Essays on Information Collection

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ABSTRACT

This thesis is devoted to the problem of information collection from theoretical and experimental perspectives.

In Chapter 2, I characterize the unique optimal learning strategy when there are two information sources, three possible states of the world, and learning is modeled as a search process. The optimal strategy consists of two phases. During the first phase, only beliefs about the state and the quality of information sources matter for the optimal choice between these sources. During the second phase, this choice also depends on how much the agent values different types of information. The information sources are substitutes when each individual source is likely to reveal the state eventually, and they are complements otherwise.

In Chapter 3, co-authored with Li Song, we conducted an experiment which demonstrates that even in a simple four person circle network people appear to fail to account for possible repetition of information they receive. Moreover, we show that this phenomenon can be partially attributed to rational considerations, which take into account other people's deviations from optimal behavior.

In Chapter 4, co-authored with Marcelo A. Fernández, we model overconfidence as if a decision maker perceives information as being more precise than it actually is. We show that the effect of overconfidence on the quality of the final decision is shaped by three forces, overestimating the precision of future information, overestimating the precision of past information and overestimating the amount of information to be collected in the future. The first force pushes an overconfident decision maker to collect more information, while the second and the third forces work in the other direction.

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Chapter 1

INTRODUCTION

In this thesis, I address several questions related to dynamics and results of the learning process. Chapter 2 gives an abstract framework for studying dynamically optimal information collection from multiple information sources. Chapter 3 explores learning in a network setting using experimental economics tools. Chapter 4 provides a theoretical background for overconfidence in learning.

In Chapter 2, I consider the situation in which a decision maker has two sources of information that she can choose to use. She is free to change the source as often as she wants as she collects information about a payoff relevant state. Each source is modeled as a search process for the proof that a certain state is realized. There are three possible state realizations, with two that are possible to verify through information sources. In the optimum, the decision maker starts with searching for the most likely out of the two verifiable states (adjusted to how easy and how costly it is to search for a given state), switching attention as she becomes more pessimistic about the state. Then at some point she changes her learning behavior to focusing on only one source until she either finds the true state or gives up the attempt to learn completely.

Theoretical paper DeMarzo, Vayanos, and Zwiebel (2003) proposes a model of information aggregation in networks when individuals are subject to *persuasion bias*. The term "persuasion bias" refers to a particular form of boundedly rational behavior when individuals connected into a network do not account for repetition in the information they acquire. In Chapter 3, co-authored with Li Song, we argue that under the assumption that agents form their beliefs as a weighted average of all information available to them, the persuasion bias assumption is equivalent to a generalized version of the famous DeGroot model (DeGroot (1974)). We test the persuasion bias hypothesis against the (generalized) Bayesian updating model and find support for the persuasion bias hypothesis. We also found a positive correlation between how well a subject fits the generalized DeGroot model, compared to the alternative generalized Bayesian updating model, and their performance in the experiment. Data suggest that the generalized DeGroot model better accommodates other subjects' deviations from equilibrium, which explains the positive correlation.

In Chapter 4, co-authored with Marcelo A. Fernández, we present a dynamic model that illustrates three forces that shape the effect of overconfidence (overprecision of consumed information) on the amount of collected information. The first force comes from overestimating the precision of the next consumed piece of information. The second force is related to overestimating the precision of already collected information. The third force reflects the discrepancy between how much information the decision maker expects to collect and how much information he actually collects in expectation, given the objective properties of information flow. The first force pushes an overconfident decision maker to collect more information, while the second and the third forces work in the other direction. We show that in the absence of the third force there is an optimal level of overconfidence that balances the first and the second forces to maximize the amount of collected information. When all three forces are active, overconfidence always has a negative effect on information investment.

Chapter 2

DYNAMIC CHOICE OF INFORMATION SOURCES

2.1 Introduction

There are many situations where an individual has the opportunity to use different sources to gather costly information before choosing among a set of alternatives. When the individual can change information sources as often as he wants, finding his optimal behavior is difficult both as a theoretical and a computational exercise. I propose a tractable way to model this problem and derive its solution.[1] I characterize a unique optimal information collection strategy when there are two information sources and three possible states of the world. The optimal strategy consists of two phases. During the first phase, the optimal choice of an information source depends only on the precision and cost of information and the agent's current beliefs about the state of the world. In this phase, the best source is the one that reveals the state most quickly, which may guide the agent to alternate between sources. During the second phase, the best choice also depends on the payoff the agent receives from choosing among the set of alternatives after the learning process. However, during this phase, the agent never alternates between sources.

This characterization of the optimal strategy can be used to deliver new insights into the market of information providers by determining when the information sources are substitutes and when they are complements. I show that sources act as substitutes when ex ante it is very likely that, given all potential information from one source, the other source cannot contribute anything new, and they are complements otherwise.

In my model, the agent must choose one of three alternatives. The payoff the agent receives from each alternative depends on the true state of the world. There are three possible states. Before making a choice, the agent can collect information about the true state from two sources. The information collection process is modeled in

[1] Che and Mierendorff (2016) independently develop a very similar approach with two main differences. First, I allow three possible states of the world, while Che and Mierendorff work with a two state version. Consequently, they do not study when the information sources are substitutes and when they are complements (for two states, I find that the sources are either independent or substitutes). Second, Che and Mierendorff propose a generalization to their benchmark with two sources by relaxing the assumption that a positive signal from a source fully reveals the state. In particular, Che and Mierendorff allow for a continuum of news sources, varying by the degree they reveal the state by a positive signal. I compare my paper with Che and Mierendorff in detail in Section 2.2.1.

continuous time to allow the agent maximal freedom in allocating his attention. At every instant, the agent can either stop information collection by choosing an alternative that maximizes his current expected utility or wait and get more information. If he wants more information, he must choose how to allocate a unit of his attention between two sources (source 1 and source 2). If a source receives a positive amount of attention, the agent observes a signal from this source and he has to pay a cost proportional to the amount of attention he pays to this source.[2,3] If the world is in state i ($i = 1, 2$), then source i sends a positive signal with probability proportional to the source intensity and attention paid to this source; otherwise the signal is 0. If the world is not in state i, then source i sends signal 0 all the time.

My main result describes the unique optimal information collection strategy.[4] Once the agent observes a positive signal, the true state is revealed and therefore the agent stops the information collection process. Conditional on receiving only 0 signals, the agent chooses what source to use and when to stop seeking additional information according to a rule defined by two thresholds. Until the agent's beliefs about the state reach the first threshold, he chooses the source that is informatively optimal (I call it the informatively optimal phase of the information collection process). Between the two thresholds, he chooses the payoff optimal source (the payoff optimal phase). Once his beliefs reach the second threshold, he stops collecting information.

During the informatively optimal phase, the agent compares the quality of both sources and chooses the one with the highest quality as measured by the ratio of the probability of receiving a positive signal from this source to the cost of observing this signal. Since new information changes the agent's beliefs about the true state, it also changes the probability of receiving a positive signal from each source. Thus, the quality of the sources changes as the agent gets more signals. The more 0 signals the agent gets from a given source, the lower the probability of receiving a positive signal from this source and the higher the probability of receiving a positive signal from the other source. Therefore, observing a signal of 0 lowers the quality of this source

[2] I assume the cost only depends on the type of source, so that it is independent of the true state and the agent's payoff from different alternatives. For example, it can be money a researcher pays to subjects of experiments or it can be a constant fee to an expert.

[3] Moscarini and Smith (2001) introduce a discounting and time-dependent information cost for the optimal experimentation model with one information source. My model serves as a benchmark for such modifications when there are two information sources.

[4] Strictly speaking, the optimal strategy is unique in the following sense. The strategy is Markovian, that is, what to do (optimal action) depends on the past only through current beliefs. For almost all parameter values (except the set of measure zero) and almost all feasible current beliefs (except the set of measure zero), the optimal action is unique.

and raises the quality of the other source. At the beginning of the informatively optimal phase, the agent starts with the source that has the highest quality. As phase one progresses, the quality of that source decreases while the quality of the other source increases. Given enough 0 signals, the qualities equalize, at which point the agent starts using both sources simultaneously by splitting his attention between two sources. Moreover, by paying more attention to the source with low intensity, the agent guarantees that the qualities stay equal after new information is received.

At the payoff optimal phase, the agent uses a constant attention allocation plan that places full attention to either source 1 or source 2. At the beginning of this phase, the agent chooses the source that gives him the highest expected payoff and then he stays with this source until the end of the information collection process. The payoff optimal phase illustrates an "elimination search" behavior. The optimal information source during this phase is sometimes that with the lowest probability of producing a positive signal. Put simply, it is sometimes optimal to search in a place where you do not think you are going to find something in order to eliminate this place.

Both thresholds that define the optimal strategy occur where the marginal cost of information equals its marginal benefit. Consider the second threshold that indicates where the payoff optimal phase ends and suppose that source 1 is used during this phase. Here, the marginal cost of information is the cost of a signal from source 1 and the marginal benefit is the expected change in the payoff from the chosen alternative. This change in payoff is positive only if the received signal is positive; otherwise, this change would be zero. Thus, the marginal benefit is the probability of receiving a positive signal from source 1 multiplied by the difference between the payoffs in state 1 from the alternatives the agent would choose after receiving a positive signal from source 1 and after receiving signal 0 from that source. I call this benefit a direct benefit since it is directly related to the final payoff.

Consider the threshold that separates the informatively optimal phase and the payoff optimal phase. Again, suppose that the source used during the payoff optimal phase is source 1. When the agent moves from one phase to another, he changes the attention allocation plan. Thus, he must use source 2 just before entering the payoff optimal phase. The marginal cost of information is the cost of a signal from source 2. The marginal benefit consists of two parts, a direct and an indirect benefit. The direct benefit is the probability of receiving a positive signal from source 2 multiplied by the difference in payoffs in state 2 from the alternatives the agent would choose after receiving a positive signal from source 2 and after receiving signal 0 from that source. This part accounts for the benefit from a positive signal. The indirect benefit comes from signal 0 and it reflects the increase in the direct benefit from all future signals the agent is going to receive from source 1 during the payoff optimal phase. Indeed, signal 0 from source 2 makes the agent more optimistic about state 1, therefore increasing the likelihood of getting a positive signal from source 1.

I use the characterization of the optimal learning strategy to find when the information sources complement each other and when they substitute for each other. I call two sources complements if raising the cost of a signal from one source leads to a lower expected attention allocated to the other source during the information collection process. I call two sources substitutes if raising the cost of a signal from one source leads to a higher expected attention allocated to the other source. I show that information sources act as substitutes when the probability of the third state is small, and they are complements otherwise. Intuitively, when the third state is unlikely, two sources provide essentially the same information since both sources separate states 1 and 2. In that case, they are substitutes. However, when there is a high likelihood that the true state is 3, the agent can only discover it when using both sources. In that case, they are complements. Taking a step back from the model, I interpret this result as follows: two ways of obtaining information substitute for each other if all information that one source can provide would most likely be enough to make a good choice of the alternative; if, having all information from one source, the agent most likely still wants to use the other source, the information sources complement each other.

The model provides a general framework for analyzing a number of economically important situations. Consider the three examples below:

1. The information sources are different directions for R&D (different directions of research, different experimental designs, or different empirical strategies). The states represent mutually exclusive hypotheses (so that each research strategy tests a different hypothesis) and the alternatives are different technological designs whose value depends on which of the hypotheses is correct.

2. The states are different causes of an accident (like whether a plane crash was caused by terrorists, mechanical failure, or pilot error). The information sources are different experts (engineers and forensic scientists) and the alternatives are different measures to prevent or mitigate the damage from similar

accidents in the future (tighten security measures or ground the whole fleet).[5]

3. The states are different causes of a disease (like infection, cancer, or something else) a doctor has in mind while choosing a treatment (chemotherapy, antibiotic drugs, or no treatment) for his patient. The information sources are different specialists (an oncologist or an infectious disease specialist) to whom the doctor can send the patient to get tests.[6]

2.2 Literature Review

My model relates to the sequential optimal experimental design problem, search problems, drift-diffusion models, rational inattention theory and multi-armed bandit literature.

Starting from Wald (1947) and Blackwell (1953), the *sequential optimal experimental design* problem (Chaloner and Verdinelli (1995); Wang, Filiba, and Camerer (2010)) is usually formulated as to construct the dynamically optimal sequence of experiments to determine which out of a number of hypotheses is true. Experiments serve as information sources while hypotheses are possible states of the world. When the number of experiments is more than one, no analytical solution that I know of has been found to this problem so far. There have only been attempts to provide numerical algorithms to construct nearly optimal strategies (Chernoff (1959); Naghshvar and Javidi (2013)). A very recent paper, Liang, Mu, and Syrgkanis (2017), is the only exception. They consider learning from a finite set of Guassian signals. Besides making different distribution assumptions, I do not limit the number of signals.

The *search problem* (Staroverov (1963); Black (1965); Ahlswede and Wegener (1987); Stone (1976)) is to find an object hidden in one of multiple locations as quickly as possible. The search in a given location acts as learning from a given source, while the actual location of the object is the true state of the world. In

[5] In many cases, the agency that leads the investigation and takes responsibility for the final decision (NTSB in U.S.) hires experts from outside to help. For example, for the crash of EgyptAir on 19 May 2016 in the Mediterranean, Egypt's Aircraft Accident Investigation Committee hired the Forensic Medicine Authority to check for a possible explosion on board. Their experts are paid based on the length of the study period (Kharoshah, Zaki, Galeb, Moulana, and Elsebaay (2011), p.11).

[6] In my model, I assume that the sources are homogeneous in terms of costs and precision. Forcing all tests to be the same in terms of cost and precision might be too heroic if one takes the example literally. However, the same cost and precision assumption is justified from the doctor's perspective. He is not the one who chooses a particular test within the group; he only identifies the need for additional testing and the direction of inspection (which cause of a disease to test for). So, for him these tests are ex ante the same. Moreover, the cost of the tests is not the cost of the tests themselves but the constant fee the doctor pays either to the oncologist or the infectious disease specialist.

the search model, the agent minimizes the cost of learning subject to revealing the state with certainty (it is assumed that the agent can search in every location). In contrast, in my model, the agent maximizes the total expected payoff. Therefore, it is sometimes optimal to stop learning before the state is revealed.[7,8]

A classical *drift-diffusion model* (Fehr and Rangel (2011); Ratcliff, Smith, Brown, and McKoon (2016); Forstmann, Ratcliff, and Wagenmakers (2016)) formalizes a learning process with several information sources, which, in contrast to my model, describe benefits and disadvantages of choosing a particular alternative directly. For example, I assume that the researcher tests different hypotheses, while the drift-diffusion model implies that he compares different designs directly (see the R&D example in the introduction).[9] Moreover, drift-diffusion models assume gradual learning modeled with Brownian motion. For example, Ke, Shen, and Villas-Boas (2016) model information sources as Brownian motions that gradually reveal the cardinal value of utility from purchasing the correspondent product.[10] Finally, many drift-diffusion models take an information acquisition strategy as an exogenously given process, that is, the agent's strategy is not derived from some utility maximization problem (for example, Krajbich and Rangel (2011)).

Nikandrova and Pancs (2017) consider a similar learning environment as Ke, Shen, and Villas-Boas (2016). However, they take an approach closer to my model by assuming a Poisson type learning instead of using Brownian motion.

Another stream of literature related to this paper is on *rational inattention theory*, proposed by Christopher Sims and surveyed in Sims (2010). In contrast to all other topics I discuss here, rational inattention theory typically uses static models. On

[7]Even if I introduce the third information source that differentiates the third state to match the assumption that the agent can search in every location, the statement that the optimal behavior sometimes leaves the state unrevealed remains true.

[8]The modified definition of the optimal strategy makes the search problems more tractable than the sequential optimal experimental design problems. I leverage this idea by splitting the solution of my model into two steps. In the first step, I make the problem easier by fixing the default alternative that the agent chooses if he stops learning before the state is revealed. One can think of this as an analog of fixing the goal of finding the state no matter what, as it is done in the search model. In the second step, the optimal strategy is found by simply comparing three strategies, each corresponding to one of three possible alternatives.

[9]There are papers, like Fisher and Rangel (2014), where information sources signal about attributes of alternatives. However, the notion of attributes is intimately connected with alternatives. By introducing the notion of a state, I leave the connection between the state and each alternative free of any assumptions.

[10]In a similar vein, Fudenberg, Strack, and Strzalecki (2015) propose an uncertainty-difference drift-diffusion model with the agent observing one stream of signals (one information source) modeled as Brownian motion with the drift proportional to the unknown difference in payoffs.

the other hand, it works with general information structures, not limited to Poisson process or Brownian motion, allowing the decision maker to choose among all of them. As an exception, a recent paper Zhong (2017) works in a dynamic setup. He shows that a Poisson process that seeks most likely state is the optimal information structure. Though working with Poisson processes, I do not have that flexibility in information structure in my model, which sometimes leads to a qualitatively different optimal strategy.

A classical *multi-armed bandit* problem (Robbins (1952); El Karoui and Karatzas (1997)) assumes that each source not only provides information but also a payoff. For example, instead of testing mutually exclusive hypotheses to decide on a technological design at the end, the researcher chooses which design (or project) to develop at every moment of time (see the R&D example in the introduction).[11] By developing a project, the agent gets a stream of random payoffs. He has some prior belief about the distribution of these payoffs at the beginning. He updates his belief after observing the realization of the payoffs. Thus, each source (project) gives him both the payoff and information about the payoff distribution. Moreover, as in the drift-diffusion model, each source provides information on an alternative (a design) directly, not through the state of the world.[12,13]

In this paper, I show that information sources are *substitutes* when the probability of the third state is small and they are *complements* otherwise. This result opens the transition from a one agent model to a multiple agents model, where information providers can choose the price of information to maximize their profits.

In contrast to my findings, Gul and Pesendorfer (2012) show that information sources

[11] This is the model of strategic experimentation in R&D (Bolton and Harris (1999); Keller, Rady, and Cripps (2005); Keller and Rady (2010); Strulovici (2010); Keller and Rady (2015)).

[12] In some cases, one can reformulate a learning model as the bandit problem but with correlated payoffs among alternatives (similar to p.70 in Gittins, Glazebrook, and Weber (2011)). Assuming the payoff distributions are independent across different alternatives, the bandit problem can be solved using the Gittins index (Jones and Gittins (1972)). This technique is generally not applicable for correlated payoffs, as demonstrated in Francetich and Kreps (2014). Klein and Rady (2011) and Francetich (2016a) depart from the standard multi-armed bandit by assuming a negative correlation between the payoffs from two projects. Both papers derive the optimal strategy from "first principles," that is, without using the Gittins index.

[13] Another example of a bandit problem is related to the clinical trials application and can be considered as a modification of the medical example from the introduction. Suppose that instead of attempting to diagnose one patient through a series of tests, the doctor is working on finding a cure to one particular disease. He has several treatments available and an infinite pool of patients. Treating the patients sequentially, he has to choose one of the treatments each time. The question is to find the optimal sequence of treatments, or clinical trials (Rosenberger and Lachin (2015)). In the clinical trials problem, there is no uncertainty about the disease but there is a systematic uncertainty about the effect each treatment has with respect to the disease.

serve as complements when the probability of the third state is zero. Two assumptions explain the difference. First, Gul and Pesendorfer (2012) assume the agent has a more passive role. Specifically, he cannot choose the information source and he cannot choose when to stop gathering information. Second, the information cost is paid by information providers who have preferences over the agent's final choice of alternative. Therefore, this is a game of persuasion. Thus, Gul and Pesendorfer (2012) place the whole burden of strategic behavior on information providers. In that case, lowering the cost for one provider incentivizes this provider to feed more information to the agent, while the other provider gives less information (information providers are strategic substitutes). Therefore, the information flow becomes less balanced, which means it becomes less desirable for the agent. Thus, in equilibrium, from the information consumer perspective, lowering the cost for one provider increases the marginal benefit of new information from the other provider. Hence, information sources serve as complements. Gul and Pesendorfer (2012) is more appropriate in political contests while my model describes a researcher's activity.

Chen and Waggoner (2016) take a different approach in defining substitutability and complementarity of information sources. They work with one information source (one homogeneous stream of signals) and study the dynamics of the marginal value of a new signal. In particular, they define substitutability of signals as diminishing marginal value of information and complementarity as increasing marginal value.

There are many papers that study the question of whether different ways of obtaining information complement each other or not from an *empirical* perspective. Taking the idea from Allen (1991) of treating R&D projects as "information acquisition activities," my model answers the question of when R&D projects behave as substitutes and when they are complements. For example, there is mixed evidence as to whether public subsidies crowd out private R&D investments (David, Hall, and Toole (2000); Lach (2002); Almus and Czarnitzki (2003); González and Pazó (2008); Aerts and Schmidt (2008)). The result from my model suggests that one needs to take into account the type of R&D project. If each separate project potentially can provide enough information, then the projects are substitutes. When projects are substitutes, investing in one decreases the marginal return from investing in another, which stimulates a crowding out effect.

De Waal, Schönbach, and Lauf (2005) found evidence supporting the substitute nature of online and printed newspapers. Nguyen and Western (2006) concluded that traditional media complement the Internet. My contribution to this discussion is that one needs to take into account the content of media sources. For example, if a newspaper tells a story from a local perspective while the Internet covers a global aspect of the event, then these media outlets may be complements because even reading the whole newspaper from cover to cover does not provide the whole picture of what happened.

2.2.1 Comparison with Che and Mierendorff "Optimal Sequential Decision with Limited Attention" (2016)

In this section, I discuss the connection between my paper and Che and Mierendorff (2016). Both papers were written independently.

Che and Mierendorff's benchmark model is almost the same as the model in Section 2.4.1. The main difference is that Che and Mierendorff model the cost of information acquisition through a discount factor instead of a per unit of time cost. The optimal strategy is almost the same in both models. The only qualitative difference is in the interpretation of source quality for the informatively optimal phase. In Che and Mierendorff, source quality incorporates the agent's possible payoffs from alternatives. In other words, a different cost structure leads to different properties of a point where the agent uses both sources simultaneously.

Che and Mierendorff take a different approach in generalizing the benchmark model. They relax the assumption that a positive signal from a source fully reveals the state by allowing a continuum of information sources. The sources vary by the degree they reveal the state by a positive signal, while the rate of a positive signal arrival is independent of the true state.

By extending the model to three states (Section 2.5), I achieve two goals. First, the optimal strategy has qualitatively different characteristics in the general model. The presence of the third state guarantees that there is always a non-zero probability that the expected utility maximizing agent stops learning before he observes a positive signal. Indeed, if the world is in state 3, both sources produce only zero signals. Consider a strategy where the agent always chooses the source with the highest quality until the state is revealed. This strategy is sometimes optimal when the probability of the third state is zero (the benchmark model). However, it cannot be optimal when the probability of the third state is positive. Second, the generalization to the three state model allows me to make a connection between substitutability / complementarity of information sources and the probability of the third state. In particular, the sources can be complements only if the probability of the third state is positive. In other words, the sources are either substitutes or independent in the benchmark model.

2.3 Setup

The problem at hand involves an agent who must choose among three alternatives, $a \in \mathcal{A} \equiv \{a_1, a_2, a_3\}$.[14] His payoff from these alternatives is uncertain; that is, it depends on what the true state of the world is. There are only three possible states of the world: 1, 2, and 3. Denote by $u_j[a]$ the payoff he gets if he chooses alternative a and the true state is j. For $j = 1, 2$, I assume that alternative a_j is the best choice when the true state is j (I do not require a_3 to be the best choice if the true state is 3):[15]

$$u_j[a_j] = \max_{i \in \{1,2,3\}} u_j[a_i], \qquad j = 1, 2. \tag{2.1}$$

The agent has some prior beliefs about the true state.[16] Denote by p_1 his belief that the state is 1 and by p_2 his belief that the state is 2 (so that state 3 occurs with probability $1 - p_1 - p_2$).

Before making a choice, the agent can collect information. Time $t \in [0, \infty)$ is continuous. At each instant of time $[t, t + dt]$, the agent chooses either to stop the information collection process by choosing an alternative or to wait and get more information.

Suppose there are two information sources available to the agent.[17] If he chooses to wait, then he has to decide how to divide his one unit of perfectly divisible attention

[14] The generalization to more than three alternatives is straightforward, as long as the number of alternatives is finite. Indeed, the statement "The optimal strategy is the best of the optimal a-type strategies, where $a \in \mathcal{A}$" remains true for any number of alternatives in the set $\mathcal{A}$.

[15] This assumption does not restrict the generality of the model. Indeed, if the same alternative is optimal for states 1 and 2, then I duplicate this alternative and solve the model with four alternatives in $\mathcal{A}$.

[16] I assume the agent's prior beliefs are correct. Fudenberg, Romanyuk, and Strack (2016) study optimal experimentation with misspecified beliefs. Studying the implications of misspecified beliefs in my model is a subject of future research.

[17] Assuming two sources is a good starting point, given the complexity of the problem. For example, for the multi-armed bandit setting with negatively correlated payoffs, Francetich (2016b) demonstrates how difficult the problem can be once more than two sources are introduced. Moreover, even with two sources, the model covers many situations. Indeed, it is not unusual to have only two possible explanations, two hypotheses in mind, two candidates for the solution. However, I cannot assume that two states are exhaustive. For example, there is always a chance that a patient has an unknown or very rare disease, or an accident happened because of a very unlikely chain of events that is almost impossible to investigate. To account for the chance that the state might never be revealed, I assume that there are three states with the third containing "everything else" that cannot be discovered by the information sources. This third state can be interpreted as "none of the above," or in other words, in the spirit of being aware of one's unawareness, as in Karni and Vierø (2017).

between two sources. Denote by $T_{t,i}$ the total amount of attention the agent paid to source i by time t. This definition implies that if the agent has not stopped collecting information by time t, the total attention he paid to both sources is $T_{t,1} + T_{t,2} = t$. This definition also implies that $\frac{dT_{t,i}}{dt}$ is the fraction of attention the agent allocates to source i and $\frac{dT_{t,1}}{dt} + \frac{dT_{t,2}}{dt} = 1$.

Using information sources is costly. If the agent chooses to wait and allocates $\frac{dT_{t,i}}{dt}$ fraction of his attention to source i, he has to pay $c_1 dT_{t,1} + c_2 dT_{t,2}$. Thus, if the information collection is still in progress by time t, the total cost the agent paid is $c_1 T_{t,1} + c_2 T_{t,2}$.[18]

If the agent chooses to wait and allocates $\frac{dT_{t,i}}{dt}$ fraction of his attention to source i, he observes a realization of an increment of a stochastic process $dX_t = (dX^{(1)}_{T_{t,1}}, dX^{(2)}_{T_{t,2}})$, where $X^{(k)}$ is a Poisson process with intensity λ_k if the state is k and $X^{(k)} \equiv 0$ if the state is not k ($X^{(1)}$ and $X^{(2)}$ are independent conditional on the state). In other words, the agent observes a pair of signals, which can take one of three possible values: (0,0), (1,0), (0,1). If the true state is 3, he always observes (0,0). If the true state is 1, he receives (1,0) with probability $\lambda_1 dT_{t,1}$ and (0,0) otherwise. If the agent observes (1,0), I say that source 1 reveals the state. Similarly for state 2.

Let $\mathcal{F}_t$ be the information available to the agent by time t.

A strategy of the agent is a triple (a^F, T, τ), where $T = \{T_t = (T_{t,1}, T_{t,2})\}_{t=0}^{+\infty}$, $dT_t \colon \mathcal{F}_t \longrightarrow \{(dT_{t,1}, dT_{t,2}) \colon dT_{t,k} \geq 0,\ dT_{t,1} + dT_{t,2} = dt\}$, is the attention allocation plan, $\tau \geq 0$ is the stopping time, and $a^F : \mathcal{F}_\tau \longrightarrow \mathcal{A}$ is the alternative chosen at the end.

The optimal strategy is the strategy that maximizes the expected payoff.[19] Formally, the agent faces the following optimization problem:

$$\sup_{(a^F, T, \tau)} \mathbb{E}\left[u_j[a^F] - c_1 T_{\tau,1} - c_2 T_{\tau,2} \mid p_1, p_2\right] \tag{2.2}$$

where j is the true state.

[18] There are different ways to make the collection of information costly. For example, one can assume discounting so that the decision maker would be impatient to make the final decision sooner. In this paper, I chose another, more direct way to impose information cost: the cost is a linear function of time. This method is consistent with optimal experimental design literature, where different information sources are different experiments.

[19] Without risk of confusion, I will sometimes mean by (a^F, T, τ) a *collection* of strategies for all possible initial beliefs.

2.4 Benchmark Models

In this section, I present the solution to two special cases of my model. The solution to the general model is based on ideas from these special cases.

In Section 2.4.1, I consider the case when the probability of the third state is 0. This benchmark demonstrates how phase one (*informatively optimal*) and phase two (*payoff optimal*) emerge in the optimal strategy. In this special case, for any parameters, the optimal strategy consists of at most one phase, either informatively optimal or payoff optimal. In the general model, the optimal strategy consists of either no phase (make the decision immediately), only the payoff optimal phase, or both phases (except for a set of parameters that has measure zero and includes this special case).

In Section 2.4.2, I revisit the case when only one information source is available (the optimal stopping problem well studied in the literature). The optimal strategy in this case consists of at most one phase, and it is always the payoff optimal phase. Since this benchmark puts no restriction on the initial beliefs, it serves as a stepping stone from the benchmark with two states in Section 2.4.1 to the general model.

