},\cdots,\frac{\partial y}{\partial x^m},\cdots,\frac{\partial^n y}{\partial x^{1^n}},\cdots,\frac{\partial^n y}{\partial x^{m^n}}\right) \quad (5.1)$$

whose argument contains the variables $x^1, \cdots, x^m$ a function $y(x^1,\cdots,x^m)$ of them and the partial derivatives of y with respect to the x's up to the n'th order.

Suppose that each of the terms in the argument of (5.1) transforms under the laws of transformation of a one-parameter continuous group of transformations with a symbol V in the canonical parameter t. Each of the arguments of (5.1) can then be considered as an independent variable under the group of transformations with the symbol V. If there are $p > m+2$ terms in the argument of (5.1) there will be no loss in generality on calling them $z^1, \cdots, z^p$.

With the above conventions we can then write:

Definition 5.2: Conformally Invariant Differential Form Φ Under the Group with Symbol V $\equiv$ The function Φ of (5.1) is such that under the transformations of the group with symbol V the following relation holds:

$$\Phi(\bar{z}^1,\cdots,\bar{z}^P) = F(z^1,\cdots,z^P;t)\,\Phi(z^1,\cdots,z^P), \qquad (5.2)$$

where $\Phi(z^1,\cdots,z^P)$ is exactly the same function of the z's that it is of the $\bar{z}$'s and $F(z^1,\cdots,z^P;t)$ is a function of the z's and the canonical parameter t which is different from the function Φ.

Definition 5.3: Constant Conformally Invariant Differential Form Φ Under the Group with Symbol V $\equiv$ The function Φ of (5.1) is such that under the transformations of the group with symbol V the relation (5.2) holds with $F(z^1,\cdots,z^P;t)$ being a function of t only.

Definition 5.4: Absolutely Invariant Differential Form Φ Under the Group with Symbol V $\equiv$ The function of Φ of (5.1) is such that under the transformations of the group with symbol V the relation (5.2) holds with $F(z^1,\cdots,z^P;t)$ being identically equal to one.

With the above definitions we can now proceed to determine some of the properties of conformally invariant differential forms.

5.20 Properties of Conformally Invariant Differential Forms

With the aid of the definitions given in Section 5.10 we can prove the following:

Theorem 5.1

Given:

G1. A one-parameter continuous group of transformations with canonical parameter t

$$\bar{z}^i = f^i(z^1, \cdots, z^p; t)\ ;\ i=1,\cdots,p\ ;$$

with symbol V.

G2. The differential form Φ is such that:

$\Phi \epsilon C^{(1)}$ or greater.

If. 1. The differential form $\Phi(z^1, \cdots, z^p)$ is conformally invariant under the transformations of G1. $\Longleftrightarrow$ R1. $V\Phi = \omega(z^1,\cdots,z^p)\Phi(z^1,\cdots,z^p)$ for some $\omega(z^1,\cdots,z^p)$.

Proof ($\Rightarrow$)

1. I1 and Df. 5.2 $\Rightarrow$ that Φ is such that:

$$\Phi(\bar{z}^1, \cdots, \bar{z}^p) = F(z^1,\cdots,z^p;t)\Phi(z^1,\cdots,z^p)\ . \tag{5.3}$$

2. On taking the derivative of (5.3) with respect to the parameter t, and evaluating the result at the identity element $t=0$, by G2 we obtain:

$$\left(\frac{\partial\Phi}{\partial\bar{z}^i}\frac{\partial\bar{z}^i}{\partial t}\right)_{t=0} = \left(\frac{\partial F}{\partial t}\right)_{t=0}\Phi(z^1,\cdots,z^p). \tag{5.4}$$

But, by elementary group theory (Ref. 5, p. 33)

$$\left.\frac{\partial\bar{z}^i}{\partial t}\right)_{t=0} = \xi^i(z^1,\cdots,z^p)\ ;\ i=1,\cdots,p\ ; \tag{5.5}$$

and

$$\left.\frac{\partial\Phi}{\partial\bar{z}^i}\right)_{t=0} = \frac{\partial\Phi(z^1,\cdots,z^p)}{\partial z^i}\ . \tag{5.6}$$

3. Remembering that

$$V = \xi^i \frac{\partial}{\partial z^i} \quad , i = 1, \cdots, p, \tag{5.7}$$

and defining

$$\left.\frac{\partial F}{\partial t}\right)_{t=0} = \omega(z^1, \cdots, z^p) \tag{5.8}$$

we obtain, on using (5.5) - (5.8) in (5.4), the result

$$V\Phi = \omega(z^1, \cdots, z^p)\, \Phi(z^1, \cdots, z^p). \tag{5.9}$$

This proves the sufficiency.

<u>Proof</u> ($\Leftarrow$)

1. R1 $\Rightarrow$:

$$V\Phi = \omega(z^1, \cdots, z^p)\, \Phi(z^1, \cdots, z^p);$$

hence:

$$\begin{aligned} V^2\Phi &= V(V\Phi) = V(\omega\Phi) \\ &= \Phi V\omega + \omega V\Phi \\ &= \Phi(\omega^2 + V\omega). \end{aligned} \tag{5.10}$$

Similarly, if:

$$V^n\Phi = \theta(z^1, \cdots, z^p)\Phi(z^1, \cdots, z^p),$$

then

$$\begin{aligned} V^{n+1}\Phi &= V(V^n\Phi) = \Phi V\theta + \theta V\Phi \\ &= \Phi(\theta\omega + V\theta). \end{aligned} \tag{5.11}$$

2. In view of the relations (5.10) and (5.11) we can write:

$$V^n\Phi = \omega^{(n)}(z^1, \cdots, z^p)\Phi(z^1, \cdots, z^p) \tag{5.12}$$

where for each n the $\omega^{(n)}(z^1, \cdots, z^p)$ are determinable functions of $z^1, \cdots, z^p$.

3. Again, by elementary group theory (Ref. 5, p. 35) we can write the following expansion for $\Phi(\bar{z}^1, \cdots, \bar{z}^p)$:

$$\Phi(\bar{z}^1,\cdots,\bar{z}^P) = \Phi(z^1,\cdots,z^P) + tV\Phi + \frac{t^2}{2}V^2\Phi + \cdots \overset{S}{=} e^{tV}\Phi . \tag{5.13}$$

Substituting (5.12) in (5.13) yields

$$\Phi(\bar{z}^1,\cdots,\bar{z}^P) = \Phi(z^1,\cdots,z^P)\left[1 + \omega^{(1)}t + \omega^{(2)}\frac{t^2}{2} + \cdots\right] . \tag{5.14}$$

4. Now, we know that the right hand side of (5.13) converges to $\Phi(\bar{z}^1,\cdots,\bar{z}^P)$ for any t. The same must hold true for (5.14), hence the expression within the brackets in the right hand side of (5.14) must converge to a function $F(z^1,\cdots,z^P;t)$ for any t. The equation (5.14) can then be written as:

$$\Phi(\bar{z}^1,\cdots,\bar{z}^P) = F(z^1,\cdots,z^P;t)\,\Phi(z^1,\cdots,z^P) . \tag{5.15}$$

By (5.15) and Df. 5.2 the function Φ is then conformally invariant under the group with the symbol V. This then proves the necessity.

From Theorem 5.1 we conclude that if Φ is to be conformally invariant under a group with symbol V then Φ must be a solution of the first order partial differential equation (5.9). Conversely, any solution of (5.9) must be conformally invariant under the group with symbol V.

With the above interpretation of Theorem 5.1 we prove the following:

<u>Theorem 5.2</u>

Given:

G1,2 of Theorem 5.1

If. 1. The differential form $\Phi(z^1,\cdots,z^P)$ is conformally invariant under the transformations of G1.	$\Longleftrightarrow$	R1. $\Phi = e^{\zeta(z^1,\cdots,z^P)}\Phi_0$ where Φ is a general absolute invariant of the group G_1 and ζ is a determinable function of $z^1,\cdots,z^P$.

<u>Proof</u> ($\Rightarrow$)

1. I1 and Theorem 5.1 $\Rightarrow$ that Φ is a solution of the first order partial differential equation

$$V\Phi = \omega(z^1, \cdots, z^p)\Phi(z^1, \cdots, z^p). \tag{5.9}$$

The general solution of (5.9) will evidently give the most general conformally invariant Φ under the group with symbol V. Thus the sufficiency will be proved if we can show that the general solution of (5.9) is of the form given in R1.

2. We proceed to determine the general solution of (5.9). Such a general solution (Ref. 8, p. 252) is given by an arbitrary function of the p independent solutions

$$\eta_i(z^1, \cdots, z^p) = c_i = \text{const.}\ , \quad i = 1, \cdots, p-1, \tag{5.16}$$

and

$$\phi(z^1, \cdots, z^p, \Phi) = c = \text{const.}\ , \tag{5.17}$$

of the system of ordinary differential equations

$$\frac{dz^1}{\xi^1} = \frac{dz^2}{\xi^2} = \cdots = \frac{dz^p}{\xi^p} = \frac{d\Phi}{\omega\Phi}\ . \tag{5.18}$$

3. Suppose that the solutions (5.16) are known and we have to determine (5.17), then from the first and last terms of (5.18) we have:

$$\frac{d\Phi}{\Phi} = \frac{\omega(z^1, \cdots, z^p)}{\xi^1(z^1, \cdots, z^p)}\, dz^1\ . \tag{5.19}$$

But, by the equations (5.16) it is possible to write:

$$z^j = f^j(z^1, c_1, \cdots, c_{p-1})\ , \quad j = 2, \cdots, p, \tag{5.20}$$

and upon substituting (5.20) in (5.19) we have:

$$\frac{d\Phi}{\Phi} = \frac{\omega(z^1, f^2(z^1, c_1, \cdots, c_{p-1}), \cdots, f^p(z^1, c_1, \cdots, c_{p-1}))}{\xi^1(z^1, f^2(z^1, c_1, \cdots, c_{p-1}), \cdots, f^p(z^1, c_1, \cdots, c_{p-1}))}\, dz^1$$

$$= \theta(z^1, c_1, \cdots, c_{p-1})\, dz^1, \tag{5.21}$$

where the definition $\theta(z^1, c_1, \cdots, c_{p-1})$ is evident from the above.

The solution of (5.21) is then:

$$\ln \Phi - \int \theta\, dz^1 = c = \text{const.}, \tag{5.22}$$

and the left hand side of (5.22) is the function ϕ of (5.17).

4. The general solution of (5.9) is then:

$$F(\eta_1, \cdots, \eta_{p-1}, \ln \Phi - \int \theta\, dz^1) = 0,$$

where F is an arbitrary function. Since F is an arbitrary function there is no loss in generality on writing the solution of (5.9) as:

$$\ln \Phi - \int \theta\, dz^1 = \ln \Phi_0, \tag{5.23}$$

where Φ_0 is an arbitrary function of $\eta_1, \cdots, \eta_{p-1}$ which is the solution of the equation $Vf=0$. With a little further manipulation (5.23) becomes:

$$\Phi = e^{\int \theta\, dz^1}\, \Phi_0$$

or

$$\Phi = e^{\varsigma(z^1, \cdots, z^p)}\, \Phi_0(\eta_1, \cdots, \eta_{p-1})\,; \tag{5.24}$$

where

$$\varsigma(z^1, \cdots, z^p) = \int \theta(z^1, c_1, \cdots, c_{p-1})\, dz^1,$$

since after the integration is performed the c's are replaced by the left hand sides of equations (5.16). Equation (5.24) then establishes the sufficiency of the condition.

Proof ($\Leftarrow$)

1. R1 $\Rightarrow$ that Φ is of the form

$$\Phi = e^{\xi(z^1,\cdots,z^P)}\Phi_0 .$$

By the construction shown Φ satisfies the equation

$$V\Phi = \omega(z^1,\cdots,z^P)\,\Phi(z^1,\cdots,z^P);$$

but, by Theorem 5.1, Φ is then conformally invariant under the group with symbol V. This proves the necessity.

Based upon the previous results it is possible to prove the following:

Theorem 5.3

Given:

G1,2 of Theorem 5.1

If. 1. $\Phi = 0$ is an equation with Φ conformally invariant under a group with symbol V. $\Rightarrow$ R1. The equation can be written as: $\Phi_0 = 0$.

Proof

1. I1 and Theorem 5.2 $\Rightarrow$

$$\Phi = e^{\xi(z^1,\cdots,z^P)}\Phi_0(\eta_1,\cdots,\eta_{P-1});$$

hence, the equation

$$\Phi = 0$$

can be written as:

$$e^{\xi(z^1,\cdots,z^P)}\Phi_0(\eta_1,\cdots,\eta_{P-1}) = 0 . \tag{5.25}$$

2. But, $e^{\xi(z^1,\cdots,z^P)} \neq 0$ for arbitrary $z^1,\cdots,z^P$, hence (5.25)

can be written as:

$$\Phi_0(\eta_1, \cdots, \eta_{p-1}) = 0.$$

This then proves the sufficiency.

Based on the above results it is then natural to construct the following:

<u>Definition 5.5:</u> <u>Partial Differential Equation $\Phi=0$ Invariant Under a Continuous One-Parameter Group of Transformations G_1</u> $\equiv$ The partial differential form Φ is conformally invariant under the transformations of G_1.

VI. SYSTEMS OF SIMULTANEOUS FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS INVARIANT UNDER CONTINUOUS ONE-PARAMETER GROUPS OF TRANSFORMATIONS

6.10 Formulation of the Problem

Suppose that we have a set of simultaneous first order partial differential equations in $m \geqslant 2$ independent variables $x^1, \cdots, x^m$ and $n \geqslant 1$ dependent variables $y_1, \cdots, y_n$ of the form:

$$\begin{aligned} &\Phi_1\left(x^1, \cdots, x^m, y_1, \cdots, y_n, \frac{\partial y_1}{\partial x^1}, \cdots, \frac{\partial y_1}{\partial x^m}, \cdots, \frac{\partial y_n}{\partial x^1}, \cdots, \frac{\partial y_n}{\partial x^m}\right) = 0, \\ &\qquad\qquad\vdots \\ &\Phi_n\left(x^1, \cdots, x^m, y_1, \cdots, y_n, \frac{\partial y_1}{\partial x^1}, \cdots, \frac{\partial y_1}{\partial x^m}, \cdots, \frac{\partial y_n}{\partial x^1}, \cdots, \frac{\partial y_n}{\partial x^m}\right) = 0, \end{aligned} \tag{6.1}$$

where each of the differential forms $\Phi_1, \cdots, \Phi_n$ is conformally invariant under the first enlargement G_1^{E1} (see Section III) of a continuous one-parameter group of transformations.

The question to be answered is: "Can the <u>invariant</u> solutions of the system (6.1) be expressed in terms of the solutions of a system of first order partial differential equations in $m-1$ independent variables?" It will be shown that the answer is in the affirmative.

6.20 Formal Investigation of the System (6.1)

Before we proceed to prove the main theorem of this section we first prove the following result:

Lemma 6.1

Given:

G1. A one-parameter continuous group of transformations of the form:

$$G_1: \quad \begin{aligned} \bar{x}^i &= f^i(x^1,\cdots,x^m;a);\ i=1,\cdots,m,\ m\geq 2, \\ \bar{y}_j &= f_j(y_j;a),\ j=1,\cdots,n,\ n\geq 1, \end{aligned}$$

where: a is a numerical parameter and $a=a_0$ (say) gives the identity transformation.

If. 1. $\eta_1(x^1,\cdots,x^m),\cdots,\eta_{m-1}(x^1,\cdots,x^m)$, $g_1(y_1,\cdots,y_n,x^1,\cdots,x^m)$, $\vdots$ $g_n(y_1,\cdots,y_n,x^1,\cdots,x^m)$ are $m+n-1$ functionally independent absolute invariants of G_1.

2. The functions $y_1(x^1,\cdots,x^m),\cdots,y_n(x^1,\cdots,x^m)$ are implicitly defined by the relations:

$$F_\delta(\eta_1,\cdots,\eta_{m-1}) = g_\delta(y_1,\cdots,y_n,x^1,\cdots,x^m)$$

where: $\delta = 1,\cdots,n$.

3. $F_1(\eta_1,\cdots,\eta_{m-1}),\cdots,F_n(\eta_1,\cdots,\eta_{m-1})$ are functions in Class $C^{(1)}$.

4. $\eta_1,\cdots,\eta_{m-1},g_1,\cdots,g_n$ are functions in Class $C^{(2)}$.

$\Longrightarrow$

R1. The quantities

$$\frac{\partial F_\delta}{\partial \eta_\omega} \quad \begin{bmatrix} \delta & 1,\cdots,n, \\ \omega & 1,\cdots,m-1 \end{bmatrix}$$

when expressed in terms of the variables $x^1,\cdots,x^m,y_1,\cdots,y_n,y_{1,\delta},\cdots,y_{n,\delta}$ of G_1^{E1}, are a set of absolute invariants of G_1^{E1}.

2. Any function

$$\mathcal{H}\left(\frac{\partial F_1}{\partial \eta_1},\cdots,\frac{\partial F_1}{\partial \eta_{m-1}},\cdots,\frac{\partial F_n}{\partial \eta_1},\cdots,\frac{\partial F_n}{\partial \eta_{m-1}}\right) \in C^{(1)}$$

is also an absolute invariant of G_1^{E1}.

Proof

1. G1, Theorem 3.1 $\Longrightarrow$ G_1 can always be expressed as:

$$G_1^c: \quad \begin{aligned} \bar{x}^i &= f^i(x^1,\cdots,x^m;t);\ i=1,\cdots,m, \\ \bar{y}_j &= e^{k_j t} y_{\underline{j}}\ ,\ j=1,\cdots,n, \end{aligned}$$

without loss in generality. Hence, by Section III, the symbol of the first enlargement, G_1^{E1}, of G_1^c can be immediately written as:

$$U^{(1)}f = \xi^i \frac{\partial f}{\partial x^i} + k_a y_a \frac{\partial f}{\partial y_a} + \left[k_a \delta_i^r - \frac{\partial \xi^r}{\partial x^i}\right] y_{a,r} \frac{\partial f}{\partial y_{a,i}} \begin{bmatrix} a=1,\cdots,n, \\ i,r=1,\cdots,m \end{bmatrix}. \quad (6.2)$$

2. I1 $\Rightarrow$ that the quantities $\eta_1, \cdots, \eta_{m-1}, g_1, \cdots, g_n$ are functionally independent solutions of the equation $Uf=0$, hence we have the following identities at our disposal:

$$\xi^i \frac{\partial \eta_\omega}{\partial x^i} = 0 \qquad \begin{bmatrix} \omega=1,\cdots,m-1, \\ i=1,\cdots,m. \end{bmatrix}, \qquad (6.3)$$

and

$$\xi^i \frac{\partial g_\delta}{\partial x^i} + k_a y_a \frac{\partial g_\delta}{\partial y_a} = 0 \qquad \begin{bmatrix} i=1,\cdots,m, \\ a,\delta=1,\cdots,n. \end{bmatrix}. \qquad (6.4)$$

3. I2,3 $\Rightarrow$ that on taking the partial derivatives of the expression

$$F_\delta(\eta_1,\cdots,\eta_{m-1}) = g_\delta(y_1,\cdots,y_n,x^1,\cdots,x^m), \quad \delta=1,\cdots,n, \quad (6.5)$$

with respect to $x^{\alpha_1},\cdots,x^{\alpha_{m-1}}$, where the numbers $\alpha_1,\cdots,\alpha_{m-1}$ are all different but are $m-1$ numbers of the set $1,\cdots,m$, we obtain the following set of simultaneous equations for $\frac{\partial F_\delta}{\partial \eta_1},\cdots,\frac{\partial F_\delta}{\partial \eta_{m-1}}$, that is:

$$\begin{aligned} \frac{\partial F_\delta}{\partial \eta_1}\frac{\partial \eta_1}{\partial x^{\alpha_1}} + \cdots + \frac{\partial F_\delta}{\partial \eta_{m-1}}\frac{\partial \eta_{m-1}}{\partial x^{\alpha_1}} &= \frac{\partial g_\delta}{\partial y_b} y_{b,\alpha_1} + \frac{\partial g_\delta}{\partial x^{\alpha_1}}, \quad b=1,\cdots,n, \\ \frac{\partial F_\delta}{\partial \eta_1}\frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} + \cdots + \frac{\partial F_\delta}{\partial \eta_{m-1}}\frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}} &= \frac{\partial g_\delta}{\partial y_b} y_{b,\alpha_{m-1}} + \frac{\partial g_\delta}{\partial x^{\alpha_{m-1}}}. \end{aligned} \qquad (6.6)$$

From (6.6) we can solve for $\frac{\partial F_\delta}{\partial \eta_\omega}$ as:

$$\frac{\partial F_\delta}{\partial \eta_\omega} = \frac{\begin{vmatrix} \frac{\partial \eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_1}} & \left(\frac{\partial g_\delta}{\partial y_b} y_{b,\alpha_1} + \frac{\partial g_\delta}{\partial x^{\alpha_1}}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_1}} \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \left(\frac{\partial g_\delta}{\partial y_b} y_{b,\alpha_{m-1}} + \frac{\partial g_\delta}{\partial x^{\alpha_{m-1}}}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1, \cdots, \eta_{m-1}}{x^{\alpha_1}, \cdots, x^{\alpha_{m-1}}}\right)}, \quad (6.7)$$

where $J\left(\frac{\eta_1, \cdots, \eta_{m-1}}{x^{\alpha_1}, \cdots, x^{\alpha_{m-1}}}\right)$ denotes the Jacobian of quantities $\eta_1, \cdots, \eta_{m-1}$ with respect to the $x^{\alpha_1}, \cdots, x^{\alpha_{m-1}}$. Since, by I1, the $\eta_1, \cdots, \eta_{m-1}$ are functionally independent it is evident that $J\left(\frac{\eta_1, \cdots, \eta_{m-1}}{x^{\alpha_1}, \cdots, x^{\alpha_{m-1}}}\right) \not\equiv 0$ *; hence, the division indicated in (6.7) is justified. In the succeeding work free indices will always be indicated by Greek lower case letters. To this effect we note that the ratio on the right hand side of (6.7) is independent of the free indices $\alpha_1, \cdots, \alpha_{m-1}$.

4. On substituting the right hand side of (6.7) into (6.2), and using the identities (6.3) and (6.4), it can be shown (Appendix I) that the quantities

$$\frac{\partial F_\delta}{\partial \eta_\omega} \qquad \begin{bmatrix} \delta = 1, \cdots, n, \\ \omega = 1, \cdots, m-1. \end{bmatrix} \qquad (6.8)$$

are a set of solutions of the first order partial differential equation $U^{(1)} f = 0$. Therefore, by Theorem 3.2, the quantities (6.8) must be a set of absolute invariants of $G_1^{E_1}$. This proves R1.

5. To prove R2 it is only necessary to note that

*That it is possible to choose the numbers $\alpha_1, \cdots, \alpha_{m-1}$ in such a way is evident from R1, Lemma 4.3.

$$U^{(1)}\mathcal{F}\left(\frac{\partial F_1}{\partial \eta_1},\ldots,\frac{\partial F_1}{\partial \eta_{m-1}},\ldots,\frac{\partial F_n}{\partial \eta_1},\ldots,\frac{\partial F_n}{\partial \eta_{m-1}}\right)$$

$$=\frac{\partial \mathcal{F}}{\partial\left(\frac{\partial F_1}{\partial \eta_1}\right)}U^{(1)}\left(\frac{\partial F_1}{\partial \eta_1}\right)+\cdot\cdot\cdot+\frac{\partial \mathcal{F}}{\partial\left(\frac{\partial F_n}{\partial \eta_{m-1}}\right)}U^{(1)}\left(\frac{\partial F_n}{\partial \eta_{m-1}}\right)\equiv 0,$$

since by R1 each of the expressions $U^{(1)}\left(\frac{\partial F_1}{\partial \eta_1}\right),\ldots,U^{(1)}\left(\frac{\partial F_n}{\partial \eta_{m-1}}\right)$ is identically zero. Therefore, by Theorem 3.2, the expression $\mathcal{F}\left(\frac{\partial F_1}{\partial \eta_1},\ldots,\frac{\partial F_n}{\partial \eta_{m-1}}\right)$ is also an absolute invariant of $G_1^{E_1}$. This proves R2.

