Aspects of Definability for Equivalence Relations

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ABSTRACT

Let F_{ω_1} be the countable admissible ordinal equivalence relation defined on ${}^{\omega}2$ by $x \; F_{\omega_1} \; y$ if and only if $\omega_1^x = \omega_1^y$. Some invariant descriptive set theoretic properties of F_{ω_1} will be explored using infinitary logic in countable admissible fragments as the main tool. Marker showed F_{ω_1} is not the orbit equivalence relation of a continuous action of a Polish group on ${}^{\omega}2$. Becker strengthened this to show F_{ω_1} is not even the orbit equivalence relation of a $\boldsymbol{\Delta}_1^1$ action of a Polish group. However, Montalbán has shown that F_{ω_1} is $\boldsymbol{\Delta}_1^1$ reducible to an orbit equivalence relation of a Polish group action; in fact, F_{ω_1} is classifiable by countable structures. It will be shown here that F_{ω_1} must be classified by structures of high Scott rank. Let E_{ω_1} denote the equivalence of order types of reals coding well-orderings. If E and F are two equivalence relations on Polish spaces X and Y, respectively, $E \leq_{\mathrm{a}\boldsymbol{\Delta}_1^1} F$ denotes the existence of a $\boldsymbol{\Delta}_1^1$ function $f : X \to Y$ which is a reduction of E to F, except possibly on countably many classes of E. Using a result of Zapletal, the existence of a measurable cardinal implies $E_{\omega_1} \leq_{\mathrm{a}\boldsymbol{\Delta}_1^1} F_{\omega_1}$. However, it will be shown that in Gödel's constructible universe L (and set generic extensions of L), $E_{\omega_1} \leq_{\mathrm{a}\boldsymbol{\Delta}_1^1} F_{\omega_1}$ is false. Lastly, the techniques of the previous result will be used to show that in L (and set generic extensions of L), the isomorphism relation induced by a counterexample to Vaught's conjecture cannot be $\boldsymbol{\Delta}_1^1$ reducible to F_{ω_1}. This shows the consistency of a negative answer to a question of Sy-David Friedman.

Let I be a σ-ideal on a Polish space X so that the associated forcing of I^+ $\boldsymbol{\Delta}_1^1$ sets ordered by $\subseteq$ is a proper forcing. Let E be a Σ_1^1 or a $\boldsymbol{\Pi}_1^1$ equivalence relation on X with all equivalence classes $\boldsymbol{\Delta}_1^1$. If for all $z \in H_{(2^{\aleph_0})^+}$, $z^\sharp$ exists, then there exists an I^+ $\boldsymbol{\Delta}_1^1$ set $C \subseteq X$ such that $E \restriction C$ is a $\boldsymbol{\Delta}_1^1$ equivalence relation.

(Joint with Magidor.) In ZFC, if there is a measurable cardinal with infinitely many Woodin cardinals below it, then for every equivalence relation $E \in L(\mathbb{R})$ on $\mathbb{R}$ with all $\boldsymbol{\Delta}_1^1$ classes and every σ-ideal I on $\mathbb{R}$ so that the associated forcing $\mathbb{P}_I$ of I^+ $\boldsymbol{\Delta}_1^1$ subsets is proper, there exists some I^+ $\boldsymbol{\Delta}_1^1$ set C so that $E \restriction C$ is a $\boldsymbol{\Delta}_1^1$ equivalence relation. In $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} + V = L(\mathscr{P}(\mathbb{R}))$, for every equivalence relation E on $\mathbb{R}$ with all $\boldsymbol{\Delta}_1^1$ classes and every σ-ideal I on $\mathbb{R}$ so that the associated forcing $\mathbb{P}_I$ is absolutely proper, there is some I^+ $\boldsymbol{\Delta}_1^1$ set C so that $E \restriction C$ is a $\boldsymbol{\Delta}_1^1$ equivalence relation.

PUBLISHED CONTENT AND CONTRIBUTIONS

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[2] William Chan and Menachem Magidor. "When an Equivalence Relation with All Borel Classes Will Be Borel Somewhere?" In: *ArXiv e-prints* (Aug. 2016). arXiv: 1608.04913 [math.LO].

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Chapter 1

INTRODUCTION

1.1 Introduction

This thesis studies some aspects of definability for equivalence relations on Polish spaces. A variety of set theoretical tools is used in this study including admissibility, constructibility, forcing, absoluteness, and large cardinals.

The thesis is a collection of three papers which have have been accepted for publication or have been submitted: [2] *The Countable Admissible Ordinal Equivalence Relation*, [1] *Equivalence Relations Which Are Borel Somewhere*, and [3] *When an Equivalence Relation with All Borel Classes will be Borel Somewhere?* (with Menachem Magidor).

For a more detailed introduction to each paper, see the introduction section of each chapter.

The second chapter is the paper *The Countable Admissible Ordinal Equivalence Relation*. The countable admissible ordinal equivalence relation is denoted F_{ω_1} and defined on ${}^\omega 2$ by $x\ F_{\omega_1}\ y$ if and only if $\omega_1^x = \omega_1^y$. Here ω_1^x is the least ordinal not isomorphic to a x-recursive well-ordering on ω and is called the Church-Kleene ordinal relative to x. Sacks [6] showed that every countable admissible ordinal is of the form ω_1^x for some $x \in {}^\omega 2$. The name of this equivalence relation is derived from this fact. The main tool used in this chapter to study this equivalence relation is infinitary logic in countable admissible fragments.

The equivalence relation F_{ω_1} is classifiable by countable structure meaning that it is $\mathbf{\Delta}_1^1$ reducible to the isomorphism relation of countable $\mathscr{L}$-structure for some countable first order language $\mathscr{L}$. For each $\mathscr{L}$-structure M, there is an ordinal known as the Scott rank of M, denoted $\mathrm{SR}(M)$, which measures the complexity of determining isomorphism with M in the sense that it corresponds to the quantifier rank of a canonical infinitary sentence that completely characterizes M up to isomorphism among the countable $\mathscr{L}$-structures. It will be shown that if a function Φ is Δ_1^1 and witnesses a reduction of F_{ω_1} to isomorphism of countable $\mathscr{L}$-structure where $\mathscr{L}$ is a recursive langauge, then $\mathrm{SR}(\Phi(x)) \geq \omega_1^x$ for all $x \in {}^\omega 2$.

The paper is also concerned with comparisons of several well-known thin Σ_1^1 equivalence relations. An equivalence relation E is thin if and only if if there is no perfect set of pairwise E-inequivalent elements. F_{ω_1} is a thin Σ_1^1 equivalence relation with all classes $\mathbf{\Delta}_1^1$. E_{ω_1} is the equivalence relation defined on ${}^\omega 2$ by $x \, E_{\omega_1} \, y$ if and only if $(x \notin \mathrm{WO} \wedge y \notin \mathrm{WO}) \vee (x \cong y)$, where WO is the set of reals coding well-orderings and $\cong$ is isomorphism as linear orders. E_{ω_1} is a thin Σ_1^1 equivalence relation with one Σ_1^1 but not $\mathbf{\Delta}_1^1$ class consisting of the non-well-orderings and all other classes $\mathbf{\Delta}_1^1$. A counterexample to Vaught's conjecture is a countable theory in $\mathscr{L}_{\omega_1,\omega}$ which has uncountably many models up to isomorphism but no perfect set of non-isomorphic models. The isomorphism relation of a counterexample to Vaught's conjecture is a thin Σ_1^1 equivalence relation with uncountably many $\mathbf{\Delta}_1^1$ classes. (It is open whether such a theory exists, and this old question is known as the Vaught's conjecture.)

The method to compare these thin Σ_1^1 equivalence relations is $\mathbf{\Delta}_1^1$ reducibility of equivalence relations, which is denoted by $\leq_{\mathbf{\Delta}_1^1}$. $F_{\omega_1} \leq_{\mathbf{\Delta}_1^1} E_{\omega_1}$ is impossible due to the boundedness principle. $E_{\omega_1} \leq_{\mathbf{\Delta}_1^1} F_{\omega_1}$ is impossible since E_{ω_1} has one Σ_1^1 but not $\mathbf{\Delta}_1^1$ class yet F_{ω_1} has only $\mathbf{\Delta}_1^1$ classes. The former failure is due to a global reason. The latter is merely a local failure due to a single equivalence class. Zapletal [7] invented the almost $\mathbf{\Delta}_1^1$ reduction to ignore countably many classes. An almost $\mathbf{\Delta}_1^1$ reduction from E to F, denoted $E \leq_{a\mathbf{\Delta}_1^1} F$, is just a $\mathbf{\Delta}_1^1$ reduction which may fail to be a reduction on at most countably many E-classes. From a general result of Zapletal about equivalence relation with infinite pinned cardinals, it follows, assuming there is a measurable cardinal, that $E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} F_{\omega_1}$ and $E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} E_T$, where T is a counterexample to Vaught's conjecture and E_T is its isomorphism equivalence relation.

This chapter show that $E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} F_{\omega_1}$ does not hold in Gödel constructible universe L and set-generic extensions of L. So relative to a measurable cardinal (in fact $0^\sharp$ is enough), the statement $E_{\omega_1} \leq_{a\mathbf{\Delta}_1^1} F_{\omega_1}$ is independent of ZFC. Similarly, it will be shown that $E_T \leq_{\mathbf{\Delta}_1^1} F_{\omega_1}$ does not hold in L or set-generic extensions of L.

The third and fourth chapters are concerned with questions of the following type: If E is an equivalence relation on ${}^\omega\omega$, is there some $\mathbf{\Delta}_1^1$ set C so that $E \restriction C$ is a $\mathbf{\Delta}_1^1$ equivalence relation?

Here $E \restriction C$ is the set $E \cap (C \times C)$. An immediate observation is that any equivalence relation E will be $\mathbf{\Delta}_1^1$ on a countable set. Hence C needs to be large in some sense. σ-ideals (containing all singletons) will always contain all countable sets. Belonging to σ-ideals is a notion of smallness. A subset that does not belong to a σ-ideal I is called an I^+ set. The above question will be studied by asking C to be large in the sense that it is I^+ for various σ-ideals I.

It is unclear how to approach this question if I is just an arbitrary σ-ideal. Some ideals of interests in mathematics are the ideals of countable sets, the ideal σ-generated by closed nowhere dense sets (the meager ideal), and the ideal σ-generated by the $\mathbf{\Delta}^1_1$ Lebesgue null sets (Lebesgue null ideal). These and many other familiar σ-ideals have a common property expressed using forcing.

Let I be a σ-ideal on ${}^\omega\omega$. Let $\mathbb{P}_I = (\mathbf{\Delta}^1_1 \setminus I, \subseteq, {}^\omega\omega)$ be the forcing of I^+ $\mathbf{\Delta}^1_1$ subsets ordered by $\subseteq$ with largest condition ${}^\omega\omega$. For all the examples of σ-ideals mentioned above, $\mathbb{P}_I$ satisfies a property isolated by Shelah known as properness. Properness is a useful notion in combinatorial set theory, since properness is preserved under countable support iteration and proper forcings do not collapse $\aleph_1$. For the descriptive set theoretic purposes of this paper, properness of $\mathbb{P}_I$ gives I^+ $\mathbf{\Delta}^1_1$ subsets consisting of points that are generic for countable elementary substructures of rank initial segments of the real universe. Because the points of the I^+ $\mathbf{\Delta}^1_1$ set are generic, many techniques of set theory such as absoluteness can be applied.

The question above is then modified to ask for an I^+ $\mathbf{\Delta}^1_1$ set C, where the ideal I has the property that $\mathbb{P}_I$ is proper. However, this is still not true. [4] showed that there is a Σ^1_1 equivalence relation E and σ-ideal I with $\mathbb{P}_I$ proper so that $E \upharpoonright C$ is Σ^1_1 but not $\mathbf{\Delta}^1_1$ for all I^+ $\mathbf{\Delta}^1_1$ sets C.

Therefore E needs to resemble $\mathbf{\Delta}^1_1$ equivalence relations in some further way for this question to possibly have a positive answer. A trivial observation is that every $\mathbf{\Delta}^1_1$ equivalence relation has all $\mathbf{\Delta}^1_1$ classes. Perhaps having all $\mathbf{\Delta}^1_1$ classes is a sufficient condition for a positive answer. The example from [4] is an Σ^1_1 equivalence relation with Σ^1_1 but not $\mathbf{\Delta}^1_1$ classes.

Zapletal, Kanovei, and Sabok [4] asked: If E is an Σ^1_1 equivalence relation with all $\mathbf{\Delta}^1_1$ classes and I is a σ-ideal with $\mathbb{P}_I$ proper, is there an I^+ $\mathbf{\Delta}^1_1$ set C so that $E \upharpoonright C$ is $\mathbf{\Delta}^1_1$?

Chapter 3 consists of the paper *Equivalence Relations Which Are Borel Somewhere*. This chapter shows that if $X^\sharp$ exists for all $X \in H_{(2^{\aleph_0})^+}$, then for every Σ^1_1 or $\mathbf{\Pi}^1_1$ equivalence relation with all $\mathbf{\Delta}^1_1$ classes and σ-ideal I with $\mathbb{P}_I$ proper, there is a I^+ $\mathbf{\Delta}^1_1$ set C so that $E \upharpoonright C$ is a $\mathbf{\Delta}^1_1$ equivalence relation. Other variations of the question for Σ^1_1 and $\mathbf{\Pi}^1_1$ equivalence relation are also shown under weaker large cardinal assumptions.

The results of this chapter give a positive answer to the question of interests when the equivalence relation belongs to the first level of the projective hierarchy. A natural question would to extend this question to projective equivalence relations with all $\mathbf{\Delta}^1_1$ classes.

Chapter 4 consists of the paper *When an Equivalence Relation with All Borel Classes Will Be Borel Somewhere?* written jointly with Menachem Magidor. This chapter will show that if there is a measurable cardinal above infinitely many Woodin cardinals, then for every equivalence relation $E \in L(\mathbb{R})$ with all classes $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$) and any σ-ideal I with $\mathbb{P}_I$ proper there is an I^+ $\mathbf{\Delta}^1_1$ set C so that $E \restriction C$ is $\mathbf{\Delta}^1_1$ is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$, respectively).

The result of Chapter 4 asserts that under the appropriate large cardinal assumptions, every equivalence relation E with all classes belonging to one of the three complexity classes $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, or $\mathbf{\Delta}^1_1$ on the first level of the projective hierarchy has an I^+ $\mathbf{\Delta}^1_1$ set C so that $E \restriction C$ belongs to the corresponding complexity class. The ultimate generalization of the results appearing above is formalized in the following question: Let Γ be some non-self-dual pointclass with some reasonable closure properties. Let E be an equivalence relation with all classes in Γ ($\check{\Gamma}$ or $\Gamma \cap \check{\Gamma}$) and I a σ-ideal with $\mathbb{P}_I$ proper, is there an I^+ $\mathbf{\Delta}^1_1$ set C so that $E \restriction C$ is in Γ ($\check{\Gamma}$ or $\Gamma \cap \check{\Gamma}$, respectively)?

With some conditions on E and the σ-ideal I, Neeman and Norwood [5] have shown under AD^+ that the answer is yes.

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Chapter 2

THE COUNTABLE ADMISSIBLE ORDINAL EQUIVALENCE RELATION

2.1 Introduction

If $x \in {}^{\omega}2$, ω_1^x denotes the supremum of the order types of x-recursive well-orderings on ω. Moreover, ω_1^x is also the minimum ordinal height of admissible sets containing x as an element. The latter definition will be more relevant for this paper.

The eponymous countable admissible ordinal equivalence relation, denoted by F_{ω_1}, is defined on ${}^{\omega}2$ by:

$$x \ F_{\omega_1} \ y \Leftrightarrow \omega_1^x = \omega_1^y.$$

It is a Σ_1^1 equivalence relation with all classes $\boldsymbol{\Delta}_1^1$. Moreover, F_{ω_1} is a thin equivalence relation, i.e., it has no perfect set of inequivalence elements. Some further properties of F_{ω_1} as an equivalence relation will be established in this paper.

Some basic results in admissibility theory and infinitary logic that will be useful throughout the paper will be reviewed in Section 2.2. This section will cover briefly topics such as KP, admissible sets, Scott ranks, and the Scott analysis. In this section, aspects of Barwise's theory of infinitary logic in countable admissible fragments, which will be the main tool in many arguments, will be reviewed. As a example of an application, a proof of a theorem of Sacks (Theorem 2.2.16), which establishes that every countable admissible ordinal is of the form ω_1^x for some $x \in {}^{\omega}2$, will be given. This proof serves as a template for other arguments. Sacks' theorem also explains why it is appropriate to call F_{ω_1} the "countable admissible ordinal equivalence relation".

There has been some early work on whether F_{ω_1} satisfies certain properties of equivalence relations related to generalization of Vaught's conjecture. For example, Marker in [14] has shown that F_{ω_1} is not induced by a continuous action of a Polish group on the Polish space ${}^{\omega}2$. Becker in [3], page 782, strengthened this to show that the equivalence relation F_{ω_1} is not an orbit equivalence relation of a $\boldsymbol{\Delta}_1^1$ group action of a Polish group. A natural question following these results would be whether F_{ω_1} is $\boldsymbol{\Delta}_1^1$ reducible to equivalence relations induced by continuous or $\boldsymbol{\Delta}_1^1$ actions of Polish groups. If such reductions do exist, another question could be what properties must these reductions have.

In Section 2.3, F_{ω_1} will be shown to be $\mathbf{\Delta}^1_1$ reducible to a continuous action of S_∞, i.e., it is classifiable by countable structures. An explicit Δ^1_1 classification of F_{ω_1} by countable structures in the language with a single binary relation symbol, due to Montalbán, will be provided. The classification of F_{ω_1} will use an effective construction of the Harrison linear ordering. This classification, denoted f, has the additional property that for all $x \in {}^\omega 2$, $\mathrm{SR}(f(x)) = \omega_1^x + 1$. This example was provided by Montalbán through communication with Marks and the author.

The explicit classification, f, mentioned above has images that are structures of high Scott rank. In Section 2.4, it will be shown that this is a necessary feature of all classification of F_{ω_1} by countable structures. The lightface version of the main result of this section is the following:

Theorem 2.4.2 *Let $\mathscr{L}$ be a recursive language. Let $S(\mathscr{L})$ denote the set of reals that code $\mathscr{L}$-structures on ω. If $f : {}^\omega 2 \to S(\mathscr{L})$ is a Δ^1_1 function such that $x\ F_{\omega_1}\ y$ if and only if $f(x) \cong_{\mathscr{L}} f(y)$, then for all x, $SR(f(x)) \geq \omega_1^x$.*

The more general form considers reductions that are $\Delta^1_1(z)$ and involves a condition on the admissible spectrum of z. Intuitively, Theorem 2.4.2 (in its lightface form as stated above) asserts that any potential classification of F_{ω_1} must have high Scott rank in the sense that the image of any real under the reduction is a structure of high Scott rank. High Scott rank means that $\mathrm{SR}(f(x))$ is either ω_1^x or $\omega_1^x + 1$.

Section 2.5 is concerned with a weak form of reduction of equivalence relations, invented by Zapletal, called the almost $\mathbf{\Delta}^1_1$ reduction. If E and F are two Σ^1_1 equivalence relations on Polish space X and Y, respectively, then E is almost $\mathbf{\Delta}^1_1$ reducible to F (in symbols: $E \leq_{\mathrm{a}\mathbf{\Delta}^1_1} F$) if and only there is a $\mathbf{\Delta}^1_1$ function $f : X \to Y$ and a countable set A such that if x and y are not E-related to any elements of A, then $x\ E\ y$ if and only if $f(x)\ F\ f(y)$.

An almost Borel reduction is simply a reduction that may fail on countably many classes. Often Σ^1_1 equivalence relations may have a few unwieldy classes. The almost Borel reduction is especially useful since it can be used to ignore these classes. One example of such a Σ^1_1 equivalence relation is E_{ω_1} which is the isomorphism relation of well-orderings with a single class of non-well-orderings. It is defined on ${}^{\omega}2$ by:

$$x \; E_{\omega_1} \; y \Leftrightarrow (x, y \notin WO) \vee (\mathrm{ot}(x) = \mathrm{ot}(y)).$$

E_{ω_1} is a thin Σ^1_1 equivalence with one Σ^1_1 class and all the other classes are $\mathbf{\Delta}^1_1$.

Zapletal isolated an invariant of equivalence relations called the pinned cardinal. This invariant involves pinned names on forcings: an idea that appears implicitly or explicitly in the works of Silver, Burgess, Hjorth, and Zapletal in the study of thin Σ^1_1 equivalence relations. Zapletal showed that there is a deep connection between E_{ω_1}, almost $\mathbf{\Delta}^1_1$ reducibilities, and pinned cardinals under large cardinal assumptions:

Theorem 2.5.7 ([21] Theorem 4.2.1, [22] Theorem 4.4.1) *If there exists a measurable cardinal and E is a Σ^1_1 equivalence relation with infinite pinned cardinal, then $E_{\omega_1} \leq_{a\mathbf{\Delta}^1_1} E$.*

Given that this result involves large cardinals, a natural question would be to explore the consistency results surrounding Zapletal's theorem. For example, a natural question is whether ZFC can prove the above result of Zapletal. More specifically, is this result true in Gödel constructible universe L? This investigation leads to F_{ω_1} in the following way: It will be shown that F_{ω_1} has infinite pinned cardinal. Hence, with a measurable cardinal, $E_{\omega_1} \leq_{\mathrm{a}\mathbf{\Delta}^1_1} F_{\omega_1}$ via the result of Zapletal. (This also holds if $0^\sharp$ exists.)

The main result of this section is

Theorem 2.5.11 *The statement $E_{\omega_1} \leq_{a\mathbf{\Delta}^1_1} F_{\omega_1}$ is not true in L (and set generic extensions of L).*

This result is proved by using infinitary logic in admissible fragments to show that if f is a $\Delta^1_1(z)$ function which witnesses $E_{\omega_1} \leq_{\mathrm{a}\mathbf{\Delta}^1_1} F_{\omega_1}$, then z has an admissibility spectrum which is full of gaps relative to the set of all admissible ordinals. No constructible real (or even a real set generic over L) can have such a property.

The final section addresses a question of Sy-David Friedman using the techniques of the previous section. Essentially, the question is:

Question 2.6.3 Is it possible that the isomorphism relation of a counterexample to

Vaught's conjecture is $\mathbf{\Delta}_1^1$ bireducible to F_{ω_1}?

The main result of this final section is:

Theorem 2.6.9 *In L (and set generic extensions of L), no isomorphism relation of a counterexample to Vaught's conjecture can be $\mathbf{\Delta}_1^1$ reducible to F_{ω_1}.*

This yields a negative answer to Friedman's question in L and set generic extensions of L.

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2.2 Admissibility and Infinitary Logic

The reader should refer to [2] for definitions and further details about admissibility.

Let $\dot{\in}$ denote a binary relation symbol. Let $\mathscr{L}$ be a language such that $\dot{\in} \in \mathscr{L}$. $\mathsf{KP}_{\mathscr{L}}$ denotes Kripke-Platek Set Theory in the language $\mathscr{L}$ with $\dot{\in}$ serving as the distinguished membership symbol. The $\mathscr{L}$ subscript will usually be concealed. $\mathsf{KP} + \mathsf{INF}$ is KP augmented with the axiom of infinity.

Definition 2.2.1. Let $\mathscr{L}$ be a language containing $\dot{\in}$. A $\mathscr{L}$-structure $\mathcal{A} = (A, \dot{\in}^{\mathcal{A}}, \dots)$ is an *admissible set* if and only if $\mathcal{A} \models \mathsf{KP}$, A is a transitive set, and $\dot{\in}^{\mathcal{A}} = \in \restriction A$.

If $\mathcal{A}$ is an admissible set, then $o(\mathcal{A}) = A \cap \mathrm{ON}$.

An ordinal α is an *admissible ordinal* if and only if there is an admissible set $\mathcal{A}$ such that $\alpha = o(\mathcal{A})$. More generally, if $x \in {}^{\omega}2$, an ordinal α is x-admissible if and only if there is an admissible set $\mathcal{A}$ such that $x \in A$ and $\alpha = o(\mathcal{A})$.

The *admissibility spectrum* of x is $\Lambda(x) = \{\alpha : \alpha \text{ is an } x\text{-admissible ordinal}\}$.

Definition 2.2.2. For $x \in {}^{\omega}2$, let $\mathcal{HYP}(x)$ denote the $\subseteq$-smallest admissible set containing x.

Definition 2.2.3. If $x \in {}^{\omega}2$, let $\omega_1^x = \min(\Lambda(x))$.

Proposition 2.2.4. *The function $(\alpha, x) \to L_\alpha(x)$, where $\alpha \in ON$ and x is a set, is a Σ_1 function in* KP. *In fact, it is Δ_1.*

Proof. See [2], Chapter II, Section 5 - 7. Also note that the function is defined on a Δ_1 set. $\square$

Proposition 2.2.5. *If $\mathcal{A}$ is an admissible set with $x \in A$ and $\alpha = o(\mathcal{A})$, then $L_\alpha(x)$ is an admissible set. In fact, $L_\alpha(x)$ is the $\subseteq$-smallest admissible set $\mathcal{A}$ such that $x \in A$ and $o(\mathcal{A}) = \alpha$.*

In particular, if α is an x-admissible ordinal, then $L_\alpha(x)$ is an admissible set.

Proof. See [2], Theorem II.5.7. $\square$

Proposition 2.2.6. *If $x \subseteq \omega$, then $\mathcal{HYP}(x) = L_{\omega_1^x}(x)$.*

Proof. See [2], Theorem II.5.9. $\square$

Definition 2.2.7. Let $x \in {}^\omega 2$. Let $\text{HYP}^x = {}^\omega 2 \cap \mathcal{HYP}(x) = {}^\omega 2 \cap L_{\omega_1^x}(x)$. HYP^x is the set of all x-hyperarithmetic reals.

In particular, x-hyperarithmetic reals are exactly those reals that appear in all admissible sets containing x.

Next, the relevant aspects of first order infinitary logic and admissible fragments will be reviewed. The detailed formalization can be found in [2], Chapter III.

Definition 2.2.8. Let $\mathscr{L}$ denote a first order language (a set of constant, relation, and function symbols). Fix a Δ_1 class $\{v_\alpha : \alpha \in \text{ON}\}$, which will represent variables. $\mathscr{L}_{\omega\omega}$ denotes the collection of finitary $\mathscr{L}$-formulas using variables from $\{v_i : i < \omega\}$. $\mathscr{L}_{\infty\omega}$ denotes the collection of all infinitary formulas with finitely many free variables.

Proposition 2.2.9. *In* $\mathsf{KP} + \mathsf{INF}$, *$\mathscr{L}_{\omega\omega}$ is a set. In* KP, *$\mathscr{L}_{\infty\omega}$ is a Δ_1 class.*

Proof. See [2], Proposition III.1.4 and page 81. $\square$

Proposition 2.2.10. (KP) *"$\mathcal{M} \models_{\mathscr{L}} \varphi(\bar{x})$" as a relation on the language $\mathscr{L}$, $\mathscr{L}$-structure $\mathcal{M}$, infinitary $\mathscr{L}$-formula φ, and tuple $\bar{x}$ of M is equivalent to a Δ_1 predicate.*

Proof. See [2], pages 82-83. $\square$

Definition 2.2.11. Let $\mathscr{L}$ be a language. Let $\mathcal{A}$ be an admissible set such that $\mathscr{L}$ is Δ_1 definable in $\mathcal{A}$. The admissible fragment of $\mathscr{L}_{\infty\omega}$ given by $\mathcal{A}$, denoted $\mathscr{L}_{\mathcal{A}}$, is defined as

$$\mathscr{L}_{\mathcal{A}} = \{\varphi \in A : \varphi \in L_{\infty\omega}\} = \{\varphi \in A : \mathcal{A} \models \varphi \in L_{\infty\omega}\}$$

The last equivalence follows from Δ_1 absoluteness.

Definition 2.2.12. Let $\mathscr{L}$ be a language consisting of a binary relation $\dot{\in}$. Let $\mathcal{M}$ be a $\mathscr{L}$-structure such that $(M, \dot{\in}^{\mathcal{M}})$ satisfies extensionality. Define WF$(\mathcal{M})$ as the substructure consisting of the well-founded elements of M. WF$(\mathcal{M})$ is called the well-founded part of $\mathcal{M}$.

$\mathcal{M}$ is called *solid* if and only if WF$(\mathcal{M})$ is transitive.

Remark 2.2.13. The notion of solid comes from Jensen's [9]. Every structure has an isomorphic solid model that is obtained by Mostowski collapsing the well-founded part.

The notion of solidness is mostly a convenience: in our usage, $\omega \subseteq M$. Therefore, Mostowski collapsing will not change reals. Transitivity is desired due to the definition of admissibility and in order to apply familiar absoluteness results. Rather than having to repeatedly Mostowski collapse WF$(\mathcal{M})$ and mention reals are not moved, one will just assume the well-founded part is transitive by demanding $\mathcal{M}$ is solid.

Lemma 2.2.14. *(Truncation Lemma) If $\mathcal{M} \models \mathsf{KP}$, then WF$(\mathcal{M}) \models \mathsf{KP}$. In particular, if $\mathcal{M}$ is a solid model, then WF$(\mathcal{M})$ is an admissible set.*

Proof. See [2], II.8.4. □

The following is the central technique used in the paper:

Theorem 2.2.15. *(Solid Model Existence Theorem) Let $\mathcal{A}$ be a countable admissible set. Let $\mathscr{L}$ be a language which is Δ_1 definable over $\mathcal{A}$ and contains a binary relation symbol $\dot{\in}$ and constant symbols $\bar{a}$ for each $a \in A$. Let T be a consistent $\mathscr{L}$-theory in the countable admissible fragment $\mathscr{L}_{\mathcal{A}}$, be Σ_1 definable over $\mathcal{A}$, and contains the following:*

(i) KP

(ii) For each $a \in A$, the sentence $(\forall v)(v \dot{\in} \bar{a} \Rightarrow \bigvee_{z \in a} v = \bar{z})$.

Then there exists a solid $\mathscr{L}$-structure $\mathcal{B}$ such that $\mathcal{B} \models T$ and $ON \cap B = ON \cap A$.

Proof. See [5] and also see [9], Section 4, Lemma 11. □

Theorem 2.2.16. *(Sacks' Theorem) If $\alpha > \omega$ is an admissible ordinal, then there exists some $x \in {}^{\omega}2$ such that $\alpha = \omega_1^x$.*

Let $z \in {}^{\omega}2$. If $\alpha \in \Lambda(z) \cap \omega_1$, then there exists $y \in {}^{\omega}2$ with $\omega_1^y = \alpha$ and $z \leq_T y$.

Proof. See [18], Corollary 3.16. The following proof is similar to [9], Section 4, Lemma 10. The second statement will be proved below:

Since $\alpha \in \Lambda(z)$, let $\mathcal{A}$ be an admissible set such that $z \in A$ and $o(\mathcal{A}) = \alpha$. (For example, $\mathcal{A} = L_\alpha(z)$ by Proposition 2.2.5.)

Let $\mathscr{L}$ be a language consisting of the following:

(I) A binary relation symbol $\dot{\in}$.

(II) Constant symbols $\bar{a}$ for each $a \in A$.

(III) One other distinguished constant symbol $\dot{c}$.

The elements of $\mathscr{L}$ can be appropriately coded as elements of A so that $\mathscr{L}$ is Δ_1 definable over $\mathcal{A}$.

Let T be a theory in the countable admissible fragment $\mathscr{L}_{\mathcal{A}}$ consisting of the following:

(i) KP

(ii) For each $a \in A$, $(\forall v)(v \dot{\in} \bar{a} \Rightarrow \bigvee_{z \in a} v = \bar{z})$.

(iii) $\dot{c} \subseteq \bar{\omega}$.

(iv) For each ordinal $\sigma \in \alpha$, "$\bar{\sigma}$ is not admissible relative to $\dot{c}$". More formally, "$L_{\bar{\sigma}}(\dot{c}) \not\models \mathsf{KP} + \mathsf{INF}$".

(v) $\bar{z} \leq_{\mathrm{T}} \dot{c}$.

T can be coded as a class in A in such a way that it is Σ_1 in $\mathcal{A}$. T is consistent: Find any $u \in {}^{\omega}2$ which codes an ordinal greater than α. Let $c = u \oplus z$. Consider the following $\mathscr{L}$-structure $\mathcal{M}$: The universe M is $H_{\aleph_1}$. For each $a \in \mathcal{A}$, $\bar{a}^{\mathcal{M}} = a$. (Since A is countable and transitive, $A \in H_{\aleph_1}$.) $\dot{\in}^{\mathcal{M}} = \in \upharpoonright H_{\aleph_1}$. $\dot{c}^{\mathcal{M}} = c$. $\mathcal{M}$ clearly satisfy (i), (ii), (iii), and (v). For (iv), suppose there is an $\sigma < \alpha$ such that $L_\sigma(c) \models \mathsf{KP}$.

Since $c \in L_\sigma(c)$ and $L_\sigma(c) \models \mathsf{KP}$, $u \in L_\sigma(c)$ because $c = u \oplus z$. Since the Mostowski collapse map is a Σ_1 definable function in KP, if reals code binary relations in the usual way, then KP proves the existence of $\mathrm{ot}(u)$. Thus $\mathrm{ot}(u) \in L_\sigma(c)$. However, $\mathrm{ot}(u) > \alpha > \sigma$. Contradiction. It has been shown that $\mathcal{M}$ also satisfy (iv). T is consistent.

The Solid Model Existence Theorem (Theorem 2.2.15) implies there is a solid $\mathscr{L}$-structure $\mathcal{B} \models T$ such that $\mathrm{ON} \cap B = \mathrm{ON} \cap A = \alpha$. Let $y = \dot{c}^{\mathcal{B}}$. The claim is that $\omega_1^y = \alpha$. By Lemma 2.2.14, $\mathrm{WF}(\mathcal{B})$ is an admissible set containing y and z. $o(\mathrm{WF}(\mathcal{B})) = \mathrm{ON} \cap \mathrm{WF}(\mathcal{B}) = \mathrm{ON} \cap B = \mathrm{ON} \cap A = \alpha$. Thus $\omega_1^y \leq \alpha$. Now suppose that $\omega_1^y < \alpha$. In V, $L_{\omega_1^y}(y) \models \mathsf{KP}$. Since the function $(\alpha, x) \mapsto L_\alpha(x)$ is Δ_1 (by Proposition 2.2.4) and the satisfaction relation is Δ_1 (by Proposition 2.2.10), by Δ_1 absoluteness between the transitive sets $\mathrm{WF}(\mathcal{B})$ and V, one has $\mathrm{WF}(\mathcal{B}) \models L_{\omega_1^y}(y) \models \mathsf{KP}$. Again by absoluteness of Δ_1 formulas between the transitive (in the sense of $\mathcal{B}$) sets $\mathrm{WF}(\mathcal{B})$ and $\mathcal{B}$, $\mathcal{B} \models L_{\omega_1^y}(y) \models \mathsf{KP}$. Letting $\sigma = \omega_1^y < \alpha$, $\mathcal{B} \models L_{\bar{\sigma}}(\dot{c}) \models \mathsf{KP}$. This contradicts $\mathcal{B} \models T$. A similar absoluteness argument shows that $z \leq_{\mathrm{T}} y$. □

Remark 2.2.17. This proof of Sacks theorem is the basic template for several other arguments throughout the paper. This proof will be frequently referred.

Next, various aspects of the Scott analysis will be reviewed. Since there are some minor variations among the definitions of Scott rank, Scott sentences, canonical Scott sentences, etc., these will be provided below. See [15], page 57-60 or [17] for more information.

Definition 2.2.18. Let $\mathscr{L}$ be a language. Define the binary relation $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{N}, \underline{b})$ where $\alpha \in \mathrm{ON}$, $\underline{a} \in {}^{<\omega}M$, and $\underline{b} \in {}^{<\omega}N$ as follows:

(i) $(\mathcal{M}, \underline{a}) \sim_0 (\mathcal{N}, \underline{b})$ if and only if for all atomic $\mathscr{L}$-formulas φ, $\mathcal{M} \models \varphi(\underline{a})$ if and only if $\mathcal{N} \models \varphi(\underline{b})$.

(ii) If α is a limit ordinal, then $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{N}, \underline{b})$ if and only if for all $\beta < \alpha$, $(\mathcal{M}, \underline{a}) \sim_\beta (\mathcal{N}, \underline{b})$.

(iii) If $\alpha = \beta + 1$, then $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{N}, \underline{b})$ if and only if for all $c \in M$, there exists a $d \in N$ such that $(\mathcal{M}, \underline{a}, c) \sim_\beta (\mathcal{N}, \underline{b}, d)$ and for all $d \in N$, there exists a $c \in M$ such that $(\mathcal{M}, \underline{a}, c) \sim_\beta (\mathcal{N}, \underline{b}, d)$.

Let $\mathcal{M}$ be a $\mathscr{L}$-structure and $\underline{a} \in {}^kM$ for some $k \in \omega$. For $\alpha \in \mathrm{ON}$, the $\mathscr{L}_{\infty\omega}$

formula $\Phi^{\mathcal{M}}_{\underline{a},\alpha}(\underline{v})$ (in variables $\underline{v}$ such that $|\underline{v}| = k$) is defined as follows:

(I) Let X be the set of all atomic and negation atomic $\mathscr{L}$-formulas with free variables $\underline{v}$ such that $|\underline{v}| = k$ which holds of $\underline{a}$. Let $\Phi^{\mathcal{M}}_{\underline{a},0}(\underline{v}) = \bigwedge X$.

(II) If α is a limit ordinal, let $X = \{\Phi^{\mathcal{M}}_{\underline{a},\beta}(\underline{v}) : \beta < \alpha\}$. Let $\Phi^{\mathcal{M}}_{\underline{a},\alpha}(\underline{v}) = \bigwedge X$.

(III) If $\alpha = \beta + 1$, then let $X = \{(\exists w)\Phi^{\mathcal{M}}_{\underline{a}b,\beta}(\underline{v}, w) : b \in M\}$ and $Y = \{\Phi^{\mathcal{M}}_{\underline{a}b,\beta}(\underline{v}, w) : b \in M\}$. Then let $\Phi^{\mathcal{M}}_{\underline{a},\alpha}(\underline{v}) = \bigwedge X \wedge (\forall w) \bigvee Y$.

For $\mathcal{M}$, a $\mathcal{L}$-structure, and $\underline{a} \in {}^kM$ (for some k), define $\rho(\mathcal{M}, \bar{a})$ to be the least $\alpha \in \mathrm{ON}$ such that for all $\underline{b} \in {}^kM$, $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{M}, \underline{b})$ if and only if for all β, $(\mathcal{M}, \underline{a}) \sim_\beta (\mathcal{M}, \underline{b})$.

Define $\mathrm{SR}(\mathcal{M}) = \sup\{\rho(\mathcal{M}, \underline{a}) + 1 : \underline{a} \in {}^{<\omega}M\}$. Define $\mathrm{R}(\mathcal{M}) = \sup\{\rho(\mathcal{M}, \underline{a}) : \underline{a} \in {}^{<\omega}M\}$.

Let $\alpha = \mathrm{R}(\mathcal{M})$. Let

$$X = \{(\forall \bar{v})(\Phi^{\mathcal{M}}_{\underline{a},\alpha}(\underline{v}) \Rightarrow \Phi^{\mathcal{M}}_{\underline{a},\alpha+1}(\underline{v})) : \underline{a} \in {}^{<\omega}M\}$$

$$\mathrm{CSS}(\mathcal{M}) = \Phi^{\mathcal{M}}_{\emptyset,\alpha} \wedge \bigwedge X.$$

$\mathrm{CSS}(\mathcal{M})$ is the *canonical Scott sentence* of $\mathcal{M}$. $\mathrm{SR}(\mathcal{M})$ is the *Scott rank* of $\mathcal{M}$.

The following are well-known results. Usually, a careful inspection of the proof indicates what can be done in $\mathsf{KP} + \mathsf{INF}$ or ZFC.

Proposition 2.2.19. *The relation $\sim$ is equivalent to a Δ_1 formula over* $\mathsf{KP} + \mathsf{INF}$.

Proof. It can be defined by Σ-recursion. □

Proposition 2.2.20. *Let $\mathcal{A}$ be an admissible set such that $\mathcal{A} \models \mathsf{INF}$. Let $\mathscr{L} \in A$ be a language. Let $\mathcal{M} \in A$ range over $\mathscr{L}$-structure, $\underline{a}$ range over elements of ${}^{<\omega}M$, and α range over $ON \cap A$. Then the function $f(\mathcal{M}, \underline{a}, \alpha) = \Phi^{\mathcal{M}}_{\underline{a},\alpha}(\underline{v})$ is Δ_1 definable in $\mathcal{A}$.*

In particular, if $R(\mathcal{M}) \in A$, then $CSS(\mathcal{M}) \in A$.

Proof. It can be defined by Σ-recursion. □

Proposition 2.2.21. ($\mathsf{KP} + \mathsf{INF}$) *Let $\mathscr{L}$ be a language. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathscr{L}$-structures, $\underline{a} \in {}^kM$, $\underline{b} \in {}^kN$ for some $k \in \omega$, and $\alpha \in ON$. Then $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{N}, \underline{b})$ if and only if $\mathcal{N} \models \Phi^{\mathcal{M}}_{\underline{a},\alpha}(\underline{b})$.*

Proof. This is proved by induction. See [15], Lemma 2.4.13. □

Proposition 2.2.22. (KP + INF) *If* $(M, \underline{a}) \equiv_{\mathscr{L}_{\infty\omega}} (N, \underline{b})$, *then for all* α, $(M, \underline{a}) \sim_\alpha (N, \underline{b})$.

Proof. $\mathcal{M} \models \varphi_{\underline{a},\alpha}(\underline{a})$. So $\mathcal{N} \models \varphi_{\underline{a},\alpha}(\underline{b})$. By Proposition 2.2.21, $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{N}, \underline{b})$. □

Definition 2.2.23. Let $\mathscr{L}$ be a language. Let φ be a formula of $L_{\infty\omega}$. The *quantifier rank* of φ, denoted $\mathrm{qr}(\varphi)$, is defined as follows:

(i) $\mathrm{qr}(\varphi) = 0$ if φ is an atomic formula.

(ii) $\mathrm{qr}(\neg\varphi) = \mathrm{qr}(\varphi)$.

(iii) $\mathrm{qr}(\bigwedge X) = \mathrm{qr}(\bigvee X) = \sup\{\mathrm{qr}(\psi) : \psi \in X\}$.

(iv) $\mathrm{qr}(\exists v\varphi) = \mathrm{qr}(\forall v\varphi) = \mathrm{qr}(\varphi) + 1$.

Proposition 2.2.24. *The relation "$qr(\varphi) = \alpha$" is Δ_1 definable in* KP + INF.

Proof. It can be defined by Σ-recursion. □

Proposition 2.2.25. (KP + INF) *Let $\mathscr{L}$ be a language. $\mathcal{M}, \mathcal{N}$ be $\mathscr{L}$-structures. $\underline{a} \in {}^kM$ and $\underline{b} \in {}^kN$ for some $k \in \omega$. Then for all $\alpha \in ON$, $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{N}, \underline{b})$ if and only if for all φ with $qr(\varphi) \leq \alpha$, $\mathcal{M} \models \varphi(\underline{a})$ if and only if $\mathcal{N} \models \varphi(\underline{b})$.*

Proof. This is proved by induction. □

Proposition 2.2.26. (ZF) *Let $\mathscr{L}$ be some language. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathscr{L}$-structures. Suppose $\mathcal{A}$ is an admissible set with $\mathscr{L}, \mathcal{M}, \mathcal{N} \in A$. Then $\mathcal{A} \models \mathcal{M} \equiv_{\mathscr{L}_{\infty\omega}} \mathcal{N}$ if and only if $\mathcal{M} \equiv_{\mathscr{L}_{\infty\omega}} \mathcal{N}$.*

Proof. See [17], Theorem 1.3. □

Remark 2.2.27. A common phenomenon is that certain properties are reflected between appropriate admissible sets and the true universe. A useful observation is that if such a property holds from the point of view of an admissible set then it is true in the universe. The above proposition asserts that infinitary elementary equivalence is such a property.

Another familiar example is the effective boundedness theorem. Suppose $\varphi : \mathrm{WO} \to \omega_1$ is a Π^1_1 rank. Let $B \subseteq \mathrm{WO}$ be Σ^1_1. Let $\mathcal{A}$ be a countable admissible set containing the parameters used to define B. Inside of $\mathcal{A}$, $\varphi(B)$ is bounded by $o(\mathcal{A})$. A priori, the true bound on $\varphi(B)$ may be higher as the true universe has more countable ordinals and more members of B. However, the effective boundedness theorem asserts that in fact, in the true universe, $\varphi(B)$ is bounded by $o(\mathcal{A})$.

The following proposition with an included proof shows countable admissible sets can also be used to produce true bounds on the Scott rank.

Proposition 2.2.28. *Let $\mathscr{L}$ be a countable language and $\mathcal{M}$ be a countable $\mathscr{L}$-structure. One may identify $\mathcal{M}$ as a real by associating it with an isomorphic structure on ω. If $\mathcal{A}$ is an admissible set with $\mathscr{L}, \mathcal{M} \in A$, then $R(\mathcal{M}) \leq ON \cap A$. In particular, $R(\mathcal{M}) \leq \omega_1^{\mathcal{M}}$ and $SR(\mathcal{M}) \leq \omega_1^{\mathcal{M}} + 1$.*

Proof. See [17], Corollary 1.

Fix an admissible set $\mathcal{A}$ as above. Let $\gamma = \mathrm{ON} \cap A$. It suffices to show that $R(M) \leq \gamma$. Suppose not. Then there exists $\underline{a}$ and $\underline{b}$ such that for all $\alpha < \gamma$, $(\mathcal{M}, \underline{a}) \sim_\alpha (\mathcal{M}, \underline{b})$ but $(\mathcal{M}, \underline{a}) \nsim_\beta (\mathcal{M}, \underline{b})$ for some $\beta > \gamma$. By Δ_1-absoluteness and Proposition 2.2.25, $A \models (\mathcal{M}, \underline{a}) \equiv_{\mathscr{L}_{\infty\omega}} (\mathcal{M}, \underline{b})$. Thus by Proposition 2.2.26, $(\mathcal{M}, \underline{a}) \equiv_{\mathscr{L}_{\infty\omega}} (\mathcal{M}, \underline{b})$. However, $(\mathcal{M}, \underline{a}) \nsim_\beta (\mathcal{M}, \underline{b})$ implies $\mathcal{M} \models \Phi^{\mathcal{M}}_{\underline{a},\beta}(\underline{a})$ and $\mathcal{M} \not\models \Phi^{\mathcal{M}}_{\underline{a},\beta}(\underline{b})$ by Proposition 2.2.21. This shows $(\mathcal{M}, \underline{a}) \not\equiv_{\mathscr{L}_{\infty\omega}} (\mathcal{M}, \underline{b})$. Contradiction. □

Definition 2.2.29. Let $\mathscr{L}$ be a language. Let $\mathcal{M}$ be a $\mathscr{L}$-structure. φ is a Scott sentence if and only if for all $\mathscr{L}$-structures $\mathcal{N}$ and $\mathcal{M}$, $\mathcal{N} \models \varphi$ and $\mathcal{M} \models \varphi$ implies $\mathcal{M} \equiv_{\mathscr{L}_{\infty\omega}} \mathcal{N}$.

Theorem 2.2.30. (ZFC) *Let $\mathscr{L}$ be a language. Let $\mathcal{M}$ be a countable $\mathscr{L}$-structure. Then there exists a $L_{\infty\omega}$-sentence φ such that for all countable $\mathscr{L}$-structure $\mathcal{N}$, $\mathcal{N} \cong_{\mathscr{L}} \mathcal{M}$ if and only if $\mathcal{N} \models \varphi$. In fact, φ can be chosen to be $CSS(\mathcal{M})$.*

(KP + INF) *If φ is a Scott sentence for a countable structure $\mathcal{M}$, then for all countable $\mathcal{N}$, $\mathcal{N} \models \varphi$ if and only if $\mathcal{N} \cong_{\mathscr{L}} \mathcal{M}$.*

Proof. Observe the first statement asserts that there exists a sentence such that whenever a countable structure satisfies this sentence, there exists an isomorphism between it and $\mathcal{M}$. The existence of this sentence requires working beyond KP + INF. The second statement asserts that KP + INF can prove that if a Scott sentence happens to exist, then for any countable structure satisfying this sentence, there is an isomorphism between it and $\mathcal{M}$.

This is the Scott's isomorphism theorem. See [15], Theorem 2.4.15 for a proof. The results in KP + INF follows essentially the same proof with the assistance of some of the above propositions proved in KP + INF. □

Definition 2.2.31. Let $\mathscr{L}$ be a countable language. Let $S(\mathscr{L})$ denote the set of all $\mathscr{L}$-structures on ω.

Definition 2.2.32. Let $\dot{\in}$ be a binary relation symbol. Let S^* denote the subset of $S(\{\dot{\in}\})$ consisting of ω-models of KP + INF.