2.4.1 Two States, Two Sources

In this section, I solve the optimization problem (2.2) for the case in which $p_2 = 1 - p_1$, that is when the ex ante probability of the third state is zero:

$$\sup_{(a^F,T,\tau)} \mathbb{E}\left[u_j[a^F] - c_1 T_{\tau,1} - c_2 T_{\tau,2} \mid p_1, p_2 = 1 - p_1\right]. \tag{2.3}$$

It is convenient to split the solution to the problem (2.3) into two steps. In the first step, I fix the function $a^F : \mathcal{F}_\tau \to \mathcal{A}$ to be

$$a^F = a\mathbf{1}\left(p_{\tau,1} \in (0,1)\right) + a_1\mathbf{1}\left(p_{\tau,1} = 1\right) + a_2\mathbf{1}\left(p_{\tau,1} = 0\right) \tag{2.4}$$

for some alternative $a \in \mathcal{A}$ and optimize over the attention allocation plan T and the stopping time:

$$V^{(a)}[p_1] \equiv$$
$$\sup_{(T,\tau)} \mathbb{E}\left[u_j\left[a\mathbf{1}\left(p_{\tau,1} \in (0,1)\right) + \sum_{k=1,2} a_k \mathbf{1}\left(p_{\tau,1} = \mathbf{1}(k=1)\right)\right] - \sum_{k=1,2} c_k T_{\tau,k} \mid p_1\right]. \tag{2.5}$$

The function a^F defined in (2.4) says that

- if the true state is revealed, the agent chooses the best alternative (according to assumption (2.1)),
- if the state has not been revealed by the stopping time τ, the agent chooses the alternative a.

I call a the *default alternative*, meaning that this is the choice the agent makes by default when he is uncertain about the state. The strategy (a^F, T, τ) is the *optimal a-type strategy* if a^F is defined by (2.4) and (T, τ) maximizes (2.5) (an a-type strategy is a strategy (a^F, T, τ), where a^F is defined by (2.4)).

It is easy to see that the optimal a^F always has the form (2.4). Indeed, for any initial belief p_1, the optimal strategy must prescribe to choose alternative a_i once the state is revealed to be i. Given that, a strategy is equivalent to a plan of what to do conditional on not receiving a positive signal. This contingency plan is defined by

- an attention plan T,
- a "give up" time, that is, a stopping time conditional on not receiving a positive signal,
- a default alternative, that is, the alternative the agent chooses if he has not received a positive signal by a "give up" time.

Thus, for any initial belief p_1, there exists $a \in \mathcal{A}$ such that the optimal a^F has the form (2.4).[20]

In the second step, I optimize over all possible default alternatives $a \in \mathcal{A}$ in (2.4). I use the same trick for the general model (2.2).

The reason to split the solution into two steps is that both steps separately are much easier to solve than the original optimization problem (2.3). In the first step, the optimization problem (2.5) has fewer parameters than (2.3). Specifically, among all payoff parameters $\{u_{i_1}[a_{i_2}]\}_{\substack{i_1=1,2,3 \\ i_2=1,2,3}}$, only $u_1[a_1]$, $u_2[a_2]$, $u_1[a]$, $u_2[a]$, and $u_3[a]$ are included in the optimization problem (2.5). In the second step, the set over which the optimization is performed is finite because $\mathcal{A}$ is finite. For any initial belief

[20] To be more precise, there exists an optimal strategy with a^F in the form (2.4). Since the optimal strategy is almost surely unique (that is, it is unique for all parameters' values except for a set of measure zero), the optimal a^F is almost surely unique as well.

p_1, the optimal strategy is the optimal a-type strategy, where $a \in \mathcal{A}$ maximizes $V^{(a)}[p_1]$.

Step 1 Theorem 1 gives the full description of the optimal a-type strategy (see Figures 2.1 and 2.2). It says that for any initial beliefs, the optimal a-type strategy has one of the following forms:

- use source 1 until either the state 1 is revealed or the belief about state 1 drops below the threshold $p_1 = R_1^{(a)}$, where

$$R_1^{(a)} = \begin{cases} \frac{c_1}{\lambda_1(u_1[a_1]-u_1[a])}, & u_1[a_1] \neq u_1[a], \\ +\infty, & u_1[a_1] = u_1[a], \end{cases} \tag{2.6}$$

- use source 2 until either the state 2 is revealed or $p_2 = 1 - p_1 = R_2^{(a)}$, where

$$R_2^{(a)} = \begin{cases} \frac{c_2}{\lambda_2(u_2[a_2]-u_2[a])}, & u_2[a_2] \neq u_2[a], \\ +\infty, & u_2[a_2] = u_2[a], \end{cases} \tag{2.7}$$

- use the source with the highest quality, until the state is revealed by a positive signal; the *quality* is defined as the probability of revealing the state to cost ratio:

$$\begin{aligned} \frac{\lambda_1 p_1}{c_1} > \frac{\lambda_2 p_2}{c_2} \equiv \frac{\lambda_2(1-p_1)}{c_2} \quad &\Rightarrow \quad \text{use source 1}, \\ \frac{\lambda_1 p_1}{c_1} < \frac{\lambda_2 p_2}{c_2} \quad &\Rightarrow \quad \text{use source 2}, \\ \frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2} \quad &\Rightarrow \quad \text{use source 1 and 2 simultaneously}, \end{aligned} \tag{2.8}$$

- do not get any information and choose the default alternative a.

Denote the set of initial beliefs for which not getting any information is optimal as Area 1.

Denote the set of initial beliefs for which "using source 1 until either the state 1 is revealed or the belief about state 1 drops below the threshold $p_1 = R_1^{(a)}$" is the optimal strategy as Area 2.1. Area 2.2 is defined in a similar way. Area 2.1 and Area 2.2 cover the set of initial beliefs for which the optimal a-type strategy consists of only the payoff optimal phase.

Denote the set of initial beliefs for which (1) using the source with the highest quality until the state is revealed is the optimal strategy and (2) $\frac{\lambda_1 p_1}{c_1} > \frac{\lambda_2 p_2}{c_2}$ as Area 3.1. Area 3.2 is defined in a similar way. Area 3.1 and Area 3.2 cover the set of initial beliefs for which the optimal a-type strategy consists of only the informatively optimal phase.

The payoff optimal phase of the optimal learning strategy is the phase when the agent will never change the source he is using, no matter what information he receives. In other words, he stays with one source (say, source k) until either the state is revealed to be k or his belief about the state to be k drops to the threshold $R_k^{(a)}$. $R_k^{(a)}$ is interpreted as the ratio of cost over benefit from information source k. Intuitively, the agent stops learning when the marginal cost of learning ($c_k \cdot dt$) is equal to its marginal benefit, that is, the probability that the state will be revealed in the next instant of time ($p_k \cdot \lambda_k \cdot dt$) multiplied by the payoff loss from the default alternative if the true state is k ($u_k[a_k] - u_k[a]$). I call this phase payoff optimal to emphasize that the optimal choice of the information source depends on the payoff parameters $\{u_{i_1}[a_{i_2}]\}_{\substack{i_1=1,2,3 \\ i_2=1,2,3}}$.

Choosing the source based on comparing $\frac{\lambda_1 p_1}{c_1}$ and $\frac{\lambda_2 p_2}{c_2}$ corresponds to the informatively optimal phase of the information collection process. I call this phase informatively optimal to emphasize that the optimal choice of the information source does *not* depend on the payoff parameters. In this benchmark case, when $p_1 + p_2 = 1$, the informatively optimal phase always ends with the full revelation of the state. This explains the rule (2.8). Indeed, conditional on the goal of knowing the state with certainty, the agent minimizes the total cost of learning:

$$C[p_1; T] = \mathbb{E}\left[c_1 T_{\tau,1} + c_2 T_{\tau,2} \mid p_1, p_2 = 1 - p_1\right],$$

over the attention allocation plan T, where τ is the first time a positive signal is observed. The objective function $C[p_1; T]$ does not depend on the payoff parameters, and therefore the optimal T does not either. In the general case, when the probability of the third state is positive, revealing the state with certainty is not feasible when the true state is 3. However, the intuition for the informatively optimal phase is similar. When the agent thinks it is highly likely he is going to know the true state at the end of the information collection process, he chooses the informatively optimal source, that is, the source with the highest quality.

At point $p_1 = \frac{c_1 \lambda_2}{c_1 \lambda_2 + c_2 \lambda_1}$, $p_2 = \frac{c_2 \lambda_1}{c_1 \lambda_2 + c_2 \lambda_1}$ both sources are used in proportion to their intensity: $\frac{dT_{t,1}}{dT_{t,2}} = \frac{\lambda_2}{\lambda_1}$. This rule guarantees that the belief p_1 does not change until the state is revealed. Moreover, at this point the probability of revealing the state at the next instant of time, $\lambda \times p \times dt$, over the cost, $c \times dt$, is the same for both sources: $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$.[21,22]

Figures 2.1 and 2.2 illustrate the optimal a-type strategy by partitioning the interval $p_1 \in (0, 1)$ into different areas (Area 1, Area 2.1, Area 2.2, Area 3.1, and Area 3.2). Each point p_1 corresponds to the agent's initial belief, and the area number determines what strategy the agent should use.

Another way to describe the optimal a-type strategy is by partitioning the interval $p_1 \in (0, 1)$ into three regions: when to stop learning, when to use source 1, and when to use source 2. This description is equivalent to the one above by the Markovian property of the problem (2.2). At any point in time t, for any initial belief $p_{0,1}$, the agent's optimal behavior depends on $p_{0,1}$ and all information received so far only through the current belief $p_{t,1}$. Thus, partitioning the interval based on the source is equivalent to partitioning the interval based on the whole strategy. Arrows on these figures show how the belief p_1 changes in the absence of a positive signal: 0 signals from source 1 decreases p_1, while 0 signals from source 2 increases p_1. By following the arrows, we can infer the strategy the agent should use. For example, take any $p_{0,1}$ in the source 2 region. If this region is located to the left of the source 1 region, $p_{0,1}$ belongs to Area 3.2. If this region is located to the left of the stop region, $p_{0,1}$ belongs to Area 2.2.

---

[21]Using both sources in proportion to their intensity leads to the following expected payoff at point $p_1 = \frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1}$, $p_2 = \frac{c_2\lambda_1}{c_1\lambda_2+c_2\lambda_1}$: for $k = 1, 2$,

$$(p_1 u_1[a_1] + p_2 u_2[a_2]) - \left(\frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}\right).$$

This expression is intuitive. Since the agent is going to collect information until the state is revealed, he always chooses the alternative that is the best in this revealed state. Thus, the benefit is his expected utility from the best alternative: $p_1 u_1[a_1] + p_2 u_2[a_2]$. The cost of learning from each source is proportional to the expected time of using this source, which is the expected waiting time for the state to be revealed, $\frac{1}{\lambda}$.

[22]Rule (2.8) can be derived from minimizing $C[p_1; T]$ over T. Below, I show a sketch of the proof. Let $C[p_1] = \min_T C[p_1; T]$. If the agent pays $x \in [0, 1]$ amount of attention to source 1 during the next instant of time and then implements the optimal allocation rule, his expected cost is equal to $c_1 x dt + c_2(1-x)dt + (1-\lambda_1 p_1 x dt - \lambda_2(1-p_1)(1-x)dt) C[p_1 - \lambda_1 p_1(1-p_1)x dt + \lambda_2 p_1(1-p_1)(1-x)dt] = C[p_1] + \left(c_1 x \left(1 - \frac{\lambda_1 p_1}{c_1}\left(C[p_1] + C'[p_1](1-p_1)\right)\right) + c_2(1-x)\left(1 - \frac{\lambda_2(1-p_1)}{c_2}\left(C[p_1] - C'[p_1]p_1\right)\right)\right) dt \equiv C[p_1] + \delta[x, p_1, C[p_1]]dt$. If $p_1 = \frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1}$, $x = \frac{\lambda_2}{\lambda_1+\lambda_2}$ and $C\left[\frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1}\right] = \frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}$, then $\delta[x, p_1, C[p_1]] = 0$. Thus, at point $p_1 = \frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1}$, the attention allocation rule $x = \frac{\lambda_2}{\lambda_1+\lambda_2}$ is optimal since following this rule does indeed give $C\left[\frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1}\right] = \frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}$.

Theorem 1 *When the probability of the third state is zero, the optimal a-type strategy is described as follows. The belief interval $p_1 \in (0,1)$ is partitioned into at most five areas ($k = 1,2$):*

Area 1 *: for beliefs in this area, stop information collection.*

Area 2.k *: for beliefs in this area, use source k until $p_k = R_k^{(a)}$ (and then stop).*

Area 3.k *: for beliefs in this area, use source k until $p_k = \frac{c_k \lambda_{3-k}}{c_1\lambda_2 + c_2\lambda_1}$ (and then use both sources in proportion to their intensity, $\frac{dT_{t,1}}{dT_{t,2}} = \frac{\lambda_2}{\lambda_1}$, until the state is revealed).*

Case 1 *If for $k = 1,2$ the following condition holds:*

$$R_k^{(a)} < 1, \ \frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k} \ \Rightarrow \ \Delta_k^{(a)} \leq 0, \tag{2.9}$$

then

- *if $1 - R_2^{(a)} < p_1 < R_1^{(a)}$, then p_1 belongs to Area 1,*
- *if $p_1 > R_1^{(a)}$, then p_1 belongs to Area 2.1,*
- *if $p_1 < 1 - R_2^{(a)}$, then p_1 belongs to Area 2.2.*

Otherwise, let $k \in \{1,2\}$ be such that:

$$R_k^{(a)} < 1, \ \frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}, \ \Delta_k^{(a)} > 0. \tag{2.10}$$

Case 2 *If* Condition Π *does not hold, then*

- *if $p_k < \frac{c_k \lambda_{3-k}}{c_1\lambda_2 + c_2\lambda_1}$, then p_1 belongs to Area 3.3-k,*
- *if $p_k > \frac{c_k \lambda_{3-k}}{c_1\lambda_2 + c_2\lambda_1}$, then p_1 belongs to Area 3.k.*

Case 3 *If* Condition Π *holds, then*

- *if $p_k < 1 - R_{3-k}^{(a)}$, then p_1 belongs to Area 2.3-k,*
- *if $1 - R_{3-k}^{(a)} < p_k < \min\left\{\bar{\pi}_k^{(a)}, R_k^{(a)}\right\}$, then p_1 belongs to Area 1,*
- *if $R_k^{(a)} < p_k < \bar{\pi}_k^{(a)}$, then p_1 belongs to Area 2.k,*
- *if $\bar{\pi}_k^{(a)} < p_k < \frac{c_k \lambda_{3-k}}{c_1\lambda_2 + c_2\lambda_1}$, then p_1 belongs to Area 3.3-k,*
- *if $p_k > \frac{c_k \lambda_{3-k}}{c_1\lambda_2 + c_2\lambda_1}$, then p_1 belongs to Area 3.k,*

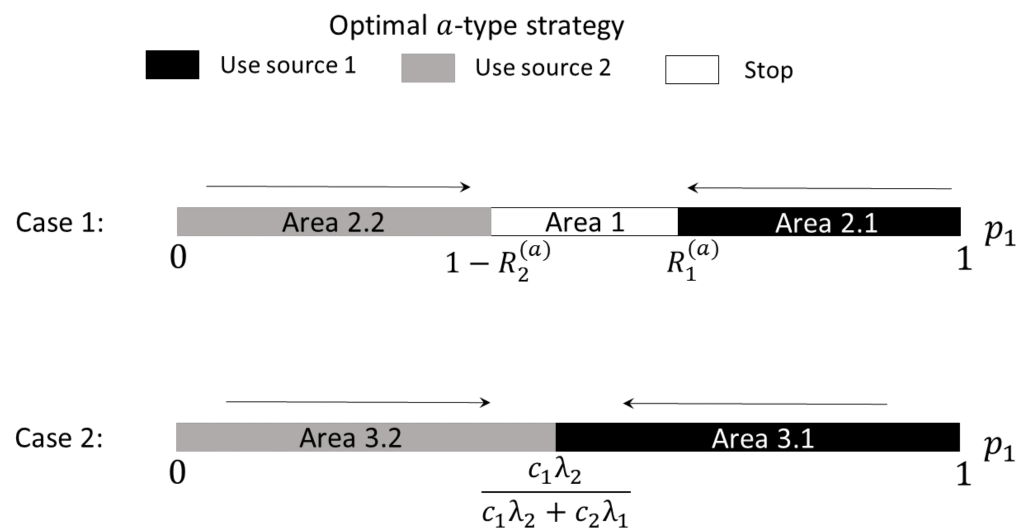

Figure 2.1: Illustration for Theorem 1, Case 1 and Case 2. The arrow shows the direction the belief vector is moving while the state is not revealed.

where

$$\Delta_k^{(a)} = \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\left(\frac{1}{R_{3-k}^{(a)}} - 1\right) + \log\left[\frac{c_k\lambda_{3-k}\left(1-R_k^{(a)}\right)}{c_{3-k}\lambda_k R_k^{(a)}}\right] - 1, \qquad (2.11)$$

and Condition Π *and* $\bar{\pi}_k^{(a)}$ *are defined in the appendix.*

Case 1 covers the set of parameters for which it is never optimal to use both sources simultaneously.[23] In Case 2, it is always optimal to learn the state with certainty. Case 3 is a mixture of Case 1 and Case 2.

Condition (2.9) for $k = 1, 2$, guarantees that it is never optimal to implement the rule (2.8).

Given expression (2.11), condition $\Delta_k^{(a)} \leq 0$ has the following interpretation. For each source j, the cost of using the other source $(\frac{c_{3-j}}{\lambda_{3-j}})$ is sufficiently large, as measured by the difference between the benefit of using that source $(u_j[a_j] - u_j[a])$ and its cost $(\frac{c_j}{\lambda_j})$. In other words, both $\frac{\frac{c_2}{\lambda_2}}{u_1[a_1]-u_1[a]-\frac{c_1}{\lambda_1}}$ and $\frac{\frac{c_1}{\lambda_1}}{u_2[a_2]-u_2[a]-\frac{c_2}{\lambda_2}}$ are large enough to make sure that when the state is likely to be

[23]Condition (2.9) guarantees that $1 - R_2^{(a)} \leq R_1^{(a)}$ in Case 1. See Remark 1 in the appendix.

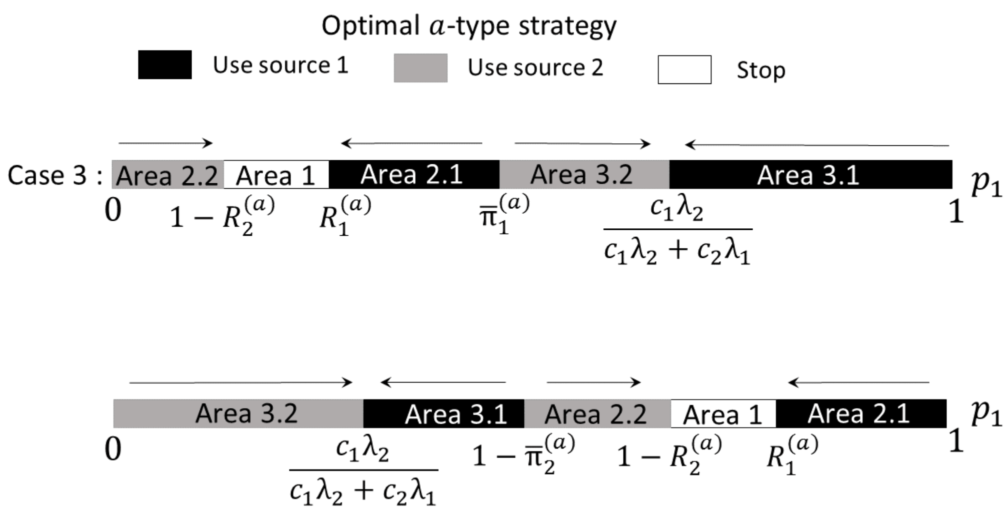

Figure 2.2: Illustration for Theorem 1, Case 3. The arrow shows the direction the belief vector is moving while the state is not revealed.

j (i.e., when $p_j \geq R_j^{(a)}$), to decide between a and a_j it is optimal to use only source j, which provides "direct" information about the state being j or not, rather than rely on "indirect" information from source $3 - j$.

Expression (2.11) can be rewritten as

$$\Delta_k^{(a)} = \frac{V^{(a)}[p_1]\Big|_{\text{Area 3.k}} - V^{(a)}[p_1]\Big|_{\text{Area 2.k}}}{\frac{c_k}{\lambda_k}(1 - p_k)},$$

where $V^{(a)}[p_1]\Big|_{\text{Area } X}$ is the expected payoff from the strategy described in Area X (see appendix for the proof). Thus, $\Delta_k^{(a)} > 0$ means that strategy described in Area 3.k delivers a higher expected payoff than the one in Area

2.k.[24,25]

At point $\bar{\pi}_k^{(a)}$, the agent is indifferent between using at most one source in his learning strategy and applying the Area 3.3-k strategy. Condition Π guarantees that such a point exists.[26]

Generally speaking, the optimal a-type strategy is unique. It means for all beliefs (except the set of measure zero) and for all other parameters' values (except the set of measure zero), the optimal action (what source to choose and when to stop) is unique. Indeed, the expected payoff from different strategies is almost never the same. The nonuniqueness can happen only at the indifference points. For some parameters' values, such indifference points might form the whole interval. For example, when $\bar{\pi}_k^{(a)} = 1 - R_{3-k}^{(a)}$, the whole interval $p_k \geq 1 - R_{3-k}^{(a)}$ is such that the agent is indifferent between Area 2.3-k and Area 3.3-k strategies. However, the set of such parameters' values has measure zero.

Step 2 . Given initial belief p_1, the optimal strategy is the optimal a-type strategy, where $a \in \mathcal{A}$ maximizes $V^{(a)}[p_1]$.

Though sufficient for computing the optimal strategy, this description is not very illustrative. Another way to describe the optimal strategy is as follows.

Denote by $V_k[p_1]$, $k = 1, 2$, the expected payoff from the optimal strategy if only source k is available. Denote this strategy as *k-strategy*. This strategy

[24] Comparing these two strategies, one must also take into account their feasibility: Strategy in Area 2.k only makes sense if $p_k \geq R_k^{(a)}$, while strategy in Area 3.k is feasible if and only if $p_k \geq \frac{c_k \lambda_{3-k}}{c_1\lambda_2 + c_2\lambda_1}$. However, condition $\Delta_k^{(a)} > 0$ does not depend on beliefs. Thus, if $\Delta_k^{(a)} > 0$, then it is optimal to use strategy from Area 3.k for all $p_k \geq \max\left\{R_k^{(a)}, 1 - R_{3-k}^{(a)}, \frac{c_k\lambda_{3-k}}{c_1\lambda_2+c_2\lambda_1}\right\}$: $p_k \geq R_k^{(a)}$ guarantees that Area 2.k strategy is feasible and is better than no learning, inequality $p_k \geq 1 - R_{3-k}^{(a)}$ excludes the Area 2.3-k strategy. By the Markovian property, it is optimal to use strategy from Area 3.k for all $p_k \geq \frac{c_k\lambda_{3-k}}{c_1\lambda_2+c_2\lambda_1}$. This reflects in Case 2 and 3, where Area 3.k covers the whole interval $p_k \in \left(1, \frac{c_k\lambda_{3-k}}{c_1\lambda_2+c_2\lambda_1}\right)$.

[25] Condition (2.9) is a more subtle than just $\Delta_k^{(a)} \leq 0$. Condition $R_k^{(a)} < 1$ is necessary for Area 2.k strategy to be feasible. Moreover, for $R_k^{(a)} \geq 1$, $\Delta_k^{(a)}$ is undefined. Suppose $R_k^{(a)} < 1$. Condition (2.9) per se does not rule out the case when $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} < \frac{\lambda_k R_k^{(a)}}{c_k}$ and $\Delta_k^{(a)} > 0$. However, condition (2.9) for $k = 1, 2$ together do imply $\Delta_i^{(a)} \leq 0$ whenever $R_i^{(a)} < 1$ for $i = 1, 2$ (see Remark 2 in the appendix for the proof).

[26] Note that point $\bar{\pi}_k^{(a)}$ must always be above $1 - R_{3-k}^{(a)}$. Indeed, by the Markovian property, for $p_k \leq 1 - R_{3-k}^{(a)}$, Area 2.3-k strategy is better than Area 3.3-k strategy if and only if it is better at point $p_k = 1 - R_{3-k}^{(a)}$.

consists of the payoff optimal phase. [27]

Denote by $V_3[p_1]$ the expected payoff from the following strategy (denote this strategy as ∞-*strategy*):[28]

- if $p_1 > \frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1}$, use source 1,
- if $p_1 < \frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1}$, use source 2,
- at point $p_1 = \frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1}$, use both sources in proportion to their intensity.

This strategy consists of the informatively optimal phase.

If $V_k[p_1] \geq \max\{V_{3-k}[p_1], V_3[p_1]\}$ for some $k = 1, 2$, then the optimal strategy is the k-strategy; if $V_3[p_1] \geq \max\{V_1[p_1], V_2[p_1]\}$, then the optimal strategy is the ∞-strategy. This description is more intuitive since it does not involve maximization over $a \in \mathcal{A}$ but rather emphasizes the strategic tradeoff the agent faces. He compares the costs and benefits of information to decide whether it is optimal to learn the state with certainty (∞-strategy) or if he should "give up" at some point if he does not observe a positive signal for a sufficiently long time (1-strategy or 2-strategy).

Figure 2.3 shows two examples of the optimal strategy defined as a partition of the belief interval into three regions (use source 1, use source 2, stop the information collection process). These examples illustrate two general features of the optimal strategy. First, if the agent is ever going to use both sources, he does it at point $p_1 = \frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1}$. Thus, if either 1-strategy or 2-strategy is better than ∞-strategy at this point, there is no initial belief such

[27] See Section 2.4.2 for the explicit form of $V_k[p_1]$.

[28] If $p_k \geq \frac{c_k\lambda_{3-k}}{c_1\lambda_2+c_2\lambda_1}$, this payoff is

$$V_3[p_1] = (p_1u_1[a_1] + p_2u_2[a_2]) - \left(p_{3-k} + \frac{c_k\lambda_{3-k}}{c_1\lambda_2 + c_2\lambda_1}\right)\left(\frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}\right) - \frac{c_k p_{3-k}}{\lambda_k} \log\left[\frac{p_k}{p_{3-k}} \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\right].$$

Part $p_1u_1[a_1] + p_2u_2[a_2]$ accounts for the utility the agent gets from the chosen alternative at the end of the information collection process. The rest is equal to the expected cost of collecting information. When $p_k = \frac{c_k\lambda_{3-k}}{c_1\lambda_2+c_2\lambda_1}$, this cost is equal to $\frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}$. When $p_k > \frac{c_k\lambda_{3-k}}{c_1\lambda_2+c_2\lambda_1}$, the middle term, $\left(p_{3-k} + \frac{c_k\lambda_{3-k}}{c_1\lambda_2+c_2\lambda_1}\right)\left(\frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}\right)$, is less than $\frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}$, while the last term, $\frac{c_k p_{3-k}}{\lambda_k} \log\left[\frac{p_k}{p_{3-k}} \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\right]$, is positive. Moreover, when $p_k > \frac{c_k\lambda_{3-k}}{c_1\lambda_2+c_2\lambda_1}$, the expected cost decreases in p_k, which means $\left(p_{3-k} + \frac{c_k\lambda_{3-k}}{c_1\lambda_2+c_2\lambda_1}\right)\left(\frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}\right) + \frac{c_k p_{3-k}}{\lambda_k} \log\left[\frac{p_k}{p_{3-k}} \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\right] < \frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}$.

Intuitively, this follows from the optimality of the strategy. The strategy to use both sources until the state is revealed is always feasible, which means the expected payoff $(p_1u_1[a_1] + p_2u_2[a_2]) - \left(\frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}\right)$ is always achieved. By definition of optimality, the expected payoff from the optimal strategy must be no less than this expression.

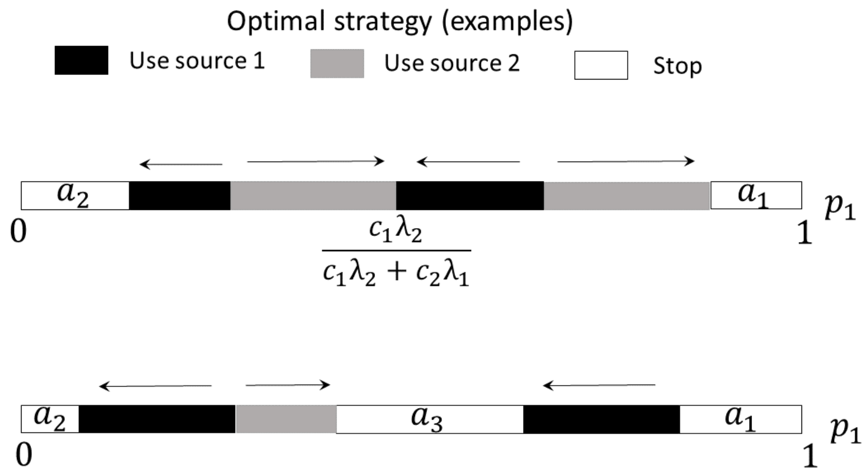

Figure 2.3: Illustration of the optimal strategy when the probability of state 3 is zero. The arrow shows the direction the belief vector is moving while the state is not revealed.

that ∞-strategy is optimal.[29] Second, if the agent chooses the k-strategy and does not "give up" immediately, the default alternative is either a_{3-k} or a_3. Indeed, the information benefit of using source k is proportional to the cost of mistake, $u_k[a_k] - u_k[a]$. If the default alternative a is a_k, then this cost is

[29] Formally, when the probability of the third state is zero and

$$R_1^{(a_2)} < 1,\ R_2^{(a_1)} < 1,\ R_k^{(a_3)} < 1,\ \Delta_1^{(a_2)} > 0,\ \Delta_2^{(a_1)} > 0,\ \Delta_k^{(a_3)} > 0, \quad \text{where } k \in \left\{ \{1,2\} : \frac{\lambda_k R_k^{(a_3)}}{c_k} \le \frac{\lambda_{3-k} R_{3-k}^{(a_3)}}{c_{3-k}} \right\}, \tag{2.12}$$

then there exist two thresholds, $0 < \underline{p}_1 < \frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1} < \bar{p}_1 < 1$ such that it is optimal to implement the ∞-strategy for all $p_1 \in \left[\underline{p}_1, \bar{p}_1\right]$; for other beliefs, it is optimal to implement the k-strategy, where $k \in \{1,2\}$ is such that $V_k[p_1] \ge V_{3-k}[p_1]$. If condition (2.12) does not hold and $k \in \{1,2\}$ is such that $V_k[p_1] \ge V_{3-k}[p_1]$, then the optimal strategy is the k-strategy.