The above result is contained in the more general Lemma 7.1. The proof has been given to illustrate the difference between the above method and that shown in Lemma 7.1.

We now proceed to consider systems of first order partial differential equations. The succeeding theorem formally states, for invariant solutions of the system, one of their properties.

Theorem 6.1

Given:

G1. A one-parameter continuous group of transformations.

$$G_1: \begin{aligned} &\bar{x}^i = f^i(x^1, \cdots, x^m; a); \; i = 1, \cdots, m, \; m \geq 2 \\ &\bar{y}_j = f_j(y_j; a); \; j = 1, \cdots, n, \; n \geq 1; \end{aligned}$$

where: a is a numerical parameter and $a = a_0$ (say) gives the identity transformation.

G2. $m+n-1$ functionally independent invariants of G_1 of the form:

$$\left.\begin{aligned} &\eta_1(x^1, \cdots, x^m), \cdots, \eta_{m-1}(x^1, \cdots, x^m), \\ &g_1(y_1, \cdots, y_n, x^1, \cdots, x^m), \cdots, g_n(y_1, \cdots, y_n, x^1, \cdots, x^m) \end{aligned}\right\} \in C^{(2)}.$$

G3. The functions $y_1(x^1, \cdots, x^m), \cdots,$ $y_n(x^1, \cdots, x^m)$ are implicitly defined by the relations:

$$F_\delta(\eta_1, \cdots, \eta_{m-1}) = g_\delta(y_1, \cdots, y_n, x^1, \cdots, x^m),$$

where: $\delta = 1, \cdots, n$.

Statement of Theorem

If. 1.
$$\left.\begin{aligned} &\Phi_1\left(x^1, \ldots, x^m, y_1, \ldots, y_n, \frac{\partial y_1}{\partial x^1}, \ldots, \frac{\partial y_n}{\partial x^m}\right) = 0, \\ &\qquad\vdots \\ &\Phi_n\left(x^1, \ldots, x^m, y_1, \ldots, y_n, \frac{\partial y_1}{\partial x^1}, \ldots, \frac{\partial y_n}{\partial x^m}\right) = 0 \end{aligned}\right.$$

is a system of partial differential equations of the first order in m independent variables $x^1, \cdots, x^m$ and n unknowns $y_1, \cdots, y_n$.

2. Each of the differential forms $\Phi_1, \cdots, \Phi_n$ is conformally invariant under the group G_1^{E1}.

$\Longrightarrow$

R1. The invariant solutions of the system of equations

$$\Phi_1 = 0, \cdots, \Phi_n = 0$$

can be expressed in terms of the solutions of the system

$$\begin{aligned} &A_1\left(\eta_1, \cdots, \eta_{m-1}, F_1, \cdots, F_n, \frac{\partial F_1}{\partial \eta_1}, \cdots, \frac{\partial F_n}{\partial \eta_{m-1}}\right) = 0, \\ &\qquad\vdots \\ &A_n\left(\eta_1, \cdots, \eta_{m-1}, F_1, \cdots, F_n, \frac{\partial F_1}{\partial \eta_1}, \cdots, \frac{\partial F_n}{\partial \eta_{m-1}}\right) = 0 \end{aligned}$$

- a system of first order partial differential equations in $m-1$ independent variables $\eta_1, \cdots, \eta_{m-1}$ and n unknowns $F_1, \cdots, F_n$.

Proof

1. G1,2 $\Longrightarrow$, on using R1 of Lemma 4.3, that " $m-1$ " of the x's, ${}^*x^i$ (say), can be expressed in terms of the " $m-1$ " absolute invariants $\eta_1, \cdots, \eta_{m-1}$ and the remaining x, ${}^*x^m$ (say). Therefore, we can write:

$$ {}^*x^i = h^i(\eta_1, \cdots, \eta_{m-1}, {}^*x^m), \; i=1, \cdots, m-1. \tag{6.9} $$

2. G1,2,3 $\Longrightarrow$, by Lemma 4.4, that the y's can be expressed as functions of the F's and the x's which in turn, by (6.9), may be written as:

$$ y_\delta = \Omega_\delta(F_1, \cdots, F_n, \eta_1, \cdots, \eta_{m-1}, {}^*x^m), \; \delta=1, \cdots, n. \tag{6.10} $$

3. The equations (6.6) can also be solved for the $y_{b,\alpha}$'s, since $J\left(\frac{g_1, \cdots, g_n}{y_1, \cdots, y_n}\right) \neq 0$ (see R3, Lemma 4.3), in terms of the $\frac{\partial F_\delta}{\partial \eta_\omega}$'s, $\frac{\partial \eta_\omega}{\partial x^\alpha}$'s and the $\frac{\partial g_\delta}{\partial y_b}$'s. But, by (6.9) and (6.10) the $\frac{\partial \eta_\omega}{\partial x^\alpha}$'s and the $\frac{\partial g_\delta}{\partial y_b}$'s may be expressed in terms of the F's, η's and ${}^*x^m$; hence, it is possible to write the $y_{b,\alpha}$'s as:

$$ y_{b,\alpha_i} = \Omega_{b\alpha_i}\left(\frac{\partial F_\delta}{\partial \eta_\omega}, F_\delta, \eta_i, {}^*x^m\right) \quad \begin{bmatrix} b=1, \cdots, n, \\ i=1, \cdots, m-1. \end{bmatrix}, \tag{6.11} $$

where the notation indicates that the functions $\Omega_{b\alpha_i}$ depend on all of the variables indicated above. This, of course, can also be seen by taking first partial derivatives with respect to the x's of the equation (6.10).

4. On substituting (6.9), (6.10) and (6.11) into the first order partial differential forms Φ_δ of I1 we obtain:

$$\Phi_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\frac{\partial y_1}{\partial x^1},\cdots,\frac{\partial y_n}{\partial x^m}\right)=$$

$$B_\delta\left({}^*x^m,\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\frac{\partial F_1}{\partial \eta_1},\cdots,\frac{\partial F_n}{\partial \eta_{m-1}}\right),$$

which can be considered as an identity in all the independent variables ${}^*x^m, \eta_1, \cdots, \eta_{m-1}$.

5. I2, Theorem 5.2 $\Longrightarrow$ that the Φ_δ can always be expressed as:

$$\Phi_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\frac{\partial y_1}{\partial x^1},\cdots,\frac{\partial y_n}{\partial x^m}\right)=$$

$$e^{\xi_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\frac{\partial y_1}{\partial x^1},\cdots,\frac{\partial y_n}{\partial x^m}\right)}\,{}_o\Phi_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\frac{\partial y_1}{\partial x^1},\cdots,\frac{\partial y_n}{\partial x^m}\right), \tag{6.13}$$

where the functions ${}_o\Phi_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\frac{\partial y_1}{\partial x^1},\cdots,\frac{\partial y_n}{\partial x^m}\right)$ are absolute invariants of the group G_1^{E1}.

6. Again, the substitution of (6.9), (6.10) and (6.11) into the right hand side of (6.13) yields:

$$\Phi_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\frac{\partial y_1}{\partial x^1},\cdots,\frac{\partial y_n}{\partial x^m}\right)=$$

$$e^{\sigma_\delta\left({}^*x^m,\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\frac{\partial F_1}{\partial \eta_1},\cdots,\frac{\partial F_n}{\partial \eta_{m-1}}\right)}\,{}_o\phi_\delta\left({}^*x^m,\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\frac{\partial F_1}{\partial \eta_1},\cdots,\frac{\partial F_n}{\partial \eta_{m-1}}\right), \tag{6.14}$$

where the functions ${}_o\phi_\delta\left({}^*x^m,\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\frac{\partial F_1}{\partial \eta_1},\cdots,\frac{\partial F_n}{\partial \eta_{m-1}}\right)$, considered in terms of the variables of G_1^{E1}, are absolute invariants of G_1^{E1} and, in particular,

$${}_o\Phi_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\frac{\partial y_1}{\partial x^1},\cdots,\frac{\partial y_n}{\partial x^m}\right)={}_o\phi_\delta\left({}^*x^m,\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\frac{\partial F_1}{\partial \eta_1},\cdots,\frac{\partial F_n}{\partial \eta_{m-1}}\right). \tag{6.15}$$

7. G1,2,3, Lemma 4.2 and Definition 4.1 $\Longrightarrow$ that for any invariant solutions of the system I1 the functions ${}_o\Phi_\delta$ can be written as:

$${}_o\Phi_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\frac{\partial y_1}{\partial x^1},\cdots,\frac{\partial y_n}{\partial x^m}\right)={}_o\Psi_\delta\left(x^1,\cdots,x^m\right) \tag{6.16}$$

and under the transformations of G_1^{E1}

$$ {}_o\Phi_\delta\left(\bar{x}^1,\dots,\bar{x}^m,\bar{y}_1,\dots,\bar{y}_n,\frac{\partial\bar{y}_1}{\partial\bar{x}^1},\dots,\frac{\partial\bar{y}_n}{\partial\bar{x}^m}\right) = {}_o\Psi_\delta\left(\bar{x}^1,\dots,\bar{x}^m\right), \tag{6.17} $$

where we have used the same functional notation ${}_o\Psi_\delta$ in (6.17) as in (6.16), since, for invariant solutions, the $\bar{y}_\delta$'s are the same functions of the $\bar{x}^1,\dots,\bar{x}^m$ that the y_δ 's are of the $x^1,\dots,x^m$. But, the functions ${}_o\Phi_\delta$ are absolute invariants of the group G_1^{E1}; therefore, from (6.16) and (6.17) we can write:

$$ {}_o\Psi_\delta\left(x^1,\dots,x^m\right) = {}_o\Psi_\delta\left(\bar{x}^1,\dots,\bar{x}^m\right). \tag{6.18} $$

Equations (6.18) tell us that the functions ${}_o\Psi_\delta$ are absolute invariants of the group S_{G_1}; hence, they must be functions of the $m-1$ functionally independent absolute invariants $\eta_1,\dots,\eta_{m-1}$ of S_{G_1}; that is:

$$ {}_o\Psi_\delta\left(x^1,\dots,x^m\right) = {}_o\psi_\delta\left(\eta_1,\dots,\eta_{m-1}\right). \tag{6.19} $$

8. Similarly, by (6.12) and (6.14), for any such invariant solutions of I1 we must have:

$$ {}_o\phi_\delta\left({}^*x^m,\eta_1,\dots,\eta_{m-1},F_1,\dots,F_n,\frac{\partial F_1}{\partial\eta_1},\dots,\frac{\partial F_n}{\partial\eta_{m-1}}\right) = {}_o\phi'_\delta\left({}^*x^m,\eta_1,\dots,\eta_{m-1}\right). \tag{6.20} $$

On using (6.19) and (6.20) in (6.15) the following relation must hold:

$$ {}_o\psi_\delta\left(\eta_1,\dots,\eta_{m-1}\right) = {}_o\phi'_\delta\left({}^*x^m,\eta_1,\dots,\eta_{m-1}\right) $$

but, this implies that the functions ${}_o\phi'_\delta$ must be independent of ${}^*x^m$; since, otherwise, the functions ${}_o\psi_\delta$ could not be absolute invariants of S_{G_1}. On retracing our steps we must then have that the functions ${}_o\phi_\delta$ of (6.14) must be independent of ${}^*x^m$.

9. The equations $\Phi_\delta = 0$ can, on using (6.12), (6.14) and the result

of Step 8, then be expressed as:

$$ {}_{o}\phi_{\delta} = A_{\delta}\left(\eta_1,\dots,\eta_{m-1},F_1,\dots,F_n,\frac{\partial F_1}{\partial \eta_1},\dots,\frac{\partial F_n}{\partial \eta_{m-1}}\right) = 0, \tag{6.21} $$

since the quantities e^{σ_δ} of (6.14) are not identically zero for arbitrary values of the indicated variables.

The invariant solutions of the system of equations I1 can then be obtained in terms of the solutions of the system of equations (6.21). These two solutions are connected, as is evident from the above proof, by the relation G3. This completes the proof of Theorem 6.1.

The following are important specializations of Theorem 6.1.

Corollary 6.1-1

If $m=2$ the invariant solutions of the system of equations (6.1) can be expressed in terms of the solutions of a system of first order ordinary differential equations in n unknowns and one independent variable.

Corollary 6.1-2

If $m=2$ and $n=1$ the system (6.1) will consist of only one first order partial differential equation. Its invariant solutions can then be expressed in terms of the solutions of a first order ordinary differential equation.

VII. SYSTEMS OF SIMULTANEOUS PARTIAL DIFFERENTIAL EQUATIONS OF ANY ORDER IN ANY NUMBER OF INDEPENDENT VARIABLES

7.10 Formulation of the Problem

Suppose we are given a system of partial differential equations of the k'th order in $m \geqslant 2$ independent variables $x^1, \cdots, x^m$ and $n \geqslant 1$ dependent variables $y_1, \cdots, y_n$ of the form:

$$\begin{aligned} &\Phi_1\left(x^1,\dots,x^m, y_1,\dots,y_n, \frac{\partial y_1}{\partial x^1},\dots,\frac{\partial y_1}{\partial x^m},\dots,\frac{\partial^k y_n}{\partial x^{1k}},\dots,\frac{\partial^k y_n}{\partial x^{mk}}\right) = 0,^{*} \\ &\qquad\vdots \\ &\Phi_n\left(x^1,\dots,x^m, y_1,\dots,y_n, \frac{\partial y_1}{\partial x^1},\dots,\frac{\partial y_1}{\partial x^m},\dots,\frac{\partial^k y_n}{\partial x^{1k}},\dots,\frac{\partial^k y_n}{\partial x^{mk}}\right) = 0, \end{aligned} \tag{7.1}$$

where the differential forms $\Phi_1, \cdots, \Phi_n$ are conformally invariant under the k'th enlargement, $G_1{}^{E}k$ (see Section 3.20), of a continuous one-parameter group of transformations.

The question to be answered is: "Can the invariant solutions of the system of equations (7.1) be expressed in terms of the solutions of a system of k'th order partial differential equations in $m-1$ independent variables?" It will be shown that the answer is in the affirmative.

7.20 Methods of Proof

It is evident, upon examination of Appendix I, that the method

*It is not essential for the succeeding investigation to make the restriction that all of the equations of the system be of the same order or that they contain all of the dependent variables.

of proof used in Section VI cannot be used in this case. In fact, for the degree of generality which we here wish to obtain, the use of the above method would make the task considerable and, perhaps, prohibitive.

In this section we shall give an alternate method of proof.

7.30 Formal Investigation of the System (7.1)

The following result will be of help in proving Theorem 7.1.

Lemma 7.1

Given:

G1. A one-parameter continuous group of transformations of the form:

$$G_1: \begin{cases} \bar{x}^i = f^i(x^1,\dots,x^m;a)\ ;\ i=1,\dots,m,\ m\geq 2, \\ \bar{y}_\sigma = f_\sigma(y_\sigma;a)\ ;\ \sigma=1,\dots,n,\ n\geq 1, \end{cases}$$

where: a is a numerical parameter and $a=a_0$ (say) gives the identity transformation.

If. 1. $\eta_1(x^1,\dots,x^m),\dots,\eta_{m-1}(x^1,\dots,x^m)$,
$g_1(y_1,\dots,y_n,x^1,\dots,x^m)$,
$\vdots$
$g_n(y_1,\dots,y_n,x^1,\dots,x^m)$
are $m+n-1$ functionally independent absolute invariants of G_1.

2. The functions

$y_1(x^1,\dots,x^m),\dots,y_n(x^1,\dots,x^m)$

are implicitly defined by the relations:

$F_\sigma(\eta_1,\dots,\eta_{m-1}) = g_\sigma(y_1,\dots,y_n,x^1,\dots,x^m)$,
$\sigma=1,\dots,n$.

3. The functions $F_\sigma(\eta_1,\dots,\eta_{m-1})$ are in Class $C^{(k)}$.

4. $\eta_1,\dots,\eta_{m-1},g_1,\dots,g_n$ are functions in Class $C^{(k)}$.

$\Rightarrow$

R1. The quantities

$$\frac{\partial F_\sigma}{\partial \eta_{j_1}},\dots,\frac{\partial^k F_\sigma}{\partial \eta_{j_1}\cdots\partial \eta_{j_k}}$$

when expressed in terms of the variables of $G_1^{E1},\dots,G_1^{Ek}$ respectively, are a set of absolute invariants of $G_1^{E1},\dots,G_1^{Ek}$ respectively.

---

* $j_1,\dots,j_k$ range over the numbers 1, ..., m-1 in such a way that all possible k'th order derivatives of the are taken into account without repetition.

Proof

1. The method of proof which shall be given does not depend explicitly on the special form of the symbols $U^{(1)}f, \cdots, U^{(n)}f$ of $G_1^{E1}, \ldots, G_1^{En}$ respectively. Nevertheless it is important to note that once the group G_1 is given the, by the construction shown in Section 3.20, the symbols of the enlargements of the group are fixed.

2. We shall restrict ourselves to illustrating the method of proof by proving that the quantities $\frac{\partial F_\delta}{\partial \eta_{j_1}}$ are a set of absolute invariants of G_1^{E1}.

3. G1, I1 $\Rightarrow$, by Lemma 4.3, that there exists a set of " $m-1$ " x 's

$$x^{\alpha_1}, \cdots, x^{\alpha_{m-1}},$$

where the indices $\alpha_1, \cdots, \alpha_{m-1}$ are $m-1$ of the numbers 1, ..., m, such that

$$J\left(\frac{\eta_1, \cdots, \eta_{m-1}}{x^{\alpha_1}, \cdots, x^{\alpha_{m-1}}}\right) \neq 0 \ . \tag{7.2}$$

Let us fix our attention on this particular set and denote them as follows:

$$^*x^i \equiv x^{\alpha_i} \ ; \quad i=1, \cdots, m-1,$$

then by (7.2) and the implicit function theorem the $^*x^i$ can be expressed as functions of the η 's and the remaining m'th variable, say $^*x^{\alpha_m}$, in some particular neighborhood of the $(^*x^1, \cdots, {}^*x^{m-1})$- space.

4. I2-4 $\Rightarrow$ that

$$F_\delta(\eta_1, \cdots, \eta_{m-1}) = g_\delta(y_1, \cdots, y_n, x^1, \cdots, x^m) \tag{7.3}$$

and that the partial derivatives of (7.3) with respect to the $^*x^i$ can be expressed as:

$$\frac{\partial F_\delta}{\partial \eta_j}\frac{\partial \eta_j}{\partial^* x^i} = \frac{\partial g_\delta}{\partial y_\alpha} y_{\alpha,i} + \frac{\partial g_\delta}{\partial^* x^i} \qquad \begin{bmatrix}\alpha,\delta = 1,\cdots,n,\\ i,j=1,\cdots,m-1.\end{bmatrix}. \tag{7.4}$$

By the results of Step 3 the equations (7.4) can be solved for the $\frac{\partial F_\delta}{\partial \eta_j}$ so that:

$$\frac{\partial F_\delta}{\partial \eta_j} = \left(\frac{\partial g_\delta}{\partial y_\alpha} y_{\alpha,i} + \frac{\partial g_\delta}{\partial^* x^i}\right)\frac{\partial^* x^i}{\partial \eta_j}. \tag{7.5}$$

In order to indicate the dependence of the $\frac{\partial F_\delta}{\partial \eta_j}$ on the variables of the group G_1^{E1} we write (7.5) as:

$$f_{\delta j}(y_{\alpha,i}, y_\alpha, x^i) \equiv \frac{\partial F_\delta}{\partial \eta_j} = \left(\frac{\partial g_\delta}{\partial y_\alpha} y_{\alpha,i} + \frac{\partial g_\delta}{\partial^* x^i}\right)\frac{\partial^* x^i}{\partial \eta_j}. \tag{7.6}$$

5. I1,2 $\Longrightarrow$ that (7.3) can also be written as:

$$F_\delta(\bar\eta_1,\cdots,\bar\eta_{m-1}) = g_\delta(\bar y_1,\cdots,\bar y_n,\bar x^1,\cdots,\bar x^m), \tag{7.7}$$

where $\bar\eta_i = \eta_i(\bar x^1,\cdots,\bar x^m)$ and (7.7) is to be considered as implicitly defining the $\bar y$'s in terms of the $\bar x$'s. Then on proceeding as in Step 4 we can write:

$$\frac{\partial \bar F_\delta}{\partial \bar\eta_j} = \left(\frac{\partial \bar g_\delta}{\partial \bar y_\alpha}\bar y_{\alpha,i} + \frac{\partial \bar g_\delta}{\partial^* \bar x^i}\right)\frac{\partial^* \bar x^i}{\partial \bar\eta_j}, \tag{7.8}$$

where:

$$\begin{aligned}\bar F_\delta &\equiv F_\delta(\bar\eta_1,\cdots,\bar\eta_{m-1}),\\ \bar g_\delta &\equiv g_\delta(\bar y_1,\cdots,\bar y_n,\bar x^1,\cdots,\bar x^m),\\ \bar y_{\alpha,i} &\equiv \frac{\partial \bar y_\alpha}{\partial^* \bar x^i},\end{aligned}$$

and the ${}^*\bar x^i$ in $\frac{\partial^* \bar x^i}{\partial \bar\eta_j}$ are to be considered as functions of the $\bar\eta$'s and the $\bar x^{\alpha_m}$.

In analogy with (7.6) equation (7.8) can be written as:

$$\bar f_{\delta j}(\bar y_{\alpha,i}, \bar y_\alpha, \bar x^i) = \left(\frac{\partial \bar g_\delta}{\partial \bar y_\alpha}\bar y_{\alpha,i} + \frac{\partial \bar g_\delta}{\partial^* \bar x^i}\right)\frac{\partial^* \bar x^i}{\partial \bar\eta_j}. \tag{7.9}$$

6. I2 $\Longrightarrow$ that (7.3) can also be written as:

$$F_\delta(\bar{\eta}_1, \cdots, \bar{\eta}_{m-1}) = g_\delta(y_1, \cdots, y_n, x^1, \cdots, x^m);$$

hence, since $\bar{\eta}_j = \eta_j$ and $\dfrac{\partial \bar{F}_\delta(\bar{\eta})}{\partial \bar{\eta}_j} = \dfrac{\partial F_\delta(\eta)}{\partial \eta_j}$,

$$\frac{\partial \bar{F}_\delta}{\partial \bar{\eta}_j} = \left(\frac{\partial g_\delta}{\partial y_\alpha} y_{\alpha,i} + \frac{\partial g_\delta}{\partial^* x^i}\right)\frac{\partial^* x^i}{\partial \eta_j}$$

and

$$\bar{f}_{\delta j}(\bar{y}_{\alpha,i}, \bar{y}_\alpha, \bar{x}^i) = \left(\frac{\partial g_\delta}{\partial y_\alpha} y_{\alpha,i} + \frac{\partial g_\delta}{\partial^* x^i}\right)\frac{\partial^* x^i}{\partial \eta_j}. \tag{7.10}$$

Therefore, on comparing (7.6) and (7.10) we can write:

$$\bar{f}_{\delta j}(\bar{y}_{\alpha,i}, \bar{y}_\alpha, \bar{x}^i) = f_{\delta j}(y_{\alpha,i}, y_\alpha, x^i). \tag{7.11}$$

7. Since the g's are absolute invariants of G_1, the relations (7.8) can also be written as:

$$\frac{\partial \bar{F}_\delta}{\partial \bar{\eta}_j} = \left(\frac{\partial g_\delta}{\partial \bar{y}_\alpha} \bar{y}_{\alpha,i} + \frac{\partial g_\delta}{\partial^* \bar{x}^i}\right)\frac{\partial^* \bar{x}^i}{\partial \bar{\eta}_j}, \tag{7.12}$$

where the g_δ are considered to be functions of the $\bar{y}_\alpha$ and $\bar{x}^i$, so that by (7.9):

$$\bar{f}_{\delta j}(\bar{y}_{\alpha,i}, \bar{y}_\alpha, \bar{x}^i) = f_{\delta j}(\bar{y}_{\alpha,i}, \bar{y}_\alpha, \bar{x}^i), \tag{7.13}$$

since the $\dfrac{\partial^* \bar{x}^i}{\partial \bar{\eta}_j}$ are the same functions of all the $\bar{x}^1, \cdots, \bar{x}^m$ that the $\dfrac{\partial^* x^i}{\partial \eta_j}$ are of all the $x^1, \cdots, x^m$.