Proposition 2.2.33. *Let $\{\phi_e : e \in \omega\}$ be a recursive enumeration of $\{\dot{\in}\}_{\omega\omega}$-formulas. The relation on $x \in S(\{\dot{\in}\})$ and $e \in \omega$ asserting "$x \models \phi_e$" is Δ^1_1.*

Also S^* is Δ^1_1.

Proof. See [13], pages 14-16 for relevant definitions and proofs. □

Remark 2.2.34. One can check that there is a Δ^1_1 function such that, given $A \in S^*$ and $n \in \omega$, the function gives the element of A which A thinks is n. Using this, one can determine in a Δ^1_1 way whether $A \in S^*$ thinks some $x \in {}^\omega 2$ exists. In the following, if $A \in S^*$ and $x \in {}^\omega 2$, the sentence "$x \in A$" should be understood as this informally described Δ^1_1 relation.

Proposition 2.2.35. *Let $\mathscr{L}$ be a recursive language. Let $\varphi \in \mathcal{HYP}(x) \cap \mathscr{L}_{\infty\omega}$. Then $Mod(\varphi) = \{s \in S(\mathscr{L}) : s \models_{\mathscr{L}} \varphi\}$ is $\Delta^1_1(x)$.*

Proof. Note that $s \in \text{Mod}(\varphi)$ if and only if

$$(\exists A)(A \in S^* \wedge x \in A \wedge s \in A \wedge A \models s \models_{\mathscr{L}} \varphi)$$

if and only if

$$(\forall A)(A \in S^* \wedge x \in A \wedge s \in A) \Rightarrow A \models s \models_{\mathscr{L}} \varphi).$$

These equivalences are established using the absoluteness of satisfaction. This shows that $\text{Mod}(\varphi)$ is $\Delta^1_1(x)$. □

Remark 2.2.36. Later, the paper will be concerned with relating countable admissible sets and isomorphism of countable structures. The second statement of Theorem 2.2.30 captures the essence of these types of arguments: Isomorphism of countable structures is reflected between the true universe and admissible sets which witness the countability of the relevant structures and possesses a Scott sentence for these structures.

The original arguments for some results of this paper used more directly the second statement of Theorem 2.2.30. The argument presented below is simpler using the Scott isomorphism theorem and Proposition 2.2.35 but may conceal this essential idea.

Now to introduce the main equivalence relation of this paper:

Definition 2.2.37. Let F_{ω_1} be the equivalence relation defined on ${}^{\omega}2$ by $x\ F_{\omega_1}\ y$ if and only if $\omega_1^x = \omega_1^y$. F_{ω_1} is a Σ_1^1 equivalence relation with all classes $\boldsymbol{\Delta}_1^1$.

The first claim from the above definition is well known and follows easily from the characterization of ω_1^x as the supremum of the x-recursive ordinals. The next proposition implies each class is $\boldsymbol{\Delta}_1^1$. There will be much to say later about the complexity of each F_{ω_1}-equivalence class.

Proposition 2.2.38. *Let α be a countable admissible ordinal and $z \in {}^{\omega}2$ be such that $\alpha < \omega_1^z$. Then the set $\{y \in {}^{\omega}2 : \omega_1^y = \alpha\}$ is $\Delta_1^1(z)$.*

Proof. If u and v are reals coding linear orderings on ω, then $u \preceq v$ means there exists an order preserving injective function f from the linear ordering coded by u to the linear ordering coded by v. $\preceq$ is a Σ_1^1 relation in the variables u and v.

Since $\alpha < \omega_1^z$, there exists some $e \in \omega$ such that $\{e\}^z$ is the characteristic function of a well-ordering isomorphic to α. Let $B = \{y \in {}^{\omega}2 : \alpha = \omega_1^y\}$. Then

$$y \in B \Leftrightarrow (\forall n)\Big((\{n\}^y \in \mathrm{WO} \Rightarrow \{n\}^y \preceq \{e\}^z)\Big) \wedge$$

$$(\forall k)(\exists j)(\{j\}^y \preceq \{e\}^z \wedge \{e\}^z \restriction k \preceq \{j\}^y)$$

B is $\Sigma_1^1(z)$. Also

$$y \notin B \Leftrightarrow (\exists j)(\forall n)(\{n\}^y \in \mathrm{WO} \Rightarrow$$

$$(\{n\}^y \preceq \{e\}^z \restriction j) \vee (\exists n)(\{n\}^y \in \mathrm{LO} \wedge \{n\}^y \preceq \{e\}^z \wedge \{e\}^z \preceq \{n\}^y)$$

B is $\Pi_1^1(z)$. Hence B is $\Delta_1^1(z)$. $\square$

2.3 Classifiable by Countable Structures

Definition 2.3.1. Let $x \in {}^{\omega}2$. A linear ordering R on ω is an x-recursive *x-pseudo-wellordering* if and only if R is an x-recursive linear ordering on ω which is not a wellordering but $L_{\omega_1^x}(x) \models R$ is a wellordering, i.e. R has no x-hyperarithmetic descending sequences.

Proposition 2.3.2. *(Harrison, Kleene) For all $x \in {}^{\omega}2$, there exists an x-recursive x-pseudo-wellordering.*

Proof. See [10] or [19], III.2.1. A generalized form of this construction will be used below.

This can also be proved using Theorem 2.2.15 and infinitary logic in admissible fragments. In the application of Theorem 2.2.15, Barwise compactness is used to show the consistency of the appropriate theory in the countable admissible fragment. See Nadel's proof given in [1] VIII, Section 5.7 for more details. □

The following characterizes the order type of x-recursive x-pseudo-wellorderings:

Theorem 2.3.3. *(Harrison) Let R be a x-recursive x-pseudo-wellordering, then $ot(R) = \omega_1^x(1 + \eta) + \rho$ where $\eta = ot(\mathbb{Q})$ and $\rho < \omega_1^x$.*

Proof. See [8] or [19], Lemma III.2.2. □

Proposition 2.3.4. *Recall if $y \in {}^{\omega}2$, then $HYP^y = L_{\omega_1^y}(y) \cap {}^{\omega}2$, the set of y-hyperarithmetic reals.*

The relation $x \in HYP^y$ is a Π^1_1 relation in the variable x and y.

Proof. This result is classical. A proof is given to illustrate some ideas from ω-models of KP.

The claim is that:

$$x \in \text{HYP}^y \Leftrightarrow (\forall A)((A \in S^* \wedge y \in A) \Rightarrow (x \in A)).$$

See Remark 2.2.34 about what "$y \in A$" should precisely mean. The latter part of the equivalence is Π^1_1. Hence the result follows from the claim.

To prove the claim:

(⇒) Suppose $A \in S^*$. Let $n \in \omega$ be the representative of y in A. Since $A \models \mathsf{KP}$, by Lemma 2.2.14 (Truncation Lemma), $\mathrm{WF}(A) \models \mathsf{KP}$. Let π be the Mostowski collapse of $\mathrm{WF}(A)$ onto an admissible set B. $y \in B$ since $y = \pi(n)$. Since $x \in \mathrm{HYP}^y$, x is in every admissible set containing y. $x \in B$. Then $\pi^{-1}(x)$ represents x in A.

(⇐) Recall $\mathcal{HYP}(y)$ is the smallest admissible set containing x and ω. The domain of $\mathcal{HYP}(y)$ is $L_{\omega_1^y}(y)$. It is countable. Let $\pi : L_{\omega_1^y}(y) \to \omega$ be any bijection. The bijection gives an element $A \in S^*$ isomorphic to $\mathcal{HYP}(y)$. $\pi(y)$ represents y in A. There exists some $n \in \omega$ such that n represents x in A, by the hypothesis. Then $x \in L_{\omega_1^y}(y)$ since $x = \pi^{-1}(n)$. $x \in \mathrm{HYP}^y$. □

The following propositions use the ideas from [19] III.1 and III.2.

Proposition 2.3.5. *There exists a recursive tree U on $2 \times \omega$ such that for all $x \in {}^\omega 2$, U^x has a path but has no x-hyperarithmetic paths.*

Proof. By Proposition 2.3.4, there is a recursive tree V on $2 \times 2 \times \omega$ such that $x \notin \mathrm{HYP}^y$ if and only if $V^{(x,y)}$ is ill-founded. Define the relation Φ on ${}^\omega 2 \times {}^\omega\omega$ by

$$\Phi(y, f) \Leftrightarrow (\forall n)((f_0(n) = 0 \vee f_0(n) = 1) \wedge V(f_0 \restriction n, y \restriction n, f_1 \restriction n))$$

where $f_i(n) = f(\langle i, n\rangle)$, for $i = 0, 1$. Φ is Π^0_1. Let U be a recursive tree on $2 \times \omega$ such that

$$\Phi(y, f) \Leftrightarrow (\forall n)((y \restriction n, f \restriction n) \in U).$$

For any y, if U^y has a path f, then $\Phi(y, f)$. Therefore, $f_1 \in [V^{(f_0,y)}]$. $f_0 \notin \mathrm{HYP}^y$. So U^y can not have a y-hyperarithmetic path f, since otherwise $f_0 \in \mathrm{HYP}^y$, which yields a contradiction. U^y has a path: Let x be any real which is not in HYP^y. $[V^{(x,y)}]$ is non-empty. Let $g \in [V^{(x,y)}]$. Let f be such that $f_0 = x$ and $f_1 = g$. Then $\Phi(y, f)$. $f \in [U^y]$. □

Definition 2.3.6. The Kleene-Brouwer ordering $<_{\mathrm{KB}}$ is defined on ${}^{<\omega}\omega$ as follows: $s <_{\mathrm{KB}} t$ if and only if

(i) $t \preceq s$ and $|t| < |s|$

or

(ii) If there exists an $n \in \omega$ such that for all $k < n$, $s(k) = t(k)$ and $s(n) < t(n)$.

Proposition 2.3.7. *Let T be a tree on ω. T is wellfounded if and only if $<_{KB}\restriction T$ is wellfounded. Moreover, if there is an x-hyperarithmetic infinite descending sequence in $<_{KB}\restriction T$, then there is an x-hyperarithmetic path through T.*

Proof. If $f \in [T]$, then $\{f \restriction n : n \in \omega\}$ is an infinite descending sequence in $<_{\mathrm{KB}}\restriction T$.

Let $S = \{s_n \in {}^{<\omega}2 : n \in \omega\}$ be an x-hyperarithmetic descending sequence in $<_{\mathrm{KB}}\restriction T$. Define $f \in {}^{\omega}\omega$ by

$$f(n) = i \Leftrightarrow (\exists p)(\forall q \geq p)(s_q(n) = i);$$

$f \in [T]$ and f is $\Sigma^0_2(S)$. f is also x-hyperarithmetic. □

Now to produce a classification of F_{ω_1} by countable structures. The idea will be to send x to an x-Harrison linear ordering. Using Proposition 2.3.5 and applying the Kleene-Brouwer ordering, one can obtain a function g such that $g(x)$ is an x-recursive x-pseudo-wellordering. Now suppose $\omega_1^x = \omega_1^y$. Let α denote this admissible ordinal. By Theorem 2.3.3, $\mathrm{ot}(g(x)) = \alpha(1 + \eta) + \rho_x$ and $\mathrm{ot}(g(y)) = \alpha(1 + \eta) + \rho_y$, where $\rho_x < \alpha$ and $\rho_y < \alpha$. However, it could happen that $\rho_x \neq \rho_y$. One way to modify g to get a classification of F_{ω_1} would be to "cut off" the recursive tail of $g(x)$. To do this, one uses a trick, as suggested by Montalbán, to cut off the recursive tail of the order type by taking a product of ω copies of $g(x)$. The details follow:

Proposition 2.3.8. *Fix $x \in {}^{\omega}2$. Let $\rho < \omega_1^x$ and $\eta = ot(\mathbb{Q})$. Then $(\omega_1^x(1+\eta)+\rho)\omega = \omega_1^x(1 + \eta)$.*

Proof. Let P be any x-recursive x-pseudo-wellorderings of order type $\omega_1^x(1+\eta)+\rho$. Let $P \times \omega$ be the x-recursive structure isomorphic to ω copies of P following each other. $P\times\omega$ is still an x-recursive x-pseudo-wellordering. It has no x-recursive tail. By Theorem 2.3.3, $\mathrm{ot}(P \times \omega) = (\omega_1^x(1 + \eta) + \rho)\omega = \omega_1^x(1 + \eta)$. □

Proposition 2.3.9. *There exists an $e \in \omega$ such that for all $x \in {}^{\omega}2$, $\{e\}^x$ is isomorphic to $(<_{KB}\restriction U^x) \cdot \omega$, where U comes from Proposition 2.3.5.*

Proof. This is basic recursion theory using the previous results. □

Theorem 2.3.10. *(Montalbán) The equivalence relation F_{ω_1} is classifiable by countable structures. In fact, there is an $e \in \omega$ such that $f(x) = \{e\}^x$ is the desired classification.*

Proof. Let $\mathscr{L} = \{\dot{R}\}$, where $\dot{R}$ is a binary relation symbol. F_{ω_1} will be classified by countable $\mathscr{L}$-structures. U^x is an x-hyperarithmetic tree with paths but no x-hyperarithmetic path. Hence $<_{\mathrm{KB}} \restriction U^x$ is an x-recursive linear ordering with infinite descending sequences but no x-hyperarithmetic infinite descending sequences. So $<_{\mathrm{KB}} \restriction U^x$ is an x-recursive x-pseudo-wellordering. It has order type $\omega_1^x(1+\eta)+\rho$ for some $\rho < \omega_1^x$. Therefore, $(<_{\mathrm{KB}} \restriction U^x) \cdot \omega$ has order type $\omega_1^x(1+\eta)$, i.e., it is an x-Harrison linear ordering. Hence $x \ F_{\omega_1} \ y$ if and only $\omega_1^x = \omega_1^x$ if and only $\omega_1^x(1+\eta) = \omega_1^y(1+\eta)$ if and only $(\leq_{\mathrm{KB}} \restriction U^x) \cdot \omega \cong_{\mathscr{L}} (\leq_{\mathrm{KB}} \restriction U^y) \cdot \omega$ if and only if $\{e\}^x \cong_{\mathscr{L}} \{e\}^y$. This gives a classification of F_{ω_1}. □

2.4 Finer Aspects of Classification by Countable Structures

The previous section provided an explicit classification $f : {}^\omega 2 \to S(\mathscr{L})$, which was Δ_1^1 and for all $x \in {}^\omega 2$, $\mathrm{SR}(f(x)) = \omega_1^x + 1$. This section will show that any classification of F_{ω_1} by countable structures must have a similar property.

The next result will calculate the complexity of each F_{ω_1} class according to effective descriptive set theory.

Theorem 2.4.1. *For any $x \in {}^\omega 2$, $[x]_{F_{\omega_1}}$ is not $\Pi_1^1(x)$.*

Proof. Suppose $[x]_{F_{\omega_1}}$ is $\Pi_1^1(x)$. Let $B = {}^\omega 2 - [x]_{F_{\omega_1}}$. B is then $\Sigma_1^1(x)$. Let U be a tree on $2 \times \omega$ recursive in x such that

$$y \in B \Leftrightarrow [U^y] \neq \emptyset.$$

Observe $U \in L_{\omega_1^x}(x)$.

Let $\mathscr{L}$ be the language consisting of the following:

(I) A binary relation symbol $\dot{\in}$.

(II) Constant symbol $\bar{a}$ for each $a \in L_{\omega_1^x}(x)$.

(III) Two other distinguished constant symbols $\dot{c}$ and $\dot{d}$.

$\mathscr{L}$ can be considered a Δ_1 definable subset of $L_{\omega_1^x}(x)$.

Let T be a theory in the countable admissible fragment $\mathscr{L}_{L_{\omega_1^x}(x)}$ consisting of the following:

(i) KP

(ii) For each $a \in L_{\omega_1^x}(x)$, $(\forall v)(v \dot{\in} \bar{a} \Rightarrow \bigvee_{z \in a} v = \bar{z})$.

(iii) $\dot{c} \subseteq \bar{\omega}$ and $\dot{d} : \bar{\omega} \to \bar{\omega}$.

(iv) For each ordinal $\sigma \in \omega_1^x$, "$\bar{\sigma}$ is not admissible relative to $\dot{c}$".

(v) $\dot{d} \in [\bar{U}^{\dot{c}}]$.

T can be considered a Σ_1 definable theory in $L_{\omega_1^x}(x)$.

T is consistent: Find any $y \in {}^\omega 2$ such that $\omega_1^y > \omega_1^x$. Then $y \in B$. There exists some $z \in {}^\omega\omega$ such that $z \in [U^y]$. Consider the $\mathscr{L}$-structure $\mathcal{M}$ defined as follows: $M = H_{\aleph_1}$. $\dot{\in}^{\mathcal{M}} = \in \restriction H_{\aleph_1}$. For each $a \in L_{\omega_1^x}(x)$, let $\bar{a}^{\mathcal{M}} = a$. Let $\dot{c}^{\mathcal{M}} = y$ and $\dot{d}^{\mathcal{M}} = z$. $\mathcal{M} \models T$.

By Theorem 2.2.15, T has a solid model $\mathcal{N}$ such that $\text{ON} \cap N = \text{ON} \cap L_{\omega_1^x}(x) = \omega_1^x$. Let $u = \dot{c}^{\mathcal{N}}$ and $v = \dot{d}^{\mathcal{M}}$. As in Theorem 2.2.16, $\omega_1^u = \omega_1^x$.

$\mathcal{N} \models v \in [U^u]$. By Δ_1 absoluteness, $\text{WF}(\mathcal{N}) \models v \in [U^u]$. Since $\mathcal{N}$ is solid, $\text{WF}(\mathcal{N})$ is transitive as viewed in V. So by Δ_1 absoluteness, $V \models v \in [U^u]$. $[U^u] \neq \emptyset$. $u \in B$. $\omega_1^u \neq \omega_1^x$. Contradiction. □

Suppose f is a classification of F_{ω_1} by countable structures in some recursive language. The Scott rank of the image of f must be high:

Theorem 2.4.2. *Let $\mathscr{L}$ be a recursive language. If $f : {}^\omega 2 \to S(\mathscr{L})$ is a $\Delta_1^1(z)$ function such that $x\ F_{\omega_1}\ y$ if and only if $f(x) \cong_{\mathscr{L}} f(y)$, then for all x such that $\omega_1^x \in \Lambda(z)$, $SR(f(x)) \geq \omega_1^x$.*

Proof. Suppose there exists an $x \in {}^\omega 2$ with $\omega_1^x \in \Lambda(z)$ and $\text{SR}(f(x)) < \omega_1^x$. Let $\alpha = \omega_1^x$. By Proposition 2.2.16, there exists a y with $z \leq_T y$ and $\omega_1^y = \alpha$. Since $\omega_1^y = \alpha = \omega_1^x$, $x\ F_{\omega_1}\ y$. This implies that $f(x) \cong_{\mathscr{L}} f(y)$. Hence $\text{SR}(f(y)) = \text{SR}(f(x)) < \omega_1^x = \alpha = \omega_1^y$. $z \leq_T y$ implies that $z \in L_{\omega_1^y}(y)$, and in particular, z is in every admissible set containing y. Since f is $\Delta_1^1(z)$, $f(y)$ is $\Delta_1^1(z, y) = \Delta_1^1(y)$ since $z \leq_T y$. $f(y)$ is hyperarithmetic in y. $f(y)$ is in every admissible set that has y as a member. Since $\text{SR}(f(y)) < \omega_1^y$ and $f(y)$ is in every admissible set containing y, $\text{CSS}(f(y))$ is in every admissible set containing y. In particular $\text{CSS}(f(y)) \in L_{\omega_1^y}(y)$.

By Proposition 2.2.35, $\text{Mod}(\text{CSS}(f(y))$ is $\Delta_1^1(y)$. Therefore,

$$v \in [y]_{F_{\omega_1}} \Leftrightarrow f(v) \in \text{Mod}(\text{CSS}(f(y))),$$

which is $\Delta_1^1(y, z) = \Delta_1^1(y)$ since $z \leq_T y$. This contradicts Theorem 2.4.1. □

Remark 2.4.3. Let f be $\Delta^1_1(z)$ as above. For all $y \in [x]_{F_{\omega_1}}$, there is an ordinal α such that $\mathrm{SR}(f(y)) = \alpha$. The previous result states that if $\omega_1^x \in \Lambda(z)$, then the Scott rank of $f(x)$ is greater than or equal to ω_1^x. So $\alpha \geq \omega_1^x$. Since $\omega_1^x \in \Lambda(z)$, by Theorem 2.2.16, there is an $x' \in {}^{\omega}2$ such that $\omega_1^{x'} = \omega_1^x$ and $z \leq_{\mathrm{T}} x'$. Then $f(x')$ is $\Delta^1_1(x', z) = \Delta^1_1(x')$. By Lemma 2.2.28, $\mathrm{SR}(f(x')) \leq \omega_1^{x'} + 1 = \omega_1^x + 1$. So one has that $\omega_1^x \leq \alpha \leq \omega_1^x + 1$. One may ask if α must take the largest possible value.

Using the methods of infinitary logic as above, there is one obvious idea to try in order to force the Scott rank to be as high as possible:

Let $\mathcal{J}$ be a countable recursive language. Suppose $f : {}^{\omega}2 \to S(\mathcal{J})$ is a $\Delta^1_1(z)$ function such that $x\ F_{\omega_1}\ y$ if and only if $f(x) \cong_{\mathcal{J}} f(y)$.

Since f is $\Delta^1_1(z)$, it is $\Sigma^1_1(z)$. There is a tree U on $2 \times 2 \times \omega$ recursive in z such that $(a, b) \in f$ if and only if $[U^{(a,b)}] \neq \emptyset$. Again, one may assume $x \geq_T z$: since one can find an x' with $\omega_1^{x'} = \omega_1^x$ and $x' \geq_T z$. This implies $\mathrm{SR}(f(x')) = \mathrm{SR}(f(x))$.

Let $\mathscr{L}$ be the language consisting of the following:

(I) A binary relation symbol $\dot{\in}$.

(II) Constant symbols $\bar{a}$ for each $a \in L_{\omega_1^x}(x)$.

(III) Four distinguished constant symbols $\dot{c}$, $\dot{d}$, $\dot{e}$, and $\dot{s}$.

Let T be a theory in the countable admissible fragment $\mathscr{L}_{L_{\omega_1^x}(x)}$ consisting of the following:

(i) KP

(ii) For each $a \in L_{\omega_1^x}(x)$, $(\forall v)(v \dot{\in} \bar{a} \Rightarrow \bigvee_{u \in a} v = \bar{u})$.

(iii) $\dot{c} \subseteq \bar{\omega}$, $\dot{d} \subseteq \bar{\omega}$, $\dot{e} : \bar{\omega} \to \bar{\omega}$, and $\dot{s} \dot{\in} {}^{<\bar{\omega}}\bar{\omega}$.

(iv) "$\bar{\alpha}$ is not admissible in $\dot{c}$" for each $\alpha < \omega_1^x$.

(v) $\dot{e} \in [\bar{U}^{(\dot{c},\dot{d})}]$.

(vi) $\rho(\dot{d}, \dot{s}) > \bar{\alpha}$ for each $\alpha < \omega_1^x$.

T can be considered a Σ_1 on $L_{\omega_1^x}(x)$ theory.

Next to show T is consistent: Find w such that $\omega_1^w > \omega_1^x$ and $w \in \Lambda(z)$. $(w, f(w)) \in f$, therefore, there exists some u such that $u \in [U^{(w,f(w))}]$. By Theorem 2.4.2, $\mathrm{SR}(f(w)) \geq \omega_1^w$. Let $k \in {}^{<\omega}\omega$ such that $\rho(w, k) > \omega_1^x$. Define $\mathcal{M}$ by $M = H_{\aleph_1}$. Let $\dot{\in}^{\mathcal{M}}$ be the $\in$ relation of $H_{\aleph_1}$. For each $a \in L_{\omega_1^x}(x)$, $\bar{a}^{\mathcal{M}} = a$. $\dot{c}^{\mathcal{M}} = w$. $\dot{d}^{\mathcal{M}} = f(w)$, and $\dot{s} = k$. Then $\mathcal{M} \models T$. T is consistent.

By Theorem 2.2.15, T has a solid model $\mathcal{N}$ such that $\mathrm{ON} \cap N = \mathrm{ON} \cap L_{\omega_1^x}(x) = \omega_1^x$. Let $v = \dot{c}^{\mathcal{N}}$, $w = \dot{d}^{\mathcal{N}}$, $u = \dot{e}^{\mathcal{N}}$, and $t = \dot{s}^{\mathcal{N}}$. As before, $\omega_1^v = \omega_1^x$. $\mathcal{N} \models u \in [U^{(v,w)}]$. $u, v, w \in \mathrm{WF}(N)$ since $\mathcal{N} \models T$ implies $v \subseteq \omega$, $w \subseteq \omega$, and u is a function from ω to ω. By Δ_1-absoluteness between transitive models, $\mathrm{WF}(\mathcal{N}) \models u \in [U^{(v,w)}]$. Since $\mathcal{N}$ is solid, by Δ_1-absoluteness between transitive models, $V \models u \in [U^{(v,w)}]$. Hence $w = f(v)$.

Now, one would like to show that $\rho((w,t)) = \omega_1^x$. The problem occurs in how $\mathcal{N}$ can satisfy (vi). It seems possible that there is an $\alpha < \omega_1^x$ such that for all (v,q) and $\beta < \omega_1^x$, $(w,t) \sim_\alpha (v,q)$ implies $(w,t) \sim_\beta (v,q)$, but there exists some ill founded ordinal $\gamma \in N$ such that $(w,t) \nsim_\gamma (v,q)$. That is, in V, $\rho((w,t)) < \omega_1^x$ but in N, $\rho((w,t)) > \alpha$ for all $\alpha < \omega_1^x$.

The natural question is whether this is actually possible: Is there a structure w on ω, a tuple $t \in {}^{<\omega}\omega$, and an ill-founded model N of KP such that $V \models \rho((w,t)) < \mathrm{ON} \cap N$ but for all $\alpha < \mathrm{ON} \cap N$, $N \models \rho((w,t)) > \alpha$.

Proposition 2.4.4. *(Makkai) There is a hyperarithmetic (or even computable) structure P such that $SR(P) = \omega_1^{\emptyset}$.*

Proof. See [12]. Also see [4], Theorem 3.6. ◻

Before this, there had not been much difficulty proving the consistency of the desired theory by exhibiting some model with domain $H_{\aleph_1}$. A model of the next theory is not as easily produced. The classical Barwise compactness theorem will be useful in showing consistency in this case.

Theorem 2.4.5. *(Barwise Compactness) Let $\mathcal{A}$ be a countable admissible set and $\mathscr{L}$ be a Δ_1 in $\mathcal{A}$ language. Let $\mathscr{L}_{\mathcal{A}}$ be the induced countable admissible fragment of $\mathscr{L}_{\infty\omega}$. Let T be a set of sentences of $\mathscr{L}_{\mathcal{A}}$ such that T is Σ_1 in $\mathcal{A}$. If every $F \subseteq T$ such that $F \in A$ has a model, then T has a model.*

Proof. See [2], Theorem III.5.6. Also see [9], Section 4, Corollary 8. ◻

Proposition 2.4.6. *Let P be a computable structure on ω such that $SR(P) = \omega_1^{\emptyset}$. Then there exists an ill-founded model N of KP and some $t \in {}^{<\omega}\omega$ such that*

(i) $N \cap ON = \omega_1^{\emptyset}$.

(ii) For all $\alpha < \omega_1^{\emptyset}$, $N \models \rho((P,t)) > \alpha$.

(iii) $V \models \rho((P,t)) < \omega_1^{\emptyset}$.

Proof. Let $\mathscr{L}$ be a language consisting of the following:

(I) A binary relation symbol $\dot{\in}$.

(II) Constant symbols $\bar{a}$ for each $a \in L_{\omega_1^\emptyset}$.

(III) A distinguished constant symbol $\dot{s}$.

$\mathscr{L}$ can be considered a Δ_1 definable subset of $L_{\omega_1^\emptyset}$.

Let T be a theory in the countable admissible fragment $\mathscr{L}_{L_{\omega_1^\emptyset}}$ consisting of the following:

(i) KP

(ii) For each $a \in L_{\omega_1^\emptyset}$, $(\forall v)(v \dot{\in} \bar{a} \Rightarrow \bigvee_{z \in a} v = \bar{z})$.

(iii) $\dot{s}$ is a finite tuple on $\bar{\omega}$.

(iv) For each $\alpha < \omega_1^\emptyset$, $\rho((P, \dot{s})) > \bar{\alpha}$.

T can be considered a Σ_1 on $L_{\omega_1^\emptyset}$ theory.

T is consistent: Let $F \subseteq T$ such that $F \in L_{\omega_1^\emptyset}$. Then there exists $\alpha < \omega_1^\emptyset$ such that all ordinals mentioned in sentences of type (iv) are less than α. Since $\mathrm{SR}(P) = \omega_1^\emptyset$, there exists some $t \in {}^{<\omega}\omega$ such that $\rho((P, t)) > \alpha$. Consider the $\mathscr{L}$-structure defined as follows: $M = H_{\aleph_1}$. $\dot{\in}^{\mathcal{M}} = \in \restriction H_{\aleph_1}$. For each $a \in L_{\omega_1^\emptyset}$, $\bar{a}^{\mathcal{M}} = a$. $\dot{s}^{\mathcal{M}} = t$. Then $\mathcal{M} \models F$. F is consistent. By Barwise compactness (Theorem 2.4.5), T is consistent.

By Theorem 2.2.15, there is a solid structure $\mathcal{N} \models T$. Let $t = \dot{s}^{\mathcal{N}}$. Since $\mathcal{N} \models T$, for all $\alpha < \omega_1^\emptyset$, $\mathcal{N} \models \rho((P, t)) > \alpha$. However, since $\mathrm{SR}(P) = \omega_1^\emptyset$, one has $V \models \rho((P, t)) < \omega_1^\emptyset$. $\mathcal{N}$ and t are as desired. □

As mentioned before, the Δ_1^1 classification f of F_{ω_1} from Theorem 2.3.10 has the property that $\mathrm{SR}(f(x)) = \omega_1^x + 1$ for all x. Given the above remarks, one can ask the following:

Question 2.4.7. Does there exists a Δ_1^1 function f classifying F_{ω_1} such that $\mathrm{SR}(f(x)) = \omega_1^x$ for all $x \in {}^\omega 2$?

[4] produced a simple computable tree of Scott rank $\omega_1^\emptyset$. However, their proof in [4], Section 2 uses Barwise-Kreisel compactness and their proof in [4], Section 4 uses an overspill into the illfounded portion of the Harrison linear ordering. It is unclear if their proof method can be made uniform enough to produce in a $\boldsymbol{\Delta}_1^1$ manner a map taking x to some x-relative version of their tree.

The distinction between structure of rank ω_1^x and $\omega_1^x + 1$ has had some role in works on the Vaught's conjecture. For example, [20], Theorem 4.2 shows that if a scattered $\varphi \in \mathscr{L}_{\omega_1\omega}$ has the property that for all countable $M \models \varphi$, $\mathrm{SR}(M) \leq \omega_1^M$, then φ has only countably many models up to isomorphism (i.e., is not a counterexample to Vaught's conjecture).

Theorem 2.4.2 is only able to provide information about $f(x)$ when $\omega_1^x \in \Lambda(z)$ with z such that f is $\Delta_1^1(z)$. Some type of condition involving $\Lambda(z)$ is required:

Lemma 2.4.8. *Suppose $x \in {}^\omega 2$ is such that ω_1^x is not a recursively inaccessible ordinal. Then there exists a $z \in {}^\omega 2$ such that z is $\Delta_1^1(x)$ and $\{ot(z^{[n]}) : n \in \omega\} = \Lambda(\emptyset) \cap \omega_1^x$, where $z^{[n]} = \{y : \langle n, y\rangle \in z\}$.*

Proof. Since ω_1^x is not recursively inaccessible let β be the largest admissible ordinal less than ω_1^x. Since $\beta + 1 < \omega_1^x$, it is an x-recursive ordinal. There is an e such that $\{e\}^x$ has order type $\beta + 1$. The set

$$B = \{n \in \omega : \{e\}^x \restriction n \text{ is an admissible ordinal}\}$$

is a set in $L_{\omega_1^x}(x)$ by Δ_1 separation. Let $f : \omega \to B$ be a bijection in $L_{\omega_1^x}(x)$. Now define z by $z^{[n]} = \{e\}^x \restriction f(n)$. $\square$

In the proof above, one needed a bijection in $L_\alpha(x)$ between ω and $\Lambda(\emptyset) \cap \alpha$. Note that by Σ_1 collection, there is no Σ_1 function $f : \gamma \to \alpha$ with $\gamma < \alpha$ and f unbounded. If α is recursively inaccessible, then $\Lambda(\emptyset) \cap \alpha$ is unbounded in α. Hence when α is recursively inaccessible, there can not exist such a bijection.

Proposition 2.4.9. *Suppose $\alpha < \omega_1$ is an admissible but not recursively inaccessible ordinal. Let $\mathscr{L} = \{\dot{<}\}$. There exists z with $\omega_1^z = \alpha$ such that:*

(i) There is an $f : {}^\omega 2 \to S(\mathscr{L})$ which is $\Delta_1^1(z)$.

(ii) For all $x, y \in {}^\omega 2$, $x \; F_{\omega_1} \; y$ if and only if $f(x) \cong_{\mathscr{L}} f(y)$.

(iii) For all x with $\omega_1^x < \omega_1^z$, $SR(f(x)) < \omega_1^x$.

Proof. By Lemma 2.4.8, let z be such that $\{\mathrm{ot}(z^{[n]}) : n \in \omega\} = \Lambda(\emptyset) \cap \alpha$ and $\alpha = \omega_1^z$. Let $f : {}^\omega 2 \to S(\mathscr{L})$ be the Δ_1^1 classification given in Theorem 2.3.10. Let $g : \omega \to S(\mathscr{L})$ be Δ_1^1 such that for all $n \in \omega$, $g(n) \cong_{\mathscr{L}} \omega + n$.

Define the set $B \subseteq \omega \times {}^{\omega}2$ by:

$$(m, x) \in B \Leftrightarrow \omega_1^x = \mathrm{ot}(z^{[m]}).$$

The claim is that B is $\Delta^1_1(z)$:

It is $\Sigma^1_1(z)$.

$$(m, x) \in B \Leftrightarrow (\forall n)\Big((\{n\}^x \in \mathrm{WO} \Rightarrow \{n\}^x \preceq z^{[m]})\Big) \wedge$$
$$(\forall k)(\exists j)(\{j\}^x \preceq z^{[m]} \wedge z^{[m]} \restriction k \preceq \{j\}^x).$$

It is $\Pi^1_1(z)$.

$$(m, x) \notin B \Leftrightarrow (\exists j)(\forall n)(\{n\}^x \in \mathrm{WO} \Rightarrow$$
$$(\{n\}^x \preceq z^{[m]} \restriction j) \vee (\exists n)(\{n\}^x \in \mathrm{LO} \wedge \{n\}^x \preceq z^{[m]} \wedge z^{[m]} \preceq \{n\}^x).$$

Now define the following function $h : {}^{\omega}2 \to S(\mathscr{L})$.

$$(x, y) \in h \Leftrightarrow (\exists n)\Big((n, x) \in B \wedge y = g(n)\Big) \vee (\forall n)\Big((n, x) \notin B \wedge y = f(x)\Big),$$

h is $\Delta^1_1(z)$. For all x, y, $x\ F_{\omega_1}\ y$ if and only if $h(x) \cong_{\mathscr{L}} h(y)$. If $\omega_1^x < \omega_1^z$, then $\mathrm{SR}(h(x)) = \mathrm{SR}(\omega + n) < \omega_1^x$, where n is such that $\mathrm{ot}(z^{[n]}) = \omega_1^x$. □

Proposition 2.4.9 asserts that for each $\alpha < \omega_1$ which is admissible but not recursively inaccessible, there exists some z with $\omega_1^z = \alpha$ and some $\Delta^1_1(z)$ classification of F_{ω_1} such that the Scott rank condition of Theorem 2.4.2 fails on all the F_{ω_1}-classes associated with admissible ordinals less than α. Can this also be achieved when α is recursively inaccessible?

The most interesting question of this kind is: Is there some classification f of F_{ω_1} which is $\Delta^1_1(z)$ and the Scott rank condition fails for some class associated with an admissible ordinal $\alpha > \omega_1^z$?

2.5 Almost Borel Reductions

Definition 2.5.1. ([21] Definition 3.1.1) Let E be a Σ^1_1 equivalence relation on a Polish space X. Let $\mathbb{P}$ be a forcing and τ be a $\mathbb{P}$-name for an element of X, i.e. $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \tau \in X$. Let τ_{left} and τ_{right} be $\mathbb{P}^2$-names for the evaluation of τ according to the left and right $\mathbb{P}$-generic coming from the $\mathbb{P}^2$-generic. τ is an E-pinned $\mathbb{P}$-name if and only if $1_{\mathbb{P}^2} \Vdash \tau_{\mathrm{left}}\ E\ \tau_{\mathrm{right}}$.

([21] Definition 4.1.1 and 4.1.2) Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcings. Let σ be an E-pinned $\mathbb{P}$ name and τ be an E-pinned $\mathbb{Q}$-name. Define the relation $\sigma\ \bar{E}\ \tau$ if and only if $1_{\mathbb{P}\times\mathbb{Q}} \Vdash \sigma\ E\ \tau$ (where σ and τ are considered $\mathbb{P}\times\mathbb{Q}$-names in the natural way). The pinned cardinal of E, denoted $\kappa(E)$, is the smallest cardinal κ such that every E-pinned $\mathbb{P}$-name is $\bar{E}$-related to an E-pinned $\mathbb{Q}$-name with $|\mathbb{Q}| < \kappa$, if this cardinal exists. Otherwise, $\kappa(E) = \infty$.

Definition 2.5.2. E_{ω_1} is the Σ^1_1 equivalence relation on ${}^{\omega}2$ defined by $x\ E_{\omega_1}\ y$ if and only if $(x \notin \mathrm{WO} \wedge y \notin \mathrm{WO}) \vee (\mathrm{ot}(x) = \mathrm{ot}(y))$.

Proposition 2.5.3. $\kappa(E_{\omega_1}) = \infty$

Proof. See [21], Example 4.1.8 or see [22], Example 4.3.1. □

Definition 2.5.4. Let E and F be two equivalence relations on Polish spaces X and Y, respectively. $E \leq_{\mathrm{a}\boldsymbol{\Delta}^1_1} F$ if and only if there is a $\boldsymbol{\Delta}^1_1$ function $f : X \to Y$ and a countable set $A \subseteq X$ such that if c and d are not E-related to any elements of A, then $c\ E\ d$ if and only if $f(c)\ F\ f(d)$. In this situation, one says E is almost $\boldsymbol{\Delta}^1_1$ reducible to F. (It is called a weak Borel reduction in [21] Definition 2.1.2.)

Proposition 2.5.5. *Let E and F be Σ^1_1 equivalence relations on Polish spaces X and Y, respectively. If $E \leq_{a\boldsymbol{\Delta}^1_1} F$, then $\kappa(E) \leq \kappa(F)$.*

Proof. See [21], Theorem 4.1.3 or see [22], Theorem 4.2.2. □

Proposition 2.5.6. $\kappa(F_{\omega_1}) = \infty$.

Proof. For any cardinal κ, consider the forcing $\mathrm{Coll}(\omega,\kappa)$. Let τ be a $\mathrm{Coll}(\omega,\kappa)$ name for a real such that $1_{\mathrm{Coll}(\omega,\kappa)} \Vdash_{\mathrm{Coll}(\omega,\kappa)} \omega_1^\tau = \check{\kappa}$. τ is a F_{ω_1}-pinned $\mathrm{Coll}(\omega,\kappa)$-name, since

$$1_{\mathrm{Coll}(\omega,\kappa)\times\mathrm{Coll}(\omega,\kappa)} \Vdash_{\mathrm{Coll}(\omega,\kappa)\times\mathrm{Coll}(\omega,\kappa)} \omega_1^{\tau_{\mathrm{left}}} = \check{\kappa} = \omega_1^{\tau_{\mathrm{right}}}.$$

Now suppose $\mathbb{Q}$ is a forcing and σ is an F_{ω_1}-pinned $\mathbb{Q}$-name with $\tau\ \bar{F}_{\omega_1}\ \sigma$. This implies that $1_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \omega_1^\sigma = \check{\kappa}$. $1_{\mathbb{Q}} \Vdash_{\mathbb{Q}} |\check{\kappa}| = \aleph_0$. Since any forcing $\mathbb{Q}$ is $|\mathbb{Q}|^+$-cc, $\mathbb{Q}$ preserves cardinals greater than or equal to $|\mathbb{Q}|^+$. Since $\mathbb{Q}$ makes κ countable, $|\mathbb{Q}| \geq \kappa$. $\kappa(F_{\omega_1}) \geq \kappa$. Since κ was arbitrary, $\kappa(F_{\omega_1}) = \infty$. □

Theorem 2.5.7. *(Zapletal) Suppose there exists a measurable cardinal. Let E be a Σ^1_1 equivalence relation. $\kappa(E) = \infty$ if and only if $E_{\omega_1} \leq_{a\boldsymbol{\Delta}^1_1} E$.*

Proof. See [21], Theorem 4.2.1 or [22] Theorem 4.4.1. □

Proposition 2.5.8. (ZFC + Measurable Cardinal) $E_{\omega_1} \leq_{a\mathbf{\Delta}^1_1} F_{\omega_1}$.

Proof. This follows from Theorem 2.5.7 and Proposition 2.5.6. □

Since Theorem 2.5.7 assumes a measurable cardinal, a natural task would be to investigate the consistency strength of the statement "For all Σ^1_1 equivalence relation, $\kappa(E) = \infty$ if and only if $E_{\omega_1} \leq_{\mathrm{a}\mathbf{\Delta}^1_1} E$".

Therefore, an interesting question is whether L satisfies the above statement. The rest of this section will consider this question.

Theorem 2.5.9. *Suppose $x \in WO$ and $y \in {}^{\omega}2$ such that $\omega_1^y < ot(x)$, then $[x]_{E_{\omega_1}}$ is not $\Sigma^1_1(y)$.*

Proof. Suppose $[x]_{E_{\omega_1}}$ was $\Sigma^1_1(y)$. Let U be a tree on $2 \times \omega$ which is recursive in y and

$$(\forall u)(u \in [x]_{E_{\omega_1}} \Leftrightarrow (\exists f)(f \in [U^u])).$$

Let $\mathscr{L}$ be a language consisting of the following:

(i) A binary relation symbol $\dot{\in}$.

(ii) For each $a \in L_{\omega_1^y}(y)$, a constant symbol $\bar{a}$.

(iii) Two distinct constant symbols $\dot{c}$ and $\dot{d}$.

$\mathscr{L}$ may be considered a Δ_1 definable language over $L_{\omega_1^y}(y)$.

Let T be a theory in the countable admissible fragment $\mathscr{L}_{L_{\omega_1^y}(y)}$ consisting of the following sentences:

(I) KP

(II) For each $a \in L_{\omega_1^y}(y)$, $(\forall v)(v \dot{\in} \bar{a} \Rightarrow \bigvee_{z \in a} v = \bar{z})$.

(III) $\dot{c} \subseteq \bar{\omega}$, $\dot{d} : \bar{\omega} \to \bar{\omega}$.

(IV) $\dot{d} \in [U^{\dot{c}}]$.

(V) For all $\alpha < \omega_1^y$, "$\bar{\alpha}$ is not admissible in $\dot{c}$".

T may be considered a Σ_1 theory in $L_{\omega_1^y}(y)$.

Next, the claim is that T is consistent. Since $x \in [x]_{E_{\omega_1}}$, there exists g such that $g \in [U^x]$. Consider the $\mathscr{L}$-structure $\mathcal{N}$ defined as follows: Let the universe N be $H_{\aleph_1}$. Let $\dot{\in}^{\mathcal{N}} = \in \restriction H_{\aleph_1}$. Let $\dot{c}^{\mathcal{N}} = x$ and $\dot{d}^{\mathcal{N}} = g$. $\mathcal{N} \models T$. For (V), observe that if $\mathscr{A}$ is an admissible set with $x \in A$, then $\text{ot}(x) \in A$. Hence $\text{ON} \cap A > \text{ot}(x) > \omega_1^y$.

By Theorem 2.2.15, let $\mathcal{M}$ be a solid model of T with $\text{ON} \cap M = \omega_1^y$. Let $z = c^{\mathcal{M}}$. $z \in [x]_{E_{\omega_1}}$ since $\dot{d}^{\mathcal{M}} \in [U^z]$. As in the proof of Sacks theorem, $\omega_1^z = \omega_1^y$. $z \in L_{\omega_1^z}(z)$. So $\text{ot}(z) \in L_{\omega_1^z}(z)$. This is impossible since $\omega_1^z = \omega_1^y < \text{ot}(x) = \text{ot}(z)$. □

Theorem 2.5.10. *If $f : {}^\omega 2 \to {}^\omega 2$ is $\Delta^1_1(y)$ and witnesses $E_{\omega_1} \leq_{a\mathbf{\Delta}^1_1} F_{\omega_1}$, then there exists a $\beta < \omega_1$ such that for all $\alpha \in \Lambda(y)$ with $\alpha > \beta$, the next admissible ordinal after α is not in $\Lambda(y)$.*

Proof. Let $f : {}^\omega 2 \to {}^\omega 2$ witness $E_{\omega_1} \leq_{\mathrm{a}\mathbf{\Delta}^1_1} F_{\omega_1}$. There exists some countable set $A \subseteq {}^\omega 2$ such that $x \; E_{\omega_1} \; y$ if and only if $f(x) \; F_{\omega_1} \; f(y)$ whenever $x, y \notin [A]_E$. Let $\beta = \sup\{\text{ot}(x) : x \in A\}$. The claim is that this β works. So suppose not. There exists $\alpha', \alpha \in \Lambda(y)$ such that $\alpha > \beta$, $\alpha' > \beta$, and α is the next admissible ordinal after α'.

Since f is $\Delta^1_1(y)$, let U be a tree on $2 \times 2 \times \omega$ such that for all $a, b \in {}^\omega 2$, $(a, b) \in f \Leftrightarrow [U^{(a,b)}]$ is ill-founded.

Claim: There exists $a, b \in {}^\omega 2$ such that $\alpha' < \text{ot}(a) < \text{ot}(b) < \alpha$, $\omega_1^{f(a)} \geq \alpha$, and $\omega_1^{f(b)} \geq \alpha$.

To prove this claim: If there exists a $c \in {}^\omega 2$ such that $\alpha' < \text{ot}(c) < \alpha$ and $\omega_1^{f(c)} = \alpha'$, then fix such a c. If not, pick any $c \in {}^\omega 2$ such that $\alpha' < \text{ot}(c) < \alpha$. In this latter case, c will essentially be ignored.

Then for any $d \in {}^\omega 2$ with $\text{ot}(d) > \beta$ and $d \notin [c]_{E_{\omega_1}}$, $\omega_1^{f(d)} \neq \alpha'$ since f is a reduction. Pick any $d \in {}^\omega 2$ with $d \notin [c]_{E_{\omega_1}}$ and $\alpha' < \text{ot}(d) < \alpha$.

Suppose that $\omega_1^{f(d)} < \alpha'$. By Proposition 2.2.16, let z be any real such that $\omega_1^z = \alpha'$ and $y \leq_T z$. $[f(d)]_{F_{\omega_1}}$ is $\Delta^1_1(z)$ by Proposition 2.2.38.

$$k \in [d]_{E_{\omega_1}} \Leftrightarrow f(k) \in [f(d)]_{F_{\omega_1}}.$$

Hence $[d]_{E_{\omega_1}}$ is $\Sigma^1_1(y, z) = \Sigma^1_1(z)$. However, $\omega_1^z = \alpha' < \text{ot}(d)$. This contradicts Theorem 2.5.9.

This shows that $\omega_1^{f(d)} \geq \alpha'$. Since $d \notin [c]_{E_{\omega_1}}$, $\omega_1^{f(d)} > \alpha'$. However, the next admissible ordinal greater than α' is α. Therefore, $\omega_1^{f(d)} \geq \alpha$.

Now let a, b be any two reals such that $a, b \notin [c]_{E_{\omega_1}}$ and $\alpha' < \text{ot}(a) < \text{ot}(b) < \alpha$. Since d in the above was arbitrary with these two properties, these two reals satisfy the claim.

Now fix $a, b \in {}^{\omega}2$ satisfying the claim. $\text{SR}(a) = \text{ot}(a)$ and $\text{SR}(b) = \text{ot}(b)$. Thus their canonical Scott sentences are both elements of L_α since $\text{ot}(a), \text{ot}(b) \in L_\alpha$, $\text{SR}(\text{ot}(a)) < \alpha$, $\text{SR}(\text{ot}(b)) < \alpha$, and Proposition 2.2.20.

Let $\mathscr{L}$ be a language consisting of:

(i) A binary relation symbol $\dot{\in}$.

(ii) For each $e \in L_\alpha(y)$, a constant symbol $\bar{e}$.

(iii) Six distinct symbols $\dot{a}, \dot{b}, \dot{c}, \dot{d}, \dot{u}, \dot{v}$.