Condition (2.12) can be rewritten as follows:

$$\frac{1}{R_1^{(a_2)}} > \frac{c_2\lambda_1}{c_1\lambda_2} e^{1+\frac{c_2\lambda_1}{c_1\lambda_2}} + 1,\ \frac{1}{R_2^{(a_1)}} > \frac{c_1\lambda_2}{c_2\lambda_1} e^{1+\frac{c_1\lambda_2}{c_2\lambda_1}} + 1,$$

$$\frac{\lambda_k R_k^{(a_3)}}{c_k} \le \frac{\lambda_{3-k} R_{3-k}^{(a_3)}}{c_{3-k}} \quad \Rightarrow \quad \frac{1}{R_k^{(a_3)}} > \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}} e^{1-\frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\left(\frac{1}{R_{3-k}^{(a_3)}}-1\right)} + 1. \tag{2.13}$$

(2.13) means the cost of mistake — that is, $u_1[a_1] - u_1[a_2]$, $u_2[a_2] - u_2[a_1]$, and $u_k[a_k] - u_k[a_3]$ for some $k \in \{1,2\}$ — is large enough to implement the ∞-strategy, which leads to the best choice of the alternative.

zero and therefore, source k offers no benefit. This feature explains why it is sometimes optimal to use the source that has low ex ante probability to reveal the state (see "use source 1" region near $p_1 = 0$ and "use source 2" region near $p_1 = 1$).

2.4.2 Three States, One Source

In this section, I present the optimal strategy when there is only one information source available to the agent (suppose it is source 1):

$$\sup_{(a,\tau)} \mathbb{E}\left[u_j[a] - c_1\tau \mid p_1, p_2\right]. \tag{2.14}$$

This is a standard optimal stopping problem (Wald and Wolfowitz (1948), Dynkin (1963), Dragalin, Tartakovsky, and Veeravalli (1999), Dragalin, Tartakovsky, and Veeravalli (2000), Shiryaev (2007)). Usually, it is formulated with two states and two alternatives. Technically, the generalization to three states and three alternatives is straightforward. However, it provides new insight to the general model I present in Section 2.5.

The Markovian property of the problem — at any point in time t, for any initial beliefs $(p_{0,1}, p_{0,2})$, the agent's optimal behavior depends on initial beliefs and all information received so far only through the current beliefs $(p_{t,1}, p_{t,2})$ — allows for representing the optimal strategy as a partition of the belief triangle

$$\{(p_1, p_2) \in [0,1]^2 \colon p_1 + p_2 \leq 1\}$$

into two regions: "use source 1" and "stop." The "use source 1" region is the set of beliefs (p_1, p_2) such that if the agent's current beliefs fall into this set, he pays his full attention to source 1. The "stop" region corresponds to the set of current beliefs where it is optimal to stop collecting information and choose an alternative.

By definition, the information source 1 separates state 1 from the other two states, meaning the information from this source cannot affect the agent's belief about the relative probabilities of states 2 and 3. Formally, the ratio $\frac{p_{t,2}}{1-p_{t,1}}$ stays constant throughout the learning process. Moreover, unless the state 1 is revealed, the belief about state 1 decreases during the learning process. Graphically, source 1 moves the belief vector along the line that holds the ratio $\frac{p_2}{1-p_1}$ constant, away from the corner $p_1 = 1$ (see Figure 2.4).

As in Section 2.4.1, a strategy is equivalent to a plan of what to do conditional on not receiving a positive signal. This contingency plan is defined by a "give up" time and the default alternative. By the Markovian property, defining a "give up" time is equivalent to defining the belief threshold $p_1 = \underline{p}_1$. Once the agent's belief reaches this threshold, he stops the information collection.

Lemma 1 gives the explicit form of the expected payoff from the strategy with a given threshold $p_1 = \underline{p}_1$ and the default alternative $a \in \mathcal{A}$.

Lemma 1 *Given the initial beliefs (p_1, p_2), any threshold $\underline{p}_1 \in (0, p_1]$ and any default alternative $a \in \mathcal{A}$, the expected payoff from using source 1 until either state 1 is revealed (and a_1 is chosen) or the belief reaches the threshold $p_1 = \underline{p}_1$ (and a is chosen), whichever happens first, is the following:*[30]

$$V_1^{(a)}\left[p_1, p_2; \underline{p}_1\right] = \frac{1-p_1}{1-\underline{p}_1}\left(u_1[a]\underline{p}_1 + \left(u_2[a]\frac{p_2}{1-p_1} + u_3[a]\left(1 - \frac{p_2}{1-p_1}\right)\right)(1-\underline{p}_1)\right)$$
$$+ \frac{p_1 - \underline{p}_1}{1-\underline{p}_1}u_1[a_1] - \frac{c_1}{\lambda_1}\left((1-p_1)\log\left[\frac{p_1\left(1-\underline{p}_1\right)}{\underline{p}_1(1-p_1)}\right] + \frac{p_1 - \underline{p}_1}{1-\underline{p}_1}\right). \quad (2.15)$$

Expression (2.15) has three terms. The first term,

$$\frac{1-p_1}{1-\underline{p}_1}\left(u_1[a]\underline{p}_1 + \left(u_2[a]\frac{p_2}{1-p_1} + u_3[a]\left(1 - \frac{p_2}{1-p_1}\right)\right)(1-\underline{p}_1)\right)$$

is the probability that state 1 is not revealed by the end of the information collection process multiplied by the expected utility from choosing the default alternative. The second term, $\frac{p_1-\underline{p}_1}{1-\underline{p}_1}u_1[a_1]$, is the probability that state 1 is revealed before the "give up" time multiplied by the payoff from choosing alternative a_1 at state 1. The last term is the expected total cost of using source 1.

As with the two states, two sources setup in Section 2.4.1, I split the solution into two steps. First, I find the optimal a-type strategy by maximizing (2.15) over $\underline{p}_1$. Then, given the optimal threshold $\underline{p}_1$, I maximize (2.15) over all possible default alternatives $a \in \mathcal{A}$.

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[30]The expected payoff from the symmetric contingency plan with source 2 instead of source 1 is

$$V_2^{(a)}\left[p_1, p_2; \underline{p}_2\right] = \frac{1-p_2}{1-\underline{p}_2}\left(u_2[a]\underline{p}_2 + \left(u_1[a]\frac{p_1}{1-p_2} + u_3[a]\left(1 - \frac{p_1}{1-p_2}\right)\right)(1-\underline{p}_2)\right)$$
$$+ \frac{p_2 - \underline{p}_2}{1-\underline{p}_2}u_2[a_2] - \frac{c_2}{\lambda_2}\left((1-p_2)\log\left[\frac{p_2\left(1-\underline{p}_2\right)}{\underline{p}_2(1-p_2)}\right] + \frac{p_2 - \underline{p}_2}{1-\underline{p}_2}\right).$$

I use the expression $V_k^{(a)}\left[p_1, p_2; \underline{p}_k\right]$, $k = 1, 2$, in Section 2.5 for the general model.

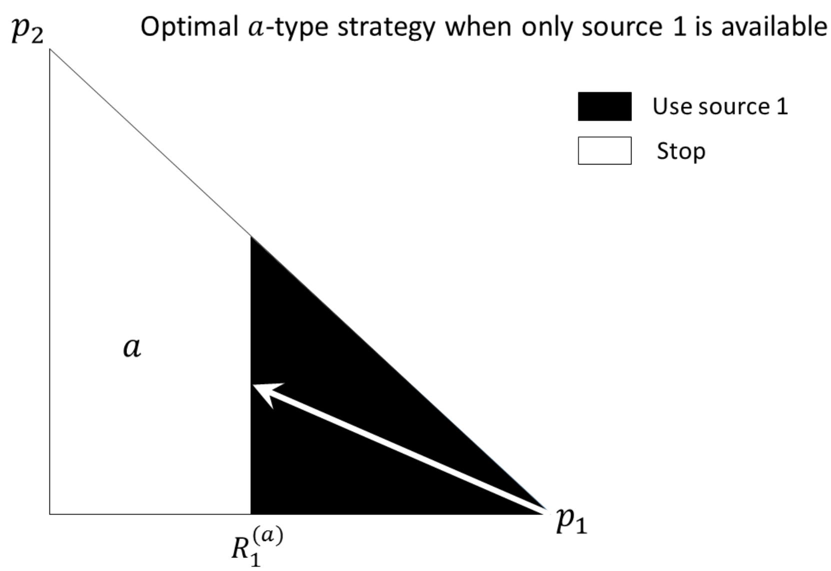

Figure 2.4: Illustration for Theorem 2. The arrow shows the direction the belief vector is moving while the state is not revealed.

Theorem 2 describes the optimal a-type strategy. It states that (2.15) achieves its maximum at $\underline{p}_1 = \min\{p_1, R_1^{(a)}\}$, where $R_1^{(a)}$ is defined by (2.6):

Theorem 2 *When only source 1 is available, for any initial beliefs, the optimal a-type strategy, $a \in \mathcal{A}$, is to use source 1 if and only if the agent is uncertain about the state and the current belief about the probability that the true state is 1 is no less than $R_1^{(a)}$.*

Theorem 2 shows that the threshold rule $p_1 = R_1^{(a)}$ derived in Theorem 1 extends to a general case when $p_1 + p_2 \leq 1$.

Figure 2.5 shows how a very simple form of the optimal a-type strategy might lead to a complex description of the optimal strategy on the belief triangle. This complexity is the main reason to split the solution into two steps and derive the optimal a-type strategy first. Figure 2.5 is derived by comparing the expected payoff from the optimal a_1-type, a_2-type, and a_3-type strategies, for each point in the belief triangle. It illustrates the optimal strategy by partitioning the belief triangle into two regions: where information collection is optimal and where it is not. For any beliefs (p_1, p_2) where information collection is optimal, the default alternative is determined by following the belief trajectory along the line with constant $\frac{p_2}{1-p_1}$ ratio away from $p_1 = 1$. For example, at point A the default alternative is a_2. See Theorem 11 in the

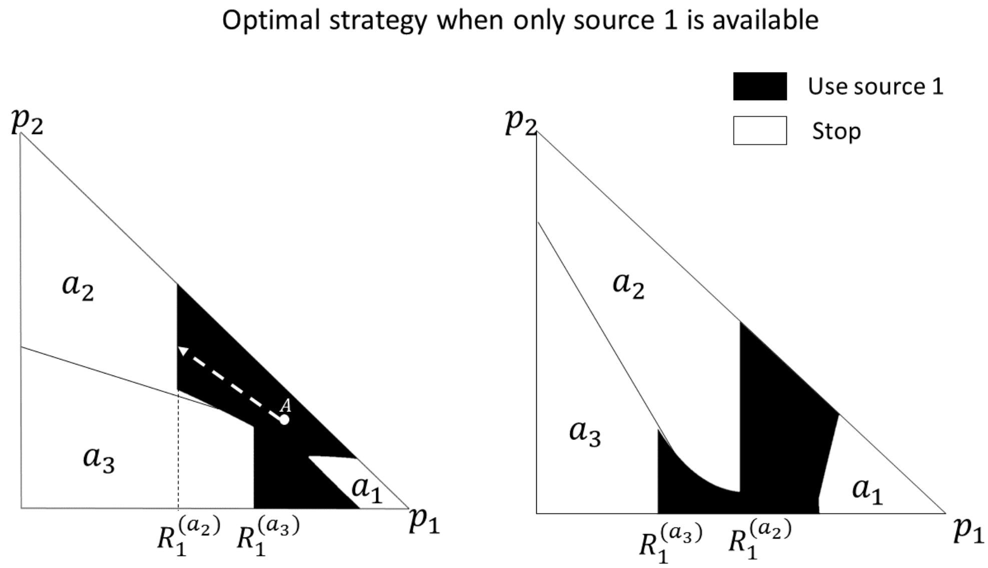

Figure 2.5: Illustration of the optimal strategy when only one source is available. "Stop" regions show which alternative is the default one.

appendix for a formal description of the optimal strategy when only one information source is available.

2.5 General Model: Three States, Two Sources

In this section, I present the solution for a general case, when all three states are possible (that is, the belief vector lies inside the belief triangle) and both sources are available.

2.5.1 a-Type Strategy

As in Section 2.4.1, I split the solution to (2.2) into two steps.

For any $a \in \mathcal{A}$, an *a-type strategy* is a strategy (a^F, T, τ), where

$$a^F = a\mathbf{1}\left(p_{\tau,1} < 1,\ p_{\tau,2} < 1\right) + a_1\mathbf{1}\left(p_{\tau,1} = 1\right) + a_2\mathbf{1}\left(p_{\tau,2} = 1\right). \tag{2.16}$$

The *optimal a-type strategy* is an a-type strategy (a^F, T, τ) such that (T, τ) maximizes

$$V^{(a)}[p_1, p_2] \equiv$$
$$\sup_{(T,\tau)} \mathbb{E}\left[u_j\left[a\mathbf{1}\left(p_{\tau,1} < 1,\ p_{\tau,2} < 1\right) + \sum_{k=1,2} a_k\mathbf{1}\left(p_{\tau,k} = 1\right)\right] - \sum_{k=1,2} c_k T_{\tau,k} \mid p_1, p_2\right].$$

The *optimal strategy* is a strategy (a^F, T, τ) that maximizes

$$V[p_1, p_2] \equiv \sup_{(a^F, T, \tau)} \mathbb{E}\left[u_j[a^F] - c_1 T_{\tau,1} - c_2 T_{\tau,2} \mid p_1, p_2\right].$$

Lemma 2 constructs the optimal strategy from the optimal a-type strategies, $a \in \mathcal{A}$.

Lemma 2 *The optimal strategy is the optimal a-type strategy where $a \in \mathcal{A}$ maximizes $V^{(a)}[p_1, p_2]$:*

$$V[p_1, p_2] = \max_{a \in \mathcal{A}} V^{(a)}[p_1, p_2].$$

Lemma 3 confirms the consistency of the strategy choice: if the agent chooses some a-type strategy, it is optimal to follow this strategy throughout the learning process. In other words, the agent will not change his mind about the optimal default alternative:

Lemma 3 *Let alternative $a \in \mathcal{A}$ and beliefs (p_1, p_2) be such that $V[p_1, p_2] = V^{(a)}[p_1, p_2]$, let (a^F, T^*, τ^*) be the optimal a-type strategy for the initial beliefs (p_1, p_2), and let $(p_{t,1}, p_{t,2})$, $t \leq \tau^*$, be the belief trajectory under the attention allocation plan T^* and the initial beliefs (p_1, p_2). Then $V[p_{t,1}, p_{t,2}] = V^{(a)}[p_{t,1}, p_{t,2}]$ for all $t \leq \tau^*$.*

2.5.2 Optimal a-Type Strategy

Similar to the benchmark case with one information source, I describe the optimal a-type strategy as a partition of the belief triangle. This partition consists of three regions: "use source 1," "use source 2," and "stop."

There are three different curves on the belief triangle along which the agent might — but not necessarily will — switch sources according to the optimal a-type strategy: a straight line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$, and two curves described by functions $p_k = \bar{p}_k^{(a)}[p_{3-k}]$, $k = 1, 2$ (I define these functions later). On $p_k = \bar{p}_k^{(a)}[p_{3-k}]$, the agent switches from source $3 - k$ to source k and never switches back (that is, the switching is irreversible). In contrast, the agent "crawls" along the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ by using both sources simultaneously.[31]

[31] This "crawling" behavior is similar to the one described in Mandelbaum, Shepp, and Vanderbei (1990) who consider optimal switching between a pair of Brownian motions within a unit square, with stopping on the boundaries.

For any initial beliefs, the optimal a-type strategy consists of at most two phases (without risk of confusion, I omit mentioning every time that once the state is revealed the agent stops information collection, so the strategy is described conditional on not knowing the state with certainty). On the first, informatively optimal phase, the agent chooses the source that has the highest quality, which is defined as the probability of revealing the state to cost ratio. More precisely, if $k \in \{1, 2\}$ is such that $\frac{\lambda_k p_k}{c_k} > \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}}$, where (p_1, p_2) is the vector of his current beliefs, then the agent chooses source k. Note that by using source k, the agent decreases his belief about state k, unless state k is revealed. Graphically, it means his beliefs are moving towards the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$. If the agent's current beliefs are on the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$, then he uses both sources simultaneously in proportion to the sources' intensity: $\frac{dT_{t,1}}{dT_{t,2}} = \frac{\lambda_2}{\lambda_1}$. This rule guarantees that the agent's beliefs stay on the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$.

On the second, payoff optimal phase, the agent's optimal a-type strategy coincides with the optimal a-type strategy in the restricted problem, when at the beginning of the game the agent has to choose which source to permanently eliminate from consideration. Thus, if the agent chooses to eliminate source $3-k$, the optimal a-type strategy on the payoff optimal phase coincides with the solution to the benchmark model in Section 2.4.2, when only source k is available: the agent uses this source until $p_k = R_k^{(a)}$.

In sum, the two phases differ in two ways. First, during the first phase, the agent might use two sources simultaneously, while during the second phase, he always allocates his full attention to one source. Second, during the first phase, the agent's attention rule is based only on his current beliefs (p_1 and p_2) and the sources' characteristics (cost and intensity), while during the second phase, his attention rule also depends on the agent's payoff ($u_1[a_1]$, $u_2[a_2]$, $u_1[a]$, $u_2[a]$). Intuitively, this happens because during the first phase, the agent takes into account only the local (short-term) benefit, which includes only the probability of the state being revealed but not the payoff from the chosen alternative. Indeed, the alternative is chosen only once during the game, so the corresponding payoff must be a global feature of the model. On the other hand, the beliefs are always changing throughout the information collection process, thus they are included in the local benefit.

Depending on initial beliefs and other parameters of the model, the informatively optimal phase might not include the simultaneous use of two sources. If the belief vector reaches the threshold between two phases sooner than it reaches the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$, only one source is used during the first phase. In this case, the

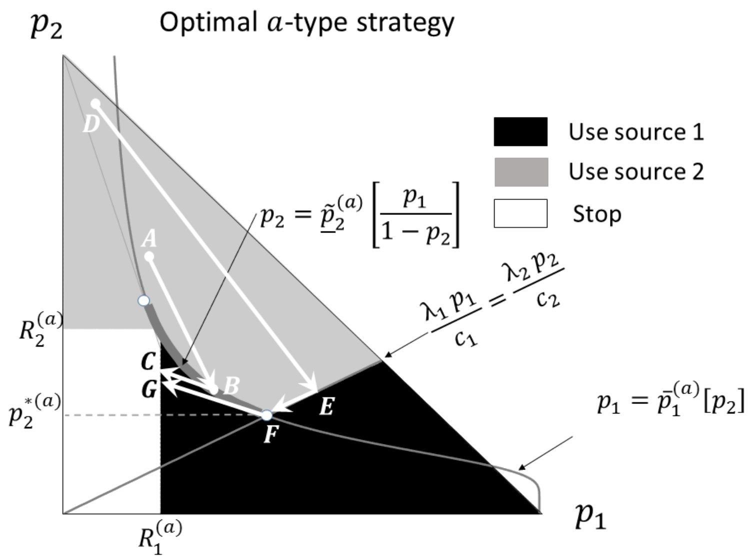

Figure 2.6: Example of the optimal a-type strategy. The paths $A \to B \to C$ and $D \to E \to F \to G$ show how the beliefs are moving in the absence of a positive signal.

switching between phases happens on one of the curves $p_k = \bar{p}_k^{(a)}[p_{3-k}]$, $k = 1, 2$. More precisely, it happens only on one part of each curve, namely $p_k = \tilde{\underline{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$. If source k is used during the first phase, then the threshold is on the curve $p_k = \tilde{\underline{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$ and the agent uses source $3-k$ at the second phase. See the path $A \to B \to C$ on Figure 2.6.

Suppose the informatively optimal phase includes the simultaneous use of both sources, resulting in clawing along the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ towards the ($p_1 = 0$, $p_2 = 0$) point. At some point along this line, the agent enters the second phase. This is the point where one of the curves, either $p_1 = \bar{p}_1^{(a)}[p_2]$ or $p_2 = \bar{p}_2^{(a)}[p_1]$, crosses the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$. For $k = 1, 2$, denote $p_k = p_k^{*(a)}$ the point where $p_{3-k} = \bar{p}_{3-k}^{(a)}[p_k]$ crosses $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$. The informatively optimal phase ends when $p_1 = p_1^{*(a)}$ if and only if $u_1[a_1] - u_1[a] \le u_2[a_2] - u_2[a]$ (which is equivalent to $\frac{\lambda_1 R_1^{(a)}}{c_1} \ge \frac{\lambda_2 R_2^{(a)}}{c_2}$).[32] Moreover, when the agent leaves the informatively optimal phase at point $p_k = p_k^{*(a)}$, he uses source $3-k$ during the payoff optimal phase. Intuitively, when $u_1[a_1] - u_1[a] < u_2[a_2] - u_2[a]$, source 2 has a utility advantage (the cost of

32 $\frac{\lambda_1 p_1^{*(a)}}{c_1} = \frac{\lambda_2 p_2^{*(a)}}{c_2}$ and $p_1^{*(a)} = R_1^{(a)}$, $p_2^{*(a)} = R_2^{(a)}$ when $u_1[a_1] - u_1[a] = u_2[a_2] - u_2[a]$. Thus, when $u_1[a_1] - u_1[a] = u_2[a_2] - u_2[a]$, the informatively optimal phase ends at point $p_1^{*(a)} = R_1^{(a)}$, $p_2^{*(a)} = R_2^{(a)}$ and the payoff optimal phase has zero length.

a mistake from a wrong alternative is higher when the state is 2). Thus, the agent should use this source at the last phase. See the path $D \to E \to F \to G$ on Figure 2.6.

Theorem 3 *The optimal a-type strategy is described as follows. The belief triangle* $\{(p_1, p_2) \in [0,1]^2 : p_1 + p_2 \leq 1\}$ *is partitioned into at most seven areas* ($k = 1, 2$)*:*

Area 1 *: for beliefs in this area, stop information collection.*

Area 2.k *: for beliefs in this area, use source* k *until* $p_k = R_k^{(a)}$ *(and then stop).*

Area 3.k.1 *: for beliefs in this area, use source* k *until* $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ *(and then use both sources in proportion to their intensity:* $\frac{dT_{t,1}}{dT_{t,2}} = \frac{\lambda_2}{\lambda_1}$ *until* $p_i = p_i^{*(a)}$*, then use source* $3-i$ *until* $p_{3-k} = R_{3-i}^{(a)}$*, where* $i \in \arg\min_{l=1,2}\{u_l[a_l] - u_l[a]\}$*, then stop).*

Area 3.k.2 *: for beliefs in this area, use source* k *until* $p_k = \underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$ *(then use source* $3-k$ *until* $p_{3-k} = R_{3-k}^{(a)}$*, then stop).*

Case 1 *If for* $k = 1, 2$ *condition (2.9) holds, then (see Figure 2.7)*

- *if* $p_1 < R_1^{(a)}$ *and* $p_2 < R_2^{(a)}$*, then* (p_1, p_2) *belongs to Area 1,*
- *if* $p_j > R_j^{(a)}$*, then* (p_1, p_2) *belongs to Area 2.j,* $j = 1, 2$.

Otherwise, let $k \in \{1, 2\}$ *be such that condition (2.10) holds. Then:*

Case 2 *If* $p_{3-k}^{**(a)} \geq R_{3-k}^{(a)}$*, then (see Figures 2.8 and 2.9)*

- *if* $\frac{\lambda_k p_k}{c_k} > \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}}$ *and* $\frac{p_{3-k}}{1-p_k} > \frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$*, then* (p_1, p_2) *belongs to Area 3.k.1,*
- *if* $\frac{\lambda_k p_k}{c_k} < \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}}$ *and* $\frac{p_k}{1-p_{3-k}} > \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$*, then* (p_1, p_2) *belongs to Area 3.3-k.1,*
- *if* $p_{3-k} > \underline{\tilde{p}}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ *and* $\frac{R_k^{(a)}}{1-R_{3-k}^{(a)}} < \frac{p_k}{1-p_{3-k}} < \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$*, then* (p_1, p_2) *belongs to Area 3.3-k.2,*
- *otherwise:*
  - *if* $p_1 < R_1^{(a)}$ *and* $p_2 < R_2^{(a)}$*, then* (p_1, p_2) *belongs to Area 1,*

– *if* $p_j > R_j^{(a)}$, *then* (p_1, p_2) *belongs to Area 2.j,* $j = 1, 2$.

Case 3 *If* $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, *then (see Figures 2.10 and 2.11)*

- *if* $\frac{\lambda_k p_k}{c_k} > \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}}$ *and* $\frac{p_{3-k}}{1-p_k} > \frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$, *then* (p_1, p_2) *belongs to Area 3.k.1,*
- *if* $\frac{\lambda_k p_k}{c_k} < \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}}$, $p_{3-k} < \bar{\pi}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ *and* $\frac{p_k}{1-p_{3-k}} > \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$, *then* (p_1, p_2) *belongs to Area 3.3-k.1,*
- *if* $\underline{\tilde{p}}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right] < p_{3-k} < \bar{\pi}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ *and* $\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{**(a)}\right]}{1-p_{3-k}^{**(a)}} < \frac{p_k}{1-p_{3-k}} < \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$, *then* (p_1, p_2) *belongs to Area 3.3-k.2,*
- *otherwise:*
  - *if* $p_1 < R_1^{(a)}$ *and* $p_2 < R_2^{(a)}$, *then* (p_1, p_2) *belongs to Area 1,*
  - *if* $p_j > R_j^{(a)}$, *then* (p_1, p_2) *belongs to Area 2.j,* $j = 1, 2$,

where

$$\bar{p}_k^{(a)}[p_{3-k}] = \frac{1}{1 + \left(\frac{1}{R_k^{(a)}} - 1\right) e^{\frac{\lambda_k c_{3-k}}{c_k \lambda_{3-k}}\left(\frac{1}{R_{3-k}^{(a)}} - \frac{1}{p_{3-k}}\right)}}, \tag{2.17}$$

$$p_{3-k}^{*(a)} \in \left(0, R_{3-k}^{(a)}\right] : \quad \frac{\lambda_{3-k} p_{3-k}^{*(a)}}{c_{3-k}} = \frac{\lambda_k \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{c_k}, \tag{2.18}$$

$$p_{3-k}^{**(a)} \in \left(p_{3-k}^{*(a)}, 1\right) : \quad \frac{c_k \left(p_{3-k}^{**(a)}\right)^2}{\left(1 - p_{3-k}^{**(a)}\right) \lambda_k} = \frac{c_{3-k}\left(1 - \bar{p}_k^{(a)}\left[p_{3-k}^{**(a)}\right]\right)}{\lambda_{3-k}}, \tag{2.19}$$

$$0 < \underline{\tilde{p}}_{3-k}^{(a)}[q_k] < p_{3-k}^{**(a)} < \bar{\tilde{p}}_{3-k}^{(a)}[q_k] < 1: \quad \frac{\bar{p}_k^{(a)}\left[\underline{\tilde{p}}_{3-k}^{(a)}[q_k]\right]}{1 - \underline{\tilde{p}}_{3-k}^{(a)}[q_k]} = \frac{\bar{p}_k^{(a)}\left[\bar{\tilde{p}}_{3-k}^{(a)}[q_k]\right]}{1 - \bar{\tilde{p}}_{3-k}^{(a)}[q_k]} = q_k, \tag{2.20}$$

and $\bar{\pi}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ *is defined in the appendix.*

Comparing Theorem 3 with Theorem 1, note that now there are four (instead of two) areas where there is a possibility of using both sources during the information collection process: Areas 3.1.1, 3.2.1, 3.1.2, and 3.2.2. The strategy in Area 3.k.1 is a generalization of the strategy in Area 3.k. The Area 3.k.2 strategy is only optimal if $p_1 + p_2 < 1$.

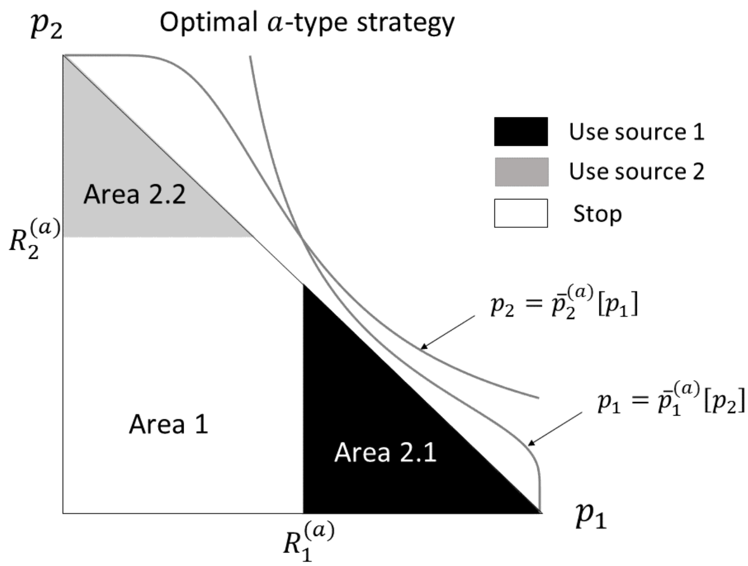

Figure 2.7: Illustration for Theorem 3, Case 1.