From (7.11) and (7.13) we then have that:

$$f_{\delta j}(\bar{y}_{\alpha,i}, \bar{y}_\alpha, \bar{x}^i) = f_{\delta j}(y_{\alpha,i}, y_\alpha, x^i). \tag{7.14}$$

The result (7.14) implies, by the definition of the functions $f_{\delta j}$, that the quantities $\dfrac{\partial F_\delta}{\partial \eta_j}$, when expressed in terms of the variables of G_1^{E1}, are absolute invariants of G_1^{E1}.

8. We shall now show that the quantities $\dfrac{\partial^2 F_\delta}{\partial \eta_{j_1} \partial \eta_{j_2}}$, when expressed in terms of the variables of G_1^{E2}, are absolutely invariant under the transformations of G_1^{E2}. On taking the partial derivatives of (7.5)

with respect to another one of the variables, $^*x^{i_2}$, considered in Step 3 and using the notation of Step 4 we obtain:

$$\frac{\partial^2 F_\delta}{\partial \eta_{j_1} \partial \eta_{j_2}} \frac{\partial \eta_{j_2}}{\partial^* x^{i_2}} = \frac{\partial}{\partial^* x^{i_2}} f_{\delta j_1} (y_{\alpha,i_1}, y_\alpha, x^i) \quad \begin{bmatrix} \alpha, \delta = 1, \cdots, n, \\ i_1, i_2, j_1, j_2 = 1, \cdots, m-1. \end{bmatrix}. \tag{7.15}$$

As in Step 4 the solutions of the equations (7.15) are:

$$\frac{\partial^2 F_\delta}{\partial \eta_{j_1} \partial \eta_{j_2}} = \left[\frac{\partial}{\partial^* x^{i_2}} f_{\delta j_1} (y_{\alpha,i_1}, y_\alpha, x^i) \right] \frac{\partial^* x^{i_2}}{\partial \eta_{j_2}} . \tag{7.16}$$

The right hand side of (7.16) can be expressed in terms of the variables of $G_1{}^{E}2$. That such is the case becomes evident on substituting the right hand side of (7.6) into (7.16) and performing the indicated differentiations. To explicitly denote this dependence define:

$$f_{\delta j_1 j_2} (y_{\alpha,i_1 i_2}, y_{\alpha,i_1}, y_\alpha, x^i) \equiv \frac{\partial^2 F_\delta}{\partial \eta_{j_1} \partial \eta_{j_2}} = \left[\frac{\partial}{\partial^* x^{i_2}} f_{\delta j_1} (y_{\alpha,i_1}, y_\alpha, x^i) \right] \frac{\partial^* x^{i_2}}{\partial \eta_{j_2}} . \tag{7.17}$$

9. Using (7.8) and (7.9) it is also possible to write:

$$\bar{f}_{\delta j_1 j_2} (\bar{y}_{\alpha,i_1 i_2}, \bar{y}_{\alpha,i_1}, \bar{y}_\alpha, \bar{x}^i) \equiv \frac{\partial^2 \bar{F}_\delta}{\partial \bar{\eta}_{j_1} \partial \bar{\eta}_{j_2}} = \left[\frac{\partial}{\partial^* \bar{x}^{i_2}} \bar{f}_{\delta j_1} (\bar{y}_{\alpha,i_1}, \bar{y}_\alpha, \bar{x}^i) \right] \frac{\partial^* \bar{x}^{i_2}}{\partial \bar{\eta}_{j_2}} , \tag{7.18}$$

where we have used the notational conventions defined in Step 5. But, since $\bar{\eta}_{j_1} = \eta_{j_1}$, $\bar{\eta}_{j_2} = \eta_{j_2}$ and $\dfrac{\partial^2 \bar{F}_\delta(\bar{\eta})}{\partial \bar{\eta}_{j_1} \partial \bar{\eta}_{j_2}} = \dfrac{\partial^2 F_\delta(\eta)}{\partial \eta_{j_1} \partial \eta_{j_2}}$, the relation (7.18) can, by (7.17), be expressed as:

$$\bar{f}_{\delta j_1 j_2} (\bar{y}_{\alpha,i_1 i_2}, \bar{y}_{\alpha,i_1}, \bar{y}_\alpha, \bar{x}^i) = f_{\delta j_1 j_2} (y_{\alpha,i_1 i_2}, y_{\alpha,i_1}, y_\alpha, x^i) . \tag{7.19}$$

Also, by (7.13), (7.18) becomes:

$$\begin{aligned} \bar{f}_{\delta j_1 j_2} (\bar{y}_{\alpha,i_1 i_2}, \bar{y}_{\alpha,i_1}, \bar{y}_\alpha, \bar{x}^i) &= \left[\frac{\partial}{\partial^* \bar{x}^{i_2}} f_{\delta j_1} (\bar{y}_{\alpha,i_1}, \bar{y}_\alpha, \bar{x}^i) \right] \frac{\partial^* \bar{x}^{i_2}}{\partial \bar{\eta}_{j_2}} \\ &= f_{\delta j_1 j_2} (\bar{y}_{\alpha,i_1 i_2}, \bar{y}_{\alpha,i_1}, \bar{y}_\alpha, \bar{x}^i) , \end{aligned} \tag{7.20}$$

since the $\dfrac{\partial^* \bar{x}^{i_2}}{\partial \bar{\eta}_{j_2}}$ are exactly the same functions of all the $\bar{x}^1, \cdots, \bar{x}^m$ that the $\dfrac{\partial^* x^{i_2}}{\partial \eta_{j_2}}$ are of the $x^1, \cdots, x^m$ and the $\dfrac{\partial}{\partial^* \bar{x}^{i_2}} f_{\delta j_1}(\bar{y}_{\alpha,i_1}, \bar{y}_\alpha, \bar{x}^i)$ are exactly the same functions of the $\bar{y}_{\alpha,i_1 i_2}, \bar{y}_{\alpha,i_1}, \bar{y}_\alpha, \bar{x}^i$ that the $\dfrac{\partial}{\partial^* x^{i_2}} f_{\delta j_1}(y_{\alpha,i_1}, y_\alpha, x^i)$ are of the $y_{\alpha,i_1 i_2}, y_{\alpha,i_1}, y_\alpha, x^i$.

Combining (7.19) and (7.20) gives us:

$$f_{\delta j_1 j_2}(\bar{y}_{\alpha,i_1 i_2}, \bar{y}_{\alpha,i_1}, \bar{y}_\alpha, \bar{x}^i) = f_{\delta j_1 j_2}(y_{\alpha,i_1 i_2}, y_{\alpha,i_1}, y_\alpha, x^i). \qquad (7.21)$$

The result (7.21) implies, by the definition of the functions $f_{\delta j_1 j_2}$, that the quantities $\dfrac{\partial^2 F_\delta}{\partial \eta_{j_1} \partial \eta_{j_2}}$ (when expressed in terms of the variables of G_1^{E2}) are absolute invariants of G_1^{E2}.

10. Now, to complete the proof, we use an induction argument.

Suppose we have shown that the result is true for $\dfrac{\partial^{k-1} F_\delta}{\partial \eta_{j_1} \cdots \partial \eta_{j_{k-1}}}$, that is, we know that:

$$\bar{f}_{\delta j_1 \cdots j_{k-1}}(\bar{y}_{\alpha,i_1 \cdots i_{k-1}}, \cdots, \bar{y}_{\alpha,i_1}, \bar{y}_\alpha, \bar{x}^i) = f_{\delta j_1 \cdots j_{k-1}}(y_{\alpha,i_1 \cdots i_{k-1}}, \cdots, y_{\alpha,i_1}, y_\alpha, x^i) \qquad (7.22)$$

and

$$\bar{f}_{\delta j_1 \cdots j_{k-1}}(\bar{y}_{\alpha,i_1 \cdots i_{k-1}}, \cdots, \bar{y}_{\alpha,i_1}, \bar{y}_\alpha, \bar{x}^i) = f_{\delta j_1 \cdots j_{k-1}}(\bar{y}_{\alpha,i_1 \cdots i_{k-1}}, \cdots, \bar{y}_{\alpha,i_1}, \bar{y}_\alpha, \bar{x}^i) \ ; \qquad (7.23)$$

where:

$$\bar{f}_{\delta j_1 \cdots j_{k-1}}(\bar{y}_{\alpha,i_1 \cdots i_{k-1}}, \cdots, \bar{y}_{\alpha,i_1}, \bar{y}_\alpha, \bar{x}^i) = \frac{\partial^{k-1} \bar{F}_\delta}{\partial \bar{\eta}_{j_1} \cdots \partial \bar{\eta}_{j_{k-1}}} ,$$

$$f_{\delta j_1 \cdots j_{k-1}}(y_{\alpha,i_1 \cdots i_{k-1}}, \cdots, y_{\alpha,i_1}, y_\alpha, x^i) = \frac{\partial^{k-1} F_\delta}{\partial \eta_{j_1} \cdots \partial \eta_{j_{k-1}}} ,$$

and we have used the notational conventions defined in Step 5.

11. Using the same method that was used to obtain (7.16) is is possible to write:

$$\frac{\partial^k F_\delta}{\partial \eta_{j_1} \cdots \partial \eta_{j_k}} = \left[\frac{\partial}{\partial^* x^{i_k}} f_{\delta j_1 \cdots j_{k-1}}(y_{\alpha, i_1 \cdots i_{k-1}}, \cdots, y_{\alpha, i_1}, y_\alpha, x^i)\right] \frac{\partial^* x^{i_k}}{\partial \eta_{j_k}} \tag{7.24}$$

and, as in the case of (7.17),

$$\frac{\partial^k \overline{F}_\delta}{\partial \overline{\eta}_{j_1} \cdots \partial \overline{\eta}_{j_k}} = \left[\frac{\partial}{\partial^* \overline{x}^{i_k}} \overline{f}_{\delta j_1 \cdots j_{k-1}}(\overline{y}_{\alpha, i_1 \cdots i_{k-1}}, \cdots, \overline{y}_{\alpha, i_1}, \overline{y}_\alpha, \overline{x}^i)\right] \frac{\partial^* \overline{x}^{i_k}}{\partial \overline{\eta}_{j_k}} \; . \tag{7.25}$$

As can be seen by inspection the right hand sides of (7.24) and (7.25) contain the variables of the group $G_1^{E_k}$. Denote this fact by writing:

$$f_{\delta j_1 \cdots j_k}(y_{\alpha, i_1 \cdots i_k}, \cdots, y_{\alpha, i_1}, y_\alpha, x^i) = \frac{\partial^k F_\delta}{\partial \eta_{j_1} \cdots \partial \eta_{j_k}}$$

and

$$\overline{f}_{\delta j_1 \cdots j_k}(\overline{y}_{\alpha, i_1 \cdots i_k}, \cdots, \overline{y}_{\alpha, i_1}, \overline{y}_\alpha, \overline{x}^i) = \frac{\partial^k \overline{F}_\delta}{\partial \overline{\eta}_{j_1} \cdots \partial \overline{\eta}_{j_k}} \; .$$

Since $\overline{\eta}_{j_\omega} = \eta_{j_\omega}$ $(\omega = 1, \cdots, k)$ and $\dfrac{\partial^k \overline{F}_\delta}{\partial \overline{\eta}_{j_1} \cdots \partial \overline{\eta}_{j_k}} = \dfrac{\partial^k F_\delta}{\partial \eta_{j_1} \cdots \partial \eta_{j_k}}$ the equations (7.24) and (7.25) yield the relation

$$\overline{f}_{\delta j_1 \cdots j_k}(\overline{y}_{\alpha, i_1 \cdots i_k}, \cdots, \overline{y}_{\alpha, i_1}, \overline{y}_\alpha, \overline{x}^i) = f_{\delta j_1 \cdots j_k}(y_{\alpha, i_1 \cdots i_k}, \cdots, y_{\alpha, i_1}, y_\alpha, x^i) \; . \tag{7.26}$$

On substituting the right hand side of (7.23) into (7.25) we can write:

$$\overline{f}_{\delta j_1 \cdots j_k}(\overline{y}_{\alpha, i_1 \cdots i_k}, \cdots, \overline{y}_{\alpha, i_1}, \overline{y}_\alpha, \overline{x}^i) = \left[\frac{\partial}{\partial^* \overline{x}^{i_k}} f_{\delta j_1 \cdots j_{k-1}}(\overline{y}_{\alpha, i_1 \cdots i_{k-1}}, \cdots, \overline{y}_{\alpha, i_1}, \overline{y}_\alpha, \overline{x}^i)\right] \frac{\partial^* \overline{x}^{i_k}}{\partial \overline{\eta}_{j_k}} \; . \tag{7.27}$$

But the $\dfrac{\partial^* \overline{x}^{i_k}}{\partial \overline{\eta}_{j_k}}$ are exactly the same functions of $\overline{x}^1, \cdots, \overline{x}^m$ that the $\dfrac{\partial^* x^{i_k}}{\partial \eta_{j_k}}$ are of the $x^1, \cdots, x^m$ similarly, the $\dfrac{\partial}{\partial^* \overline{x}^{i_k}} f_{\delta j_1 \cdots j_{k-1}}(\overline{y}_{\alpha, i_1 \cdots i_{k-1}}, \cdots, \overline{y}_{\alpha, i_1}, \overline{y}_\alpha, \overline{x}^i)$ are exactly the same functions of $\overline{y}_{\alpha, i_1 \cdots i_k}, \cdots, \overline{y}_{\alpha, i_1}, \overline{y}_\alpha, \overline{x}^i$ that the $\dfrac{\partial}{\partial^* x^{i_k}} f_{\delta j_1 \cdots j_{k-1}}(y_{\alpha, i_1 \cdots i_{k-1}}, \cdots, y_{\alpha, i_1}, y_\alpha, x^i)$ are of $y_{\alpha, i_1 \cdots i_k}, \cdots, y_{\alpha, i_1}, y_\alpha, x^i$. Hence, in accordance with our previous notational conventions, the following relation must hold:

$$\bar{f}_{\delta j_1 \cdots j_k}\left(\bar{y}_{\alpha, i_1 \cdots i_k}, \cdots, \bar{y}_{\alpha, i_1}, \bar{y}_\alpha, \bar{x}^i\right) = f_{\delta j_1 \cdots j_k}\left(\bar{y}_{\alpha, i_1 \cdots i_k}, \cdots, \bar{y}_{\alpha, i_1}, \bar{y}_\alpha, \bar{x}^i\right). \qquad (7.28)$$

The relations (7.26) and (7.28) imply that

$$f_{\delta j_1 \cdots j_k}\left(\bar{y}_{\alpha, i_1 \cdots i_k}, \cdots, \bar{y}_{\alpha, i_1}, \bar{y}_\alpha, \bar{x}^i\right) = f_{\delta j_1 \cdots j_k}\left(y_{\alpha, i_1 \cdots i_k}, \cdots, y_{\alpha, i_1}, y_\alpha, x^i\right),$$

but, by the definition of absolute invariants and our notational conventions the above relation also implies that the $\dfrac{\partial^k F_\delta}{\partial \eta_{j_1} \cdots \partial \eta_{j_k}}$ are absolute invariants of the group $G_1^{E_k}$. This fact has been shown to hold for (say) $\ell = 1, 2$ and for $\ell = k$, if it holds for $\ell = k-1$. In particular, if we take $\ell = 2$ it must hold for $\ell = 3$, etc.

Hence, the quantities in each of the sets:

$$\frac{\partial F_\delta}{\partial \eta_{j_1}}, \; \cdots, \; \frac{\partial^k F_\delta}{\partial \eta_{j_1} \cdots \partial \eta_{j_k}} \qquad \begin{bmatrix} \delta = 1, \cdots, n, \\ j_1, \cdots, j_k = 1, \cdots, m-1. \end{bmatrix},$$

when expressed in terms of the variables of $G_1^{E_1}$, ... $G_1^{E_k}$, are absolute invariants of the groups $G_1^{E_1}$, ..., $G_1^{E_k}$ respectively. This completes the proof of Lemma 7.1.

Theorem 7.1

Given:

G1. A one-parameter continuous group of transformations:

$$G_1: \begin{array}{l} \bar{x}^i = f^i(x^1,\cdots,x^m;a);\ i=1,\cdots,m,\ m\geq 2, \\ \bar{y}_\delta = f_\delta(y_\delta;a);\ \delta=1,\cdots,n;\ n\geq 1, \end{array}$$

where: a is a numerical parameter and $a=a_0$ (say) gives the identity transformation.

G2. $m+n-1$ functionally independent absolute invariants of G_1 of the form:

$$\left.\begin{array}{l} \eta_1(x^1,\cdots,x^m),\ \cdots,\ \eta_{m-1}(x^1,\cdots,x^m), \\ g_1(y_1,\cdots,y_n,x^1,\cdots,x^m),\cdots,g_n(y_1,\cdots,y_n,x^1,\cdots,x^m) \end{array}\right\} \epsilon C^{(k)}.$$

G3. The functions $y_1(x^1,\cdots,x^m),\cdots,y_n(x^1,\cdots,x^m)$ are implicitly defined by the relations:

$$F_\delta(\eta_1,\cdots,\eta_{m-1}) = g_\delta(y_1,\cdots,y_n,x^1,\cdots,x^m),$$

where: $F_\delta \in C^{(k)},\ \delta=1,\cdots,n.$

Statement of Theorem:

If. 1.

$$\Phi_1\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\cdots,\frac{\partial^k y_n}{\partial x^{1k}},\cdots,\frac{\partial^k y_n}{\partial x^{mk}}\right)=0,$$
$$\vdots$$
$$\Phi_n\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\cdots,\frac{\partial^k y_n}{\partial x^{1k}},\cdots,\frac{\partial^k y_n}{\partial x^{mk}}\right)=0$$

is a system of partial differential equations of the k'th ($k\geq 1$) order in $m\geq 2$ independent variables $x^1,\cdots,x^m$ and $n\geq 1$ dependent varaibles $y_1,\cdots,y_n$.

2. Each of the differential forms $\Phi_1,\cdots,\Phi_n$ is conformally invariant under the k'th enlargement, G_1E_k, of the group G_1.

$\Rightarrow$

R1. The invariant solutions of the system of equations $\Phi_\delta=0$ can be expressed in terms of the solutions of the system:

$$A_1\left(\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\cdots,\frac{\partial^k F_n}{\partial \eta_1^k},\cdots,\frac{\partial^k F_n}{\partial \eta_{m-1}^k}\right)=0,$$
$$\vdots$$
$$A_n\left(\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\cdots,\frac{\partial^k F_n}{\partial \eta_1^k},\cdots,\frac{\partial^k F_n}{\partial \eta_{m-1}^k}\right)=0$$

- a system of k'th order partial differential equations in $m-1$ independent variables $\eta_1,\cdots,\eta_{m-1}$.

Proof

1. G1,2 $\Rightarrow$, on using the notation of Step 3, Lemma 7.1; that " $m-1$ " of the x's can be expressed in terms of the " $m-1$ " absolute invariants $\eta_1, \cdots, \eta_{m-1}$ and the remaining x, say ${}^*x^m$. Hence, it is possible to write

$$ {}^*x^i = h^i(\eta_1, \cdots, \eta_{m-1}, {}^*x^m) \, , \; i=1, \cdots, m-1. \tag{7.29} $$

2. G1,2,3 $\Rightarrow$ that the y's can be expressed as functions of the F's and the x's, that is:

$$ y_\delta = \lambda_\delta (F_1, \cdots, F_n, x^1, \cdots, x^m) \, , \; \delta=1, \cdots, n, \tag{7.30} $$

since, by Lemma 4.3, $J\left(\dfrac{g_1, \cdots, g_n}{y_1, \cdots, y_n}\right) \neq 0$. The equations (7.30) may, on using (7.29), also be written as:

$$ y_\delta = \Omega_\delta (F_1, \cdots, F_n, \eta_1, \cdots, \eta_{m-1}, {}^*x^m). \tag{7.31} $$

3. On taking the first partial derivatives of (7.31) with respect to the x's and using (7.29) it is possible to write:

$$ y_{\alpha, j_1} = \Omega_{\alpha j_1} \left(\frac{\partial F_\delta}{\partial \eta_{i_1}}, F_\delta, \eta_{i_1}, {}^*x^m \right) \qquad \begin{bmatrix} \alpha, \delta = 1, \cdots, n, \\ i_1 = 1, \cdots, m-1, \\ j_1 = 1, \cdots, m. \end{bmatrix}, \tag{7.32} $$

where the notation indicates that the functions $\Omega_{\alpha j_1}$ may depend on all the variables $\dfrac{\partial F_\delta}{\partial \eta_{i_1}}$, F_δ , η_{i_1} and ${}^*x^m$. Similarly, the k'th order partial derivatives of the y's with respect to the x's may also be expressed as:

$$ y_{\alpha, j_1 \cdots j_k} = \Omega_{\alpha j_1 \cdots j_k} \left(\frac{\partial^k F_\delta}{\partial \eta_{i_1} \cdots \partial \eta_{i_k}}, \cdots, \frac{\partial F_\delta}{\partial \eta_{i_1}}, F_\delta, \eta_{i_1}, {}^*x^m \right). \tag{7.33} $$

4. The substitution of (7.29), (7.31) and expressions of the form (7.33) into each of the partial differential equations Φ_δ of I1 yields the following relations:

$$\Phi_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\cdots,\frac{\partial^k y_n}{\partial x^{1k}},\cdots,\frac{\partial^k y_n}{\partial x^{mk}}\right)= \tag{7.34}$$
$$B_\delta\left({}^*x^m,\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\cdots,\frac{\partial^k F_n}{\partial\eta_1^k},\cdots,\frac{\partial^k F_n}{\partial\eta_{m-1}^k}\right),$$

which may be considered as identities in all of the independent variables ${}^*x^m, \eta_1, \cdots, \eta_{m-1}$.

5. I2, Theorem 5.2 $\Rightarrow$ that the Φ_δ can always be expressed as:

$$\Phi_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\cdots,\frac{\partial^k y_n}{\partial x^{1k}},\cdots,\frac{\partial^k y_n}{\partial x^{mk}}\right)= \tag{7.35}$$
$$e^{\zeta_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\cdots,\frac{\partial^k y_n}{\partial x^{1k}},\cdots,\frac{\partial^k y_n}{\partial x^{mk}}\right)}\,{}_0\Phi_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\cdots,\frac{\partial^k y_n}{\partial x^{1k}},\cdots,\frac{\partial^k y_n}{\partial x^{mk}}\right),$$

where the functions ${}_0\Phi_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\cdots,\frac{\partial^k y_n}{\partial x^{1k}},\cdots,\frac{\partial^k y_n}{\partial x^{mk}}\right)$ are absolute invariants of the group $G_1{}^{E_k}$.