$\mathscr{L}$ may be considered as a Δ_1 definable language in $L_\alpha(y)$.

Let T be a theory in the countable admissible fragment $\mathscr{L}_{L_\alpha(y)}$ consisting of the following sentences:

(I) KP in the symbol $\dot{\in}$.

(II) For each $e \in L_\alpha(y)$, $(\forall v)(v \dot{\in} \bar{e} \Rightarrow \bigvee_{z \in e} v = \bar{z})$.

(III) $\dot{a}, \dot{b}, \dot{c}, \dot{d} \subseteq \bar{\omega}$. $\dot{u}, \dot{v}$ are functions from $\bar{\omega} \to \bar{\omega}$.

(IV) $\dot{u} \in [U^{(\dot{a},\dot{c})}]$ and $\dot{v} \in [U^{(\dot{b},\dot{d})}]$.

(V) $\dot{a} \models \text{CSS}(a)$ and $\dot{b} \models \text{CSS}(b)$.

(VI) For all $\beta < \alpha$, $\bar{\beta}$ is not admissible in $\dot{c}$ and $\bar{\beta}$ is not admissible in $\dot{d}$.

T may be considered a Σ_1 theory in $L_\alpha(y)$.

Since $(a, f(a)) \in f$ and $(b, f(b)) \in f$, let $u, v \in {}^{\omega}\omega$, be such that $u \in [U^{(a,f(a))}]$ and $v \in [U^{(b,f(b))}]$.

To show that T is consistent: consider the following model of $\mathcal{N}$: The universe N is $H_{\aleph_1}$. $\dot{\in}^{\mathcal{N}} = \in \restriction H_{\aleph_1}$. For each $e \in L_\alpha(y)$, $\bar{e}^{\mathcal{N}} = e$. $\dot{a}^{\mathcal{N}} = a$, $\dot{b}^{\mathcal{N}} = b$. Let $\dot{c}^{\mathcal{N}} = f(a)$ and $\dot{d}^{\mathcal{N}} = f(b)$. Let $\dot{u}^{\mathcal{N}} = u$ and $\dot{v}^{\mathcal{N}} = v$. Then $\mathcal{N} \models T$.

By Theorem 2.2.15, there exists a solid model $\mathcal{M} \models T$ with $M \cap \text{ON} = \alpha$. Let $a' = \dot{a}^{\mathcal{M}}$, $b' = \dot{b}^{\mathcal{M}}$. $f(a') = \dot{c}^{\mathcal{M}}$ and $f(b') = \dot{d}^{\mathcal{M}}$ since $\dot{u}^{\mathcal{M}} \in [U^{(a',\dot{c}^{\mathcal{M}})}]$ and $\dot{v}^{\mathcal{M}} \in [U^{(b',\dot{d}^{\mathcal{M}})}]$. As in the proof of Sacks theorem, $\omega_1^{f(a')} = \omega_1^{f(b')} = \alpha$. By absoluteness of satisfaction from $\mathcal{M}$ to $\text{WF}(\mathcal{M})$ to V, $a' \models \text{CSS}(a)$ and $b' \models \text{CSS}(b)$. Hence in V, $\text{ot}(a') = \text{ot}(a)$ and $\text{ot}(b') = \text{ot}(b)$. In particular, $\text{ot}(a') \neq \text{ot}(b')$. Hence

$\neg(a' \ E_{\omega_1} \ b')$. However, $\omega_1^{f(a')} = \omega_1^{f(b')} = \alpha$ implies $f(x) \ F_{\omega_1} \ f(y)$. This contradicts f being a reduction.

This proves the theorem for those $\alpha \in \Lambda(y) \cap \omega_1$. Note the statement that f and countable $A \subseteq {}^{\omega}2$ witnesses $E_{\omega_1} \leq_{\mathrm{a}\boldsymbol{\Delta}_1^1} F_{\omega_1}$ can be written as

$$(\forall x)(\forall y)((x \notin [A]_{E_{\omega_1}} \wedge y \notin [A]_{E_{\omega_1}}) \Rightarrow (x \ E_{\omega_1} \ y \Leftrightarrow f(x) \ F_{\omega_1} \ f(y))).$$

This is $\Pi_2^1(y, A)$ and so holds in all generic extensions by Schoenfield's absoluteness. To show the theorem holds for all $\alpha \in \Lambda(y)$ and $\alpha \geq \omega_1$, let $G \subseteq \mathrm{Coll}(\omega, \alpha)$ be $\mathrm{Coll}(\omega, \alpha)$-generic over V. In $V[G]$, let $\beta = \sup\{\mathrm{ot}(x) : x \in A\}$ be the same ordinal as before. Since $\beta < \omega_1^V \leq \alpha < \omega_1^{V[G]}$, the result above, applied in $V[G]$ for $\Lambda(y) \cap \omega_1^{V[G]}$, will show the theorem holds for α. This concludes the proof. □

Theorem 2.5.11. $L \models \neg(E_{\omega_1} \leq_{a\boldsymbol{\Delta}_1^1} F_{\omega_1})$. *This also holds in set generic extensions of L.*

Proof. In L, for all $x \in {}^{\omega}2$, there exists some $\alpha < \omega_1$ such that $x \in L_\alpha$. Then $\Lambda(x) - \alpha = \Lambda(\emptyset) - \alpha$. Hence there are no reals with admissible spectrum as described in Theorem 2.5.10. □

2.6 Counterexamples to Vaught's Conjecture and F_{ω_1}

Definition 2.6.1. Let $\mathscr{L}$ be a recursive language. Let $\varphi \in \mathscr{L}_{\omega_1\omega}$. Define $E_{\mathscr{L}}^{\varphi}$ to be the Σ_1^1 equivalence relation on $S(\mathscr{L})$ defined by

$$x \ E_{\mathscr{L}}^{\varphi} \ y \Leftrightarrow (x \not\models \varphi \wedge y \not\models \varphi) \vee (x \cong_{\mathscr{L}} y).$$

See Proposition 2.2.35 for the $\boldsymbol{\Delta}_1^1$ definability of $x \not\models \varphi$.

Definition 2.6.2. A *counterexample to Vaught's conjecture* is a $\varphi \in \mathscr{L}_{\omega_1\omega}$ (for some recursive language $\mathscr{L}$) such that $E_{\mathscr{L}}^{\varphi}$ is a thin equivalence relation with uncountably many classes.

From a list of questions from the Vaught's Conjecture Workshop 2015 at the University of California at Berkeley, Sy-David Friedman asked the following question:

Question 2.6.3. (Sy-David Friedman) Is there some recursive language $\mathscr{L}$ such that F_{ω_1} is $\boldsymbol{\Delta}_1^1$ bireducible to the $\mathscr{L}$-isomorphism relation restricted to some $\boldsymbol{\Delta}_1^1$ invariant set?

Every invariant $\mathbf{\Delta}^1_1$ set for the $\mathscr{L}$-isomorphism relation is of the form $\text{Mod}(\varphi)$ for some $\varphi \in \mathscr{L}_{\omega_1\omega}$ (see [6], Theorem 11.3.6). Therefore, the above question is equivalent to whether there exists some φ such that $F_{\omega_1} \equiv_{\mathbf{\Delta}^1_1} E^{\varphi}_{\mathscr{L}}$.

$E^{\varphi}_{\mathscr{L}} \leq_{\mathbf{\Delta}^1_1} F_{\omega_1}$ implies that $E^{\varphi}_{\mathscr{L}}$ is thin. $F_{\omega_1} \leq_{\mathbf{\Delta}^1_1} E_{\mathscr{L}}$ implies that $E^{\varphi}_{\mathscr{L}}$ has uncountably many classes. Hence any such φ is a counterexample to Vaught's conjecture.

Using the ideas from the previous section, it will be shown that in L, no counterexample φ of Vaught's conjecture has the property that $E^{\varphi}_{\mathscr{L}} \leq_{\mathbf{\Delta}^1_1} F_{\omega_1}$. Hence, Friedman's question has a negative answer in L.

Theorem 2.6.4. *Let $\mathscr{L}$ be a recursive language. Let $M \in S(\mathscr{L})$ and $y \in {}^{\omega}2$ be such that $\omega_1^y < R(M)$. Then $[M]_{\cong_{\mathscr{L}}}$ is not $\Sigma^1_1(y)$. ($\cong_{\mathscr{L}}$ denotes the equivalence relation of $\mathscr{L}$-isomorphism. Recall R is defined in Definition 2.2.18.)*

Proof. Suppose $[M]_{\cong_{\mathscr{L}}}$ is $\Sigma^1_1(y)$. Let U be a tree on $2 \times \omega$ which is recursive in y and

$$(\forall N)(N \in [M]_{\cong_{\mathscr{L}}} \Leftrightarrow (\exists f)(f \in [U^N])).$$

Let $\mathscr{U}$ be the language consisting of the following:

(i) A binary relation symbol $\dot{\in}$.

(ii) For each $a \in L_{\omega_1^y}(y)$, a constant symbol $\bar{a}$.

(iii) Two distinct constant symbols $\dot{c}$ and $\dot{d}$.

$\mathscr{U}$ may be considered a Δ_1 definable class in $L_{\omega_1^y}(y)$.

Let T be a theory in the countable admissible fragment $\mathscr{U}_{L_{\omega_1^y}(y)}$ consisting of the following sentences:

(I) KP

(II) For each $a \in L_{\omega_1^y}(y)$, $(\forall v)(v \dot{\in} \bar{a} \Rightarrow \bigvee_{z \in a} v = \bar{z})$.

(III) $\dot{c} \subseteq \bar{\omega}$, $\dot{d} : \bar{\omega} \to \bar{\omega}$.

(IV) $\dot{d} \in [U^{\dot{c}}]$.

(V) For all $\alpha < \omega_1^y$, $\bar{\alpha}$ is not admissible in $\dot{c}$.

T may be considered a Σ_1 definable theory in $L_{\omega_1^y}(y)$.

T is consistent: Since $M \in [M]_{\cong_{\mathscr{L}}}$, there is some g such that $g \in [U^M]$. Define a $\mathscr{U}$-structure $\mathcal{N}$ as follows: Let the universe N be $H_{\aleph_1}$. Let $\dot{\in}^{\mathcal{N}} = \dot{\in} \restriction H_{\aleph_1}$. Let

$\dot{c}^{\mathcal{N}} = M$ and $\dot{d}^{\mathcal{N}} = g$. $\mathcal{N}$ is a model of T. To see (V), note that $\omega_1^M > \omega_1^y$. This is because, if otherwise, M would be an element of some admissible set $\mathcal{A}$ such that $o(\mathcal{A}) \leq \omega_1^y$. By Proposition 2.2.28, $\mathrm{R}(M) \leq \mathrm{ON} \cap A \leq \omega_1^y$, which is a contradiction.

By Theorem 2.2.15, let $\mathcal{M}$ be a solid model of T. Let $P = \dot{c}^{\mathcal{M}}$. $P \in [M]_{\cong_{\mathscr{L}}}$ since $\dot{d} \in [U^P]$. Like in the proof of Sacks theorem, $\omega_1^P = \omega_1^y$. Therefore, $P \in L_{\omega_1^y}(P)$. By Proposition 2.2.28, $\mathrm{R}(P) \leq \omega_1^y$. However $P \in [M]_{\cong_{\mathscr{L}}}$ implies that $P \cong_{\mathscr{L}} M$. $\mathrm{R}(P) \leq \omega_1^y < R(M)$. Contradiction. □

Fact 2.6.5. *Let $\mathscr{L}$ be a recursive language. If φ is a counterexample to the Vaught conjecture, then for all limit ordinals $\beta > qr(\varphi)$, φ has a model of Scott rank β.*

Proof. See [11], Theorem 10.8. A similar result is also shown in [16], page 19. □

Fact 2.6.6. *Let $\mathcal{A}$ be a countable admissible set. Let $\alpha < o(\mathcal{A})$. Then there exists a countable admissible set $\mathcal{B}$ extending $\mathcal{A}$ with $o(\mathcal{B}) = o(\mathcal{A})$ and such that there exists a $c \in {}^{\omega}2 \cap B$ with $ot(c) = \alpha$.*

Proof. This can be proved using the techniques of infinitary logic in the countable admissible fragment $\mathcal{A}$ using a Scott sentence for α as a linear ordering. Since this is similar to several previous arguments, the details are omitted. □

Fact 2.6.7. *Let E be a $\Pi_1^1(z)$ equivalence relation on a Polish space ${}^{\omega}2$ with countably many classes. Then for all $x \in {}^{\omega}2$, there is a $\Delta_1^1(z)$ set U such that $x \in U \subseteq [x]_E$.*

Proof. See [7].

In the effective proof of Silver's dichotomy for $\mathbf{\Pi}_1^1$ equivalence using the Gandy-Harrington topology, the two outcomes depend on whether the set

$$V = \{x \in {}^{\omega}2 : \text{There exists } \Delta_1^1(z) \text{ set } U \text{ with } x \in U \subseteq [x]_E\}$$

is equal to ${}^{\omega}2$. If $V = {}^{\omega}2$, then E has only countable many classes. This gives the desired result above. See [6], Theorem 5.3.5 for a presentation of the effective proof of Silver's theorem. □

Fact 2.6.8. *Let $\mathscr{L}$ be a recursive language. Let $\varphi \in \mathscr{L}_{\omega_1\omega}$ be a counterexample to Vaught's conjecture. Let $z \in {}^{\omega}2$ be such that $\varphi \in L_{\omega_1^z}(z)$. Let β be a z-admissible ordinal. Let $M, N \in S(\mathscr{L})$ be such that $M \models \varphi$, $N \models \varphi$, $R(M) < \beta$ and $R(N) < \beta$. Then there exists a countable admissible set $\mathcal{A}$ extending $L_\beta(z)$ such that $o(\mathcal{A}) = \beta$ and $CSS(M), CSS(N) \in A$.*

Proof. Let $\alpha < \beta$ be such that $\mathrm{R}(M) < \alpha$ and $\mathrm{R}(N) < \alpha$. $L_\beta(z)$ is an admissible set. By Fact 2.6.6, there exists some countable admissible set $\mathcal{B}$ extending $L_\beta(z)$ containing some real c which codes α and $o(\mathcal{B}) = \beta$.

Let $\equiv_\alpha^{\mathscr{L}}$ be the relation of $\mathscr{L}$-elementary equivalence with respect to just the formulas of quantifier rank less than α. $\equiv_\alpha^{\mathscr{L}} \restriction \mathrm{Mod}(\varphi)$ is a $\Delta_1^1(c, z)$ equivalence relation. (Proposition 2.2.35 is used here.) Since φ is a counterexample to Vaught's conjecture, $\equiv_\alpha^{\mathscr{L}} \restriction \mathrm{Mod}(\varphi)$ has only countably many classes. By Fact 2.6.7, there exists $\Delta_1^1(c, z)$ sets U_M and U_N such that $M \in U_M \subseteq [M]_{\equiv_\alpha^{\mathscr{L}} \restriction \mathrm{Mod}(\varphi)}$ and $N \in U_N \subseteq [N]_{\equiv_\alpha^{\mathscr{L}} \restriction \mathrm{Mod}(\varphi)}$. Let T_M and T_N be the $c \oplus z$ recursive trees such that for all P

$$P \in U_M \Leftrightarrow T_M^P \text{ is illfounded}$$

$$P \in U_N \Leftrightarrow T_N^P \text{ is illfounded.}$$

Let $\mathscr{U}$ be the language consisting of the following:

(i) A binary relation symbol $\dot{\in}$.

(ii) For each $a \in B$, a constant symbol $\bar{a}$.

(iii) Four new constant symbols, $\dot{R}$, $\dot{S}$, $\dot{e}$, and $\dot{f}$.

$\mathscr{U}$ may be considered a Δ_1 definable language in $\mathcal{B}$.

Let T be a theory in the countable admissible fragment $\mathscr{U}_{\mathcal{B}}$ consisting of the following sentences:

(I) KP

(II) For each $a \in B$, $(\forall v)(v \dot{\in} \bar{a} \Rightarrow \bigvee_{z \in a} v = \bar{z})$.

(III) $\dot{R} \subseteq \bar{\omega}$, $\dot{S} \subseteq \bar{\omega}$, $\dot{e} \in [T_M^{\dot{R}}]$, and $\dot{f} \in [T_N^{\dot{R}}]$.

T may be considered as a Σ_1 definable set in $\mathcal{A}$.

T is consistent: Since $M \in U_M$ and $N \in U_N$, find some $v \in [T_M^M]$ and $w \in [T_N^N]$. Consider the $\mathscr{U}$ structure $\mathcal{I}$ defined as follows: Let the universe I be $H_{\aleph_1}$. Let $\dot{\in}^{\mathcal{I}} = \in \restriction H_{\aleph_1}$. Let $\dot{R}^{\mathcal{I}} = M$, $\dot{S}^{\mathcal{I}} = N$, $\dot{e}^{\mathcal{I}} = v$, and $\dot{f}^{\mathcal{I}} = w$. Then $\mathcal{I} \models T$.

By Theorem 2.2.15, let $\mathcal{J}$ be a solid model of T with $o(\mathcal{J}) = o(\mathcal{B})$. Let $R = \dot{R}^{\mathcal{J}}$, $S = \dot{S}^{\mathcal{J}}$, $e = \dot{e}^{\mathcal{J}}$, and $f = \dot{f}^{\mathcal{J}}$. By Δ_1 absoluteness (first between $\mathcal{J}$ and $\mathrm{WF}(\mathcal{J})$ and then between $\mathrm{WF}(\mathcal{J})$ and V), $e \in [T_M^R]$ and $f \in [T_M^S]$. Hence $R \in U_M$ and $S \in U_N$.

Let $\mathcal{A} = \mathrm{WF}(\mathcal{J})$. By Lemma 2.2.14, $\mathcal{A}$ is an admissible set. It has been shown that $\mathcal{A}$ has two elements R and S such that $R \in [M]_{\equiv_\alpha^{\mathscr{L}} \restriction \mathrm{mod}(\varphi)}$ and $S \in [N]_{\equiv_\alpha^{\mathscr{L}} \restriction \mathrm{Mod}(\varphi)}$.

Since CSS(M) and CSS(N) have quantifier rank less than α, $M \equiv_\alpha^{\mathscr{L}} R$, and $N \equiv_\alpha^{\mathscr{L}} S$, the following must hold: $R \models \text{CSS}(M)$ and $S \models \text{CSS}(N)$. Hence CSS(R) = CSS(M) and CSS(S) = CSS(N).

Since $R(M), R(N) < \alpha$, Proposition 2.2.20 implies that CSS(R) $\in A$ and CSS(S) $\in A$. Therefore, CSS(M) $\in A$ and CSS(N) $\in A$. This completes the proof. □

Theorem 2.6.9. *Let $\mathscr{L}$ be a recursive language. Let $\varphi \in \mathscr{L}_{\omega_1\omega}$ be a counterexample to Vaught's conjecture. Suppose f is a $\boldsymbol{\Delta}_1^1$ function witnessing $E_{\mathscr{L}}^{\varphi} \leq_{\boldsymbol{\Delta}_1^1} F_{\omega_1}$, then there exists some ordinal γ and real z such that for all $\alpha \in \Lambda(z)$ with $\alpha > \gamma$, the next admissible ordinal greater than α is not in $\Lambda(z)$.*

Proof. First, the theorem will be shown for $\Lambda(z) \cap \omega_1$. At the end, this result will be used to obtain the theorem for the full $\Lambda(z)$.

Let $f : S(\mathscr{L}) \to {}^\omega 2$ be $\Delta_1^1(r)$ witnessing $E_{\mathscr{L}}^{\varphi} \leq_{\boldsymbol{\Delta}_1^1} F_{\omega_1}$ where r is some real. Find any $s \in {}^\omega 2$ such that $\varphi \in L_{\omega_1^s}(s)$. Let $z = r \oplus s$. Note that f is $\Delta_1^1(z)$. Let $\gamma = \omega_1^z$. Certainly $\gamma > \text{qr}(\varphi)$.

Now suppose there exists some $\alpha, \beta \in \Lambda(z) \cap \omega_1$ with $\gamma < \alpha$ and β is the next admissible ordinal greater than α.

Between two consecutive admissible ordinals, there are infinitely many limit ordinals. Since φ is a counterexample to the Vaught's conjecture, Fact 2.6.5 implies that there are infinitely many models of φ with Scott ranks between α and β. Let P, M, and N be three models of φ with distinct Scott rank between α and β. Since f is a reduction of $E_{\mathscr{L}}^{\varphi}$ to F_{ω_1}, at most one $X \in \{P, M, N\}$ has the property that $\omega_1^{f(X)} = \alpha$. If such an X among these three exists, then without loss of generality, assume it was P. (If no X among these three has this property, then one can just ignore P for the rest of the proof.)

Now to show that $\omega_1^{f(M)} \geq \beta$ and $\omega_1^{f(N)} \geq \beta$: Suppose $\omega_1^{f(M)} < \beta$. Since P and M are not $\mathscr{L}$-isomorphic and f is a reduction to F_{ω_1}, $\omega_1^{f(M)} \neq \alpha$ (since one assumed that $\omega_1^{f(P)} = \alpha$, if this could occur among the three models). Thus, $\omega_1^{f(M)} < \alpha$ since β is the next admissible ordinal after α. Observe that

$$X \in [M]_{E_{\cong_{\mathscr{L}}}^{\varphi}} \Leftrightarrow f(X) \in [f(M)]_{F_{\omega_1}}.$$

Let $y \in {}^\omega 2$ be such that $z \leq_T y$ and $\omega_1^y = \alpha$ (which exists due to Theorem 2.2.16). $[f(M)]_{F_{\omega_1}}$ is $\Delta_1^1(y)$ by Proposition 2.2.38. This shows that $[M]_{\cong_{\mathscr{L}}}$ is $\Sigma_1^1(y, z) = \Sigma_1^1(y)$. $\omega_1^y = \alpha < R(M)$. This contradicts Theorem 2.6.4.

So it has been shown that $\omega_1^{f(M)} > \alpha$. But since β is the smallest admissible ordinal greater than α, $\omega_1^{f(M)} \geq \beta$. The same exact argument shows $\omega_1^{f(N)} \geq \beta$.

By Fact 2.6.8, let $\mathcal{A}$ be a countable admissible set extending $L_\beta(z)$ containing $\mathrm{CSS}(M)$ and $\mathrm{CSS}(N)$ with $o(\mathcal{A}) = \beta$.

Since f is $\Delta_1^1(z)$, let U be a z-recursive tree on $2 \times 2 \times \omega$ such that for all $X \in S(\mathscr{L})$ and $r \in {}^\omega 2$,

$$(X, r) \in f \Leftrightarrow [U^{(X,r)}] \neq \emptyset.$$

Let $\mathscr{U}$ be a language consisting of:

(i) A binary relation symbol $\dot{\in}$.

(ii) For each $e \in A$, a constant symbol $\bar{e}$.

(iii) Six distinct symbols $\dot{R}$, $\dot{S}$, $\dot{c}$, $\dot{d}$, $\dot{u}$, and $\dot{v}$.

$\mathscr{U}$ may be considered as a Δ_1 definable language in A.

Let T be the theory in the countable admissible fragment $\mathscr{U}_{\mathcal{A}}$ consisting of the following sentences:

(I) KP

(II) For each $e \in A$, $(\forall v)(v \dot{\in} \bar{e} \Rightarrow \bigvee_{z \in e} v = \bar{z})$.

(III) $\dot{R}, \dot{S}, \dot{c}, \dot{d} \subseteq \bar{\omega}$. $\dot{u}$ and $\dot{v}$ are functions from $\bar{\omega} \to \bar{\omega}$.

(IV) $\dot{u} \in [U^{(\dot{R},\dot{c})}]$ and $\dot{v} \in [U^{(\dot{S},\dot{d})}]$.

(V) $\dot{R} \models \mathrm{CSS}(M)$ and $\dot{S} \models \mathrm{CSS}(N)$.

(VI) For all $\xi < \beta$, $\bar{\xi}$ is not admissible in $\dot{c}$ and $\bar{\xi}$ is not admissible in $\dot{d}$.

T may be considered a Σ_1 definable theory in $\mathcal{A}$. Note that $\mathcal{A}$ was chosen so that (V) would be expressible.

Since $(M, f(M)) \in f$ and $(N, f(N)) \in f$, let $u, v \in {}^\omega\omega$ be such that $u \in [U^{(M,f(M))}]$ and $v \in [U^{(N,f(N))}]$.

Now to show T is consistent: Consider the following $\mathscr{U}$-structure $\mathcal{G}$. The domain of $\mathcal{G}$ is $G = H_{\aleph_1}$. For each $e \in A$, $\bar{e}^{\mathcal{G}} = e$. $\dot{R}^{\mathcal{G}} = M$. $\dot{S}^{\mathcal{G}} = N$, $\dot{c}^{\mathcal{G}} = f(M)$, $\dot{d}^{\mathcal{G}} = f(N)$, $\dot{u}^{\mathcal{G}} = u$, and $\dot{v}^{\mathcal{G}} = v$. Then $\mathcal{G} \models T$.

By Theorem 2.2.15, there exists a solid model $\mathcal{H} \models T$ with $o(\mathcal{H}) = o(\mathcal{A})$. Let $R = \dot{R}^{\mathcal{H}}$ and $S = \dot{S}^{\mathcal{H}}$. Then $f(R) = \dot{c}^{\mathcal{H}}$ and $f(S) = \dot{d}^{\mathcal{H}}$ since $\dot{u}^{\mathcal{H}} \in [U^{(R,\dot{c}^{\mathcal{H}})}]$

and $\dot{v}^{\mathcal{H}} \in [U^{(S,\dot{d}^{\mathcal{H}})}]$. As in the proof of Sacks' theorem, $\omega_1^{f(R)} = \omega_1^{f(S)} = \beta$. By the absoluteness of satisfaction (from $\mathcal{H}$ to $\mathrm{WF}(\mathcal{H})$ to V), $R \models \mathrm{CSS}(M)$ and $S \models \mathrm{CSS}(N)$. Hence in V, R and S are not $\mathscr{L}$-isomorphic. However, $\omega_1^{f(R)} = \omega_1^{f(S)} = \beta$ implies that $f(R)\ F_{\omega_1}\ f(S)$. This contradicts f being a reduction.

This proves the theorem for $\alpha \in \Lambda(y) \cap \omega_1$. The statement f witnesses $E^{\varphi}_{\mathscr{L}} \leq_{\mathbf{\Delta}^1_1} F_{\omega_1}$ is $\mathbf{\Pi}^1_2$. So the same argument as at the end of the proof of Theorem 2.5.10 shows the result holds for all $\alpha \in \Lambda(y)$. □

Corollary 2.6.10. *In L (and any set generic extension of L), there are no recursive language $\mathscr{L}$ and counterexample to Vaught's conjecture $\varphi \in \mathscr{L}_{\omega_1\omega}$ such that $E^{\varphi}_{\mathscr{L}} \leq_{\mathbf{\Delta}^1_1} F_{\omega_1}$.*

Proof. There is no $z \in {}^{\omega}2$ having the property of Theorem 2.6.9 in L or set generic extensions of L. □

Corollary 2.6.11. *In L (and set generic extensions of L), Question 2.6.3 has a negative answer.*

Proof. This follows from Corollary 2.6.11 and the remarks following Question 2.6.3. □

This leaves open whether there is an answer to Question 2.6.3 in ZFC.

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Chapter 3

EQIVALENCE RELATIONS WHICH ARE BOREL SOMEWHERE

3.1 Introduction

The basic question addressed here in its most naive form is:

Question: If E is an equivalence relation on a Polish space X, is there a large and nice set $C \subseteq X$ such that $E \restriction C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

Here, $E \restriction C = E \cap (C \times C)$.

Two immediate concerns about the question arise from the phrase “large and nice":

The basic idea of the question is that given an equivalence relation E, can one find a subset C such that $E \restriction C$ is a simpler equivalence relation, in particular $\mathbf{\Delta}^1_1$. One does not want to hide any complexity of $E \restriction C$ inside the set C. Therefore C should be “nice” in the sense that it is a $\mathbf{\Delta}^1_1$ subset of X.

Every equivalence relation restricted to a countable set is a $\mathbf{\Delta}^1_1$ equivalence relation. Conditions must be imposed on C to make the question meaningful. σ-ideals on the Polish space X would include all countable subsets of X. So if one demands that C be a non-small set according to a σ-ideal I on X, then the most egregious trivialities vanish. Subsets C of X with $C \notin I$ are called I^+ sets. In the question, a reasonable largeness requirement on C should be that it is I^+ and $\mathbf{\Delta}^1_1$.

Without I having some useful properties, there seems to be no particular reason to expect any interesting answer. Some conditions should be imposed on I: Given a σ-ideal I on a Polish space X, there is a natural forcing $\mathbb{P}_I$ associated with I that has been used extensively in descriptive set theory and cardinal characteristics of the continuum. $\mathbb{P}_I$ consists of all I^+ $\mathbf{\Delta}^1_1$ subsets of X ordered by $\subseteq$. Motivated by works in cardinal characteristics, one could require I to have the property that $\mathbb{P}_I$ is a proper forcing.

In cardinal characteristics, properness is used for preservation of certain properties under countable support iterations. This will not be how properness is used in this paper. Rather, properness will be used to produce I^+ $\mathbf{\Delta}^1_1$ subsets for which forcing and absoluteness can be used to derive meaningful information. The main tool that makes this approach possible is the following result:

Fact 4.2.4. (Zapletal, [34] Proposition 2.2.2.) *Let I be a σ-ideal on a Polish space X. The following are equivalent:*

(i) $\mathbb{P}_I$ is a proper forcing.

(ii) For any sufficiently large cardinal Θ, for every $B \in \mathbb{P}_I$, and for every countable $M \prec H_\Theta$ with $\mathbb{P}_I \in M$ and $B \in M$, the set $C := \{x \in B : x \text{ is } \mathbb{P}_I\text{-generic over } M\}$ is I^+ $\mathbf{\Delta}^1_1$.

With this result, the question is now asked with respect to a σ-ideal such that $\mathbb{P}_I$ is proper. Beyond properness of $\mathbb{P}_I$, the ideal does not necessarily have any definability restrictions. The desired set C of the question should be I^+ and $\mathbf{\Delta}^1_1$ but not necessarily E-invariant.

A natural place to begin exploring this question is with the simplest class of definable equivalence relations just beyond $\mathbf{\Delta}^1_1$ equivalence relations: If I is a σ-ideal such that $\mathbb{P}_I$ is proper and E is an Σ^1_1 equivalence relation, is there an I^+ $\mathbf{\Delta}^1_1$ set C such that $E \restriction C$ is $\mathbf{\Delta}^1_1$?

Unfortunately, the answer is no.

Fact 3.1.1. ([19]) *There exists an Σ^1_1 equivalence relation E and a σ-ideal I with $\mathbb{P}_I$ proper such that for all I^+ $\mathbf{\Delta}^1_1$ set C, $E \restriction C$ is not $\mathbf{\Delta}^1_1$.*

Proof. (See [19], Example 4.25.) Let K be a Σ^1_1 but not $\mathbf{\Delta}^1_1$ ideal on ω. Define E_K on ${}^\omega({}^\omega 2)$ by $x \, E_K \, y$ if and only if $\{n \in \omega : x(n) \neq y(n)\} \in K$. E_K is a Σ^1_1 but not $\mathbf{\Delta}^1_1$ equivalence relation. (Note that it has non-$\mathbf{\Delta}^1_1$ classes.)

Let $\mathbb{S}^\omega$ denote countable product of Sacks forcing, i.e. $p \in \mathbb{S}^\omega$ if and only if p is a function on ω so that for each n, $p(n)$ is a perfect tree. If $p \in \mathbb{S}^\omega$, let $[p] = \{x \in {}^\omega({}^\omega 2) : (\forall n)(x(n) \in [p(n)])\}$. Let I be the σ-ideal generated by $\mathbf{\Delta}^1_1$ sets that do not contain $[p]$ for any $p \in \mathbb{S}^\omega$. [19] Fact 9.25 (iii) shows that any I^+ $\mathbf{\Delta}^1_1$ set contains $[p]$ for some $p \in \mathbb{S}^\omega$. This can be used to show that if B is I^+ $\mathbf{\Delta}^1_1$, then $E_K \leq_{\mathbf{\Delta}^1_1} E_K \restriction B$. Hence for every I^+ $\mathbf{\Delta}^1_1$ B, $E_K \restriction B$ is not a $\mathbf{\Delta}^1_1$ equivalence relation. □

This suggests that in order to possibly obtain a positive answer, the equivalence relation considered should more closely resemble $\mathbf{\Delta}^1_1$ equivalence relations. An obvious feature of $\mathbf{\Delta}^1_1$ equivalence relations is that all their equivalence classes are $\mathbf{\Delta}^1_1$. Kanovei, Sabok, and Zapletal then asked the following question of Σ^1_1 equivalence relations which share this feature:

Question 3.1.2. ([19] Question 4.28) If I is a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper and E is a Σ^1_1 equivalence relation with all classes $\mathbf{\Delta}^1_1$, then is there an I^+ $\mathbf{\Delta}^1_1$ set C such that $E \restriction C$ is $\mathbf{\Delta}^1_1$?

Similarly, the question can be asked for the dual class of equivalence relations on the same projective level:

Question 3.1.3. If I is a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper and E is a $\mathbf{\Pi}^1_1$ equivalence relation with all classes $\mathbf{\Delta}^1_1$, then is there an I^+ $\mathbf{\Delta}^1_1$ set C such that $E \restriction C$ is $\mathbf{\Delta}^1_1$?

Again having all $\mathbf{\Delta}^1_1$ classes is necessary for a positive answer by considering the example from the proof of Fact 3.1.1 using a $\mathbf{\Pi}^1_1$ ideal on ω.

With these restrictions, the initial naive question becomes a rather robust question. Throughout the paper, the term "main question" will refer to questions of the former type for various classes of definable equivalence relations on Polish spaces. For concreteness, the reader should perhaps keep in mind the following explicit instance of the main question: If E is a Σ^1_1 equivalence relation with all classes $\mathbf{\Delta}^1_1$, is there a nonmeager or positive measure $\mathbf{\Delta}^1_1$ set C such that $E \restriction C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

Section 3.2 will provide the basic concepts from idealized forcing including the main tool about proper idealized forcings used throughout the paper. Some useful notations for expressing the main question are also introduced. The main question in a slightly stronger form is formalized.

Section 3.3 will provide known results and examples to show that the main question has a positive answer for the most natural Σ^1_1 equivalence relations with all $\mathbf{\Delta}^1_1$ classes. In particular, [19] showed that Σ^1_1 equivalence relations with all classes countable and equivalence relations $\mathbf{\Delta}^1_1$ reducible to orbit equivalence relations of Polish group actions have a positive answer to the main question.

Section 3.4 will show that a positive answer to the main question for Σ^1_1 equivalence relations with all $\mathbf{\Delta}^1_1$ classes follows from some large cardinal assumptions. In particular, it can be proved from iterability principles (such as the existence of a measurable cardinal):

Theorem 3.4.22. *Let I be a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper. Let E be a Σ^1_1 equivalence relation on X with all classes $\mathbf{\Delta}^1_1$. If for all $z \in {}^\omega\omega$, $z^\sharp$ exists and $(\chi^I_E)^\sharp$ exists, then there is an I^+ $\mathbf{\Delta}^1_1$ set C such that $E \restriction C$ is $\mathbf{\Delta}^1_1$.*

Here χ^I_E is a set depending on I and E. This set χ^E_I is in $H_{(2^{\aleph_0})^+}$ so it is a fairly small set. More explicitly, χ^I_E is a triple $\langle \mathbb{P}_I, \mu^I_E, \sigma^I_E \rangle$, where $\mu^I_E, \sigma^I_E \in V^{\mathbb{P}_I}$ are names that witness two existential formulas. In fact, these two names can be chosen a bit more constructively using the fullness or maximality property of forcing. In particular, there is a positive answer to the main question for Σ^1_1 equivalence relations with all $\mathbf{\Delta}^1_1$ classes if there exists a Ramsey cardinal.

After showing the positive answer follows from certain large cardinal principles, a natural question would be whether the negative answer to the main question holds in L or forcing extensions of L, only assuming mild large cardinal assumptions if necessary. The next sections give partial results for a positive answer using different and weaker consistency assumptions for a restricted class of equivalence relations or ideals. Although these results are inherently interesting, these sections should be understood as an attempt to find situations that can not be used to produce a counterexample to a positive answer to the main question. These results seem to enforce the intuition that a universe with very weak large cardinals may be the ideal place to search for such a counterexample.

In Section 3.5, it will be shown that $I_{\text{countable}}$, the ideal of countable sets, and I_{E_0}, the σ-ideal generated by $\mathbf{\Delta}^1_1$ sets on which E_0 is smooth, will always give a positive answer to the main question for Σ^1_1 equivalence relation with all $\mathbf{\Delta}^1_1$ classes. The associated forcings for these two σ-ideals are Sacks forcing and Prikry-Silver forcing, respectively. The meager ideal, I_{meager}, and the Lebesgue null ideal, I_{null}, have, as their associated forcing, the Cohen forcing and random real forcing, respectively. Under $\mathsf{MA} + \neg\mathsf{CH}$, the main question has a positive answer for the meager and null ideal:

Theorem 3.5.15. ($\mathsf{ZFC} + \mathsf{MA} + \neg\mathsf{CH}$) *Let I be either I_{null} or I_{meager}. Let E be a Σ^1_1 equivalence relation with all classes $\mathbf{\Delta}^1_1$. Then there exists an I^+ $\mathbf{\Delta}^1_1$ set C such that*

$E \upharpoonright C$ is $\mathbf{\Delta}_1^1$.

Section 3.6 will consider thin Σ_1^1 equivalence relations, i.e., equivalence relations with no perfect set of inequivalent elements. Burgess showed that such equivalence relations have at most $\aleph_1$ many equivalence classes. This suggests that the main question for thin Σ_1^1 equivalence relations with all $\mathbf{\Delta}_1^1$ classes should be approached combinatorially using covering numbers and the properness of $\mathbb{P}_I$. For example assuming PFA, there is a positive answer for all σ-ideals I with $\mathbb{P}_I$ proper and E a thin Σ_1^1 equivalence relation with all classes $\mathbf{\Delta}_1^1$. However, the combinatorial approach is not the right one. Using definability ideas, the main question for thin Σ_1^1 equivalence relations (even without all $\mathbf{\Delta}_1^1$ classes) has a strong positive answer:

Theorem 3.6.8. (ZFC) *If I is a σ-ideal such that $\mathbb{P}_I$ is proper and E is a thin Σ_1^1 equivalence relation, then there exists a I^+ $\mathbf{\Delta}_1^1$ set C such that C is contained in a single E-class.*

Section 3.7 will show that a positive answer for $\mathbf{\Pi}_1^1$ equivalence relations with all $\mathbf{\Delta}_1^1$ classes follows from sharps in much the same way as in the Σ_1^1 case:

Theorem 3.7.12. *Let I be a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper. Let E be a $\mathbf{\Pi}_1^1$ equivalence relation on X with all classes $\mathbf{\Delta}_1^1$. If for all $z \in {}^\omega\omega$, $z^\sharp$ exists and $(\chi_E^I)^\sharp$ exists, then there is an I^+ $\mathbf{\Delta}_1^1$ set C such that $E \upharpoonright C$ is $\mathbf{\Delta}_1^1$.*

The set χ_E^I is defined similarly to the Σ_1^1 case.

Section 3.8 will consider $\mathbf{\Pi}_1^1$ equivalence relations with all classes countable. As mentioned above, ZFC can provide a positive answer to the main question for Σ_1^1 equivalence relation with all countable classes. In the $\mathbf{\Pi}_1^1$ case, there is insufficient absoluteness to carry out the same proof. However, from the consistency of a remarkable cardinal, one can obtain the consistency of a positive answer to the main question for $\mathbf{\Pi}_1^1$ equivalence relation with all countable classes:

Theorem 3.8.10. *Let κ be a remarkable cardinal in L. Let $G \subseteq Coll(\omega, < \kappa)$ be $Coll(\omega, < \kappa)$ generic over L. In $L[G]$, if I is a σ-ideal with $\mathbb{P}_I$ proper and E is a $\mathbf{\Pi}_1^1$ equivalence relation with all classes countable, then there exists some I^+ $\mathbf{\Delta}_1^1$ set C*

such that $E \restriction C$ is $\mathbf{\Delta}^1_1$.

There is also a similar result using a weakly compact cardinal but $\mathbb{P}_I$ must be a $\aleph_1$-c.c. forcing.

In fact, much more holds in the model from the above theorem. Using some ideas of Neeman and Norwood, there is actually a positive answer to the main question for all equivalence relations in $L(\mathbb{R})$ with all classes $\mathbf{\Delta}^1_1$ in the above model. (See [6].)

Section 3.9 will show that in L, the main question for $\mathbf{\Delta}^1_2$ equivalence relations with all classes $\mathbf{\Delta}^1_1$ (in fact countable) is false.

Theorem 3.9.9. *In L, there is a $\mathbf{\Delta}^1_2$ equivalence relation with all classes countable such that for all σ-ideals I and all I^+ $\mathbf{\Delta}^1_1$ sets C, $E \restriction C$ is not $\mathbf{\Delta}^1_1$.*

[8] has shown that a positive answer to the main question for projective equivalence relations with all classes $\mathbf{\Delta}^1_1$ holds under strong large cardinal assumptions.

Finally, the last section will summarize the work of the paper from the point of view of showing the consistency of a negative answer to the main question. Related questions will be introduced. Some dubious speculations about how a negative answer could be obtained will be discussed.

Drucker, in [11], has independently obtained some results that are very similar to those which appear in this paper: He has shown that a positive answer to the main question follows from a measurable cardinal using similar ideas to those appearing in Section 3.4. He has obtained results for σ-ideals whose forcings are provably $\aleph_1$-c.c. which are similar to Section 3.5. Drucker also proved the results of Section 3.9 of this paper using a very similar equivalence relation. In [11], Drucker also considers more general forms of canonization than that which appears in this paper.

Since this paper, more progress has been made concerning positive answers to the main question for larger classes of equivalence relations. [8] used arguments involving homogeneous trees and games to establish a positive answer to the main question for projective equivalence relations (and beyond) with all classes $\mathbf{\Delta}^1_1$ under strong large cardinal and determinacy assumptions. [6] shows that an upper on the consistency strength of a positive answer to the main question for $L(\mathbb{R})$ equivalence relations with all classes $\mathbf{\Delta}^1_1$ is the existence of a remarkable cardinal. [27] has shown that a positive answer holds for more general forms of the main question for more abstract pointclasses under AD^+.

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3.2 Basic Concepts

This section reviews the basics of idealized forcing and formalizes the main question of interest.

Definition 3.2.1. Let I be a σ-ideal on a Polish space X. Let $\mathbb{P}_I$ be the collection of all I^+ $\mathbf{\Delta}^1_1$ subsets of X. Let $\leq_{\mathbb{P}_I} = \subseteq$. Let $1_{\mathbb{P}_I} = X$. $(\mathbb{P}_I, \leq_{\mathbb{P}_I}, 1_{\mathbb{P}_I})$ is the forcing associated with the ideal I.

Fact 3.2.2. *Let I be a σ-ideal on a Polish space X. There is a $\mathbb{P}_I$-name $\dot{x}_{\text{gen}}$ such that for all $\mathbb{P}_I$-generic filters G over V and all B which are $\mathbf{\Delta}^1_1$ coded in V, $B \in G$ if and only if $\dot{x}_{\text{gen}}[G] \in B$.*

Proof. See [34], Proposition 2.1.2. $\square$

Definition 3.2.3. Let I be a σ-ideal on a Polish space X. Let $M \prec H_\Theta$ be a countable elementary substructure for some cardinal Θ. $x \in X$ is $\mathbb{P}_I$-generic over M if and only if the set $\{A \in \mathbb{P}_I \cap M : x \in A\}$ is a $\mathbb{P}_I$-generic filter over M.

Fact 3.2.4. *Let I be a σ-ideal on a Polish space X. The following are equivalent:*

(i) $\mathbb{P}_I$ is a proper forcing.

(ii) For any sufficiently large cardinal Θ, for every $B \in \mathbb{P}_I$, and for every countable $M \prec H_\Theta$ with $\mathbb{P}_I \in M$ and $B \in M$, the set $C := \{x \in B : x \text{ is } \mathbb{P}_I\text{-generic over } M\}$ is I^+ $\mathbf{\Delta}^1_1$.

Proof. See [34], Proposition 2.2.2. Since this is the most important tool in this paper, a proof will be sketched:

(i) $\Rightarrow$ (ii) Let $B \in \mathbb{P}_I \cap M$ be arbitrary. It is straightforward to show that C is $\mathbf{\Delta}^1_1$. Suppose $C \in I$. Then by Fact 3.2.2, $B \Vdash^V_{\mathbb{P}_I} \dot{x}_{\text{gen}} \notin C$. This implies that there is some $D \subseteq \mathbb{P}_I$ which is dense, $D \in M$, and $B \Vdash^V_{\mathbb{P}_I} \check{M} \cap \check{D} \cap \dot{G} = \emptyset$. Therefore, there can be no $(M, \mathbb{P}_I)$-generic condition below B. $\mathbb{P}_I$ is not proper.

(ii) $\Rightarrow$ (i) Let $B \in \mathbb{P}_I \cap B$ be arbitrary. Suppose $C \notin I$. Then $C \Vdash^V_{\mathbb{P}_I} \dot{x}_{\text{gen}} \in C$. So for all $D \subseteq \mathbb{P}_I$ with D dense and $D \in M$, $C \Vdash^V_{\mathbb{P}_I} \check{M} \cap \check{D} \cap \dot{G} \neq \emptyset$. C is an $(M, \mathbb{P}_I)$-generic condition below B. $\mathbb{P}_I$ is proper. $\square$

The following is some convenient notation:

Definition 3.2.5. ([19] Definition 1.15) Let Λ and Γ be classes of equivalence relations defined on $\mathbf{\Delta}_1^1$ subsets of Polish spaces. Let I be a σ-ideal on a Polish space X. Define $\Lambda \to_I \Gamma$ to mean: for all B which are I^+ $\mathbf{\Delta}_1^1$ subsets of X and every equivalence relation E defined on X such that $E \restriction B \in \Lambda$, there exists an I^+ $\mathbf{\Delta}_1^1$ set $C \subseteq B$ such that $E \restriction C \in \Gamma$.

The following are some of the classes of equivalence relations that will appear later.

Definition 3.2.6. For any Polish space X, ev denotes the full equivalence relation on X consisting of a single class.

For any Polish space X, id is the equality equivalence relation.

$\mathbf{\Delta}_1^1$ denotes the class of all $\mathbf{\Delta}_1^1$ equivalence relations defined on $\mathbf{\Delta}_1^1$ subsets of Polish spaces. (In context, it should be clear when $\mathbf{\Delta}_1^1$ refers to the class of equivalence relations or just the $\mathbf{\Delta}_1^1$ definable subsets.)

${\Sigma_1^1}^{\mathbf{\Delta}_1^1}$ is the class of all Σ_1^1 equivalence relations defined on $\mathbf{\Delta}_1^1$ subsets of Polish spaces with all classes $\mathbf{\Delta}_1^1$.

${\mathbf{\Pi}_1^1}^{\mathbf{\Delta}_1^1}$ is the class of all $\mathbf{\Pi}_1^1$ equivalence relations defined on $\mathbf{\Delta}_1^1$ subsets of Polish spaces with all classes $\mathbf{\Delta}_1^1$.

${\mathbf{\Delta}_2^1}^{\mathbf{\Delta}_1^1}$ is the class of all $\mathbf{\Delta}_2^1$ equivalence relations defined on $\mathbf{\Delta}_1^1$ subsets of Polish spaces with all classes $\mathbf{\Delta}_1^1$.

A thin equivalence relation is an equivalence relation with no perfect set of inequivalent elements.

${\Sigma_1^1}^{\text{thin}}$ is the class of all thin Σ_1^1 equivalence relations defined on $\mathbf{\Delta}_1^1$ subsets of Polish spaces.

${\Sigma_1^1}^{\text{thin}\ \mathbf{\Delta}_1^1}$ is the class of all thin Σ_1^1 equivalence relations defined on $\mathbf{\Delta}_1^1$ subsets of Polish spaces and have all classes $\mathbf{\Delta}_1^1$.

${\mathbf{\Pi}_1^1}^{\aleph_0}$ denotes the class of all $\mathbf{\Pi}_1^1$ equivalence relations with all classes countable and defined on $\mathbf{\Delta}_1^1$ subsets of Polish spaces.

A thin set is a set without a perfect subset.

Let ${\mathbf{\Pi}_1^1}^{\text{thin}}$ denote the class of all $\mathbf{\Pi}_1^1$ equivalence relations with all classes thin and defined on $\mathbf{\Delta}_1^1$ subsets of Polish spaces.

Kanovei, Sabok, and Zapletal asked the following question:

Question 3.2.7. ([19] Question 4.28) If I is a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper, then does $\Sigma_1^{1\boldsymbol{\Delta}_1^1} \to_I \boldsymbol{\Delta}_1^1$ hold?

This paper will address this question and its various related forms for other classes of definable equivalence relations.

3.3 Examples

This section gives known results concerning the main question and some examples.