When current beliefs are in Area 2.1 or in Area 2.2, the information collection process is on the payoff optimal phase. Areas 3.1.1, 3.2.1, 3.1.2, and 3.2.2 correspond to the informatively optimal phase.

Case 1 in Theorem 3 is a generalization of Case 1 in Theorem 1. It covers the set of parameters' values for which the information collection process never involves using both sources, either simultaneously or sequentially. The conditions on parameters are exactly the same. See Figure 2.7 for illustration of Case 1.

Cases 2 and 3 involve switching sources. They come together as a generalization of Cases 2 and 3 in Theorem 1.[33]

The description of Cases 2 and 3 uses the indifference curve $p_k = \bar{p}_k^{(a)}[p_{3-k}]$. This curve is defined by (2.17) for all parameters' values. Alternatively, when $R_k^{(a)} < 1$, (2.17) can be rewritten as

$$c_{3-k} = \lambda_{3-k} p_{3-k} \left(u_{3-k}[a_{3-k}] - u_{3-k}[a] + \frac{c_k}{\lambda_k} \log \left[\frac{\bar{p}_k^{(a)}[p_{3-k}] \left(1 - R_k^{(a)}\right)}{R_k^{(a)} \left(1 - \bar{p}_k^{(a)}[p_{3-k}]\right)} \right] \right). \quad (2.21)$$

[33]However, the way I partition the set of parameters' values between Case 2 and Case 3 is not the same as in Theorem 1, but the principle is the same: in Case 2, the description does not involve the indifference curve $p_{3-k} = \bar{\pi}_{3-k}^{(a)} \left[\frac{p_k}{1-p_{3-k}} \right]$. The subset of parameters' values in Case 3 in Theorem 3 when $\bar{\pi}_{3-k}^{(a)}[1] = 1$ is included in Case 2 in Theorem 1.

Condition (2.21) says that at point (p_1, p_2) such that $p_k = \bar{p}_k^{(a)}[p_{3-k}]$, the cost of using source $3-k$ is equal to the benefit of using that source. Recall that $\lambda_{3-k}p_{3-k}dt$ is the probability that source $3-k$ reveals the state. The first part of the benefit, $\lambda_{3-k}p_{3-k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a]\right)$, is a "direct" benefit from using source $3-k$: if state $3-k$ is revealed, the utility benefit from choosing the best alternative instead of the default is $u_{3-k}[a_{3-k}] - u_{3-k}[a]$. The second part is an "indirect" benefit that accounts for changes in usefulness of source k. $\frac{c_k}{\lambda_k}\log\left[\frac{p_k\left(1-R_k^{(a)}\right)}{R_k^{(a)}(1-p_k)}\right]$ increases with p_k, the probability that the state is k. The higher that probability, the greater the probability that source k reveals the state, and therefore, the more valuable source k is. What is more, $\frac{c_k}{\lambda_k}\log\left[\frac{p_k\left(1-R_k^{(a)}\right)}{R_k^{(a)}(1-p_k)}\right]$ is equal to 0 when $p_k = R_k^{(a)}$, which is a threshold for using source k. To sum up, (2.21) says that at point (p_1, p_2) such that $p_k = \bar{p}_k^{(a)}[p_{3-k}]$, the cost of using source $3-k$ is equal to the sum of the direct benefit of using that source (choosing the best alternative if the state is $3-k$) and the indirect benefit (increasing the usefulness of source k through shifting the belief vector in the direction of the point $p_k = 1$). The direct benefit is the benefit of a positive signal while the indirect benefit is the benefit of a zero signal, which increases the direct benefit of all future signals.

Since $\bar{p}_k^{(a)}[p_{3-k}]$ is a threshold between two phases, (2.21) has the meaning of the first order condition for the optimal stopping problem for using source $3-k$ under the condition that upon stopping, source k will be used until point $p_k = R_k^{(a)}$.

Condition (2.9) is equivalent to the requirement that when $R_k^{(a)} < 1$ and $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, the curve $p_k = \bar{p}_k^{(a)}[p_{3-k}]$ lies outside of the belief triangle (see appendix for the proof). Figure 2.7 shows that both curves, $p_1 = \bar{p}_1^{(a)}[p_2]$ and $p_2 = \bar{p}_2^{(a)}[p_1]$, lie outside the belief triangle for Case 1.[34]

When $\frac{\lambda_2 R_2^{(a)}}{c_2} > \frac{\lambda_1 R_1^{(a)}}{c_1}$, Cases 2 and 3 are illustrated in Figures 2.8, 2.9, 2.10 and 2.11. Requirement $\frac{\lambda_2 R_2^{(a)}}{c_2} > \frac{\lambda_1 R_1^{(a)}}{c_1}$ guarantees that point $(R_1^{(a)}, R_2^{(a)})$ lies above the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$, which means that at the point p_2^* the agent switches to source 1.

Recall that in Area 3.2.2, the belief trajectory moves along a straight line that holds $\frac{p_1}{1-p_2}$ constant. This line intersects the curve $p_1 = \bar{p}_1^{(a)}[p_2]$ at two points,

[34]This is not a coincidence. The following statement is true: conditions (2.9) for $k = 1$ and 2 together are equivalent to the requirement that both curves, $p_1 = \bar{p}_1^{(a)}[p_2]$ and $p_2 = \bar{p}_2^{(a)}[p_1]$, lie outside the belief triangle.

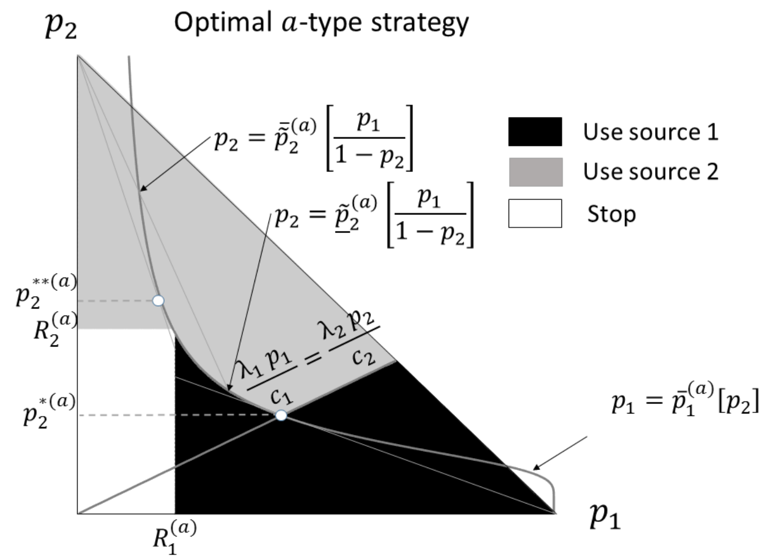

Figure 2.8: Illustration for Theorem 3, Case 2.

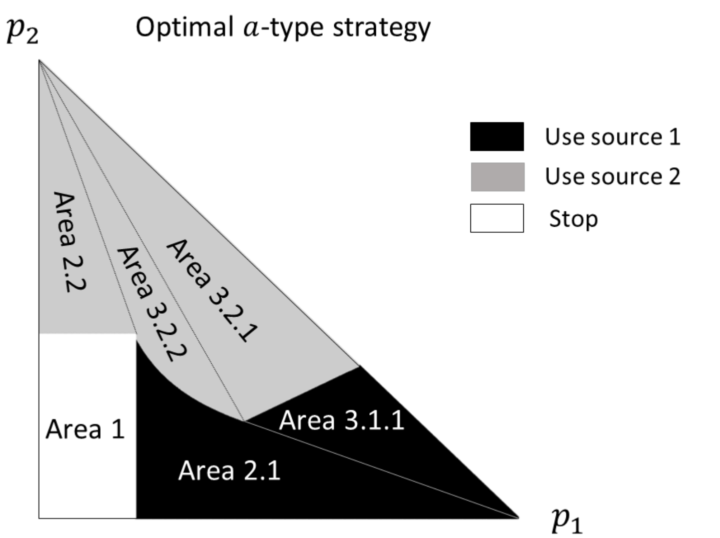

Figure 2.9: Illustration for Theorem 3, Case 2.

$p_2 = \underline{\tilde{p}}_2^{(a)}\left[\frac{p_1}{1-p_2}\right]$ and $p_2 = \bar{\tilde{p}}_2^{(a)}\left[\frac{p_1}{1-p_2}\right]$ (see Figure 2.8). Such points are well-defined only for $\frac{p_1}{1-p_2} \geq \frac{\bar{p}_1^{(a)}\left[p_2^{**(a)}\right]}{1-p_2^{**(a)}}$.

Condition $p_2^{**(a)} > R_2^{(a)}$ means that the curve $p_2 = \bar{\tilde{p}}_2^{(a)}\left[\frac{p_1}{1-p_2}\right]$ lies above the line $p_2 = R_2^{(a)}$ and to the left of the line $p_1 = R_1^{(a)}$. This means that both strategies

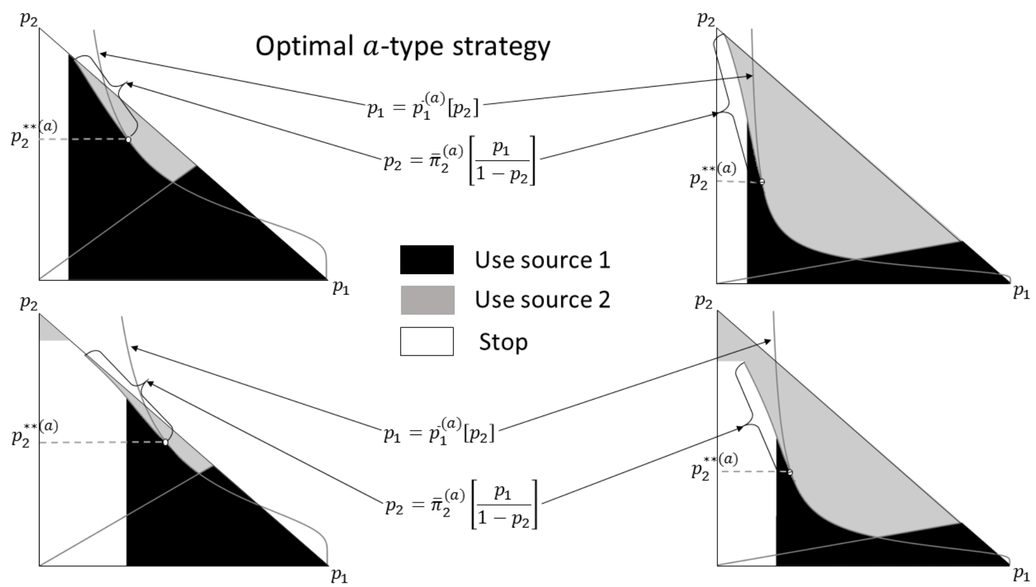

Figure 2.10: Illustration for Theorem 3, Case 3.

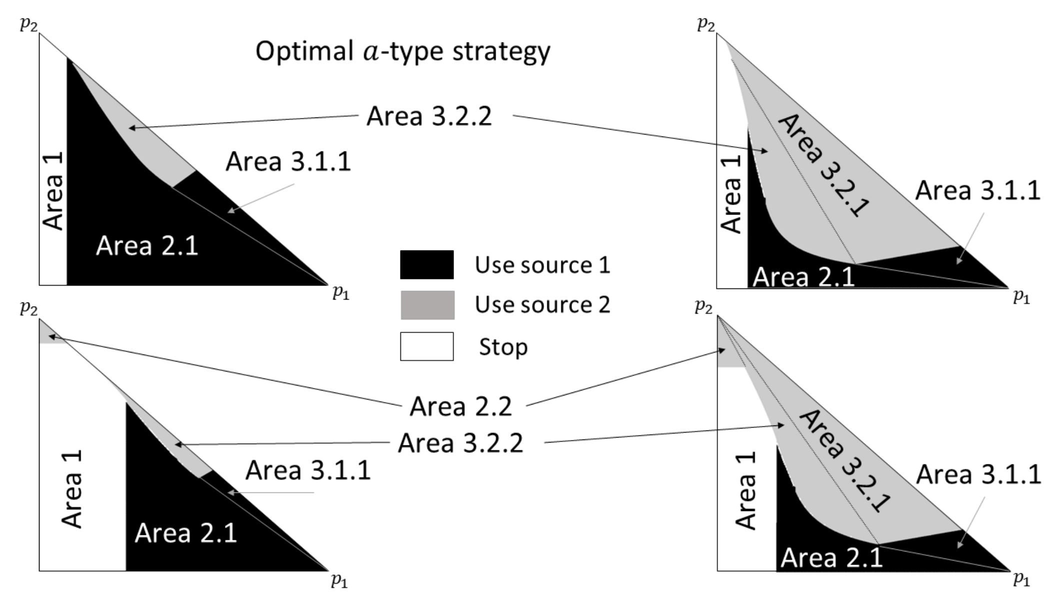

Figure 2.11: Illustration for Theorem 3, Case 3.

— the best a-type strategy conditional on using at most one source throughout the information collection process and the Area 3.2.2 strategy — imply using only source 2 for all points $p_2 \geq \bar{\bar{p}}_2^{(a)}\left[\frac{p_1}{1-p_2}\right]$ with a fixed ratio $\frac{p_1}{1-p_2}$. Thus, there is no point $\bar{\pi}_2^{(a)}\left[\frac{p_1}{1-p_2}\right] \geq \bar{\bar{p}}_2^{(a)}\left[\frac{p_1}{1-p_2}\right]$ where the agent is indifferent between using source 2 and either source 1 or no learning.

When $p_2^{**(a)} < R_2^{(a)}$, the agent is indifferent along the curve $p_2 = \pi_2^{(a)}\left[\frac{p_1}{1-p_2}\right]$ between using source 2 and either using source 1 or stopping. Note that the belief trajectory never crosses this curve: for all initial beliefs at this curve, the belief vector either never moves (no learning) or moves away from it. Thus, in contrast to the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ and two curves described by functions $p_k = \bar{p}_k^{(a)}[p_{3-k}]$, $k = 1, 2$, the curves $p_{3-k} = \bar{\pi}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$, $k = 1, 2$, are not switching curves; that is, the agent never switches sources at these curves since his beliefs never pass through them. However, all five curves, together with $p_k = R_k^{(a)}$, $k = 1, 2$, can be called the indifference curves, since when the current beliefs are located on these curves, the agent is indifferent between at least two out of three actions (use source 1, use source 2, and stop).

2.5.3 Optimal Strategy

Lemma 2 offers a way to construct the optimal strategy from the optimal a_1-type, a_2-type, and a_3-type strategies. For every point (p_1, p_2) in the belief triangle, the optimal action (use source 1, use source 2, or stop) coincides with the optimal action at this point according to the optimal a-type strategy, where $a \in \mathcal{A}$ is such that $V[p_1, p_2] = V^{(a)}[p_1, p_2]$. Thus, to compute the optimal strategy, I need to compare three numbers — $V^{(a_1)}[p_1, p_2]$, $V^{(a_2)}[p_1, p_2]$ and $V^{(a_3)}[p_1, p_2]$, — for every point (p_1, p_2). While this algorithm is computationally easy, it is not very illustrative. The purpose of this section is to provide some intuition about how that optimal strategy looks.

Similar to Section 2.4.1, the optimal strategy can be represented as the best strategy out of a finite number of simple strategies (by “simple,” I mean generally simpler than the optimal a-type strategy). Specifically, there are only five strategies that can potentially be optimal: 1-strategy, 2-strategy, 3.a_1-strategy, 3.a_2-strategy, and 3.a_3-strategy.[35]

For $k = 1, 2$, denote by $V_k[p_1, p_2]$ the expected payoff from the k-strategy, the optimal strategy if only source k is available (see Section 2.4.2). Note that such a strategy is feasible (well-defined) for any initial beliefs.

For $k = 1, 2$, let the 3.a_k-strategy be the strategy with the default alternative a_k and the contingency learning plan described as follows:

[35]If one assumes gradual learning instead of “breakthrough” learning, that is, the Brownian motion instead of the Poisson process, the two-step approach with a-type strategies no longer works. However, I conjecture that this other description of the optimal strategy through a number of simple strategies presented in Section 2.5.3 has an analog in the gradual learning case.

- if
$$\frac{\lambda_k p_k}{c_k} \le \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}} \quad \text{and} \quad \frac{p_k}{1 - p_{3-k}} > \frac{p_k^{*(a_k)}}{1 - \bar{p}_{3-k}^{(a_k)}\left[p_k^{*(a_k)}\right]}, \tag{2.22}$$
then use source $3-k$ until $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$, then use both sources in proportion to their intensity until $p_k = p_k^{*(a_k)}$, then use source $3-k$ until $p_{3-k} = R_{3-k}^{(a_k)}$;

- if
$$\frac{\lambda_k p_k}{c_k} \ge \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}} \quad \text{and} \quad \frac{p_{3-k}}{1 - p_k} > \frac{\bar{p}_{3-k}^{(a_k)}\left[p_k^{*(a_k)}\right]}{1 - p_k^{*(a_k)}}, \tag{2.23}$$
then use source k until $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$, then use both sources in proportion to their intensity until $p_k = p_k^{*(a_k)}$, then use source $3-k$ until $p_{3-k} = R_{3-k}^{(a_k)}$ (see the path $D \to E \to F \to G$ on Figure 2.6);

- if
$$p_k \ge \tilde{\underline{p}}_k^{(a_k)}\left[\frac{p_{3-k}}{1-p_k}\right] \quad \text{and} \quad \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{**(a_k)}\right]}{1 - p_k^{**(a_k)}} \le \frac{p_{3-k}}{1-p_k} \le \frac{\bar{p}_{3-k}^{(a_k)}\left[p_k^{*(a_k)}\right]}{1 - p_k^{*(a_k)}}, \tag{2.24}$$
then use source k until $p_k = \tilde{\underline{p}}_k^{(a_k)}\left[\frac{p_{3-k}}{1-p_k}\right]$, then permanently switch to source $3-k$ and use it until $p_{3-k} = R_{3-k}^{(a_k)}$ (see the path $A \to B \to C$ on Figure 2.6).

Note that this strategy is well-defined if and only if $R_{3-k}^{(a_k)} < 1$, $\Delta_{3-k}^{(a_k)} > 0$, and either (2.22), or (2.23), or (2.24) holds.

Similarly, let the $3.a_3$-strategy be the strategy with the default alternative a_3 and the contingency learning plan described as follows: let $k \in \{1, 2\}$ be such that
$$R_{3-k}^{(a_3)} < 1, \; \frac{\lambda_{3-k} R_{3-k}^{(a_3)}}{c_{3-k}} \le \frac{\lambda_k R_k^{(a_3)}}{c_k}, \; \Delta_{3-k}^{(a_3)} > 0, \tag{2.25}$$
then

- if
$$\frac{\lambda_k p_k}{c_k} \le \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}} \quad \text{and} \quad \frac{p_k}{1 - p_{3-k}} > \frac{p_k^{*(a_3)}}{1 - \bar{p}_{3-k}^{(a_3)}\left[p_k^{*(a_3)}\right]}, \tag{2.26}$$
then use source $3-k$ until $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$, then use both sources in proportion to their intensity until $p_k = p_k^{*(a_3)}$, then use source $3-k$ until $p_{3-k} = R_{3-k}^{(a_3)}$;

- if
$$\frac{\lambda_k p_k}{c_k} \geq \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}} \quad \text{and} \quad \frac{p_{3-k}}{1-p_k} > \frac{\bar{p}^{(a_3)}_{3-k}\left[p^{*(a_3)}_k\right]}{1-p^{*(a_3)}_k}, \tag{2.27}$$
then use source k until $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$, then use both sources in proportion to their intensity until $p_k = p^{*(a_3)}_k$, then use source $3-k$ until $p_{3-k} = R^{(a_3)}_{3-k}$;

- if
$$p_k \geq \underline{\tilde{p}}^{(a_3)}_k\left[\frac{p_{3-k}}{1-p_k}\right] \quad \text{and} \quad \left.\frac{\bar{p}^{(a_3)}_{3-k}\left[p_k\right]}{1-p_k}\right|_{p_k=\min\left\{p^{**(a_3)}_k, R^{(a_3)}_k\right\}} \leq \frac{p_{3-k}}{1-p_k} \leq \frac{\bar{p}^{(a_3)}_{3-k}\left[p^{*(a_3)}_k\right]}{1-p^{*(a_3)}_k}, \tag{2.28}$$
then use source k until $p_k = \underline{\tilde{p}}^{(a_3)}_k\left[\frac{p_{3-k}}{1-p_k}\right]$, then permanently switch to source $3-k$ and use it until $p_{3-k} = R^{(a_3)}_{3-k}$.

Again, this strategy is well-defined if and only if condition (2.25) holds, and either (2.26), or (2.27), or (2.28) holds.

Denote by $V_{3.a}[p_1, p_2]$ the expected payoff from the $3.a$-strategy, $a \in \mathcal{A}$.

Thus, the expected payoff from the optimal strategy is
$$V[p_1, p_2] = \max\left\{V_1[p_1, p_2], V_2[p_1, p_2], \max_{a\in\mathcal{A}} V_{3.a}[p_1, p_2]\right\}.$$
The optimal strategy is the k-strategy if $V_k[p_1, p_2] = V[p_1, p_2]$; the optimal strategy is the $3.a$-strategy if $V_{3.a}[p_1, p_2] = V[p_1, p_2]$.

For $k = 1, 2$, the curve
$$\underline{\mathcal{P}}_k = \left\{p_k = \underline{\tilde{p}}^{(a_k)}_k\left[\frac{p_{3-k}}{1-p_k}\right] : \frac{\bar{p}^{(a)}_{3-k}\left[p^{**(a_k)}_k\right]}{1-p^{**(a_k)}_k} \leq \frac{p_{3-k}}{1-p_k} \leq \frac{\bar{p}^{(a_k)}_{3-k}\left[p^{*(a_k)}_k\right]}{1-p^{*(a_k)}_k}\right\}$$
and the point $p_1 = \frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1}$, $p_2 = \frac{c_2\lambda_1}{c_1\lambda_2+c_2\lambda_1}$ together serve as a generalization of the point $p_1 = \frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1}$, $p_2 = \frac{c_2\lambda_1}{c_1\lambda_2+c_2\lambda_1}$ in Section 2.4.1. Specifically, if there exist initial beliefs (p_1, p_2) such that condition (2.24) holds for them and $3.a_k$-strategy is optimal, this strategy must be optimal at point (p'_1, p'_2) such that $p'_k = \underline{\tilde{p}}^{(a_k)}_k\left[\frac{p'_{3-k}}{1-p'_k}\right]$ and $\frac{p'_{3-k}}{1-p'_k} = \frac{p_{3-k}}{1-p_k}$ (see Lemma 3). Similarly, if there exist initial beliefs (p_1, p_2) such that either (2.22) or (2.23) holds for them, $p_1 + p_2 < 1$, and the $3.a_k$-strategy is optimal, this strategy must be optimal at point (p_1, p_2) such that $p_k = \underline{\tilde{p}}^{(a_k)}_k\left[\frac{\bar{p}^{(a_k)}_{3-k}\left[p^{*(a_k)}_k\right]}{1-p^{*(a_k)}_k}\right] = p^{*(a_k)}_k$ and $p_{3-k} = \bar{p}^{(a_k)}_{3-k}\left[p^{*(a_k)}_k\right] = \frac{c_{3-k}\lambda_k p^{*(a_k)}_k}{c_k\lambda_{3-k}}$.

The curve

$$\underline{\mathcal{P}}_3 = \left\{ p_k = \underline{\tilde{p}}_k^{(a_3)} \left[\frac{p_{3-k}}{1-p_k} \right] : \frac{\bar{p}_{3-k}^{(a_3)}[p_k]}{1-p_k} \Bigg|_{p_k = \min\left\{p_k^{**(a_3)}, R_k^{(a_3)}\right\}} \leq \frac{p_{3-k}}{1-p_k} \leq \frac{\bar{p}_{3-k}^{(a_3)}\left[p_k^{*(a_3)}\right]}{1-p_k^{*(a_3)}} \right\},$$

where $k \in \{1,2\}$ is such that condition (2.25) holds, plays a similar role for the 3.a_3-strategy.

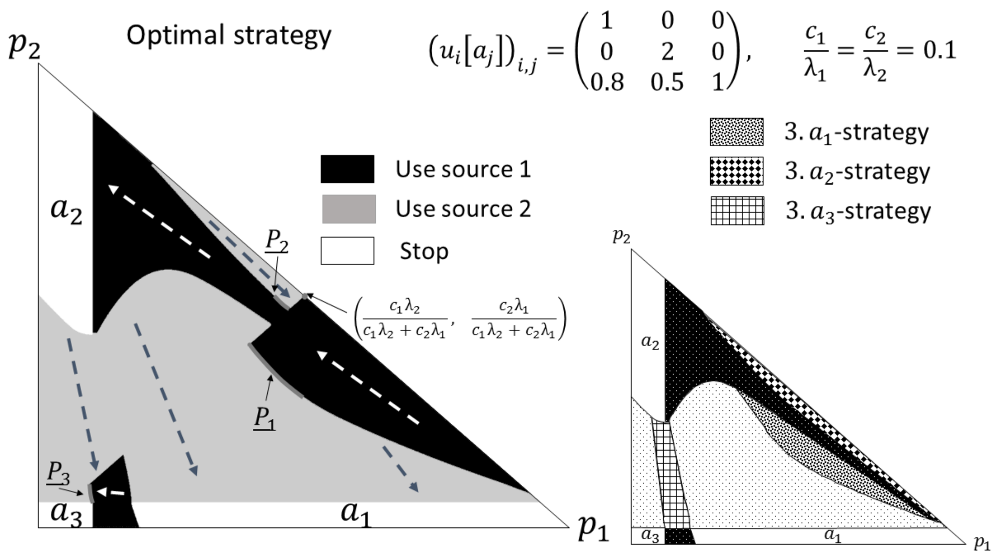

Figure 2.12: The optimal strategy when $u_1[a_1] = u_3[a_3] = 1$, $u_2[a_2] = 2$, $u_3[a_1] = 0.8$, $u_3[a_2] = 0.5$, $u_1[a_2] = u_1[a_3] = u_2[a_1] = u_2[a_3] = 0$, $\frac{c_1}{\lambda_1} = \frac{c_2}{\lambda_2} = 0.1$. On the left, the arrows show the direction the belief vector is moving until the state is revealed. The right figure shows what strategy is optimal given the initial beliefs: 3.a_1-strategy, 3.a_2-strategy, 3.a_3-strategy, use source 1 only, use source 2 only, or no learning.

Figure 2.12 shows the optimal strategy for certain parameters' values.

Another way to look at the optimal strategy is to return to its description in Lemma 2,

$$V[p_1, p_2] = \max_{a \in \mathcal{A}} V^{(a)}[p_1, p_2],$$

and note that the optimal a_1-type, a_2-type, and a_3-type strategies do not depend on $u_3[a_1]$, $u_3[a_2]$, and $u_3[a_3]$, while the expected payoffs $V^{(a_1)}[p_1, p_2]$, $V^{(a_2)}[p_1, p_2]$, and $V^{(a_3)}[p_1, p_2]$ do. Moreover, this dependence is very simple:

$$V^{(a)}[p_1, p_2] = \frac{u_3[a] + f^{(a)}[p_1, p_2]}{1 - p_1 - p_2},$$

where function $f^{(a)}[p_1, p_2]$ does not depend on $u_3[a_1]$, $u_3[a_2]$, and $u_3[a_3]$. Thus, the optimal strategy at beliefs (p_1, p_2), $p_1 + p_2 < 1$, is the optimal a-type strategy, where $a \in \mathcal{A}$ maximizes $u_3[a] + f^{(a)}[p_1, p_2]$. It means that for any beliefs (p_1, p_2), $p_1 + p_2 < 1$, and any other parameters' values except $u_3[a_1]$, $u_3[a_2]$, and $u_3[a_3]$, the optimal default alternative is a if it gives a high enough payoff at state 3, that is, $u_3[a]$ is high enough. For example, Figure 2.12 shows the optimal strategy when $u_3[a_3] > u_3[a_1] > u_3[a_2]$. When the probability of the third state, $1 - p_1 - p_2$, is high, the optimal default alternative is a_3.

2.6 Information Sources as Complements and Substitutes

In this section, I find the conditions under which the information sources are substitutes or complements. If we consider that the cost the agent pays to use an information source is what the information provider gets, c_1 and c_2 serve as prices for different types of information, and the total time the agent uses each source, $T_{\tau,1}$ and $T_{\tau,2}$, is the demand for information of a given type.

Denote by $T_1 = \mathbb{E}\left[T_{\tau,1}\right]$ the expected total time the agent uses source 1 (the expected total attention the agent pays to source 1) under the optimal strategy. Similarly, $T_2 = \mathbb{E}\left[T_{\tau,2}\right]$. For the rest of this section, I treat T_1 and T_2 as functions of c_1 and c_2.