On using the relations (7.29), (7.31) and expressions of the form (7.33) we obtain, by (7.34), that:

$$B_\delta\left({}^*x^m,\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\cdots,\frac{\partial^k F_n}{\partial\eta_1^k},\cdots,\frac{\partial^k F_n}{\partial\eta_{m-1}^k}\right)= \tag{7.36}$$
$$e^{\sigma_\delta\left({}^*x^m,\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\cdots,\frac{\partial^k F_n}{\partial\eta_1^k},\cdots,\frac{\partial^k F_n}{\partial\eta_{m-1}^k}\right)}\,{}_0\phi_\delta\left({}^*x^m,\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\cdots,\frac{\partial^k F_n}{\partial\eta_1^k},\cdots,\frac{\partial^k F_n}{\partial\eta_{m-1}^k}\right),$$

where the functions ${}_0\phi_\delta$ of the indicated arguments, when considered in terms of the variables $G_1{}^{E_k}$, are absolute invariants of the group $G_1{}^{E_k}$ and, in particular,

$${}_0\Phi_\delta\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\cdots,\frac{\partial^k y_n}{\partial x^{1k}},\cdots,\frac{\partial^k y_n}{\partial x^{mk}}\right)={}_0\phi_\delta\left({}^*x^m,\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\cdots,\frac{\partial^k F_n}{\partial\eta_1^k},\cdots,\frac{\partial^k F_n}{\partial\eta_{m-1}^k}\right). \tag{7.37}$$

6. G1,2,3, Lemma 4.2 and Definition 4.1 $\Rightarrow$ that a repetition of the arguments of Steps 7 and 8 of Theorem 6.1 will show that the functions ${}_0\phi_\delta$ of (7.37) are independent of ${}^*x^m$ in the case where we are considering the invariant solutions of the system I1. Hence, the

equations $\Phi_\delta = 0$ can, on using (7.34) and (7.36) be written as:

$$ {}_0\phi_\delta = A_\delta\left(\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\cdots,\frac{\partial^k F_n}{\partial \eta_1^k},\cdots,\frac{\partial^k F_n}{\partial \eta_{m-1}^k}\right) = 0 \,, \qquad (7.38) $$

since the functions e^{σ_δ} of (7.36) are not identically zero for arbitrary values of the indicated variables.

The invariant solutions of the system of partial differential equations I1 can then be obtained in terms of the solutions of the system of equation (7.38). These two solutions are connected by the relation

$$ F_\delta(\eta_1,\cdots,\eta_{m-1}) = g_\delta(y_1,\cdots,y_n,x^1,\cdots,x^m) $$

as is evident from the above proof. This completes the proof of Theorem 7.1.

The following important specializations of Theorem 7.1 are immediately obtained.

<u>Corollary 7.1-1</u>

If $m=2$ and $n=1$ the invariant solutions of the equations (7.1) can be expressed in terms of the solutions of a k'th order ordinary differential equation with independent variable η and dependent variable $F(\eta)$.

<u>Corollary 7.1-2</u>

If $m=2$ the invariant solutions of the system of partial differential equations (7.1) can be expressed in terms of the solutions of a system of ordinary differential equations of the k'th order with independent variable η and dependent variables $F_1(\eta),\cdots,F_n(\eta)$.

The following theorem establishes the connection between a system of partial differential equations in " $m-1$ " independent variables and a system of the form (7.1).

Theorem 7.2

Given:

G1. A one-parameter continuous group of transformations of the form:

$$G_1: \begin{array}{l} \bar{x}^i = f^i(x^1,\cdots,x^m;a)\,;\ i=1,\cdots,m,\ m\geqslant 2, \\ \bar{y}_\delta = f_\delta(y_\delta;a)\,;\ \delta=1,\cdots,n,\ n\geqslant 1, \end{array}$$

where: a is a numerical parameter and $a=a_0$ (say) gives the identity transformation.

G2. $m+n-1$ functionally independent absolute invariants of G_1 of the form:

$$\left.\begin{array}{l} \eta_1(x^1,\cdots,x^m),\cdots,\eta_{m-1}(x^1,\cdots,x^m) \\ g_1(y_1,\cdots,y_n,x^1,\cdots,x^m),\cdots,g_n(y_1,\cdots,y_n,x^1,\cdots,x^m) \end{array}\right\}\in C^{(k)}.$$

G3. The functions $y_1(x^1,\cdots,x^m),\cdots,y_n(x^1,\cdots,x^m)$ are implicitly defined by the relations

$$F_\delta(\eta_1,\cdots,\eta_{m-1}) = g_\delta(y_1,\cdots,y_n,x^1,\cdots,x^m),$$

where $F_\delta\ \epsilon\ C^{(k)},\ \delta=1,\cdots,n.$

Statement of Theorem

If. 1.

$$\left.\begin{array}{l} A_1\left(\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\cdots,\dfrac{\partial^k F_n}{\partial\eta_1^k},\cdots,\dfrac{\partial^k F_n}{\partial\eta_{m-1}^k}\right)=0, \\ \vdots \\ A_n\left(\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\cdots,\dfrac{\partial^k F_n}{\partial\eta_1^k},\cdots,\dfrac{\partial^k F_n}{\partial\eta_{m-1}^k}\right)=0 \end{array}\right.$$

is a system of partial differential equations of the k'th order in $m-1$ independent variables $\eta_1,\cdots,\eta_{m-1}$ and $n\geqslant 1$ dependent variables $F_1,\cdots,F_n$.

$\rightarrow$

R1. To the system of partial differential equations of I1 there corresponds a system of k'th order partial differential equations:

$$\begin{array}{l} \Phi_1\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\cdots,\dfrac{\partial^k y_n}{\partial x^{1k}},\cdots,\dfrac{\partial^k y_n}{\partial x^{mk}}\right)=0, \\ \vdots \\ \Phi_n\left(x^1,\cdots,x^m,y_1,\cdots,y_n,\cdots,\dfrac{\partial^k y_n}{\partial x^{1k}},\cdots,\dfrac{\partial^k y_n}{\partial x^{mk}}\right)=0 \end{array}$$

in $m\geqslant 2$ independent variables $x^1,\cdots,x^m$ and the $n\geqslant 1$ dependent variables $y_1,\cdots,y_n$.

2. Each of the differential forms $\Phi_1,\cdots,\Phi_n$ is absolutely invariant under the transformations of G_1^{Ek}.

Proof

Proof of R1

1. G3 $\Rightarrow$ that the expressions $\frac{\partial F_\delta}{\partial \eta_{j_1}}, \cdots, \frac{\partial^k F_\delta}{\partial \eta_{j_1} \cdots \partial \eta_{j_k}}$ can be constructed in terms of the variables of $G_1{}^{E}1$, ..., $G_1{}^{E}k$ respectively.

2. G2 $\Rightarrow$ that the η's are known functions of $x^1, \cdots, x^m$, hence their partial derivatives, up to any required order, are also known functions of the x's.

3. By Steps 1 and 2 the left hand sides of the equations I1 can then be expressed in terms of the x's, y's and the partial derivatives of the y's with respect to the x's up to the k'th order. The resulting expressions can then be written as:

$$\begin{aligned} \Phi_1\left(x^1, \cdots, x^m, y_1, \cdots, y_n, \cdots, \frac{\partial^k y_n}{\partial x^{1k}}, \cdots, \frac{\partial^k y_n}{\partial x^{mk}}\right) &= 0, \\ &\vdots \\ \Phi_n\left(x^1, \cdots, x^m, y_1, \cdots, y_n, \cdots, \frac{\partial^k y_n}{\partial x^{1k}}, \cdots, \frac{\partial^k y_n}{\partial x^{mk}}\right) &= 0 \end{aligned} \tag{7.39}$$

- a system of k'th order partial differential equations with $m \geqslant 2$ independent variables $x^1, \cdots, x^m$ and $n \geqslant 1$ dependent variables $y_1, \cdots, y_n$.

4. If the $F_\delta(\eta_1, \cdots, \eta_{m-1})$, $\delta = 1, \cdots, n$, are a set of solutions of the system of equations I1 it is evident from the construction described in Steps 1-3 that the functions $y_\delta(x^1, \cdots, x^m)$, implicitly defined by the relations of G3, must satisfy the system of partial differential equations (7.39).

<u>Thus the system of equations I1 is transformed into the system of partial differential equations (7.39) through the relations G3.</u> This proves R1.

Proof of R2

1. To see that each of the differential forms $\Phi_1, \cdots, \Phi_n$ of (7.39) is absolutely invariant under the transformations of $G_1^{E_k}$ we proceed as follows:

By Lemma 7.1 each of the sets of quantities $\frac{\partial F_\delta}{\partial \eta_{j_1}}, \cdots, \frac{\partial^k F_\delta}{\partial \eta_{j_1} \cdots \partial \eta_{j_k}}$ forms a set of absolute invariants of $G_1^{E_1}, \ldots, G_1^{E_k}$ respectively. Hence, in view of G2, each of the differential forms $A_1, \cdots, A_n$ of the system I1 is absolutely invariant under the transformations of $G_1^{E_k}$. Since the partial differential forms Φ_δ of (7.39) were constructed from the forms A_δ of I1 it is then evident that the Φ_δ must also be absolutely invariant under the transformations of $G_1^{E_k}$. This proves R2.

Corollary 7.2-1

Under the conditions of Theorem 7.2 the correspondence $A_\delta \rightarrow \Phi_\delta$ $(\delta = 1, \cdots, n)$ is unique with respect to fixed invariants $\eta_1, \cdots, \eta_{m-1}$, $g_1, \cdots, g_n$ of G_1.

Proof

1. We proceed by considering the manner in which the system of partial differential equations (7.39) has been constructed. The following facts are evident:

(a) For any set of solutions $F_\delta(\eta_1, \cdots, \eta_{m-1})$ of the equations $A_\delta = 0$ $(\delta = 1, \cdots, n)$ and for the given η's the quantities $\frac{\partial F_\delta}{\partial \eta_{j_1}}, \cdots, \frac{\partial^k F_\delta}{\partial \eta_{j_1} \cdots \partial \eta_{j_k}}$ are fixed functions of the x's.

(b) By (a) the form of the Φ_δ of (7.39) can only depend on the form of the absolute invariants $g_\delta(y_1, \cdots, y_n, x^1, \cdots, x^m)$ of G_1. Hence, for given absolute invariants g_δ, the differential forms Φ_δ are fixed.

2. Combining the facts stated in 1(a) and (b) then shows that the method of Theorem 7.2 establishes a unique correspondence $A_\delta \rightarrow \Phi_\delta$ once the absolute invariants $\eta_1, \cdots, \eta_{m-1}, g_1, \cdots, g_n$ are given. This proves Corollary 7.2-1.

7.40 The Effect of a Change in the Invariants

Suppose that the relations G3 of Theorem 7.1 are replaced by:

$$G_\delta(\xi_1, \cdots, \xi_{m-1}) = g_\delta(y_1, \cdots, y_n, x^1, \cdots, x^m), \tag{7.40}$$

where

$$\xi_j = f_j(\eta_1, \cdots, \eta_{m-1}), \; j=1, \cdots, m-1, \tag{7.41}$$

$$J\left(\frac{\xi_1, \cdots, \xi_{m-1}}{\eta_1, \cdots, \eta_{m-1}}\right) \neq 0,$$

and the functions G_δ are in Class $C^{(k)}$ with respect to their arguments.

It is then natural to ask: "What is the effect of using the relations (7.40) instead of the relations G3 of Theorem 7.1 on the system of partial differential equations obtained from (7.1)?" This question is answered by the following theorem.

Theorem 7.3

If the relations G3 of Theorem 7.1 are replaced by (7.40) then the system of partial differential equations corresponding to I1 of Theorem 7.1 is of the form:

$$B_\delta\left(\xi_1, \cdots, \xi_{m-1}, G_1, \cdots, G_n, \cdots, \frac{\partial^k G_n}{\partial \xi_1^k}, \cdots, \frac{\partial^k G_n}{\partial \xi_{m-1}^k}\right) = 0, \; \delta=1, \cdots, n. \tag{7.42}$$

Also, the system (7.42) is related to the system

$$A_\delta\left(\eta_1,\cdots,\eta_{m-1},F_1,\cdots,F_n,\cdots,\frac{\partial^k F_n}{\partial\eta_1^k},\cdots,\frac{\partial^k F_n}{\partial\eta_{m-1}^k}\right)=0 \qquad (7.43)$$

of Theorem 7.1 by the change of independent variables (7.41).

Proof

1. The proof of the first part of the theorem follows immediately on consideration of the method of proof used in Lemma 7.1. Clearly, as a consequence of our construction, the result of Lemma 7.1 will remain unaltered on replacing the F's by the G's and the η's by the ξ's since the ξ's are also absolute invariants of G_1; that is, they also satisfy the equation $U^{(0)}f=0$. Consequently, the result of Theorem 7.1 will also remain unaltered on replacing the F's by the G's and the η's by the ξ's. The system of partial differential equations (7.1) can then be expressed as:

$$B_\delta\left(\xi_1,\cdots,\xi_{m-1};G_1,\cdots,G_n,\cdots,\frac{\partial^k G_n}{\partial\xi_1^k},\cdots,\frac{\partial^k G_n}{\partial\xi_{m-1}^k}\right)=0\ ,$$

where the functional relations (7.42) have been denoted by B_δ instead of A_δ, as in Theorem 7.1, since they are not necessarily the same.

2. To prove the second part of the theorem we note that the equations (7.42) can, by (7.41), be expressed entirely in terms of the η's, that is, as:

$$C_\delta\left(\eta_1,\cdots,\eta_{m-1},\overline{G}_1,\cdots,\overline{G}_n,\cdots,\frac{\partial^k\overline{G}_n}{\partial\eta_1^k},\cdots,\frac{\partial^k\overline{G}_n}{\partial\eta_{m-1}^k}\right)=0\ , \qquad (7.44)$$

where the $\overline{G}$'s are now considered as functions of the η's, that is:

$$\overline{G}_\delta(\eta_1,\cdots,\eta_{m-1}) = G_\delta\left(f_1(\eta_1,\cdots,\eta_{m-1}),\cdots,f_{m-1}(\eta_1,\cdots,\eta_{m-1})\right).$$

The functional relations among the arguments of (7.44) are denoted by $\mathcal{C}_\delta$ instead of $\mathcal{B}_\delta$, as in (7.42), to emphasize their difference. It should be noted that the systems of equations (7.42) and (7.44) are related by the change of independent variables (7.41).

3. Suppose now that the functions $\mathcal{C}_\delta$ denote different functional relations among their arguments than those indicated by the functions $\mathcal{A}_\delta$. By Corollary 7.2-1 there corresponds to (7.44) a different system of partial differential equations than that which corresponds to (7.43). This contradicts the hypothesis that (7.42) and (7.43) were obtained from the same system of partial differential equations. Therefore, the systems of partial differential equations (7.42) and (7.43) must, by Step 2, be related by the change of independent variables (7.41). This completes the proof of Theorem 7.3.

The above result assures us that, under the conditions of Theorem 7.1, the invariant solutions of the system (7.1) can always be expressed in terms of the solutions of a k'th order system of partial differential equations in $m-1$ independent variables irrespective of the choice of the $m-1$ absolute invariants of S_{G_1}. Furthermore these equations are related by changes of the independent variables.

7.50 Practical Application of the Previously Developed Theory

For purposes of discussion let us restrict ourselves to the consideration of a single partial differential equation of any order in two independent variables x^1, x^2 and one dependent variable y.

To determine the similarity equation which corresponds to the given partial differential equation we proceed as follows:

1. Determine a continuous one-parameter group of transformations under which the partial differential equation is invariant.*

2. Determine two functionally independent absolute invariants of the group obtained in Step 1 of the form:

$$\eta(x^1, x^2) \quad \text{and} \quad g(y, x^1, x^2)$$

and the solution

$$y = \lambda\left(F(\eta), x^1, x^2\right) \tag{7.45}$$

of the equation

$$g(y, x^1, x^2) = F(\eta).$$

That such an inversion can, in principle, be performed is assured by Lemma 4.4.

3. Check the boundary conditions and see if they can be expressed in terms of $F(\eta)$ and its derivatives. If such is the case the problem can be solved in terms of the solutions of the similarity equation and we can proceed to Step 4.

4. The substitution of (7.45) into the partial differential equation will, by Corollary 7.1-1, give an ordinary differential equation - the desired similarity equation. Such a procedure will, by Lemma 4.2 and Definition 4.1, determine the invariant (under the group obtained in Step 1) solutions of the given partial differential equation.

<u>It is then clear that the preceeding theory establishes a</u>

*Depending on the boundary conditions which must be satisfied it is advantageous in some cases to demand constant conformal, instead of absolute, invariance of the differential form Φ in the equation $\Phi = 0$.

practical and rigorous method of determining similarity equations and that such a method has marked advantages over any of the previously known trial and error methods. The procedure outlined above will be illustrated in Section VIII.

VIII. ILLUSTRATION OF THE GENERAL THEORY BY SOME KNOWN EXAMPLES OF SIMILARITY SOLUTIONS IN HYDRODYNAMICS

In this section we shall give a few examples of some known similarity solutions in hydrodynamics in order to illustrate the general theory developed in the previous sections. The treatment will reveal the very simple group theoretical motivation which exists for the choice of similarity parameters. The table at the end of Section IX supplements the succeeding examples.

8.10 The Boundary Layer Over a Flat Plate with Pressure Gradient Produced by an External Velocity of the Form $U=Ax^q$

This problem was first posed by Falkner and Skan (Ref. 9) and the numerical solutions of the resulting ordinary differential equation have been exhaustively studied by Hartree (Ref. 10). A short account of this work is given by Goldstein (Ref. 11, p. 140).

We wish to find a solution of the Prandtl boundary layer equations (Ref. 11, p. 118):

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = U\frac{\partial U}{\partial x} + \nu \psi_{yyy}\,, \tag{8.1}$$

where:

$\psi(x,y)$ = Stream function.

U = Ax^q (A,q real constants),

= The velocity of the fluid in the x-direction external to the boundary layer,

$U \;=\; \frac{\partial \psi}{\partial y} \;=\; x$ -velocity component

of fluid within the boundary layer,

$V \;=\; -\frac{\partial \psi}{\partial x} \;=\; y$ -velocity component

of fluid within the boundary layer,

ν = Kinematic coefficient of viscosity,

satisfying the boundary conditions:

$$U = 0 \quad at \quad y = 0\,,\; x \geqslant 0\,, \tag{8.2}$$

$$V = 0 \quad at \quad y = 0\,,\; x \geqslant 0\,, \tag{8.3}$$

$$\lim_{y \to \infty} U = A x^{q}. \tag{8.4}$$

For the particular case under consideration (8.1) can be written as:

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} - q A^2 x^{2q-1} = 0 \tag{8.5}$$

The equation (8.5) is constant conformally invariant under the continuous one-parameter group of transformations

$$\bar{x} = a^m x\,,\quad \bar{y} = a^n y\,,\quad \bar{\psi} = a^p \psi\,, \tag{8.6}$$

where:

a = the numerical parameter $(\neq 0)$,

m, n, p = constants to be determined,

if

$$\frac{n}{m} = \frac{1}{2}(1-q) \tag{8.7}$$

and

$$\frac{p}{m} = \frac{1}{2}(1+q)\,. \tag{8.8}$$

Now, a set of absolute invariants of (8.6), corresponding to $\eta(x^1, x^2)$ and $g(y, x^1, x^2)$ of the general theory, is given by:

$$\frac{y}{x^{n/m}} \tag{8.9}$$

and

$$\frac{\psi}{x^{P/m}} \; . \tag{8.10}$$

Hence, by Corollary 7.1-1, (8.5) will be reduced to an ordinary differential equation (which, by Lemma 4.2, defines the invariant solutions of (8.5)) on using the change of variables:

$$\eta = \frac{y}{x^{\frac{1}{2}(1-q)}} \tag{8.11}$$

and

$$\psi = x^{\frac{1}{2}(1+q)} f(\eta) \; , \tag{8.12}$$

obtained by substituting (8.7) into (8.9) and (8.8) into (8.10).

The ordinary differential (similarity) equation corresponding to (8.5) need not be derived at this stage. Of greater practiaal importance is the checking of the boundary conditions in order to see whether the problem can be solved in terms of the solutions of the similarity equation. This can easily be done, for on using (8.11) and (8.12) the boundary conditions (8.2)-(8.4) can be expressed as:

$$\left.\frac{df}{d\eta}\right)_{\eta=0} = 0 \; ,$$

$$f(0) = 0 \; ,$$

$$\lim_{\eta \to \infty} \frac{df}{d\eta} = A \; .$$

The above conditions are in terms of $f(\eta)$ and its derivatives; therefore, the problem can be solved by means of the solutions of the similarity equation

$$\nu \frac{d^3 f}{d\eta^3} + \frac{1}{2}(1+q) f(\eta) \frac{d^2 f}{d\eta^2} - q\left(\frac{df}{d\eta}\right)^2 = -qA^2 \; , \tag{8.13}$$

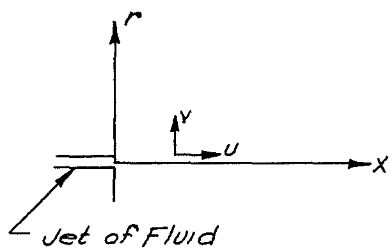
obtained from (8.5) on using (8.11) and (8.12). As a check we note that for $q=0$ the equation (8.13) reduces to the Blasius similarity equation for a boundary layer with no pressure gradient.

8.20 The Spread of a Cylindrical Jet

This problem was originally formulated and solved by Schlichting (Ref. 12) and an account of it is also given by Goldstein (Ref. 11, p. 145) and Durand (Ref. 13, p. 175).

The situation is shown in Fig. 1.

FIG. 1.



The equations applicable in this case are the Prandtl boundary layer equations expressed in cylindrical coordinates, namely:

$$\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)-\frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)=\frac{\nu}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right\}, \tag{8.14}$$

where:

$\psi(x,r)$ = Stream function,

$u = \frac{1}{r}\frac{\partial\psi}{\partial r}$ = the axial velocity component of the fluid,

$v = -\frac{1}{r}\frac{\partial\psi}{\partial x}$ = the radial velocity component of the fluid,

ν = Kinematic coefficient of viscosity,

ρ = Density of the fluid.

The cylindrical symmetry of the problem has already been taken into account in writing (8.14).

The appropriate boundary conditions are:

$$V = 0 \quad at \quad r = 0 \;, \quad x \geqslant 0 \;, \tag{8.15}$$

$$\frac{\partial U}{\partial r} = 0 \quad at \quad r = 0 \;, \quad x \geqslant 0 \;, - \tag{8.16}$$

$$\lim_{r \to \infty} U = 0 \;, \tag{8.17}$$

and the rate, $\mathcal{M}$, at which momentum flows across a section of the jet,

$$\mathcal{M} = 2\pi\rho \int_0^\infty U^2 r\,dr \;, \tag{8.18}$$

must be the same for all sections.

Equation (8.14) is constant conformally invariant under the one-parameter continuous group of transformations:

$$\bar{x} = a^m x \;, \quad \bar{r} = a^n r \;, \quad \bar{\psi} = a^p \psi \quad (a \neq 0) \tag{8.19}$$

if

$$\frac{p}{m} = 1 \tag{8.20}$$

When the transformations (8.19) are applied to (8.18) we obtain:

$$\mathcal{M} = 2\pi\rho\, a^{2n-2p} \int_0^\infty \bar{U}^2 \bar{r}\, d\bar{r} = a^{2n-2p} \bar{\mathcal{M}} ,$$

but, $\mathcal{M}$ must be an absolute invariant of the group (8.19); therefore

$$p = n .$$

Now, two absolute invariants of (8.19), corresponding to $\eta(x^1, x^2)$ and $g(y, x^1, x^2)$ of the general theory, are:

$$\eta = \frac{r}{x^{n/m}}$$

and

$$\frac{\psi}{x^{p/m}} ,$$

hence, by Corollary 7.1-1, the substitution of:

$$\eta = \frac{r}{x}$$

and

$$\psi = x\, f(\eta)$$

will transform (8.14) into a similarity equation. The variables introduced by Schlichting, without motivation, are:

$$\eta = \frac{1}{\sqrt{\nu}}\,\frac{r}{x} \quad \text{and} \quad \psi = \nu x\, f(\eta). \tag{8.21}$$

It is easily verified that the boundary conditions (8.15)-(8.17) can be expressed entirely in terms of $f(\eta)$ and its derivatives. The problem can then be solved entirely in terms of the solution of the similarity equation:

$$-\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{df}{d\eta}\right) = \frac{d}{d\eta}\left(\frac{d^2f}{d\eta^2} - \frac{1}{\eta}\frac{df}{d\eta}\right)$$

obtained from (8.14) on using the substitutions (8.21). The invariant solutions of (8.14) (see Lemma 4.2) are given by $\psi = \nu x\, f(\eta)$, where $f(\eta)$ is the general solution of the above ordinary differential equation.