Proposition 3.3.1. *Let Γ_1 denote the class of equivalence relations $\boldsymbol{\Delta}_1^1$ reducible to orbit equivalence relations of Polish group actions. Then $\Gamma_1 \to_I \boldsymbol{\Delta}_1^1$ for any σ-ideal I on X such that $\mathbb{P}_I$ is proper.*

Proof. See [19], Theorem 4.26. $\square$

The main question also holds for Σ_1^1 equivalence relations with all classes countable. The following is an example of a Σ_1^1 but not $\boldsymbol{\Delta}_1^1$ equivalence relation with classes of size at most two: Let $A \subseteq {}^\omega 2$ be any Σ_1^1 but not $\boldsymbol{\Delta}_1^1$ set. Define the equivalence relation E on ${}^\omega 2 \times 2$ by

$$(x, i)\ E\ (y, j) \Leftrightarrow (x = y \wedge i = j) \vee (x = y \wedge x \in A).$$

E is Σ_1^1. For all $z \in {}^\omega 2$, $z \in A$ if and only if $(z, 0)\ E\ (z, 1)$. E can not be $\boldsymbol{\Delta}_1^1$.

Proposition 3.3.2. *Let Γ_2 denote the class of Σ_1^1 equivalence relations with all classes countable. Then $\Gamma_2 \to_I \boldsymbol{\Delta}_1^1$ for any σ-ideal I on X such that $\mathbb{P}_I$ is proper.*

Proof. See [19], Theorem 4.27. The proof is provided below to emphasize a particular observation.

Fix a $B \subseteq X$ which is I^+ $\boldsymbol{\Delta}_1^1$. As E is Σ_1^1, there exists some $z \in {}^\omega 2$ such that E is $\Sigma_1^1(z)$. For each $x \in X$, $[x]_E$ is $\Sigma_1^1(x, z)$. Since every $\Sigma_1^1(x, z)$ set with a non-$\Delta_1^1(x, z)$ element has a perfect subset (see [24] Theorem 6.3), the statement "all E-classes are countable" is equivalent to

$$(\forall x)(\forall y)(y\ E\ x \Rightarrow y \in \Delta_1^1(x, z)).$$

As the relation "$y \in \Delta_1^1(x, z)$" in variables x and y is $\Pi_1^1(z)$, the above is $\Pi_1^1(z)$. By Mostowski absoluteness, $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I}$ "all E-classes are countable". There is some

$\mathbb{P}_I$-name τ such that $B \Vdash_{\mathbb{P}_I} \tau \in {}^\omega X \wedge \tau$ enumerates $[\dot{x}_{\text{gen}}]_E$. By [34] Proposition 2.3.1, there exists some $B' \subseteq B$ with $B' \in \mathbb{P}_I$ and a $\mathbf{\Delta}^1_1$ function f such that $B' \Vdash_{\mathbb{P}_I} f(\dot{x}_{\text{gen}}) = \tau$. Choose $M \prec H_\Theta$ with Θ sufficiently large and M contains $\mathbb{P}_I$, B', τ, and the code for f. By Fact 4.2.4, let $C \subseteq B'$ be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic over M elements in B'.

The claim is that for $x, y \in C$, $x \; E \; y$ if and only if $(\exists n)(f(n) = y)$. This is because, for all $x \in C$, $M[x] \models f(x)$ enumerates $[x]_E$. Let N be the Mostowski collapse of $M[x]$. One can always assume the transitive closure of elements of X is a subset of M (for instance, one could have identified X with ${}^\omega\omega$). Therefore the Mostowski collapse map does not move elements of X. Hence $N \models f(x)$ enumerates $[x]_E$. The statement "$f(x)$ enumerates $[x]_E$" is the conjunction of a Σ^1_1 and $\mathbf{\Pi}^1_1$ formula coded in N. By Mostowski absoluteness, $f(x)$ enumerates $[x]_E$ in V. This proves the claim. Thus $E \restriction C$ is a $\mathbf{\Delta}^1_1$ equivalence relation. $\square$

By the work above, for each x which is $\mathbb{P}_I$-generic over M, $M[x] \models [x]_E$ is countable. So in $M[x]$, there exists some real u_x such that u_x codes an enumeration of $[x]_E$. In $M[x]$, $[x]_E$ is $\Delta^1_1(u_x)$. In the above proof, one showed that u_x remains an enumeration of $[x]_E$ even in V. So $[x]_E$ is $\Delta^1_1(u_x)$ even in V. This observation is the quintessential idea of the proof of the positive answer for the main question assuming large cardinal properties. Note that in the above proof, there was a $\mathbf{\Delta}^1_1$ function f which uniformly provided the enumeration of $[x]_E$ for each $x \in C$. This feature is not necessary.

Below, positive answers to the main question will be demonstrated for some specific equivalence relations.

Definition 3.3.3. For each $x \in {}^\omega 2$, ω_1^x is the least x-admissible ordinal above ω. Define the equivalence relation F_{ω_1} on ${}^\omega 2$ by $x \; F_{\omega_1} \; y$ if and only if $\omega_1^x = \omega_1^y$.

F_{ω_1} is an Σ^1_1 equivalence relation with all classes $\mathbf{\Delta}^1_1$.

Example 3.3.4. Let I be a σ-ideal on ${}^\omega 2$ with $\mathbb{P}_I$ proper. Then $\{F_{\omega_1}\} \to_I \{\text{ev}\}$, i.e. there is an I^+ class.

Proof. Let B be an arbitrary I^+ $\mathbf{\Delta}^1_1$ set. Choose $M \prec H_\Theta$ where Θ is a sufficiently large cardinal and $\mathbb{P}_I, B \in M$. By Fact 4.2.4, let $C \subseteq B$ be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic over M reals in B. For each $x \in C$, $\omega_1^x \in M[x] \cap \text{ON}$. Since the ground model and the forcing extension have the same ordinals (by properness), $\omega_1^x \in M \cap \text{ON}$. For each $\alpha \in \text{ON}$, let $F_{\omega_1}^\alpha = \{x \in {}^\omega 2 : \omega_1^x = \alpha\}$. Each $F_{\omega_1}^\alpha = \emptyset$ or is an F_{ω_1}-class.

$C = \bigcup_{\alpha \in M \cap \mathrm{ON}} F_{\omega_1}^{\alpha} \cap C$. Since F_{ω_1}-classes are $\mathbf{\Delta}_1^1$, $F_{\omega_1}^{\alpha} \cap C$ is $\mathbf{\Delta}_1^1$ for all α. As M is countable, $M \cap \mathrm{ON}$ is countable. There exists some $\alpha \in M \cap \mathrm{ON}$ such that $F_{\omega_1}^{\alpha} \cap C$ is I^+ since I is a σ-ideal. For this α, $F_{\omega_1} \upharpoonright F_{\omega_1}^{\alpha} \cap C = \mathrm{ev} \upharpoonright F_{\omega_1}^{\alpha} \cap C$. □

Since $M \prec H_\Theta$, for each $x \in C$, there exists a countable admissible ordinal $\alpha > \omega_1^x$ with $\alpha \in M$. By Sacks theorem applied in M, let $y \in M$ be such that $\omega_1^y = \alpha$. Then $[x]_{F_{\omega_1}}$ is $\Delta_1^1(y)$. Again the phenomenon described above occurs: there exist some $y \in M[x]$ (in fact $y \in M$) such that $M[x] \models [x]_{F_{\omega_1}}$ is $\Delta_1^1(y)$ and $V \models [x]_{F_{\omega_1}}$ is $\Delta_1^1(y)$.

Actually, F_{ω_1} is classifiable by countable structures. Proposition 3.3.1 would have already shown $\{F_{\omega_1}\} \to_I \mathbf{\Delta}_1^1$. See [7] for more information about F_{ω_1}.

Definition 3.3.5. Define the equivalence relation E_{ω_1} on ${}^\omega 2$ by

$$x \; E_{\omega_1} \; y \Leftrightarrow (x \notin \mathrm{WO} \wedge y \notin \mathrm{WO}) \vee (\mathrm{ot}(x) = \mathrm{ot}(y))$$

where WO is the set of reals coding well-orderings and, for $x \in \mathrm{WO}$, $\mathrm{ot}(x)$ is the order type of the linear order coded by x.

E_{ω_1} is a Σ_1^1 equivalence relation with all classes $\mathbf{\Delta}_1^1$ except for one Σ_1^1 class consisting of the reals that do not code wellfounded linear orderings.

Example 3.3.6. Let I be a σ-ideal on ${}^\omega 2$ with $\mathbb{P}_I$ proper. Then $\{E_{\omega_1}\} \to_I \{\mathrm{ev}\}$.

Proof. Let $B \subseteq {}^\omega 2$ be I^+ $\mathbf{\Delta}_1^1$.

(Case I) There exists some $B' \leq_{\mathbb{P}_I} B$ such that $B' \Vdash_{\mathbb{P}_I} \dot{x}_{\mathrm{gen}} \notin \mathrm{WO}$: Let $M \prec H_\Theta$ be a countable elementary structure with Θ a sufficiently large cardinal and $\mathbb{P}_I, B' \in M$. By Fact 4.2.4, let $C \subseteq B'$ be the I^+ $\mathbf{\Delta}_1^1$ set of all $\mathbb{P}_I$-generic over M reals in B'. Let $x \in C$. By Fact 3.2.2, let $G_x \subseteq \mathbb{P}_I$ be the generic filter associated with x. $B' \in G_x$ since $x \in B'$. $B' \Vdash_{\mathbb{P}_I} \dot{x}_{\mathrm{gen}} \notin \mathrm{WO}$ implies that $M[x] \models x \notin \mathrm{WO}$. Let N be the Mostowski collapse of $M[x]$. Since the Mostowski collapse map does not move reals, $x \in N$. Also $N \models x \notin \mathrm{WO}$. Since WO is Π_1^1, $V \models x \notin \mathrm{WO}$. Hence $E \upharpoonright C$ consists of a single class. So $E \upharpoonright C = \mathrm{ev} \upharpoonright C$.

(Case II) $B \Vdash_{\mathbb{P}_I} \dot{x}_{\mathrm{gen}} \in \mathrm{WO}$: Then choose $M \prec H_\Theta$, a countable elementary substructure, and Θ a sufficiently large cardinal. By Fact 4.2.4, let $C \subseteq B$ be the I^+ $\mathbf{\Delta}_1^1$ set of $\mathbb{P}_I$-generic over M reals in B. As in Case I, $B \Vdash_{\mathbb{P}_I} \dot{x}_{\mathrm{gen}} \in \mathrm{WO}$ implies that $V \models x \in \mathrm{WO}$. So when $x \in C$, $\mathrm{ot}(x) \in M[x] \cap \mathrm{ON}$. For each ordinal $\alpha < \omega_1$, $E_{\omega_1}^{\alpha} := \{x \in \mathrm{WO} : \mathrm{ot}(x) = \alpha\}$ is a $\mathbf{\Delta}_1^1$ set. Since $M[x]$ and M have the same ordinals and M is countable, $M[x]$ has only countably many ordinals. $C = \bigcup_{\alpha \in M \cap \mathrm{ON}} E_{\omega_1}^{\alpha} \cap C$.

$E^\alpha_{\omega_1} \cap C$ is $\mathbf{\Delta}^1_1$ for each α. Since I is a σ-ideal, there is some $\alpha \in M \cap \mathrm{ON}$ such that $E^\alpha_{\omega_1} \cap C$ is I^+. So for this α, $E_{\omega_1} \upharpoonright E^\alpha_{\omega_1} \cap C = \mathrm{ev} \upharpoonright E^\alpha_{\omega_1} \cap C$. □

Note that E_{ω_1} does not have all classes $\mathbf{\Delta}^1_1$. However, it is a thin Σ^1_1 equivalence relation. It will be shown later that the main question can be answered positively for thin Σ^1_1 equivalence relation regardless of whether the classes are all $\mathbf{\Delta}^1_1$.

Next, there is one further enlightening example which does not fall under the scope of Proposition 3.3.1 or Proposition 3.3.2.

Fact 3.3.7. *There exist a Π^1_1 set $D \subseteq {}^\omega\omega$, a Π^1_1 set $P \subseteq ({}^\omega\omega)^3$, and a Σ^1_1 set $S \subseteq ({}^\omega\omega)^3$ such that:*

(1) If $z \in D$, then for all $x, y \in {}^\omega\omega$, $P(z, x, y) \Leftrightarrow S(z, x, y)$.

(2) For all $z \in D$, the relation $x \; E_z \; y$ if and only $P(z, x, y)$ is an equivalence relation, which is $\mathbf{\Delta}^1_1$ by (1).

(3) If E is a $\mathbf{\Delta}^1_1$ equivalence relation, then there is a z such that $x \; E \; y \Leftrightarrow P(z, x, y)$.

Proof. See [9], Definition 14. □

Definition 3.3.8. ([9] Definition 29) Define the equivalence relation $E_{\mathbf{\Delta}^1_1}$ on $({}^\omega\omega)^2$ by

$$(z_1, x_1) \; E_{\mathbf{\Delta}^1_1} \; (z_2, x_2) \Leftrightarrow (z_1 = z_2) \wedge (\neg D(z_1) \vee S(z_1, x_1, x_2)).$$

$E_{\mathbf{\Delta}^1_1}$ is a Σ^1_1 equivalence relation. For each z, define E_z by $x \; E_z \; y$ if and only if $(z, x) \; E_{\mathbf{\Delta}^1_1} \; (z, y)$. For all z, E_z is a $\mathbf{\Delta}^1_1$ equivalence relation. If E is a $\mathbf{\Delta}^1_1$ equivalence relation, then there exists a $z \in D$ such that $E = E_z$. $E_{\mathbf{\Delta}^1_1}$ has all classes $\mathbf{\Delta}^1_1$.

Fact 3.3.9. *If E is a $\mathbf{\Delta}^1_1$ equivalence relation, then $E \leq_{\mathbf{\Delta}^1_1} E_{\mathbf{\Delta}^1_1}$.*

Proof. (See [9]) Let $z \in D$ such that $x \; E \; y \Leftrightarrow S(z, x, y)$ for all $x, y \in {}^\omega\omega$. Then $f : {}^\omega\omega \to ({}^\omega\omega)^2$ defined by $f(x) = (z, x)$ is the desired reduction. □

Proposition 3.3.10. *$E_{\mathbf{\Delta}^1_1}$ is a Σ^1_1 equivalence relation with all classes $\mathbf{\Delta}^1_1$, has uncountable classes, and is not reducible to the orbit equivalence relation of a Polish group action.*

Proof. All except the last statement have been mentioned above. Let E_1 be the equivalence relation on ${}^\omega({}^\omega 2)$ defined by $x \; E_1 \; y$ and and only if $(\exists m)(\forall n \geq m)(x(n) =$

$y(n))$. E_1 is a Δ^1_1 equivalence relation so by Fact 3.3.9, $E_1 \leq_{\mathbf{\Delta}^1_1} E_{\mathbf{\Delta}^1_1}$. E_1 is not $\mathbf{\Delta}^1_1$ reducible to any orbit equivalence relation of a Polish group action, by [22] Theorem 4.2. $\square$

$E_{\mathbf{\Delta}^1_1}$ is a Σ^1_1 equivalence relation which does not fall under Proposition 3.3.1 or Proposition 3.3.2. Next, it will be shown that the main question formulated for $E_{\mathbf{\Delta}^1_1}$ has a positive answer.

Theorem 3.3.11. *Let I be a σ-ideal on $({}^\omega\omega)^2$ such that $\mathbb{P}_I$ is a proper forcing, then $\{E_{\mathbf{\Delta}^1_1}\} \to_I \mathbf{\Delta}^1_1$.*

Proof. Since D is Π^1_1, let T be a recursive tree on $\omega \times \omega$ such that $x \in D$ if and only if T^x is well-founded. Define

$$D_\alpha := \{x : \mathrm{rk}(T^x) < \alpha\}.$$

Each D_α is $\mathbf{\Delta}^1_1$. Define

$$(z_1, x_1)\ E^\alpha_{\mathbf{\Delta}^1_1}(z_2, x_2) \Leftrightarrow z_1 = z_2 \wedge (\neg D_\alpha(z_1) \vee P(z_1, x_1, x_2))$$

$$\Leftrightarrow z_1 = z_2 \wedge (\neg D_\alpha(z_1) \vee (S(z_1, x_1, x_2)))$$

Each $E^\alpha_{\mathbf{\Delta}^1_1}$ is $\mathbf{\Delta}^1_1$ and $E_{\mathbf{\Delta}^1_1} = \bigcap_{\alpha<\omega_1} E^\alpha_{\mathbf{\Delta}^1_1}$.

Let $B \subseteq ({}^\omega\omega)^2$ be a $\mathbf{\Delta}^1_1$ I^+ set. Let $\pi_1 : ({}^\omega\omega)^2 \to {}^\omega\omega$ be the projection onto the first coordinate.

(Case I) $B \not\Vdash_{\mathbb{P}_I} \pi_1(\dot{x}_{\mathrm{gen}}) \in D$: Then there exists some $B' \leq_{\mathbb{P}_I} B$ such that $B' \Vdash_{\mathbb{P}_I} \pi_1(\dot{x}_{\mathrm{gen}}) \notin D$. Now let $M \prec H_\Theta$ be a countable elementary substructure with $B', \mathbb{P}_I \in M$ and Θ some sufficiently large cardinal. By Fact 4.2.4, let $C \subseteq B'$ be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic over M elements in B'. By elementarity, $B' \Vdash^M_{\mathbb{P}_I} \pi_1(\dot{x}_{\mathrm{gen}}) \notin D$. So for all $x \in C$, $M[x] \models \pi_1(x) \notin D$. For each $x \in C$, let N_x denote the Mostowski collapse of $M[x]$. Note that the Mostowski collapse map does not move reals. Hence $N_x \models \pi_1(x) \notin D$. By Mostowski absoluteness, $\pi_1(x) \notin D$. So for all $(z_1, x_1), (z_2, x_2) \in C$, $(z_1, x_2)\ E_{\mathbf{\Delta}^1_1}(z_2, x_2) \Leftrightarrow z_1 = z_2$. So $E_{\mathbf{\Delta}^1_1} \restriction C$ is $\mathbf{\Delta}^1_1$.

(Case II) Otherwise $B \Vdash_{\mathbb{P}_I} \pi_1(\dot{x}_{\mathrm{gen}}) \in D$: Let $M \prec H_\Theta$ be a countable elementary substructure with $B, \mathbb{P}_I \in M$ and Θ some sufficiently large cardinal. By Fact 4.2.4, let $C \subseteq B$ be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic over M elements in B. By elementarity, $B \Vdash^M_{\mathbb{P}_I} \pi_1(\dot{x}_{\mathrm{gen}}) \in D$. For all $x \in C$, $M[x] \models \pi_1(x) \in D$. $M[x] \models (\exists\alpha < \omega_1)(\mathrm{rk}(T^{\pi_1(x)}) < \alpha)$. Let $\beta = M \cap \omega_1$. Since $\mathbb{P}_I$ preserves $\aleph_1$, $\omega_1^{M[x]} = \omega_1^M$ for all $x \in C$. For each $x \in C$, let N_x be the Mostowski collapse of $M[x]$. Note that the Mostowski collapse map does not move any reals. Then for all $x \in C$, $N_x \models (\exists \alpha < \omega_1^{N_x} = \beta)(\mathrm{rk}(T^{\pi_1(x)}) < \alpha)$. For each $x \in C$, there is some $\alpha < (\omega_1)^{N_x}$ such that $N_x \models \mathrm{rk}(T^{\pi_1(x)}) < \alpha$. After expressing this statement using a real in N_x that codes the countable (in N_x) ordinal α, Mostowski absoluteness implies that $\mathrm{rk}(T^{\pi_1(x)}) < \alpha$. It has been shown that for all $x \in C$, $\mathrm{rk}(T^{\pi_1(x)}) < \beta$. For all $x \in C$, $\pi_1(x) \in D_\beta$. $E_{\mathbf{\Delta}_1^1} \upharpoonright C = E^\beta_{\mathbf{\Delta}_1^1} \upharpoonright C$. The latter is $\mathbf{\Delta}_1^1$. □

The above proof motivates the ideas used in the next section.

3.4 Positive Answer for $\Sigma_1^{1\ \mathbf{\Delta}_1^1}$

Using some of the ideas from the earlier examples, it will be shown that a positive answer to the main question follows from large cardinals. Avoiding any explicit mention of iteration principles, a crude result for the positive answer is first given assuming some generic absoluteness and the existence of tree representations that behave very nicely with generic extensions. This result will illustrate all the main ideas before going into the more optimal but far more technical proof using interable structures.

For simplicity, assume that E is a Σ_1^1 equivalence relation on ${}^\omega\omega$.

First, a classical result about Σ_1^1 equivalence relations.

Fact 3.4.1. *Let E be a $\Sigma_1^1(z)$ equivalence relation on ${}^\omega\omega$. Then there exists $\mathbf{\Delta}_1^1$ relations E_α, for $\alpha < \omega_1$, with the property that if $\alpha < \beta$, then $E_\alpha \supseteq E_\beta$, $E = \bigcap_{\alpha<\omega_1} E_\alpha$, and there exists a club set $C \subseteq \omega_1$ such that for all $\alpha \in C$, E_α is an equivalence relation.*

Proof. See [3]. Since E is $\Sigma_1^1(z)$, let T be a z-recursive tree on $\omega \times \omega \times \omega$ such that $(x, y) \in E$ if and only if $T^{(x,y)}$ is illfounded. For each $\alpha < \omega_1$, define E_α by $(x, y) \in E_\alpha \Leftrightarrow \mathrm{rk}(T^{(x,y)}) > \alpha$. Observe that E_α is $\Delta_1^1(z, c)$ for any c which codes the ordinal α.

The verification of the rest of the theorem is an application of the boundedness theorem and can be found in any reference on the descriptive set theory of equivalence relations. (See Lemma 3.7.2 for a similar result in the $\mathbf{\Pi}_1^1$ case.) □

For the rest of this section, fix a z-recursive tree T as in the proof above. $\{E_\alpha : \alpha < \omega_1\}$ will refer to the sequence of $\mathbf{\Delta}_1^1$ equivalence relations obtained from T.

Lemma 3.4.2. *Let E be a $\Sigma^1_1(z)$ equivalence relation. Let $x, y \in {}^\omega\omega$ be such that $[x]_E$ is $\Pi^1_1(y)$. Let δ be an ordinal such that $\omega_1^{x\oplus y\oplus z} < \delta$ and E_δ is an equivalence relation. Then $[x]_E = [x]_{E_\delta}$.*

Proof. Define $E' \subseteq ({}^\omega\omega)^2$ by

$$a\ E'\ b \Leftrightarrow (a \in [x]_E \wedge b \in [x]_E) \vee (a \notin [x]_E \wedge b \notin [x]_E).$$

E' is $\Pi^1_1(x \oplus y \oplus z)$. $({}^\omega\omega)^2 - E'$ is then $\Sigma^1_1(x \oplus y \oplus z)$. $({}^\omega\omega)^2 - E' \subseteq ({}^\omega\omega)^2 - E$. By the effective boundedness theorem, there exists an $\alpha < \omega_1^{x\oplus y\oplus z} < \delta$ such that for all $(x, y) \in ({}^\omega\omega)^2 - E'$, $\mathrm{rk}(T^{(x,y)}) \leq \alpha$. Hence $E' \supseteq E_\alpha$.

Since $E \subseteq E_\delta$, $[x]_E \subseteq [x]_{E_\delta}$. Since $E_\delta \subseteq E_\alpha \subseteq E'$, $[x]_{E_\delta} = \{u : (u, x) \in E_\delta\} \subseteq \{u : (u, x) \in E'\} = [x]_{E'} = [x]_E$. Therefore, $[x]_E = [x]_{E_\delta}$. □

Lemma 3.4.2 gives an upper bound on the ordinal level of the sequence $\{E_\alpha : \alpha < \omega_1\}$ where a $\mathbf{\Pi}^1_1$ E-class stabilizes. According to this lemma, a crucial piece of information in finding this bound is a real y which can be used as a parameter in some Π^1_1 definition of the $\mathbf{\Pi}^1_1$ E-class. Rather than knowing a particular Π^1_1 code, it suffices to know where some particular code lives:

Lemma 3.4.3. *Let E be a $\Sigma^1_1(z)$ equivalence relation. Let $\mathbb{P}$ be a forcing in M which adds a generic real. Choose Θ to be a regular cardinal greater than $|\mathbb{P}|^+$. Let $M \prec H_\Theta$ be a countable elementary substructure with $z, \mathbb{P} \in M$ and contains $|\mathbb{P}|^+$. Suppose for all g which are $\mathbb{P}$-generic over M, there exists a $y \in M[g]$ such that $V \models [g]_E$ is $\Pi^1_1(y)$. Then there exists a countable ordinal α such that for all $\mathbb{P}$-generic over M reals g, $[g]_E = [g]_{E_\alpha}$.*

Proof. Let M' be the Mostowski collapse of M. Let α be the image under the Mostowski collapse map of $|\mathbb{P}|^+$. α is an uncountable successor cardinal in $M'[g]$ for all g which are $\mathbb{P}$-generic over M. Let γ be the image of $|\mathbb{P}|$ under this Mostowski collapse. Of course, $\alpha = \gamma^+$.

Now fix such a g. Let h be $\mathrm{Coll}(\omega, \gamma)$-generic over $M[g]$. Note that $\aleph_1^{M[g][h]} = \alpha$. α is a $(g \oplus y \oplus z)$-admissible ordinal greater than $\omega_1^{g\oplus y\oplus z}$ since α is a cardinal of $M'[g][h]$. Note that since $M'[g][h]$ is countable, $\alpha < \omega_1^V$.

E is a $\Sigma^1_1(z)$ equivalence relation in M'. Therefore, the statement "E is an equivalence relation" is $\Pi^1_2(z)$. By Schoenfield absoluteness, this statement is absolute to any forcing extension of M'. So E remains an equivalence relation in $M'[g][h]$. Therefore, Fact 3.4.1 holds in $M'[g][h]$. There exists a club set of $\beta < (\aleph_1)^{M'[g][h]} = \alpha$ so that E_β is an equivalence relation in $M'[g][h]$. As β is countable in $M'[g][h]$, there is some real $c \in M'[g][h]$ which codes β. E_β is $\Delta^1_1(z, c)$. Therefore, the statement "E_β is an equivalence relation" is $\mathbf{\Pi}^1_1$ in $M'[g][h]$. By Mostowski absoluteness between $M'[g][h]$ and V, E_β is an equivalence relation in V. $E_\alpha = \bigcap_{\beta<\alpha} E_\beta$ is an equivalence relation in V since it is an intersection of equivalence relations in V.

Now Lemma 3.4.2 can be applied to show that $[g]_E = [g]_{E_\alpha}$. This α is as required and the proof is complete.

A close inspection of the argument shows that if there was a common ordinal α so that $\alpha = \aleph_1^{M'[g]}$ for all g's which are $\mathbb{P}$-generic over M, then one could do the argument above using α and $M[g]$ without introducing any Lévy collapses. For instance, this would hold if $\mathbb{P}$ was $\aleph_1$-preserving.

The argument above actually shows that any $\alpha \in M'$ which is a successor cardinal in M' greater than the image of $|\mathbb{P}|$ under the Mostowski collapse map would also work. Therefore, if the initial model M was chosen as above with the additional property that M has no largest cardinal, then $\alpha = M' \cap \mathrm{ON}$ would also work. $\square$

Remark 3.4.4. Also there are more careful versions of Lemma 3.4.1 in which all the E_α's are equivalence relations which could be used to avoid this issue entirely. However, the simpler form of Lemma 3.4.1 was used so that it could be more easily applied to the less familiar $\mathbf{\Pi}^1_1$ setting in Lemma 3.7.2.

Now returning to the setting of the main question: Suppose E is a $\Sigma^1_1(z)$ equivalence relation with all classes $\mathbf{\Delta}^1_1$. Let I be a σ-ideal with $\mathbb{P}_I$-proper. According to Lemma 3.4.3, if one could find some $M \prec H_\Theta$ such that whenever x is $\mathbb{P}_I$-generic over M, a Π^1_1 code for $[x]_E$ resides inside $M[x]$, then letting C be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic reals over M, there would exist some $\alpha < \omega_1$ such that $E \restriction C = E_\alpha \restriction C$. Hence $E \restriction C$ is $\mathbf{\Delta}^1_1$.

A plausible candidate for the Π^1_1 code of $[x]_E$ which is an element of $M[x]$ would be some y such that $M[x] \models [x]_E$ is $\Pi^1_1(y)$. However, $M[x]$ may not think $[x]_E$ is $\mathbf{\Delta}^1_1$. The statement "all E-class are $\mathbf{\Delta}^1_1$" is $\Pi^1_4(z)$. If V satisfies $\mathbf{\Pi}^1_4$-generic absoluteness, one can choose $M \prec H_\Theta$ such that some particular $\Pi^1_4(z)$ statement becomes absolute between M and all its generic extensions. So in such a structure M, $M[x]$ will think $[x]_E$ is $\mathbf{\Delta}^1_1$.

Now in $M[x]$, there is some y such $M[x] \models [x]_E$ is $\Pi_1^1(y)$. In general, it is not clear if $[x]_E$ is $\Pi_1^1(y)$ in V. The formula "$[x]_E$ is $\Pi_1^1(y)$" is $\Pi_2^1(z)$. One can not use Schoenfield absoluteness between $M[x]$ (or rather its transitive collapse) and V since it is not the case that $\omega_1^V \subseteq M[x]$ because $M[x]$ is countable in V. So what is needed is some $M \prec H_\Theta$ such that for all $\mathbb{P}_I$-generic over M real x, a certain $\Pi_2^1(z)$ formula is absolute between $M[x]$ and V. The concept of universal Baireness can be used to remedy this issue.

Definition 3.4.5. ([12]) $A \subseteq {}^\omega\omega$. A is universally Baire if and only if there exists $\alpha, \beta \in \mathrm{ON}$ and trees U on $\omega \times \alpha$ and W on $\omega \times \beta$ such that:

(1) $A = p[U]$. ${}^\omega\omega - A = p[W]$.

(2) For all $\mathbb{P}$, $1_\mathbb{P} \Vdash_\mathbb{P} p[\check{U}] \cup p[\check{W}] = {}^\omega\omega$,

where p of a tree denotes the projection of the tree.

Fact 3.4.6. *Suppose A is a Σ_2^1 set defined by a Σ_2^1 formula $\varphi(x)$. Let U and W be trees witnessing that A is universally Baire. Then $1_\mathbb{P} \Vdash_\mathbb{P} (\forall x)(\varphi(x) \Leftrightarrow x \in p[\check{U}])$.*

Proof. See [12], page 221-222. □

Definition 3.4.7. Let E be a $\Sigma_1^1(z)$ equivalence relation. Define the set D by

$$(x, T) \in D \Leftrightarrow (T \text{ is a tree on } \omega \times \omega) \wedge (\forall y)(y\ E\ x \Leftrightarrow T^y \in \mathrm{WF}).$$

D is $\Pi_2^1(z)$.

Finally, the first result showing that a positive answer follows from some strong set theoretic assumptions:

Proposition 3.4.8. *Assume all $\boldsymbol{\Pi}_2^1$ sets are universally Baire and $\boldsymbol{\Pi}_4^1$-generic absoluteness holds. Let I be a σ-ideal such that $\mathbb{P}_I$ is proper. Then $\Sigma_1^{1\,\boldsymbol{\Delta}_1^1} \to_I \boldsymbol{\Delta}_1^1$.*

Proof. Let E be a $\Sigma_1^1(z)$ equivalence relation. Since all $\boldsymbol{\Pi}_2^1$ sets are universally Baire, let U and W be trees on $\omega \times \omega \times \alpha$ and $\omega \times \omega \times \beta$, respectively, where $\alpha, \beta \in \mathrm{ON}$, giving the universally Baire representations for the $\Pi_2^1(z)$ set D from Definition 3.4.7.

Suppose $B \in \mathbb{P}_I$. Using the reflection theorem, choose Θ large enough so that B, $\mathbb{P}_I$, z, U, and W are contained in H_Θ and H_Θ satisfies $\boldsymbol{\Pi}_4^1$-generic absoluteness for the statement "$(\forall x)(\exists T)((x,T) \in D)$". Let $M \prec H_\Theta$ be a countable elementary substructure containing B, $\mathbb{P}_I$, z, U, and W.

By Fact 4.2.4, let C be the I^+ $\mathbf{\Delta}^1_1$ subset of $\mathbb{P}_I$-generic over M reals in B. Let $g \in C$. Since E has all classes $\mathbf{\Delta}^1_1$, M satisfies $(\forall x)(\exists T)((x,T) \in D)$. Because M has generic absoluteness for this formula, $M[g] \models (\forall x)(\exists T)((x,T) \in D)$. There exists some $T \in M[g]$ such that $M[g] \models (g,T) \in D$.

By Fact 3.4.6, $M[g] \models (g,T) \in p[U]$. There exists $\Phi : \omega \to \alpha$ with $\Phi \in M[g]$ such that $(g,T,\Phi) \in [U]$. For each $n \in \omega$, $(g \restriction n, T \restriction n, \Phi \restriction n) \in M$. For each n, $M[g] \models (g \restriction n, T \restriction n, \Phi \restriction n) \in U$. By absoluteness, for each n, $M \models (g \restriction n, T \restriction n, \Phi \restriction n) \in U$. Since $M \prec H_\Theta$, for all n, $(g \restriction n, T \restriction n, \Phi \restriction n) \in U$ in the true universe V. Therefore, in V, $(g,T,\Phi) \in [U]$. $(g,T) \in p[U]$. $(g,T) \in D$. Note that $(g,T) \in D$ implies that $[g]_E$ is $\Pi^1_1(T)$.

It has been shown that for the chosen M, whenever g is $\mathbb{P}_I$-generic over M, there exists some $z \in M[g]$ such that $[g]_E$ is $\Pi^1_1(z)$. By Lemma 3.4.3, there is a countable ordinal α such that for all $\mathbb{P}_I$-generic over M reals g, $[g]_E = [g]_{E_\alpha}$. Hence $E \restriction C = E_\alpha \restriction C$. $E \restriction C$ is $\mathbf{\Delta}^1_1$. □

Remark 3.4.9. By [2], Theorem 8, $\mathbf{\Pi}^1_4$ generic absoluteness is equiconsistent with every set having a sharp and the existence of a reflecting cardinal. The proof of [2], Theorem 8, shows that any structure satisfying Σ^1_4 generic absoluteness is closed under sharps. By [12], Theorem 3.4, all $\mathbf{\Pi}^1_2$ sets are universally Baire is equivalent to the existence of sharps for all sets. Hence, the hypothesis of Proposition 3.4.8 is equiconsistent with all sets having sharps and the existence of a reflecting cardinal.

Observe that $\mathbf{\Pi}^1_4$ generic absoluteness can be avoided for those Σ^1_1 equivalence relation such that the statement "all E-classes are $\mathbf{\Delta}^1_1$" hold in any model of ZFC containing the defining parameters for E.

Proposition 3.4.10. *The consistency of* ZFC*, sharps of all sets exists, and there exists a reflecting cardinal implies the consistency of* $\Sigma_1^{1^{\mathbf{\Delta}^1_1}} \to_I \mathbf{\Delta}^1_1$ *for all* σ*-ideal* I *on a Polish space such that* $\mathbb{P}_I$ *is proper.*

Proof. See Remark 3.4.9. □

Next, a positive answer to the main question will be obtained from assumptions with weaker consistency strength. The result above illustrates all the main ideas but used stronger than necessary assumptions: $\mathbf{\Pi}^1_4$ generic absolutness and all $\mathbf{\Pi}^1_2$ sets are universally Baire. $\mathbf{\Pi}^1_4$ generic absoluteness was used to preserve the statement "all E-classes are $\mathbf{\Delta}^1_1$." Below, it will be shown how sharps can be used to give a $\mathbf{\Pi}^1_3$ statement which is equivalent. Sharps will also be used to make the statement "all E-classes are $\mathbf{\Delta}^1_1$" true in the desired generic extensions, which is more subtle than just applying Martin-Solovay absoluteness. As observed above, sharps play an important role in $\mathbf{\Pi}^1_2$ sets being universally Baire. In the following, a much more careful analysis will be given to determine exactly which sharps are needed.

For the more optimal proof, iterable structures will be the main tools. Familar examples of iterable structures are V itself when V has a measurable cardinal, certain elementary substructures of V_Θ when V contains a measurable cardinal, and mice that come from the existence of sharps. In the first two, the measure exists in the structure, but in the latter, the measure is external. Some references for this material are [28], [1], and any text on inner model theory.

Let X be a set. A simple formulation of the statement "$X^\sharp$ exists" is that there is an elementary embedding $j : L[X] \to L[X]$ which fixes the elements of the transitive closure of X. Another classical formulation is that there is a closed unbounded class of indiscernible (called the Silver's indiscernibles) for $L[X]$. When $x \in {}^\omega 2$, the object $x^\sharp$ can be considered as a real coding statements about indiscernibles (in a language with countably many new constant symbols to be interpreted as a countably infinite subset of indiscernibles) true in $L[x]$. Another very useful characterization of $X^\sharp$ is given by mice:

Definition 3.4.11. (See [28], Definitions 10.18, 10.30, and 10.37.) Let $\mathscr{L} = \{\dot\in, \dot E, \dot U\}$ where $\dot\in$ is a binary relation symbol, and both $\dot E$ and $\dot U$ are unary predicates. Let X be a set. An X-mouse is a $\mathscr{L}$-structure, $\mathcal{M} = \langle J_\alpha[X], \in, X, U\rangle$, where $J_\alpha[X]$ is the α^{th} level of Jensen's fine structural hierarchy of $L[X]$, $\dot E^{\mathcal{M}} = X$, and $\dot U^{\mathcal{M}} = U$, with the following additional properties:

(a) $\mathcal{M}$ is an amenable structure, i.e., for all $z \in J_\alpha[X]$, $z \cap X \in J_\alpha[X]$ and $z \cap U \in J_\alpha[X]$.

(b) In the language $\{\dot\in\}$, $(J_\alpha[X], \in) \models$ "$\mathsf{ZFC} - \mathsf{P}$ and there is a largest cardinal."

(c) If κ is the largest cardinal of $(J_\alpha[X], \in)$, then $\mathcal{M} \models U$ is a κ-complete normal non-trivial ultrafilter on κ.

(d) $\mathcal{M}$ is iterable, i.e., every structure appearing in any putative linear iteration of

$\mathcal{M}$ (by U) is well-founded.

The statement $X^\sharp$ exists is also equivalent to the existence of an X-mouse. $X^\sharp$ will sometimes also denote the smallest X-mouse $\mathcal{M}$ in the sense that if $\mathcal{N}$ is an X-mouse, then there is an α such that the α^{th} iteration $\mathcal{M}_\alpha$ is $\mathcal{N}$.

Under the condition that sharps of all reals exist, the statement "all E-classes are $\boldsymbol{\Delta}^1_1$" will be shown to be $\boldsymbol{\Pi}^1_3$. This is a significant improvement since $\boldsymbol{\Pi}^1_3$ generic absoluteness is much easier to obtain.

Proposition 3.4.12. *Let E be a $\Sigma^1_1(z)$ equivalence relation. There is a $\Pi^1_3(z)$ formula $\varpi(v)$ in free variable v such that:*

Let $x \in {}^\omega 2$. If $(x \oplus z)^\sharp$ exists, then the statement "$[x]_E$ is $\boldsymbol{\Delta}^1_1$" is equivalent to $\varpi(x)$.

Assume for all $r \in {}^\omega\omega$, $r^\sharp$ exists. The statement "all E-classes are $\boldsymbol{\Delta}^1_1$" is equivalent to $(\forall x)\varpi(x)$. In particular, this statement is $\Pi^1_3(z)$.

Proof. For simplicity, assume E is a Σ^1_1 equivalence relation on ${}^\omega\omega$. Let T be a recursive tree on $\omega \times \omega \times \omega$ such that

$$(x, y) \in E \Leftrightarrow T^{x,y} \text{ is illfounded.}$$

Claim : Assume $x^\sharp$ exists, then

$$\text{“}V \models [x]_E \text{ is } \boldsymbol{\Delta}^1_1\text{”} \Leftrightarrow$$

$$\text{“}1_{\mathrm{Coll}(\omega,<c_1)} \Vdash_{\mathrm{Coll}(\omega,<c_1)} (\exists c)(c \in \mathrm{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c)))\text{”} \in x^\sharp.$$

Here c_1 comes from $\{c_n : n \in \omega\}$, which is a collection of constant symbols used to denote indiscernibles. In the above, $x^\sharp$ is considered as a theory consisting of the statements about indiscernibles true in $L[x]$.

Proof of Claim: Assume $[x]_E$ is $\boldsymbol{\Delta}^1_1$. Then

$$(\exists \xi < \omega_1)(\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \xi).$$

Since $x^\sharp$ exists, ω_1 is inaccessible in $L[x]$ and $|\mathscr{P}^{L[x]}(\mathrm{Coll}(\omega, \xi))| = \aleph_0$. In V, there is a $g \subseteq \mathrm{Coll}(\omega, \xi)$ which is $\mathrm{Coll}(\omega, \xi)$ generic over $L[x]$. Since $g \in V$, there is a $c \in L[x][g] \subseteq V$ such that $c \in \mathrm{WO}$ and $\mathrm{ot}(c) = \xi$.

$$V \models (\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c)).$$

Since this statement above is Π^1_2, Schoenfield absoluteness gives

$$L[x][g] \models (\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c)).$$

Using the weak homogeneity of $\mathrm{Coll}(\omega,\xi)$,

$$L[x] \models 1_{\mathrm{Coll}(\omega,\xi)} \Vdash_{\mathrm{Coll}(\omega,\xi)} (\exists c)(c \in \mathrm{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c))).$$

The statement forced above is Σ^1_3. By upward absoluteness of Σ^1_3 statements

$$L[x] \models 1_{\mathrm{Coll}(\omega,<\omega_1)} \Vdash_{\mathrm{Coll}(\omega,<\omega_1)} (\exists c)(c \in \mathrm{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c))).$$

$$\text{“}1_{\mathrm{Coll}(\omega,<c_1)} \Vdash_{\mathrm{Coll}(\omega,<c_1)} (\exists c)(c \in \mathrm{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c)))\text{”} \in x^\sharp.$$

($\Leftarrow$) Assume

$$\text{“}1_{\mathrm{Coll}(\omega,<c_1)} \Vdash_{\mathrm{Coll}(\omega,<c_1)} (\exists c)(c \in \mathrm{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c)))\text{”} \in x^\sharp.$$

Let $\xi < \omega_1$ be a Silver indiscernible for $L[x]$. Then

$$L[x] \models 1_{\mathrm{Coll}(\omega,<\xi)} \Vdash_{\mathrm{Coll}(\omega,<\xi)} (\exists c)(c \in \mathrm{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c))).$$

Again since $\xi < \omega_1$ and ω_1 is inaccessible in $L[x]$, $\mathscr{P}^{L[x]}(\mathrm{Coll}(\omega,<\xi))$ is countable in V. In V, there exists $g \subseteq \mathrm{Coll}(\omega,<\xi)$ which is $\mathrm{Coll}(\omega,<\xi)$-generic over V.

$$L[x][g] \models (\exists c)(c \in \mathrm{WO} \wedge (\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c))).$$

Since $g \in V$, $L[x][g] \subseteq V$ and there exists a $c \in L[x][g]$ such that

$$L[x][g] \models (\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c)).$$

This statement is $\mathbf{\Pi}^1_2$. Since $L[x][g] \subseteq V$, Schoenfield absoluteness can be applied to give

$$V \models (\forall y)(\neg(y\ E\ x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c)).$$

Therefore,

$$V \models [x]_E \text{ is } \mathbf{\Delta}^1_1.$$

This concludes the proof of the claim.

The statement in variables v and w expressing “$w = v^\sharp$” is Π^1_2. Therefore

$$[x]_E \text{ is } \mathbf{\Delta}^1_1$$

if and only if

$$(\forall y)((y = x^\sharp) \Rightarrow$$

$$\text{“}1_{\mathrm{Coll}(\omega,<c_1)} \Vdash_{\mathrm{Coll}(\omega,<c_1)} (\exists c)(c \in \mathrm{WO} \wedge (\forall y)(\neg(y \, E x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c)))\text{”} \in y).$$

The latter is $\Pi^1_3(x)$.

Similarly

$$(\forall x)([x]_E \text{ is } \mathbf{\Delta}^1_1)$$

if and only if

$$(\forall x)(\forall y)((y = x^\sharp) \Rightarrow$$

$$\text{“}1_{\mathrm{Coll}(\omega,<c_1)} \Vdash_{\mathrm{Coll}(\omega,<c_1)} (\exists c)(c \in \mathrm{WO} \wedge (\forall y)(\neg(y \; E \; x) \Rightarrow \mathrm{rk}(T^{x,y}) < \mathrm{ot}(c)))\text{”} \in y).$$

The latter is Π^1_3.

Let $\varpi(v)$ be the statement:

$$(\forall y)(y = v^\sharp \Rightarrow$$

$$\text{“}1_{\mathrm{Coll}(\omega,<c_1)} \Vdash_{\mathrm{Coll}(\omega,<c_1)} (\exists c)(c \in \mathrm{WO} \wedge (\forall y)(\neg(y \; E \; v) \Rightarrow \mathrm{rk}(T^{v,y}) < \mathrm{ot}(c)))\text{”} \in y).$$

By the above results, this works. $\square$

So assuming for all $x \in {}^\omega 2$, $x^\sharp$ exists, the statement “all E-classes are $\mathbf{\Delta}^1_1$” is $\mathbf{\Pi}^1_3$. Below, some conditions for $\mathbf{\Pi}^1_3$ generic absoluteness will be explored. However, there is still a subtle point to be noted. Assume all sharps of reals exist and generic $\mathbf{\Pi}^1_3$ absoluteness holds for a forcing $\mathbb{P}$. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Then in $V[G]$, the statement $(\forall x)(\varpi(x))$ remains true by $\mathbf{\Pi}^1_3$ absoluteness. But if $V[G]$ does not satisfy all sharps of reals exist, then it may not be true that $(\forall v)(\varpi(v))$ is equivalent to the statement “all E-classes are $\mathbf{\Delta}^1_1$”. For the main question in the case of $\mathbb{P}_I$, only the fact that $[\dot{x}_{\mathrm{gen}}]_E$ is $\mathbf{\Delta}^1_1$ will be of any concern. In $V[G]$, one has $(\forall x)(\varpi(x))$. In particular, if g is the generic real added by G, then $\varpi(g)$ holds. If $(g \oplus z)^\sharp$ exists, then Proposition 3.4.12 implies “$[g]_E$ is $\mathbf{\Delta}^1_1$” is equivalent to $\varpi(g)$. Hence $[g]_E$ is $\mathbf{\Delta}^1_1$ in $V[G]$. The following is a situation (applicable later) for which $(g \oplus z)^\sharp$ exists.

Fact 3.4.13. *Let A be a set. Suppose $A^\sharp$ exists and $j : L[A] \to L[A]$ is a nontrivial elementary embedding fixing the elements in the transitive closure of A. Suppose $\mathbb{P} \in L[A]$ is a forcing in $(V_{crit(j)})^{L[A]}$. Suppose $G \subseteq \mathbb{P}$ is generic over V (or just $L[A]$), then $\langle A, G\rangle^\sharp$ exists in $V[G]$.*

Proof. Since $\mathbb{P} \in (V_{\mathrm{crit}(j)})^{L[A]}$, $j''\,\mathrm{tc}(\{\mathbb{P}\}) = \mathrm{tc}(\{\mathbb{P}\})$. Define the lift $\tilde{j} : L[\langle A, G\rangle] \to L[\langle A, G\rangle]$ by

$$\tilde{j}(\tau[G]) = j(\tau)[G].$$

By the usual arguments, $\tilde{j}$ is a nontrivial elementary embedding definable in $V[G]$. Hence $\langle A, G\rangle^{\sharp}$ exists. □

Next, a few more basic properties of iterable structures:

Fact 3.4.14. *Let $\mathscr{L} = \{\dot{\in}, \dot{U}\}$. Suppose $\mathcal{N} = (N, \in, V)$ is an iterable structure. Suppose $\mathcal{M} = (M, \in, U)$ is an£ $\mathscr{L}$-structure such that there exists an $\mathscr{L}$-elementary embedding $j : \mathcal{M} \to \mathcal{N}$. Then $\mathcal{M}$ is iterable.*

Proof. See [1], Lemma 18 for a proof. □

Fact 3.4.15. *Let $M \prec H_\Theta$ be a countable elementary substructure where Θ is some sufficienty large cardinal. Let $\mathcal{U}$ be an iterable structure and $\mathcal{U} \in M$. Let $\mathcal{U}^M = \mathcal{U} \cap M$. Then $\mathcal{U}^M$ is iterable.*

Proof. Let φ be some $\mathscr{L} = \{\dot{\in}, \dot{U}\}$ sentence. For any $x \in \mathcal{U} \cap M$, $\mathcal{U}^M \models \varphi(x)$ if and only if $M \models \mathcal{U} \models \varphi(x)$. Since $M \prec H_\Theta$, if and only if $V \models \mathcal{U} \models \varphi(x)$. Hence $\mathcal{U}^M \prec \mathcal{U}$ as an $\mathscr{L}$-structure. By Fact 3.4.14, $\mathcal{U}^M$ is iterable. □

As mentioned above, it is not possible in general to claim that $\mathbf{\Pi}^1_2$ statements are absolute between a countable model M and the universe V since Schoenfield absoluteness can not be applied when it is not the case that $\omega_1^V \subseteq M$. However, ω_1-iterable structures can be used to solve this problem by applying Schoenfield absoluteness in the ω_1 iteration.