Definition 1 *Two sources are* ***substitutes (complements, independent)*** *if the expected time of using one source increases (decreases, does not change) when the cost of the other source increases. Formally:*

$$\begin{aligned}
\frac{dT_1[c_1,c_2]}{dc_2} > 0, \quad \frac{dT_2[c_1,c_2]}{dc_1} > 0 \quad &\Rightarrow \quad \textit{substitutes},\\
\frac{dT_1[c_1,c_2]}{dc_2} < 0, \quad \frac{dT_2[c_1,c_2]}{dc_1} < 0 \quad &\Rightarrow \quad \textit{complements},\\
\frac{dT_1[c_1,c_2]}{dc_2} = 0, \quad \frac{dT_2[c_1,c_2]}{dc_1} = 0 \quad &\Rightarrow \quad \textit{independent}.
\end{aligned}$$

Note that if source 1 is a substitute for source 2 ($\frac{dT_1[c_1,c_2]}{dc_2} > 0$), then source 2 is a substitute for source 1 ($\frac{dT_2[c_1,c_2]}{dc_1} > 0$). Moreover,[36]

$$\frac{dT_1[c_1,c_2]}{dc_2} = \frac{dT_2[c_1,c_2]}{dc_1}. \tag{2.29}$$

[36](2.29) is akin to the well-known fact that the Slutsky matrix is symmetric. One way to prove (2.29) directly is by using the Envelope Theorem argument (see page 69 in Mas-Colell, Whinston,

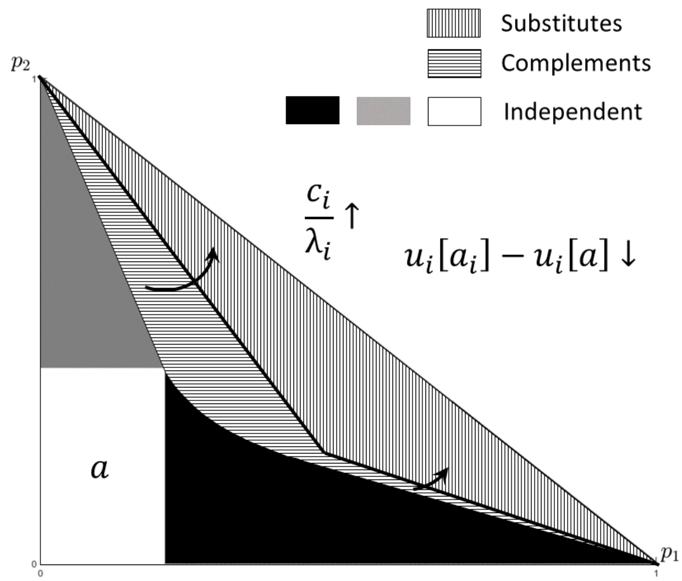

Figure 2.13: Illustration for Theorem 4. Partition of the triangle of initial beliefs into three regions: where the sources are substitutes, where they are complements, and where they are independent (all that is conditional on the agent using the optimal a-type strategy).

If the agent uses either 1-strategy or 2-strategy, the sources are independent since the agent uses at most one source throughout the information collection process and therefore $\frac{dT_1[c_1,c_2]}{dc_2} = \frac{dT_2[c_1,c_2]}{dc_1} = 0$.

Theorem 4 states that if the sources are not independent, they are substitutes when the probability of the third state is low and they are complements otherwise.

Theorem 4 *Suppose the agent uses the* 3.a*-strategy. Let* $k \in \{1,2\}$ *be such that*

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and Green (1995)):

$$V[p_1,p_2] = \sup_{(a^F,T,\tau)} \mathbb{E}\left[u_j[a^F] - c_1T_{\tau,1} - c_2T_{\tau,2}\right].$$

By the Envelope theorem,

$$\frac{\partial V[p_1,p_2]}{\partial c_1} = -T_1[c_1,c_2],$$

and therefore

$$\frac{\partial^2 V[p_1,p_2]}{\partial c_1 \partial c_2} = -\frac{\partial T_1[c_1,c_2]}{\partial c_2}.$$

Similarly,

$$\frac{\partial^2 V[p_1,p_2]}{\partial c_1 \partial c_2} = -\frac{\partial T_2[c_1,c_2]}{\partial c_1}.$$

(2.29) follows.

$\frac{\lambda_k R_k^{(a)}}{c_k} \geq \frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}}$. *Then there exists a threshold* $\hat{q}_{3-k}^{(a)} \in \left[\left. \frac{\bar{p}_{3-k}^{(a)}[p_k]}{1-p_k} \right|_{p_k=\min\left\{p_k^{**(a)}, R_k^{(a)}\right\}}, 1 \right)$ *such that if* [37]

$$\frac{\lambda_k p_k}{c_k} \leq \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}} \quad \text{and} \quad \frac{p_k}{1-p_{3-k}} > \max\left\{ \frac{1}{1+\frac{c_{3-k}\lambda_k}{c_k \lambda_{3-k}} \frac{1-\hat{q}_{3-k}^{(a)}}{\hat{q}_{3-k}^{(a)}}}, \frac{p_k^{*(a)}}{1-\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]} \right\},$$

or

$$\frac{\lambda_k p_k}{c_k} \geq \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}} \quad \text{and} \quad \frac{p_{3-k}}{1-p_k} > \max\left\{ \hat{q}_{3-k}^{(a)}, \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}} \right\},$$

or

$$p_k \geq \underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right] \quad \text{and} \quad \hat{q}_{3-k}^{(a)} \leq \frac{p_{3-k}}{1-p_k} \leq \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}},$$

then the sources are substitutes, and they are complements otherwise. Moreover, $\hat{q}_{3-k}^{(a)}$ *is a function of* $\frac{c_1}{\lambda_1}$, $\frac{c_2}{\lambda_2}$, $u_1[a_1]-u_1[a]$ *and* $u_2[a_2]-u_2[a]$ *(and it does not depend on anything else); it is nondecreasing in* $\frac{c_1}{\lambda_1}$ *and in* $\frac{c_2}{\lambda_2}$*, and it is nonincreasing in* $u_1[a_1]-u_1[a]$ *and in* $u_2[a_2]-u_2[a]$.

Figure 2.13 illustrates Theorem 4 on the belief triangle. For a fixed default alternative, the belief triangle represents the optimal a-type strategy as partitioned into two regions: where the agent uses 3.a-strategy and the rest, where he uses at most one source. In the former region, the sources are substitutes on one side of the threshold lines and they are complements on the other side. In the latter region, the sources are independent.

Theorem 4 also gives the comparative statics of the threshold $\hat{q}_{3-k}^{(a)}$. As cost of information increases, the sources are more likely to become complements. The reverse is true for the benefit from information. Intuitively, when information becomes more costly, on average, the agent stops the learning process sooner. When the planning horizon is short, but the agent finds it optimal to use both sources, it means they give him sufficiently different information. In other words, they complement each other.

[37] $\hat{q}_k^{(a)} = \frac{1}{1+\frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\frac{1-\hat{q}_{3-k}^{(a)}}{\hat{q}_{3-k}^{(a)}}}$ solves the system of equations with respect to $(\hat{q}_k^{(a)}, p_k, p_{3-k})$: $\frac{p_{3-k}}{1-p_k} = \hat{q}_{3-k}^{(a)}$, $\frac{p_k}{1-p_{3-k}} = \hat{q}_k^{(a)}$, $\frac{\lambda_k p_k}{c_k} = \frac{\lambda_{3-k}p_{3-k}}{c_{3-k}}$.

2.7 Conclusion

I proposed a tractable model for dynamic information collection from multiple sources. I fully characterized the unique optimal strategy for the agent who at each instant in time can choose between two information sources modeled with Poisson processes. This strategy consists of at most two phases. In the first phase, the agent chooses the informatively superior source, while in the second phase, he always pays attention to the same source until the end of the information collection process. I demonstrated how this characterization of the optimal strategy helps to categorize the information sources as substitutes and complements.

There are many potential applications of my model that suggest potential directions for future research.

The inclusion of many decision makers in the model allows the study of the *free-riding problem in information collection*. It is well-known that when there is only one way to collect information (one information source), we observe free-riding on the amount of information collected in equilibrium. Bolton and Harris (1999) and Keller, Rady, and Cripps (2005) showed that this is the case in the bandit setting when agents can observe each other's actions. Mukhopadhaya (2003) demonstrated the same phenomenon for information aggregation in the committee. My model provides the basis to study the free-riding problem not only on the amount of information but also on the type of information being collected.

This paper also contributes to the literature of *optimal delegation* in principal-agent problems. Szalay (2005) considers only one information source and shows that elimination of an intermediate alternative improves the agent's incentives for information collection. My model allows the study of not only the elimination of an alternative, but also how the number of information sources changes the quality of the final choice from the principal's perspective. Guo (2016) considers a one-armed bandit learning environment and allows the principal to control the learning process dynamically. She shows that the optimal delegation rule is a cutoff rule: once the principal's current belief reaches the cutoff, the principal requires the agent to stop the experimentation. My model potentially permits extension of the analysis to a setting where the agent can choose between several information sources and therefore the principal has another dimension of control besides the stopping time — the control of the source.

The interpretation of the information sources as media firms relates my paper to the *media market* literature (Gentzkow and Shapiro (2006); Mullainathan and Shleifer

(2005); Anderson and Coate (2005); Godes, Ofek, and Sarvary (2009); Perego and Yuksel (2015)). One important question this literature asks is how the competitive structure of the media market affects social welfare.

Another interpretation of the information sources, as experts, relates my model to the literature on *persuasion* (Gentzkow and Kamenica (2011); Gentzkow and Kamenica (2014); Li and Norman (2015)). This literature suggests studying incentives of information providers who have preferences over the alternatives among which the agent has to choose.

Chapter 3

ARE PEOPLE SUBJECT TO PERSUASION BIAS? TEST OF DEGROOT MODEL

3.1 Introduction

Communication and learning are essential elements to opinion formation. Studying communication leads to studying social networks through which communication occurs. In this paper we perceive communication as transmission of opinion about the state of the world from one person in a directed network to another along a directed edge. The evolution of a person's opinion over time constitutes learning.

The primary object of this paper is the learning process, namely how people change their opinions over time as they are keeping sharing their opinions with each other. We focus on two alternative models of opinion formation and test them against each other in an experiment with a simple directed circle network. The first model formalized a so called "persuasion bias" effect in the process of opinion formation. The second model is a generalization of a classical Bayesian updating model that describes people's behavior that leads to the Bayesian Nash equilibrium. The contribution of this paper is threefold. First, we introduce the persuasion bias model and its alternative model. Second, we conduct the experiment and show that the first model fits our data better. Finally, we use our data to get some insight about why the first model is better. Specifically, we show that the persuasion bias effect has underlying rational reasoning.

Section 3.3 presents a theoretical foundation for this paper. Communication is restricted to occur according to a circle directed graph which represents social network. An edge from one individual to another indicates the direction of information flow. Each individual has exactly one neighbor to whom he listens. Thus, information flows in the same direction along the circle. The nature and timing of information is as follows. At the beginning, everybody receives some private information about the true state of the world. This information comes in the form of a normally distributed signal. Then in each stage, everybody reports their beliefs by guessing the true state. The guess of individual i in a previous stage is visible to individual j if and only if there is an edge from i to j. So, every agent forms his beliefs about the true state based on his private signal and the history of his neighbor' guesses.

We consider two models of how people change their beliefs about some state of the world as they communicate. The persuasion bias hypothesis (PBH) model says that an agent's new guess is a weighted sum of his own guess in the previous stage and the guess of his neighbor in the previous stage. The Bayesian update hypothesis (BUH) model says that each new guess of agent i must be a weighted sum of agent i's private signal and his neighbor's guess from the previous stage. So, the PBH model weights the agent's previous stage guess, while the BUH model takes the agent's private signal. The BUH model generalizes the Bayesian Nash equilibrium prediction for this game. The equilibrium prediction for stage t is to place weight $1/t$ to agent i's private signal and weight $(t-1)/t$ to agent i's neighbor's guess. To match the number of parameters in both models, we generalize the ration model prediction by allowing non-zero weights to be estimated.

We assume that signals are distributed normally to exploit the following property of the normal distribution. A normally distributed prior about the state, combined with a normally distributed signal, drawn independently (conditional on the state) from all previous information, leads to the optimal guess about the state equal to the weighted sum of the mean of the prior and the signal value. However, if the signal is correlated (conditional on the state) with the prior, this rule does not hold in general. So, if the agent takes a weighted sum of two normally distributed signals, we assume that he treats them as independent. All signals and therefore posteriors after Bayesian updating are normally distributed in our setup. The BUH model says that the agent treats his own private signal and his neighbor's last guess as independent, which is objectively true. The PBH model says that the agent treats his own last guess and his neighbor's last guess as independent, implying that he ignores the correlation between the two.

The term "persuasion bias" was introduced by DeMarzo, Vayanos, and Zwiebel (2003) defining it as a phenomenon according to which people "fail to account for possible repetition in the information they receive." For example, if a person talks with the same individual two days in a row, today she "forgets" that she already talked to him yesterday and therefore treats whatever she hears as new information, independent of yesterday's information. Therefore, anyone who is subject to persuasion bias, can be "persuaded" by repetition of the same information. The PBH model captures the persuasion bias phenomenon by modeling the neglection of correlation among different pieces of incoming information.

Section 3.4 describes the experiment design. We consider a network with 4 people, making 5 subsequent guesses (5 stages of the game) about the true state. We ran 6 sessions, 20 games in each session, getting data from 24 subjects in total.

In Section 3.7, we test the models by pooling all subjects together and show that the persuasion bias model describes a "representative agent" behavior better. Testing the models for each subject separately in Section 3.8, we found that the persuasion bias model describes the behavior of a larger number of subjects.

In Section 3.9 we make an attempt to understand why the boundedly rational model performs better in a very simple learning environment. First, we study the correlation between how well a subject fits the persuasion bias model and his performance in the experiment. As a measure of how well a subject fits a model we take the log-likelihood function evaluated at the estimated parameters' values. We found that this correlation is positive for those subjects who fits the persuasion bias model better than the generalized rational one. This observation is surprising given that the game has a unique equilibrium and this equilibrium is a special case of the generalized rational model but not the persuasion bias model. To understand this phenomenon, we study correlation between how well a subject actually fits the persuasion bias model and how well a rational subject theoretically should fit that model, given the behavior of all other subjects fixed. We find that for those subjects whose rational alter-ego fits the persuasion bias model better, this correlation is positive. That means that when the persuasion bias model gives a higher profit, people tend to detect it and make their strategy closer to that model.

Summarizing, we come to the conclusion that the superior performance of the persuasion bias model has a more deep explanation than just appealing to the simplicity of that model due to its ad hoc nature. The persuasion bias behavior might give a higher profit than the equilibrium strategy because it takes into account other subjects' deviations from the equilibrium. This makes the persuasion bias model a competitive alternative to the quantal response equilibrium (McKelvey and Palfrey (1995)) or the cognitive hierarchical model (Camerer, Ho, and Chong (2004)) for network games.

3.2 Literature Review

The topic of opinion formation in networks receives a lot of attention in literature (see Acemoglu and Ozdaglar (2011) and Jackson and Yariv (2011) for review). Roughly speaking, there are two approaches to model opinion formation. The first approach is to assume that agents use the rational updating rule predicted by

Bayesian Nash equilibrium (see, for example, Gale and Kariv (2003) and Acemoglu, Dahleh, Lobel, and Ozdaglar (2011)). The other approach is to assume that agents use some ad hoc updating rule, which is computationally much simpler than the rational updating rule (for example, DeGroot (1974)). Our paper belongs to the literature that compares ad hoc updating rules with the rational one.

The persuasion bias phenomenon is a special case of *correlation neglect*, applied to a network setting. Enke and Zimmermann (2013) experimentally study correlation neglect in an abstract setting and found support for this phenomenon. In contrast to our experimental design, there are no actual interactions among subjects, all signals and guesses are generated by a computer with known algorithm. Thus, correlation neglect is unambiguously suboptimal behavior. Our findings go in line with it, as the success of the PBH model can only be partially explained by the rational reasoning.

DeMarzo, Vayanos, and Zwiebel (2003) were the first formally introducing the persuasion bias phenomenon and they model this effect using the DeGroot model of belief formation (DeGroot (1974)). In contrast to the PBH model we are proposing in this paper, the DeGroot model restricts the weights the agent puts on his own guess and his neighbor's guess to be time-invariant, that is to be the same in each stage.[1] Therefore, the PBH model can be called the generalized DeGroot model. This generalization gives more flexibility in fitting the model while keeping the essence of the persuasion bias phenomenon. Indeed, the PBH model allows the precision of incoming information to be different, while keeping the independence assumption.

We are not the first to discover the good properties of the DeGroot model. Golub and Jackson (2010) study theoretically the convergence of beliefs in the DeGroot model. In particular, they show that in the setup we consider in this paper, the DeGroot model predicts the convergence of guesses. In general, Golub and Jackson (2010) provide an argument why it might be not a bad idea to use a naive updating rule formalized by the DeGroot model by giving conditions when in the time limit such rule leads to a consensus belief that is arbitrarily close to the true state for large networks.

The closest papers to ours are Corazzini, Pavesi, Petrovich, and Stanca (2012),

[1] More precisely, they include time-variant distortion weight, but they restrict this weight to be the same for all individuals in a network. For example, at stage t, agent i places weight $(1-\lambda_t)\alpha_i$ to his own previous stage guess and $\lambda_t\beta_i$ to his neighbor's guess, while agent j places weight $(1-\lambda_t)\alpha_j$ to her own previous stage guess and $\lambda_t\beta_j$ to her neighbor's guess. Moreover, they restrict the non-distorted weights to be positive and their sum to be 1 (for example, $\alpha_i + \beta_i = 1$).

Battiston and Stanca (2014) and Brandts, Giritligil, and Weber (2015). Corazzini, Pavesi, Petrovich, and Stanca (2012) test the DeGroot model with uniform weights against the equilibrium prediction by comparing the limiting beliefs between two networks. The first network coincides with ours. The second network adds a couple of directed links making it asymmetric across agents. All signals have the same conditional distribution, so the equilibrium prediction for the consensus belief is the average of all signals in either network. In contrast, the DeGroot model with uniform weights predicts a non-uniform weighted sum of signals for the asymmetric network, with higher weights given to the agents with most outgoing links. What they see in experimental data speaks against both models' predictions: the weights are not uniform, with higher weights given to the agents with most incoming links. This observation is supported by the DeGroot model with non-uniform weights, where agent i weights agent j's guess based on agent j's indegree. This idea is studied further in Battiston and Stanca (2014), who in particular also found experimental evidence of an indegree effect. However, in contrast to these results, Brandts, Giritligil, and Weber (2015)'s experimental data show that the higher weights are given to the agents with most outgoing links, which supports the DeGroot model with uniform weights.

In sum, all three papers, Corazzini, Pavesi, Petrovich, and Stanca (2012), Battiston and Stanca (2014) and Brandts, Giritligil, and Weber (2015), test different versions of the DeGroot model, and each version is characterized by its own weight matrix. Note that equilibrium prediction for the asymmetric network can be generated by some version of the DeGroot model, as shown in Battiston and Stanca (2014, Theorem 1) (the weights would be network-dependent). In contrast, we compare two classes of models. The persuasion bias model class incorporates all different versions of the DeGroot model. The Bayesian update model class includes the equilibrium model, but as we mentioned before, the consensus guess prediction of the equilibrium model is the same as a certain version of the DeGroot model.

3.3 Model

3.3.1 Setup

Consider a finite set of agents $\mathcal{N} = \{1, \ldots, n\}$ that are connected in a directed circle graph, so that agent 1 listens to agent n, agent 2 listens to agent 1, agent 3 listens to agent 2, etc. For each agent $i \in \mathcal{N}$, call agent

$$\mathcal{N}(i) = \begin{cases} i-1, & i > 1 \\ n, & i = 1 \end{cases}$$

his or her neighbor.

Suppose all agents want to estimate an unknown parameter θ (which we call "state of the world"). Suppose the prior distribution of θ is uniform over the real line (and it is common knowledge among all agents). At the beginning of the game each agent i receives a signal s_i about the state that is distributed according to a normal distribution with state-dependent mean θ and state-independent variance σ^2:

$$s_i|\theta \sim \mathcal{N}\left(\theta, \sigma^2\right).$$

All signals are independent conditional on the state. The game proceeds in several stages. In the first stage, agents observe their own signals and make their guesses about the state θ. In the second stage, agents observe the guesses of their neighbors made in the first stage and make a second guess. In the third stage, agents can see their neighbors' guesses from the second stage and guess again, etc. There are T stages in total.

Denote by x_i^t the guess of agent i made in stage t. Then the information set of agent i at stage t is $\mathcal{I}_i^t = \left\{s_i, \{x_{\mathcal{N}(i)}^\tau\}_{\tau=1}^{t-1}\right\}$. To simplify notations, denote $y_i^t = x_{\mathcal{N}(i)}^t$ agent i's neighbor's guess in stage t. Note that the network structure, agents' identities, and the precision of all signals are common knowledge.

The payoff agent i gets is

$$-\sum_{t=1}^{T} \left|x_i^t - \theta\right|.$$

This payoff is maximized at $x_i^t = \theta$. However, the parameter θ is generally unknown. So, the question is, How do the agents choose their guesses?

3.3.2 Rational Benchmark

In this section we assume that all agents are rational, meaning they want to maximize their expected payoff.[2]

First of all, note that a rational agent wants to maximize her expected payoff in *each* stage. This means we can safely ignore any complex strategic considerations

[2] Results are robust to some specification of risk-aversion, for example, if the agents use CARA utility.

she might have.[3] Indeed, agent i's own guesses stay independent from incoming information up until stage $t = n$. In stage $t = n$, agent i infers all signals (as we will show below) and therefore she gets the maximum possible payoff from every stage $t \geq n$.

Formally, a rational agent i chooses x_i^t that minimizes the expected value of $\left|x_i^t - \theta\right|$ given her current beliefs. This means a rational agent i must guess x_i^t equal to the median of the probability distribution of θ that corresponds to agent i's beliefs in stage t:

$$x_i^t = \text{Median}\left(\theta|\mathcal{I}_i^t\right).$$

In the first stage, the updated belief is a normal distribution with mean s_i and variance σ^2. Thus, a rational agent must guess $x_i^1 = s_i$.

In the second stage, agent i knows the guess of her neighbor in the first stage. Assuming her neighbor is rational, agent i infers her neighbor's signal. So, in the second stage the updated belief is a normal distribution with mean $\frac{1}{2}\left(s_i + y_i^1\right)$ and variance $\frac{\sigma^2}{2}$. Thus, agent i must guess $x_i^2 = \frac{1}{2}\left(s_i + y_i^1\right)$.

In the third stage, agent i knows the guess of her neighbor in the second stage. Assuming the neighbor of her neighbor is rational and assuming her neighbor believes that, agent i can infer her neighbor's neighbor's signal.[4] So, in the third stage the updated belief is a normal distribution with mean equal to the average of all inferred signals.

In sum, if all agents are rational and this is common knowledge, then in every stage every agent must guess an average of all signals available to him by that time. Formally:

Theorem 5 *Bayesian Nash equilibrium of this game is unique and leads to the following prediction:*

$$\begin{aligned}
x_i^1 &= s_i, \\
x_i^t &= \frac{t-1}{t} y_i^{t-1} + \frac{1}{t} s_i, \quad t = 2, \ldots, n \\
x_i^t &= x_i^n, \quad t = n+1, \ldots
\end{aligned}$$

[3] An example of a complex strategic consideration can be an attempt to sacrifice the current stage benefit in order to influence others' beliefs, which might lead to a higher payoff in the future stages.

[4] Note that this statement does not hold for *any* network. For example, if agent i listens to two agents who listen to some common source, then agent i might not be able to infer a signal from that common source, provided that both agents also have access to some private sources.

This prediction holds for any number of stages T.

3.3.3 Generalized DeGroot Model

The DeGroot model postulates the existence of a matrix $W \in \mathbb{R}^{n\times n}$ such that for all $t \geq 1$

$$x^{t+1} = Wx^t, \quad W_{ij} = 0 \text{ if } j \neq (\mathcal{N}(i) \cup \{i\}),$$

where x^t is a vector of all agents' guesses in stage t. Equivalently, for all $t \geq 1$ and all $i \in \mathcal{N}$,

$$x_i^{t+1} = w_{ii}x_i^t + w_{i,\mathcal{N}(i)}y_i^t.$$

We generalize this model by allowing weights to vary over time. The generalized DeGroot model postulates the existence of a sequence of matrices $\{W^t\}_{t=1}^{T-1}$, $W^t \in \mathbb{R}^{n\times n}$, such that for all $t \geq 1$

$$x^{t+1} = W^t x^t, \quad W_{ij}^t = 0 \text{ if } j \neq (\mathcal{N}(i) \cup \{i\}). \tag{3.1}$$

Equivalently, for all $t \geq 1$ and all $i \in \mathcal{N}$,

$$x_i^{t+1} = w_{ii}^t x_i^t + w_{i,\mathcal{N}(i)}^t y_i^t.$$

3.3.4 Estimation Strategy

The goal of the experiment we conducted is to test the Persuasion Bias Hypothesis. This hypothesis is formalized by the generalized DeGroot model.

An alternative hypothesis we chose satisfies three properties. First, it falls into the same class of linear models by requiring x_i^{t+1} be a linear function of $s_i, y_i^1, \ldots, y_i^t$. Second, it has the same number of parameters (weights). Finally, it embeds the rational benchmark from Theorem 5.

We call this alternative hypothesis the Bayesian Update Hypothesis, or the generalized rational model. In contrast to Theorem 5, it allows the weights in front of y_i^{t-1} and s_i to be arbitrary.

We start our analysis by making two assumptions about the updating process.

Assumption 1 *Agents' guesses at first stage coincide with their signals:*

$$x_i^1 = s_i. \tag{3.2}$$

The first assumption is that agents update rationally in the first stage by choosing their best guesses equal to their signals. In experiments, any deviation from this assumption tells us that subjects have trouble with processing information with fixed properties. If a subject fails to update rationally in the first stage, we have an identification problem. Specifically, we cannot distinguish whether deviations from a prediction come from failed updating process or they come from wrong perception of information properties. We assume that subjects update rationally, that is, according to the Bayes rule. Thus, any deviations from rational behavior are attributed to subjects' perception of information properties. [5]

Assumption 2 *There exist* $\{\mu_i^{\tau,t} : i \in \mathcal{N},\ \ \tau = 1, \ldots, t-1,\ \ t = 2, 3, \ldots, T\}$ *and* $\{\lambda_i^t : i \in \mathcal{N},\ t = 2, 3, \ldots, T\}$ *such that for every* $i \in \mathcal{N}$ *and every* $2 \le t \le T$*:*

$$x_i^t = \lambda_i^t s_i + \sum_{\tau=1}^{t-1} \mu_i^{\tau,t} \cdot y_i^\tau. \tag{3.3}$$

Compared to the general form of updating function

$$x_i^t = f_i^t(I_i^t),$$

Assumption 2 requires linearity. In particular, it requires that each subject's weights placed on her neighbor's guesses as well as her signal are independent of the realization of her signal and her neighbor's guesses. This assumption holds for both hypotheses we test.

We estimate the model (3.2)-(3.3) and test two mutually exclusive hypotheses, the Persuasion Bias Hypothesis and the Bayesian Update Hypothesis.

Persuasion Bias Hypothesis: *For any* i*,* t_1 *and* t_2*, there exists a constant* $c_i^{t_1,t_2}$ *such that*

$$\lambda_i^{t_1} = c_i^{t_1,t_2} \lambda_i^{t_2}, \quad \mu_i^{\tau,t_1} = c_i^{t_1,t_2} \mu_i^{\tau,t_2}, \quad \forall \tau < \min\{t_1, t_2\}. \tag{3.4}$$

Theorem 6 says that the Persuasion Bias Hypothesis is equivalent to the generalized DeGroot model (3.1).

[5] We even include this question in a training test before an experiment (see Appendix B.4). This makes it essentially common knowledge that the first guess is equal to a private signal and therefore it gives an advantage to the rational model. The rejection of the rational model in this setting makes our results more robust.

Theorem 6 *Under Assumptions 1 and 2, the Persuasion Bias Hypothesis is equivalent to the existence of a sequence of matrices $\{W^t\}_{t=1}^{T-1}$, $W^t \in \mathbb{R}^{n\times n}$, such that (3.1) holds for every $t \geq 1$.*

The Persuasion Bias Hypothesis reflects the original idea formulated by DeMarzo, Vayanos, and Zwiebel (2003): agents "fail to adjust for possible repetitions of information they receive, instead treating all information as new." Indeed, if an agent is subject to persuasion bias, she will treat her signal and other agents' beliefs available to her as a set of independent signals about the unknown parameter. Therefore, as she moves from round t to round $t+1$, she will not reconsider the relative weights she placed on the signals available to her in round t. In other words, in round $t+1$, she will treat her belief in round t as a sufficient statistic for all information available to her in round t and she will update her belief by taking a weighted average of her old belief and new information. This gives us model (3.1).

More formally, note that if one faces a series of independent signals drawn from (possibly different) distributions with mean θ, the expected value or the mode of θ given these signals (the best guess of θ) is not necessarily a linear combination of the signals. For example, the expression for condition mean:

$$E(\theta|s_1, ..., s_n) = \sum_{\theta} \theta P(s_1|\theta)...P(s_n|\theta)P(\theta)$$

does not imply any linearity, unless additional assumptions are added. By property (B.1) of normal distribution, we conclude that if the agent has normal prior about the state, his signal is distributed normally and he treats all guesses of his neighbors as independent normally distributed signals, his updating rule would follow DeGroot's model.