8.30 Spiral Viscous Flows

So far we have given examples in which the partial differential equations are constant conformally invariant under generalizations of the group of uniform expansions (8.6 and 8.19). In this section we shall give an example in which another type of group is considered.

On taking the "curl" of the Navier-Stokes equations for an incompressible viscous flow we obtain a partial differential equation for the vorticity vector. If we now restrict ourselves to considering two-dimensional viscous flows and satisfy the continuity equation by using the stream function $\psi(r,\theta)$, in polar coordinates r, θ, we obtain the following equation:

$$\nu \nabla^2(\nabla^2\psi) = \frac{1}{r}\left\{\frac{\partial\psi}{\partial r}\,\frac{\partial(\nabla^2\psi)}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\,\frac{\partial(\nabla^2\psi)}{\partial r}\right\}, \tag{8.22}$$

where:

$\nabla^2 \quad = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$,

$\nu \quad$ = Kinematic coefficient of viscosity,

$r, \theta \quad$ = Polar coordinates.

Equation (8.22) is constant conformally invariant under the spiral group of transformations:

$$\bar{r} = e^{\alpha} r \;,\; \bar{\theta} = \theta + c\alpha \,,\; \bar{\psi} = \psi + C\alpha, \tag{8.23}$$

where:

$\alpha \quad$ = The numerical parameter,

$C, c \quad$ = Numerical constants.

Two absolute invariants of the group (8.23), corresponding to $\eta(x^1, x^2)$ and $g(y, x^1, x^2)$ of the general theory, are:

$$\theta - c \ln r$$

and

$$\psi - C \ln r .$$

Hence, by Corollary 7.1-1, the substitution of

$$\eta = \theta - c \ln r$$

and

$$\psi = C \ln r + f(\eta) \tag{8.24}$$

will give an ordinary differential (similarity) equation, it is

$$\nu\left[(c^2+1)\frac{d^4f}{d\eta^4} + 4c\frac{d^3f}{d\eta^3} + 4\frac{d^2f}{d\eta^2}\right] = C\frac{d^3f}{d\eta^3} + 2\frac{df}{d\eta}\,\frac{d^2f}{d\eta^2}, \tag{8.25}$$

this is an ordinary differential equation obtained by Oseen (Ref. 14).

The equation (8.25) has also been derived by Birkhoff (Ref. 3,

p. 124) on using only the subgroup $\bar{r} \to r$ and $\bar{\theta} \to \theta$ of (8.23) and an extremely complex argument to determine the form (8.24) of the dependent variable.

It should be remarked that (8.24), when $f(\eta)$ is replaced by the general solution of (8.25), determines the invariant solutions (see Lemma 4.2) of (8.22) under the group (8.23).

IX. SOME NEW EXAMPLES OF PROBLEMS TO BE SOLVED BY SIMILARITY CONSIDERATIONS

9.10 The Uniformly Loaded Semi-Infinite Wedge-Shaped Plate

The problem under consideration arises quite naturally as the simplest case of a sweptback wing, it is that of one-half of a delta wing attached to a semi-infinite fuselage. Although the problem, when considered in this manner, bears no relation to practical experience it must be noted that such solutions are nevertheless useful since the main region of interest is near the corner of the wedge-shaped plate. The problem we wish to solve is therefore related to the extensive experimental and theoretical work which has been and is being done at GALCIT (Refs. 15, 16 and 17) on the structural problem of sweptback wings.

We wish to determine the deflections of a uniformly loaded semi-infinite wedge-shaped plate which is clamped at one edge and free at the other as shown in Fig. 2.

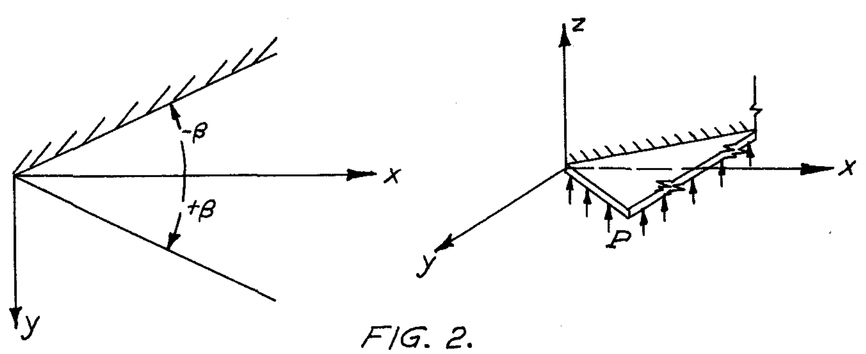


FIG. 2.

In Fig. 2 the orientation of axes has been chosen to correspond with the notation of Timoshenko (Ref. 18).

The small deflection theory of plates (Refs. 18, p. 88) predicts that the deflection $w(x,y)$ of the plate from its neutral x,y -plane is given by the solution of the equation:

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{P(x,y)}{D}\,, \tag{9.1}$$

where:

$P(x,y)$ = Normal loading on the plate,

$D = \dfrac{Et^3}{12(1-\nu^2)}$ = plate stiffness factor,

t = Thickness of plate,

E = Young's modulus for the plate material,

ν = Poisson's constant,

under appropriate boundary conditions.

Since the edges of the plate (Fig. 2) are invariant under the group of uniform expansions one might suspect that, for the problem at hand, there exist possible similarity solutions to (9.1) which will satisfy the boundary conditions.

9.11 The Conditions for Invariance of the Plate Equation

Let us consider the conditions for invariance of (9.1) under a group which is a generalization of the group of uniform expansions, namely:

$$\bar{x} = a^m x\,, \quad \bar{y} = a^n y\,, \quad \bar{w} = a^p w\,, \tag{9.2}$$

where:

a = The numerical parameter $(\neq 0)$,

m, n, p = Real numbers to be determined.

We note that two invariants of the group (9.2) are:

$$\frac{y}{x^{n/m}} \quad \text{and} \quad \frac{w}{x^{p/m}}\,. \tag{9.3}$$

Under (9.2) the equation (9.1) transforms into

$$a^{4m-p}\frac{\partial^4 \overline{w}}{\partial \overline{x}^4}+2a^{2m+2n-p}\frac{\partial^4 \overline{w}}{\partial \overline{x}^2 \partial \overline{y}^2}+a^{4n-p}\frac{\partial^4 \overline{w}}{\partial \overline{y}^4}=\frac{P(\overline{x},\overline{y})}{D}. \tag{9.4}$$

For (9.1) to be invariant under (9.2) the left hand side of (9.4) must be independent of a if $P(\overline{x},\overline{y})=P(x,y)$ - as must be the case for uniform loading (P = constant). This condition is satisfied by setting each of the exponents of a in (9.4) equal to zero, which gives:

$$\frac{n}{m}=1 \quad \text{and} \quad \frac{p}{m}=4. \tag{9.5}$$

If $m=n=1$ then (9.2) becomes the group of uniform expansions, which substantiates our previous remark.

By (9.3) and (9.5) the absolute invariants of the special group which leaves (9.1) invariant are:

$$\frac{y}{x} \tag{9.6}$$

and

$$\frac{w}{x^4}. \tag{9.7}$$

The quantities $\frac{y}{x}$ and $\frac{w}{x^4}$ correspond to the functions $\eta(x^1,\cdots,x^m)$ and $g\,(y, x^1,\cdots,x^m)$ of Theorem 7.1 of the general theory. By Corollary 7.1-1 the substitution of

$$\eta(x,y)=\frac{y}{x} \tag{9.8}$$

and

$$w=x^4 f(\eta) \tag{9.9}$$

into (9.1) will give us the ordinary differential equation

$$(\eta^2+1)^2\frac{d^4 f}{d\eta^4}-4\eta(\eta^2+1)\frac{d^3 f}{d\eta^3}+4(3\eta^2+1)\frac{d^2 f}{d\eta^2}-24\eta\frac{df}{d\eta}+24f=\frac{P}{D}. \tag{9.10*}$$

*As has been previously remarked, (9.9) determines the invariant solutions (see Lemma 4.2) of (9.1) under the group (9.2).

The appropriate boundary conditions for (9.10) are determined in the next section.

9.12 Boundary Conditions

The edges of the plate shown in Fig. 2 correspond to the following values of η :

$$\eta = -k = \tan(-\beta) \qquad \text{for the clamped edge}$$

and

$$\eta = k = \tan\beta \qquad \text{for the free edge.}$$

The boundary conditions for the clamped edge of the plate are:

$$w = 0 \quad \sim \quad \text{zero deflection} \tag{9.11}$$

$$\frac{\partial w}{\partial \vec{n}} = \left[\frac{\partial w}{\partial x}\tan\beta - \frac{\partial w}{\partial y}\right]\cos\beta = 0 \quad \sim \quad \text{zero slope} \tag{9.12}$$

where $\vec{n}$ is the outward pointing normal to the edge of the plate.

From Ref. 18 (p. 94) we have that, for our case, the boundary conditions at the free edge can be expressed as:

$$\nu \nabla^2 w + (1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\cos^2\beta + \frac{\partial^2 w}{\partial y^2}\sin^2\beta - \frac{\partial^2 w}{\partial x \partial y}\sin 2\beta\right\} = 0 \tag{9.13}$$

and

$$-\frac{\partial}{\partial x}\nabla^2 w \sin\beta + \frac{\partial}{\partial y}\nabla^2 w \cos\beta - (1-\nu)\frac{\partial}{\partial \vec{s}}\left\{\frac{\partial^2 w}{\partial x \partial y}\cos 2\beta + \frac{1}{2}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right)\right\} = 0, \tag{9.14}$$

where:

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2},$$

$\vec{s}$ = Tangential direction at free edge,

$$\frac{\partial}{\partial \vec{s}} = -\frac{\partial}{\partial x}\cos\beta - \frac{\partial}{\partial y}\sin\beta .$$

The boundary conditions (9.13) and (9.14) correspond, respectively, to the requirements that the bending moment and shear force must vanish at free edge.

9.13 The Mathematical Problem to be Solved

In this section we summarize our previous results.

The solution of the ordinary differential equation

$$(\eta^2+1)^2 \frac{d^4f}{d\eta^4} - 4\eta(\eta^2+1)\frac{d^3f}{d\eta^3} + 4(3\eta^2+1)\frac{d^2f}{d\eta^2} - 24\eta\frac{df}{d\eta} + 24f = \frac{P}{D} \tag{9.10}$$

must be found such that it satisfies the boundary conditions:

$$f(-k) = 0 \ , \tag{9.15}$$

$$\left.\frac{df}{d\eta}\right)_{\eta=-k} = 0 \ , \tag{9.16}$$

$$12(1+\nu k^2)f(k) - 6k\{\nu k^2+(2-\nu)\}\left(\frac{df}{d\eta}\right)_{\eta=+k} + \{\nu k^4+2(2-\nu)k^2+\nu\}\left(\frac{d^2f}{d\eta^2}\right)_{\eta=+k} = 0 \ , \tag{9.17}$$

$$24k\left\{1+\frac{(1-\nu)}{1+k^2}\right\}f(k) + 6\{3k^2+(2-\nu)\}\left(\frac{df}{d\eta}\right)_{\eta=+k} - 6\{1+k^2\}\left(\frac{d^2f}{d\eta^2}\right)_{\eta=+k} + (1+k^2)^2\left(\frac{d^3f}{d\eta^3}\right)_{\eta=+k} = 0. \tag{9.18}$$

The boundary conditions (9.15)-(9.18) were determined by substituting (9.9) into the expressions (9.11)-(9.14) and noting that resulting expressions must hold for arbitrary x .

9.14 Solution of the Ordinary Differential Equation

The complementary solution of (9.10) can be determined by obtaining the general solution of:

$$L(f) \equiv (\eta^2+1)^2\frac{d^4f}{d\eta^4} - 4\eta(\eta^2+1)\frac{d^3f}{d\eta^3} + 4(3\eta^2+1)\frac{d^2f}{d\eta^2} - 24\eta\frac{df}{d\eta} + 24f = 0 \ , \tag{9.19}$$

where $L(f)$ denotes the differential operator defined by the linear differential equation (9.19).

The general solution of (9.19) is obtained by noting that:

$$L(\eta^m) = f_0(m)\eta^{m-4} + f_1(m)\eta^{m-2} + f_2(m)\eta^m, \tag{9.20}$$

where:

$$\left.\begin{aligned} f_0(m) &= m(m-1)(m-2)(m-3)\ , \\ f_1(m) &= 2m(m-1)(m-3)(m-4)\ , \\ f_2(m) &= \quad (m-1)(m-2)(m-3)(m-4)\ . \end{aligned}\right\} \tag{9.21}$$

From (9.21) we immediately note that η and η^3 must be solutions of (9.19). In the same manner we see that the two linear combinations

$$a\eta^4 + b\eta^2 \tag{9.22}$$

and

$$c\eta^2 + d \tag{9.23}$$

must be solutions of (9.19) for appropriate ratios of $\frac{a}{b}$ and $\frac{c}{d}$. These ratios are determined by substitution of (9.22) and (9.23) into (9.19). Since $L(\eta^2)=8$ we see that a particular solution of (9.10) can also be obtained.

It is evident by inspection that the four complementary solutions given above are linearly independent, hence the solution of (9.10) is:

$$f(\eta) = C_1\eta + C_2(\eta^2 - \tfrac{1}{3}) + C_3\eta^3 + C_4(\eta^4 - 3\eta^2) + \frac{1}{8}\frac{P}{D}\eta^2\ . \tag{9.24}$$

That such a solution satisfies the original partial differential equation can be checked by substituting (9.24) into (9.1).

9.15 Solution of the Problem

For purposes of illustration we shall consider a case which is slightly simpler than that shown in Fig. 2. That is, the case where the clamped edge coincides with the x-axis as illustrated below.

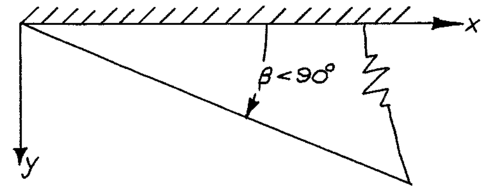


FIG. 3.

The boundary conditions (9.17) and (9.18) remain unaltered but (9.15) and (9.16) must now be written as:

$$f(0)=0 \tag{9.25}$$

and

$$\left.\frac{df}{d\eta}\right)_{\eta=0}=0\,. \tag{9.26}$$

By (9.25) and (9.26) we see that (9.24) must then be expressed as:

$$f(\eta)=C_3\eta^3+C_4(\eta^4-3\eta^2)+\frac{1}{8}\frac{P}{D}\eta^2\,, \tag{9.27}$$

and the deflection of the plate is given by $w=x^4 f(\eta)$.

The substitution of (9.27) into (9.17) and (9.18) gives two simultaneous equations for C_3 and C_4, namely:

$$6\nu k(k^2+1)C_3+2\left\{k^4(6-\nu)+2k^2(1+\nu)-\nu\right\}C_4=\frac{1}{4}\frac{P}{D}\left\{2k^2(1-2\nu)-\nu k^4-\nu\right\} \tag{9.28}$$

and

$$\left\{8k^6+7k^4(2-\nu)+3k^2(1-\nu)+1\right\}C_3+2k\left\{k^2(5-3\nu)+(1+\nu)\right\}C_4=-\frac{k}{4}\frac{P}{D}\left\{4k^4+k^2(5-\nu)+2k(1-\nu)+(1-\nu)\right\}. \tag{9.29}$$

From (9.28) and (9.29) the expressions for C_3 and C_4 can be written in determinant form, that is:

$$C_3=\frac{1}{2}\frac{P}{D}\frac{1}{\Delta}\begin{vmatrix} 2k^2(1-2\nu)-\nu k^4-\nu & k^4(6-\nu)+2k^2(1+\nu)-\nu \\ -\left\{4k^5+k^3(5-\nu)+2k^2(1-\nu)+k(1-\nu)\right\} & k^3(5-3\nu)+k(1+\nu)\end{vmatrix} \tag{9.30}$$

and

$$C_4 = \frac{1}{4}\frac{P}{D}\frac{1}{\Delta}\begin{vmatrix} 6\nu k(k^2+1) & 2k^2(1-2\nu)-\nu k^4-\nu \\ 8k^6+7k^4(2-\nu)+3k^2(1-\nu)+1 & -\{4k^5+k^3(5-3\nu)+2k^2(1-\nu)+k(1-\nu)\} \end{vmatrix}, \quad (9.31)$$

where:

$$\Delta = 2\begin{vmatrix} 6\nu k(k^2+1) & k^4(6-\nu)+2k^2(1+\nu)-\nu \\ 8k^6+7k^4(2-\nu)+3k^2(1-\nu)+1 & k^3(5-3\nu)+k(1+\nu) \end{vmatrix} .$$

It is then seen from (9.30) and (9.31) that the effect of the angle β ($k=\tan\beta$) on the deflection pattern of the plate (Fig. 3) is evidenced only through the constants C_3 and C_4. It is evident from the above that the variation of the deflection pattern with the angle β obeys a very complex law.

9.16 The Stresses at the Corner

The stresses at the upper surface of the plate are given by (Ref. 18, p. 40)

$$\sigma_x = -\frac{Et}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right),$$

$$\sigma_y = -\frac{Et}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right),$$

and

$$\tau_{xy} = -\frac{Et}{(1+\nu)}\frac{\partial^2 w}{\partial x\partial y},$$

which in terms of $f(\eta)$ and its derivatives can be expressed as:

$$\sigma_x = -\frac{Etx^2}{(1-\nu^2)}\left\{12f(\eta)-6\eta\frac{df}{d\eta}+(\eta^2+\nu)\frac{d^2f}{d\eta^2}\right\},$$

$$\sigma_y = -\frac{Etx^2}{(1-\nu^2)}\left\{12\nu f(\eta)-6\nu\eta\frac{df}{d\eta}+(1+\nu\eta^2)\frac{d^2f}{d\eta^2}\right\},$$

$$\tau_{xy} = -\frac{Etx^2}{(1+\nu)}\left\{3\frac{df}{d\eta} - \eta\frac{d^2f}{d\eta^2}\right\}.$$

Now, the above expressions show that on any line η = constant the stresses will only depend on x. Hence as $x \to 0$ (the corner) we have $\sigma_x, \sigma_y, \tau_{xy} \to 0$ at the same rate that $x^2 \to 0$ for any wedge angle less than 180°.*

9.17 Conclusions

The defledtion patterns for uniformly loaded semi-infinite wedge shaped plates satisfying different boundary conditions (clamped-clamped, clamped-simply supported, simply supported-simply supported, etc.) can also be determined by the method outlined above since all the boundary conditions can be expressed in terms of $f(\eta)$ and its derivatives.

The 90° plate can be treated by applying the boundary conditions as shown in Fig. 2 with β = 45°, thus avoiding the difficulty that η is infinite for β = 90° when the plate is placed as shown in Fig. 3. This case, then, creates no essentially difficulty. The solution obtained above may also be considered to give the stress distributions in a finite triangular plate. Such will be the case when the stresses at the edge crosswise to the two radial edges are distributed in the manner given by the second set of equations of Section 9.16. For such a stress distribution the stresses at the corner are

*Plates with a wedge angle greater than 180° cannot be considered by this method since there is no mathematical distinction between the right and left hand sides of the (x,y)-plane.

zero for any wedge angle less than 180° (as shown in Section 9.16). Williams (Ref. 19) claims that the stress can only be zero at the corner, in the case of a finite plate with the same boundary conditions on the radial edges, for wedge angles which are less than 90° and arbitrary boundary conditions at the circumferential edge.

It should be noted that the method is also applicable when the load is not uniform but is a function of η alone (that is, $P(x,y)$ is invariant under the group (9.2)). Especially tractable will be the case when $P(\eta)$ is a polynomial in η since in this case the particular solutions of (9.10) can be obtained with ease.

9.20 The Accelerated Boundary Layer with Pressure Gradient Over a Semi-Infinite Flat Plate

The problem of unsteady boundary layers is of interest since, in some instances, it is necessary to know the time dependence of the skin friction coefficient when determining the drag coefficient of airborne vehicles which may undergo sudden changes of altitude or velocity along their flight path. The problem, as posed above, is frought with considerable geometrical complexities; consequently, to gain an understanding of the underlying physical phenomena, it behooves us to first study the simplest case - the accelerated boundary layer over a semi-infinite plate.

9.21 Formulation of the Problem

Consider a semi-infinite flat plate to be placed in an incompressible fluid of small viscosity with a free stream velocity which is a function of distance along the plate and time. This situation is illustrated in Fig. 4.

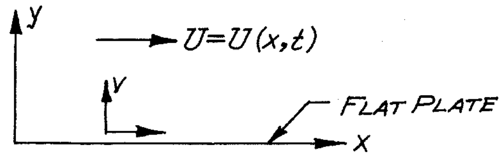


FIG. 4.

In terms of the stream function $\psi(x,y,t)$ the Prandtl boundary layer equations may be written as:

$$\psi_{yt} + \psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} = \frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial U^2}{\partial x}, \tag{9.32}$$

where the significance of the notation has been given in Section 8.10.

The appropriate boundary conditions are:

$$u = 0 \quad at \quad y=0,\ x \geqslant 0 \quad for\ any\ t, \tag{9.33}$$

$$v = 0 \quad at \quad y=0,\ x \geqslant 0 \quad for\ any\ t, \tag{9.34}$$

$$\lim_{y \to \infty} u(x,y,t) = U(x,t), \tag{9.35}$$

and the initial conditions may be written as:

$$u(x,y,t) = v(x,y,t) = 0 \quad at\ t=0 \quad for\ any\ x,y. \tag{9.36}$$

9.22 Similarity Solutions of (9.32)

Evidently equation (9.32) is strongly non-linear and there is at present no method of determining its exact solution under the boundary conditions (9.30)-(9.36). A considerable simplification can be achieved by considering the possible similarity solutions of (9.32).

Consider the continuous one-parameter group of transformations:

$$\bar{x} = a^m x, \quad \bar{y} = a^n y, \quad \bar{t} = a^p t, \quad \bar{\psi} = a^q \psi, \tag{9.37}$$

where:

a = The numerical parameter $(\neq 0)$,

m, n, p, q = Numerical constants to be determined.

Almost by inspection, one set of functionally independent absolute invariants of (9.37), corresponding to $\eta_1(x^1, x^2, x^3)$, $\eta_2(x^1, x^2, x^3)$ and $g(y, x^1, x^2, x^3)$ of the general theory, is:

$$\eta_1 = \frac{y}{x^{n/m}} ,$$

$$\eta_2 = \frac{t}{x^{p/m}} ,$$

and

$$g = \frac{\psi}{x^{q/m}} .$$

It is therefore necessary to determine the ratios $\frac{n}{m}$, $\frac{p}{m}$ and $\frac{q}{m}$ subject to the conditions that (9.32) be invariant under (9.37) and that

$$U = fnc.(\eta_1, \eta_2) .$$

These conditions will be satisfied if

$$\frac{n}{m} = \frac{q}{m} = \frac{1}{2} , \quad \frac{p}{m} = 1$$

and the free stream velocity is such that

$$U(x,t) = U\left(\frac{t}{x}\right) , \tag{9.38}$$

which, it should be emphasized, is a definite restriction on the free stream flow.