Fact 3.4.16. *Let X be a set. Suppose $\mathcal{M} = (J_\alpha[X], \in, X, U)$ is an X-mouse. Then $J_\alpha[X]$ is Π^1_2-correct, that is, if φ is a $\mathbf{\Pi}^1_2$ sentence with parameters in $J_\alpha[X]$, then $J_\alpha[X] \models \varphi$ if and only if $V \models \varphi$.*

Let κ be the largest cardinal of $(J_\alpha[X], \in)$. Suppose $\mathbb{P} \in J_\alpha[X]$ is a forcing such that $(J_\alpha[X], \in) \models \mathbb{P} \in V_\kappa$. Then $J_\alpha[X]$ is $\mathbb{P}$-generically Π^1_2-correct, that is, for all $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over $J_\alpha[X]$ and $G \in V$, and any $\mathbf{\Pi}^1_2$-formula coded in $J_\alpha[X][G]$, $J_\alpha[X][G] \models \varphi$ if and only if $V \models \varphi$.

Proof. Let $\mathcal{M}_0 = \mathcal{M}$. Let $j_{0,\omega_1} : \mathcal{M}_0 \to \mathcal{M}_{\omega_1}$ denote the ω_1-iteration. $\mathcal{M}_{\omega_1}$ is well-founded, so let $\mathcal{M}_{\omega_1} = (J_\beta[X], \in, X, U_{\omega_1})$. Note that $\beta \geq \omega_1$. By [28], Lemma 10.21 (d), j_{0,ω_1} is a full (Σ_ω) elementary embedding in the language $\{\dot{\in}, \dot{E}\}$. So if φ is a $\mathbf{\Pi}^1_2$ statement with parameter in $J_\alpha[X]$ then $J_\alpha[X] \models \varphi$ if and only if $J_\beta[X] \models \varphi$. Since $\omega_1^V \subseteq J_\beta[X]$, $J_\beta[X] \models \varphi$ if and only if $V \models \varphi$.

For the second statement: Since $(J_\alpha[X], \in) \models \mathbb{P} \in V_\kappa$, j_{0,ω_1} does not move any elements in the transitive closure of $\mathbb{P}$. Also no new subsets of $\mathbb{P}$ appear in $J_\beta[X]$. Thus if G is $\mathbb{P}$-generic over $J_\alpha[X]$, then G is $\mathbb{P}$-generic over $J_\beta[X]$. Lift the elementary embedding $j_{0,\omega_1} : J_\alpha[X] \to J_\beta[X]$ to $\tilde{j}_{0,\omega_1} : J_\alpha[X][G] \to J_\beta[X][G]$ in the usual way: if $\tau \in (J_\alpha[X])^{\mathbb{P}}$, then

$$\tilde{j}_{0,\omega_1}(\tau[G]) = j_{0,\omega_1}(\tau)[G].$$

$\tilde{j}_{0,\omega_1}$ is a well-defined elementary embedding. Let φ be a Π^1_2 formula coded in $J_\alpha[X][G]$. Using this elementary embedding, $J_\alpha[X][G] \models \varphi$ if and only if $J_\beta[X][G] \models \varphi$. Since $\omega_1^V \subseteq J_\beta[X][G]$ and using Schoenfield absoluteness, $J_\beta[X][G] \models \varphi$ if and only if $V \models \varphi$. □

Fact 3.4.17. *Let $\mathbb{P}$ be a forcing. Suppose $\varphi(v)$ is a formula with one free variable. By fullness or the maximality principle (see [23] Theorem IV.7.1), there exists a name $\tau^{\mathbb{P}}_\varphi$ such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\exists v)(\varphi(v))$ if and only if $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \varphi(\tau^{\mathbb{P}}_\varphi)$. If $\tau^{\mathbb{P}}_\varphi$ is a name for a real, one may assume that it is a nice name for a real.*

Note that $\tau^{\mathbb{P}}_\varphi$ is not unique. $\tau^{\mathbb{P}}_\varphi$ will just refer to any $\mathbb{P}$-name that satisfies the above property.

Fact 3.4.18. *Consider the Σ^1_3 sentence $(\exists v)(\varphi(v))$ where $\varphi(v)$ is Π^1_2. If $\langle \mathbb{P}, \tau^{\mathbb{P}}_\varphi \rangle^\sharp$ exists, then $(\exists v)(\varphi(v))$ is absolute between the ground model and $\mathbb{P}$-extensions.*

Proof. This is originally proved using the Martin-Solovay tree, which were implicit in [25]. The proof from [5] Theorem 3 is sketched below to make explicit what sharps are necessary.

Suppose $1_{\mathbb{P}} \Vdash^V_{\mathbb{P}} (\exists v)\varphi(v)$. Then $1_{\mathbb{P}} \Vdash^V_{\mathbb{P}} \varphi(\tau^{\mathbb{P}}_\varphi)$.

Note that $\mathbb{P} \in \langle \mathbb{P}, \tau^{\mathbb{P}}_\varphi \rangle^\sharp$ (where $\langle \mathbb{P}, \tau^{\mathbb{P}}_\varphi \rangle^\sharp$ is considered as a mouse as in Definition 3.4.11) and $\langle \mathbb{P}, \tau^{\mathbb{P}}_\varphi \rangle^\sharp \models \mathbb{P} \in V_\kappa$, where κ is the largest cardinal of $\langle \mathbb{P}, \tau^{\mathbb{P}}_\varphi \rangle^\sharp$.

Using some standard way of coding, let T be a tree of attempts to build a tuple $(\mathcal{M}, \mathbb{Q}, H, y, j)$ with the following properties:

(1) $\mathcal{M}$ is a countable structure satisfying (a), (b), and (c) from Definition 3.4.11.

(2) $\mathbb{Q} \in \mathcal{M}$ is a forcing.

(3) H is $\mathbb{Q}$-generic over $\mathcal{M}$.

(4) y is a real in $\mathcal{M}[H]$ and $\mathcal{M}[H] \models \varphi(y)$.

(5) $j : \mathcal{M} \to \langle \mathbb{P}, \tau_\varphi^{\mathbb{P}} \rangle^\sharp$ is an elementary embedding in the language $\{\dot{\in}, \dot{E}, \dot{U}\}$ with $j(\mathbb{Q}) = \mathbb{P}$.

Let G be an arbitrary $\mathbb{P}$-generic over V. Since $V[G] \models (\exists v)\varphi(v)$, $V[G] \models \varphi(\tau_\varphi^{\mathbb{P}}[G])$. By the downward absoluteness of Π^1_2 statements (which follows from Mostowski absoluteness), $\langle \mathbb{P}, \tau_\varphi^{\mathbb{P}} \rangle^\sharp[G] \models \varphi(\tau_\varphi^{\mathbb{P}}[G])$. By Downward-Lowenheim-Skolem, let $\mathcal{N}$ be a countable $\{\dot{\in}, \dot{E}, \dot{U}\}$ elementary substructure of $\langle \mathbb{P}, \tau_\varphi^{\mathbb{P}} \rangle^\sharp$ containing $\mathbb{P}$ and $\tau_\varphi^{\mathbb{P}}$. Let $\mathcal{M}$ be the Mostowski collapse of $\mathcal{N}$, and $j : \mathcal{M} \to \langle \mathbb{P}, \tau_\varphi^{\mathbb{P}} \rangle^\sharp$ be the induced elementary embedding. Let $\mathbb{Q} = j^{-1}(\mathbb{P})$. Let $H = j^{-1}[G]$. Let $y = j^{-1}(\tau_\varphi^{\mathbb{P}})[H]$. So in $V[G]$, $(\mathcal{M}, \mathbb{Q}, H, y, j)$ is a path through T.

Therefore, in $V[G]$, the tree T is illfounded. Hence it is illfounded in V by Δ_1-absoluteness. In V, let $(\mathcal{M}, \mathbb{Q}, H, y, j)$ be such a path. By Fact 3.4.14, $\mathcal{M}$ is iterable. By Fact 3.4.16, $\mathcal{M}[H] \models \varphi(y)$ implies $V \models \varphi(y)$. This establishes that $(\exists v)\varphi(v)$ is downward absolute from $V[G]$ to V. This completes the proof. $\square$

Definition 3.4.19. Let I be a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper. If ϖ is the formula from Proposition 3.4.12, then $\neg(\forall v)\varpi(v)$ is Σ^1_3 and can be written as $(\exists v)\zeta(v)$ where ζ is $\mathbf{\Pi}^1_2$. Let μ^I_E be $\tau_\zeta^{\mathbb{P}_I}$ from Fact 3.4.17.

Definition 3.4.20. Let I be a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is proper. Consider the formula "$(\exists y)([\dot{x}_{\text{gen}}]_E$ is $\Pi^1_1(y))$". Write it as $(\exists y)\psi(y)$. By Fact 3.4.17, let σ^I_E be $\tau_\psi^{\mathbb{P}_I}$.

Definition 3.4.21. Suppose I is a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is proper. Let $E \in \Sigma_1^{1\mathbf{\Delta}^1_1}$. Define $\chi^I_E = \langle \mathbb{P}_I, \mu^I_E, \sigma^I_E \rangle$.

Despite the notation, χ^I_E is not unique since μ^I_E and σ^I_E are not unique.

The following result gives a positive answer to the main question for $\Sigma_1^{1\mathbf{\Delta}^1_1}$ using sharps for some small sets.

Theorem 3.4.22. *Suppose I is a σ ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is proper. If for all $x \in {}^\omega 2$, $x^\sharp$ exists and $(\chi^I_E)^\sharp$ exists for all $E \in \Sigma_1^{1\mathbf{\Delta}^1_1}$, then $\Sigma_1^{1\mathbf{\Delta}^1_1} \to_I \mathbf{\Delta}^1_1$.*

Proof. Let Θ be sufficiently large and $M \prec H_\Theta$ be countable elementary with $(\chi_E^I)^\sharp \in M$. Note that M thinks $(\chi_E^I)^\sharp$ exists and for all $x \in {}^\omega 2$, $x^\sharp$ exists.

First to show that $(\forall v)(\varpi(v))$ is $\mathbb{P}_I$-generically absolute for M: Since M satisfies all sharps of reals exist, Proposition 3.4.12 implies "all E-classes are $\mathbf{\Delta}_1^1$" is equivalent to $(\forall v)(\varpi(v))$. The latter is $\mathbf{\Pi}_3^1$ and so its negation is Σ_3^1. Since $M \models \langle \mathbb{P}_I, \mu_E^I \rangle^\sharp$ exists, Fact 3.4.18 implies the statement, $(\forall v)(\varpi(v))$ is absolute between M and $\mathbb{P}_I$ extensions of M. Since M satisfies "all E-classes are $\mathbf{\Delta}_1^1$" and M satisfies all sharps of reals exist, M satisfies $(\forall v)(\varpi(v))$. Therefore, all $\mathbb{P}_I$ extensions of M satisfy the formula $(\forall v)(\varpi(v))$.

Since $\mathbb{P}_I^\sharp$ exists, there exists a $j : L[\mathbb{P}_I] \to L[\mathbb{P}_I]$ with $\mathbb{P}_I \in (V_{\text{crit}(j)})^{L[\mathbb{P}]}$. Therefore, Fact 3.4.13 implies $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} \dot{x}_{\text{gen}}^\sharp$ exists . So $M \models 1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} \dot{x}_{\text{gen}}^\sharp$ exists . By Proposition 3.4.12,

$$M \models 1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} ([\dot{x}_{\text{gen}}]_E \text{ is } \mathbf{\Delta}_1^1) \Leftrightarrow \varpi(\dot{x}_{\text{gen}}).$$

Since all $\mathbb{P}_I$ extensions of M satisfy $(\forall v)(\varpi(v))$, all $\mathbb{P}_I$ extensions of M satisfy $[\dot{x}_{\text{gen}}]_E$ is $\mathbf{\Delta}_1^1$.

By the result of the previous paragraph, $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I}^M (\exists y)\psi(y)$, where ψ is the formula from Definition 3.4.20. Therefore, $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I}^M \psi(\sigma_E^I)$. Note that $\psi(y)$ is actually $\psi'(\dot{x}_{\text{gen}}, y)$, where ψ' is $\mathbf{\Pi}_2^1$ with parameters from M asserting that $[\dot{x}_{\text{gen}}]$ is $\Pi_1^1(y)$. Since $\dot{x}_{\text{gen}}$ is constructible from $\mathbb{P}_I$, the existence of $\langle \mathbb{P}_I, \dot{x}_{\text{gen}}, \sigma_E^I \rangle^\sharp$ follows from the existence of $\langle \mathbb{P}_I, \sigma_E^I \rangle^\sharp$. Applying the downward absoluteness of $\mathbf{\Pi}_2^1$ statements from $M[x]$ to $(\langle \mathbb{P}_I, \sigma_E^I \rangle^\sharp)^M[x]$ (where x is any $\mathbb{P}_I$-generic over M real) gives $(\langle \mathbb{P}_I, \sigma_E^I \rangle^\sharp)^M[x] \models \psi'(x, \sigma_E^I[x])$. By Fact 3.4.15, $(\langle \mathbb{P}_I, \sigma_E^I \rangle^\sharp)^M$ is still iterable. Applying Fact 3.4.16 (generic Π_2^1-correctness) to $(\langle \mathbb{P}_I, \sigma_E^I \rangle^\sharp)^M[x]$ and V, one has $V \models \psi'(x, \sigma_E^I[x])$, where x is any $\mathbb{P}_I$-generic over M. So it has been shown that $M[x] \models [x]_E$ is $\Pi_1^1(\sigma_E^I[x])$ and $V \models [x]_E$ is $\Pi_1^1(\sigma_E^I[x])$.

Lemma 3.4.3 implies that there is some countable α such that for all x which are $\mathbb{P}_I$-generic over M, $[x]_{E_\alpha} = [x]_E$. Therefore if B is an arbitrary I^+ $\mathbf{\Delta}_1^1$ subset and C is the I^+ $\mathbf{\Delta}_1^1$ set of $\mathbb{P}_I$-generic over M reals in B, then $E \restriction C = E_\alpha \restriction C$. $\{E\} \to_I \mathbf{\Delta}_1^1$. $\square$

Since $\mathbb{P}_I$ is a collection of subsets of ${}^\omega\omega$ and both μ_E^I and σ_E^I can be taken to be nice names for reals, χ_E^I is an element of $H_{(2^{\aleph_0})^+}$. Therefore, if there is a measurable or a Ramsey cardinal, then $(\chi_E^I)^\sharp$ will exist.

Corollary 3.4.23. *If $z^\sharp$ exists for all $z \in H_{(2^{\aleph_0})^+}$, then $\Sigma_1^{1\mathbf{\Delta}_1^1} \to_I \mathbf{\Delta}_1^1$ for all σ-ideal I such that $\mathbb{P}_I$ is proper.*

Corollary 3.4.24. *If there exists a Ramsey cardinal, then* ${\Sigma_1^1}^{\mathbf{\Delta}_1^1} \rightarrow_I \mathbf{\Delta}_1^1$ *for all* σ*-ideal* I *such that* $\mathbb{P}_I$ *is proper.*

3.5 Σ_1^1 Equivalence Relations and Some Ideals

Some partial results about the main question for Σ_1^1 equivalence relations with all classes $\mathbf{\Delta}_1^1$ will be provided in this and the next section. These are proved using various different techniques and different set theoretic assumptions (usually of lower consistency strength than the full answer of the previous section). These results may be useful in understanding what combination of universes, Σ_1^1 equivalence relations, and σ-ideals can not be used to demonstate the consistency of a negative answer to the main question.

In this section, the focus will be on the main question in the case of some classical ideals I with $\mathbb{P}_I$ proper.

Definition 3.5.1. Let X be a Polish space. Let $I_{\text{countable}} := \{A \subseteq X : |A| \leq \aleph_0\}$. $\mathbb{P}_{I_{\text{countable}}}$ is forcing equivalent to Sacks forcing.

Proposition 3.5.2. $\Sigma_1^1 \rightarrow_{I_{countable}} \mathbf{\Delta}_1^1$.

Proof. Let E be any Σ_1^1 equivalence relation. (Note that there is no condition on the classes being $\mathbf{\Delta}_1^1$ for this proposition.) Let B be $I^+_{\text{countable}}$ $\mathbf{\Delta}_1^1$ set, i.e. an uncountable $\mathbf{\Delta}_1^1$ set.

Suppose there is some $x \in B$ such that $[x]_E \cap B$ is uncountable. The perfect set property for the Σ_1^1 set $[x]_E \cap B$ implies that $[x]_E \cap B$ has a perfect subset C. Then $E \upharpoonright C = \text{ev} \upharpoonright C$. So $\{E\} \rightarrow_{I_{\text{countable}}} \mathbf{\Delta}_1^1$.

Otherwise, $[x]_E \cap B$ is countable for all $x \in B$. $E \upharpoonright B$ is a Σ_1^1 equivalence relation with all classes countable. Then $\{E\} \rightarrow_I \mathbf{\Delta}_1^1$ follows from Fact 3.3.2. □

Definition 3.5.3. Let I_{E_0} denote the σ-ideal σ-generated by the $\mathbf{\Delta}_1^1$ sets on which E_0 is smooth.

Fact 3.5.4. $\mathbb{P}_{I_{E_0}}$ *is forcing equivalent to Prikry-Silver forcing. Hence* $\mathbb{P}_{I_{E_0}}$ *is a proper forcing.*

Proof. See [33], Lemma 2.3.37. □

Fact 3.5.5. $\Sigma_1^1 \rightarrow_{I_{E_0}} \{id, ev, E_0\}$.

Proof. See [19], Theorem 7.1.1. □

Corollary 3.5.6. $\Sigma^1_1 \rightarrow_{I_{E_0}} \mathbf{\Delta}^1_1$.

Proof. This follows immediately from Fact 3.5.5 since id, ev, and E_0 are all $\mathbf{\Delta}^1_1$ equivalence relations. □

Definition 3.5.7. Let I_{meager} be the σ-ideal σ-generated by the meager subsets of ${}^\omega\omega$ (or more generally any Polish space).

Let I_{null} be the σ-ideal σ-generated by the Lebesgue null subsets of ${}^\omega\omega$.

Kechris communicated to the author the following results concerning the meager ideal. Define a set to be I_{meager} measurable if and only if that set has the Baire property. Define a set to be I_{null} measurable if and only if that set is Lebesgue measurable. First, a well-known result on the additivity of the meager ideal and null ideal under certain types of unions.

Fact 3.5.8. *Let I be I_{meager} or I_{null}. Let $\{A_\eta\}_{\eta<\xi}$ be a sequence of sets in I. Define a prewellordering $\sqsubseteq$ on $\bigcup_{\eta<\xi} A_\eta$ by: $x \sqsubseteq y$ if and only if the least η such that $x \in A_\eta$ is less than or equal to the least η such that $y \in A_\eta$. If $\sqsubseteq$ is I-measurable (with the version of I defined on the product space), then $\bigcup_{\eta<\xi} A_\eta$ is in I.*

Proof. See [21], Proposition 1.5.1 for a proof. □

Theorem 3.5.9. *Let I be I_{meager} or I_{null}. If all $\mathbf{\Pi}^1_3$ sets are I measurable, then $\Sigma^{1\ \mathbf{\Delta}^1_1}_1 \rightarrow_I \mathbf{\Delta}^1_1$. Moreover, if E is a Σ^1_1 equivalence relation with all classes $\mathbf{\Delta}^1_1$ and B is I^+ $\mathbf{\Delta}^1_1$, then there exists a I^+ $\mathbf{\Delta}^1_1$ $C \subseteq B$ with $B \setminus C \in I$ and $E \upharpoonright C$ is a $\mathbf{\Delta}^1_1$ equivalence relation.*

Proof. (Kechris) For simplicity, assume E is an equivalence relation on ${}^\omega\omega$. Let $B \subseteq {}^\omega\omega$ be I^+ $\mathbf{\Delta}^1_1$. For simplicity, assume $B = {}^\omega\omega$. $({}^\omega\omega)^2 \setminus E$ is a $\mathbf{\Pi}^1_1$ set. Let T be a tree on $\omega \times \omega \times \omega$ such that

$$\neg(x\ E\ y) \Leftrightarrow T^{(x,y)} \text{ is well-founded.}$$

For each $\alpha < \omega_1$, let $A_\alpha = \{x : (\forall y)((x,y) \notin E \Rightarrow \text{rk}(T^{(x,y)}) < \alpha)\}$. First, the claim is that for all $x \in {}^\omega\omega$, there exists some $\alpha < \omega_1$ with $x \in A_\alpha$: to see this, fix x and let $L_x = \{(x,y) : y \notin [x]_E\}$. Since $[x]_E$ is $\mathbf{\Delta}^1_1$, L_x is $\mathbf{\Delta}^1_1$.

$L_x \subseteq ({}^\omega\omega)^2 \setminus E$. By the boundedness theorem, there exists some $\alpha < \omega_1$ such that $\sup\{\mathrm{rk}(T^{(x,y)}) : (x, y) \in L_x\} < \alpha$. $x \in A_\alpha$. It has been shown that ${}^\omega\omega = \bigcup_{\alpha<\omega_1} A_\alpha$.

The next claim is that there exists some $\alpha < \omega_1$ such that A_α is I^+: Suppose that for all $\alpha < \omega_1$, $A_\alpha \in I$. Note that there is a $\mathbf{\Pi}^1_2$ formula $\Phi(x, c)$ (using the tree T as a parameter) such that if $c \in \mathrm{WO}$, then

$$\Phi(x, c) \Leftrightarrow (\forall y)(\mathrm{rk}(T^{(x,y)}) < \mathrm{ot}(c)).$$

Define $\sqsubseteq$ using the sequence $\{A_\alpha : \alpha < \omega_1\}$. Then

$$x \sqsubseteq y \Leftrightarrow (\forall c)(c \in \mathrm{WO} \Rightarrow (\Phi(y, c) \Rightarrow \Phi(x, c))).$$

$\sqsubseteq$ is $\mathbf{\Pi}^1_3$ on ${}^\omega\omega \times {}^\omega\omega$. By Fact 3.5.8, ${}^\omega\omega = \bigcup_{\alpha<\omega_1} A_\alpha$ is in I. Contradiction.

Choose an $\alpha < \omega_1$ such that A_α is I^+. Since $\mathbf{\Pi}^1_2$ sets are I-measurable, let C be $\mathbf{\Delta}^1_1$ I^+ such that $A_\alpha \triangle C \in I$. Thus $C \setminus A_\alpha \in I$. Since I is the σ-generated by certain $\mathbf{\Delta}^1_1$ sets, there exists a $\mathbf{\Delta}^1_1$ set $D \in I$ such that $C \setminus A_\alpha \subseteq D$. Let $B_0 = C \setminus D$. Note that B_0 is I^+ $\mathbf{\Delta}^1_1$ and $B_0 \subseteq A_\alpha$.

Now suppose $\xi < \omega_1$ and a sequence $\{B_\eta : \eta < \xi\}$ of $\mathbf{\Delta}^1_1$ I^+ sets has been defined with the property that if $\eta_1 \neq \eta_2$ then $B_{\eta_1} \cap B_{\eta_2} \in I$. Let $K_\xi = \bigcup_{\eta<\xi} B_\eta$. Define $A^\xi_\alpha = A_\alpha \setminus K_\xi$. ${}^\omega\omega \setminus K_\xi = \bigcup_{\alpha<\omega_1} A^\xi_\alpha$. If ${}^\omega\omega \setminus K_\xi$ is I^+, then repeating the above procedure produces some I^+ $\mathbf{\Delta}^1_1$ B_ξ with the property that for all $\eta < \xi$, $B_\eta \cap B_\xi \in I$ and for some $\alpha < \omega_1$, $B_\xi \subseteq A^\xi_\alpha \subseteq A_\alpha$.

Observe that for some $\xi < \omega_1$, ${}^\omega\omega \setminus K_\xi$ must be in I. This is because otherwise the construction succeeds in producing an antichain $\{B_\eta : \eta < \omega_1\}$ of cardinality $\aleph_1$ in $\mathbb{P}_I$. However, $\mathbb{P}_I$ has the $\aleph_1$-chain condition. Contradiction.

So choose ξ such that ${}^\omega\omega \setminus K_\xi \in I$. By construction, for each $\eta < \xi$, there is some $\alpha_\eta < \omega_1$ such that $B_\eta \subseteq A^\eta_{\alpha_\eta} \subseteq A_{\alpha_\eta}$. Since $\xi < \omega_1$, there is some $\mu < \omega_1$ such that $\sup\{\alpha_\eta : \eta < \xi\} < \mu$. Then $K_\xi = \bigcup_{\eta<\xi} A_{\alpha_\eta} \subseteq A_\mu$. Hence for all $x, y \in K_\xi$,

$$x \ E \ y \Leftrightarrow \mathrm{rk}(T^{(x,y)}) \geq \mu.$$

So K_ξ is I^+ $\mathbf{\Delta}^1_1$ with ${}^\omega\omega \setminus K_\xi \in I$ and $E \restriction K_\xi$ is a $\mathbf{\Delta}^1_1$ equivalence relation. $\square$

Theorem 3.5.10. *The consistency of* ZFC *implies the consistency of* ZFC *and* $\Sigma^1_1 \overset{\mathbf{\Delta}^1_1}{\longrightarrow}_{I_{meager}} \mathbf{\Delta}^1_1$.

The consistency of ZFC + Inaccessible Cardinal *implies the consistency of* ZFC *and* $\Sigma^1_1 \overset{\mathbf{\Delta}^1_1}{\longrightarrow}_{I_{null}} \mathbf{\Delta}^1_1$.

Proof. By [31], from a model ZFC, one can obtain a model of ZFC in which all $\mathrm{OD}_{{}^{\omega}\omega}$ subsets of ${}^{\omega}\omega$ have the Baire property.

By [32], from a model of ZFC with an inaccessible cardinal, one can obtain a model of ZFC in which all $\mathrm{OD}_{{}^{\omega}\omega}$ subsets of ${}^{\omega}\omega$ are Lebesgue measurable.

Then both results follow from Theorem 3.5.9. □

Let κ be an inaccessible cardinal. $\mathrm{Coll}(\omega, < \kappa)$ denotes the Lévy collapse of κ to ω_1. Since the generic extension of the Lévy collapse of an inaccessible to ω_1 (and the related Solovay's model) appears often in descriptive set theory, the following is worth mentioning:

Corollary 3.5.11. *Let κ be an inaccessible cardinal in V. Let $G \subseteq Coll(\omega, < \kappa)$ be $Coll(\omega, < \kappa)$-generic over V. Then in $V[G]$, ${\Sigma^1_1}^{\mathbf{\Delta}^1_1} \to_{I_{meager}} \mathbf{\Delta}^1_1$ and ${\Sigma^1_1}^{\mathbf{\Delta}^1_1} \to_{I_{null}} \mathbf{\Delta}^1_1$.*

Proof. [32] shows that in this model, all $\mathrm{OD}_{{}^{\omega}\omega}$ subsets of ${}^{\omega}\omega$ have the Baire property and are Lebesgue measurable. As above, the result follows from Theorem 3.5.9. □

[31] shows that the existence of an inaccessible cardinal and the statement that all $\mathbf{\Pi}^1_3$ sets are Lebesgue measurable are equiconsistent.

To show that the above statement even for I_{null} is consistent relative to ZFC will require a slight modification of the above proof using a different set theoretic assumption.

Definition 3.5.12. Let I be a σ-ideal on a Polish space X. $\mathrm{cov}(I)$ is the smallest cardinal κ such that there exists a set $U \subseteq I$ with $\bigcup U = X$ and $|U| = \kappa$.

Proposition 3.5.13. *Let I be I_{meager} or I_{null}. If all $\mathbf{\Pi}^1_2$ sets are I-measurable and $cov(I) > \aleph_1$, then ${\Sigma^1_1}^{\mathbf{\Delta}^1_1} \to_I \mathbf{\Delta}^1_1$.*

Proof. The proof is similar to Theorem 3.5.9. In this case, one can conclude that for some $\alpha < \omega_1$, A_α is I^+ from the fact that $\mathrm{cov}(I) > \aleph_1$ and ${}^{\omega}\omega = \bigcup_{\alpha<\omega_1} A_\alpha$. The I-measurability of $\mathbf{\Pi}^1_2$ sets is needed to find some $C \subseteq A_\alpha$ which is I^+ $\mathbf{\Delta}^1_1$. □

Fact 3.5.14. *Let I be I_{meager} or I_{null}.* MA + ¬CH *implies all $\mathbf{\Pi}^1_2$ sets are I-measurable and $cov(I) = 2^{\aleph_0} > \aleph_1$.*

Proof. See [26]. □

Theorem 3.5.15. *The consistency of* ZFC *implies the consistency of* ZFC *and* ${\Sigma^1_1}^{\mathbf{\Delta}^1_1} \to_I \mathbf{\Delta}^1_1$ *where* I *is* I_{meager} *or* I_{null}.

Proof. The consistency of ZFC implies the consistency of $\mathsf{ZFC} + \mathsf{MA} + \neg\mathsf{CH}$ by a well-known iterated forcing argument. □

3.6 Thin Σ^1_1 Equivalence Relations

Definition 3.6.1. An equivalence relation E on a Polish space X is thin if and only if there does not exists a perfect set of pairwise E-inequivalent elements.

Let ${\Sigma^1_1}^{\text{thin}}$ denote the class of thin Σ^1_1 equivalence relations defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces.

Let ${\Sigma^1_1}^{\text{thin } \mathbf{\Delta}^1_1}$ denote the class of thin Σ^1_1 equivalence relations with all classes $\mathbf{\Delta}^1_1$ defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces.

Fact 3.6.2. *Suppose E is a thin Σ^1_1 equivalence relation, then E has at most $\aleph_1$ many equivalence classes.*

Proof. See [3]. □

The above fact may suggest that the properness of $\mathbb{P}_I$ should be used with countable support iterations to change covering numbers. It will be shown below that descriptive set theoretic techniques will give a stronger result in just ZFC. However, in the context of proper forcing, the following combinatorial approach is worth mentioning:

Definition 3.6.3. Let I be a σ-ideal on a Polish space X. $\text{cov}^*(I)$ is the smallest cardinal κ such that there exists some I^+ $\mathbf{\Delta}^1_1$ $B \subseteq X$ and a set $U \subseteq I$ with $|U| = \kappa$ and $B \subseteq \bigcup U$.

Proposition 3.6.4. *Suppose I is a σ-ideal such that $\text{cov}^*(I) > \omega_1$. Then ${\Sigma^1_1}^{thin\ \mathbf{\Delta}^1_1} \to_I \{ev\}$.*

Proof. Let $E \in {\Sigma^1_1}^{\text{thin } \mathbf{\Delta}^1_1}$. Let $\{C_\alpha : \alpha < \omega_1\}$ enumerate all the E-classes in order type ω_1, using Fact 3.6.2. Each C_α is $\mathbf{\Delta}^1_1$ since C_α is an equivalence class of E.

Let B be an arbitrary I^+ $\mathbf{\Delta}^1_1$ set. $B = \bigcup_{\alpha<\omega_1} B \cap C_\alpha$. Since $\text{cov}^*(I) > \aleph_1$, there is some α such that $B \cap C_\alpha$ is I^+ $\mathbf{\Delta}^1_1$. Then $E \restriction (B \cap C_\alpha) = \text{ev} \restriction B \cap C_\alpha$. □

Proposition 3.6.5. *If* PFA *holds, then for all I such that $\mathbb{P}_I$ is proper,* $\Sigma_1^1 \overset{thin\ \mathbf{\Delta}_1^1}{\longrightarrow}_I \{ev\}$.

Proof. Let B be a I^+ $\mathbf{\Delta}_1^1$ set. Let $U = \{C_\beta : \beta < \omega_1\}$ be a collection of $\mathbf{\Delta}_1^1$ sets in I. $\mathbb{P}_I$ being proper implies that $\mathbb{P}_I \restriction B$ is proper. Let $D_\beta := \{F \in \mathbb{P}_I \restriction B : F \cap C_\beta = \emptyset\}$. D_β is dense in $\mathbb{P}_I \restriction B$. By PFA, there is a filter $G \subseteq \mathbb{P}_I \restriction B$ which is generic for $\{D_\beta : \beta < \omega_1\}$. H constructs a real $x_H \in B$. By genericity, $x_H \notin C_\beta$ for all $\beta < \omega_1$. So U can not cover B. $\text{cov}^*(I) > \aleph_1$. The result follows from Proposition 3.6.4. □

The results are unsatisfactory in several ways. Models of PFA satisfy ¬CH and this was an essential fact since the proof used $\text{cov}^*(I) > \aleph_1$. Definability of the equivalence relation was not used in any deep way. The core of the proofs was combinatorial, using $\text{cov}^*(I) > \aleph_1$.

The rest of this section provides results addressing the main question for thin Σ_1^1 equivalence relations which rely on definability properities of these equivalence relations. The best validation of the definability approach to thin Σ_1^1 equivalence is that a stronger result will be proved with weaker assumptions (just ZFC).

Fact 3.6.6. *Let E be a thin Σ_1^1 equivalence relation on a Polish space X. Let $\mathbb{P}$ be a forcing. Suppose $\tau \in V^{\mathbb{P}}$ is such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \tau \in X$. Then there is a dense set D_τ^E such that for all $p \in D_\tau^E$, $(p,p) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau_{left}\ E\ \tau_{right}$, where τ_{left} and τ_{right} denote the $\mathbb{P} \times \mathbb{P}$ names for the evaluation of τ according to the left and right generic for $\mathbb{P}$, respectively, coming from a generic for $\mathbb{P} \times \mathbb{P}$.*

Proof. This is due to Silver. See [4], Lemma 2.1 or the proof of [15], Theorem 2.3.

A sketch of the result is provided:

Suppose not. Then there exists some $u \in \mathbb{P}$ such that for all $q \leq_{\mathbb{P}} u$, $(q,q) \nVdash_{\mathbb{P}\times\mathbb{P}} \tau_{\text{left}}\ E\ \tau_{\text{right}}$. Hence, there is some u such that for all $q \leq_{\mathbb{P}} u$, there exists $q_0, q_1 \leq_{\mathbb{P}} q$ with $(q_0, q_1) \Vdash_{\mathbb{P}\times\mathbb{P}} \neg(\tau_{\text{left}}\ E\ \tau_{\text{right}})$.

Suppose E is a thin $\Sigma_1^1(z)$ equivalence relation. Let Θ be some large ordinal such that V_Θ reflects the necessary statements to perform the proof below:

Let $N \prec V_\Theta$ be a countable elementary substructure with $z, \mathbb{P}, u, \tau \in N$. Let M be the Mostowski collapse of N and $\pi : N \to M$ be the Mostowski collapsing map. One may assume that for all $x \in X$, $\text{tc}(x) \subseteq \omega$. So π does not move reals or elements of X. In particular $\pi(z) = z$. Let $\mathbb{Q} = \pi(\mathbb{P})$, $v = \pi(u)$, and $\sigma = \pi(\tau)$.

By elementarity, M satisfies that for all $q \leq_{\mathbb{Q}} v$, there exists $q_0, q_1 \leq_{\mathbb{Q}} q$ such that $(q_0, q_1) \Vdash_{\mathbb{Q}\times\mathbb{Q}} \neg(\sigma_{\text{left}}\ E\ \sigma_{\text{right}})$.

Let $(D_n : n \in \omega)$ enumerate all the dense open sets in $\mathbb{Q} \times \mathbb{Q}$ of M. One may assume that $D_{n+1} \subseteq D_n$, by replacing D_n with $E_n = \bigcap_{m \leq n} D_m$. Next, a function $f : {}^{<\omega}2 \to \mathbb{Q}$ will be constructed with the following properties:

(1) If $s \subseteq t$, then $f(t) \leq_{\mathbb{Q}} f(s)$.

(2) For all $n \in \omega$, if $|s| = |t| = n$ and $s \neq t$, then $(f(s), f(t)) \in D_n$.

(3) For all $s \in {}^{<\omega}2$, $(f(s0), f(s1)) \Vdash_{\mathbb{Q}\times\mathbb{Q}} \neg(\sigma_{\text{left}}\ E\ \sigma_{\text{right}})$.

To construct this f: Let $f(\emptyset) = v$.

Suppose for all $s \in {}^{n}2$, $f(s)$ has been constructed with the above properties. For each $s \in {}^{n}2$, find some $q^{s0}, q^{s1} \leq (f(s), f(s))$ such that $(q^{s0}, q^{s1}) \Vdash_{\mathbb{Q}\times\mathbb{Q}} \neg(\sigma_{\text{left}}\ E\ \sigma_{\text{right}})$. Using the fact that D_{n+1} is dense open, find $\{r^t : t \in {}^{n+1}2\}$ with the property that for all $t \in {}^{n+1}2$, $r^t \leq_{\mathbb{Q}} q^t$ and for all $a, b \in {}^{n+1}2$ with $a \neq b$, $(r^a, r^b) \in D_{n+1}$. For $t \in {}^{n+1}2$, define $f(t) = r^t$.

For each $x \in {}^{\omega}2$, let $G_x := \{p \in \mathbb{Q} : (\exists n)(f(x \upharpoonright n) \leq_{\mathbb{Q}} p)\}$. If $x, y \in {}^{\omega}2$ and $x \neq y$, then $G_x \times G_y$ is $\mathbb{Q} \times \mathbb{Q}$-generic over M, using (2) and the assumption that for all $n \in \omega$, $D_{n+1} \subseteq D_n$. So let n be largest such that $x \upharpoonright n = y \upharpoonright n$. Let $s = x \upharpoonright n$. Without loss of generality, suppose $x(n) = 0$ and $y(n) = 1$. Then $f(s0) \in G_x$ and $f(s1) \in G_y$. Also $(f(s0), f(s1)) \Vdash_{\mathbb{Q}\times\mathbb{Q}} \neg(\sigma_{\text{left}}\ E\ \sigma_{\text{right}})$. By the forcing theorem applied in M, $M[G_x][G_y] \models \neg(\sigma[G_x]\ E\ \sigma[G_y])$. By Mostowski absoluteness, $V \models \neg(\sigma[G_x]\ E\ \sigma[G_y])$.

Define $\Phi : {}^{\omega}2 \to X$ by $\Phi(x) = \sigma[G_x]$. By an appropriate coding, Φ is a $\boldsymbol{\Delta}^1_1$ function. $\Phi[{}^{\omega}2]$ is a Σ^1_1 set of pairwise disjoint E-inequivalent elements. By the perfect set property for Σ^1_1 sets, there is a perfect set of pairwise E-inequivalent elements. This contradicts E being a thin equivalence relation. $\square$

Fact 3.6.7. *Let E be a thin Σ^1_1 equivalence relation on a Polish space X. Let $\mathbb{P}$ be some forcing and $\tau \in V^{\mathbb{P}}$ be such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \tau \in X$. Suppose $p \in D^E_\tau$. Let $M \prec H_\Theta$ be a countable elementary substructure with Θ sufficiently large and $\mathbb{P}, p, \tau \in M$. Then for all $G, H \in V$ such that $p \in G$, $p \in H$, and G and H are $\mathbb{P}$-generic over M, $V \models \tau[G]\ E\ \tau[H]$.*

Proof. This is due to Silver. See [4], Lemma 2.4.

Suppose G and H are any two such generics. Let K be such that it is $\mathbb{P}$-generic over $M[G][H]$. Then $M[G][K] \models \tau[G]\ E\ \tau[K]$ and $M[H][K] \models \tau[H]\ E\ \tau[K]$. By Mostowski absoluteness, $M[G][H][K] \models \tau[G]\ E\ \tau[H]$. By Mostowski absoluteness again, $V \models \tau[G]\ E\ \tau[H]$. $\square$

Theorem 3.6.8. *$\Sigma_1^{1\ thin} \to_I \{ev\}$ whenever I is a σ-ideal such that $\mathbb{P}_I$ is proper.*

Proof. Let $B \in \mathbb{P}_I$. Since $D^E_{\dot{x}_{\text{gen}}}$ from Fact 3.6.7 is dense, there exists some $B' \leq_{\mathbb{P}_I} B$ such that $B' \in D^E_{\dot{x}_{\text{gen}}}$. So $(B', B') \Vdash_{\mathbb{P}_I \times \mathbb{P}_I} (\dot{x}_{\text{gen}})\ E\ (\dot{x}_{\text{gen}})$. Let $M \prec H_\Theta$ with Θ sufficiently large and $\mathbb{P}, B' \in M$. By Fact 4.2.4, the set $C \subseteq B'$ of $\mathbb{P}_I$-generic over M reals is an I^+ $\boldsymbol{\Delta}^1_1$ set. For $x \in C$, if G_x denotes the $\mathbb{P}_I$-generic over M filter constructed from x, then $\dot{x}_{\text{gen}}[G_x] = x$. Note that for all $x \in C$, $B' \in G_x$. By Fact 3.6.7, for all $x, y \in C$, $V \models x\ E\ y$. Hence $E \restriction C = \text{ev} \restriction C$. $\square$

Note that in this result, E does not need to be an equivalence relation with all $\boldsymbol{\Delta}^1_1$ classes.

3.7 Positive Answer for $\boldsymbol{\Pi}^1_1$ Equivalence Relations

The variant of Question 3.2.7 for $\boldsymbol{\Pi}^1_1$ equivalence relations can be phrased as follows:

Question 3.7.1. Let $\boldsymbol{\Pi}_1^{1\ \boldsymbol{\Delta}^1_1}$ be the class of $\boldsymbol{\Pi}^1_1$ equivalence relations with all classes $\boldsymbol{\Delta}^1_1$ defined on $\boldsymbol{\Delta}^1_1$ subsets of Polish spaces. If I is a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper, then does $\boldsymbol{\Pi}_1^{1\ \boldsymbol{\Delta}^1_1} \to_I \boldsymbol{\Delta}^1_1$ hold?

A positive answer for the $\boldsymbol{\Pi}^1_1$ case follows from the same assumptions as the main question for Σ^1_1 in almost the exact same manner as above:

Lemma 3.7.2. *Let E be a $\Pi^1_1(z)$ equivalence relation on ${}^\omega\omega$. Then there exists $\boldsymbol{\Delta}^1_1$ relations E_α, for $\alpha < \omega_1$, with the property that if $\alpha < \beta$, then $E_\alpha \subseteq E_\beta$, $E = \bigcup_{\alpha<\omega_1} E_\alpha$, and there exists a club set $C \subseteq \omega_1$ such that for all $\alpha \in C$, E_α is an equivalence relation.*

Proof. Let T be a z-recursive tree on $\omega \times \omega \times \omega$ such that $(x, y) \in E \Leftrightarrow T^{(x,y)}$ is wellfounded. For each $\alpha < \omega_1$, define $E_\alpha := \{(x, y) : \text{rk}(T^{(x,y)}) < \alpha\}$. Each E_α is $\boldsymbol{\Delta}^1_1$. If $\alpha < \beta$, $E_\alpha \subseteq E_\beta$. $E = \bigcup_{\alpha<\omega_1} E_\alpha$.

Let C be the set of all α such that E_α is an equivalence relation. Increasing union of equivalence relations are equivalence relations so C is closed. Fix $\alpha < \omega_1$. The set $D = \{(x, x) : x \in {}^\omega\omega\}$ is Σ^1_1. So by the boundedness theorem, there exist some

$\delta < \omega_1$ such that $\mathrm{rk}(T^{(x,x)}) < \delta$ for all $x \in {}^\omega\omega$. Let $\beta_0 = \max\{\alpha, \delta\}$. Suppose β_n has been defined. The set $G = \{(x, y) : (y, x) \in E_{\beta_n}\}$ is Σ^1_1. By the boundedness theorem, there exists some $\beta' > \beta_n$ such that for all $(x, y) \in G$, $\mathrm{rk}(T^{(x,y)}) < \beta'$. The set $H = \{(x, z) : (\exists y)((x, y) \in E_{\beta_n} \wedge (y, z) \in E_{\beta_n}\}$ is Σ^1_1. Again by the boundedness theorem, there exists some $\beta_{n+1} > \beta'$ such that for all $(x, z) \in H$, $\mathrm{rk}(T^{(x,z)}) < \beta_{n+1}$. One has constructed an increasing sequence $\{\beta_n : n \in \omega\}$. Let $\beta = \sup\{\beta_n : n \in \omega\}$. Then E_β is an equivalence relation. C is unbounded. $\square$

Lemma 3.7.3. *Let E be a $\Pi^1_1(z)$ equivalence relation. Let $x, y \in {}^\omega\omega$ be such that $[x]_E$ is $\Sigma^1_1(y)$. Let δ be an ordinal such that $\omega_1^{y\oplus z} \leq \delta$ and E_δ is a equivalence relation. Then $[x]_E = [x]_{E_\delta}$.*

Proof. Define $E' \subseteq ({}^\omega\omega)^2$ by

$$a\ E'\ b \Leftrightarrow (a \in [x]_E \wedge b \in [x]_E) \vee (a = b).$$

E' is $\Sigma^1_1(y)$. $E' \subseteq E$. By the effective boundedness theorem, there exists a $\alpha < \omega_1^{y\oplus z} \leq \delta$ such that for all $(x, y) \in E'$, $\mathrm{rk}(T^{(x,y)}) < \alpha$. Hence $E' \subseteq E_\alpha$.

Since $E_\delta \subseteq E$, $[x]_{E_\delta} \subseteq [x]_E$. Also $[x]_E = [x]_{E'} \subseteq [x]_{E_\alpha} \subseteq [x]_{E_\delta}$. Therefore, $[x]_E = [x]_{E_\delta}$. $\square$

Note that x is not used as a parameter in the above lemma. This is in contrast to Lemma 3.4.2. This observation will be used later. (See Proposition 3.10.6.)

Lemma 3.7.4. *Let E be a $\Pi^1_1(z)$ equivalence relation. Let $\mathbb{P}$ be a forcing in M which adds a generic real. Choose Θ to be a regular cardinal greater than $|\mathbb{P}|^+$. Let $M \prec H_\Theta$ be a countable elementary substructure with $z, \mathbb{P} \in M$ and contains $|\mathbb{P}|^+$. Suppose for all g which are $\mathbb{P}$-generic over M, there exists a $y \in M[g]$ such that $V \models [g]_E$ is $\Sigma^1_1(y)$. Then there exists a countable ordinal α such that for all $\mathbb{P}$-generic over M reals g, $[g]_E = [g]_{E_\alpha}$.*

Proof. The proof is almost the same as the proof for Lemma 3.4.3 using Lemma 3.7.3 in place of Lemma 3.4.2. $\square$

These previous results can be used to give a positive answer for a specific Π^1_1 equivalence relation in ZFC.

Example 3.7.5. Let H be an equivalence relation on ${}^\omega\omega$ defined by $x\ H\ y$ if and only if $x \in L_{\omega_1^y}(y) \wedge y \in L_{\omega_1^x}(x)$.

H is a Π^1_1 equivalence relation with all classes countable. H is the equivalence relation of hyperarithmetic equivalence.

If I is a σ-ideal on ${}^\omega\omega$ with $\mathbb{P}_I$ proper, then $\{H\} \to_I \mathbf{\Delta}^1_1$.

Proof. Fix B an I^+ $\mathbf{\Delta}^1_1$ set. Choose $M \prec H_\Theta$ with Θ sufficiently large and $B, \mathbb{P}_I \in M$. By Fact 4.2.4, let $C \subseteq B$ be the set of $\mathbb{P}_I$-generic over M elements in B. Let $x \in C$. ω_1^x is a countable ordinal in $M[x]$. In $M[x]$, $L_{\omega_1^x}(x)$ is countable. $[x]_E^{M[x]} \subseteq L_{\omega_1^x}(x)$. Therefore, in $M[x]$, there is a function $f : \omega \to {}^\omega\omega$ such that f enumerates $[x]_H^{M[x]}$. By absoluteness, $[x]_H = [x]_H^{M[x]}$. So $[x]_H$ is $\Delta^1_1(f)$ and $f \in M[x]$. By Lemma 3.7.4, there is some countable ordinal α such that $[x]_E = [x]_{E_\alpha}$ for all $x \in C$. So $E \restriction C = E_\alpha \restriction C$. $E \restriction C$ is $\mathbf{\Delta}^1_1$. $\square$

Definition 3.7.6. Let E be a $\Pi^1_1(z)$ equivalence relation. Define the set D by

$$(x, T) \in D \Leftrightarrow (T \text{ is a tree on } \omega \times \omega) \wedge (\forall y)(y\ E\ x \Leftrightarrow T^y \notin \mathrm{WF}).$$

D is $\Pi^1_2(z)$.