An equivalent way to write the Persuasion Bias Hypothesis is

$$\frac{\lambda_i^t}{\lambda_i^{t-1}} = \frac{\mu_i^{\tau,t}}{\mu_i^{\tau,t-1}}, \quad 1 \leq \tau \leq t-2, \quad 3 \leq t \leq T, \quad i \in \mathcal{N}. \tag{3.5}$$

From (3.5), it is easy to see that the number of restrictions the Persuasion Bias Hypothesis places on the general model (3.2)-(3.3) is $n \cdot \sum_{t=3}^{T}(t-2) = \frac{n(T-1)(T-2)}{2}$, or $\frac{(T-1)(T-2)}{2}$ restrictions for each agent.

Bayesian Update Hypothesis:

$$\mu_i^{\tau,t} = 0, \quad 1 \leq \tau \leq t-2, \quad 3 \leq t \leq T, \quad i \in \mathcal{N}. \tag{3.6}$$

Comparing (3.5) and (3.6), it is easy to see that the number of restrictions is the same in both hypotheses. Moreover, comparing (3.6) with Theorem 5, we conclude that the Bayesian Update Hypothesis embeds the Bayesian Nash equilibrium prediction.[6]

3.4 Experiment

Our experiment tests learning in a network of four people, connected in a circle (see Figure 3.1). We ran 6 experimental sessions. Each session consists of 20 periods.

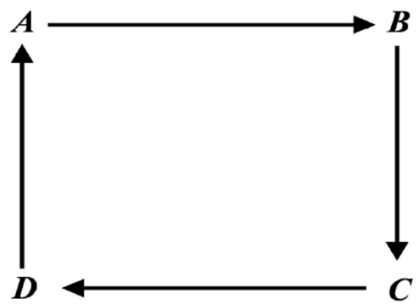

Figure 3.1: The network structure applied in the experiment

In each session, the number of participants is 4. At the beginning of each session, all participants are assigned a position in the network (A, B, C, or D). Positions do not change throughout the whole session.

There are 5 stages in each period of the game. At the beginning of each period, a new state is drawn, signals are generated, and the subjects privately observe their signals. In each stage, the subjects make their guesses about the state. Starting from the second stage, they observe the guesses of their neighbors from previous states. At the end of each period, each subject can see all four signals, the true state, his payment for this period and his total payment for all periods so far. The exact instructions are in Appendix B.3.

The state θ is drawn from a uniform distribution over the integers from 0 to 1000. Each signal is drawn from the discretized approximation of the normal distribution with mean θ. Specifically, each signal is an integer from 0 to 1000, drawn from

[6]Note that starting from stage n, all guesses are the same and equal to the average of all n signals. This fact not only allows us to generalize the rational model with (3.6), but also raises the question of identification of model (3.2)-(3.3). To address the identification concern, we limit the number of periods to $T = n + 1$. In experiment, we rarely observe a convergence by the last period, which allows us to estimate all parameters in model (3.2)-(3.3).

discrete distribution with mean θ. This distribution has a bell-shape form with variance $\frac{616}{53}$ (see Appendix B.3).[7]

To make sure our subjects understand the experiment setup, we asked them to answer some questions about the experiment before the experiment started and after the instructions were read (see Appendix B.4). Before the experiment started, all answers were checked and explanations were provided where needed.

One session was run in the EEPS Lab in California Institute of Technology (subjects were undergraduate students of various majors). The other five sessions were run in the Laboratory for Experimental and Behavioral Economics at the High School of Economics in Moscow (subjects were people of different backgrounds, not students).

3.5 Data Statistic

The average payoff from the whole game was 794 points from all 20 periods, with the minimal profit 713 points (subject D in session 1) and maximal profit was 834 points (subject 2 in session 4). The maximal feasible profit was 1000. The average profit in the first session was 773 points and the average profit in the other five sessions was 799 points. This means there is no significant difference in performance between undergraduate students in CalTech and subjects from Moscow.

Figure 3.2 shows how performance changes over time. We use this figure to decide how many periods we should devote to the learning process. Ideally, we would like to use all 20 periods to increase statistical power. However, Figure 3.2 shows that the per stage profit distribution becomes relatively stable only after period 5, which implies a 5-period learning time.

Figure 3.3 shows how guesses converge over time. In agreement with previous literature, we see a clear pattern suggesting a monotonic convergence. However, the convergence is rarely full, and we see differences in guesses even in the last stage. Moreover, we do not see much difference between stages 4 and 5. Since studying convergence per se is not our goal, we cut the game after stage 5.

Figure 3.4 provides guess distributions separately for periods 1-5, 6-10, 11-15 and 16-20. Comparing periods 1-5 with 6-10, we see a learning pattern. For the first five

[7] We do not use normal distribution directly in the experiment because we want to avoid any confusion subjects might have because of it. Our concern is that not all people are familiar with normal distribution and they would unlikely be able to fully understand it during the experiment. Discretization makes the setup easy to explain and understand. Moreover, the equilibrium in the discrete setup is *essentially* the same as in Theorem 5: if everybody behaves optimally, then fitting the linear model (3.3) gives the same weights as in Theorem 5 (see Appendix B.6).

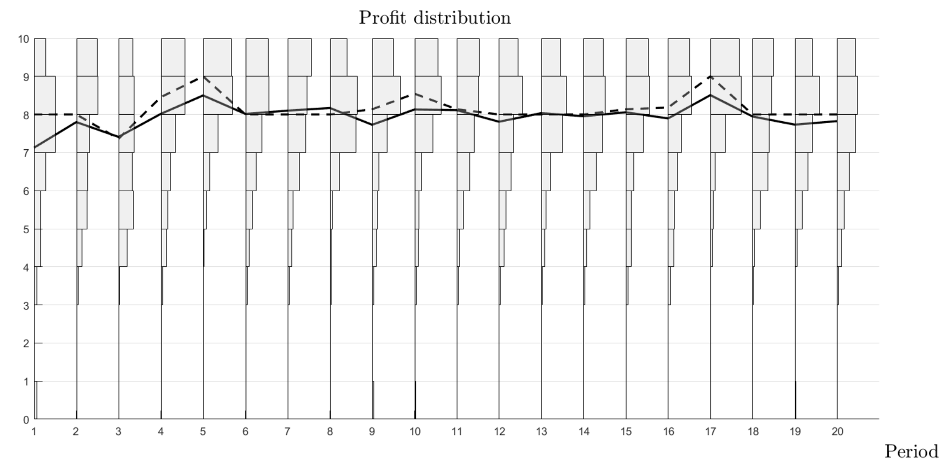

Figure 3.2: Profit distribution for each period (each distribution comes from 120 data points = 24 subjects × 5 stages). A solid line connects the means, a dashed line connects the medians.

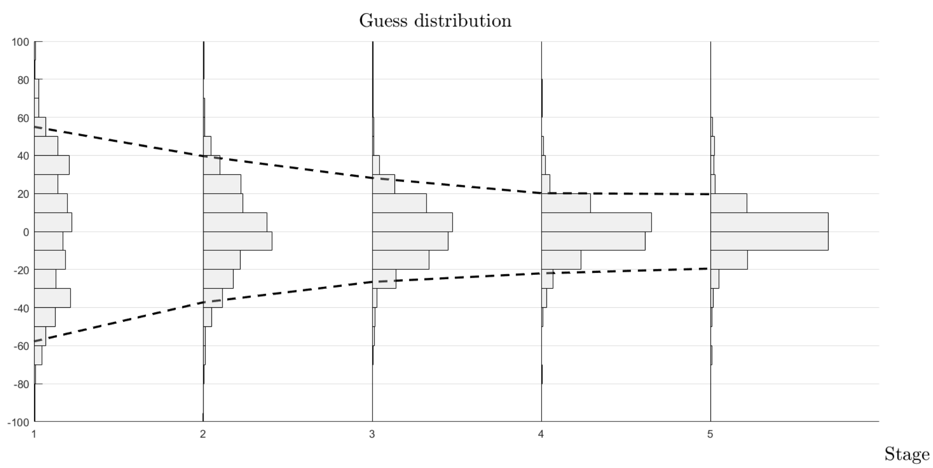

Figure 3.3: The distributions of guess deviations $x_i^t - \frac{1}{4}\sum_{i=A,B,C,D} x_i^t$ for each stage $t = 1, 2, 3, 4, 5$ (each distribution comes from 480 data points = 24 subjects × 20 periods). Dashed lines connect the 5% and 95% quantiles across distributions.

periods, the distributions have larger variance and the convergence is not as clean as for the next five periods.

3.6 Test of Assumptions 1 and 2

In this section, we check whether:

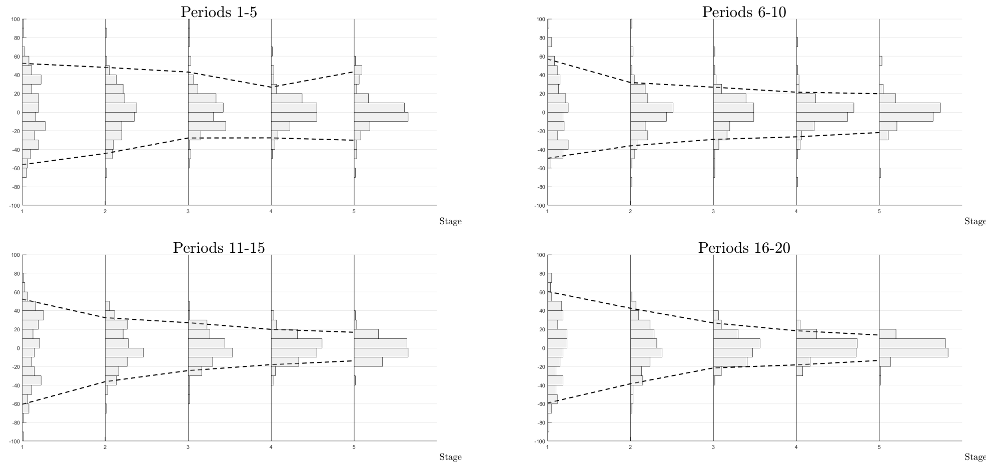

Figure 3.4: The distributions of guess deviations $x_i^t - \frac{1}{4} \sum_{i=A,B,C,D} x_i^t$ for each stage $t = 1, 2, 3, 4, 5$ (each distribution comes from 120 data points = 24 subjects $\times$ 5 periods). Dashed lines connect the 5% and 95% quantiles across distributions.

1. agents' guesses at the first stage coincide with their signals (Assumption 1),

2. the weights in (3.3) are independent of signals realization (Assumption 2).

Only 4.6% of observations (22 out of 480=6 sessions $\times$ 20 periods $\times$ 4 subjects) do not satisfy Assumption 1. Moreover, it is obvious that some of them are mistakes (for example, the first guess is 136 when the signal was 459).

To test Assumption 2, we use various fluctuation tests, which are basically more sophisticated versions of the Chow test for a structural change, with the null hypothesis being that a subject does not change his or her behavior over time (for details, see Zeileis, Leisch, Hornik, and Kleiber (2002)). Specifically, we use Rec-CUSUM, OLS-CUSUM, Rec-MOSUM, and OLS-MOSUM tests for each subject separately (24 subjects in total).[8] We test linearity for each stage separately (stages 2,3,4 and 5) and all together. So, for each subject we have 5 specifications, with 4 different tests for each specification (20 tests in total for each subject). If we take all 20 periods, then we get that 11 out 24 people do not satisfy Assumption 2 according to at least one of these 20 tests, at significance level 5%. However, if we take only the last 15 periods allowing for a 5-period learning time, we get that only 8 people out of 24 do

[8]These tests form different processes, all based on the residuals. Under the null hypothesis, the limiting processes for these empirical processes are the Wiener Process (Rec-CUSUM), the standard Brownian bridge (OLS-CUSUM), the increments of the Wiener Process (Rec-MOSUM) and the increments of the standard Brownian bridge (OLS-MOSUM).

not satisfy Assumption 2 according to at least one of these 20 tests, at significance level 5%. Moreover, if we lower the significance level to 1%, then at the last 15 periods only 4 people fail Assumption 2 according to at least one of these 20 tests.

In the remainder of the paper, we treat the first five periods as learning time and exclude them from our analysis.

3.7 Persuasion Bias Hypothesis (PBH) vs Bayesian Update Hypothesis (BUH) on Pooled Data

There are two ways to perform econometric analysis of our data. One approach is to pool all data together and force all subjects to use the same updating rule. The central question we address through this approach is what updating rule describes a "representative subject" behavior better. Another approach is to analyze each subject completely separately and therefore study individual updating rules. Then the question is what updating rule describes the behavior of a larger number of subjects. This section is devoted to the first approach, while the next section implements the second one.

In both sections we focus on comparing two models, the Persuasion Bias Hypothesis (3.5) and the Bayesian Update Hypothesis (3.6). Both models are special cases of a more general model (3.2)-(3.3).

If we pool all 24 subjects for 15 last periods, we get Table 3.1. This table shows the estimation result for three models:

the Persuasion Bias Hypothesis (PBH) model: (nonlinear)

$$\begin{cases} x_i^2 = \lambda^2 s_i + \mu^{1,2} y_i^1 \\ x_i^3 = c^3\lambda^2 s_i + c^3\mu^{1,2} y_i^1 + \mu^{2,3} y_i^2 \\ x_i^4 = c^4c^3\lambda^2 s_i + c^4c^3\mu^{1,2} y_i^1 + c^4\mu^{2,3} y_i^2 + \mu^{3,4} y_i^3 \\ x_i^5 = c^5c^4c^3\lambda^2 s_i + c^5c^4c^3\mu^{1,2} y_i^1 + c^5c^4\mu^{2,3} y_i^2 + c^5\mu^{3,4} y_i^3 + \mu^{4,5} y_i^4 \end{cases} \tag{3.7}$$

the Bayesian Update Hypothesis (BUH) model:

$$\begin{cases} x_i^2 = \lambda^2 s_i + \mu^{1,2} y_i^1 \\ x_i^3 = \lambda^3 s_i + \mu^{2,3} y_i^2 \\ x_i^4 = \lambda^4 s_i + \mu^{3,4} y_i^3 \\ x_i^5 = \lambda^5 s_i + \mu^{4,5} y_i^4 \end{cases} \tag{3.8}$$

Model	PBH	BUH	GM
	x_i^t	x_i^t	x_i^t
$s_i \cdot \mathbf{1}(t=2)\ (\lambda^2)$	0.609*** (0.061)	0.593*** (0.044)	0.593*** (0.064)
$y_i^1 \cdot \mathbf{1}(t=2)\ (\mu^{1,2})$	0.389*** (0.063)	0.405*** (0.046)	0.405*** (0.066)
$\left(\lambda^2 s_i + \mu^{1,2} y_i^1\right) \cdot \mathbf{1}(t=3)\ (c^3)$	0.740*** (0.085)		
$s_i \cdot \mathbf{1}(t=3)\ (\lambda^3 = c^3\lambda^2)$	*0.451*	0.491*** (0.066)	0.430*** (0.043)
$y_i^1 \cdot \mathbf{1}(t=3)\ (\mu^{1,3} = c^3\mu^{1,2})$	*0.288*		0.275*** (0.074)
$y_i^2 \cdot \mathbf{1}(t=3)\ (\mu^{2,3})$	0.263*** (0.083)	0.510*** (0.066)	0.296*** (0.086)
$\left(\lambda^3 s_i + \sum_{\tau=1,2} \mu^{\tau,3} y_i^\tau\right) \cdot \mathbf{1}(t=4)\ (c^4)$	0.462*** (0.042)		
$s_i \cdot \mathbf{1}(t=4)\ (\lambda^4 = c^4\lambda^3)$	*0.208*	0.240*** (0.014)	0.239*** (0.021)
$y_i^1 \cdot \mathbf{1}(t=4)\ (\mu^{1,4} = c^4\mu^{1,3})$	*0.133*		0.099*** (0.035)
$y_i^2 \cdot \mathbf{1}(t=4)\ (\mu^{2,4} = c^4\mu^{2,3})$	*0.122*		0.058* (0.033)
$y_i^3 \cdot \mathbf{1}(t=4)\ (\mu^{3,4})$	0.537*** (0.042)	0.759*** (0.015)	0.604*** (0.033)
$\left(\lambda^4 s_i + \sum_{\tau=1,2,3} \mu^{\tau,4} y_i^\tau\right) \cdot \mathbf{1}(t=5)\ (c^5)$	0.642*** (0.080)		
$s_i \cdot \mathbf{1}(t=5)\ (\lambda^5 = c^5\lambda^4)$	*0.134*	0.164*** (0.027)	0.186*** (0.025)
$y_i^1 \cdot \mathbf{1}(t=5)\ (\mu^{1,5} = c^5\mu^{1,4})$	*0.085*		0.057*** (0.013)
$y_i^2 \cdot \mathbf{1}(t=5)\ (\mu^{2,5} = c^5\mu^{2,4})$	*0.078*		0.034 (0.022)
$y_i^3 \cdot \mathbf{1}(t=5)\ (\mu^{3,5} = c^5\mu^{3,4})$	*0.345*		0.410*** (0.090)
$y_i^4 \cdot \mathbf{1}(t=5)\ (\mu^{4,5})$	0.356*** (0.080)	0.834*** (0.028)	0.311*** (0.095)
AIC	12743.43	12936.21	12737.01
BIC	12790.88	12983.66	12816.1
$\ln(L)$	-6362.713	-6459.105	-6353.505

Signif. codes: *** 1%, ** 5%, * 10%

Table 3.1: All subjects are pooled, standard errors are clustered by subjects, periods and stages. Only the last 15 periods are used. *Italic entries* are derived from the estimated coefficients.

the general model (GM):

$$\begin{cases} x_i^2 &= \lambda^2 s_i + \mu^{1,2} y_i^1 \\ x_i^3 &= \lambda^3 s_i + \mu^{1,3} y_i^1 + \mu^{2,3} y_i^2 \\ x_i^4 &= \lambda^4 s_i + \mu^{1,4} y_i^1 + \mu^{2,4} y_i^2 + \mu^{3,4} y_i^3 \\ x_i^5 &= \lambda^5 s_i + \mu^{1,5} y_i^1 + \mu^{2,5} y_i^2 + \mu^{3,5} y_i^3 + \mu^{4,5} y_i^4 \end{cases} \tag{3.9}$$

First of all, note that the PBH model and the BUH model are not nested but they have the same number of parameters. So, we can compare them using maximum likelihood function. The last row in Table 3.1 shows that the PBH model has a larger maximum likelihood function, which makes it a better model.

The advantage of the PBH model can be demonstrated in a less rigorous but more intuitive way by just looking at the estimated coefficients. First, let's look at stage 3. The vector of estimated weights $(c^3\lambda^2, c^3\mu^{1,2}, \mu^{2,3})$ from the PBH model is much closer to $(\lambda^3, \mu^{1,3}, \mu^{2,3})$ from the GM model than $(\lambda^3, 0, \mu^{2,3})$ from the BUH model:

$$\begin{aligned} PBH: \quad x_i^3 &= 0.451 s_i + 0.288 y_i^1 + 0.263 y_i^2, \\ BUH: \quad x_i^3 &= 0.491 s_i + 0.510 y_i^2, \\ GM: \quad x_i^3 &= 0.430 s_i + 0.275 y_i^1 + 0.296 y_i^2. \end{aligned}$$

In other words, in stage 3 the PBH restrictions distort the unrestricted estimated model less than the BUH restrictions. Another way to see it is to compare $x_i^2 = \lambda^2 s_i + \mu^{1,2} y_i^1$ and $x_i^3 = \lambda^3 s_i + \mu^{1,3} y_i^1 + \mu^{2,3} y_i^2$ from the GM model:

$$\frac{\lambda^3}{\lambda^2} = 0.725, \quad \frac{\mu^{1,3}}{\mu^{1,2}} = 0.680.$$

The PBH model requires $\frac{\lambda^3}{\lambda^2} = \frac{\mu^{1,3}}{\mu^{1,2}}$. Since the difference between 0.725 and 0.680 is small, we can say that we have evidence in favor of the PBH model for stage 3.

Stage 4 shows a little more ambiguous picture. $(\lambda^4, \mu^{1,4}, \mu^{2,4}, \mu^{3,4})$ from the GM model is somewhere in between $(c^4 c^3 \lambda^2, c^4 c^3 \mu^{1,2}, c^4 \mu^{2,3}, \mu^{3,4})$ from the PBH model and $(\lambda^4, 0, 0, \mu^{3,4})$ from the BUH model, with the PBH model being a little closer to the GM model:

$$\begin{aligned} PBH: \quad x_i^4 &= 0.208 s_i + 0.133 y_i^1 + 0.122 y_i^2 + 0.537 y_i^3, \\ BUH: \quad x_i^4 &= 0.240 s_i + 0.759 y_i^3, \\ GM: \quad x_i^4 &= 0.239 s_i + 0.099 y_i^1 + 0.058 y_i^2 + 0.604 y_i^3. \end{aligned}$$

Stage 5 points out to a seemingly "weak" point of the BUH model, that is y_i^3 has a large effect on x_i^5:

$$\begin{array}{rcccccccccccc} PBH: & x_i^5 & = & 0.134s_i & + & 0.085y_i^1 & + & 0.078y_i^2 & + & 0.345y_i^3 & + & 0.356y_i^4, \\ BUH: & x_i^5 & = & 0.164s_i & & & & & & & + & 0.834y_i^4, \\ GM: & x_i^5 & = & 0.186s_i & + & 0.057y_i^1 & + & 0.034y_i^2 & + & 0.410y_i^3 & + & 0.311y_i^4. \end{array}$$

However, a large correlation between the last two guesses, y_i^3 and y_i^4, makes this argument questionable.

Besides comparing the PBH and BUH models, we can make two additional observations from Table 3.1. First, we compare the GM model with the other two. The GM model has more parameters, so we have to use either AIC or BIC. BIC penalizes the number of parameters more severely, working against the GM model. According to BIC, the PBH model is the best. However, according to AIC, the GM model is the best one.

Second, we use the estimated coefficients to analyze how close they are to the equilibrium prediction from Theorem 5. Nonlinear distortion of the PBH model does not affect the coefficients for stage 2 too much (λ^2 and $\mu^{1,2}$). All models show that subjects slightly overweight their signals in stage 2 compared to the equilibrium response, which is $0.5s_i + 0.5y_i^1$. This means that (1) subjects do not deviate too far away from the equilibrium response, (2) they deviate in the direction of their own private signal, so that they rely more on the information that comes to them directly, not through their neighbor. In stage 3, the BUH model does not lead to the equilibrium response ($0.33s_i + 0.66y_i^2$). Moreover, we see a significant weight placed on y_i^1, which also goes against the equilibrium prediction. However, in stage 4, the weights in the BUH model come pretty close to the equilibrium, which is $0.25s_i + 0.75y_i^3$. Moreover, although statistically significant, the effect from the neighbor's guesses in stage 1 and 2 is small in the general model, which also goes along with Theorem 5. Since stage 5 has an identification issue ($y_i^4 = x_i^4 = x_i^5$ in equilibrium), we do not analyze this stage here. In sum, equilibrium prediction is good for stage 4 and probably for stage 2, but not for stage 3.

Models (3.7), (3.8) and (3.9) use all data, forcing subjects to behave according to a particular model in all stages. Moreover, these models implicitly put the error term at the end of each row, assuming these errors are independent of each other. This error structure is not the only one that is consistent with theoretical analogs of these models. For example, stage 3 of the PBH model can look like this:

$$x_i^3 = c^3 x_i^2 + \mu^{2,3} y_i^2, \tag{3.10}$$

implying a different error structure. We performed various robustness checks with respect to the number of stages used in the analysis and the error structure (see Appendix B.5). We conclude that the conclusion that the PBH model is better than the BUH model holds, with one exception. If x_i^2 is used as a proxy for $\lambda^2 s_i + \mu^{1,2} y_i^1$ (like in (3.10)), then the PBH model is worse than the BUH model in stage 3.

3.8 Persuasion Bias Hypothesis (PBH) vs Bayesian Update Hypothesis (BUH): Subjects' Classification

In this section we test the Persuasion Bias Hypothesis (3.5) against the Bayesian Update Hypothesis (3.6) by looking at each subject separately, therefore allowing for heterogeneous behavior.

For each subject, we compare the PBH model and the BUH model based on maximum likelihood across different specifications, varying the number of stages and the error structure (see Appendix B.7). Table 3.2 summarizes this comparison by classifying each subject into one of three groups, PBH, BUH or NA. Out of 24 subjects, we get 12 subjects whose behavior is better described by the PBH model, 11 subjects left unclassified (NA), and only 1 subject fits BUH model unambiguously better than the PBH model. If we limit the comparison to stage 3 only, out of 11 NA subjects, 3 subjects fit the PBH model, 7 subjects fit the BUH model and the remaining subject is unclassified. If we limit the comparison to stage 4 only, out of 11 NA subjects, 6 subjects fit the PBH model, 4 subjects fit the BUH model and the remaining subject is unclassified. Finally, if we limit the comparison to stage 5 only, out of 11 NA subjects, 8 subjects fit the PBH model, 2 subjects fit the BUH model and the remaining subject is unclassified. In sum,

1. more subjects fit the PBH model better than the BUH model,
2. relative to the BUH model, the PBH model is getting better with the stages.

3.9 Performance

In this section we ask whether the better fit of the PBH model means that the persuasion bias has some rationale besides being just a convenient rule of thumb. More precisely, we first look at the correlation between how well a subject fits the

Session	Subject	Classification By Stage			Classification
		Stage 3	Stage 4	Stage 5	
1	A	PBH	BUH	PBH	NA
1	B	BUH	BUH	PBH	NA
1	C	PBH	PBH	PBH	PBH
1	D	PBH	PBH	PBH	PBH
2	A	PBH	BUH	PBH	NA
2	B	BUH	PBH	BUH	NA
2	C	PBH	PBH	PBH	PBH
2	D	BUH	PBH	PBH	NA
3	A	PBH	PBH	PBH	PBH
3	B	BUH	PBH	BUH	NA
3	C	PBH	BUH	PBH	NA
3	D	BUH	BUH	BUH	BUH
4	A	PBH	PBH	PBH	PBH
4	B	PBH	PBH	PBH	PBH
4	C	PBH	PBH	PBH	PBH
4	D	BUH	PBH	PBH	NA
5	A	PBH	PBH	PBH	PBH
5	B	NA	NA	NA	NA
5	C	BUH	PBH	PBH	NA
5	D	PBH	PBH	PBH	PBH
6	A	PBH	PBH	PBH	PBH
6	B	BUH	PBH	PBH	NA
6	C	PBH	PBH	PBH	PBH
6	D	PBH	PBH	PBH	PBH

Table 3.2: Subject classification into groups based on the difference $LL_{\text{PBH-BUH}}$ between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model for all specifications (see Appendix B.7). Only the last 15 periods are used.

PBH model and his performance in the experiment. We'll show that under some conditions this correlation is positive.

Next, we rationalize this positive correlation by demonstrating that a profit-maximizing subject might sometimes be classified as PBH-subject, indicating that the PBH model might sometimes be more effective in accomodating inefficiency in other subjects' behavior. This result relates to two fundamental assumptions that the equilibrium notion is based on, that is, each subject maximizes his payoff and correctly predicts other subjects' behavior. We argue that the PBH model relaxes both assumptions by introducing small mistakes in payoff maximization as well as understanding other subjects' behavior.

3.9.1 Observation

In this section we focus on finding the sign of the correlation between how well a subject fit the PBH model and his performance in the experiment, leaving all explanations for this sign to the next section.

As a measure of how well a subject fits the PBH model, we use the difference between log-likelihood functions for PBH and BUH models in different specifications, $LL_{\text{PBH-BUH}}$. This measure is especially high for two subjects, subject D in session 5 and subject C in session 6 (see Appendix B.7). We dropped these two outliers.

To control for signal realizations, we include a term which we call *IdealProfit*. For each subject, we calculate his *Ideal Guess* in each period and in each stage. This guess is based on the theoretical analysis in Theorem 5 and is equal to the average of all inferred signals:

$$\begin{array}{llll} x_A^1 = s_A, & x_B^1 = s_B, & x_C^1 = s_C, & x_D^1 = s_D, \\ x_A^2 = \frac{s_A+s_D}{2}, & x_B^2 = \frac{s_A+s_B}{2}, & x_C^2 = \frac{s_B+s_C}{2}, & x_D^2 = \frac{s_C+s_D}{2}, \\ x_A^3 = \frac{s_A+s_C+s_D}{3}, & x_B^3 = \frac{s_A+s_B+s_D}{3}, & x_C^3 = \frac{s_A+s_B+s_C}{3}, & x_D^3 = \frac{s_B+s_C+s_D}{3}, \end{array}$$
$$x_i^4 = x_i^5 = \frac{s_A + s_B + s_C + s_D}{4}, \quad i = A, B, C, D.$$

Then we calculate the *IdealProfit* for each subject in each period:

$$-\sum_{t=1}^{5} \left| x_i^t - \theta \right|,$$

where x_i^t is subject i's *Ideal Guess* in stage t. The actual *Profit* is calculated by the same formula, with x_i^t being the actual guesses.