By Theorem 7.1 a similarity equation will then be obtained from (9.32) on using the substitutions:

$$\left.\begin{aligned} \eta_1 &= \frac{y}{\sqrt{x}} , \\ \eta_2 &= \frac{t}{x} , \\ \text{and} \quad \psi &= \sqrt{x}\, f(\eta_1, \eta_2). \end{aligned}\right\} \tag{9.39}$$

For suitable $f(\eta_1, \eta_2)$ (that is: given by the solutions of the similarity equation (9.40)) the last of (9.39) gives the invariant solutions of (9.32) (see Lemma 4.2) under the group (9.37).

9.23 The Mathematical Problem to be Solved

The third order partial differential equation in two independent variables:

$$\nu\frac{\partial^3 f}{\partial \eta_1^3}-\frac{\partial^2 f}{\partial\eta_1\partial\eta_2}+\frac{1}{2}f\frac{\partial^2 f}{\partial\eta_1^2}-\eta_2\left\{\frac{\partial f}{\partial\eta_2}\,\frac{\partial^2 f}{\partial\eta_1^2}-\frac{\partial f}{\partial\eta_1}\,\frac{\partial^2 f}{\partial\eta_1\partial\eta_2}\right\}=\frac{\partial U}{\partial\eta_2}+\frac{1}{2}\eta_2\frac{\partial U^2(\eta_2)}{\partial\eta_2}, \quad (9.40)$$

obtained by substituting (9.38) and (9.39) in (9.32), must be solved subject to the conditions:

$$\left.\frac{\partial f}{\partial\eta_1}\right)_{\eta_1=0}=0 \quad \text{for arbitrary } \eta_2,$$

$$f(0,\eta_2)=\eta_2\left.\frac{\partial f}{\partial\eta_2}\right)_{\eta_1=0},$$

$$\lim_{\eta_1\to\infty}\frac{\partial f}{\partial\eta_1}=U(\eta_2),$$

$$\left.\frac{\partial f}{\partial\eta_1}\right)_{\eta_2=0}=0 \quad \text{for arbitrary } \eta_1,$$

and

$$f(\eta_1,0)=0 \quad \text{for arbitrary } \eta_1.$$

As a check we note that (9.40) reduces to the Blasius similarity equation for the boundary layer when f is independent of η_2 and U = constant.

9.24 Two Special Cases

Consider the physical significance of the variable η_2 . This variable can be written as:

$$\eta_2=\frac{1}{\frac{x}{t}}$$

and we notice that $\frac{x}{t}$ has the dimensions of velocity and hence may be looked upon as an average velocity within the boundary layer $\overline{U}$ (say) - a measure of the velocity which a particle, placed at the leading edge of the plate, has to attain in order to reach a point x (say) within the boundary layer. Now, if the external velocity is suddenly applied it is evident that the particles near the plate will experience large changes in velocity, η_2 will then be very small ($\eta_2 \ll 1$). If the external velocity is applied very slowly the situation is reversed and $\eta_2 \gg 1$.

Two extreme cases can then be investigated, namely:

Case I, $\eta_2 \ll 1$:

Here most of the non-linear terms in (9.40) can be neglected so that the equation becomes:

$$\nu \frac{\partial^3 f}{\partial \eta_1^3} - \frac{\partial^2 f}{\partial \eta_1 \partial \eta_2} + \frac{1}{2} f \frac{\partial^2 f}{\partial \eta_1^2} = \frac{\partial U}{\partial \eta_2} \quad . \tag{9.41}$$

It should be noted that the time dependency enters into (9.41) only through the linear terms $\frac{\partial^2 f}{\partial \eta_1 \partial \eta_2}$ and $\frac{\partial U}{\partial \eta_2}$.

Case II, $\eta_2 \gg 1$:

In this case (9.40) reduces to:

$$\frac{\partial f}{\partial \eta_2} \frac{\partial^2 f}{\partial \eta_1^2} - \frac{\partial f}{\partial \eta_1} \frac{\partial^2 f}{\partial \eta_1 \partial \eta_2} = \frac{1}{2} \frac{\partial U^2(\eta_2)}{\partial \eta_2} \, , \tag{9.42}$$

since the terms not containing η_2 as a factor can now be neglected.

9.25 Conclusions

No attempt will be made at this time to obtain solutions to the equations (9.40), (9.41) or (9.42) since such a task is properly part of another complete investigation.

Besides the problem considered above equation (9.40) may be applicable to the investigation of boundary layers in two-dimensional channels with linearly varying area and in which the sides of the channel are approaching each other at a constant rate. Such a situation would occur in the flow within the reed valves of a pulse-jet engine. Another case would be that in which the channel walls are stationary but in which the fluid velocity within the channel is time dependent.

9.30 Further Known and New Examples of Similarity Equations in Aeronautics

Further examples of the variety of similarity equations which can be obtained by use of the general theory are given in Table I. Not all of the similarity equations listed in Table I necessarily have physically significant solutions and so no attempt has been made to carry the examples to completion.

The title of Column 3, Table I, has been couched in standard aeronautical terminology. To impose similarity conditions on different physical quantities is, group-theoretically, equivalent to saying that these quantities be absolutely invariant under a particular group of transformations.

It should be noted that only one group of transformations, a generalization of the group of uniform expansions, has been used in Table I. Other groups which may be used in the same manner are listed in Table II. Some of the groups listed in this table are given in Ref. 4 (p. 3).

TABLE I

FURTHER KNOWN AND NEW EXAMPLES OF SIMILARITY EQUATIONS

A.I. = ABSOLUTELY INVARIANT (SEE DEFINITION 5.4)
C.C.I = CONSTANT CONFORMALLY INVARIANT (SEE DEFINITION 5.3)

COLUMN NO.				①	②	③	④	⑤	⑥	⑦	⑧	⑨
PARTIAL DIFFERENTIAL EQUATION	GROUP (THE PARAMETER $a \neq 0$)	INVARIANTS OF THE GROUP	CASE NO.	TYPE OF INVARIANCE OF Φ	CONDITION ON GROUP TO SATISFY ①	SIMILARITY IMPOSED ON:	CONDITION ON GROUP TO SATISFY ③	GROUP REQUIRED TO SATISFY ① & ③	NEW INDEPENDENT VARIABLE	NEW DEPENDENT VARIABLE	THE RESULTANT SIMILARITY EQUATION	REMARKS
THE TRANSONIC EQUATION $\Phi = \varphi_x \varphi_{xx} + \varphi_{yy} = 0$ $u = \varphi_x \;,\; v = \varphi_y$	$\bar{x} = a^m x$ $\bar{y} = a^n y$ $\bar{\varphi} = a^p \varphi$	$\eta = \frac{y}{x^{n/m}}$ $g = \frac{\varphi}{x^{p/m}}$	1	A.I.	$3m = 2p$ $2n = p$	—	—	$\frac{n}{m} = \frac{3}{4} , \frac{p}{m} = \frac{3}{2}$	$\eta = \frac{y}{x^{3/4}}$	$\varphi = x^{3/2} f(\eta)$	$\left(\frac{8}{9} + \frac{3}{2}\eta^2 f - \frac{3}{4}\eta^3 \frac{df}{d\eta}\right)\frac{d^2f}{d\eta^2} + \frac{1}{2}\left(3\eta f - \eta^2 \frac{df}{d\eta}\right)\frac{df}{d\eta} + f^2(\eta) = 0$	1. Φ differs from the standard transonic equation by a transformation of variable. 2. CASE 2-⑧ Integration of each of the factors is possible in terms of simple functions.
			2	C.C.I	$\frac{p}{m} + 2\frac{n}{m} = 3$	u, v	$\frac{p}{m} - \frac{n}{m} = 0$	$\frac{n}{m} = 1 , \frac{p}{m} = 1$	$\eta = \frac{y}{x}$	$\varphi = x f(\eta)$	$\left(1 + \eta^2 f + \eta^3 \frac{df}{d\eta}\right)\frac{d^2f}{d\eta^2} = 0$	
			3	C.C.I.	$\frac{p}{m} + 2\frac{n}{m} = 3$	φ	$p = 0$	$\frac{n}{m} = \frac{3}{2} , \frac{p}{m} = 0$	$\eta = \frac{y}{x^{3/2}}$	$\varphi = f(\eta)$	$\left(\frac{8}{9} - 3\eta^3 \frac{df}{d\eta}\right)\frac{d^2f}{d\eta^2} - \eta^2\left(\frac{df}{d\eta}\right)^2 = 0$	
THE BLASIUS BOUNDARY LAYER EQUATION $\Phi = \psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} = 0$ $u = \frac{\partial \psi}{\partial y} \;,\; v = -\frac{\partial \psi}{\partial x}$	$\bar{x} = a^m x$ $\bar{y} = a^n y$ $\bar{\psi} = a^p \psi$	$\eta = \frac{y}{x^{n/m}}$ $g = \frac{\psi}{x^{p/m}}$	1	C.C.I	$\frac{n}{m} + \frac{p}{m} = 1$	u	$\frac{p}{m} - \frac{n}{m} = 0$	$\frac{n}{m} = \frac{1}{2} , \frac{p}{m} = \frac{1}{2}$	$\eta = \frac{y}{\sqrt{x}}$	$\psi = \sqrt{x} f(\eta)$	$\nu \frac{d^3f}{d\eta^3} + \frac{1}{2} f \frac{d^2f}{d\eta^2} = 0$	1. CASE 1-⑧ is the standard Blasius similarity equation for the boundary layer, with external pressure gradient, over a flat plate. 2. CASE 3-⑧ can be integrated in terms of elliptic functions
			2	C.C.I	$\frac{n}{m} + \frac{p}{m} = 1$	v	$\frac{p}{m} - 1 = 0$	$\frac{n}{m} = 0 , \frac{p}{m} = 1$	$\eta = y$	$\psi = x f(\eta)$	$\nu \frac{d^3f}{d\eta^3} + f \frac{d^2f}{d\eta^2} - \left(\frac{df}{d\eta}\right)^2 = 0$	
			3	C.C.I.	$\frac{n}{m} + \frac{p}{m} = 1$	ψ	$p = 0$	$\frac{n}{m} = 1 , \frac{p}{m} = 0$	$\eta = \frac{y}{x}$	$\psi = f(\eta)$	$\nu \frac{d^3f}{d\eta^3} + \left(\frac{df}{d\eta}\right)^2 = 0$	
			4	C.C.I.	$\frac{n}{m} + \frac{p}{m} = 1$	$\frac{\partial u}{\partial y}$	$\frac{p}{m} - 2\frac{n}{m} = 0$	$\frac{n}{m} = \frac{1}{3} , \frac{p}{m} = \frac{2}{3}$	$\eta = \frac{y}{x^{1/3}}$	$\psi = x^{2/3} f(\eta)$	$3\nu \frac{d^3f}{d\eta^3} + 2f \frac{d^2f}{d\eta^2} - \left(\frac{df}{d\eta}\right)^2 = 0$	
THE ACCELERATED BOUNDARY LAYER EQUATION $\Phi = \psi_{yt} + \psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} = 0$ $u = \frac{\partial \psi}{\partial y} \;,\; v = -\frac{\partial \psi}{\partial x}$	$\bar{x} = a^m x$ $\bar{y} = a^n y$ $\bar{t} = a^p t$ $\bar{\psi} = a^q \psi$	$\eta_1 = \frac{y}{x^{n/m}}$ $\eta_2 = \frac{t}{x^{p/m}}$ $g = \frac{\psi}{x^{q/m}}$	1	A.I.	$m = 4n$ $p = 2n$ $q = 3n$	—	—	$\frac{q}{m} = \frac{3}{4} , \frac{n}{m} = \frac{1}{4} , \frac{p}{m} = \frac{1}{2}$	$\eta_1 = \frac{y}{\sqrt[4]{x}} , \eta_2 = \frac{t}{\sqrt{x}}$	$\psi = x^{3/4} f(\eta_1, \eta_2)$	$4\nu \frac{\partial^3 f}{\partial \eta_1^3} + 3f \frac{\partial^2 f}{\partial \eta_1^2} 4\frac{\partial^2 f}{\partial \eta_1 \partial \eta_2} - 2\left(\frac{\partial f}{\partial \eta_1}\right)^2 + 2\eta_2\left(\frac{\partial f}{\partial \eta_1}\frac{\partial^2 f}{\partial \eta_1 \partial \eta_2} - \frac{\partial f}{\partial \eta_2}\frac{\partial^2 f}{\partial \eta_1^2}\right) = 0$	1. It is assumed that there is no external pressure gradient (i.e.: $\frac{\partial p(x,t)}{\partial x} = 0$)
			2	C.C.I.	$p = 2n$ $q = m - n$	u	$\frac{n}{m} = \frac{1}{2}$	$\frac{q}{m} = \frac{1}{2} , \frac{n}{m} = \frac{1}{2} , \frac{p}{m} = 1$	$\eta_1 = \frac{y}{\sqrt{x}} , \eta_2 = \frac{t}{x}$	$\psi = \sqrt{x} f(\eta_1, \eta_2)$	$\nu \frac{\partial^3 f}{\partial \eta_1^3} - \frac{\partial^2 f}{\partial \eta_1 \partial \eta_2} + \frac{1}{2} f \frac{\partial^2 f}{\partial \eta_1^2} + \eta_2\left(\frac{\partial f}{\partial \eta_1}\frac{\partial^2 f}{\partial \eta_1 \partial \eta_2} + \frac{\partial f}{\partial \eta_2}\frac{\partial^2 f}{\partial \eta_1^2}\right) = 0$	
			3	C.C.I.	$p = 2n$ $q = m - n$	t	$p = 0$	$\frac{q}{m} = 1 , \frac{n}{m} = 0 , \frac{p}{m} = 0$	$\eta_1 = y , \eta_2 = t$	$\psi = x f(\eta_1, \eta_2)$	$\nu \frac{\partial^3 f}{\partial y^3} + f \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial y \partial t} - \left(\frac{\partial f}{\partial y}\right)^2 = 0$	
			4	C.C.I.	$p = 2n$ $q = m - n$	ψ	$\frac{m}{p} = \frac{n}{2n}$	$\frac{q}{m} = 0 , \frac{n}{m} = 1 , \frac{p}{m} = 2$	$\eta_1 = \frac{y}{x} , \eta_2 = \frac{t}{x^2}$	$\psi = f(\eta_1, \eta_2)$	$\nu \frac{\partial^3 f}{\partial \eta_1^3} - \frac{\partial^2 f}{\partial \eta_1 \partial \eta_2} - \left(\frac{\partial f}{\partial \eta_1}\right)^2 + 2\eta_2\left(\frac{\partial f}{\partial \eta_1}\frac{\partial^2 f}{\partial \eta_1 \partial \eta_2} + \frac{\partial f}{\partial \eta_2}\frac{\partial^2 f}{\partial \eta_1^2}\right) = 0$	
THE LAPLACE EQUATION $\Phi = \varphi_{xx} + \varphi_{yy} = 0$	$\bar{x} = a^m x$ $\bar{y} = a^n y$ $\bar{\varphi} = a^p \varphi$	$\eta = \frac{y}{x^{n/m}}$ $g = \frac{\varphi}{x^{p/m}}$	1	A.I.	$p = 2n$ $p = 2m$	—	—	$\frac{n}{m} = 1 , \frac{p}{m} = 2$	$\eta = \frac{y}{x}$	$\varphi = x^2 f(\eta)$	$(\eta^2 + 1)\frac{d^2f}{d\eta^2} = 0$	1. If the equation is non-homogeneous then the term on the right hand side of the equation will have to be a function of η.
			2	C.C.I.	$n = m$	$\frac{\partial \varphi}{\partial x} , \frac{\partial \varphi}{\partial y}$	$p = m$	$\frac{n}{m} = 1 , \frac{p}{m} = 1$	$\eta = \frac{y}{x}$	$\varphi = x f(\eta)$	$(\eta^2 + 1)\frac{d^2f}{d\eta^2} - 2\eta \frac{df}{d\eta} + 2f(\eta) = 0$	
THE WAVE EQUATION $\Phi = \varphi_{xx} + \varphi_{yy} - \varphi_{tt} = 0$	$\bar{x} = a^m x$ $\bar{y} = a^n y$ $\bar{t} = a^p t$ $\bar{\varphi} = a^q \varphi$	$\eta_1 = \frac{y}{x^{n/m}}$ $\eta_2 = \frac{t}{x^{p/m}}$ $g = \frac{\varphi}{x^{q/m}}$	1	A.I.	$q = 2m$ $q = 2n$ $q = 2p$	—	—	$\frac{q}{m} = 2 , \frac{n}{m} = 1 \; \frac{p}{m} = 1$	$\eta_1 = \frac{y}{x} , \eta_2 = \frac{t}{x}$	$\varphi = x^2 f(\eta_1, \eta_2)$	$(\eta_1^2 + 1)\frac{\partial^2 f}{\partial \eta_1^2} + 2\eta_1\eta_2 \frac{\partial^2 f}{\partial \eta_1 \partial \eta_2} + (\eta_2^2 - 1)\frac{\partial^2 f}{\partial \eta_2^2} - 2\eta_1 \frac{\partial f}{\partial \eta_1} - 2\eta_2 \frac{\partial f}{\partial \eta_2} + 2f = 0$	1. The partial differential equation is expressed in normalized form. 2. The same procedure is applicable in more than two space dimensions.
			2	C.C.I.	$2m = 2n = 2p$	$\frac{\partial \varphi}{\partial x}$	$q = m$	$\frac{q}{m} = 1 , \frac{n}{m} = 1 , \frac{p}{m} = 1$	$\eta_1 = \frac{y}{x} , \eta_2 = \frac{t}{x}$	$\varphi = x f(\eta_1, \eta_2)$	$(\eta_1^2 + 1)\frac{\partial^2 f}{\partial \eta_1^2} + 2\eta_1\eta_2 \frac{\partial^2 f}{\partial \eta_1 \partial \eta_2} + (\eta_2^2 - 1)\frac{\partial^2 f}{\partial \eta_2^2} = 0$	
THE HEAT EQUATION $\Phi = \varphi_{xx} - \varphi_t = 0$	$\bar{x} = a^m x$ $\bar{y} = a^n y$ $\bar{\varphi} = a^p \varphi$	$\eta = \frac{t}{x^{n/m}}$ $g = \frac{\varphi}{x^{p/m}}$	1	A.I.	$p = 2m$ $p = n$	—	—	$\frac{n}{m} = 2 , \frac{p}{m} = 2$	$\eta = \frac{t}{x^2}$	$\varphi = x^2 f(\eta)$	$4\eta^2 \frac{d^2f}{d\eta^2} - (2\eta + 1)\frac{df}{d\eta} + 2f(\eta) = 0$	1. The partial differential equation is expressed in normalized form.
			2	C.C.I.	$n = 2m$	$\frac{\partial \varphi}{\partial x}$	$p = m$	$\frac{n}{m} = 2 , \frac{p}{m} = 1$	$\eta = \frac{t}{x^2}$	$\varphi = x f(\eta)$	$4\eta^2 \frac{d^2f}{d\eta^2} + (2\eta - 1)\frac{df}{d\eta} = 0$	

TABLE II

A PARTIAL LIST OF CONTINUOUS ONE-PARAMETER GROUPS

OF TRANSFORMATIONS AND THEIR INVARIANTS

GROUP	ABSOLUTE INVARIANTS OF THE GROUP	REMARKS
I. Generalization of Group of Uniform Expansions $\bar{x}^i = a^{m_i} x^i,$ $\bar{y}_\delta = a^{p_\delta} y_\delta,$ $i = 1, \cdots, m,$ $\delta = 1, \cdots, n.$	$\eta_k = \dfrac{x^{k+1}}{(x^1)^{\frac{m_{k+1}}{m_1}}},$ $g_\delta = \dfrac{y_\delta}{(x^1)^{\frac{p_\delta}{m_1}}},$ $k = 1, \cdots, m-1.$	1. This is the generalization of the groups used in Table I. 2. The set of absolute invariants listed is only one of the possible sets. 3. The m_i and p_δ are fixed numerical constants.
II. The Spiral Group $\bar{x}^1 = e^a x^1,$ $\bar{x}^i = x^i + c^i a,$ $\bar{y}_\delta = y_\delta + C_\delta a,$ $i = 1, \cdots, m-1,$ $\delta = 1, \cdots, n.$	$\eta_i = x^i - c^i \ln x^1,$ $g_\delta = y_\delta - C_\delta \ln x^1$	1. This is the generalization of the group used in Section 9.20. 2. a is the parameter and c^i and C_δ are constants.
III. The Translation Group $\bar{x}^i = x^i + a,$ $\bar{y}_\delta = y_\delta + a,$ $i = 1, \cdots, m,$ $\delta = 1, \cdots, n.$	$\eta_k = x^k - x^1,$ $g_\delta = y_\delta - x^1.$ $k = 1, \cdots, m-1$	1. The set of absolute invariants listed is only one of the possible sets.

Notes: 1. The notation used is that of the general theory.

2. Group I includes the one-parameter affine and projective groups as special cases.

X. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

It has been shown that the general theory, developed in Sections II-VII, provides a rigorous motivation for all previously known similarity solutions in Aeronautics and a practical way of determining new solutions of the same type.

As shown in Section 9.10 problems in classical Elasticity are also susceptible to treatment by similarity considerations. It might, therefore, be deemed worthwhile to examine the partial differential equations of Elasticity from the point of view developed in this thesis. Such a procedure, it is very likely, will reveal some new physically significant solutions of the relevant problems in the fields of elasticity. Further study of the equations developed in Section 9.20 will certainly provide additional insight into non-stationary boundary layer phenomena. The theory developed in Section VII is sufficiently general so that a thorough study of the possible invariant solutions of the Navier-Stokes equations can now be made.

The methods developed in the preceding sections are certainly applicable to physics in general and it is hoped that some new solutions to problems in physics will be found in this manner.

From the mathematical point of view there are certain interesting possibilities. By Theorem 7.2 known existance theorems for certain types of partial differential equations under special types of boundary conditions may be translated into existance theorems for the invariant solutions of partial differential equations with an additional independent variable. The situation appears to be particularly attractive in the case of non-linear ordinary differential equations since it then affords a study of the existence of invariant solutions of non-linear partial differential equations in two independent variables. Conversely, by Theorem 7.1, the study of the invariant solutions of non-linear partial differential equations in two independent variables may be translated into the study of corresponding non-linear ordinary differential equations about which more is relatively known.

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*Theoretical and Experimental Effect of Sweep Upon the Stress and Deflection Distribution in Aircraft Wings of High Solidity.