Theorem 3.7.7. *Assume all $\mathbf{\Pi}^1_2$ sets are universally Baire and $\mathbf{\Pi}^1_4$-generic absoluteness holds. Let I be a σ-ideal such that $\mathbb{P}_I$ is proper. Then $\mathbf{\Pi}^1_1{}^{\mathbf{\Delta}^1_1} \to_I \mathbf{\Delta}^1_1$.*

Proof. The proof is the same as Theorem 3.4.8 with the required change. $\square$

A similar argument using iterable structures as in the Σ^1_1 case yields a positive answer from a more precise assumption with lower consistency strength.

Proposition 3.7.8. *Let E be a $\Pi^1_1(z)$ equivalence relation. There is a $\Pi^1_3(z)$ formula $\varpi(v)$ in free variable v such that:*

Let $x \in {}^\omega\omega$. If $(x \oplus z)^\sharp$ exists, then the statement "$[x]_E$ is $\mathbf{\Delta}^1_1$" is equivalent to $\varpi(x)$.

Assume for all $r \in {}^\omega\omega$, $r^\sharp$ exists. The statement "all E-classes are $\mathbf{\Delta}^1_1$" is equivalent to $(\forall x)\varpi(x)$. In particular, this statement is $\Pi^1_3(z)$.

Proof. Assume for simplicity, E is a Π^1_1 equivalence relation on ${}^\omega\omega$. Let T be a tree on $\omega \times \omega \times \omega$ such that

$$(x, y) \in E \Leftrightarrow T^{x,y} \text{ is wellfounded.}$$

Let $\varpi(v)$ be the statement:

$$(\forall y)(y = v^\sharp \Rightarrow$$

$$\text{“}1_{\mathrm{Coll}(\omega,<c_1)} \Vdash_{\mathrm{Coll}(\omega,<c_1)} (\exists c)(c \in \mathrm{WO} \wedge (\forall y)((y\ E\ v) \Rightarrow \mathrm{rk}(T^{v,y}) < \mathrm{ot}(c)))\text{”} \in y).$$

The rest of the argument is the same as in Proposition 3.4.12. □

Definition 3.7.9. Let I be a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper. Let μ_E^I be $\tau_{\neg\varpi}^{\mathbb{P}_I}$ from Fact 3.4.17.

Definition 3.7.10. Let I be a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is proper. Consider the formula “$(\exists y)([\dot{x}_{\mathrm{gen}}]_E$ is $\Sigma_1^1(y))$”. Write it as $(\exists y)\psi(y)$. By Fact 3.4.17, let σ_E^I be $\tau_\psi^{\mathbb{P}_I}$.

Definition 3.7.11. Suppose I is a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is proper. Let $E \in \mathbf{\Pi}_1^{1\,\mathbf{\Delta}_1^1}$. Define $\chi_E^I = \langle \mathbb{P}_I, \mu_E^{\mathbb{P}_I}, \sigma_E^{\mathbb{P}_I}\rangle$.

Theorem 3.7.12. *Suppose I is a σ ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is proper. If for all $x \in {}^\omega 2$, $x^\sharp$ exists and $(\chi_E^I)^\sharp$ exists for all $E \in \mathbf{\Pi}_1^{1\,\mathbf{\Delta}_1^1}$, then $\mathbf{\Pi}_1^{1\,\mathbf{\Delta}_1^1} \rightarrow_I \mathbf{\Delta}_1^1$.*

Proof. This is similar to Theorem 3.4.22. □

Corollary 3.7.13. *If $z^\sharp$ exists for all $z \in H_{(2^{\aleph_0})^+}$, then $\mathbf{\Pi}_1^{1\,\mathbf{\Delta}_1^1} \rightarrow_I \mathbf{\Delta}_1^1$ for all σ-ideal I such that $\mathbb{P}_I$ is proper.*

3.8 $\mathbf{\Pi}_1^1$ Equivalence Relations with Thin or Countable Classes

The preservation of the statement “all classes are $\mathbf{\Delta}_1^1$” played an important role in the consistency results above. Next, one will consider $\mathbf{\Pi}_1^1$ equivalence relations which are very sensitive to set theoretic assumptions and generic extensions.

Definition 3.8.1. Let X be a Polish space. $A \subseteq X$ is thin if and only if it does not contain a perfect set.

Fact 3.8.2. *For each $z \in {}^\omega 2$, define $Q_z := \{x \in {}^\omega 2 : x \in L_{\omega_1^{z\oplus x}}(z)\}$. Q_z is the largest thin $\Pi_1^1(z)$ set in the sense that if S is a thin $\Pi_1^1(z)$ set, then $S \subseteq Q_z$. Moreover, for each $\alpha < \omega_1^{L[z]}$, there exists some $x \in Q_z$ such that $\alpha < \omega_1^x$. Therefore, if $\omega_1^{L[z]} = \omega_1$, then Q_z is an uncountable thin $\Pi_1^1(z)$ set. It is consistent that $\mathbf{\Pi}_1^1$ sets do not have the perfect set property.*

Proof. See [24], pages 83-87. One will give the $\Pi_1^1(z)$ definition to get a better understanding of what Q_z is:

$$x \in Q_z \Leftrightarrow (\forall M)((M \text{ is an } \omega\text{-model of } \mathsf{KP} \wedge z \in M \wedge x \in M) \Rightarrow M \models x \in L[z]).$$

So $x \in Q_z$ if and only if $L_{\omega_1^{z\oplus x}}(z \oplus x) = L_{\omega_1^{z\oplus x}}(z)$. Or put another way, the smallest admissible set containing $z \oplus x$ is a model of $V = L[z]$. Certainly $Q_z \subseteq L[z]$. So Q_z can also be thought of as the set of reals that appear in $L[z]$ very quickly in the sense that $x \in Q_z$ if and only if the first ordinal α such that $L_\alpha(z \oplus x)$ is admissible is also the first z-admissible ordinal α such that $x \in L_\alpha[z]$. □

Now one can give a simple example of an equivalence relation E, a model of ZFC, and forcing which does not preserve the statement "all E classes are $\mathbf{\Delta}^1_1$." Note that this statement is $\mathbf{\Pi}^1_4$ so it will be preserved if the universe satisfies $\mathbf{\Pi}^1_4$-generic absoluteness. The desired example will necessarily have to reside in a universe with weak large cardinals.

Definition 3.8.3. ω_1 is inaccessible to reals if and only if for all $x \in {}^\omega\omega$, $L[x] \models (\omega_1^V$ is inaccessible) if and only if for all $x \in {}^\omega\omega$, $\omega_1^{L[x]} < \omega_1$.

Proposition 3.8.4. *Let E and F be equivalence relations on $({}^\omega\omega)^2$ defined by*

$$(a, x)\ E\ (b, y) \Leftrightarrow (a = b) \wedge (x, y \in Q_a \vee x = y)$$

$$(a, x)\ F\ (b, y) \Leftrightarrow (a = b) \wedge (x, y \notin Q_a \vee x = y).$$

E is a Π^1_1 equivalence relation and F is a Σ^1_1 equivalence relation.

Let κ be an inaccessible cardinal in L. Let $G \subseteq Coll(\omega, <\kappa)$ be $Coll(\omega, <\kappa)$-generic over L. Then $L[G] \models E$ and F have all classes $\mathbf{\Delta}^1_1$. Let $g \subseteq Coll(\omega, \kappa)$ be $Coll(\omega, \kappa)$-generic over $L[G]$, then $L[G][g] \models$ not all E and F classes are $\mathbf{\Delta}^1_1$.

Proof. The formula provided in the proof of Fact 3.8.2 shows that the formula "$x \in Q_z$" is Π^1_1 in variables x and z. From this, it follows that E and F are Π^1_1 and Σ^1_1, respectively.

In $L[G]$, ω_1 is inaccessible to reals ([28] Theorem 8.20). For each (a, b), $[(a, b)]_E$ is either a singleton or in bijection with Q_a. Since $Q_a \subseteq ({}^\omega\omega)^{L[a]}$, in all cases, $[(a, b)]_E$ is countable and hence $\mathbf{\Delta}^1_1$. F-classes are then singletons or complements of countable sets. All F-classes are $\mathbf{\Delta}^1_1$.

$\text{Coll}(\omega, <\kappa) * \dot{\text{Coll}}(\omega, \check{\kappa})$ is a forcing (in L) of size κ which collapses κ to ω. Such forcings are forcing equivalent to $\text{Coll}(\omega, \kappa)$ by [16] Lemma 26.7. Let $h \subseteq \text{Coll}(\omega, \kappa)$ which is $\text{Coll}(\omega, \kappa)$-generic over L with $L[h] = L[G][g]$. $L[G][g] \models \omega_1^{L[h]} = \omega_1$. ω_1 is not inaccessible to reals in $L[G][g]$. Moreover, $[(h, h)]_E$ is not $\mathbf{\Delta}^1_1$ in $L[G][g]$ as it is an uncountable thin set and the perfect set property holds for Σ^1_1 sets. Similarly, F has a class which is not $\mathbf{\Delta}^1_1$. □

In the previous example, in $L[G]$, $\mathrm{Coll}(\omega, \omega_1^{L[G]}) = \mathrm{Coll}(\omega, \kappa)$ is not a proper forcing. One may ask whether there is an Σ^1_1 or $\mathbf{\Pi}^1_1$ equivalence relation with all classes $\mathbf{\Delta}^1_1$ and a proper forcing coming from a σ-ideal on a Polish space such that in the induced generic extension, the statement that "all classes are $\mathbf{\Delta}^1_1$" is false. Sy-David Friedman's forcing to code subsets of ω_1 is an $\aleph_1$-c.c. forcing which can be repesented as an idealized forcing which (like in the proof of the above proposition) adds a real r such that $L[A][r] = L[r]$. The two equivalence relations from the above proposition can be used with this forcing to give a similar result. See Section 3.10 for more details about this forcing.

For Σ^1_1 equivalence relations with all classes countable, Proposition 3.3.2 shows that the main question has a positive answer without additional set theoretic assumptions. There were two important aspects of the proof. First, the countability of all classes of a Σ^1_1 equivalence relation is Σ^1_1 and hence remains true in all generic extensions. This fact is used to give an enumeration f of $[x]_E$ in $M[x]$. Secondly, the statement that f enumerates $[x]_E$ is $\mathbf{\Pi}^1_1$ and hence absolute between (the countable model) $M[x]$ and V.

One can ask the same question for $\mathbf{\Pi}^1_1$ equivalence relations with all classes countable. However, the above proof can not be applied. First, the countability of all classes of a $\mathbf{\Pi}^1_1$ equivalence relation is $\mathbf{\Pi}^1_4$. Secondly, the statement that some function f enumerates $[x]_E$ is $\mathbf{\Pi}^1_2$; hence, it does not necessarily persist from $M[x]$ to V.

The $\mathbf{\Pi}^1_1$ equivalence relations where these issues are most perceptible are the equivalence relations E with all classes countable but for some x, $L[x] \models [x]_E$ is uncountable . It is not provable that all E-classes are countable; however, all the E-classes are thin.

Proposition 3.8.5. *Let A be a $\mathbf{\Pi}^1_1$ set. The statement "A is thin" is $\mathbf{\Pi}^1_2$. Let E be a $\mathbf{\Pi}^1_1$ equivalence relation. The statement "all E-classes are thin" is $\mathbf{\Pi}^1_2$. Both of these statements are absolute to generic extensions.*

Proof.

$$(\forall T)(T \text{ is perfect tree } \Rightarrow ((\exists x)((\forall n)(x \upharpoonright n \in T) \wedge x \notin A)))$$

$$(\forall x)(\forall T)(T \text{ is perfect tree } \Rightarrow ((\exists y)((\forall n)(y \upharpoonright n \in T) \wedge \neg(x \mathrel{E} y))))$$

These two $\mathbf{\Pi}^1_2$ formulas are equivalent to "A is thin" and "all E classes are thin", respectively. □

Definition 3.8.6. Let $\mathbf{\Pi}^{1\ \aleph_0}_1$ denote the class of all $\mathbf{\Pi}^1_1$ equivalence relations with all classes countable defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces. Let $\mathbf{\Pi}^{1\ \mathrm{thin}}_1$ denote the class of all $\mathbf{\Pi}^1_1$ equivalence relations with all classes thin defined on $\mathbf{\Delta}^1_1$ subsets of Polish spaces.

Theorem 3.8.7. *If $\omega_1^L < \omega_1$, then $\Pi^{1\ thin}_1 \to_{I_{meager}} \mathbf{\Delta}^1_1$.*

Proof. Let E be a Π^1_1 equivalence relation. Fix a non-meager $\mathbf{\Delta}^1_1$ set B. Let $\mathbb{C}$ denote Cohen forcing, i.e., finite partial functions from ω into 2. Let U be the set of all constructible dense subsets of $\mathbb{C}$. Since $\omega_1^L < \omega_1$, $|U| = \aleph_0$.

For a sufficiently large cardinal Θ, let $M \prec H_\Theta$ be a countable elementary substructure with $B, \mathbb{P}_{I_{\mathrm{meager}}}, \omega_1^L, U \in M$, $\omega_1^L \subseteq M$, and $U \subseteq M$. By Fact 4.2.4, let C be the set of $\mathbb{P}_{I_{\mathrm{meager}}}$-generic over M reals in B.

Take $x \in C$. Since Cohen forcing $\mathbb{C}$ and $\mathbb{P}_{I_{\mathrm{meager}}}$ are forcing equivalent and $U \subseteq M$, x is also $\mathbb{C}$-generic over L. Since $\mathbb{C}$ satisfies the $\aleph_1$-chain condition, $\omega_1^{L[x]} = \omega_1^L < \omega_1$. Since ω_1^L is countable in M, $L_{\omega_1^{L[x]}}[x] = L_{\omega_1^L}[x] \subseteq M[x]$ and is countable there. Since $[x]_E$ is thin, $[x]_E \subseteq L_{\omega_1^{L[x]}}[x]$. In $M[x]$, there is an enumeration $f : \omega \to ([x]_E)^{M[x]}$. The claim is that $([x]_E)^{M[x]} = [x]_E$: since $([x]_E)^V \subseteq L_{\omega_1^{L[x]}}[x] \subseteq M[x]$, $M[x] \models y\ E\ x \Leftrightarrow (L[x])^{M[x]} \models y\ E\ x \Leftrightarrow L[x] \models y\ E\ x \Leftrightarrow V \models y\ E\ x$, by Mostowski absoluteness.

Therefore, in V, $[x]_E$ is $\Delta^1_1(f)$ and $f \in M[x]$. By Lemma 3.7.4, there is some countable $\alpha < \omega_1$, such that $E \restriction C = E_\alpha \restriction C$. The latter is $\mathbf{\Delta}^1_1$. □

If ω_1 is inaccessible to reals, then $\mathbf{\Pi}^{1\ \mathrm{thin}}_1 = \mathbf{\Pi}^{1\ \aleph_0}_1$. Familiar models that satisfy ω_1 is inaccessible to reals include generic extensions of the Lévy collapse of an inaccessible cardinal to ω_1. Next, one will consider the main question for $\mathbf{\Pi}^{1\ \aleph_0}_1$ in models of this type and obtain some improved consistency results.

The main large cardinal useful here is the remarkable cardinal isolated in [30] to understand absoluteness for proper forcing in $L(\mathbb{R})$. It is a fairly weak large cardinal. Its existence is consistent relative to ω-Erdös cardinals. If $0^\sharp$ exists, then all Silver's indiscernibles are remarkable cardinals in L. Also if κ is a remarkable cardinal, then κ is a remarkable cardinal in L.

Definition 3.8.8. ([30] Definition 1.1) A cardinal κ is a remarkable cardinal if and only if for all regular cardinals $\theta > \kappa$, there exists $M, N, \pi, \sigma, \bar{\kappa}$ and $\bar{\theta}$ such that the following holds:

(i) M and N are countable transitive sets.

(ii) $\pi : M \to H_\theta$ is an elementary embedding.

(iii) $\pi(\bar{\kappa}) = \kappa$.

(iv) $\sigma : M \to N$ is an elementary embedding with $\mathrm{crit}(\sigma) = \bar{\kappa}$.

(v) $\bar{\theta} = \mathrm{ON} \cap M$, $\sigma(\bar{\kappa}) > \bar{\theta}$, and $N \models \theta$ is a regular cardinal.

(vi) $M \in N$ and $N \models M = H_{\bar{\theta}}$.

Fact 3.8.9. *(Schindler) Let κ be a remarkable cardinal in L. Let $G \subseteq Coll(\omega, <\kappa)$ be $Coll(\omega, <\kappa)$-generic over L. Let $\mathbb{P} \in L[G]$ be a proper forcing. Let $H \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter over $L[G]$. If $x \in ({}^\omega\omega)^{L[G][H]}$, then there exists a forcing $\mathbb{Q} \in L_\kappa$ and a $K \subseteq \mathbb{Q}$ in $L[G][H]$ which is $\mathbb{Q}$-generic over L and $x \in L[K]$.*

Proof. See [30], Lemma 2.1. □

Theorem 3.8.10. *Let κ be a remarkable cardinal in L. Let $G \subseteq Coll(\omega, <\kappa)$ be $Coll(\omega, <\kappa)$ generic over L. In $L[G]$, if I is a σ-ideal such that $\mathbb{P}_I$ is proper, then $\mathbf{\Pi}^1_1{}^{\aleph_0} \to_I \mathbf{\Delta}^1_1$.*

Proof. Working in $L[G]$, let E be an equivalence relation in $\mathbf{\Pi}^1_1{}^{\aleph_0}$. For simplicity, assume E is Π^1_1 (otherwise one should include the parameter defining E in all the discussions below). In particular, all E-classes are thin, and this statement will be absolute to all generic extensions.

Let B be an I^+ $\mathbf{\Delta}^1_1$ set. Let $M \prec H_\Theta$ be a countable elementary substructure with Θ a sufficiently large cardinal and $B, \mathbb{P}_I, G \in M$. $H_\Theta = L_\Theta[G]$. Therefore, $M = L^M[G]$. Note that from the point of view of M, G is L^M generic for $\mathrm{Coll}(\omega, <\kappa)^M$. Using Fact 4.2.4, let $C \subseteq B$ be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic over M reals in B.

Fix $x \in C$. Applying Fact 3.8.9 in $M[x] = L^M[G][x]$, there exist some $\mathbb{Q} \in (L_\kappa)^M$ and $K \subseteq \mathbb{Q}$ in $M[x]$ which is $\mathbb{Q}$-generic over L^M such that $x \in L[K]$. Since M satisfies $\mathbb{Q} \in L_\kappa$ and κ is a remarkable cardinal (in particular inaccessible) in L, M thinks that $\mathscr{P}^L(\mathbb{Q}) \in L_\kappa$. Since $M = L[G]$, $M \models \mathscr{P}^L(\mathbb{Q})$ is countable. Let $f : \omega \to \mathscr{P}^L(\mathbb{Q})$ be a function in M such that M thinks it surjects onto $\mathscr{P}^L(\mathbb{Q})$. Since $M \prec (H_\Theta)^{L[G]}$, f really is a surjection onto $\mathscr{P}^L(\mathbb{Q})$ in the real universe $L[G]$. This establishes that $\mathscr{P}^L(\mathbb{Q}) \subseteq M$. In particular, $\mathscr{P}^L(\mathbb{Q}) \subseteq L^M$. This and the fact that K is generic over L^M imply that K is $\mathbb{Q}$-generic over the real L. Since $\mathbb{Q} \in L_\kappa$, all cardinals of L greater than $|\mathbb{Q}|$ are preserved in $L[K]$.

Therefore, $\omega_1^{L[x]} \leq \omega_1^{L[K]} \leq (|\mathbb{Q}|^+)^L$. Since $\mathbb{Q} \in M$ and $M \models \mathbb{Q} \in L$, there is some ordinal α such that $M \models L \models |\mathbb{Q}|^+ = \alpha$. Because $M \prec H_\Theta$, the real universe $L[G]$ satisfies $L \models |\mathbb{Q}|^+ = \alpha$. This establishes that $(|\mathbb{Q}|^+)^L \in M$. Since $\mathbb{Q} \in L_\kappa$ and κ is inaccessible, $(|\mathbb{Q}|^+)^L < \kappa$. Since $M = L^M[G]$, $(|\mathbb{Q}|^+)^L$ is a countable ordinal in M. As shown above, $\omega_1^{L[x]} < (|\mathbb{Q}|^+)^L$, so in M, $\omega_1^{L[x]}$ is countable. Since $[x]_E$ is thin, $[x]_E \subseteq ({}^\omega\omega)^{L[x]}$. As $\omega_1^{L[x]}$ is countable in $M[x]$, $M[x] \models [x]_E$ is countable. There exists some surjection $h : \omega \to ([x]_E)^{M[x]}$. The claim is that $([x]_E)^{L[G]} = ([x]_E)^{M[x]}$: since $([x]_E)^{L[G]} \subseteq L_{\omega_1^{L[x]}}[x] \subseteq M[x]$, $M[x] \models y\ E\ x \Leftrightarrow (L[x])^{M[x]} \models y\ E\ x \Leftrightarrow (L[x])^{L[G]} \models y\ E\ x \Leftrightarrow L[G] \models y\ E\ x$, by Mostowski absoluteness.

Therefore, in $L[G]$, $[x]_E$ is $\Delta_1^1(h)$ and $h \in M[x]$. By Lemma 3.7.4, there is a countable $\alpha < \omega_1^{L[G]}$, such that $E \upharpoonright C = E_\alpha \upharpoonright C$. $E_\alpha \upharpoonright C$ is $\boldsymbol{\Delta}_1^1$. $\square$

It is now known that much more holds: in the model $L[G]$ of Theorem 3.8.10, the main question has a positive answer for all equivalence relations $E \in L(\mathbb{R})$ with all classes $\boldsymbol{\Delta}_1^1$. See [6]. This shows that the consistency strength of a remarkable cardinal is an upper bound on the consistency strength of a positive answer to the main question for projective equivalence relations with all classes $\boldsymbol{\Delta}_1^1$.

Using some well-known results of Kunen, a similar proof shows that the consistency of $\boldsymbol{\Pi}_1^{1\ \aleph_0} \to_I \boldsymbol{\Delta}_1^1$ for I such that $\mathbb{P}_I$ is $\aleph_1$-c.c. follows from the consistency of a weakly compact cardinal.

Fact 3.8.11. *(Kunen) Let κ be a weakly compact cardinal. Let $\mathbb{P}$ be a κ-c.c. forcing. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. If $x \in H_\kappa^{V[G]}$, then there exists a forcing $\mathbb{Q} \in V_\kappa$ and a $K \subseteq \mathbb{Q}$ which is generic over V such that $x \in V[K]$.*

Proof. This is due to Kunen. See [14], Lemma 5.3 for a proof. $\square$

Theorem 3.8.12. *Let κ be a weakly compact cardinal in L. Let $G \subseteq Coll(\omega, <\kappa)$ be $Coll(\omega, <\kappa)$-generic over L. In $L[G]$, if I is a σ-ideal such that $\mathbb{P}_I$ is $\aleph_1$-c.c., then $\boldsymbol{\Pi}_1^{1\ \aleph_0} \to_I \boldsymbol{\Delta}_1^1$.*

Proof. Let $\dot{\mathbb{P}}_I$ be a name for $\mathbb{P}_I$ in $L[G]$. $\mathrm{Coll}(\omega, <\kappa)$ satisfies the κ-chain condition. Since $\aleph_1^{L[G]} = \kappa$, for some $p \in G$, $p \Vdash_{\mathrm{Coll}(\omega,<\kappa)} \dot{\mathbb{P}}_I$ satisfies the $\check{\kappa}$-chain condition. By considering the forcing of conditions below p, one may as well assume $p = 1_{\mathrm{Coll}(\omega,<\kappa)}$. Then $\mathrm{Coll}(\omega, <\kappa) * \dot{\mathbb{P}}_I$ satisfies the κ-chain condition. Now use Fact 3.8.11 and finish the proof much like in Theorem 3.8.10. $\square$

3.9 $\boldsymbol{\Delta}^1_2$ Equivalence Relations with all Classes $\boldsymbol{\Delta}^1_1$

One can ask the same question for $\boldsymbol{\Delta}^1_2$ equivalence relations with all classes $\boldsymbol{\Delta}^1_1$: If E is a $\boldsymbol{\Delta}^1_2$ equivalence relation with all classes $\boldsymbol{\Delta}^1_1$ and I is a σ-ideal such that $\mathbb{P}_I$ is proper, does $\{E\} \to_I \boldsymbol{\Delta}^1_1$ hold?

It will be shown that in L there is a $\boldsymbol{\Delta}^1_2$ equivalence relation E_L such that $\{E_L\} \to_I \boldsymbol{\Delta}^1_1$ does not hold for any σ-ideal I.

Definition 3.9.1. Let E_L be the equivalence relation defined on ${}^\omega 2$ by $x\ E_L\ y$ if and only if

$$(\forall A)((A \text{ is a well-founded model of } \mathsf{KP} + \mathsf{V} = \mathsf{L} + \mathsf{INF}) \Rightarrow (x \in A \Leftrightarrow y \in A))$$

where INF asserts that ω exists. E_L is a Π^1_2 equivalence relation.

Here A is considered as a structure with domain ω. As A thinks ω exists, there is an isomorphic copy of ω in A. The statement "$x \in A$" should be understood using this copy of ω in A.

Rather than $\mathsf{KP} + \mathsf{V} = \mathsf{L}$, one could also use some $\Upsilon + \mathsf{V} = \mathsf{L}$ where Υ is a large enough fragment of ZFC to prove all the familiar properties about perfect sets and constructibility needed below. If one is willing to assume that there exists a transitive model of ZFC, then one can replace the above with $\mathsf{ZFC} + \mathsf{V} = \mathsf{L}$ and be in the familiar setting.

Definition 3.9.2. Assume $V = L$. Let $\iota : {}^\omega 2 \to \omega_1$ be the function such that $\iota(x)$ is the smallest admissible ordinal α such that $x \in L_\alpha$.

Proposition 3.9.3. *For all $x, y \in {}^\omega 2$, $x\ E_L\ y$ if and only if $\iota(x) = \iota(y)$.*

Proof. Assume $\iota(x) = \iota(y)$. Let A be a wellfounded model of $\mathsf{KP} + \mathsf{V} = \mathsf{L}$ such that $x \in A$. There is some β such that L_β is the Mostowski collapse of A. L_β is transitive and satisfies KP, so it is an admissible set. β is an admissible ordinal. $\iota(x) \le \beta$. $y \in L_{\iota(y)} = L_{\iota(x)} \subseteq L_\beta$. So $y \in A$. Hence $x \in A$ implies $y \in A$. By a symmetric argument, $y \in A$ implies $x \in A$. $x\ E_L\ y$.

Assume $x\ E_L\ y$. Suppose $\alpha < \omega_1$ with $L_\alpha \models \mathsf{KP}$ and $x \in L_\alpha$. Since L_α is countable, there is a countable structure A with domain ω isomorphic to L_α. $A \models \mathsf{KP}$, A is an ω-model, and $x \in A$. $x\ E_L\ y$ implies that $y \in A$. Therefore, $y \in L_\alpha$. Hence $\iota(x) \le \iota(y)$. By a symmetric argument, $\iota(y) \le \iota(x)$. $\iota(x) = \iota(y)$. $\square$

Earlier drafts of this paper only asserted that E_L was Π^1_2. Drucker observed that a very similar equivalence relation to E_L was actually Δ^1_2:

Proposition 3.9.4. *(Drucker) E_L is Δ^1_2.*

Proof. The claim is that

$$x\, E_L\, y \Leftrightarrow H_{\aleph_1} \models (\exists M)((M \text{ is transitive}) \wedge (x, y \in M) \wedge (M \models \mathsf{KP} + \mathsf{V} = \mathsf{L}) \wedge (M \models \psi(x, y)))$$

where

$$\psi(x, y) \Leftrightarrow (\forall A)((A \text{ is transitive} \wedge A \models \mathsf{KP} + \mathsf{V} = \mathsf{L}) \Rightarrow (x \in A \Leftrightarrow y \in A)).$$

To see this: ($\Rightarrow$) By Proposition 3.9.3, $\iota(x) = \iota(y)$. Then $H_{\aleph_1}$ satisfies the above formula using $L_{\iota(x)}$.

($\Leftarrow$) Suppose $\neg(x\, E_L\, y)$. Let M witness the negation of the statement from Defintion 3.9.1. Without loss of generality, $\iota(x) < \iota(y)$. By Δ_1 absoluteness, if $H_{\aleph_1}$ thinks M is transitive and satisfies $\mathsf{KP} + \mathsf{V} = \mathsf{L}$, then M really is transitive and satisfies $\mathsf{KP} + \mathsf{V} + \mathsf{L}$. So $M = L_\alpha$ for some $\alpha < \omega_1$. Since $x, y \in M = L_\alpha$, $\alpha \geq \iota(y)$. Then $M \models \neg(\psi(x, y))$ since $L_{\iota(x)} \in L_\alpha = M$, $x \in L_{\iota(x)}$, and $y \notin L_{\iota(x)}$.

$\psi(x, y)$ is a first order formula in the language of set theory. First order satisfaction is Δ_1. The above shows that $x\, E_L\, y$ is equivalent to a formula which is Σ_1 over $H_{\aleph_1}$. Hence E_L is Σ^1_2. ◻

Assuming $\mathsf{V} = \mathsf{L}$, Proposition 3.9.3 associates each E_L class with a countable ordinals. This suggests that E_L is thin. However, the complexity of the statement that a particular $\mathbf{\Delta}^1_2$ equivalence relation is thin is beyond the scope of Shoenfield absoluteness. Therefore the usual argument of passing to a forcing extension satisfying $\neg\mathsf{CH}$ will not work. Morever, E_L looks quite different in models that do not satisfy $\mathsf{V} = \mathsf{L}$. Thinness will be proved more directly.

The following fact will be useful. It implies that if $\alpha < \beta$ are admissible ordinals and a new real appears in L_β which was not in L_α, then L_α is countable from the view of L_β.

Fact 3.9.5. *If $\omega < \alpha < \beta$ are admissible ordinals and $({}^\omega 2)^{L_\beta} \not\subseteq L_\alpha$, then there is an $f \in L_\beta$ such that $f : \omega \to \alpha$ is a surjection. In particular, $L_\beta \models |L_\alpha| = \aleph_0$.*

Proof. This is essentially a result of Putnam. Below, a brief sketch of the proof is given using some elementary fine structure theory. (See [17], [29], or [10].)

Note that if α is admissible, then $\omega \cdot \alpha = \alpha$. [17] Lemma 2.15 shows that $L_\alpha = J_\alpha$, if α is admissible.

Now suppose $\alpha < \beta$ are admissible ordinals. Since $({}^\omega 2)^{J_\beta} \nsubseteq L_\alpha$, there is some $x \in \mathscr{P}(\omega)$ such that $x \in J_\beta$ and $x \notin J_\alpha$. Then there is some $\alpha < \gamma < \beta$ and some $n \in \omega$ such that x is Σ_n definable over J_γ but not in J_γ. [17] Lemma 3.4 (i) shows that all J_γ are Σ_n-uniformizable for all n. Then [17] Lemma 3.1 can be applied to show that there is a Σ_n in J_γ surjection f of ω onto J_γ. f is definable in J_γ and so $f \in J_{\gamma+1} \subseteq J_\beta$. Since $J_\alpha \subseteq J_\gamma$, using this f, one can construct a surjection in J_β from ω onto J_α. □

Lemma 3.9.6. *Suppose α is an ordinal such that there exists an $x \in {}^\omega 2$ with $\iota(x) = \alpha$, then there exists a greatest $\beta < \alpha$ such that there exists a $y \in {}^\omega 2$ with $\iota(y) = \beta$.*

Proof. Fix an x such that $\iota(x) = \alpha$. If the result was not true, then there exists a sequences of reals $(x_n : n \in \omega)$ such that $\iota(x_n) < \alpha$ and $\alpha = \lim_{n\in\omega} \iota(x_n)$. $L_{\iota(x)} = \bigcup_{n\in\omega} L_{\iota(x_n)}$. $x \in L_{\iota(x)}$. This implies $x \in L_{\iota(x_n)}$ for some $n \in \omega$. This contradicts $\iota(x) = \alpha$ being the smallest admissible ordinal γ such that $x \in L_\gamma$. □

Proposition 3.9.7. ($\mathsf{V} = \mathsf{L}$) *E_L is a thin equivalence relation.*

Proof. Let $T \subseteq {}^{<\omega}2$ be an arbitrary perfect tree. Let $\alpha = \iota(T)$. L_α satisfies that there are no functions from ω taking reals as images which enumerate all paths through T. By Lemma 3.9.6, let $\beta < \alpha$ be greatest such that there is a y with $\iota(y) = \beta$. By Fact 3.9.5, $L_\alpha \models |L_\beta| = \aleph_0$. However, since L_α satisfies no function from ω into the reals enumerates the paths through T, there exists $v, w \in L_\alpha$ such that in L_α, v and w are paths through T and $v, w \notin L_\beta$. By the choice of β, $\iota(v) = \iota(w) = \alpha$. By Proposition 3.9.3, $v \; E_L \; w$. By Δ_1-absoluteness, $v, w \in [T]$. It has been shown that every perfect set has E_L equivalent elements. □

Proposition 3.9.8. *If E is a thin equivalence relation with all classes countable, then for any σ-ideal I, $\{E\} \rightarrow_I \boldsymbol{\Delta}^1_1$ fails.*

Proof. Suppose there exists some $\boldsymbol{\Delta}^1_1$ I^+ B such that $E \restriction B$ is $\boldsymbol{\Delta}^1_1$. By Silver's Dichotomy for $\boldsymbol{\Pi}^1_1$ equivalence relations, either $E \restriction B$ has countably many classes or a perfect set of pairwise E-inequivalent elements. The former is not possible since this would imply the I^+ set B is a countable union of countable sets. The latter is also not possible since E is thin. Contradiction. $\square$

Theorem 3.9.9. *($\mathsf{V} = \mathsf{L}$) For any σ-ideal I on ${}^\omega 2$, $\{E_L\} \to_I \mathbf{\Delta}^1_1$ fails.*

In particular in L, ${\mathbf{\Delta}^1_2}^{\mathbf{\Delta}^1_1} \to_I \mathbf{\Delta}^1_1$ for σ-ideal I with $\mathbb{P}_I$ proper is not true. (${\mathbf{\Delta}^1_2}^{\mathbf{\Delta}^1_1}$ is the class of $\mathbf{\Delta}^1_2$ equivalence relation with all classes $\mathbf{\Delta}^1_1$.)

Proof. E_L is thin and has all classes countable. Use Proposition 3.9.8. $\square$

A positive answer to the main question for $\mathbf{\Delta}^1_2$ equivalence relations with all classes $\mathbf{\Delta}^1_1$ is now known to follow from large cardinals. In fact, [8] has shown that if there is a measureable cardinal above infinity many Woodin cardinals then a positive answer holds for all equivalence relations in $L(\mathbb{R})$ with all classes $\mathbf{\Delta}^1_1$.

3.10 Conclusion

This last section will put the results of this paper into perspective. Some questions will be raised and some speculations will be made.

Large cardinal assumptions were used throughout the paper to obtain a positive answer to the main question in its various forms. In the most general case for Σ^1_1 or $\mathbf{\Pi}^1_1$ equivalence relations with all $\mathbf{\Delta}^1_1$ classes, iterability assumptions were used to get a positive answer. Iterability is a fairly strong large cardinal assumption: for example, it requires the universe to transcend L in a way set forcing extensions can never do.

However, this paper leaves open the possibility that even the most general form of this question for ${\Sigma^1_1}^{\mathbf{\Delta}^1_1}$ and ${\mathbf{\Pi}^1_1}^{\mathbf{\Delta}^1_1}$ could be provable in just ZFC.

Question 3.10.1. Is it consistent (relative large cardinals) that there is a σ-ideal I on a Polish space with $\mathbb{P}_I$ proper and $E \in {\Sigma^1_1}^{\mathbf{\Delta}^1_1}$ such that $\{E\} \to_I \mathbf{\Delta}^1_1$ is false?

Same question for ${\mathbf{\Pi}^1_1}^{\mathbf{\Delta}^1_1}$.

The results of this paper provide limitations to any attempt to produce a counterexample to a positive answer to the main question.

The results of the paper seem to suggest a universe with few and very weak large cardinals is the ideal place to consider finding a counterexample. For example, Theorems 3.4.22 and 3.7.12 show that any universe that has sharps for sets in $H_{(2^{\aleph_0})^+}$ will always give a positive answer to the main question.

The following is perhaps the main open question:

Question 3.10.2. Is there a negative answer to the main question for the Σ_1^1 or $\boldsymbol{\Pi}_1^1$ case in L?

Cohen forcing ($\mathbb{P}_{I_{\text{meager}}}$) is perhaps the simpliest of all forcings. This paper leaves open the possibility that Cohen forcing in L could be used to produce a counterexample to the main question.

Question 3.10.3. Can Cohen forcing (meager ideal) be used with some Σ_1^1 or $\boldsymbol{\Pi}_1^1$ equivalence relation with all classes $\boldsymbol{\Delta}_1^1$ to produce a counterexample to the main question?

Propositions 3.5.2 and 3.5.4 show that the ideal of countable sets (Sacks forcing) and the E_0-ideal (Prikry-Silver forcing) can never be used to produce a counterexample to the main question in the Σ_1^1 case.

One of the most common forcing extensions in descriptive set theory is the extension by the (gentle) Lévy collapse $\text{Coll}(\omega, < \kappa)$, where κ is some inaccessible cardinal. Here there is a partial answer to Question 3.10.3: Corollary 3.5.11 shows that the meager ideal and null ideal cannot be used in an extension by the Lévy collapse of an inaccessible to produce a counterexample to the main question in the Σ_1^1 case. Moreover, Fact 3.5.14 implies that these two ideals cannot be used for a counterexample if $\mathsf{MA} + \neg\mathsf{CH}$ holds.

Propositions 3.3.2 and 3.3.1 assert that Σ_1^1 equivalence relations with all classes countable or are $\boldsymbol{\Delta}_1^1$ reducible to orbit equivalence relations of Polish group actions can not be used to show the consistency of a negative answer. One may suspect that an unusual $\Sigma_1^{1\ \boldsymbol{\Delta}_1^1}$ equivalence relation may be necessary. Thin equivalence relations include somewhat unusual objects such as F_{ω_1}, E_{ω_1}, and any potential counterexamples to Vaught's conjecture. However, Theorem 3.6.8 shows that thin Σ_1^1 equivalence relations have the strongest form of canonization in the sense that one of their classes is in I^+.

It seems that one has reached an impasse in regard to the main question for $\Sigma_1^{1\ \boldsymbol{\Delta}_1^1}$. There is a lack of interesting examples of $\Sigma_1^{1\ \boldsymbol{\Delta}_1^1}$ equivalence relations which may be useful for producing a consistency result for a negative answer to the main question for $\Sigma_1^{1\ \mathbf{\Delta}_1^1}$.

Here is where $\mathbf{\Pi}_1^{1\ \mathbf{\Delta}_1^1}$ becomes much more interesting and provides a possible path forward. What appears to be promising is that $\mathbf{\Pi}_1^1$ equivalence relations seem to be much more suspectible to set theoretic assumptions.

One difficulty in producing the appropriate type of Σ_1^1 equivalence relation is the requirement that all classes be $\mathbf{\Delta}_1^1$. In the $\mathbf{\Pi}_1^1$ case, one situation in which this requirement is easily obtained is by considering $\mathbf{\Pi}_1^1$ equivalence relations with all classes thin and assume ω_1 is inaccessible to reals, i.e., the class $\mathbf{\Pi}_1^{1\ \text{thin}}$.

Even in this case, one must still limit the universe to one in which only weak large cardinals exist: The easiest way to obtain ω_1 is inaccessible to real is via a Lévy collapse. Theorem 3.8.10 shows that this attempt will never work if one uses a Lévy collapse extension of a remarkable cardinal. Moreover, Theorem 3.8.12 shows that using $\mathbf{\Pi}_1^{1\ \text{thin}}$ with a $\aleph_1$-c.c. forcing will never work in a Lévy collapse extension of a weakly compact cardinal.

A closer look at the proofs of Lemmas 3.7.3 and 3.7.4 shows the following:

Definition 3.10.4. Let $E \in \mathbf{\Pi}_1^{1\ \mathbf{\Delta}_1^1}$. Let $r(x) = \min\{\omega_1^z : [x]_E \text{ is } \Sigma_1^1(z)\}$.

Proposition 3.10.5. *Let $E \in \mathbf{\Pi}_1^{1\ \mathbf{\Delta}_1^1}$ and I be a σ-ideal such that $\mathbb{P}_I$ is proper. Suppose, for all $B \in \mathbb{P}_I$, there exists some $C \subseteq B$ with $C \in \mathbb{P}_I$ and* $\sup\{r(x) : x \in C\} < \omega_1$. *Then $\{E\} \to_I \mathbf{\Delta}_1^1$.*

Therefore, any counterexample to a positive answer for the main question for $\mathbf{\Pi}_1^{1\ \mathbf{\Delta}_1^1}$ must violate the hypothesis of this proposition. The next result gives a hypothetical condition under which this happens:

Proposition 3.10.6. *Suppose ω_1 is inaccessible to reals. Let I be a σ-ideal on a Polish space such that $\mathbb{P}_I$ is proper and whenever g is $\mathbb{P}_I$-generic over V, $V[g] = L[g]$. Let $E \in \Pi_1^{1\ \mathit{thin}}$ with the property that for all x, $L[x] \models [x]_E$ is uncountable thin. Then for all $C \in \mathbb{P}_I$,* $\sup\{r(x) : x \in C\} = \omega_1$.

Proof. The first claim is that $[x]_E$ can not be $\Delta_1^1(z)$ for any z such that $\omega_1^z < \omega_1^{L[x]}$. (Note that ω_1^z refers to the least z-admissible ordinal and $\omega_1^{L[x]}$ is the least uncountable cardinal of $L[x]$.)

Suppose otherwise: $[x]_E$ is $\Sigma^1_1(z)$ and $\omega_1^z < \omega_1^{L[x]}$. As in Lemma 3.7.3, define

$$a\ E'\ b \Leftrightarrow (a \in [x]_E \wedge b \in [x]_E) \vee (a = b).$$

E' is $\Sigma^1_1(z)$. $E' \subseteq E$. By the effective bounding theorem, there are some $\alpha < \omega_1^z$ such that $E' \subseteq E_\alpha$. Now, applying Lemma 3.7.2 in $L[x]$ and the fact that $\alpha < \omega_1^z < \omega_1^{L[x]}$, there exists some β such that $\alpha < \beta < \omega_1^{L[x]}$ such that E_β is an equivalence relation. Using the argument in Lemma 3.7.3, $[x]_E = [x]_{E_\beta}$. E_β is $\mathbf{\Delta}^1_1(c)$ for any $c \in {}^\omega 2$ such that $\mathrm{ot}(c) = \beta$. Since $\beta < \omega_1^{L[x]}$, there exists such a $c \in L[x]$. Hence $[x]_{E_\beta}$ is $\Delta^1_1(x, c)$.

$$V \models (\forall a)(a\ E\ x \Leftrightarrow a\ E_\beta\ x)$$

Since $x, c \in L[x]$ and this statement is $\Pi^1_2(x, c)$, by Schoenfield absoluteness

$$L[x] \models (\forall a)(a\ E\ x \Leftrightarrow y\ E_\beta\ x).$$

So $L[x] \models [x]_E$ is $\mathbf{\Delta}^1_1$. However, the assumption was that $L[x] \models [x]_E$ is uncountable thin. ZFC proves that no $\mathbf{\Delta}^1_1$ set can be uncountable thin. Contradiction. This proves the claim.

So now let $\alpha < \omega_1$. Let $M \prec H_\Theta$ with $\alpha \subseteq M$ and $C, \mathbb{P}_I \in M$. Note that $\omega_1^M \geq \alpha$. Let $x \in C$ be $\mathbb{P}_I$-generic over M. Then $M[x] \models \omega_1^{L[x]} = \omega_1^{M[x]} = \omega_1^M \geq \alpha$, using the fact that $\mathbb{P}_I$ has the property $V[g] = L[g]$, wherever g is $\mathbb{P}_I$ generic over V. Certainly, the real $(\omega_1^{L[x]})^V$ is greater than or equal to $(\omega_1^{L[x]})^M \geq \alpha$. So $\omega_1^{L[x]} \geq \alpha$. By the claim above, $r(x) \geq \alpha$. Hence $\sup\{r(x) : x \in C\} = \omega_1$. □

Note that if V satisfies ω_1 is inaccessible to reals and $V[g] = L[g]$ whenever g is $\mathbb{P}_I$-generic over V, then "ω_1 is inaccessible to reals" is not preserved into the extension $V[g] = L[g]$. Compare this to what happens in the $\mathrm{Coll}(\omega, <\kappa)$ extension of L when κ is a remarkable cardinal in L (see Theorem 3.8.10).

Given this result, the natural questions are whether such an ideal exists and whether such a Π^1_1 thin equivalence relation exists.

First consider the following: Suppose $\kappa \in L$ and κ is not Mahlo in L. Let $G \subseteq \mathrm{Coll}(\omega, <\kappa)$. In $L[G]$, $\omega_1^{L[G]}$ is not Mahlo and $L[G]$ satisfies ω_1 is inaccessible to reals. By [28] Exercise 8.7, there is an $A \subseteq \omega_1$ in $L[G]$, which is reshaped, i.e., for all $\xi < \omega_1$, $L[A \cap \xi] \models |\xi| = \aleph_0$. Since $L[A] \subseteq L[G]$, $\omega_1^{L[A]} \leq \omega_1^{L[G]}$. Since A is reshaped, $L[A] \models \omega_1^{L[A]} \geq \omega_1^{L[G]}$. So $\omega_1^{L[A]} = \omega_1^{L[G]}$. Since $L[G]$ satisfies ω_1 is inaccessible to reals, $L[A]$ also satisfies ω_1 is inaccessible to reals.

In [13] Section 1, it is shown that in $L[A]$, where A is a reshaped subset of ω_1, there is an $\aleph_1$-c.c. forcing which adds a real g such that $L[A][g] = L[g]$. This forcing consists of perfect trees. By [34] Corollary 2.1.5, there is a σ-ideal I_F such that $\mathbb{P}_{I_F}$ is forcing equivalent to Sy-David Friedman's forcing to code subsets of ω_1. In $L[A]$, I_F would be a σ-ideal that satisfies the property of Proposition 3.10.6.

It is not known whether $\sup\{r(x) : x \in C\} = \omega_1$ for all I^+ set C is enough for a negative answer to the main question for $\mathbf{\Pi}^{1\ \text{thin}}_1$. It could be possible that there is a C such that for all $x \in C$, $[x]_E$ is very complicated as x ranges over C, but C consists of pairwise E-inequivalent elements (or even C is a single E-class).

In L, Jensen's minimal nonconstructible Δ^1_3 real forcing (see [18] and [16], Chapter 28) is also a forcing consisting of perfect trees. Again by [34] Corollary 2.1.5, there is a σ-ideal I_J such that $\mathbb{P}_{I_J}$ is forcing equivalent to Jensen's forcing. $\mathbb{P}_{I_J}$ is $\aleph_1$-c.c. by [16] Lemma 28.4. Moreover, by [16] Corollary 28.6, if g, h are $\mathbb{P}_{I_J}$-generic over L, then $g \times h$ is $\mathbb{P}_{I_J} \times \mathbb{P}_{I_J}$ generic over L. Hence, below any B such that $B \Vdash_{\mathbb{P}_{I_J}} (\dot{x}_{\text{gen}})_{\text{left}}\ E(\dot{x}_{\text{gen}})_{\text{right}}$ (or $B \Vdash_{\mathbb{P}_{I_J}} \neg((\dot{x}_{\text{gen}})_{\text{left}}\ E\ (\dot{x}_{\text{gen}})_{\text{right}})$), if C is the I^+ set of $\mathbb{P}_{I_J}$-generic real over M in B (for some $M \prec H_\Theta$), then B consists of pairwise E-inequivalent (or pairwise E-equivalent) reals. But of course, this example does not satisfy all of the conditions of Proposition 3.10.6.

It is not known whether the $\mathbf{\Pi}^{1\ \text{thin}}_1$ equivalence relations needed in Proposition 3.10.6 exist.

Question 3.10.7. Let κ be inaccessible but not Mahlo in L. Suppose $G \subseteq \text{Coll}(\omega, <\kappa)$ be generic over L. Let $A \subseteq \omega_1$ with $A \in L[G]$ be a reshaped subset of ω_1. Then is there a Π^1_1 equivalence relation E such that for all $x \in ({}^\omega\omega)^{L[A]}$, $L[x] \models [x]_E$ is uncountable thin?

This leads to an interesting related question about whether it is possible to partition ${}^\omega\omega$ in a Π^1_1 way into $\mathbf{\Pi}^1_1$ pieces that are all uncountable thin:

Question 3.10.8. In L, is there a Π^1_1 equivalence relation E such that $L \models (\forall x)([x]_E$ is uncountable thin)?

Sy-David Friedman has communicated to the author a solution to this last question. See the appendix below for more information.

3.11 Appendix

This appendix includes some remarks of Sy-David Friedman.

Sy-David Friedman and Törnquist, using some ideas of Miller and Conley, have given a solution to Question 3.10.8.

Theorem 3.11.1. *(Friedman, Törnquist) In L, there exists a* $\mathbf{\Pi}^1_1$ *equivalence relation* E *such that* $L \models (\forall x)([x]_E$ *is uncountable thin*).