Table 3.3 shows the regression of the profit each subject gets during the experiment on how well a subject fits the PBH model. This table uses the following specification:

the Persuasion Bias Hypothesis (PBH) model:

$$\begin{cases} x_i^2 = \lambda^2 s_i + \mu^{1,2} y_i^1, \\ x_i^3 = c^3 \lambda^2 s_i + c^3 \mu^{1,2} y_i^1 + \mu^{2,3} y_i^2, \\ x_i^4 = c^4 c^3 \lambda^2 s_i + c^4 c^3 \mu^{1,2} y_i^1 + c^4 \mu^{2,3} y_i^2 + \mu^{3,4} y_i^3; \end{cases} \tag{3.11}$$

the Bayesian Update Hypothesis (BUH) model:

$$\begin{cases} x_i^2 = \lambda^2 s_i + \mu^{1,2} y_i^1, \\ x_i^3 = \lambda^3 s_i + \mu^{2,3} y_i^2, \\ x_i^4 = \lambda^4 s_i + \mu^{3,4} y_i^3 \end{cases} \tag{3.12}$$

	Profit	*Profit*	*Profit*	*Profit*
Intercept		-444* (251)	-311 (296)	
$\mathrm{LL}_{\text{PBH-BUH}}$			4.99 (4.61)	3.24 (4.31)
$\mathrm{LL}_{\text{PBH-BUH}} \cdot \mathbf{1}(\mathrm{LL}_{\text{PBH-BUH}} > 0)$	10.1** (4.6)	13.4** (4.7)		
$\mathrm{LL}_{\text{PBH-BUH}} \cdot \mathbf{1}(\mathrm{LL}_{\text{PBH-BUH}} < 0)$	-28.5** (12.6)	-29.5** (12.0)		
IdealProfit	1.24*** (0.04)	0.90*** (0.19)	0.93*** (0.23)	1.17*** (0.04)
Number of observations	22	22	22	22
Adjusted R^2	0.986	0.643	0.489	0.982

Signif. codes: *** 1%, ** 5%, * 10%

Table 3.3: $\mathrm{LL}_{\text{PBH-BUH}}$ is the difference in log-likelihood functions for (3.11) and (3.12). *Profit* sums the actual profit from stages 1,2,3 and 4. *IdealProfit* sums the profit based on *Ideal Guesses* from stages 1,2,3 and 4. Only the last 15 periods are used. One data point corresponds to one subject. Two subjects are dropped: subject D in session 5 and subject C in session 6.

to calculate $\mathrm{LL}_{\text{PBH-BUH}}$ and therefore it ignores the behavior on the last stage. Table 3.3 demonstrates the main conclusion of this section:

1. when $\mathrm{LL}_{\text{PBH-BUH}} > 0$, the correlation between $\mathrm{LL}_{\text{PBH-BUH}}$ and *Profit* is positive;

2. when $\mathrm{LL}_{\text{PBH-BUH}} > 0$, the correlation between $\mathrm{LL}_{\text{PBH-BUH}}$ and *Profit* is negative.

Indeed, this is clear from the first two columns (the second column adds an intercept to the regression). The other two columns forces a linear regression with respect to $\mathrm{LL}_{\text{PBH-BUH}}$, which leads to a positive but insignificant coefficient for $\mathrm{LL}_{\text{PBH-BUH}}$.

Appendix B.8 lists similar regressions for other specifications of the PBH and the BUH models (varying the number of stages and the error structure) as well as other definitions of the performance measure (varying how many stages are used to calculate the profit). In general, across all specifications,

1. term $\mathrm{LL}_{\text{PBH-BUH}} \cdot \mathbf{1}(\mathrm{LL}_{\text{PBH-BUH}} > 0)$ has either significantly positive effect or an insignificant effect (p-value > 10%) on the profit,

2. term $LL_{\text{PBH-BUH}} \cdot \mathbf{1}(LL_{\text{PBH-BUH}} < 0)$ has either significantly negative effect or an insignificant effect (p-value > 10%) on the profit.

3.9.2 Explanation

Session	Subject	Classification based on *Ideal Guess* Stage 3	Stage 4
1	A	PBH	PBH
1	B	BUH	BUH
1	C	BUH	BUH
1	D	NA	NA
2	A	NA	NA
2	B	PBH	BUH
2	C	PBH	PBH
2	D	BUH	PBH
3	A	BUH	BUH
3	B	NA	NA
3	C	BUH	BUH
3	D	BUH	PBH
4	A	BUH	PBH
4	B	BUH	PBH
4	C	BUH	BUH
4	D	NA	NA
5	A	BUH	BUH
5	B	BUH	BUH
5	C	BUH	BUH
5	D	BUH	BUH
6	A	PBH	BUH
6	B	BUH	PBH
6	C	BUH	PBH
6	D	BUH	PBH

Table 3.4: Subject classification into groups based on the difference $LL_{\text{PBH-BUH}}$ between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model for all specifications (see Appendix B.9). Only the last 15 periods are used.

In the previous section we gave some evidence that the superior performance of the PBH model has a more deep explanation than just appealing to the simplicity of the PBH rule. To investigate this idea, we perform the same analysis as in Sections 3.7 and 3.8 but now using subject i's *Ideal Guess* in stage t instead of the actual guess x_i^t. That will show which model is better if each subject behaved optimally, given the other subjects' behavior is fixed.

Estimating (3.7) and (3.8), we get a significantly better fit for the persuasion bias model. Going through all robustness checks, we get that all specifications but one rank the PBH model as a better model than the BUH model (see Appendix B.9).

Table 3.4 is the analog of Table 3.2, but now this classification is based on *Ideal Guesses*, not the actual guesses. We omit stage 5 since the PBH model has an automatic advantage in this stage: when working with *Ideal Guesses*, $x_i^5 = x_i^4$ by construction. Table 3.4 shows that the BUH model is better, especially for stage 3. Specifically, 9 out of 24 subjects fit the BUH model better in both stages 3 and 4, while only 2 subjects fit the PBH model better. Out of the rest, 7 subjects follow the BUH model in stage 3 and they follow the PBH model in stage 4. Other subjects either follow the PBH model in stage 3 and the BUH model in stage 4 (2 subjects) or are left unclassified (4 subjects). This indicates that it is pooling that gives an advantage to the PBH model. In light of that, we grouped subjects based on their individual classifications and compared the models for each group. This grouped-based pooling supports the individual classification (see Appendix B.9).

Thus, based on the ideal guesses,

1. most subjects fit the BUH model, especially in stage 3, but
2. some subjects demonstrate the advantage of the PBH model.

The first observation is not surprising, given that the BUH model serves as a rational benchmark. What is interesting is that the PBH model is sometimes better than the BUH model even when a subject behaves optimally. This indicates that the PBH model superior performance phenomenon could be partly explained by some kind of rational reasoning.

To explore this reasoning even further, we run the following two regressions:

$$\mathrm{LL}^{\mathrm{real}}_{\mathrm{PBH\text{-}BUH}} = \alpha_0 + \alpha_1 \mathrm{LL}^{\mathrm{ideal}}_{\mathrm{PBH\text{-}BUH}},$$

$$\begin{aligned}\mathrm{LL}^{\mathrm{real}}_{\mathrm{PBH\text{-}BUH}} = \beta_0 + \beta_1 \mathrm{LL}^{\mathrm{ideal}}_{\mathrm{PBH\text{-}BUH}} \cdot \mathbf{1}\left(\mathrm{LL}^{\mathrm{ideal}}_{\mathrm{PBH\text{-}BUH}} > 0\right) \\ + \beta_2 \mathrm{LL}^{\mathrm{ideal}}_{\mathrm{PBH\text{-}BUH}} \cdot \mathbf{1}\left(\mathrm{LL}^{\mathrm{ideal}}_{\mathrm{PBH\text{-}BUH}} < 0\right),\end{aligned}$$

where $\mathrm{LL}^{\mathrm{real}}_{\mathrm{PBH\text{-}BUH}}$ is the difference in log-likelihood functions for two models based on the actual data, while $\mathrm{LL}^{\mathrm{ideal}}_{\mathrm{PBH\text{-}BUH}}$ is the difference in log-likelihood functions for two models based on the data where *Ideal Guesses* are used instead of the actual guesses x_i^t.

	$\mathrm{LL}^{\mathrm{real}}_{\text{PBH-BUH}}$	$\mathrm{LL}^{\mathrm{real}}_{\text{PBH-BUH}}$
Intercept	5.12^{**} (2.15)	-0.35 (2.86)
$\mathrm{LL}^{\mathrm{ideal}}_{\text{PBH-BUH}}$	0.16 (0.10)	
$\mathrm{LL}^{\mathrm{ideal}}_{\text{PBH-BUH}} \cdot \mathbf{1}\left(\mathrm{LL}^{\mathrm{ideal}}_{\text{PBH-BUH}} > 0\right)$		0.54^{***} (0.17)
$\mathrm{LL}^{\mathrm{ideal}}_{\text{PBH-BUH}} \cdot \mathbf{1}\left(\mathrm{LL}^{\mathrm{ideal}}_{\text{PBH-BUH}} < 0\right)$		-0.14 (0.15)
Number of observations Adjusted R^2	22 0.07	22 0.27

Signif. codes: *** 1%, ** 5%, * 10%

Table 3.5: $\mathrm{LL}_{\text{PBH-BUH}}$ is the difference in log-likelihood functions for (3.11) and (3.12). Only the last 15 periods are used. One data point corresponds to one subject. Two subjects are dropped: subject D in session 5 and subject C in session 6.

Based on estimation of (3.11) and (3.12), we get positive but insignificant α_1, positive and significant β_1, negative and insignificant β_2 (see Table 3.5). Appendix B.10 verifies this result for all other specifications and shows that

1. α_1 is usually positive across different specifications, but always insignificant;

2. β_1 is positive and sometimes significant.

Positive α_1 implies that there is a positive correlation between how well a subject fits the PBH model and how well he should have fitted the PBH model if he had been maximizing his profit. However, this effect is insignificant, so we cannot say that such a correlation exists. Nevertheless, once we restrict the sample to the subjects that should have fitted the PBH model better than the BUH model (i.e., $\mathrm{LL}^{\mathrm{ideal}}_{\text{PBH-BUH}} > 0$), this correlation becomes not only positive but also significant in many specifications. That means that in the environment where the PBH model is more efficient than the BUH model, subjects tend to detect this feature and take it into account in their strategy.

Intuitively, subjects that use the PBH model correct their neighbors' mistakes by relying more on their own previous guesses. In other words, the optimal behavior must adapt to other players' mistakes, and the PBH model captures this adaptation. More precisely, each subject faces a tradeoff between relying on his neighbor's last guess efficiency as ideally the optimal thing to do and hedging with his own last guess against his neighbor's nonoptimal behavior. Depending on how rational the neighbor is, this tradeoff can be optimally resolved in favor of one or the other model.

Moreover, Table 3.4 shows that in the earlier stages, it is easier to be more rational: for most subjects, the BUH model leads to a better performance in stage 3. However, in the later stages, when calculating the optimal guess is harder, PBH-type behavior has a higher profit.

3.10 Conclusion

We tested the persuasion bias hypothesis against a Bayesian updating model in a 5-stage game with a 4-person circle network. In the horse race of two models, the persuasion bias hypothesis model gets stronger support. Moreover, the PBH model might even be more optimal than the BUH model in practice. Intuitively, the PBH model is more robust to non-optimality in others' behavior. Ideally, a subject wants to guess an average of t signals. The BUH model logic heavily relies on the assumption that the neighbor can optimally form the best guess each time. So, a BUH-subject ignores his own guess in the previous stage and adjusts his neighbor's guess by taking into account the private signal. In other words, he treats his neighbor's guess as a sufficient statistic for $t-1$ signals, while his own private information is a sufficient statistic for 1 signal. In contrast, a PBH-subject adjusts his own guess every time he forms a new guess. Thus, he treats his neighbor's guess as a source for 1 new signal, while his own private information is a sufficient statistic for $t-1$ signal.

In this paper we chose a directed circle network and the normal distribution. Both elements are essential for a rational model predicting that updated beliefs should be a linear function of available information (Assumption 2). A more complicated network and / or different distribution might lead to a non-linear prediction by Bayesian Nash equilibrium. This might be a potential source for an identification argument for testing the generalized DeGroot model. Our approach however has the advantage of being more robust to model misspecifications, inheriting this feature from linear econometric models.

In light of our conjecture about the rationality of some deviations from the Bayesian

Nash equilibrium, we propose the following modification of our experiment. In stage 3, each subject is asked to report what he thinks his neighbor's neighbor signal was. In stage 4 and 5, each subject should say what he thinks about all the subjects' signals. This additional information would allow us to go beyond the two models we consider in this paper and study the updating process more closely. We leave the running of such an experiment for future research.

Chapter 4

IMPLICATIONS OF OVERCONFIDENCE ON INFORMATION INVESTMENT

4.1 Introduction

Overconfidence is a well documented behavioral bias by which an individual believes to have better information or perform at a task better than he actually does. On the other hand, a large body of literature has focused on understanding the decision to invest in information made by purely rational agents. This issue is relevant to several applications, such as bench trials, where a judge decides how much effort he is going to spend to collect information relevant to the case. Whether the results of this last strand of literature remain valid when agents are overconfident is an open question.

Moore and Healy (2008) classify overconfidence into three categories: overestimation, overplacement and overprecision. Overestimation and overplacement refer to cases where an individual thinks he has performed in a task better than he actually did or better than others, respectively. Overprecision is the case when an individual believes her information is more precise than it actually is. This paper deals with overprecision, the most prevalent and the least reversed phenomenon of the three (Ortoleva and Snowberg (2015), Benoît and Dubra (2011), and Moore and Healy (2008)). In the reminder of the paper we refer to the overprecision-type bias as overconfidence.

The leading example we are using in this paper is about a judge who must decide whether to convict or acquit a defendant who can be either innocent or guilty. In this example, what the judge decides affects not only the judge but a lot of people, the defendant including. If we take the society as a whole, it cares about the quality of the verdict, that is, it wants to acquit innocent and convict guilty defendant. The question we address is how overconfidence in the judge's perception of information affects the probability that he reaches the correct verdict.

According to Moore and Healy (2008)'s informal definition, overconfidence is the "excessive precision in one's beliefs." In general, there are two types of beliefs — prior and posterior. Suppose the overconfidence occurs in the prior belief. For example, the judge thinks he knows a lot about the case while he actually knows less. Then his incentive to acquire additional information is lower than it should be. So, the overconfidence in the prior belief leads to the lower quality of the verdict.

Now suppose the overconfidence occurs in the posterior belief. By definition, the posterior belief is the updated belief after acquiring information. So, the overconfidence in posterior means the excessive precision of that information. If the judge thinks he gets information of higher quality, his incentives to acquire that information is higher. This means that the overconfidence in the information quality leads to the higher quality of the verdict.

Note that the notion of prior and posterior is relative, since today's posterior becomes tomorrow's prior. In this paper we assume that the judge starts with a uniform prior and consider a dynamic model of information collection. We model overconfidence as the distortion in the perceived precision of information flow.[1] With this model, we argue that the total effect that overconfidence has on the probability of the judge reaching the correct decision depends on the nature of the information collection process. More specifically, it depends on how much control the judge has over the amount of information he collects.

First, suppose an unbiased judge chooses whether to collect information or not, that is the information investment choice space is binary. For example, he decides whether to hold the trial or announce the verdict right away. Then overestimating the quality of information (or equivalently, the ability to perceive information) leads to higher willingness to pay for that information. This means that overconfidence has a positive effect on the probability of the correct verdict.

Now suppose the judge can decide how much information to collect. For example, he decides on the length of the trial. Then the effect of overconfidence can be either positive or negative, depending on how much a rational judge would invest in information. If the rational judge invests very little (he holds a very short trial), then overconfidence increases information investment. On the other hand, if the rational judge invests a lot, overconfidence has the opposite effect. In general, the effect is shaped by two forces. The first is the only active force in the binary investment choice scenario. It comes from increasing the marginal benefit from each hour of trial. The second force comes from increasing the total benefit from a fixed investment. When the rational judge's investment is high, the second force

[1] Perceiving information as being more precise than it actually is can be seen as a sign of excessive gullibility of the judge. In a static model, this phenomenon is the opposite of the most common formalization of overconfidence as overprecision of the prior. In a dynamic model, however, this is no longer an issue due to prior and posterior being relative notions.

prevails. Moreover, we show that there is an optimal level of belief distortion (either to over-precision or to under-precision) that maximizes the probability of the correct decision by balancing the two forces. This optimal level is higher when the judge does not care too much about choosing the correct sentence.

Finally, suppose the judge can decide how much information to collect dynamically. In contrast to the previous scenario, the judge does not have to decide upfront how long the trial would be and can stop the trial at any moment in time. In this setting we find the effect of overconfidence to be detrimental to the quality of the judgment. The dynamic nature of information collection in this scenario introduces a third force that pushes investment down. This third force describes an excessive sensitivity to the noise in information flow. By overestimating the quality of information, the judge treats unexpected noise as a meaningful signal and therefore his belief about the defendant's innocence reaches his desired standard of proof threshold belief sooner than he (ex ante) expects. It turns out that under the assumptions of a normally distributed information flow and of a symmetric payoff, the net effect from all three forces is negative, meaning that having an underconfident judge is always better for the quality of the judgement. Intuitively, when the second force is weak, that is, when the accumulated information is low, the judge is very sensitive to noise, which makes the third force strong.

We can look at the forces from the prior-posterior perspective at a given moment in time. The first force corresponds to having excessive precision in posterior, as it comes from increasing the precision of information the decision maker is about to collect. The other two forces come from excessive precision in prior, as they both come from overestimating the quality of information already collected.

The rest of the paper is organized as follows. Section 4.3 gives a general setup, leaving aside the question of how the judge collects information. Section 4.4 introduces the dynamic information collection scenario and present all three forces. Section 4.5 elaborates on the nature of these forces by requiring the judge to commit in advance to the amount of information he is going to collect. This requirement essentially makes the model static. Under the commitment restriction, we generalize the model presented in Section 4.4 by relaxing distributional assumptions and demonstrate the first and the second forces in this general setup.

4.2 Related Literature

The overconfidence phenomenon has been studied in many settings, including (but not limited to) financial markets (Scheinkman and Xiong (2003)), medicine (Berner and Graber (2008)), war (Johnson (2009)), political behavior (Ortoleva and Snowberg (2015)). Much evidence that people are prone to overconfidence has been documented in literature. Barber and Odean (2001), Chuang and Lee (2006), and Goetzmann and Huang (2015) found empirical support for overconfidence in financial environments. One of the earliest studies, Oskamp (1965), experimentally demonstrates overconfidence among actual judges when they are presented with information about published cases. Klayman, Soll, González-Vallejo, and Barlas (1999) and Soll and Klayman (2004) provide more recent experimental evidence for judges' overconfidence. Using actual data on bail decisions made by judges in New York City between 2008 and 2013, Kleinberg, Lakkaraju, Leskovec, Ludwig, and Mullainathan (2017) show that judges appear to "respond to 'noise' as if it were signal." From the perspective of Moore and Healy (2008)'s classification of overconfidence, this observation can be interpreted as overprecision. By mixing the actual signal with noise, judges are effectively boosting the perceived precision of all incoming information as a whole.

An alternative way to model overconfidence as misperception of the quality of information is through correlation neglect. Ortoleva and Snowberg (2015) find theoretically and verify empirically that overconfidence (modeled as correlation neglect) leads to ideological extremeness, increased voter turnout and stronger partisan identification. Levy and Razin (2013) focus on information aggregation, as opposed to information investment, and find conditions under which correlation neglect can lead to increased information aggregation. In contrast Glaeser and Sunstein (2009) study a "credulous Bayesian" that neglects correlation in a context where there is no cost of information acquisition and find overconfidence to be detrimental to information aggregation. We depart from these studies in modeling overconfidence as a misperception of the precision parameter.[2] In contrast to correlation neglect, misperception of the precision can potentially occur in any setting, even if there is actually no correlation in the incoming information (conditional on the true state). In fact, we mostly focus on that case in this paper, though Section 4.5 presents more general results as well.

[2] Dubra (2004) defines overconfidence as an optimistic bias in prior beliefs. This interpretation of overconfidence is orthogonal to a more popular definition of overconfidence as underestimating the volatility (Alpert and Raiffa (1982)). We used the latter one.

Our discussion about an overconfident judge can also be applied to a jury room, where a juror misperceives the quality of her own information. This relates our paper to the literature studying information acquisition or investment by committees. Martinelli (2006) considers a setup when each committee member chooses how much to invest in the precision of a binary signal when costs are convex. This setup is very similar to our model in Section 4.5. Chan, Lizzeri, Suen, and Yariv (2017) work with a dynamic setup that are close to our model in Section 4.4. All papers assume that the jurors are rational.

Scheinkman and Xiong (2003) explain speculative bubbles using overconfidence. Overconfidence as misperception of information quality generates disagreement about fundamentals which results in a price bubble. They model overconfidence in a way similar to our model in Section 4.4. However, they did not allow for the agents to choose whether to observe information flow or not.

4.3 Setup

Consider a single decision maker who has to decide between two actions. The payoff from these actions depends on the true state of the world. For example, suppose this decision maker is a judge who decides whether to acquit ($v = A$) or convict ($v = C$) a defendant. The defendant might be either innocent ($z = I$) or guilty ($z = G$). In this case z plays the role of the true state of the world.

We assume the decision maker gets a payoff $u(v, z)$ from action $v \in \{A, C\}$ if the true state is $z \in \{I, G\}$. At the beginning, the decision maker has some prior beliefs about the true state, $p_0 = \mathbb{P}(z = I)$. Given belief p, his expected utility from action v is $U(v, p) = pu(v, I) + (1 - p)u(v, G)$.

Naturally, we assume that when the defendant is innocent, it is better to acquit her, and when she is guilty, it is better to convict her. Moreover, for simplicity, we focus on the symmetric case, where the judge gets utility R from the incorrect verdict and utility $Q + R$, $Q > 0$, from the correct verdict. Formally,

Assumption 3

$$u(A, I) = u(C, G) > u(C, I) = u(A, G).$$

Denote

$$Q = u(A, I) - u(C, I) = u(C, G) - u(A, G) > 0, \quad R = u(C, I) = u(A, G).$$

Then

$$\max_{v\in\{A,C\}} U(v,p) = \begin{cases} pQ + R, & p \geq \frac{1}{2}, \\ (1-p)Q + R, & p \leq \frac{1}{2}. \end{cases}$$

Before deciding on the action, the decision maker can collect more information about the true state. For simplicity, we restrict our analysis to a symmetric case when the decision maker has no initial bias in his belief:

Assumption 4 (Uniform prior belief) $p_0 = 0.5$.

4.4 Dynamic Model

The decision maker collects information by observing the change in a Brownian motion process with state-dependent drift:

$$dX_t = \mu_z dt + \sigma dW_t, \quad \mu_z = \begin{cases} 1, & \text{if } z = I, \\ -1, & \text{if } z = G. \end{cases} \tag{4.1}$$

where W_t is the standard Wiener process.

Information collection is costly. In a dynamic setting, we can differentiate two types of costs, attention cost and time cost. The time cost is formalized through a discount factor $\delta \geq 0$. For simplicity, we focus on the no-discounting case here ($\delta = 0$). Appendix C.5 shows that the case $\delta > 0$ leads to the same conclusions.

The attention cost is proportional to the amount of time the decision maker spends on the information collection. Formally, the decision maker chooses a stopping time $\tau \geq 0$ (which is path-dependent, that is, whether or not the decision maker stops by t depends on X_t) and, upon stopping, a verdict $v \in \{A, C\}$ (which depends on X_τ). The utility he eventually gets is equal to $u(v,z) - \kappa\tau$, where $\kappa > 0$ is a parameter of the model.

The problem the decision maker faces is called an optimal stopping problem. It has already been studied in the literature, so we can take the solution off-the-shelf:

Theorem 7 (Shiryaev (2007), Chapter 4, Theorem 5, p.185) *The optimal strategy exists and is given by*

$$\tau = \inf\{t \geq 0 \colon p_t \notin (\lambda, 1-\lambda)\}, \quad v = \begin{cases} A, & p_\tau \geq 1-\lambda, \\ C, & p_\tau \leq \lambda. \end{cases} \tag{4.2}$$

where p_t is the belief that the true state is I at time t. Threshold $\lambda \in (0, 0.5)$ is uniquely defined by

$$\frac{1-2\lambda}{2\lambda(1-\lambda)} - \log\left(\frac{\lambda}{1-\lambda}\right) = \frac{Q}{\kappa\sigma^2}. \tag{4.3}$$

For completeness, we include the proof in Appendix C.1.

When observing (4.1), the decision maker updates his belief about the state[3]

$$p_t = \mathbb{P}\left[z = I \mid X_t\right] = \frac{1}{1 + e^{-\frac{2X_t}{\sigma^2}}}, \tag{4.4}$$

which is equivalent to

$$X_t = \frac{\sigma^2}{2} \log\left(\frac{p_t}{1-p_t}\right). \tag{4.5}$$

So, another way to write the optimal strategy (4.2) is

$$\tau = \inf\left\{t \geq 0 \colon X_t \notin (-\chi, \chi)\right\}, \quad \nu = \begin{cases} A, & X_\tau \geq \chi, \\ C, & X_\tau \leq -\chi. \end{cases} \tag{4.6}$$

where $\chi = \frac{\sigma^2}{2}\log\left(\frac{1-\lambda}{\lambda}\right) > 0$. The advantage of this representation is that it expresses the strategy in terms of an external (observable) variable X_t and not in terms of a mental quantity which is the decision maker's belief. This distinction is important to us since overconfidence introduces a distortion in the belief updating rule, so that an overconfident person would form a different belief than a rational one, given the same observed process X_t.

Definition 2 *An* η***-type*** *decision maker updates his belief according to*

$$p_t = \frac{1}{1 + e^{-\frac{2X_t\eta}{\sigma^2}}} \tag{4.7}$$

while observing

$$dX_t = \mu_z dt + \sigma dW_t. \tag{4.8}$$

In other words, the η-type decision maker believes the variance is η times lower than it actually is. Parameter η captures the level of overconfidence, with $\eta = 1$ corresponding to the rational case. Thus, the η-type decision maker is overconfident when $\eta > 1$ and he is underconfident (he underestimates the precision of information) when $\eta < 1$.

[3] We assume $X_0 = 0$.

Given the observed process (4.8) and the strategy (4.6) with a fixed $\chi > 0$, the probability of the correct decision (the probability of acquittal if the defendant is innocent and of conviction if the defendant is guilty) is

$$\mathbb{P}\left(v = \begin{cases} A, & z = I \\ C, & z = G \end{cases}\right) = \mathbb{P}\left[v = A \mid z = I\right] = \mathbb{P}\left[v = C \mid z = G\right] = \frac{1}{1 + e^{-\frac{2\chi}{\sigma^2}}}. \quad (4.9)$$

Indeed, this probability is equal to the probability that the decision is correct at the time when this decision is made.

Consider the η-type decision maker. His optimal strategy is (4.6) with threshold $\chi = \mathcal{X}\left(\frac{\sigma^2}{\eta}\right)$, where

$$\mathcal{X}\left(\sigma^2\right) = \frac{\sigma^2}{2} \log\left(\frac{1 - \lambda\left(\sigma^2\right)}{\lambda\left(\sigma^2\right)}\right), \quad (4.10)$$

where $\lambda\left(\sigma^2\right) \in (0, 0.5)$ is the solution to (4.3).

Note that the probability of the correct decision (4.9) is increasing in the threshold χ. Theorem 8 states that the higher the overconfidence level η, the lower the probability of the correct decision.

Theorem 8 *$\mathcal{X}\left(\sigma^2\right)$ defined by (4.10) is increasing in σ^2.*

Intuitively, the expression $\chi = \frac{\sigma^2}{2} \log\left(\frac{1-\lambda}{\lambda}\right)$ shows that increasing σ^2 has two effects on χ. The direct effect increases χ. This effect comes from the attempt to keep the same standard of proof by collecting more information that is less precise. The indirect effect decreases χ through λ (λ is increasing in σ^2). This effect comes from the attempt to keep the same total cost of information by lowering the standard of proof $1 - \lambda$. Theorem 8 states that the first effect always dominates the second.[4]

While being explicitly connected to the formula for $\mathcal{X}\left(\sigma^2\right)$, these two effects give an ambiguous prediction for the expected stopping time. On the one hand, increasing χ without changing the variance increases the expected stopping time. On the other hand, increasing the variance without changing the threshold decreases the expected stopping time. Thus, if the variance is actually changing, the first effect has an unclear prediction for whether the expected stopping time increases or decreases. If the variance is not actually changing, yet the decision maker thinks it increases, the second effect decreases the stopping time by lowering the perceived standard

[4] This result is not robust to relaxing Assumption 3, see Appendix C.6.

of proof and increases it by not updating aggressively enough. This distinction is important for us because it illuminates a commitment aspect of the information collection problem.

Suppose the decision maker has to decide ex ante when to stop information collection, that is, he has to commit on the stopping time τ at time $t = 0$. Then there are two forces that shape the overall effect on τ from increasing σ^2. The first force comes from decreasing the benefit of the marginal information piece dX_t and therefore it lowers τ. The second force comes from decreasing the benefit of information already collected, X_t, and therefore it increases τ. Once we drop the commitment restriction, another force arises. This third force captures the discrepancy between what the decision maker expects to see (information flow with a high variance) and what he actually observes (information flow with a low variance). Though not changing his perception of the variance once observing X_t, the decision maker bases his stopping decision on the low variance information flow. Thus, the third force increases τ since the decision maker does not update enough thinking he observes more noise than he actually does.

We elaborate on the commitment model and the first two forces in the next section. We conclude this section by expanding the intuition for the third force.

Suppose that at time $t = 0$ the decision maker commits to stop collecting information at a certain time τ. The commitment prevents the decision maker to collect more information when $|X_\tau|$ is too small (which corresponds to low standard of proof). It also prevents him from stopping the information collection process earlier when $|X_\tau|$ is too large (high standard of proof). The optimal τ balances out these two events based on distribution X_τ. An η-type decision maker expects to observe $\mu_z\tau + \frac{\sigma}{\sqrt{\eta}}W_\tau$ distributed as $\mathcal{N}\left(\mu_z\tau, \frac{\sigma^2}{\eta}\right)$, while he actually observes $\mu_z\tau + \sigma W_\tau$ distributed as $\mathcal{N}\left(\mu_z\tau, \sigma^2\right)$. Thus, the η-type decision maker, $\eta > 1$, underestimates the probability $|X_\tau|$ being large. In other words, the η-type decision maker wants to stop the information collection before the committed stopping time with higher probability than he believes at $t = 0$. Thus, in the absence of commitment the η-type decision maker stops sooner. This effect is captured by the third force.