APPENDIX I

Proof of R1 Lemma 6.1

1. Taking partial derivatives, as indicated below, of each of the relations (6.3) and (6.4) we obtain the following identities.

$$\frac{\partial}{\partial x^\alpha}\left[\xi^j\frac{\partial g_\delta}{\partial x^j}+k_a y_a\frac{\partial g_\delta}{\partial y_a}\right]=\frac{\partial \xi^j}{\partial x^\alpha}\frac{\partial g_\delta}{\partial x^j}+\xi^j\frac{\partial^2 g_\delta}{\partial x^\alpha\partial x^j}+k_a y_a\frac{\partial^2 g_\delta}{\partial x^\alpha\partial y_a}\equiv 0 \qquad \begin{bmatrix} j,\alpha=1,\cdots,m, \\ a,\delta=1,\cdots,n. \end{bmatrix}, \quad (\text{I.1})$$

$$\frac{\partial}{\partial y_b}\left[\xi^j\frac{\partial g_\delta}{\partial x^j}+k_a y_a\frac{\partial g_\delta}{\partial y_a}\right]=\xi^j\frac{\partial^2 g_\delta}{\partial y_b\partial x^j}+k_{\underline{b}}\frac{\partial g_\delta}{\partial y_{\underline{b}}}+k_a y_a\frac{\partial^2 g_\delta}{\partial y_b\partial y_a}\equiv 0 \qquad \begin{bmatrix} j=1,\cdots,m, \\ a,b,\delta=1,\cdots,n. \end{bmatrix}, \quad (\text{I.2})$$

$$\frac{\partial}{\partial x^\alpha}\left[\xi^j\frac{\partial \eta_\omega}{\partial x^j}\right]=\frac{\partial \xi^j}{\partial x^\alpha}\frac{\partial \eta_\omega}{\partial x^j}+\xi^j\frac{\partial^2 \eta_\omega}{\partial x^\alpha\partial x^j}\equiv 0 \qquad \begin{bmatrix} \alpha,j=1,\cdots,m \\ \omega=1,\cdots,m-1 \end{bmatrix}. \quad (\text{I.3})$$

2. Since, for the purposes of the group theory we can consider each of the quantities $x^1,\cdots,x^m,\ y_1,\cdots,y_n,\ y_{1,\delta},\cdots,y_{n,\delta}$ as independent variables, the derivatives of $\dfrac{\partial F_\delta}{\partial \eta_\omega}$* can be written as:

$$\frac{\partial}{\partial x^j}\left(\frac{\partial F_\delta}{\partial \eta_\omega}\right)=\frac{\partial}{\partial x^j}\frac{\begin{vmatrix} \frac{\partial \eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_1}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_1}+\frac{\partial g_\delta}{\partial x^{\alpha_1}}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_1}} \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_{m-1}}+\frac{\partial g_\delta}{\partial x^{\alpha_{m-1}}}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1,\cdots,\eta_{m-1}}{x^{\alpha_1},\cdots,x^{\alpha_{m-1}}}\right)}$$

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*The derivative of an $m\times m$ determinant is obtained as a sum of $m\times m$ determinants in which each successive row (column) is differentiated while the elements in the rest of the rows (columns) remain the same as in the original determinant. This result is proved in Ref. 20, p. 8.

$$
\frac{\begin{vmatrix}
\frac{\partial^2\eta_1}{\partial x^j\partial x^{\alpha_1}} & \cdots & \frac{\partial^2\eta_{\omega-1}}{\partial x^j\partial x^{\alpha_1}} & \left(\frac{\partial^2 g_\delta}{\partial x^j\partial y_b}y_{b,\alpha_1}+\frac{\partial^2 g_\delta}{\partial x^j\partial x^{\alpha_1}}\right) & \frac{\partial^2\eta_{\omega+1}}{\partial x^j\partial x^{\alpha_1}} & \cdots & \frac{\partial^2\eta_{m-1}}{\partial x^j\partial x^{\alpha_1}} \\
\frac{\partial\eta_1}{\partial x^{\alpha_2}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_2}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_2}+\frac{\partial g_\delta}{\partial x^{\alpha_2}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_2}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_2}} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial\eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_{m-1}}+\frac{\partial g_\delta}{\partial x^{\alpha_{m-1}}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix}}{J\left(\frac{\eta_1,\ \cdots,\ \eta_{m-1}}{x^{\alpha_1},\ \cdots,\ x^{\alpha_{m-1}}}\right)} + \cdots +
$$

$$
\frac{\begin{vmatrix}
\frac{\partial\eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_1}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_1}+\frac{\partial g_\delta}{\partial x^{\alpha_1}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_1}} \\
\vdots & & \vdots & \vdots & \vdots & & \\
\frac{\partial\eta_1}{\partial x^{\alpha_{m-2}}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_{m-2}}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_{m-2}}+\frac{\partial g_\delta}{\partial x^{\alpha_{m-2}}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_{m-2}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-2}}} \\
\frac{\partial^2\eta_1}{\partial x^j\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial^2\eta_{\omega-1}}{\partial x^j\partial x^{\alpha_{m-1}}} & \left(\frac{\partial^2 g_\delta}{\partial x^j\partial y_b}y_{b,\alpha_{m-1}}+\frac{\partial^2 g_\delta}{\partial x^j\partial x^{\alpha_{m-1}}}\right) & \frac{\partial^2\eta_{\omega+1}}{\partial x^j\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial^2\eta_{m-1}}{\partial x^j\partial x^{\alpha_{m-1}}}
\end{vmatrix}}{J\left(\frac{\eta_1,\ \cdots,\ \eta_{m-1}}{x^{\alpha_1},\ \cdots,\ x^{\alpha_{m-1}}}\right)}
$$

$$
-\frac{\partial F_\delta}{\partial\eta_\omega}\,
\frac{\begin{vmatrix}
\frac{\partial^2\eta_1}{\partial x^j\partial x^{\alpha_1}} & \cdots & \frac{\partial^2\eta_{m-1}}{\partial x^j\partial x^{\alpha_1}} \\
\frac{\partial\eta_1}{\partial x^{\alpha_2}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_2}} \\
\vdots & & \vdots \\
\frac{\partial\eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix} + \cdots + \begin{vmatrix}
\frac{\partial\eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_1}} \\
\vdots & & \vdots \\
\frac{\partial\eta_1}{\partial x^{\alpha_{m-2}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-2}}} \\
\frac{\partial^2\eta_1}{\partial x^j\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial^2\eta_{m-1}}{\partial x^j\partial x^{\alpha_{m-1}}}
\end{vmatrix}}{J\left(\frac{\eta_1,\ \cdots,\ \eta_{m-1}}{x^{\alpha_1},\ \cdots,\ x^{\alpha_{m-1}}}\right)} \tag{I.4}
$$

By using Theorem 1.5, Ref. 7, p. 647, (I.4) can be written as:

$$
\frac{\partial}{\partial x^j}\left(\frac{\partial F_\delta}{\partial\eta_\omega}\right) =
$$

$$
\frac{\begin{vmatrix}
\frac{\partial\eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_1}} & \left(\frac{\partial^2 g_\delta}{\partial x^j\partial y_b}y_{b,\alpha_1}+\frac{\partial^2 g_\delta}{\partial x^j\partial x^{\alpha_1}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_1}} \\
\frac{\partial\eta_1}{\partial x^{\alpha_2}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_2}} & \left(\frac{\partial^2 g_\delta}{\partial x^j\partial y_b}y_{b,\alpha_2}+\frac{\partial^2 g_\delta}{\partial x^j\partial x^{\alpha_2}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_2}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_2}} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial\eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \left(\frac{\partial^2 g_\delta}{\partial x^j\partial y_b}y_{b,\alpha_{m-1}}+\frac{\partial^2 g_\delta}{\partial x^j\partial x^{\alpha_{m-1}}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix}}{J\left(\frac{\eta_1,\ \cdots,\ \eta_{m-1}}{x^{\alpha_1},\ \cdots,\ x^{\alpha_{m-1}}}\right)}
$$

$$
+\frac{\begin{vmatrix} \frac{\partial^2\eta_1}{\partial x^j\partial x^{\alpha_1}} \cdots \frac{\partial^2\eta_{\omega-1}}{\partial x^j\partial x^{\alpha_1}} & 0 & \frac{\partial^2\eta_{\omega+1}}{\partial x^j\partial x^{\alpha_1}} \cdots \frac{\partial^2\eta_{m-1}}{\partial x^j\partial x^{\alpha_1}} \\ \frac{\partial\eta_1}{\partial x^{\alpha_2}} \cdots \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_2}} & \left(\frac{\partial g_\delta}{\partial y_b} y_{b,\alpha_2} + \frac{\partial g_\delta}{\partial x^{\alpha_2}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_2}} \cdots \frac{\partial\eta_{m-1}}{\partial x^{\alpha_2}} \\ \vdots & \vdots & \vdots \\ \frac{\partial\eta_1}{\partial x^{\alpha_{m-1}}} \cdots \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \left(\frac{\partial g_\delta}{\partial y_b} y_{b,\alpha_{m-1}} + \frac{\partial g_\delta}{\partial x^{\alpha_{m-1}}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} \cdots \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1, \cdots, \eta_{m-1}}{x^{\alpha_1}, \cdots, x^{\alpha_{m-1}}}\right)} + \cdots
$$

$$
+\frac{\begin{vmatrix} \frac{\partial\eta_1}{\partial x^{\alpha_1}} \cdots \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_1}} & \left(\frac{\partial g_\delta}{\partial y_b} y_{b,\alpha_1} + \frac{\partial g_\delta}{\partial x^{\alpha_1}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_1}} \cdots \frac{\partial\eta_{m-1}}{\partial x^{\alpha_1}} \\ \vdots & \vdots & \vdots \\ \frac{\partial\eta_1}{\partial x^{\alpha_{m-2}}} \cdots \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_{m-2}}} & \left(\frac{\partial g_\delta}{\partial y_b} y_{b,\alpha_{m-2}} + \frac{\partial g_\delta}{\partial x^{\alpha_{m-2}}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_{m-2}}} \cdots \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-2}}} \\ \frac{\partial^2\eta_1}{\partial x^j\partial x^{\alpha_{m-1}}} \cdots \frac{\partial^2\eta_{\omega-1}}{\partial x^j\partial x^{\alpha_{m-1}}} & 0 & \frac{\partial^2\eta_{\omega+1}}{\partial x^{\alpha_{m-1}}\partial x^j} \cdots \frac{\partial^2\eta_{m-1}}{\partial x^j\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1, \cdots, \eta_{m-1}}{x^{\alpha_1}, \cdots, x^{\alpha_{m-1}}}\right)} \tag{I.5}
$$

$$
-\frac{\partial F_\delta}{\partial \eta_\omega} \frac{\begin{vmatrix} \frac{\partial^2\eta_1}{\partial x^j\partial x^{\alpha_1}} \cdots \frac{\partial^2\eta_{m-1}}{\partial x^j\partial x^{\alpha_1}} \\ \frac{\partial\eta_1}{\partial x^{\alpha_2}} \cdots \frac{\partial\eta_{m-1}}{\partial x^{\alpha_2}} \\ \vdots \\ \frac{\partial\eta_1}{\partial x^{\alpha_{m-1}}} \cdots \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-1}}} \end{vmatrix} + \cdots + \begin{vmatrix} \frac{\partial\eta_1}{\partial x^{\alpha_1}} \cdots \frac{\partial\eta_{m-1}}{\partial x^{\alpha_1}} \\ \vdots \\ \frac{\partial\eta_1}{\partial x^{\alpha_{m-2}}} \cdots \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-2}}} \\ \frac{\partial^2\eta_1}{\partial x^j\partial x^{\alpha_{m-1}}} \cdots \frac{\partial^2\eta_{m-1}}{\partial x^j\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1, \cdots, \eta_{m-1}}{x^{\alpha_1}, \cdots, x^{\alpha_{m-1}}}\right)},
$$

$$
\frac{\partial}{\partial y_a}\left(\frac{\partial F_\delta}{\partial \eta_\omega}\right) = \frac{\begin{vmatrix} \frac{\partial\eta_1}{\partial x^{\alpha_1}} \cdots \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_1}} & \left(\frac{\partial^2 g_\delta}{\partial y_a\partial y_b} y_{b,\alpha_1} + \frac{\partial^2 g_\delta}{\partial y_a\partial x^{\alpha_1}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_1}} \cdots \frac{\partial\eta_{m-1}}{\partial x^{\alpha_1}} \\ \frac{\partial\eta_1}{\partial x^{\alpha_2}} \cdots \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_2}} & \left(\frac{\partial^2 g_\delta}{\partial y_a\partial y_b} y_{b,\alpha_2} + \frac{\partial^2 g_\delta}{\partial y_a\partial x^{\alpha_2}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_2}} \cdots \frac{\partial\eta_{m-1}}{\partial x^{\alpha_2}} \\ \vdots & \vdots & \vdots \\ \frac{\partial\eta_1}{\partial x^{\alpha_{m-1}}} \cdots \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \left(\frac{\partial^2 g_\delta}{\partial y_a\partial y_b} y_{b,\alpha_{m-1}} + \frac{\partial^2 g_\delta}{\partial y_a\partial x^{\alpha_{m-1}}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} \cdots \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1, \cdots, \eta_{m-1}}{x^{\alpha_1}, \cdots, x^{\alpha_{m-1}}}\right)}, \tag{I.6}
$$

$$\frac{\partial}{\partial y_{a,j}}\left(\frac{\partial F_\delta}{\partial \eta_\omega}\right) = \frac{\begin{vmatrix} \frac{\partial \eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_1}} & \frac{\partial g_\delta}{\partial y_b}\delta_a^b \delta_j^{\alpha_1} & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_1}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \frac{\partial g_\delta}{\partial y_b}\delta_a^b \delta_j^{\alpha_{m-1}} & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1, \ldots, \eta_{m-1}}{x^{\alpha_1}, \ldots, x^{\alpha_{m-1}}}\right)}$$

$$= \frac{\begin{vmatrix} \frac{\partial \eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_1}} & \frac{\partial g_\delta}{\partial y_a}\delta_j^{\alpha_1} & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_1}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \frac{\partial g_\delta}{\partial y_a}\delta_j^{\alpha_{m-1}} & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1, \ldots, \eta_{m-1}}{x^{\alpha_1}, \ldots, x^{\alpha_{m-1}}}\right)}. \tag{I.7}$$

3. The quantity $\left[k_a y_{a,j} - \frac{\partial \xi^r}{\partial x^j} y_{a,r}\right]\frac{\partial}{\partial y_{a,j}}\left(\frac{\partial F_\delta}{\partial \eta_\omega}\right)$ contained in (6.2) can, on using (I.7), be written as:

$$\left[k_a y_{a,j} - \frac{\partial \xi^r}{\partial x^j} y_{a,r}\right]\frac{\partial}{\partial y_{a,j}}\left(\frac{\partial F_\delta}{\partial \eta_\omega}\right) =$$

$$\frac{\begin{vmatrix} \frac{\partial \eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_1}} & k_a\frac{\partial g_\delta}{\partial y_a} y_{a,\alpha_1} & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_1}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & k_a\frac{\partial g_\delta}{\partial y_a} y_{a,\alpha_{m-1}} & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1, \ldots, \eta_{m-1}}{x^{\alpha_1}, \ldots, x^{\alpha_{m-1}}}\right)} \tag{I.8}$$

$$- \frac{\begin{vmatrix} \frac{\partial \eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_1}} & \frac{\partial \xi^r}{\partial x^{\alpha_1}}\frac{\partial g_\delta}{\partial y_a} y_{a,r} & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_1}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \frac{\partial \xi^r}{\partial x^{\alpha_{m-1}}}\frac{\partial g_\delta}{\partial y_a} y_{a,r} & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1, \ldots, \eta_{m-1}}{x^{\alpha_1}, \ldots, x^{\alpha_{m-1}}}\right)}$$

4. Substituting the expressions (I.5), (I.6) and (I.8) into (6.2) and at the same time regrouping the terms, we obtain:

$$\mathcal{U}^{(1)}\left(\frac{\partial F_\delta}{\partial \eta_\omega}\right) =$$

$$
\frac{\begin{vmatrix}
\frac{\partial \eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_1}} & \left(\xi^j \frac{\partial^2 g_\delta}{\partial x^j \partial y_b} + k^b_a y_a \frac{\partial^2 g_\delta}{\partial y_a \partial y_b} + k_b \frac{\partial g_\delta}{\partial y_b}\right) y_{b,\alpha_1} & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_1}} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \left(\xi^j \frac{\partial^2 g_\delta}{\partial x^j \partial y_b} + k^b_a y_a \frac{\partial^2 g_\delta}{\partial y_a \partial y_b} + k_b \frac{\partial g_\delta}{\partial y_b}\right) y_{b\alpha_{m-1}} & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix}}{\mathcal{J}\left(\frac{\eta_1, \ldots, \eta_{m-1}}{x^{\alpha_1}, \ldots, x^{\alpha_{m-1}}}\right)}
$$

$$
+\frac{\begin{vmatrix}
\frac{\partial \eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_1}} & \left(\xi^j \frac{\partial^2 g_\delta}{\partial x^j \partial x^{\alpha_1}} + k^b_a y_a \frac{\partial^2 g_\delta}{\partial y_a \partial x^{\alpha_1}} + \frac{\partial \xi^j}{\partial x^{\alpha_1}} \frac{\partial g_\delta}{\partial x^j}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_1}} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \left(\xi^j \frac{\partial^2 g_\delta}{\partial x^j \partial x^{\alpha_{m-1}}} + k^b_a y_a \frac{\partial^2 g_\delta}{\partial y_a \partial x^{\alpha_{m-1}}} + \frac{\partial \xi^j}{\partial x^{\alpha_{m-1}}} \frac{\partial g_\delta}{\partial x^j}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix}}{\mathcal{J}\left(\frac{\eta_1, \ldots, \eta_{m-1}}{x^{\alpha_1}, \ldots, x^{\alpha_{m-1}}}\right)}
$$

$$
+\frac{\begin{vmatrix}
\xi^j \frac{\partial^2 \eta_1}{\partial x^j \partial x^{\alpha_1}} & \cdots & \xi^j \frac{\partial^2 \eta_{\omega-1}}{\partial x^j \partial x^{\alpha_1}} & 0 & \xi^j \frac{\partial^2 \eta_{\omega+1}}{\partial x^j \partial x^{\alpha_1}} & \cdots & \xi^j \frac{\partial^2 \eta_{m-1}}{\partial x^j \partial x^{\alpha_1}} \\
\frac{\partial \eta_1}{\partial x^{\alpha_2}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_2}} & \left(\frac{\partial g_\delta}{\partial y_b} y_{b,\alpha_2} + \frac{\partial g_\delta}{\partial x^{\alpha_2}}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_2}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_2}} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \left(\frac{\partial g_\delta}{\partial y_b} y_{b,\alpha_{m-1}} + \frac{\partial g_\delta}{\partial x^{\alpha_{m-1}}}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix}}{\mathcal{J}\left(\frac{\eta_1, \ldots, \eta_{m-1}}{x^{\alpha_1}, \ldots, x^{\alpha_{m-1}}}\right)} + \cdots \qquad \text{(I.9)}
$$

$$
+\frac{\begin{vmatrix}
\frac{\partial \eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_1}} & \left(\frac{\partial g_\delta}{\partial y_b} y_{b,\alpha_1} + \frac{\partial g_\delta}{\partial x^{\alpha_1}}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_1}} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial \eta_1}{\partial x^{\alpha_{m-2}}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_{m-2}}} & \left(\frac{\partial g_\delta}{\partial y_b} y_{b\alpha_{m-2}} + \frac{\partial g_\delta}{\partial x^{\alpha_{m-2}}}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_{m-2}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}} \\
\xi^j \frac{\partial^2 \eta_1}{\partial x^j \partial x^{\alpha_{m-1}}} & \cdots & \xi^j \frac{\partial^2 \eta_{\omega-1}}{\partial x^j \partial x^{\alpha_{m-1}}} & 0 & \xi^j \frac{\partial^2 \eta_{\omega+1}}{\partial x^j \partial x^{\alpha_{m-1}}} & \cdots & \xi^j \frac{\partial^2 \eta_{m-1}}{\partial x^j \partial x^{\alpha_{m-1}}}
\end{vmatrix}}{\mathcal{J}\left(\frac{\eta_1, \ldots, \eta_{m-1}}{x^{\alpha_1}, \ldots, x^{\alpha_{m-1}}}\right)}
$$

$$
-\frac{\partial F_\delta}{\partial \eta_\omega}\,\frac{\begin{vmatrix}
\xi^j \frac{\partial^2 \eta_1}{\partial x^j \partial x^{\alpha_1}} & \cdots & \xi^j \frac{\partial^2 \eta_{m-1}}{\partial x^j \partial x^{\alpha_1}} \\
\frac{\partial \eta_1}{\partial x^{\alpha_2}} & & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_2}} \\
\vdots & & \vdots \\
\frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix} + \cdots + \begin{vmatrix}
\frac{\partial \eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_1}} \\
\vdots & & \vdots \\
\frac{\partial \eta_1}{\partial x^{\alpha_{m-2}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-2}}} \\
\xi^j \frac{\partial^2 \eta_1}{\partial x^j \partial x^{\alpha_{m-1}}} & \cdots & \xi^j \frac{\partial^2 \eta_{m-1}}{\partial x^j \partial x^{\alpha_{m-1}}}
\end{vmatrix}}{\mathcal{J}\left(\frac{\eta_1, \ldots, \eta_{m-1}}{x^{\alpha_1}, \ldots, x^{\alpha_{m-1}}}\right)}
$$

$$
-\frac{\begin{vmatrix}
\frac{\partial \eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_1}} & \frac{\partial \xi^j}{\partial x^{\alpha_1}}\left(\frac{\partial g_\delta}{\partial y_b} y_{b,j} + \frac{\partial g_\delta}{\partial x^j}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_1}} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \frac{\partial \xi^j}{\partial x^{\alpha_{m-1}}}\left(\frac{\partial g_\delta}{\partial y_b} y_{b,j} + \frac{\partial g_\delta}{\partial x^j}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix}}{J\left(\frac{\eta_1, \cdots, \eta_{m-1}}{x^{\alpha_1}, \cdots, x^{\alpha_{m-1}}}\right)}
$$

Each of the terms in the ω'th column of the first two determinants in the above expression is identically zero by (I.1) and (I.2) hence the said determinants are identically zero.