Proof. E will be an equivalence relation on $\mathbb{R}$. Consider $\mathbb{R}$ with its usual $\mathbb{Q}$-vector space structure. By [20] Exercise 19.2 (i), let C be a perfect $\mathbf{\Pi}^0_1$ $\mathbb{Q}$-linearly independent set of reals. Let $P \subseteq C$ be an uncountable thin $\mathbf{\Pi}^1_1$ subset. Let $\langle C \rangle$ and $\langle P \rangle$ denote the additive subgroups of $\mathbb{R}$ generated by C and P, respectively.

Since C consists of $\mathbb{Q}$-linearly independent reals, each element of $\langle C \rangle$ has a unique representation as $\mathbb{Z}$-linear combinations of elements of C. By Lusin-Novikov (countable section) uniformization, $\langle C \rangle$ is $\mathbf{\Delta}^1_1$. Also by Lusin-Novikov, there is a $\mathbf{\Delta}^1_1$ function Φ on $\mathbb{R}$ such that if $r \in \langle C \rangle$, then $\Phi(r)$ is a representation of r as a $\mathbb{Z}$-linear combination of elements of C, and if $r \notin \langle C \rangle$, then $\Phi(r)$ is some default value.

Then $\langle P \rangle$ has the following definition: $r \in \langle P \rangle$ if and only if $r \in \langle C \rangle$ and $\Phi(r)$ consists of only elements from P. The latter is $\mathbf{\Pi}^1_1$. Hence $\langle P \rangle$ is a coanalytic subgroup of $\mathbb{R}$.

By definition, $\langle P \rangle$ is the set of $\mathbb{Z}$-linear combinations of elements of P. Since P is thin, by Mansfield-Solovay, P consists entirely of constructible reals. In particular, in any forcing extension $L[G]$ of L, $P^L = P^{L[G]}$. So, $\langle P \rangle^{L[G]}$ consists of $\mathbb{Z}$-linear combinations of elements of $P^{L[G]} = P^L$. Hence, $\langle P \rangle^{L[G]} = \langle P \rangle^L$. If $\langle P \rangle^L$ had a perfect subset, then by Schoenfield's absoluteness, $\langle P \rangle^{L[G]}$ would have a perfect subset. If G was generic for a forcing which makes $(2^{\aleph_0})^{L[G]} > \aleph_1^L$, then $|\langle P \rangle|^{L[G]} = (2^{\aleph_0})^{L[G]} > \aleph_1^L = |\langle P \rangle|^L$. This contradicts $\langle P \rangle^{L[G]} = \langle P \rangle$. This shows that in L, $\langle P \rangle$ is uncountable thin.

Let E be the coset equivalence relation of $\mathbb{R}/\langle P \rangle$: $r \mathrel{E} s \Leftrightarrow (r - s) \in \langle P \rangle$. E is $\mathbf{\Pi}^1_1$. For all r, $[r]_E$ is in bijection with $\langle P \rangle$. Hence $[r]_E$ is uncountable thin. □

At the time of asking Question 3.10.8, there was hope that any natural constructibly coded $\mathbf{\Pi}^1_1$ equivalence relation which witnessed a positive answer to Question 3.10.8 would also serve as a witness to a positive answer to Question 3.10.7.

Unfortunately, the equivalence relation E of Theorem 3.11.1 does not work. The definition of E has a particular constructibly coded thin $\mathbf{\Pi}^1_1$ group built into it. E, as a coset relation, copies this thin uncountable (in L) set throughout the reals. Now suppose V is some universe such that $\omega_1^L < \omega_1^V$. In V, choose some $z \in \mathbb{R}$ such that $L[z] \models \omega_1^L < \omega_1$. Since $[z]_E$ is in bijection with $\langle P \rangle$ (which is in bijection with ω_1^L), $L[z] \models [z]_E$ is countable.

It seems any possible solution to Question 3.10.7 will need to be defined without using any explicit definition of a thin $\mathbf{\Pi}_1^1$ set.

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Chapter 4

WHEN AN EQUIVALENCE RELATION WITH ALL BOREL CLASSES WILL BE BOREL SOMEWHERE?

(With Menachem Magidor)

4.1 Introduction

The basic question of interest is:

Question 4.1.1. If E is an equivalence relation on ${}^{\omega}\omega$, is E a simpler equivalence relation when restricted to some subset?

This question can also be asked for equivalence relations on arbitrary Polish spaces, but for simplicity, this paper will only consider equivalence relations on ${}^{\omega}\omega$. Usually, descriptive set theoretic results about ${}^{\omega}\omega$ have proofs that can be transfered to arbitrary Polish spaces.

What should be the measure of complexity and what should be the paragon of simplicity? The measure of complexity will vaguely be definability and there is no need to explicitly state what it is since the paper will only strive to reach the base of complexity. However, there are various useful notions of definability given by considerations in topology, recursion theory, logical complexity, and set theory. The base of definable complexity needs to be explicitly stated. The class of Borel sets (denoted $\boldsymbol{\Delta}_1^1$) is chosen to be this base since it is a simple class characterized by all the notions of definability mentioned above. Moreover, many natural mathematical concerns appear at this level, and $\boldsymbol{\Delta}_1^1$ objects seem to be well behaved and relatively well understood.

Now the question can be more precisely formulated:

Question 4.1.2. If E is an equivalence relation on ${}^{\omega}\omega$, is there a $\boldsymbol{\Delta}_1^1$ set $C \subseteq {}^{\omega}\omega$ so that $E \upharpoonright C$ is a $\boldsymbol{\Delta}_1^1$ equivalence relation?

Here, $E \upharpoonright C = E \cap (C \times C)$. However, there is one obvious triviality. If C is countable, then any equivalence relation restricted to C is $\boldsymbol{\Delta}_1^1$. Since countable subsets of ${}^{\omega}\omega$ belong to any σ-ideal on ${}^{\omega}\omega$ which contains all singletons, this egregious triviality disappears if one asks that, in the above question, C be $\boldsymbol{\Delta}_1^1$ and non-trivial according to a σ-ideal on ${}^\omega\omega$. Subsets of ${}^\omega\omega$ that are not in the ideal I are called I^+ sets. In this paper, σ-ideals will always contain all the singletons.

However, it is unclear how to approach this question for arbitrary σ-ideals. The collection of available techniques is greatly enriched by considering σ-ideals on ${}^\omega\omega$ so that the associated forcing $\mathbb{P}_I$ of $\boldsymbol{\Delta}_1^1$ I^+ sets is a proper forcing. Considering such σ-ideals makes available powerful tools from models of set theory and absoluteness. (In fact, the questions below all have negative answers when considering arbitrary σ-ideals. See Section 4.2.) Now a test question can be posed for a slightly more complicated class of equivalence relations than the $\boldsymbol{\Delta}_1^1$ equivalence relations: Analytic (denoted Σ_1^1) sets are continuous images of $\boldsymbol{\Delta}_1^1$ or even closed sets.

Question 4.1.3. Let E be a Σ_1^1 equivalence relation on ${}^\omega\omega$. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is a proper forcing. Is there an I^+ $\boldsymbol{\Delta}_1^1$ set C so that $E \upharpoonright C$ is a $\boldsymbol{\Delta}_1^1$ equivalence relation?

Note that questions like the above are very familiar. For example, the ideal of Lebesgue null set and the ideal of meager sets have the property that their associated forcings are proper forcings. It is very common in mathematics to ask questions about properties that hold on positive measure sets (or Lebesgue almost everywhere) or on non-meager (or comeager) sets.

Unfortunately, Question 4.1.3 has a negative answer:

Proposition 4.1.4. *There is a Σ_1^1 equivalence relation E and a σ-ideal I with $\mathbb{P}_I$ proper so that for all $\boldsymbol{\Delta}_1^1$ I^+ set C, $E \upharpoonright C$ is not $\boldsymbol{\Delta}_1^1$.*

Proof. See [6], Example 4.25. □

So a positive answer is not even possible for the simplest class of equivalence relations in the projective hierarchy just above $\boldsymbol{\Delta}_1^1$. A positive answer to any variation of the basic question will likely only be feasible if the equivalence relations bear at least some resemblance to $\boldsymbol{\Delta}_1^1$ equivalence relations. [6] then proved that a positive answer does hold for Σ_1^1 equivalence relations with all countable classes and equivalence relations $\boldsymbol{\Delta}_1^1$ reducible to orbit equivalence relations of Polish group actions. In both these examples, the equivalence relations have all $\boldsymbol{\Delta}_1^1$ classes. Of course, $\boldsymbol{\Delta}_1^1$ equivalence relations have all $\boldsymbol{\Delta}_1^1$ classes. Perhaps those two examples give evidence that a sufficient resemblance for a positive answer is the property of having all $\boldsymbol{\Delta}_1^1$ classes. [6] asked the following question:

Question 4.1.5. ([6] Question 4.28) Let E be a Σ^1_1 equivalence relation on ${}^\omega\omega$ with all $\mathbf{\Delta}^1_1$ classes. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is a proper forcing. Let B be an I^+ $\mathbf{\Delta}^1_1$ set. Is there some $C \subseteq B$ which is I^+ $\mathbf{\Delta}^1_1$ so that $E \restriction C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

Under large cardinal assumptions, this question has a positive answer: Here coanalytic sets (denoted $\mathbf{\Pi}^1_1$) are complements of Σ^1_1 sets.

Theorem 4.1.6. *Suppose for all $X \in H_{(2^{\aleph_0})^+}$, $X^\sharp$ exists. Then for all Σ^1_1 and $\mathbf{\Pi}^1_1$ equivalence relations with all $\mathbf{\Delta}^1_1$ classes, any σ-ideal I on ${}^\omega\omega$ with $\mathbb{P}_I$ proper, and B an I^+ $\mathbf{\Delta}^1_1$ set, there exists some I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $E \restriction C$ is $\mathbf{\Delta}^1_1$.*

Proof. See [1]. Also see [2] for a similar result proved using a measurable cardinal. $\square$

It should be noted that the proofs of Theorem 4.1.6 in both [1] and [2] use an approximation of Σ^1_1 equivalence relations by $\mathbf{\Delta}^1_1$ equivalence relations: Burgess showed that for every Σ^1_1 equivalence relation E there is (in a uniform way) an ω_1-length decreasing sequence $(E_\alpha : \alpha < \omega_1)$ of $\mathbf{\Delta}^1_1$ equivalence relations so that $E = \bigcap_{\alpha<\omega_1} E_\alpha$. The strategy of the proof is to find some countable elementary $M \prec H_\Xi$, where Ξ is large enough to contain certain desired objects, and some countable ordinal α so that if C is the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic reals over M (which exists by properness of $\mathbb{P}_I$), then $E \restriction C = E_\alpha \restriction C$. The sharps are used to obtain the absoluteness necessary to determine the countable level α at which the E classes and E_α classes of all generic reals stabilize.

In conversation with the first author, Neeman asked the following generalization of Question 4.1.5: Projective sets are those obtainable by applying finitely many applications of complements and continuous images starting with the $\mathbf{\Delta}^1_1$ sets.

Question 4.1.7. Assume some large cardinal hypotheses. Let E be a projective equivalence relation with all $\mathbf{\Delta}^1_1$ classes. Let I be a σ-ideal on ${}^\omega\omega$ with $\mathbb{P}_I$ proper. Let $B \subseteq {}^\omega\omega$ be an I^+ $\mathbf{\Delta}^1_1$ subset. Does there exist some I^+ $\mathbf{\Delta}^1_1$ $C \subseteq B$ so that $E \restriction C$ is $\mathbf{\Delta}^1_1$?

It is unclear if the proofs of Theorem 4.1.6 can be generalized to give an answer to this question since there does not appear to be any form of $\mathbf{\Delta}^1_1$ approximation to arbitrary projective equivalence relations. Moreover, it is known to be consistent that there is a negative answer to Question 4.1.7 even when restricted to the next level of the projective hierarchy above Σ_1^1 and $\mathbf{\Pi}_1^1$. A Σ_2^1 set is a continuous image of a $\mathbf{\Pi}_1^1$ set. A $\mathbf{\Pi}_2^1$ set is the complement of a Σ_2^1 set. A $\mathbf{\Delta}_2^1$ set is a set that is both Σ_2^1 and $\mathbf{\Pi}_2^1$:

Proposition 4.1.8. *In the constructible universe L, there is a $\mathbf{\Delta}_2^1$ equivalence relation with all classes countable so that for every σ-ideal I and every I^+ $\mathbf{\Delta}_1^1$ set B, $E \upharpoonright B$ is not $\mathbf{\Delta}_1^1$.*

Proof. See [1] or [2]. □

In fact, it is not even known what is the status of Question 4.1.5 or its $\mathbf{\Pi}_1^1$ analog in L. Perhaps the most interesting open question in this area is whether it is consistent that Question 4.1.5 or its $\mathbf{\Pi}_1^1$ analog has a negative answer. See the conclusion section of [1] for some discussions on this question.

This paper will be concerned with extending a positive answer to these types of questions to larger classes of equivalence relations on ${}^\omega\omega$ with all $\mathbf{\Delta}_1^1$ classes. As mentioned above, some new methods will need to be developed to take the role of Burgess's approximation in Theorem 4.1.6. A certain game will be used to fulfill this role.

Question 4.1.7 will be answered by an even more general result. Like in Theorem 4.1.6, the results of this paper will be proved in an extension of ZFC, the standard axiom system of set theory. Here, ZFC will be augmented by large cardinal axioms. The large cardinal axioms used here are well accepted and have proven to be very useful in descriptive set theory.

The model $L(\mathbb{R})$ is the smallest inner model of ZF (possibly without the axiom of choice) containing all the reals of the original universe. It contains all the sets which are "constructible" (in the sense of Gödel) from the reals of the original universe. Nearly all objects of ordinary mathematics can be found in $L(\mathbb{R})$. In particular, all projective subsets of ${}^\omega\omega$ belong to $L(\mathbb{R})$. A main result of the paper is:

Theorem 4.4.22. Suppose there is a measurable cardinal with infinitely many Woodin cardinals below it. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is a proper forcing. Let $E \in L(\mathbb{R})$ be an equivalence relation on ${}^\omega\omega$.

If E has all Σ^1_1 ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$) classes, then for every I^+ $\mathbf{\Delta}^1_1$ set B, there is an I^+ $\mathbf{\Delta}^1_1$ $C \subseteq B$ so that $E \upharpoonright C$ is Σ^1_1 ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$, respectively).

This gives a positive answer to Question 4.1.7. Moreover, it shows that for a large class of equivalence relations on ${}^\omega\omega$ so that all the equivalences classes belong to a particular pointclass of the first level of the projective hierarchy, the equivalence relation somewhere is as simple as its equivalence classes.

Having answered Question 4.1.7 positively and even given a positive answer for the larger class of $L(\mathbb{R})$ equivalence relation with all $\mathbf{\Delta}^1_1$ classes, the ultimate natural question is the following:

Question 4.1.9. Is it consistent relative to some large cardinals, that (the axiom of choice fails and) for every equivalence relation E with all $\mathbf{\Delta}^1_1$ classes and every σ-ideal I on ${}^\omega\omega$ such that $\mathbb{P}_I$ is a proper forcing, there is an I^+ $\mathbf{\Delta}^1_1$ set C so that $E \upharpoonright C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

As it is often the case for various regularity properties like the perfect set property, Lebesgue measurability, or the property of Baire, the axiom of choice can be used with a diagonalization argument to produce a failure of this property. In fact, using the axiom of choice, there is an equivalence relation with classes of size at most two so that for any σ-ideal I and any I^+ $\mathbf{\Delta}^1_1$ set C, $E \upharpoonright C$ is not $\mathbf{\Delta}^1_1$.

For the regularity properties mentioned above, it is consistent that all sets have these properties in a choiceless model of ZF, like the model $L(\mathbb{R})$. For instance, if the axiom of determinacy, AD, holds then all sets are Lebesgue measurable and have the property of Baire.

Assuming determinacy for certain games on the reals, every equivalence relation with all $\mathbf{\Delta}^1_1$ classes can be canonicalized by certain σ-ideals whose associated forcings are proper:

Theorem 4.5.13. Assume $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_\mathbb{R}$. Let E be an equivalence relation on ${}^\omega\omega$. If E has all Σ^1_1 ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$) classes, then for every nonmeager $\mathbf{\Delta}^1_1$ set B, there is a $\mathbf{\Delta}^1_1$ set $C \subseteq B$ which is comeager in B so that $E \upharpoonright C$ is Σ^1_1 ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$, respectively).

Theorem 4.5.14. Assume $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_\mathbb{R} + V = L(\mathscr{P}(\mathbb{R}))$. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is absolutely proper. Let E be an equivalence relation on ${}^\omega\omega$. If E has all Σ^1_1 ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$) classes, then for every I^+ $\mathbf{\Delta}^1_1$ set B, there is an I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $E \restriction C$ is Σ^1_1 ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$, respectively).

The notion of an absolutely proper forcing is defined in [12]. Definable forcings which are proper under AC often are absolutely proper under AD. The proof of the above theorem requires some absoluteness given by an embedding theorem for absolutely proper forcing under determinacy assumptions which is analogous to the proper forcing embedding theorem shown in [13] which holds under AC with large cardinals. [12] has also used a stronger form of the embedding theorem for absolutely proper forcings to establish a positive answer under AD^+ to a more general form of Question 4.1.7 for σ-ideals with associated forcing absolutely proper.

Section 4.2 will review the basics of idealized forcing, the theory of measure, and homogeneous trees. The relevant game concepts will be introduced here.

Section 4.3 will prove that certain types of equivalence relations can be $\mathbf{\Delta}^1_1$, Σ^1_1, or $\mathbf{\Pi}^1_1$ equivalence relations on I^+ $\mathbf{\Delta}^1_1$ subsets of ${}^\omega\omega$ for any σ-ideal I so that $\mathbb{P}_I$ is proper, under three assumptions about absoluteness and tree representations. The main results of this section will be proved using a certain game. This section can be understood with just basic knowledge of set theory and forcing. The results of this section hold more generally for relations with Σ^1_1 ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$) sections. Therefore, all the theorems in this paper have an analogous statement for graphs G so that for all $x \in {}^\omega\omega$, the set $G_x = \{y : x \ G \ y\}$ is Σ^1_1 ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$). However, this paper will focus mostly on equivalence relations.

Section 4.4 will mostly assume axiom of choice and will give a general situation in which the three assumptions used in the previous section hold. This section will give a very brief survey of the theory of generic absoluteness and tree representations of subsets of ${}^\omega\omega$, especially the Martin-Solovay tree construction. Theorem 4.4.22 will be presented.

Section 4.5 will assume a bit more than the axiom of determinacy for the reals and will mention the necessary results about tree representations and generic absoluteness to show that the three assumptions from Section 4.3 hold for every equivalence relation with all Σ^1_1, $\mathbf{\Pi}^1_1$, or $\mathbf{\Delta}^1_1$ classes. Finally, Theorem 4.5.13 and Theorem 4.5.14 will be presented.

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4.2 Basics

In this paper, σ-ideals always contain all the singleton.

Definition 4.2.1. Let I be a σ-ideal on ${}^\omega\omega$. Let $\mathbb{P}_I = (\mathbf{\Delta}^1_1 \setminus I, \subseteq, {}^\omega\omega)$ be the forcing of I^+ $\mathbf{\Delta}^1_1$ subsets of ${}^\omega\omega$ ordered by $\leq_{\mathbb{P}_I} = \subseteq$ and has largest element $1_{\mathbb{P}_I} = {}^\omega\omega$. Often $\mathbb{P}_I$ is identified with $\mathbf{\Delta}^1_1 \setminus I$.

Fact 4.2.2. *Let I be a σ-ideal on ${}^\omega\omega$. There is a name $\dot{x}_{\text{gen}} \in V^{\mathbb{P}_I}$ so that for all $\mathbb{P}_I$-generic filters G over V and all $\mathbf{\Delta}^1_1$ sets B coded in V, $V[G] \models B \in G \Leftrightarrow \dot{x}_{\text{gen}}[G] \in B$.*

Proof. See [17], Proposition 2.1.2. □

Definition 4.2.3. Let I be a σ-ideal on ${}^\omega\omega$. Let $M \prec H_\Xi$ be a countable elementary substructure for some sufficiently large cardinal Ξ. $x \in {}^\omega\omega$ is $\mathbb{P}_I$-generic over M if and only if the collection $\{B \in \mathbb{P}_I \cap M : x \in B\}$ is a $\mathbb{P}_I$-generic filter over M.

The following results make available some very useful techniques for handling ideals whose associated forcings are proper forcings. For the purpose of this paper, the following may as well be taken as the definition of properness:

Proposition 4.2.4. *Let I be a σ-ideal on ${}^\omega\omega$. The following are eqivalent:*

(i) $\mathbb{P}_I$ is a proper forcing.

(ii) For any sufficiently large cardinal Ξ, every $B \in \mathbb{P}_I$, and every countable $M \prec H_\Xi$ with $\mathbb{P}_I \in M$ and $B \in M$, the set $C = \{x \in B : x \text{ is } \mathbb{P}_I\text{-generic over } M\}$ is an I^+ $\mathbf{\Delta}^1_1$ set.

Proof. See [17], Proposition 2.2.2. □

This proposition shows that σ-ideals whose associated forcing is proper may be useful for answering Question 4.1.5 since it indicates how to produce I^+ $\mathbf{\Delta}^1_1$ sets. It is should be noted that some restrictions on the type of σ-ideals considered in Question 4.1.5 are necessary:

Let F_{ω_1} denote the countable admissible ordinal equivalence relation defined by $x \ F_{\omega_1} \ y$ if and only if $\omega_1^x = \omega_1^y$. F_{ω_1} is a thin Σ^1_1 equivalence relation with all $\mathbf{\Delta}^1_1$ classes. Thin means that F_{ω_1} does not have a perfect set of pairwise F_{ω_1}-inequivalent elements. Let I be the σ-ideal which is σ-generated by the F_{ω_1}-classes. Suppose there was an I^+ $\mathbf{\Delta}^1_1$ set C so that $F_{\omega_1} \restriction C$ is $\mathbf{\Delta}^1_1$. By definition of I, each F_{ω_1}-class is in I. So since C is I^+, C must intersect nontrivially uncountably many classes of F_{ω_1}. So $F_{\omega_1} \upharpoonright C$ has uncountable many classes. Since F_{ω_1} is thin, there is also no perfect set of $F_{\omega_1} \upharpoonright C$ inequivalent elements. This contradicts Silver's dichotomy (see Fact 4.5.2).

Of course, I is not proper or even ω_1-preserving: Let $G \subseteq \mathbb{P}_I$ be a $\mathbb{P}_I$-generic filter over V. Fact 4.2.2 implies that $\dot{x}_{\text{gen}}[G]$ is not in any ground model coded $\mathbf{\Delta}^1_1$ set in I. $\omega_1^{\dot{x}_{\text{gen}}[G]}$ can not be a countable admissible ordinal of V since if it was countable then a theorem of Sacks shows that there is a $z \in ({}^\omega\omega)^V$ so that $\omega_1^z = \omega_1^{\dot{x}_{\text{gen}}[G]}$. Then $x \in [z]_{F_{\omega_1}}$. By definition of I, $[z]_{F_{\omega_1}}$ is a $\mathbf{\Delta}^1_1$ set coded in V that belongs to I. Hence $\omega_1^{\dot{x}_{\text{gen}}[G]}$ must be an uncountable admissible ordinal of V, but in $V[G]$, $\omega_1^{\dot{x}_{\text{gen}}[G]}$ is a countable admissible ordinal. Hence $\mathbb{P}_I$ collapses ω_1.

Definition 4.2.5. A measure μ on a set X is a nonprincipal ultrafilter on X. Nonprincipal means for all $x \in X$, $\{x\} \notin \mu$.

If κ is a cardinal, then μ is κ-complete if and only if for all $\beta < \kappa$ and sequences $(A_\alpha : \alpha < \beta)$ with each $A_\alpha \in \mu$, $\bigcap_{\alpha<\beta} A_\alpha \in \mu$. $\aleph_1$-completeness is often called countably completeness.

Let $\text{meas}_\kappa(X)$ be the set of all κ-complete ultrafilters on X.

Suppose $\mu \in \text{meas}_{\aleph_1}({}^{<\omega}X)$. By countably completeness, there is a unique m so that ${}^mX \in \mu$. In this case, m is called the dimension of μ and this is denoted $\dim(\mu) = m$.

Definition 4.2.6. Let X be a set. For $m \leq n < \omega$, let $\pi_{n,m} : {}^nX \to {}^mX$ be defined by $\pi_{n,m}(f) = f \upharpoonright m$.

Let $m \leq n < \omega$. Let ν be a measure of dimension m and μ be a measure of dimension n. μ is an extension of ν (or ν is a projection of μ) if and only if for all $A \in \nu$ with $A \subseteq {}^mX$, $\pi_{n,m}^{-1}[A] \in \mu$.

A tower of measures over X is a sequence $(\mu_n : n \in \omega)$ so that:

(i) For all n, $\mu_n \in \text{meas}_{\aleph_1}({}^{<\omega}X)$ and $\dim(\mu_n) = n$.

(ii) For all $m \leq n < \omega$, μ_n is an extension of μ_m.

A tower of measures over X, $(\mu_n : n \in \omega)$, is countably complete if and only if for all sequence $(A_n : n \in \omega)$ with the property that for $n \in \omega$, $A_n \in \mu_n$, there exists a $f : \omega \to X$ so that for all $n \in \omega$, $f \upharpoonright n \in A_n$.

Definition 4.2.7. A tree T on X is a subset of ${}^{<\omega}X$ so that if $s \subseteq t$ and $t \in T$, then $s \in T$.

If $s \in {}^n(X \times Y)$ where $n \in \omega$, then in a natural way, s be may be considered as a pair (s_0, s_1) with $s_0 \in {}^nX$ and $s_1 \in {}^nY$.

Let T be a tree on X. The body of T, denoted $[T]$, is the set of infinite paths through T, that is $[T] = \{f \in {}^\omega X : (\forall n \in \omega)(f \restriction n \in T)\}$.

Suppose T is a tree on $X \times Y$. For each $s \in {}^{<\omega}X$, define $T^s = \{t \in {}^{|s|}Y : (s,t) \in T\}$. If $f \in {}^\omega X$, then define $T^f = \bigcup_{n \in \omega} T^{f \restriction n}$.

Let T be a tree on $X \times Y$, then

$$p[T] = \{f \in {}^\omega X : T^f \text{ is ill-founded}\} = \{f \in {}^\omega X : [T^f] \neq \emptyset\}.$$

Definition 4.2.8. For any $k \in \omega$, $A \subseteq {}^k({}^\omega\omega)$ is Σ^1_1 if and only if there exists a tree on ${}^k\omega \times \omega$ so that $A = p[T]$. $A \subseteq {}^k({}^\omega\omega)$ is $\mathbf{\Pi}^1_1$ if and only if $A = {}^k({}^\omega\omega) \setminus B$ for some Σ^1_1 set $B \subseteq {}^k({}^\omega\omega)$. $A \subseteq {}^k({}^\omega\omega)$ is $\mathbf{\Delta}^1_1$ if and only if A is both Σ^1_1 and $\mathbf{\Pi}^1_1$.

Definition 4.2.9. (i) For each $s \in {}^{<\omega}({}^k\omega)$, $\mu_s \in \text{meas}_{\aleph_1}({}^{<\omega}\gamma)$ and concentrates on T^s (that is, $T^s \in \mu_s$).

(ii) For all $s, t \in {}^{<\omega}({}^k\omega)$, if $s \subseteq t$, then μ_t is an extension of μ_s.

(iii) For all $f \in p[T]$, $(u_{f \restriction n} : n \in \omega)$ is a countably complete tower of measures on γ.

A collection $(u_s : s \in {}^{<\omega}({}^k\omega))$ which witnesses the homogeneity of T is called a homogeneity system for T.

Let κ be a cardinal. The homogeneous tree T is κ-homogeneous if and only if each μ_s is κ-complete.

Definition 4.2.10. For any $k \in \omega$, $A \subseteq {}^k({}^\omega\omega)$ is homogeneously Suslin if and only if there exists an ordinal γ and a homogeneous tree on ${}^k\omega \times \gamma$ so that $A = p[T]$.

If the tree T is κ-homogeneous, then A is said to be κ-homogeneously Suslin.

Homogeneously Suslin sets have an important role in the theory of determinacy. In particular, games on ${}^\omega\omega$ associated with homogeneously Suslin sets are determined. Later, the homogeneity system of homogeneous trees will be used to show a certain player has a winning strategy in a particular game using techniques that are very similar to the Martin proof of Σ^1_1 determinacy from a measurable cardinal.

Below, the basic setting of the relevant games will be described:

Definition 4.2.11. Let X be some set. Let $A \subseteq {}^{\omega}X$. The game associated to A, denoted G_A, is the following: The game has two players, Player 1 and Player 2, who alternatingly take turns playing elements of X with Player 1 playing first. The picture below denotes a partial play where Player 1 plays the sequence $(a_i : i \in \omega)$ and Player 2 plays the sequence $(b_i : i \in \omega)$.

$$\begin{array}{cccccccc} a_0 & & a_1 & & \ldots & a_{k-1} & & \\ \hline & b_0 & & b_1 & \ldots & & b_{k-1} & \end{array}$$

Player 2 is said to win this play of G_A if and only if the infinite sequence $(a_0 b_0 a_1 b_1 \ldots) \in A$. Otherwise Player 1 wins.

A function $\tau : {}^{<\omega}X \to X$ is a winning strategy for Player 1 if and only if for all sequence $(b_i : i \in \omega)$ played by Player 2, Player 1 wins by playing $(a_i : i \in \omega)$ where this sequence is defined recursively by $a_0 = \tau(\emptyset)$ and $a_{k+1} = \tau(a_0 b_0 \ldots a_k b_k)$.

A winning stategy $\tau : {}^{<\omega}X \to X$ for Player 2 is defined similarly.

The game G_A is determined if Player 1 or Player 2 has a winning strategy.

Let X be a set. ${}^{\omega}X$ is given the topology with basis $\{U_s : s \in {}^{<\omega}X\}$, where $U_s = \{f \in {}^{\omega}X : s \subseteq f\}$.

Fact 4.2.12. *([4], Gale-Stewart) If $A \subseteq {}^{\omega}X$ is open, then G_A is determined. Hence if A is closed, then G_A is also determined.*

4.3 The Game

Definition 4.3.1. Let R be a relation on $({}^{\omega}\omega)^2$. Let $R_x = \{y : (x, y) \in R\}$ and $R^x = \{y : (y, x) \in R\}$

The following results will be stated using the vertical sections R_x; however, the results hold using horizontal sections with the appropriate changes.

Definition 4.3.2. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$, where γ is some ordinal.

Denote $p[S]$ by R_S. Denote $({}^{\omega}\omega \times {}^{\omega}\omega) \setminus p[S] = R^S$.

Definition 4.3.3. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$ for some ordinal γ. Let I be a σ-ideal on ${}^{\omega}\omega$ so that $\mathbb{P}_I$ is proper.

Assumption A_Σ asserts that $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} \check{S}$ is a homogeneous tree.

Assumption A_Π asserts $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} \check{S}$ is a homogeneous tree.

Assumption A_Σ and A_Π just assert that the tree S remains homogeneous in $\mathbb{P}_I$-generic extensions. (Since the completeness of countably complete measures is a measurable cardinal and $|\mathbb{P}_I|$ is always less than a measurable cardinal under AC, this is always true under AC.)

Definition 4.3.4. Let S be a homogeneously Suslin tree on $\omega \times \omega \times \gamma$ for some ordinal γ. Let I be a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is a proper forcing.

Let D_Σ be the formula on ${}^\omega\omega \times {}^\omega\omega$ asserting:

$$D_\Sigma(x, T) \Leftrightarrow (T \text{ is tree on } \omega \times \omega) \wedge (\forall y)(R_S(x, y) \Leftrightarrow T^y \text{ is ill-founded}).$$

Let D_Π be the formula on ${}^\omega\omega \times {}^\omega\omega$ asserting:

$$D_\Pi(x, T) \Leftrightarrow (T \text{ is a tree on } \omega \times \omega) \wedge (\forall y)(\neg(R^S(x, y)) \Leftrightarrow T^y \text{ is ill-founded}).$$

If $D_\Sigma(x, T)$ holds, then T is a tree which witnesses $(R_S)_x$ is Σ^1_1. Similarly, if $D_\Pi(x, T)$ holds, then T is a tree which witnesses ${}^\omega\omega \setminus (R^S)_x$ is Σ^1_1, i.e. $(R^S)_x$ is $\boldsymbol{\Pi}^1_1$.

Definition 4.3.5. Let S be a homogeneously Suslin tree on $\omega \times \omega \times \gamma$ for some ordinal γ. Let I be a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is a proper forcing.

Let assumption B_Σ say: $(\forall x)(\exists T)D_\Sigma(x, T)$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} (\forall x)(\exists T)D_\Sigma(x, T)$.

Let assumption B_Π say: $(\forall x)(\exists T)D_\Pi(x, T)$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} (\forall x)(\exists T)D_\Pi(x, T)$.

Assumption B_Σ states that all R_S sections are Σ^1_1 and all R_S sections remain Σ^1_1 in $\mathbb{P}_I$-generic extensions. Similarly, assumption B_Π states that all R^S sections are $\boldsymbol{\Pi}^1_1$ and all R^S sections remain $\boldsymbol{\Pi}^1_1$ in $\mathbb{P}_I$-generic extensions.

Definition 4.3.6. Let S be a homogeneously Suslin tree on $\omega \times \omega \times \gamma$ for some ordinal γ. Let I be a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is a proper forcing.

Let assumption C_Σ state: There is an ordinal ϵ and a tree U on $\omega \times \omega \times \epsilon$ so that $p[U] = \{(x, T) : D_\Sigma(x, T)\}$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} p[\check{U}] = \{(x, T) : D_\Sigma(x, T)\}$.

Let assumption C_Π state: There is an ordinal ϵ and a tree U on $\omega \times \omega \times \epsilon$ so that $p[U] = \{(x, T) : D_\Pi(x, T)\}$ and $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} p[\check{U}] = \{(x, T) : D_\Pi(x, T)\}$.

Assumption C_Σ states that the set defined by D_Σ has a tree representation that continues to represent the formula D_Σ in $\mathbb{P}_I$-generic extensions. C_Π is similar.

The following shows under certain assumptions a more general canonicalization property holds for relations. [2] defines this phenomenon as the rectangular canonization property.

Theorem 4.3.7. *Let γ be an ordinal. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$. Let I be a σ-ideal on ${}^{\omega}\omega$ so that $\mathbb{P}_I$ is proper. Assume A_Σ, B_Σ, and C_Σ hold for S and I.*

Then for any I^+ $\mathbf{\Delta}^1_1$ set $B \subseteq {}^{\omega}\omega$, there exists an I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $R_S \cap (C \times {}^{\omega}\omega)$ is an Σ^1_1 relation.

Proof. Let U be the tree on $\omega \times \omega \times \epsilon$ witnessing C_Σ for S and I.

Let $M \prec H_\Xi$ be a countable elementary substructure with Ξ sufficiently large and $B, I, \mathbb{P}_I, S, U \in M$.

Claim 1 : Let g be $\mathbb{P}_I$-generic over M. If $x, T \in M[g]$ and $M[g] \models D_\Sigma(x, T)$, then $V \models D_\Sigma(x, T)$.

Proof of Claim 1: By assumption C_Σ for S and I and the fact that $M \prec H_\Xi$, $M[g] \models D_\Sigma(x, T)$ implies $M[g] \models (x, T) \in p[U]$. There exists some $f \in M[g]$ with $f : \omega \to \epsilon$ so that $M[g] \models (x, T, f) \in [U]$. Hence for each $n \in \omega$, $M[g] \models (x \restriction n, T \restriction n, f \restriction n) \in U$. For each $n \in \omega$, $(x \restriction n, T \restriction n, f \restriction n) \in M$. So by absoluteness, $M \models (x \restriction n, T \restriction n, f \restriction n) \in U$. For all $n \in \omega$, $V \models (x \restriction n, T \restriction n, f \restriction n) \in U$. $V \models (x, T) \in p[U]$. $V \models D_\Sigma(x, T)$.

Now fix a $g \in {}^{\omega}\omega$ so that g is $\mathbb{P}_I$-generic over M.

As $M \prec H_\Xi$, $M \models (\forall x)(\exists T) D_\Sigma(x, T)$. $M[g] \models (\forall x)(\exists T) D_\Sigma(x, T)$ by assumption B_Σ and the fact that $M \prec H_\Xi$. So fix a tree T on $\omega \times \omega$ so that $M[g] \models D_\Sigma(g, T)$.

Consider the following game $G^{g,T}$:

$$\begin{array}{c|ccccccc} & m_0, n_0 & & m_1, n_1 & & \ldots & m_{k-1}, n_{k-1} & \\ \hline & & \alpha_0 & & \alpha_1 & \ldots & & \alpha_{k-1} \end{array}$$

The rules are:

(1) Player 1 plays $m_i, n_i \in \omega$. Player 2 plays $\alpha_i < \gamma$.

(2) $(m_0 \ldots m_{k-1}, n_0 \ldots n_{k-1}) \in T$.

(3) $(g \restriction k, m_0 \ldots m_{k-1}, \alpha_0 \ldots \alpha_{k-1}) \in S$.

The first player to violate these rules loses. If the game continues forever, then Player 2 wins.

Claim 2 : In $M[g]$, Player 2 has a winning strategy in the game $G^{g,T}$.

Proof of Claim 2: By an appropriate coding, $G^{g,T}$ is equivalent to a game G_A, where $A \subseteq {}^{\omega}\gamma$ is a closed subset.

Suppose Player 2 does not have a winning strategy. By closed determinacy (Fact 4.2.12), Player 1 must have a winning strategy τ^*.

By assumption A_Σ, S is a homogeneous tree in $M[g]$. Let $(\mu_t : t \in {}^{<\omega}(\omega \times \omega))$ be a homogeneity system witnessing the homogeneity of S.

Now two sequences of natural numbers, $(a_i : i \in \omega)$ and $(b_i : i \in \omega)$, and a sequence $(A_n : n \in \omega)$ so that $A_n \subseteq {}^{n}\gamma$ will be constructed by recursion:

Let $a_0, b_0 \in \omega$ so that $(a_0, b_0) = \tau^*(\emptyset)$. Let $A_0 = \{\emptyset\}$.

Suppose $a_0, ..., a_{k-1}$, $b_0, ..., b_{k-1}$, and $A_0, ..., A_{k-1}$ has been constructed. Define the function

$$h_k : S^{(g \restriction k, a_0 \ldots a_{k-1})} \longrightarrow \omega \times \omega$$

defined by

$$h_k(\beta_0 \ldots \beta_{k-1}) = \tau^*(a_0, b_0, \beta_0, ..., a_{k-1}, b_{k-1}, \beta_{k-1}).$$

$\mu_{(g \restriction k, a_0 \ldots a_{k-1})}$ concentrates on $S^{(g \restriction k, a_0 \ldots a_{k-1})}$ and is countably complete; therefore, there is a unique (a_k, b_k) so that

$$h_k^{-1}[\{(a_k, b_k)\}] \in \mu_{(g \restriction k, a_0 \ldots a_{k-1})}.$$

Let $A_k = h_k^{-1}[\{(a_k, b_k)\}]$.

This completes the construction of $(a_i : i \in \omega)$, $(b_i : i \in \omega)$, and $(A_i : i \in \omega)$.

Let $L \in {}^{\omega}(\omega \times \omega)$ be such that for all $i \in \omega$, $L(i) = (a_i, b_i)$. Note that $L \in [T]$. To see this, suppose not. Then there is some least $k \in \omega$ so that $L \restriction (k+1) = (a_0 \ldots a_k, b_0 \ldots b_k) \notin T$. For $i \leq k$, define $\mu_i = \mu_{g \restriction i, a_0 \ldots a_{i-1}}$. For $0 \leq i \leq j \leq k$, let $\pi_{j,i} : {}^{j}\gamma \to {}^{i}\gamma$ be defined by $\pi_{j,i}(s) = s \restriction i$. By definition of the homogeneity system for S, for $0 \leq i \leq j \leq k$, μ_j is an extension of μ_i. Hence for all $0 \leq i \leq k$, $\pi_{k,i}^{-1}[A_i] \in \mu_k$. By countable completeness of μ_k, $\bigcap_{0 \leq i \leq k} \pi_{k,i}^{-1}[A_i] \in \mu_k$. Let $(\beta_0 \ldots \beta_{k-1}) \in \bigcap_{0 \leq i \leq k} \pi_{k,i}^{-1}[A_i]$. Consider the following play of $G^{g,T}$ where player 1 uses the strategy τ^* and Player 2 plays $(\beta_0 \ldots \beta_{k-1})$:

$$\begin{array}{ccccccccc} a_0, b_0 & & a_1, b_1 & & \ldots & a_{k-1}, b_{k-1} & & a_k, b_k \\ \hline & \beta_0 & & \beta_1 & \ldots & & \beta_{k-1} & \end{array}$$

Note that for all $0 \le i \le k$, $(\beta_0 \ldots \beta_{i-1}) \in A_i = h_i^{-1}[\{(a_i, b_i)\}] \subseteq S^{(g \restriction i, a_0 \ldots a_{i-1})}$. So rule (3) of the game $G^{g,T}$ is not violated by Player 2. However, $(a_0 \ldots a_k, b_0 \ldots b_k) = L \restriction (k+1) \notin T$. Player 1 violates rule (2) and is the first player to violate any rules. Player 1 loses this game. This contradicts the assumption that τ^* is a winning strategy for Player 1. So this completes the proof that $L \in [T]$.

Let $\mathfrak{a} = (a_i : i \in \omega)$. Since $L \in [T]$ and $D_\Sigma(g, T)$, this implies that $R_S(g, \mathfrak{a})$.

Now let $J \in {}^\omega(\omega \times \omega)$ be such that for all $k \in \omega$, $J \restriction k = (g \restriction k, a_0 \ldots a_{k-1})$. Then by definition of S, $J \in p[S]$. Since S is a homogeneous tree via $(u_t : t \in {}^{<\omega}(\omega \times \omega))$, $(\mu_{J \restriction k} : k \in \omega)$ is a countably complete tower of measures.

Each $A_k \in \mu_{g \restriction k, a_0 \ldots a_{k-1}} = \mu_{J \restriction k}$. So by the countable completeness of the tower, there exists some $\Phi : \omega \to \gamma$ so that for all $k \in \omega$, $\Phi \restriction k \in A_k$. Now consider the play of $G^{g,T}$ where Player 1 uses its winning strategy τ^* and Player 2 plays Φ. By construction of the sequences $(a_i : i \in \omega)$, $(b_i, i \in \omega)$, and $(A_i : i \in \omega)$, the game looks as follows:

$$\begin{array}{ccccccc} a_0, b_0 & & a_0, b_1 & & \ldots & a_{k-1}, b_{k-1} & \\ \hline & \Phi(0) & & \Phi(1) & \ldots & & \Phi(k-1) \end{array}$$

Neither players violate any rules in this play. Hence the game continues forever, and so Player 2 wins this play of $G^{g,T}$. This contradicts the fact that τ^* was a winning strategy for Player 1.

So Player 1 could not have had a winning strategy. Player 2 must have a winning strategy in $G^{g,T}$. This completes the proof of Claim 2.

By Claim 2, Player 2 has a winning strategy $\tau \in M[g]$.

<u>Claim 3</u> : τ is a winning strategy for $G^{g,T}$ in V.

Proof of Claim 3: Suppose the following is a play of $G^{g,T}$ in which Player 2 uses τ and loses:

$$\begin{array}{ccccccc} m_0, n_0 & & m_1, n_1 & & \ldots & m_{k-1}, n_{k-1} & \\ \hline & \alpha_0 & & \alpha_1 & \ldots & & \alpha_{k-1} \end{array}$$

Since $\tau \in M[g]$ and ${}^{<\omega}\omega \subseteq M[g]$, this entire finite play belongs to $M[g]$. So, Player 2 loses this game in $M[g]$, as well. This contradicts τ being a winning strategy in $M[g]$. This completes the proof of Claim 3.

<u>Claim 4</u> : For all $y \in {}^{\omega}\omega$, $R_S(g, y)$ if and only if $(S \cap M)^{(g,y)}$ is ill-founded.

Proof of Claim 4: By Claim 1, $M[g] \models D_\Sigma(g, T)$ implies $V \models D_\Sigma(g, T)$. Hence in V, T gives the Σ^1_1 definition of $(R_S)_g$.

Suppose $R_S(g, y)$. Then T^y is ill-founded. Let $f \in [T^y]$. Consider the following play of the game $G^{g,T}$ where Player 1 plays y and f, and Player 2 responds using its winning strategy τ.

$$\begin{array}{c|c|c|c|c|c|c} y(0), f(0) & & y(1), f(1) & & \ldots & y(k-1), f(k-1) & \\ \hline & \alpha_0 & & \alpha_1 & \ldots & & \alpha_{k-1} \end{array}$$

Since $f \in [T^y]$, Player 1 can not lose. Since τ is a winning strategy for Player 2, Player 2 also does not lose at a finite stage. Hence Player 2 wins by having the game continue forever. Let $\Phi : \omega \to \gamma$ be the sequence coming from Player 2's response, i.e. for all k, $\Phi(k) = \alpha_k$.

Since $\tau \in M[g]$ and ${}^{<\omega}\omega \subseteq M[g]$, each finite partial play of $G^{g,T}$ above belongs to $M[g]$. Hence $\Phi \restriction k \in M[g]$ for all $k \in \omega$. As $\mathrm{On}^M = \mathrm{On}^{M[g]}$ because $\mathbb{P}_I$ is proper, $(g \restriction k, y \restriction k, \Phi \restriction k) \in (S \cap M)$ for all $k \in \omega$.

It has been shown that $R_S(g, y)$ implies $(S \cap M)^{(g,y)}$ is ill-founded.

Of course, if $(S \cap M)^{(g,y)}$ is ill-founded, then $S^{(g,y)}$ is ill-founded. By definition, $R_S(g, y)$.

This completes the proof of Claim 4.

Let $b : \omega \to \mathrm{On}^M$ be a bijection. Define a new tree S' on $\omega \times \omega \times \omega$ by $(s_1, s_2, s_3) \in S' \Leftrightarrow (s_1, s_2, b \circ s_3) \in S$.

By Fact 4.2.4, let $C \subseteq B$ be the I^+ $\mathbf{\Delta}^1_1$ set of $\mathbb{P}_I$-generic reals over M inside B. By Claim 4, for all $y \in {}^{\omega}\omega$, $R_S(g, y) \Leftrightarrow (S')^{(g,y)}$ is ill-founded. $R_S \cap (C \times {}^{\omega}\omega)$ is Σ^1_1. The proof of the theorem is complete. $\square$

Theorem 4.3.8. *Let γ be an ordinal. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$. Let I be a σ-ideal on ${}^{\omega}\omega$ so that $\mathbb{P}_I$ is proper. Assume A_Π, B_Π, and C_Π hold for S and I.*

Then for any I^+ $\mathbf{\Delta}^1_1$ set $B \subseteq {}^\omega\omega$, there exists an I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $R^S \cap (C \times {}^\omega\omega)$ is a $\mathbf{\Pi}^1_1$ relation.

Proof. The proof of this is very similar to the proof of Theorem 4.3.7. ◻

Theorem 4.3.9. *Let γ and ν be ordinals. Let S be a homogeneous tree on $\omega \times \omega \times \gamma$. Let U be a homogeneous tree on $\omega \times \omega \times \nu$. Suppose $p[S] = ({}^\omega\omega \times {}^\omega\omega) \setminus p[U]$. Let $R = R_S = R^U$. Let I be a σ-ideal on ${}^\omega\omega$ such that $\mathbb{P}_I$ is a proper forcing. Suppose A_Σ, B_Σ, and C_Σ holds for S and I. Suppose A_Π, B_Π, and C_Π holds for U and I.*

Then for any I^+ $\mathbf{\Delta}^1_1$ set $B \subseteq {}^\omega\omega$, there exists an I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq B$ so that $R \cap (C \times {}^\omega\omega)$ is a $\mathbf{\Delta}^1_1$ relation.

Proof. By Theorem 4.3.7, there is some I^+ $\mathbf{\Delta}^1_1$ set $C' \subseteq B$ so that $R \cap (C' \times {}^\omega\omega)$ is Σ^1_1. By Theorem 4.3.8, there is some I^+ $\mathbf{\Delta}^1_1$ set $C \subseteq C'$ so that $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Pi}^1_1$. Therefore, $R \cap (C \times {}^\omega\omega)$ is $\mathbf{\Delta}^1_1$. ◻

If the above assumptions holds and $R_S = E$ defines an equivalence relation with all Σ^1_1 classes, then there is some I^+ $\mathbf{\Delta}^1_1$ set so that $E \upharpoonright C$ is an Σ^1_1 equivalence relation.

Simialarly, suppose $R_S = G$ is a graph on ${}^\omega\omega$. Then $G_x = \{y : x\ G\ y\}$ is the set of neighbors of x. Suppose G_x is Σ^1_1 for all x. Then there is an I^+ $\mathbf{\Delta}^1_1$ set C so that the induced subgraph $G \upharpoonright C$ is an Σ^1_1 graph.