5. We now want to show that the rest of the expression (I.9) is identically zero. To do this we use the fact that the expression (6.7) for $\frac{\partial F_\delta}{\partial \eta_\omega}$ is independent of the indices $\alpha_1, \cdots, \alpha_{m-1}$, hence on using Theorem 1.5, Ref. 7, p. 647, again we obtain the following:

$$
\frac{\partial \xi^j}{\partial x^{\alpha_1}}
\begin{vmatrix}
\frac{\partial \eta_1}{\partial x^j} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^j} \\
\frac{\partial \eta_1}{\partial x^{\alpha_2}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_2}} \\
\vdots & & \vdots \\
\frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix}
\frac{\partial F_\delta}{\partial \eta_\omega} =
$$

$$
\begin{vmatrix}
\frac{\partial \eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_1}} & \frac{\partial \xi^j}{\partial x^{\alpha_1}}\left(\frac{\partial g_\delta}{\partial y_b} y_{b,j} + \frac{\partial g_\delta}{\partial x^j}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_1}} \\
\frac{\partial \eta_1}{\partial x^{\alpha_2}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_2}} & 0 & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_2}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_2}} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & 0 & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix}
\tag{I.10}
$$

$$
+\begin{vmatrix}
\frac{\partial \xi^j}{\partial x^{\alpha_1}}\frac{\partial \eta_1}{\partial x^j} & \cdots & \frac{\partial \xi^j}{\partial x^{\alpha_1}}\frac{\partial \eta_{\omega-1}}{\partial x^j} & 0 & \frac{\partial \xi^j}{\partial x^{\alpha_1}}\frac{\partial \eta_{\omega+1}}{\partial x^j} & \cdots & \frac{\partial \xi^j}{\partial x^{\alpha_1}}\frac{\partial \eta_{m-1}}{\partial x^j} \\
\frac{\partial \eta_1}{\partial x^{\alpha_2}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_2}} & \left(\frac{\partial g_\delta}{\partial y_b} y_{b,\alpha_2} + \frac{\partial g_\delta}{\partial x^{\alpha_2}}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_2}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_2}} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial \eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \left(\frac{\partial g_\delta}{\partial y_b} y_{b,\alpha_{m-1}} + \frac{\partial g_\delta}{\partial x^{\alpha_{m-1}}}\right) & \frac{\partial \eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial \eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix}
$$

It is evident from the above, that, on adding $m-1$ relations of the type (I.10) it is possible to write the following equation

$$
\left[\begin{vmatrix}
\frac{\partial\xi^j}{\partial x^{\alpha_1}}\frac{\partial\eta_1}{\partial x^j} & \cdots & \frac{\partial\xi^j}{\partial x^{\alpha_1}}\frac{\partial\eta_{m-1}}{\partial x^j} \\
\frac{\partial\eta_1}{\partial x^{\alpha_2}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_2}} \\
\vdots & & \vdots \\
\frac{\partial\eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix}
+\cdots+
\begin{vmatrix}
\frac{\partial\eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_1}} \\
\vdots & & \vdots \\
\frac{\partial\eta_1}{\partial x^{\alpha_{m-2}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-2}}} \\
\frac{\partial\xi^j}{\partial x^{\alpha_{m-1}}}\frac{\partial\eta_1}{\partial x^j} & \cdots & \frac{\partial\xi^j}{\partial x^{\alpha_{m-1}}}\frac{\partial\eta_{m-1}}{\partial x^j}
\end{vmatrix}\right]\frac{\partial F_\delta}{\partial\eta_\omega} =
$$

$$
\begin{vmatrix}
\frac{\partial\eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_1}} & \frac{\partial\xi^j}{\partial x^{\alpha_1}}\left(\frac{\partial g_\delta}{\partial y_b}y_{b,j}+\frac{\partial g_\delta}{\partial x^j}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_1}} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial\eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \frac{\partial\xi^j}{\partial x^{\alpha_{m-1}}}\left(\frac{\partial g_\delta}{\partial y_b}y_{b,j}+\frac{\partial g_\delta}{\partial x^j}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix}
$$

$$
+\begin{vmatrix}
\frac{\partial\xi^j}{\partial x^{\alpha_1}}\frac{\partial\eta_1}{\partial x^j} & \cdots & \frac{\partial\xi^j}{\partial x^{\alpha_1}}\frac{\partial\eta_{\omega-1}}{\partial x^j} & 0 & \frac{\partial\xi^j}{\partial x^{\alpha_1}}\frac{\partial\eta_{\omega+1}}{\partial x^j} & \cdots & \frac{\partial\xi^j}{\partial x^{\alpha_1}}\frac{\partial\eta_{m-1}}{\partial x^j} \\
\frac{\partial\eta_1}{\partial x^{\alpha_2}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_2}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_2}+\frac{\partial g_\delta}{\partial x^{\alpha_2}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_2}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_2}} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial\eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_{m-1}}+\frac{\partial g_\delta}{\partial x^{\alpha_{m-1}}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-1}}}
\end{vmatrix}+\cdots \tag{I.11}
$$

$$
+\begin{vmatrix}
\frac{\partial\eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_1}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_1}+\frac{\partial g_\delta}{\partial x^{\alpha_1}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_1}} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{\partial\eta_1}{\partial x^{\alpha_{m-2}}} & \cdots & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_{m-2}}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_{m-2}}+\frac{\partial g_\delta}{\partial x^{\alpha_{m-2}}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_{m-2}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-2}}} \\
\frac{\partial\xi^j}{\partial x^{\alpha_{m-1}}}\frac{\partial\eta_1}{\partial x^j} & \cdots & \frac{\partial\xi^j}{\partial x^{\alpha_{m-1}}}\frac{\partial\eta_{\omega-1}}{\partial x^j} & 0 & \frac{\partial\xi^j}{\partial x^{\alpha_{m-1}}}\frac{\partial\eta_{\omega+1}}{\partial x^j} & \cdots & \frac{\partial\xi^j}{\partial x^{\alpha_{m-1}}}\frac{\partial\eta_{m-1}}{\partial x^j}
\end{vmatrix}
$$

6. Solving (I.11) for the first determinant on the right hand side and substituting the resulting expression into (I.9) we obtain:

$$U^{(1)}\left(\frac{\partial F_\delta}{\partial \eta_\omega}\right) =$$

$$\frac{\begin{vmatrix} \frac{\partial\xi^j}{\partial x^{\alpha_1}}\frac{\partial\eta_1}{\partial x^j}+\xi^j\frac{\partial^2\eta_1}{\partial x^j\partial x^{\alpha_1}} & \cdots & \frac{\partial\xi^j}{\partial x^{\alpha_1}}\frac{\partial\eta_{\omega-1}}{\partial x^j}+\xi^j\frac{\partial^2\eta_{\omega-1}}{\partial x^j\partial x^{\alpha_1}} & 0 & \frac{\partial\xi^j}{\partial x^{\alpha_1}}\frac{\partial\eta_{\omega+1}}{\partial x^j}+\xi^j\frac{\partial^2\eta_{\omega+1}}{\partial x^j\partial x^{\alpha_1}} & \cdots & \frac{\partial\xi^j}{\partial x^{\alpha_1}}\frac{\partial\eta_{m-1}}{\partial x^j}+\xi^j\frac{\partial^2\eta_{m-1}}{\partial x^j\partial x^{\alpha_1}} \\ \frac{\partial\eta_1}{\partial x^{\alpha_2}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_2}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_2}+\frac{\partial g_\delta}{\partial x^{\alpha_2}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_2}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_2}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial\eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_{m-1}}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_{m-1}}+\frac{\partial g_\delta}{\partial x^{\alpha_{m-1}}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1, \ldots, \eta_{m-1}}{x^{\alpha_1}, \ldots, x^{\alpha_{m-1}}}\right)} + \cdots$$

$$+\frac{\begin{vmatrix} \frac{\partial\eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_1}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_1}+\frac{\partial g_\delta}{\partial x^{\alpha_1}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_1}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_1}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial\eta_1}{\partial x^{\alpha_{m-2}}} & \cdots & \frac{\partial\eta_{\omega-1}}{\partial x^{\alpha_{m-2}}} & \left(\frac{\partial g_\delta}{\partial y_b}y_{b,\alpha_{m-2}}+\frac{\partial g_\delta}{\partial x^{\alpha_{m-2}}}\right) & \frac{\partial\eta_{\omega+1}}{\partial x^{\alpha_{m-2}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-2}}} \\ \frac{\partial\xi^j}{\partial x^{\alpha_{m-1}}}\frac{\partial\eta_1}{\partial x^j}+\xi^j\frac{\partial^2\eta_1}{\partial x^j\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\xi^j}{\partial x^{\alpha_{m-1}}}\frac{\partial\eta_{\omega-1}}{\partial x^j}+\xi^j\frac{\partial^2\eta_{\omega-1}}{\partial x^j\partial x^{\alpha_{m-1}}} & 0 & \frac{\partial\xi^j}{\partial x^{\alpha_{m-1}}}\frac{\partial\eta_{\omega+1}}{\partial x^j}+\xi^j\frac{\partial^2\eta_{\omega+1}}{\partial x^j\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\xi^j}{\partial x^{\alpha_{m-1}}}\frac{\partial\eta_{m-1}}{\partial x^j}+\xi^j\frac{\partial^2\eta_{m-1}}{\partial x^j\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1, \ldots, \eta_{m-1}}{x^{\alpha_1}, \ldots, x^{\alpha_{m-1}}}\right)}$$

$$+\frac{\begin{vmatrix} \frac{\partial\xi^j}{\partial x^{\alpha_1}}\frac{\partial\eta_1}{\partial x^j}+\xi^j\frac{\partial^2\eta_1}{\partial x^j\partial x^{\alpha_1}} & \cdots & \frac{\partial\xi^j}{\partial x^{\alpha_1}}\frac{\partial\eta_{m-1}}{\partial x^j}+\xi^j\frac{\partial^2\eta_{m-1}}{\partial x^j\partial x^{\alpha_1}} \\ \frac{\partial\eta_1}{\partial x^{\alpha_2}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_2}} \\ \vdots & & \vdots \\ \frac{\partial\eta_1}{\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1, \ldots, \eta_{m-1}}{x^{\alpha_1}, \ldots, x^{\alpha_{m-1}}}\right)} + \cdots$$

$$+\frac{\begin{vmatrix} \frac{\partial\eta_1}{\partial x^{\alpha_1}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_1}} \\ \vdots & & \vdots \\ \frac{\partial\eta_1}{\partial x^{\alpha_{m-2}}} & \cdots & \frac{\partial\eta_{m-1}}{\partial x^{\alpha_{m-2}}} \\ \frac{\partial\xi^j}{\partial x^{\alpha_{m-1}}}\frac{\partial\eta_1}{\partial x^j}+\xi^j\frac{\partial^2\eta_1}{\partial x^j\partial x^{\alpha_{m-1}}} & \cdots & \frac{\partial\xi^j}{\partial x^{\alpha_{m-1}}}\frac{\partial\eta_{m-1}}{\partial x^j}+\xi^j\frac{\partial^2\eta_{m-1}}{\partial x^j\partial x^{\alpha_{m-1}}} \end{vmatrix}}{J\left(\frac{\eta_1, \ldots, \eta_{m-1}}{x^{\alpha_1}, \ldots, x^{\alpha_{m-1}}}\right)} .$$

One row in each of the above determinants is identically zero by (I.3) hence we immediately have the result that:

$$U^{(1)}\left(\frac{\partial F_\delta}{\partial \eta_\omega}\right) \equiv 0 \qquad \left[\begin{array}{l}\delta = 1, \cdots, n, \\ \omega = 1, \cdots, m-1.\end{array}\right]; \tag{I.12}$$

therefore,we have shown that the quantities $\frac{\partial F_\delta}{\partial \eta_\omega}$ are absolute invariants of the enlargement $G_1^{E_1}$ of the group Gl.

APPENDIX II

GROUP-THEORETIC PROOF OF THE BUCKINGHAM PI-THEOREM

II.1 Introduction

In the preceding sections it has been shown that the group-theoretic reasoning provides a foundation for a similarity theory of partial differential equations. In this Appendix we shall show that group-theoric methods will also yield a rigorous proof of the Buckingham Pi-Theorem.

II.2 Notation

In addition to the symbols listed at the beginning of this thesis we shall need to define the following:

$q_1, \cdots, q_r$	Fundamental units (that is: mass, length, time).
$Q_1, \cdots, Q_m$	Physical quantities (density, viscosity, etc.).
G_r	Any continuous r-parameter group of transformations.
G_r'	The particular continuous r-parameter group of transformations obtained by changes in the fundamental units.
G_r''	The particular continuous r-parameter group of transformations induced in the physical quantities by changes in the fundamental units.
$\Pi_i(Q_1, \cdots, Q_m)$	Dimensionless functions of the physical quantities, that is: absolutely invariant under G_r''.
$F(Q_1, \cdots, Q_m)$	Any function of the physical quantities.

$G(\Pi_1, \cdots, \Pi_n)$ A dimensionless function of $\Pi_1, \cdots, \Pi_n$; that is, absolutely invariant under G_r'' .

Bar under repeated index The summation convention is inoperative for the index.

II.3 Formulation of the Problem

The Buckingham Pi-Theorem is the fundamental result which underlies all of dimensional analysis. It is usually stated loosely in the following form:

"If r is the number of fundamental units and $F(Q_1, \cdots, Q_m) = 0$ is an equation expressing a relation among m physical quantities then there exist $m-r$ dimensionless quantities $\Pi_1, \cdots, \Pi_{m-r}$ such that $F=0$, in dimensionless form, can be expressed as

$$G(\Pi_1, \cdots, \Pi_{m-r}) = 0 \text{."}$$

The above statement is not precise since no conditions have been imposed on $F(Q_1, \cdots, Q_m)$.

It is our purpose here to provide explicit conditions under which Buckingham's Pi-Theorem will hold. The theorem which shall be given has the nature of an existence theorem. It does not provide a means of determining the dimensionless quantities $\Pi_1, \cdots, \Pi_{m-r}$ but does assure us that, under very general conditions, they exist.

The m physical quantities $Q_1, \cdots, Q_m$ when expressed in terms of the r fundamental units take the form:

$$Q_j = \prod_{i=1}^{r} q_{\underline{i}}^{c_{ij}} \quad , \ j = 1, \cdots, m. \tag{II.1}$$

Any change from one system of fundamental units to another introduces positive numerical factors (α_i) such that the relation among different systems of fundamental units can be expressed as:

$$G_r' : \quad \overline{q}_i = \alpha_i q_i \;\; ; \;\; i=1,\cdots,r \; , \; \alpha_i > 0 \; . \qquad \text{(II.2)}$$

Evidently (II.2) is a continuous r-parameter group of transformations. The inverse is given by $\frac{1}{\alpha_i}$, the unit by $\alpha_i = 1$ and the closure and associativity conditions hold. Since the α_i can range over the positive real numbers it is evident that the group is continuous.

Note, in addition, that (II.2) induces the following continuous r-parameter group of transformations in (II.1):

$$G_r'' : \quad \overline{Q}_j = Q_j \prod_{i=1}^{r} \alpha_i^{\,c_{ij}} \;\; ; \; j=1,\cdots,m, \; m>r . \qquad \text{(II.3)}$$

Clearly, any theorem on $F(Q_1,\cdots,Q_m)$ must then depend on the properties of G_r'' and not those of G_r' . The group G_r' then plays a subsidiary role in the proof of the Buckingham Pi-Theorem.

Requiring that the equation $F(Q_1,\cdots,Q_m)=0$ be dimensionless is equivalent to saying that it be invariant under the group G_r'' . The properties of such an equation will be investigated in Section II.5.

II.4 Review of Previous Proofs of Buckingham's Pi-Theorem

Buckingham (Ref. 21) in his original proof of the Pi-Theorem effectively assumes that the function $F(Q_1,\cdots,Q_m)$ is analytic* in $Q_1,\cdots,Q_m$. This assumption is not explicitly given in his statement of the theorem, but it is used in his proof of the theorem and as such drastically restricts the class of functions which can be

*It can be expanded in a Taylor series about the origin.

treated by dimensional analysis.

Birkhoff (Ref. 2) on using the notation of "sets of transitivity" relaxes the assumptions on $F(Q,\cdots,Q_m)$ by only requiring that it and its first partial derivatives be continuous. He does not succeed completely since he effectively proves Theorem II.1 of Section II.6. Thus, he can show that there exist "$m-s$ $(s\leq r)$" dimensionless quantities $\Pi_1,\cdots,\Pi_{m-s}$ but does not prove that $s=r$ as is required by the statement of the Pi-Theorem.

II.5 Conformally Invariant Functions Under Continuous r-Parameter Groups of Transformations.

To prove Buckingham's Pi-Theorem we first need to consider the conditions which must be satisfied by $F(Q_1,\cdots,Q_m)$ so that the <u>equation</u>

$$F(Q_1,\cdots,Q_m)=0 \tag{II.5}$$

is invariant under a continuous r-parameter group of transformations.

This investigation will parallel that given in Section V. We begin by considering the most general type of invariance - conformal invariance - of the function F under the transformations of a G_r.

By the method of proof used in Theorem 5.1 it can be shown that the necessary and sufficient condition for the function $F(Q_1,\cdots,Q_m)$ to be conformally invariant (see Definition 5.2) under a G_r is that

$$X_a F \equiv \xi_a^i(Q\text{'s})\frac{\partial F}{\partial Q_i} = f_a(Q_1,\cdots,Q_m)F(Q_1,\cdots,Q_m) \quad \left[\begin{array}{l} a=1,\cdots,r \\ i=1,\cdots,m,\ m>r \end{array}\right], \tag{II.6}$$

where $X_a F$ are the symbols of G_r (see Ref. 5, p. 72) and the $f_a(Q_1,\cdots,Q_m)$ are some functions of $Q_1,\cdots,Q_m$.

The general form of F can now be determined by obtaining the general solution of (II.6). First, we note that the general solution of the homogeneous system (II.6) defines the absolute invariants of G_r (Ref. 5, p. 72) - call this solution $F_0(Q_1,\cdots,Q_m)$. Since $F(Q_1,\cdots,Q_m)$, for arbitrary $Q_1,\cdots,Q_m$, is not identically zero (II.6) can be written as:

$$X_a \ln F = f_a(Q_1,\cdots,Q_m) \qquad \text{(II.7)}$$

- denote a particular integral of (II.7) by $\varsigma(Q_1,\cdots,Q_m)$. That such a particular integral exists is assured by the theory of systems of first order partial differential equations.

By a theorem contained in Ref. 22 (p. 90) the general solution of (II.7) can then be expressed as:

$$\ln F = \varsigma(Q_1,\cdots,Q_m) + \ln F_0(Q_1,\cdots,Q_m);$$

therefore,

$$F = e^{\varsigma(Q_1,\cdots,Q_m)} F_0(Q_1,\cdots,Q_m) \qquad \text{(II.8)}$$

is the general form of a function F conformally invariant under a G_r.

On using (II.8) the equation (II.5) can be written as

$$F_0(Q_1,\cdots,Q_m) = 0 \qquad \text{(II.9)}$$

since the function $e^{\varsigma(Q_1,\cdots,Q_m)}$, for arbitrary $Q_1,\cdots,Q_m$, is not identically zero.

The above arguments show that if the equation (II.5) is invariant under a G_r then its further properties can be obtained by investigating the functions $F_0(Q_1,\cdots,Q_m)$. This will be done in what follows.

II.6 Results Leading to the Proof of the Pi-Theorem

The following theorem can be proved for any function which is absolutely invariant under a continuous r-parameter group of transformations.

Theorem II.1

If the function $F_o(Q_1,\cdots,Q_m)$ is absolutely invariant under a G_r then, for $F_o(Q_1,\cdots,Q_m)\in C^{(1)}$ and $m>r$, there exist " $m-s$ ", $1\leqslant s\leqslant r$, absolute invariant functions of the Q's

$$\Pi_k(Q_1,\cdots,Q_m), \quad k=1,\cdots,m-s,$$

such that

$$F_o(Q_1,\cdots,Q_m)=G(\Pi_1,\cdots,\Pi_{m-s})$$

in some neighborhood of the $Q_1,\cdots,Q_m$ space.

Proof

1. The proof is evident if we make use of the following two theorems:

Theorem A (Ref. 5, p. 62)

"A necessary and sufficient condition that the function $F_o(Q_1,\cdots,Q_m)\in C^{(1)}$ be absolutely invariant under a G_r is that it satisfy the system of first order partial differential equations

$$X_a F_o \equiv \xi_a^i(Q's)\frac{\partial F_o}{\partial Q_i}=0 \qquad \begin{bmatrix} a=1,\cdots,r, \\ i=1,\cdots,m,\ m>r \end{bmatrix}, \qquad \text{(II.10)}$$

where the $\xi_a^i(Q$'s) are the vectors of the group G_r."

and

Theorem B (Ref. 22, p. 72)

"If the set of first order partial differential equations $X_aF=0$, $a=1,\cdots,q$, is complete then it has " $m-q$ " functionally independent solutions and its general solution is an arbitrary function of these solutions."

2. The theorem will be proved if we determine the number of equations in (II.10) which form a complete system.

Evidently the rank ($\mathcal{R}$) of the matrix

$$M = \left\| \xi_a^i (Q\text{'s}) \right\| \qquad \begin{bmatrix} a=1,\cdots,r, \\ i=1,\cdots,m,\ m>r \end{bmatrix}$$

will be such that

$$\mathcal{R}(M) = s \leqslant r . \tag{II.11}$$

3. The result (II.11) implies that s of the equations $X_a F_0 = 0$ are independent, say:

$$X_\sigma F_0 = 0 , \quad \sigma = 1,\cdots,s . \tag{II.12}$$

The system (II.12) then forms a complete system (Ref. 5, p. 69).

4. The proof of the theorem is then obtained on combining Theorems A and B and using the result (II.12).

It is also necessary to know under what conditions the " r " parameters of the group G_r'' are essential. To this end we state:

Theorem II.2

A necessary and sufficient condition for the parameters $\alpha_1,\cdots,\alpha_r$ of the group G_r'' be essential is that, for $\alpha_i > 0$, $i=1,\cdots,r$, $\mathcal{R}\| c_{ij} \| = r$.

Proof

1. The proof will consist of showing that there does not exist a set of functions $\varphi^\beta(\alpha\text{'s})$ $(\beta=1,\cdots,r)$, other than the trivial $\varphi^\beta(\alpha\text{'s}) \equiv 0$, such that the functions $\overline{Q_j}$ satisfy an equation of the type (Ref. 5, p. 9):

$$\sum_{\beta=1}^{r} \varphi^{\beta}(\alpha\text{'s}) \frac{\partial \overline{Q}_j}{\partial \alpha_\beta} = 0 \quad ; \; j=1,\cdots,m, \; m>r. \tag{II.13}$$

But, from (II.3):

$$\frac{\partial \overline{Q}_j}{\partial \alpha_\beta} = \frac{c_{\beta j}}{\alpha_\beta} \overline{Q}_j \; ;$$

therefore, (II.13) can, for arbitrary $\overline{Q}_j$, be written as:

$$\sum_{\beta=1}^{r} \frac{c_{\beta j}}{\alpha_\beta} \varphi^{\beta}(\alpha\text{'s}) = 0 . \tag{II.14}$$

2. By hypothesis $R \|c_{\beta j}\| = r$. Hence, by suitable renumbering, the principal r-rowed minor of the matrix of coefficients of (II.14) can be written so that:

$$\left|\frac{c_{\delta k}}{\alpha_\delta}\right| = \alpha_1^{-1} \cdots \alpha_r^{-1} \; |c_{\delta k}| \neq 0 . \tag{II.15}$$

Therefore, by (II.15) and since the $\alpha_i > 0$, we have:

$$R \; \left\|\frac{c_{\beta j}}{\alpha_\beta}\right\| = r . \tag{II.16}$$

3. The result (II.16) then tells us that the only solutions of (II.14) are:

$$\varphi^{\beta}(\alpha\text{'s}) \equiv 0 \; , \quad \beta=1,\cdots,r.$$

This completes the proof of Theorem II.2

II.7 Proof of the Buckingham Pi-Theorem

Buckingham's Pi-Theorem can now be precisely stated in the following form:

Theorem II.3

If the equation $F(Q_1,\cdots,Q_m)=0$ is invariant under G_r'' and:

(a) $F(Q_1,\cdots,Q_m) \; \varepsilon \; C^{(1)}$

(b) $\quad R\|c_{ij}\| = r \qquad \begin{bmatrix} i=1,\cdots,r, \\ j=1,\cdots,m,\ m>r. \end{bmatrix}$

then there exist " $m-r$ " functionally independent absolute invariants of G_r''

$$\Pi_k(Q_1,\cdots,Q_m)\ ,\quad k=1,\cdots,m-r$$

such that in some neighborhood of the $(Q_1,\cdots,Q_m)$ -space, exclusive of the origin, it can be expressed as

$$G(\Pi_1,\cdots,\Pi_{m-r}) = 0\ .$$

Proof

1. The combination of Theorems II.1, II.2 and the result (II.9) will give the required result if we can show that:

$$R\left\|\xi_a^i(Q's)\right\| = r. \tag{II.17}$$

But, for G_r'' we have (Ref. 5, p. 20)

$$\xi_a^i(Q's) = -c_{a\underline{i}}Q_{\underline{i}}\ ;$$

therefore,

$$R\left\|\xi_a^i(Q's)\right\| = R\left\|-c_{a\underline{i}}Q_{\underline{i}}\right\|. \tag{II.18}$$

2. On repeating the argument used in the proof of Theorem II.2 we find that if the $Q_i \neq 0$ then the hypothesis (b) implies that (II.17) holds. The application of Theorems II.1, II.2 and the result of Section II.5 then completes the proof of Theorem II.3.

We have shown that the Pi-Theorem holds for all functions of class greater than or equal to $C^{(1)}$. Thus Buckingham's implicit assumption of analyticity of the function $F(Q_1,\cdots,Q_m)$ can be considerably relaxed. Furthermore, the hypothesis (b) is one which is not contained in Buckingham's statement of the Pi-Theorem. In practice this assumption provides a convenient test for determining the correctness of the initial set-up of a problem in dimensional analysis.