Since equivalence relations were the original motivation, the rest of the paper will focus on equivalence relations; however, all the results hold for graphs and relations with the appropriate sections.

4.4 Canonicalization for Equivalence Relations in $L(\mathbb{R})$

This section will provide a brief description of the theory of tree representations of subsets of ${}^\omega\omega$ and absoluteness. This will be used to indicate some circumstances in which the assumptions A_Σ, B_Σ, C_Σ, A_Π, B_Π, and C_Π hold. The results of the previous section will be applied to some familiar classes of equivalence relations. The following discussion is in $\mathsf{ZF} + \mathsf{DC}$ until it is explicitly mentioned that AC will be assumed.

Definition 4.4.1. Let κ be a cardinal. A κ-weak homogeneity system with support some ordinal γ is a sequence of κ-complete measures on ${}^{<\omega}\gamma$, $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$, so that:

(i) If $s \neq t$, then $\mu_s \neq \mu_t$.

(ii) $\dim(\mu_s) \leq |s|$.

(iii) If μ_s is an extension of some measure ν, then there exists some $k < |s|$ so that $\mu_{s \restriction k} = \nu$.

Define $W_{\bar{\mu}}$ by

$$W_{\bar{\mu}} = \{x \in {}^{\omega}\omega : (\exists f \in {}^{\omega}\omega)(f \text{ is an increasing sequence}$$

$$\wedge (\mu_{x \restriction f(k)} : k \in \omega) \text{ is a countably complete tower})\}$$

A set $A \subseteq {}^{\omega}\omega$ is κ-weakly homogeneous if and only there is a κ-weak homogenity system $\bar{\mu}$ so that $A = W_{\bar{\mu}}$.

Definition 4.4.2. Let γ be an ordinal. A tree on $\omega \times \gamma$ is κ-weakly homogeneous if and only there is some κ-weak homogeneity system $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ so that $p[T] = W_{\bar{\mu}}$ and for all $s \in {}^{<\omega}\omega$, there is some $k \leq |s|$ so that μ_s concentrates on $T^{s \restriction k}$.

$A \subseteq {}^{\omega}\omega$ is κ-weakly homogeneously Suslin if and only if $A = p[T]$ for some tree T which is κ-weakly homogeneous.

Fact 4.4.3. *If $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ is a κ-weak homogeneity system with support γ, then there is a tree T on $\omega \times \gamma$ so that $\bar{\mu}$ witnesses T is κ-weakly homogeneously Suslin.*

Hence a set is κ-weakly homogeneous if and only if it is κ-weakly homogeneously Suslin.

Proof. See [16], Proposition 1.12. □

Definition 4.4.4. Let μ be a countably complete measure on ${}^{<\omega}X$. Let M_μ be the Mostowski collapse of the ultrapower $\mathrm{Ult}(V, \mu)$. Let $j_\mu : V \to M_\mu$ be the composition of the ultrapower map and the Mostowski collapse map.

Suppose ν and μ are countably complete measures on ${}^{<\omega}X$. Suppose for some $m \leq n$, $\dim(\mu) = m$ and $\dim(\nu) = n$, and ν is an extension of μ. Define $\Lambda_{m,n} : {}^{{}^{m}X}V \to {}^{{}^{n}X}V$ by $\Lambda_{m,n}(f)(s) = f(s \restriction m)$ for each $s \in {}^{n}X$. Define an elementary embedding $\mathrm{Ult}(V, \nu) \to \mathrm{Ult}(V, \mu)$ by $[f]_\nu \mapsto [\Lambda_{m,n}(f)]_\mu$. This induces an elementary embedding $j_{\nu,\mu} : M_\nu \to M_\mu$.

Definition 4.4.5. Let γ and θ be ordinals. Let $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ be a weak homogeneity system with support γ. The Martin-Solovay tree with respect to $\bar{\mu}$ below θ, denoted $\mathrm{MS}_\theta(\bar{\mu})$, is a tree on $\omega \times \theta$ defined by: for all $s \in {}^{<\omega}\omega$ and $h \in {}^{|s|}\theta$

$$(s, h) \in \mathrm{MS}_\theta(\bar{\mu})$$

$$\Leftrightarrow (\forall i < j < |s|)(\mu_{s\restriction j} \text{ is an extension of } \mu_{s\restriction i} \Rightarrow j_{\mu_{s\restriction i},\mu_{s\restriction j}}(h(i)) > h(j))$$

If $(u_n : n \in \omega)$ is a tower of measure, then the tower is countably complete if and only if the directed limit of the directed system $(M_{\mu_i} : j_{\mu_i,\mu_j} : i < j < \omega)$ is well-founded. Suppose $x \in p[\mathrm{MS}_\theta(\bar{\mu})]$. If $(x, \Phi) \in [\mathrm{MS}_\theta(\bar{\mu})]$, then Φ witnesses in a continuous way that the directed limit model is ill-founded. This shows that $x \in p[\mathrm{MS}_\theta(\bar{\mu})]$ implies that $x \notin W_{\bar{\mu}}$. In fact, the converse is also true giving the following result:

Fact 4.4.6. *($\mathsf{ZF} + \mathsf{DC}$) Let κ be a cardinal. Suppose $\bar{\mu}$ is a κ-weak homogeneity system with support γ. Then if $\theta > |\gamma|^+$, then $p[\mathrm{MS}_\theta(\bar{\mu})] = {}^\omega\omega \setminus W_{\bar{\mu}}$.*

Proof. See [16] Lemma 1.19, [7] Fact 1.3.12, or [5] Theorem 4.10. □

Let μ be a κ-complete ultrafilter on some set X. Let $\mathbb{P}$ be a forcing with $|\mathbb{P}| < \kappa$. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. It can be shown that if $f^* : X \to V$ is a function in $V[G]$, then there is a function $f \in V$ and $A \in \mu$ so that $V[G] \models (\forall x \in A)(f(x) = f^*(x))$.

In $V[G]$, define $\mu^* \subseteq \mathcal{P}(X)$ by $A \in \mu^*$ if and only there exists a $B \in \mu$ so that $B \subseteq A$. In $V[G]$, μ^* is a κ-complete ultrafilter on X. Let M_{μ^*} denote the Mostowski collapse of $\mathrm{Ult}(V[G], \mu^*)$. Let $j^*_{\mu^*} : V[G] \to M_{\mu^*}$ be the induced elementary embedding.

In $V[G]$, $\mathrm{Ult}(V, \mu)$ can be embedded into $\mathrm{Ult}(V[G], \mu^*)$ as follows: for all $f \in ({}^X V) \cap V$, $[f]_\mu \mapsto [f]_{\mu^*}$. If $f \in ({}^X V) \cap V$ and $g' \in {}^X V[G]$ are such that $\mathrm{Ult}(V[G], \mu^*) \models [g']_{\mu^*} \in [f]_{\mu^*}$, then $\{x \in X : g'(x) \in f(x)\} \in \mu^*$. Therefore, one can find a $g^* \in V[G]$ so that $g^* : X \to V$ and $[g']_{\mu^*} = [g^*]_{\mu^*}$. By the above observation, one can find a $g \in V$ so that $[g]_{\mu^*} = [g^*]_{\mu^*} = [g']_{\mu^*}$. This shows that $\mathrm{Ult}(V, \mu)$ is identified (via the embedding above) as an $\in^{\mathrm{Ult}(V[G],\mu^*)}$-initial segment of $\mathrm{Ult}(V[G], \mu^*)$. After Mostowski collapsing the ultrapowers, it can be seen that $j_{\mu^*} \restriction M_\mu = j_\mu$.

Suppose $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ is a κ-weak homogeneity system. Denote $\bar{\mu}^* = (\mu^*_s : s \in {}^{<\omega}\omega)$. $\bar{\mu}^*$ is a κ-weak homogeneity system. From the construction, the Martin-Solovay trees depends only on $j_{\mu^*_s} \restriction \mathrm{ON}$. So by the above discussion, $\mathrm{MS}_\theta(\bar{\mu})^V = \mathrm{MS}_\theta(\bar{\mu}^*)^{V[G]}$. Hence Fact 4.4.6 implies that $V[G] \models p[\mathrm{MS}_\theta(\bar{u})] = {}^\omega\omega \setminus W_{\bar{\mu}^*}$.

(The above argument can be applied to a κ-homogeneous tree S and its witnessing κ-homogeneity system $\bar{\mu}$ to show that if $|\mathbb{P}| < \kappa$, then $\bar{\mu}^*$ is a κ-weak homogeneity system for S in $V[G]$. Assuming the axiom of choice, this shows assumption A_Σ and A_Π.)

Now suppose that T is a κ-weakly homogeneous tree on $\omega \times \alpha$ witnessed by the κ-weak homogeneity system $\bar{\mu}$. This gives that $p[T] = W_{\bar{\mu}}$. One seeks to show that $p[\mathrm{MS}_\theta(\bar{\mu})^V]$ continues to represent ${}^\omega\omega \setminus p[T]$ in $V[G]$. By the previous paragraph, it suffices to show that $V[G] \models p[T] = W_{\bar{\mu}^*}$: If $x \in W_{\bar{\mu}^*}$, then there is an increasing function $f : \omega \to \omega$ so that $(\mu^*_{x\restriction f(n)} : n \in \omega)$ is a countably complete tower. For all n, $\mu^*_{x\restriction f(n)}$ concentrates on $T^{x\restriction n}$. So by countably completeness, there is a path $\Phi \in [T^x]$. So $x \in p[T]$. Conversely, suppose $x \in p[T]$. Fact 4.4.6 implies that in V, T and $\mathrm{MS}_\theta(\bar{\mu})$ are complementing trees. By the absoluteness of well-foundedness, $V[G] \models \emptyset = p[T] \cap p[\mathrm{MS}_\theta(\bar{\mu})] = p[T] \cap p[\mathrm{MS}_\theta(\bar{\mu}^*)]$. So $x \notin p[\mathrm{MS}_\theta(\bar{\mu}^*)]$. Then applying Fact 4.4.6 in $V[G]$ to the weak homogeneity system $\bar{\mu}^*$, one obtains that $x \in W_{\bar{\mu}^*}$.

So in summary:

Fact 4.4.7. ($\mathsf{ZF} + \mathsf{DC}$) *Let κ be a cardinal. Let T be a κ-weakly homogeneous tree on $\omega \times \gamma$, for some ordinal γ, with κ-weak homogeneity system $\bar{\mu}$. Let $\theta > |\gamma|^+$. Let $\mathbb{P}$ be a forcing with $|\mathbb{P}| < \kappa$ and $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V.*

$V[G] \models \mathrm{MS}_\theta(\bar{\mu}^*) = \mathrm{MS}_\theta(\bar{\mu})^V$. $V[G] \models p[\mathrm{MS}_\theta(\bar{\mu})^V] = {}^\omega\omega \setminus p[T]$.

Proof. See [16], Section 1 and especially Lemma 1.19. Also see [7], Section 1.3. $\square$

So if T is κ-weakly homogeneous, an appropriate Martin-Solovay tree will continue to represent the complement of $p[T]$ in generic extensions by forcings of cardinality less than κ. The Martin-Solovay trees give the generically-correct tree representations for complements of κ-weakly homogeneously Suslin sets. However, the formulas D_Σ and D_Π involve more negations and quantifications over ${}^\omega\omega$. Multiple iterations of the Martin-Solovay construction will be needed. The following results are useful for continuing the Martin-Solovay construction of generically-correct tree representation for more complex sets. In addition, these results will also imply that these representations are also homogeneously Suslin. Until the end of this section, the axiom of choice will be assumed.

Definition 4.4.8. If $B \subseteq {}^k({}^\omega\omega) \times {}^\omega\omega$, denote

$$\exists^{\mathbb{R}} B = \{x : (\exists y)((x, y) \in B)\}$$

$$\forall^{\mathbb{R}} B = \{x : (\forall y)((x, y) \in B)\}$$

If $A \subseteq {}^k({}^\omega\omega)$, then denote

$$\neg A = {}^k({}^\omega\omega) \setminus A$$

Fact 4.4.9. *Let $A \subseteq {}^\omega\omega$. A is κ-weakly homogeneously Suslin if and only if there is a κ-homogeneously Suslin set $B \subseteq {}^\omega\omega \times {}^\omega\omega$ so that $A = \exists^{\mathbb{R}} B$.*

Proof. See [16], Proposition 1.10. □

A Woodin cardinal is a technical large cardinal which has been very useful in descriptive set theory. (See [7], Section 1.5 for more information about Woodin cardinals.)

Fact 4.4.10. *Let δ be a Woodin cardinal. Let $\bar{\mu} = (\mu_s : s \in {}^{<\omega}\omega)$ be a δ^+-weak homogeneity system with support $\gamma \in$ ON. Then for sufficiently large θ, $\mathrm{MS}_\theta(\bar{\mu})$ is κ-homogeneous for all $\kappa < \delta$.*

Proof. See [8]. □

Definition 4.4.11. If κ is a cardinal, then let Hom_κ be the collection of κ-homogeneously Suslin subsets of ${}^\omega\omega$. Let $\mathrm{Hom}_{<\kappa} = \bigcap_{\gamma<\kappa} \mathrm{Hom}_\gamma$.

The following are some well-known results on what sets can be in $\mathrm{Hom}_{<\lambda}$ when λ is limit of Woodin cardinals.

Fact 4.4.12. *(Martin-Steel) Let λ be a limit of Woodin cardinals. Then $\mathrm{Hom}_{<\lambda}$ is closed under complements and $\forall^{\mathbb{R}}$.*

Proof. Let $A \in \mathrm{Hom}_{<\lambda}$. Let $\kappa < \lambda$. Let δ be a Woodin cardinal so that $\kappa < \delta < \lambda$. Let $A = p[T]$ for some δ^+-weakly homogeneous tree via a δ^+-weak homogeneity system $\bar{\mu}$. By Fact 4.4.6 and Fact 4.4.10, $\neg A = p[\mathrm{MS}_\theta(\bar{\mu})]$ and $\mathrm{MS}_\theta(\bar{\mu})$ is κ-homogeneous.

Let $A \subseteq {}^\omega\omega \times {}^\omega\omega$ be in $\mathrm{Hom}_{<\lambda}$. Let $\kappa < \lambda$. Let δ be a Woodin cardinal so that $\kappa < \delta < \lambda$. By Fact 4.4.9, $\exists^{\mathbb{R}} A$ is δ^+-weakly homogeneously Suslin via a δ^+-weak homogeneity system $\bar{\mu}$. By Fact 4.4.6 and Fact 4.4.10, $\mathrm{MS}_\theta(\bar{\mu})$ is κ-homogeneously Suslin and $\forall^{\mathbb{R}} A = \neg\exists^{\mathbb{R}} A = p[\mathrm{MS}_\theta(\bar{\mu})]$. □

Fact 4.4.13. *(Martin) If κ is a measurable cardinal, then every $\mathbf{\Pi}^1_1$ set is κ-homogeneously Suslin.*

Proof. See [11], Theorem 4.15. □

Fact 4.4.14. *(Martin-Steel) Let λ be a limit of Woodin cardinals, then all projective sets are in* $\mathrm{Hom}_{<\lambda}$.

Proof. Every Woodin cardinal has a stationary set of measurable cardinals below it. Hence every $\mathbf{\Pi}^1_1$ set is κ-homogeneously Suslin for all $\kappa < \lambda$. That is, all $\mathbf{\Pi}^1_1$ sets are in $\mathrm{Hom}_{<\lambda}$. Then by closure given by Fact 4.4.12, all projective sets are in $\mathrm{Hom}_{<\lambda}$. □

In fact, an even larger class of sets of reals can be homogeneously Suslin: $L(\mathbb{R})$ is the smallest transitive class model of ZF containing all the reals of V, i.e. $({}^{\omega}\omega)^V \subseteq L(\mathbb{R})$.

Fact 4.4.15. *(Woodin) Suppose λ is a limit of Woodin cardinals and there is a measurable cardinal greater than λ. Then every subset of ${}^{\omega}\omega$ in $L(\mathbb{R})$ is in* $\mathrm{Hom}_{<\lambda}$.

In the previous section, sets given by projections of certain trees were essentially identified with their trees. Homogeneously Suslin sets were defined to be those sets that can be presented as projections of some trees satisfying certain properties. In the ground model, there could be many homogeneous trees representing the same homogeneously Suslin set A. When considering generic extensions of the ground model, there is a question of which tree should be used to represent A in the generic extension. For instance, suppose $\kappa_1 < \kappa_2$. In the ground model, suppose $A = p[T_1]$ where T_1 is a κ_1-homogeneous tree and $A = p[T_2]$ where T_2 is a κ_2-homogeneous tree. Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are two different forcing. Which tree should represent A in each forcing extension? Are there circumstances in which one tree may be preferable over another? What are the relations between $p[T_1]$ and $p[T_2]$ in various forcing extensions?

Absolutely complemented trees and universally Baireness provide a way to interpret homogeneously Suslin sets in a way which is independent of the homogeneous tree representation in some sense:

Definition 4.4.16. (See [3]) Let κ be an ordinal. Let T be a tree on $\omega \times X$ and let U be a tree on $\omega \times Y$, for some sets X and Y. T and U are κ-absolute complements if and only if for all forcings $\mathbb{P} \in V_\kappa$ and all $G \subseteq \mathbb{P}$ which are $\mathbb{P}$-generic over V, $V[G] \models p[T] = {}^\omega\omega \setminus p[U]$.

A tree T on $\omega \times X$ is κ-absolutely complemented if and only if there exists some tree U on $\omega \times Y$ (for some set Y) so that T and U are κ-absolute complements.

A set $A \subseteq {}^\omega\omega$ is κ-universally Baire if and only if $A = p[T]$ for some tree T which is κ-absolutely complemented.

Fact 4.4.17. *Let T_1 and T_2 be trees on $\omega \times \gamma_1$ and $\omega \times \gamma_2$ which are κ-absolutely complemented. If $\mathbb{P} \in V_\kappa$ and $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V, then $V[G] \models p[T_1] = p[T_2]$.*

Proof. Let U_1 and U_2 be trees witnessing that T_1 and T_2 are κ-absolutely complemented, respectively. Suppose without loss of generality that $V[G] \models p[T_1] \cap ({}^\omega\omega \setminus p[T_2]) \neq \emptyset$. Since T_2 and S_2 are κ-absolutely complementing, $V[G] \models p[T_1] \cap p[S_2] \neq \emptyset$. Define a tree $T_1 \otimes S_2$ by

$$(s, h, g) \in T_1 \otimes S_2 \Leftrightarrow (s, h) \in T_1 \wedge (s, g) \in S_2.$$

In $V[G]$, $T_1 \otimes S_2$ is ill-founded. By the absoluteness of well-foundedness, $V \models T_1 \otimes S_2$ is ill-founded. So $V \models p[T_1] \cap p[S_2] \neq \emptyset$. This is impossible since in V, $p[T_1] = p[T_2]$, $p[S_1] = p[S_2]$, $p[T_1] = {}^\omega\omega \setminus p[S_1]$, and $p[T_2] = {}^\omega\omega \setminus p[S_2]$. $\square$

So if A is a κ-universally Baire set and if T_1 and T_2 are two κ-absolutely complemented trees so that $V \models A = p[T_1] = p[T_2]$, then either tree can be used to represent A in forcing extensions by forcings in V_κ. As a matter of convention, if A is κ-universally Baire and $\mathbb{P} \in V_\kappa$, the set A will always refer to $p[T]$ for some and any κ-absolutely complemented tree $T \in V$ so that $V \models p[T] = A$.

Fact 4.4.18. *Let κ be a cardinal. κ-weakly homogenously Suslin sets are κ-universally Baire.*

Proof. (See [16], Corollary 1.21) Let $A = p[T]$ where T is a κ-weakly homogeneously Suslin set via κ-weak homogeneity system $\bar{\mu}$. Fact 4.4.7 implies that for an appropriate θ, $\mathrm{MS}_\theta(\bar{\mu})$ witnesses that T is κ-absolutely complemented. $\square$

In particular, κ-homogeneously Suslin sets can be interpreted unambiguously in $\mathbb{P}$-extensions whenever $\mathbb{P} \in V_\kappa$.

Let λ be a limit of Woodin cardinals. Let $\dot{A}$ be a new unary relation symbol. Let $A \subseteq ({}^\omega\omega)^n$ be such that $A \in \mathrm{Hom}_{<\lambda}$. Let $(H_{\aleph_1}, \in, A)$ be the $\{\dot{\in}, \dot{A}\}$-structure with domain $H_{\aleph_1}$ (the hereditarily countable sets) and with $\dot{A}$ interpreted as A. Now let $\mathbb{P} \in V_\lambda$ be some forcing and $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter over V. $\mathbb{P} \in V_\kappa$ for some $\kappa < \lambda$. The structure $(H_{\aleph_1}, \in, A^{V[G]})$ is understood in the following way: It is a structure with domain $H_{\aleph_1}^{V[G]}$ (the hereditarily countable subsets of $V[G]$) and $A^{V[G]}$ is $p[T]^{V[G]}$ for any γ-homogeneous tree T so that $V \models A = p[T]$ and $\gamma \geq \kappa$. By the above discussion, this is independent of which tree T is chosen. Actually, in the proof of the fact below, depending on the quantifier complexity of a particular formula φ involving $\dot{A}$, A will be considered as $p[T]$ for a sufficiently homogeneous tree T so that after the appropriate number of applications of the Martin-Solovay tree construction, the resulting tree representation of φ will be at least κ-homogeneous.

Using ideas very similar to the proof of Fact 4.4.12 (also see the proof of Fact 4.5.12 for Cohen forcing), one has the following absoluteness result:

Fact 4.4.19. *(Woodin) Let λ be a limit of Woodin cardinals. Let $A \in Hom_{<\lambda}$. Let $\mathbb{P} \in V_\lambda$ and $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Then $(H_{\aleph_1}^V, \in, A)$ and $(H_{\aleph_1}^{V[G]}, \in, A^{V[G]})$ are elementarily equivalent.*

Proof. See [16], Theorem 2.6. □

In this setting, V and $V[G]$ satisfy the same formulas involving $\dot{A}$ and quantifications over the reals with the above intended interpretation. In particular, V and $V[G]$ satisfy the same projective formulas.

Now, the above discussion will be applied to indicate when assumptions A_Σ, B_Σ, C_Σ, A_Π, B_Π, and C_Π hold.

Let λ be a limit of Woodin cardinals. By the above discussion about universal Baireness, one may speak about an equivalence relation $E \in \text{Hom}_{<\lambda}$ without explicit reference to a fix tree defining E. By Fact 4.4.12, if $E \in \text{Hom}_{<\lambda}$, then ${}^\omega\omega \setminus E \in \text{Hom}_{<\lambda}$. Given an κ-weakly homogeneous tree representation of E for sufficiently large γ, the associated Martin-Solovay tree will be a sufficiently homogeneous tree representation of ${}^\omega\omega \setminus E$ by Fact 4.4.10. Hence in this setting, $E_S = E^T$, where T is the appropriate Martin-Solovay tree using the homogeneity system on S. (So if E_S has all $\mathbf{\Pi}^1_1$ classes, then the results of Section 4.3 should be applied to E^T using assumption A_Π, B_Π, C_Π for T and I.) Fix a σ-ideal I on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper.

The formula D_Σ and D_Π both involve complements and real quantification over the homogeneously Suslin set E. By Fact 4.4.12, $D_\Sigma, D_\Pi \in \text{Hom}_{<\lambda}$. Starting with an appropriate weakly homogeneous tree representation of E, the process described in the proof of Fact 4.4.12 produces a tree U representing D_Σ or D_Π that is generically correct for $\mathbb{P}_I$, in the sense that $1_{\mathbb{P}_I} \Vdash_{\mathbb{P}_I} p[\check{U}] = \{(x,T) : D_\Sigma(x,T)\}$. So assumption C_Σ holds for E and I. (A similar argument holds for C_Π.)

E having all Σ^1_1 classes can be expressed as a formula using some real quantifiers over the equivalence relation $E \in \mathrm{Hom}_{<\lambda}$. Fact 4.4.19 implies that these statements are absolute to the $\mathbb{P}_I$-extension. The tree S remains homogeneous in the $\mathbb{P}_I$-extension by the remark mentioned before Fact 4.4.7. This shows that A_Σ and B_Σ holds for E and I.

Finally, using the above discussion and results of the previous section, the following can be obtained:

Theorem 4.4.20. *Let λ be a limit of Woodin cardinals. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Let $E \in \mathrm{Hom}_{<\lambda}$ be an equivalence relation on ${}^\omega\omega$.*

If E has all Σ^1_1 ($\boldsymbol{\Pi}^1_1$ or $\boldsymbol{\Delta}^1_1$) classes, then for every I^+ $\boldsymbol{\Delta}^1_1$ set B, there is an I^+ $\boldsymbol{\Delta}^1_1$ $C \subseteq B$ so that $E \restriction C$ is Σ^1_1 ($\boldsymbol{\Pi}^1_1$ or $\boldsymbol{\Delta}^1_1$, respectively).

Theorem 4.4.21. *Suppose there are infinitely many Woodin cardinals. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Let E be a projective equivalence relation on ${}^\omega\omega$.*

If E has all Σ^1_1 ($\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Delta}^1_1$) classes, then for every I^+ $\boldsymbol{\Delta}^1_1$ set B, there is an I^+ $\boldsymbol{\Delta}^1_1$ $C \subseteq B$ so that $E \restriction C$ is Σ^1_1 ($\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Delta}^1_1$, respectively).

Theorem 4.4.22. *Suppose there is a measurable cardinal with infinitely many Woodin cardinals below it. Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is proper. Let $E \in L(\mathbb{R})$ be an equivalence relation on ${}^\omega\omega$.*

If E has all Σ^1_1 ($\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Delta}^1_1$) classes, then for every I^+ $\boldsymbol{\Delta}^1_1$ set B, there is an I^+ $\boldsymbol{\Delta}^1_1$ $C \subseteq B$ so that $E \restriction C$ is Σ^1_1 ($\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Delta}^1_1$, respectively).

With the appropriate assumptions, even more sets of reals are homogeneously Suslin and these canonicalization results would hold for equivalence relations in those classes. For example, Chang's model $L({}^\omega\mathrm{ON}) = \bigcup_{\alpha\in\mathrm{ON}} L({}^\omega\alpha)$ is the smallest inner model of ZF containing all the countable sequences of ordinals of V. Woodin has shown that with a proper class of Woodin cardinals, every set of reals in $L({}^\omega\mathrm{ON})$ is ∞-homogeneously Suslin. Hence under this assumption, the above result would hold for equivalence relations in $L({}^\omega\mathrm{ON})$ with all Σ^1_1, $\boldsymbol{\Pi}^1_1$, or $\boldsymbol{\Delta}^1_1$ classes.

All the above theorems also hold for graphs G so that for all x, G_x is Σ^1_1 ($\mathbf{\Pi}^1_1$ or $\mathbf{\Delta}^1_1$). (See the end of Section 4.3.)

4.5 Canonicalization for All Equivalence Relations

This section will consider Question 4.1.9: Is it consistent that for every equivalence relation E with all $\mathbf{\Delta}^1_1$ classes and every σ-ideal I such that $\mathbb{P}_I$ is proper, there is an I^+ $\mathbf{\Delta}^1_1$ subset C such that $E \restriction C$ is a $\mathbf{\Delta}^1_1$ equivalence relation?

As with other regularity properties, this question has a negative answer if the axiom of choice holds. First, a definition and a property of all $\mathbf{\Pi}^1_1$ equivalence relations:

Definition 4.5.1. An equivalence relation E on ${}^\omega\omega$ is thin if and only if there does not exist a perfect set $P \subseteq {}^\omega\omega$ such that $\neg(x \; E \; y)$ for all $x, y \in P$ with $x \neq y$.

There are Σ^1_1 thin equivalence relation with uncountably many classes. In fact, there are Σ^1_1 thin equivalence relation with all $\mathbf{\Delta}^1_1$ classes and uncountably many classes: for example, the countable admissible ordinal equivalence relation, F_{ω_1}, and any counterexamples to Vaught's conjecture (if they exist). The Silver's dichotomy imply that there are no $\mathbf{\Pi}^1_1$ thin equivalence relations:

Fact 4.5.2. *(Silver) If E is a $\mathbf{\Pi}^1_1$ equivalence relation on ${}^\omega\omega$, then either E has countably many classes or there exists a perfect set of pairwise E-inequivalent elements.*

Proof. See [14]. ◻

Proposition 4.5.3. (ZF) *If there is a well-ordering of ${}^\omega\omega$, then there is a thin equivalence relation E^* on ${}^\omega\omega$ with equivalence classes of size at most two.*

For any σ-ideal I on ${}^\omega\omega$ and any I^+ $\mathbf{\Delta}^1_1$ set C, $E^* \restriction C$ is not $\mathbf{\Delta}^1_1$.

Proof. First a remark: Proposition 4.1.8 is proved in a similar way by showing that in L, there is a thin Δ^1_2 equivalence relation with all countable classes.

Now the proof of the proposition: Using the well-ordering of ${}^\omega\omega$, let $\Phi : 2^{\aleph_0} \to {}^\omega\omega$ be bijection and let $\Psi : 2^{\aleph_0} \to {}^\omega\omega$ be an enumeration of all the perfect trees on ω.

The equivalence E^* is defined by stages through transfinite recursion as follows:

Let $A_0 = \emptyset$. $E^*_0 = \emptyset$.

Stage $\xi + 1$: Suppose A_ξ and E^*_ξ have been defined with $|A_\xi| < 2^{\aleph_0}$. Find some reals r_ξ and s_ξ so that $r_\xi, s_\xi \notin A_\xi$, $r_\xi \neq s_\xi$, and $r_\xi, s_\xi \in [\Psi(\xi)]$.

If $\Phi(\xi) \in A_\xi \cup \{r_\xi, s_\xi\}$, then define $A_{\xi+1} = A_\xi \cup \{r_\xi, s_\xi\}$ and

$$E^*_{\xi+1} = E^*_\xi \cup \{(r_\xi, r_\xi), (s_\xi, s_\xi), (r_\xi, s_\xi), (s_\xi, r_\xi)\}.$$

If $\Phi(\xi) \notin A_\xi \cup \{r_\xi, s_\xi\}$, then define $A_{\xi+1} = A_\xi \cup \{r_\xi, s_\xi, \Phi(\xi)\}$ and

$$E^*_{\xi+1} = E^*_\xi \cup \{(\Phi(\xi), \Phi(\xi)), (r_\xi, r_\xi), (s_\xi, s_\xi), (r_\xi, s_\xi), (s_\xi, r_\xi)\}.$$

At limit stage ξ: Let $A_\xi = \bigcup_{\eta<\xi} A_\eta$ and $E^*_\xi = \bigcup_{\eta<\xi} E^*_\eta$.

Note that $A_{2^{\aleph_0}} = {}^\omega\omega$. Let $E^* = E^*_{2^{\aleph_0}}$. E^* is an equivalence relation on ${}^\omega\omega$. E^* has classes of size at most two. E^* is thin: Suppose T is a perfect tree on ω. Then $T = \Psi(\xi)$ for some $\xi < 2^{\aleph_0}$. Then $r_\xi \ E^* \ s_\xi$ and $r_\xi, s_\xi \in [\Psi(\xi)] = [T]$.

Now let I be a σ-ideal on ${}^\omega\omega$. Suppose there was some I^+ $\mathbf{\Delta}^1_1$ set C so that $E^* \restriction C$ is $\mathbf{\Delta}^1_1$. Since C is I^+ and I is a σ-ideal, C must be uncountable. Since E^* has classes of size at most two, $E \restriction C$ cannot have only countably many classes. Since $\mathbf{\Delta}^1_1$ equivalence relations are $\mathbf{\Pi}^1_1$, the Silver's dichotomy (Fact 4.5.2) implies that there is a perfect set $P \subseteq C$ of E^*-inequivalent elements. There is a perfect tree T so that $[T] = P$. Let $\xi < 2^{\aleph_0}$ be so that $\Psi(\xi) = T$. Then $r_\xi, s_\xi \in [T] = P \subseteq C$ and $r_\xi \ E^* \ s_\xi$. Contradiction. □

Hence to get a positive answer to Question 4.1.9, there can not exist a well-ordering of the reals, so the full axiom of choice must fail.

First, the immediate concern in the choiceless setting is the definition of properness: Since set may not have a cardinality, it is preferable to use V_Ξ rather than H_Ξ. Recall in ZFC, for any σ-ideal I on ${}^\omega\omega$, $\mathbb{P}_I$ was proper if and only if for all sufficiently large cardinals Ξ, any $B \in \mathbb{P}_I$, and all countable elementary $M \prec V_\Xi$ with $\mathbb{P}_I, B \in M$, the set $\{x \in B : x \text{ is } \mathbb{P}_I\text{-generic over } M\}$ is I^+ $\mathbf{\Delta}^1_1$. Without the axiom of choice, the downward Lowenheim-Skolem theorem may fail for structures in countable languages and so there may be no countable elementary substructure. Moreover, in the previous section, it was also important to be able to choose countable elementary substructures containing certain homogeneously Suslin trees.

However, only dependence choice (DC) is needed to prove the following form of the downward Lowenheim-Skolem theorem: Let $\mathscr{L}$ be a countable language. Let M

be an $\mathscr{L}$-structure. Let $A \subseteq M$ be countable. Then there exists an $\mathscr{L}$-elementary substructure N of M so that $A \subseteq N$.

Hence with DC, the definition of properness and the ability to construct elementary substructure of V_Ξ with certain desired objects inside are still available.

Without the axiom of choice, determinacy for various games are useful for settling many questions in descriptive set theory: The axiom of determinacy (AD) asserts that all games of the form in Definition 4.2.11 where the moves are elements of ω are determined. The axiom of determinacy for the reals ($\mathsf{AD}_\mathbb{R}$) asserts that all games of the form in Definition 4.2.11 where the moves are elements of ${}^\omega\omega$ are determined.

In terms of large cardinals, the consistency of AD follows from the consisteny of infinitely many Woodin cardinals. The consistency of $\mathsf{AD}_\mathbb{R}$ follows from the consistency of the existence of a cardinal λ which is both a limit of Woodin cardinals and $< \lambda$-strong cardinals.

Θ denotes the supremum of the ordinals which are surjective images of $\mathbb{R}$. As described above, DC would be useful for carrying out arguments from the earlier sections. A result of Solovay shows that $\mathsf{ZF} + \mathsf{AD}_\mathbb{R} + V = L(\mathscr{P}(\mathbb{R})) + \mathrm{cof}(\Theta) > \omega$ can prove DC. It should be noted that Solovay has also shown that $\mathsf{ZF} + \mathsf{AD}_\mathbb{R} + \mathsf{DC}$ can prove the consistency of $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$; hence $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$ is strictly stronger than $\mathsf{AD}_\mathbb{R}$ in terms in consistency. (See [15] for these results concerning $\mathsf{AD}_\mathbb{R}$ and DC.)

$\mathsf{AD}_\mathbb{R}$ is preferable over AD since $\mathsf{AD}_\mathbb{R}$ can prove that every subset of ${}^\omega\omega$ is homogeneously Suslin and can prove a strong form of absoluteness for proper forcings:

Fact 4.5.4. *(Martin) Under* $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$*, every tree on $\omega \times \lambda$, where λ is an ordinal, is weakly homogeneously Suslin.*

Proof. See [10]. □

Fact 4.5.5. *(Martin) Under* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$*, for every $A \subseteq {}^\omega\omega$, A is homogeneously Suslin if and only if if A and ${}^\omega\omega \setminus A$ are Suslin. Moreover, one can find a homogeneously Suslin tree T on $\omega \times \kappa$, for $\kappa < \Theta$, so that $A = p[T]$.*

Proof. See [9]. □

Fact 4.5.6. *(Martin, Woodin) Under* $\mathsf{ZF} + \mathsf{AD}$, $\mathsf{AD}_\mathbb{R}$ *is equivalent to the statement that every subset of ${}^\omega\omega$ is Suslin.*

Combining the last two facts gives:

Fact 4.5.7. *Under* $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$*, every subset of the* ${}^{\omega}\omega$ *is homogeneously Suslin.*

In the previous section, an important aspect of analyzing tree representations in generic extensions was the fact that any κ-complete measure μ could be naturally extended to a κ-complete measure μ^* in a forcing extension by $\mathbb{P}$, whenever $|\mathbb{P}| < \kappa$.

Let I be a σ-ideal on ${}^{\omega}\omega$. $\mathbb{P}_I$ is in bijection with ${}^{\omega}\omega$ and hence is not well-ordered under AD. Also note that the measures produced using AD to witness homogeneity and weak homogeneity are $\aleph_1$-complete. For the general σ-ideal, it is not clear how to modify the arguments of the previous section in the context of $\mathsf{AD}_{\mathbb{R}}$.

However, there is one important σ-ideal for which the previous arguments will work with minor modifications: For the meager ideal, $\mathbb{P}_{I_{\text{meager}}}$ is forcing equivalent to Cohen forcing, denoted $\mathbb{C}$, which is a countable forcing.

Let T be an $\aleph_1$-weakly homogeneous tree on $\omega \times \gamma$ witnessed by the weak homogeneity system $\bar{\mu}$. Fact 4.4.6, which is provable in $\mathsf{ZF} + \mathsf{DC}$, implies that $V \models p[T] = {}^{\omega}\omega \setminus \mathrm{MS}_{\gamma^+}(\bar{u})$.

Since $|\mathbb{C}| = \aleph_0 < \aleph_1$, any $\aleph_1$-complete measure can be extended to an $\aleph_1$-complete measure in the $\mathbb{C}$-forcing extension. Likewise, every $\aleph_1$-weak homogeneity system $\bar{\mu}$ can be extended to an $\aleph_1$-weak homogeneity system. Fact 4.4.7 and the discussion before it holds when $\kappa = \aleph_1$ and $\mathbb{P} = \mathbb{C}$:

Fact 4.5.8. *Assume* $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$*. Let* T *be an* $\aleph_1$*-weakly homogeneous tree on* $\omega \times \gamma$ *witnessed by the weak homogeneity system* $\bar{\mu}$*. If* $G \subseteq \mathbb{C}$ *is* $\mathbb{C}$*-generic over* V*, then* $V[G] \models \mathrm{MS}_{\gamma^+}(\bar{\mu}^V) = \mathrm{MS}_{\gamma^+}(\bar{\mu}^*)$ *and* $V[G] \models p[\mathrm{MS}_{\gamma^+}(\bar{\mu})^V] = {}^{\omega}\omega \setminus p[T]$.

The notion of an absolutely proper forcing is defined in [12]. A strong absoluteness result for absolutely proper forcing due to Neeman and Norwood can be used to show that T and $\mathrm{MS}_{\gamma^+}(\bar{\mu})$ continue to complement each other in the $\mathbb{P}_I$ generic extension for an arbitrary σ-ideal I so that $\mathbb{P}_I$ is absolutely proper. This result is similar to [13]. A more general version of the following result proved under AD^+ appears in [12].

Fact 4.5.9. *(Neeman and Norwood) Under* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} + V = L(\mathscr{P}(\mathbb{R}))$*, for every absolutely proper forcing* $\mathbb{P}$*, and* $G \subseteq \mathbb{P}$ *which is* $\mathbb{P}$*-generic over* V*, there is an elementary embedding* $j : L(\mathscr{P}(\mathbb{R})) \to L(\mathscr{P}(\mathbb{R})^{V[G]})$ *so that* j *does not move ordinals or reals.*

Fact 4.5.10. *Assume* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} + V = L(\mathscr{P}(\mathbb{R}))$. *Suppose T and S are trees on $\omega \times \gamma$ and $\omega \times \delta$ so that $p[T] = {}^{\omega}\omega \setminus p[S]$. Then $V[G] \models p[T] = {}^{\omega}\omega \setminus p[S]$.*

In particular, if T is a weakly homogeneous tree on $\omega \times \gamma$ witnessed by the weak homogeneity system $\bar{\mu}$, then $V[G] \models p[T] = {}^{\omega}\omega \setminus p[\mathrm{MS}_{\gamma^+}(\bar{\mu})]$.

Proof. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Let $j : L(\mathscr{P}(\mathbb{R})) \to L(\mathscr{P}(\mathbb{R})^{V[G]})$ be an elementary embedding which does not move ordinals or reals. Note that if T is a tree on $\omega \times \gamma$, then $j(T)$ is a tree on $j(\omega) \times j(\gamma) = \omega \times \gamma$ and for all $s \in {}^{<\omega}(\omega \times \gamma)$, $s \in T$ if and only if $j(s) \in j(T)$ if and only if $s \in j(T)$. Hence $T = j(T)$ and similarly $S = j(S)$. So by elementarity, $L(\mathscr{P}(\mathbb{R})^{V[G]}) \models p[T] = {}^{\omega}\omega \setminus p[S]$. As $V[G]$ and $L(\mathscr{P}(\mathbb{R})^{V[G]})$ have the same reals, $V[G] \models p[T] = {}^{\omega}\omega \setminus p[S]$.

For the second statement, note that under $\mathsf{ZF}+\mathsf{DC}$, Fact 4.4.6 implies that $L(\mathscr{P}(\mathbb{R})) \models p[T] = {}^{\omega}\omega \setminus p[\mathrm{MS}_{\gamma^+}(\bar{\mu})]$. The rest follows by applying the first part. □

Fact 4.5.11. *Assume* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} + V = L(\mathscr{P}(\mathbb{R}))$ *(or just* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$ *for Cohen forcing, $\mathbb{C}$). Let $\mathbb{P}$ be an absolutely proper forcing. Suppose T is a tree on ${}^{k}\omega \times \gamma$ for some cardinal γ and $k \in \omega$. If $A \subseteq {}^{j}({}^{\omega}\omega)$, for some $j \leq k$, is defined by applying complementation and $\exists^{\mathbb{R}}$ over $p[T]$, then there is some tree U on ${}^{j}\omega \times \delta$ for some cardinal δ so that $A = p[U]$ and $1_{\mathbb{P}} \Vdash_{\mathbb{P}} A = p[\check{U}]$.*

Proof. This is proved by induction. Suppose B is some set defined by real quantifiers over $p[T]$ such that there is some tree L on ${}^{l}\omega \times \epsilon$ so that $B = p[L]$ and $1_{\mathbb{P}} \Vdash_{\mathbb{P}} B = p[L]$.

For the $\exists^{\mathbb{R}}$ case: Suppose $l = i + 1$. Define the tree U on ${}^{i}\omega \times \epsilon$ as the induced tree defined by considering the tree L on ${}^{i+1}\omega \times \omega$ as a tree on ${}^{i}\omega \times (\omega \times \epsilon)$ with ϵ and $\omega \times \epsilon$ identified by some bijection. Then $\exists^{\mathbb{R}} B = p[U]$.

For complementation: By Fact 4.5.4, L is weakly homogeneously Suslin. Let $\bar{\mu}$ be some weak homogeneity system witnessing this for L. By Fact 4.4.6, $p[T] = {}^{\omega}\omega \setminus p[\mathrm{MS}_{\epsilon^+}(\bar{\mu})]$. By Fact 4.5.10 (or Fact 4.5.8), $1_{\mathbb{P}} \Vdash_{\mathbb{P}} p[T] = {}^{\omega}\omega \setminus p[\mathrm{MS}_{\epsilon^+}(\bar{\mu})]$. □

Fact 4.5.12. *Assume* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} + V = L(\mathscr{P}(\mathbb{R}))$ *(or just* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$ *in the case of $\mathbb{C}$). Let $\mathbb{P}$ be a absolutely proper forcing. Let T be a tree on ${}^{k}\omega \times \gamma$ for some cardinal γ. Let A denote a predicate symbol for $p[T]$ which will always be interpreted as $p[T]$ in forcing extensions. Let φ be a formula on $\mathbb{R}$ using predicate A, complementation, and $\exists^{\mathbb{R}}$. Then for all $r \in \mathbb{R}^V$, $V \models \varphi(r) \Leftrightarrow V[G] \models \varphi(r)$, whenever $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V.*

Proof. In the $V = L(\mathscr{P}(\mathbb{R}))$ case, this is essentially immediate from the absoluteness result of Fact 4.5.9 and the fact that $L(\mathscr{P}(\mathbb{R})^{V[G]}) \models \varphi(r) \Leftrightarrow V[G] \models \varphi(r)$.

So consider the case for $\mathbb{C}$: Let $G \subseteq \mathbb{C}$ be a $\mathbb{C}$-generic over V. By Fact 4.5.11, for some tree U, $V \models (\forall x)(\varphi(x) \Leftrightarrow x \in p[U])$ and $V[G] \models (\forall x)(\varphi(x) \Leftrightarrow x \in p[U])$. Then for any $x \in \mathbb{R}^V$,

$$V \models \varphi(x) \Leftrightarrow V \models x \in p[U] \Leftrightarrow V[G] \models x \in p[U] \Leftrightarrow V[G] \models \varphi(x)$$

where the second equivalence follows from the absoluteness of well-foundedness. $\square$

Now assume $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} + V = L(\mathscr{P}(\mathbb{R}))$ (or just $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$ when working with the meager ideal). Let E be an equivalence relation on ${}^{\omega}\omega$ with all Σ^1_1 (or $\mathbf{\Pi}^1_1$ classes). Let I be a σ-ideal on ${}^{\omega}\omega$ so that the associated forcing $\mathbb{P}_I$ is a absolutely proper forcing.

By Fact 4.5.7, E is homogeneously Suslin. Let S be a homogeneously Suslin tree so that $E = p[S]$. In the case of the meager ideal and under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$: By the countability of $\mathbb{C}$, the argument above about weak homogeneity systems would show that the homogeneity system for S would lift to a homogeneity system for S in the $\mathbb{C}$-extension. Thus S would still be a homogeneous tree in the $\mathbb{C}$-extension. Under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} + V = L(\mathscr{P}(\mathbb{R}))$, for the general σ-ideal I with $\mathbb{P}_I$ absolutely proper, Fact 4.5.9 gives an elementary embedding $j : L(\mathscr{P}(\mathbb{R})) \to L(\mathscr{P}(\mathbb{R})^{V[G]})$. So $L(\mathscr{P}(\mathbb{R})^{V[G]}) \models S$ is homogeneously Suslin. This is not exactly the requirement of A_{Σ} or A_{Π} so the proof of Theorem 4.3.7 needs to be slightly modified: To prove Claim 2, first use the same argument with the fact that S is a homogeneous tree in $L(\mathscr{P}(\mathbb{R})^{M[g]})$ to show that Player 2 has a winning strategy in $L(\mathscr{P}(\mathbb{R})^{M[g]})$. This strategy is still a winning strategy for Player 2 in $M[g]$. This proves Claim 2 and the rest of the argument remains unchanged.

A_{Σ} (and similarly A_{Π}) holds for S and I, except for the minor point of the previous paragraph. The formula $D_{\Sigma}(x, T)$ from Definition 4.3.5 can be expressed as a statement involving a predicate for $p[S]$, complementation, and real quantifiers. Fact 4.5.11 shows that there is some tree U representing D_{Σ} in V and in $\mathbb{P}_I$-extensions. Statement C_{Σ} (and similarly C_{Π}) holds for S and I. The statement $(\forall x)(\exists T) D_{\Sigma}(x, T)$ is true in V since E is an equivalence relation with all Σ^1_1 classes. This formula is also expressed as a statement involving a predicate for $p[S]$, complementation, and real quantifiers, so Fact 4.5.12 implies that this statement remains true in the $\mathbb{P}_I$-extension. B_{Σ} (and similarly B_{Π}) holds for S and I.

As Section 4.3 works in $\mathsf{ZF} + \mathsf{DC}$, the arguments of that section can be carried out in the present context, with the changes mentioned above. (Recall the discussion earlier in this section about properness and elementary substructures under DC.)

Since Cohen forcing satisfies the $\aleph_1$-chain condition, one can obtain more than just canonicalization on a nonmeager set but in fact on a comeager set.

Finally the following results are obtained. Again, the analogous result for graphs also hold:

Theorem 4.5.13. *Assume* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$. *Let E be an equivalence relation on ${}^\omega\omega$. If E has all Σ_1^1 ($\boldsymbol{\Pi}_1^1$ or $\boldsymbol{\Delta}_1^1$) classes, then for every nonmeager $\boldsymbol{\Delta}_1^1$ set B, there is a $\boldsymbol{\Delta}_1^1$ set $C \subseteq B$ which is comeager in B so that $E \restriction C$ is Σ_1^1 ($\boldsymbol{\Pi}_1^1$ or $\boldsymbol{\Delta}_1^1$, respectively).*

Theorem 4.5.14. *Assume* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} + V = L(\mathscr{P}(\mathbb{R}))$. *Let I be a σ-ideal on ${}^\omega\omega$ so that $\mathbb{P}_I$ is absolutely proper. Let E be an equivalence relation on ${}^\omega\omega$. If E has all Σ_1^1 ($\boldsymbol{\Pi}_1^1$ or $\boldsymbol{\Delta}_1^1$) classes, then for every I^+ $\boldsymbol{\Delta}_1^1$ set B, there is an I^+ $\boldsymbol{\Delta}_1^1$ set $C \subseteq B$ so that $E \restriction C$ is Σ_1^1 ($\boldsymbol{\Pi}_1^1$ or $\boldsymbol{\Delta}_1^1$, respectively).*

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