4.5 General Static Model

The decision maker collects information by acquiring a signal $S \in \mathcal{S}$ that has state-dependent distribution $F_z(\cdot)$. Upon observing $S = s$, the optimal verdict is $v = A$ if $dF_I(s) > dF_G(s)$ and $v = C$ otherwise. Denote set $\mathcal{S}_A = \{s\colon dF_I(s) > dF_G(s)\}$

all realizations of S that leads to an acquittal decision. Similarly, denote $\mathcal{S}_C = \{s\colon dF_I(s) < dF_G(s)\}$ all realizations of S that leads to conviction. We assume that the measure of $\mathcal{S} \setminus (\mathcal{S}_A \cup \mathcal{S}_C)$ is zero under any state, so that the decision maker is (almost) never indifferent between the two verdicts. Denote $p_{z,v} = \mathbb{P}[S \in \mathcal{S}_v \mid z]$ the probability of making the decision $v \in \{A, C\}$, given state z. Then the probability of the correct decision is $\frac{1}{2}\left(p_{I,A} + p_{G,C}\right)$ and therefore the expected utility from signal S is $\frac{1}{2}\left(p_{I,A} + p_{G,C}\right) Q + R$. When the decision maker does not use the signal, his expected utility is $\frac{Q}{2} + R$. Thus, the quality of signal S can be summarized by

$$\frac{1}{2}\left(p_{I,A} + p_{G,C}\right) - \frac{1}{2}.$$

We assume that the decision maker can increase the quality by paying more for the signal. Formally, the quality of the signal is an increasing function of cost, $h(c)$. Thus, the expected utility is

$$\left(\frac{1}{2} + h(c)\right) Q + R - c$$

and the decision maker chooses cost $c > 0$. The first order condition is

$$h'(c)Q = 1. \tag{4.11}$$

To guarantee that the solution to (4.11) exists, is unique and maximizes the expected utility, we assume

Assumption 5 *$h\colon (0, +\infty) \to \left[0, \frac{1}{2}\right]$ is such that $h(0) = 0$, $\lim_{c \to 0} h'(c) = +\infty$, $\lim_{c \to +\infty} h'(c) = 0$, $h''(c) < 0$.*

Given that general model, we impose the following definition of overconfidence:

Definition 3 *An **η-type** decision maker perceives the quality of the signal being $h(\eta c)$ while paying c.*

Consider the η-type decision maker. His expected utility from signal S is

$$\left(\frac{1}{2} + h(\eta c)\right) Q + R - c. \tag{4.12}$$

Example (Normal distribution) *When $S \equiv X_t \sim \mathcal{N}\left(\mu_z t, \sigma^2 t\right)$, its quality is equal to $f\left(\frac{t}{\sigma^2}\right)$, where*

$$f(\rho) = \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{\frac{\rho}{2}}} e^{-x^2} dx. \tag{4.13}$$

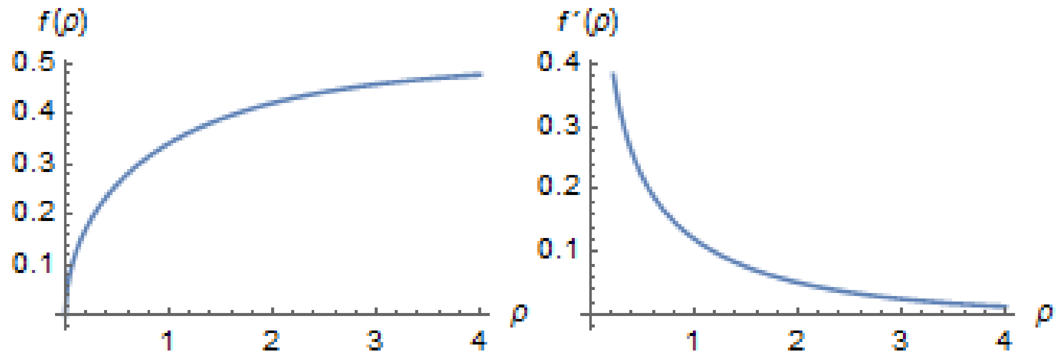

Figure 4.1: Function $f(\rho) = \frac{1}{\sqrt{\pi}} \int\limits_0^{\sqrt{\frac{\rho}{2}}} e^{-x^2} dx$ and its derivative.

Then a linear cost function $c(t) = \kappa \cdot t$ implies $h(c) = f\left(\frac{c}{\kappa\sigma^2}\right)$. Figure 4.1 shows that $h(c)$ satisfies Assumption 5. Moreover, it is easy to see that Definition 3 is the analog of Definition 2 for the static case.

Optimizing (4.12) over $c > 0$, we get

$$c > 0: \; \eta h'(\eta c) Q = 1. \tag{4.14}$$

Treating the solution to (4.14) as a function of the overconfidence level η, we have

$$c'(\eta) = -\frac{h'(\eta c) + \eta c h''(\eta c)}{\eta^2 h''(\eta c)}. \tag{4.15}$$

A higher c means higher probability of the correct decision, $\frac{1}{2} + h(c)$. As we increase the level of overconfidence η, c increases if and only if $h'(\eta c) + \eta c h''(\eta c)$ is positive. From Assumption 5, $h'(\eta c)$ is always positive, while $\eta c h''(\eta c)$ is always negative. The term $h'(\eta c)$ corresponds to the **first force**: higher effective cost ηc increases the quality of the signal because h is increasing. The term $\eta c h''(\eta c)$ corresponds to the **second force**: higher effective cost ηc decreases the marginal benefit of information because h' is decreasing.

The total effect is captured by the behavior of function $xh'(x)$, which derivative is equal to $h'(x) + xh''(x)$. If it increases at point $x = \eta c(\eta)$, then increasing the level of overconfidence makes the final decision better. If it decreases, the final decision becomes worse with increased overconfidence.

Note the interpretation of $xh'(x)$, $x = \eta c$, as the marginal benefit of information. The first x corresponds to the first force, which increases function $xh'(x)$ as we increase x through the level of overconfidence η. The second x corresponds to the second force, which decreases function $xh'(x)$.

Without making any additional assumptions, it is hard to say more about the behavior of $c(\eta)$. One interesting special case is when the following assumptions holds:

Assumption 6 *There exists $\hat{c} > 0$ such that $ch'(c)$ increases for $c < \hat{c}$ and it decreases for $c > \hat{c}$.*

This assumption says that when the amount of collected information is below some threshold, the second force is weaker than the first force, and vice versa. Recall that the second force comes from changing the benefit of already collected information. As we increase the amount of collected information is small, this force becomes stronger. On the other hand, the first force comes from changing the benefit of the marginal information piece and therefore it depends on the amount of collected information only through the non-stationary properties of the information flow. Assumption 6 says that the information flow is stationary enough to make sure that there is a unique threshold such that the second force prevails if and only if the amount of collected information is above that threshold.

Under Assumption 6, we can prove that there exists a *unique* optimal level of overconfidence η^* (which can be less than 1, which corresponds to underconfidence) such that more overconfidence is good below that level and it is bad for all η above η^*. Formally:

Theorem 9 *The probability of choosing the correct action is increasing in $\eta \in (0, \eta^*)$ and it is decreasing in $\eta \in (\eta^*, +\infty)$, where $\eta^* = \frac{1}{Qh'(\hat{c})}$.*

Example (Normal distribution) *When $S \equiv X_t \sim \mathcal{N}\left(\mu_z t, \sigma^2 t\right)$, function $ch'(c) = \frac{c}{\kappa\sigma^2} f'\left(\frac{c}{\kappa\sigma^2}\right)$ satisfies Assumption 6 (see Figure 4.2).*

Here is an example where Assumption 6 is violated.

Example (Binary distribution) *Suppose $S \in \{I, G\}$, $\mathbb{P}[S = I \mid z = I] = \mathbb{P}[S = G \mid z = G] \geq \frac{1}{2}$. Then its quality $h = \mathbb{P}[S = z \mid z] - \frac{1}{2}$. This means that the binary distribution does not imply any specific form of function $h(c)$.*

1. *Suppose the agent payment is some decreasing function of the variance of the random variable $\mathbf{1}(S = z)$, for example, $c = -\log\left(4\mathbb{P}[S = z \mid z]\left(1 - \mathbb{P}[S = z \mid z]\right)\right)$,*

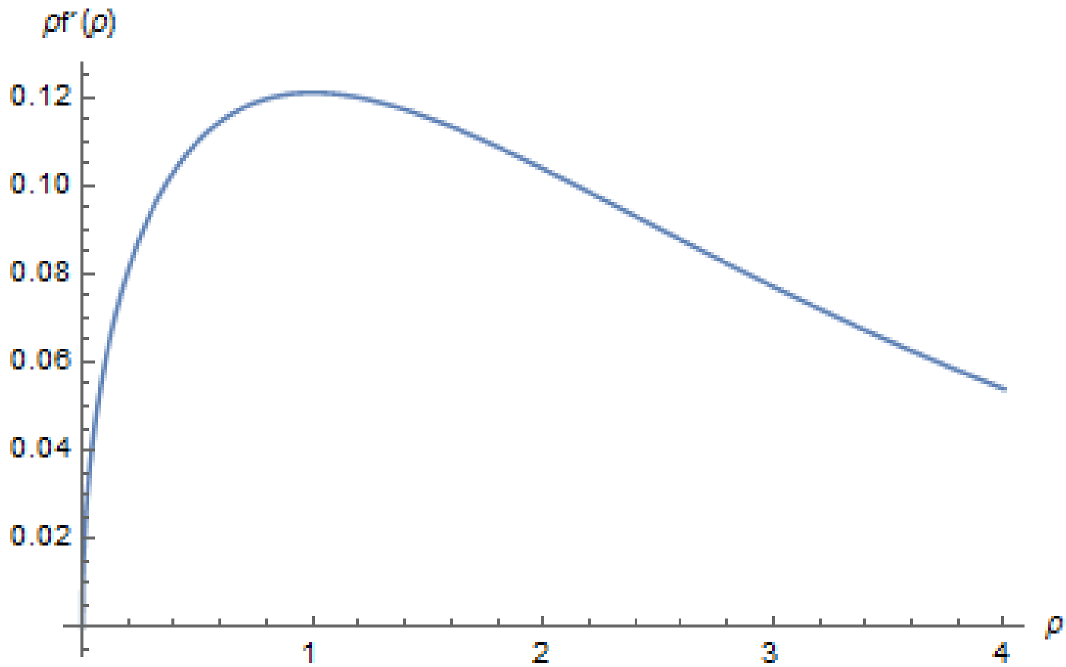

Figure 4.2: Function $\rho f'(\rho) = \frac{e^{-\frac{\rho}{2}}\sqrt{\rho}}{2\sqrt{2\pi}}$ defines how the marginal benefit of information changes with the level of overconfidence under normal distribution assumption.

$\kappa > 0$. Then $h(c) = \mathbb{P}[S = z \mid z] - \frac{1}{2} = \frac{1}{2}\sqrt{1 - e^{-c}}$ satisfies both Assumptions 5 and 6.

2. *Function $h(c) = \left(2c + \frac{1}{c}\sin(c)\right)^{\frac{1}{4}} - 1$ satisfies Assumption 5 but not Assumption 6. See Figure 4.3.*

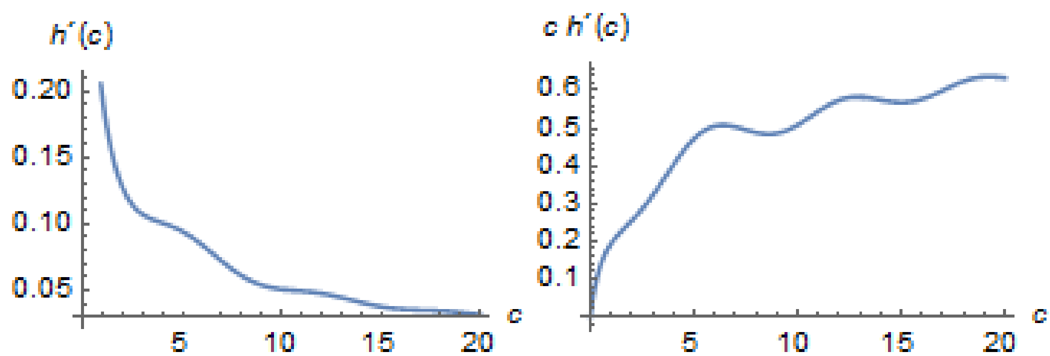

Figure 4.3: Functions $h'(c)$ and $ch'(c)$, where $h(c) = \left(2c + \frac{1}{c}\sin(c)\right)^{\frac{1}{4}} - 1$.

Note that the optimal level of overconfidence is decreasing in Q. This means that overconfidence is bad when the benefit from choosing the correct action is high. Intuitively, when the benefit from choosing the correct action is very low, the rational agent collects very little information. A distortion in his incentives by increasing the perceived quality of information always has a positive effect. Indeed, an increase in the quality of already collected information (the second force) does not have a large effect since the amount of this information is small. So, overconfidence is good for low Q. On the other hand, when the benefit from choosing the correct action is very high, the rational agent collects a lot of information. This means that the second force has a lot of power as it works with a large amount of information. Consequently, overconfidence is bad in this case.

At the end of this section we give an example of a model when only the first force is active.

4.5.1 Binary Information Acquisition Decision

Suppose the decision maker can choose only between two values, $c \in \{0, \bar{c}\}$. This describes the situation when the agent simply has to decide whether to acquire information or not. In this case the optimality condition (4.14) should be changed to

$$c = \bar{c} \quad \Leftrightarrow \quad h(\eta\bar{c})Q > \bar{c}. \tag{4.16}$$

Definition 4 *The maximum cost the decision maker is ready to pay for the signal is called the* ***willingness to pay****.*

Given condition (4.16) and Assumption 5, the willingness to pay is equal to

$$c > 0: \quad h(\eta c)Q = c, \quad \eta h'(\eta c)Q < 1. \tag{4.17}$$

Treating the solution to (4.17) as a function of the overconfidence level η, we have $c'(\eta) > 0$.

Theorem 10 *The willingness to pay is increasing in η.*

In this model, there is no "already collected information" since the choice is binary, either buy the signal or not. Thus, the second force is absent here and Theorem 10 result is driven by the first force.

4.6 Conclusion

We presented three forces that shape the effect that overconfidence has on the quality of the final decision, or equivalently, on the amount of information collected in equilibrium. Two aspects of the information collection process are crucial to understand the effect that misperception of information quality has on information investment in a particular scenario. First, whether the information investment choice space is binary (collect or not) or continuous (how much to collect). In the latter case, there is a trade-off between increased quality of already collected information, which pushes the overconfident agent to collect less information, and increased quality of the marginal piece of information, which pushes him to collect more information. Second, if the information investment choice is continuous, whether information has been collected all at once or not. In the latter case, misperception of information quality creates a systematic bias between how much information the decision maker expects to collect and how much information he actually collects. An overconfident agent overestimates the expected amount of information he is going to collect in the future.

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Appendix A

APPENDIX FOR CHAPTER 2

A.1 Proof of Theorem 1.

Lemma 4 gives sufficient conditions for a function to be equal to $V^{(a)}[p_1]$. Moreover, it provides an algorithm to construct an optimal a-type strategy from $V^{(a)}[p_1]$ for any p_1.

Lemma 4 *If function $V: [0,1] \to \mathbb{R}$ is such that there exists a set of a finite number of points $\mathcal{S} \subset [0,1]$ such that*

1. $V[1] = u_1[a_1]$, $V[0] = u_2[a_2]$,
2. *V is continuous everywhere; moreover, it is continuously differentiable everywhere except at points $\mathcal{S}$,*
3. *for all $p_1 \in (0,1) \setminus \mathcal{S}$,*

$$\min\left\{\mathcal{L}_1[p_1], \mathcal{L}_2[p_1], V[p_1] - U^{(a)}[p_1]\right\} = 0, \tag{A.1}$$

where

$$U^{(a)}[p_1] = \begin{cases} u_1[a]p_1 + u_2[a](1-p_1), & 0 < p_1 < 1, \\ u_1[a_1], & p_1 = 1, \\ u_2[a_2], & p_1 = 0, \end{cases} \tag{A.2}$$

$$\mathcal{L}_1[p_1] = \frac{c_1}{\lambda_1 p_1} + V'[p_1](1-p_1) - (u_1[a_1] - V[p_1]), \tag{A.3}$$

$$\mathcal{L}_2[p_1] = \frac{c_2}{\lambda_2(1-p_1)} - V'[p_1]p_1 - (u_2[a_2] - V[p_1]), \tag{A.4}$$

4. *there exists an a-type strategy (a^F, T^*, τ^*) such that*[1]

   a) *it is Markovian,*

[1] The optimal strategy is Markovian, that is, the optimal choice of the information source at any given time depends only on the beliefs at this point in time. More precisely, it depends on the initial beliefs and the signals observed so far only through the current beliefs. Abusing notations, strategy (a^F, T^*, τ^*) is defined as a plan of actions for every belief $p_1 \in [0,1]$. Strictly speaking, I need to define a strategy for each $p_1 \in [0,1]$ separately. Markovian property makes two definitions equivalent.

b) the stopping time τ^* is almost surely finite and whenever the state is revealed, the agent stops the information collection process,

c) any trajectory of beliefs (a^F, T^*, τ^*) can generate (for any initial belief and any realization of signals) does not spend any nontrivial time (that is the time period of non-zero length) on set $\mathcal{S}$,

d) the following conditions hold:

$$dT^*_{t,1} > 0,\ t < \tau^* \quad \Rightarrow \quad \mathcal{L}_1[p_{t,1}] = 0, \tag{A.5}$$

$$dT^*_{t,2} > 0,\ t < \tau^* \quad \Rightarrow \quad \mathcal{L}_2[p_{t,1}] = 0, \tag{A.6}$$

$$V[p_{\tau^*,1}] \quad = \quad U^{(a)}[p_{\tau^*,1}], \tag{A.7}$$

then for any initial belief p_1,

1. $V[p_1] = V^{(a)}[p_1]$,
2. (a^F, T^*, τ^*) *is the optimal a-type strategy.*

Lemma 4 is not constructive, it can only be used to check whether a given function is indeed equal to $V^{(a)}[p_1]$. Boundary conditions $V[1] = u_1[a_1]$ and $V[0] = u_2[a_2]$ follow immediately from the definition (2.5). For example, when $p_1 = 1$, the true state is 1 and therefore no additional information is needed to choose the best alternative, which is a_1. Equation (A.1) describes the first order condition for the optimization problem (2.5).[2] When the agent believes the state is 1 with probability p_1 and he pays attention to source 1, his expected payoff is changing according to the differential equation $\mathcal{L}_1[p_1] = 0$. This action is optimal if and only if $\mathcal{L}_2[p_2] \geq 0$ and $V[p_1] \geq U^{(a)}[p_1]$. Similarly for source 2. When the agent believes the state is 1 with probability p_1 and he chooses to stop the information collection process, his expected payoff is equal to the expected payoff from choosing a^F defined by (2.4):

$$V[p_1] = U^{(a)}[p_1] = \mathbb{E}\left[u_j\left[a\mathbf{1}\left(p_{\tau,1} \in (0,1)\right) + a_1\mathbf{1}\left(p_{\tau,1} = 1\right) + a_2\mathbf{1}\left(p_{\tau,1} = 0\right)\right] \mid p_1\right].$$

Conditional on the restriction (2.4), this choice is optimal if and only if $\mathcal{L}_1[p_1] \geq 0$ and $\mathcal{L}_2[p_2] \geq 0$. Lemma 4 says that if a function $V[p_1]$ satisfies the boundary conditions and the first order conditions, and there exists a strategy (a^F, T^*, τ^*) such that the expected payoff from this strategy is equal to $V[p_1]$, then the strategy (a^F, T^*, τ^*) is the optimal a-type strategy.

[2] Equation (A.1) is called the Hamilton-Jacobi-Bellman equation.

Proof of Lemma 4 Using the Bayes formula, we have

$$dp_{t,1} = -\lambda_1 p_{t,1}(1-p_{t,1})dT_{t,1} + \lambda_2 p_{t,1}(1-p_{t,1})dT_{t,2} + (1-p_{t,1})dX^{(1)}_{T_{t,1}} - p_{t,1}dX^{(2)}_{T_{t,2}}.$$

Take any a-type strategy (a^F, T, τ) and any initial belief $p_{0,1} \in (0,1)$. Let $\{p_{t,1}\}_{t\geq 0}$ be the realization of the belief process, given that the agent follows strategy (a^F, T, τ) and has initial belief $p_{0,1}$. Suppose that after observing a positive signal, the agent stops the information collection process: if $t < \tau$, then no positive signals have been observed so far. Let $\tau' \in [0, \tau]$ be any time such that $V \in C^1$ along the belief trajectory so far (that is, $p_{t,1} \notin \mathcal{S}$ for all $0 < t < \tau'$). Then Ito's formula gives

$$V[p_{\tau',1}] = V[p_{0,1}] + \int_0^{\tau'} p_{t,1}(1-p_{t,1})V'[p_{t,1}]\left(\lambda_2 dT_{t,2} - \lambda_1 dT_{t,1}\right)$$
$$+ \int_0^{\tau'} \left(u_1[a_1] - V[p_{t,1}]\right) dX^{(1)}_{T_{t,1}} + \int_0^{\tau'} \left(u_2[a_2] - V[p_{t,1}]\right) dX^{(2)}_{T_{t,2}}. \quad \text{(A.8)}$$

Let $\tau' = \sup\{t \leq \tau : p_{t',1} \notin \mathcal{S}, t' \leq t\}$. If the learning process always stops eventually (that is, $\mathbb{P}\left[\tau < +\infty \mid p_{0,1}\right] = 1$), then τ' is always finite and therefore $V[p_{\tau',1}]$ is well-defined. Thus, if $\mathbb{P}\left[\tau < +\infty \mid p_{0,1}\right] = 1$, integration (A.8) over all possible realizations of signals gives

$$\mathbb{E}\left[V[p_{\tau',1}] \mid p_{0,1}\right] = V[p_{0,1}] + \mathbb{E}\left[\int_0^{\tau'} p_{t,1}(1-p_{t,1})V'[p_{t,1}]\left(\lambda_2 dT_{t,2} - \lambda_1 dT_{t,1}\right) \mid p_{0,1}\right]$$
$$+\mathbb{E}\left[\int_0^{\tau'} p_{t,1}\left(u_1[a_1] - V[p_{t,1}]\right)\lambda_1 dT_{t,1} + \int_0^{\tau'} (1-p_{t,1})\left(u_2[a_2] - V[p_{t,1}]\right)\lambda_2 dT_{t,2} \mid p_{0,1}\right]$$

or equivalently,

$$V[p_{0,1}] = \mathbb{E}\left[V[p_{\tau',1}] \mid p_{0,1}\right] + \lambda_1 \mathbb{E}\left[\int_0^{\tau'} p_{t,1}\mathcal{L}_1[p_{t,1}]dT_{t,1} \mid p_{0,1}\right] - \mathbb{E}\left[\int_0^{\tau'} c_1 dT_{t,1} \mid p_{0,1}\right]$$
$$+ \lambda_2 \mathbb{E}\left[\int_0^{\tau'} (1-p_{t,1})\mathcal{L}_2[p_{t,1}]dT_{t,2} \mid p_{0,1}\right] - \mathbb{E}\left[\int_0^{\tau'} c_2 dT_{t,2} \mid p_{0,1}\right], \quad \text{(A.9)}$$

where $\tau' = \sup\{t \leq \tau : p_{t',1} \notin \mathcal{S}, t' \leq t\}$.

Suppose $\tau' < \tau$ is the time when the belief trajectory goes past a point in $\mathcal{S}$, that is $p_{\tau',1} \in \mathcal{S}$ and there exists $\delta > 0$ such that $p_{\tau'+t,1} \notin \mathcal{S}$ for $t \in (0, \delta)$. Then (A.9) holds for $p_{\tau',1}$ as the initial belief:

$$V[p_{\tau',1}] = \mathbb{E}\left[V[p_{\tau'',1}] \mid p_{\tau',1}\right] + \lambda_1 \mathbb{E}\left[\int_{\tau'}^{\tau''} p_{t,1}\mathcal{L}_1[p_{t,1}]dT_{t,1} \mid p_{\tau',1}\right] - \mathbb{E}\left[\int_{\tau'}^{\tau''} c_1 dT_{t,1} \mid p_{\tau',1}\right]$$
$$+ \lambda_2 \mathbb{E}\left[\int_{\tau'}^{\tau''} (1-p_{t,1})\mathcal{L}_2[p_{t,1}]dT_{t,2} \mid p_{\tau',1}\right] - \mathbb{E}\left[\int_{\tau'}^{\tau''} c_2 dT_{t,2} \mid p_{\tau',1}\right], \quad \text{(A.10)}$$

where $\tau'' = \sup\{t \in (\tau', \tau] \colon p_{t',1} \notin \mathcal{S}, t' \in (\tau', t)\}$. Combining (A.9) and (A.10), I get that (A.9) holds for $\tau' = \sup\{t \leq \tau \colon$ the belief trajectory does not spend any nontrivial time on set $\mathcal{S}$ on $(0,t)\}$.

(A.1) and (A.9) give

$$V[p_{0,1}] \geq \mathbb{E}\left[V[p_{\tau',1}] - \int_0^{\tau'} c_1 dT_{t,1} + c_2 dT_{t,2} \mid p_{0,1}\right], \quad \text{(A.11)}$$

where $\tau' = \sup\{t \leq \tau \colon$ the belief trajectory does not spend any nontrivial time on set $\mathcal{S}$ on $(0,t)\}$.

Claim 1 *If* $\mathbb{P}\left[\tau < +\infty \mid p_{0,1}\right] = 1$ *and the information collection stops once a positive signal is observed, then*

$$V[p_{0,1}] \geq \mathbb{E}\left[V[p_{\tau,1}] - \int_0^{\tau} c_1 dT_{t,1} + c_2 dT_{t,2} \mid p_{0,1}\right]. \quad \text{(A.12)}$$

Proof: Suppose $\tau' < \tau$ is such that there exists $\delta > 0$ so that $p_{t,1} \in \mathcal{S}$ for all $t \in [\tau', \min\{\tau' + \delta, \tau\}]$. It is sufficient to show that

$$V[p_{\tau',1}] \geq \mathbb{E}\left[V[p_{\min\{\tau'+\delta,\tau\},1}] - \int_{\tau'}^{\min\{\tau'+\delta,\tau\}} c_1 dT_{t,1} + c_2 dT_{t,2} \mid p_{\tau',1}\right]. \quad \text{(A.13)}$$

Consider an infinite sequence of strategies $(a^F, T^{(m)}, \tau^{(m)})$, $m = 1, 2, 3, \ldots$, such that strategy $(a^F, T^{(m)}, \tau^{(m)})$ deviates from strategy (a^F, T, τ) in a way that at the moment τ' it chooses only source 2 for $\Delta_m > 0$ amount of time:

- $dT^{(m)}_{\tau'+t,2} = dt$ for $t \in [0, \Delta_m]$ and $dT^{(m)}_{\tau'+\Delta_m+t,k} = dT_{\tau'+t,k}$, $k = 1, 2$,
- $\tau^{(m)} = \begin{cases} \tau' + \Delta_m, & \text{if source 2 produces a positive signal during } t \in [\tau', \tau' + \Delta_m], \\ \tau + \Delta_m, & \text{otherwise.} \end{cases}$

The idea is to "move" the belief trajectory away from $\mathcal{S}$. Since $\mathcal{S}$ is a finite set of points, paying attention to only one of the sources shifts the belief $p_{t,1}$ away from $\mathcal{S}$ either to the right (if source 2 is used) or to the left (if source 1 is used). Using (A.11) with strategy $(a^F, T^{(m)}, \tau^{(m)})$:

$$V[p_{\tau',1}] \geq \mathbb{P}\left[\tau^{(m)} = \tau' + \Delta_m \mid p_{\tau'}\right]\mathbb{E}\left[V[p^{(m)}_{\tau^{(m)},1}] - c_2\Delta_m \mid \tau^{(m)} = \tau' + \Delta_m,\ p_{\tau'}\right]$$
$$+ \mathbb{P}\left[\tau^{(m)} > \tau' + \Delta_m \mid p_{\tau'}\right]\times$$
$$\mathbb{E}\left[V[p^{(m)}_{\min\{\tau'+\delta,\tau\}+\Delta_m,1}] - c_2\Delta_m - \int_{\tau'}^{\min\{\tau'+\delta,\tau\}} c_1 dT_{t,1} + c_2 dT_{t,2} \mid \tau^{(m)} > \tau' + \Delta_m,\ p_{\tau'}\right]. \tag{A.14}$$

As $\Delta_m \to 0$ with $m \to +\infty$, the right hand side of (A.14) converges to the right hand side of (A.13). Thus, (A.13) is true. ■

(A.1) and (A.12) together give

$$V[p_{0,1}] \geq \mathbb{E}\left[U^{(a)}[p_{\tau,1}] - \int_0^{\tau} c_1 dT_{t,1} + c_2 dT_{t,2} \mid p_{0,1}\right]. \tag{A.15}$$

for any initial belief $p_{1,0}$ and any strategy (a^F, T, τ) such that $\mathbb{P}\left[\tau < +\infty \mid p_{1,0}\right] = 1$ and the information collection stops once a positive signal is observed.

By definition, the right hand side of (A.15) is the expected payoff from the strategy (a^F, T, τ):

$$\mathbb{E}\left[U^{(a)}[p_{\tau,1}] - \int_0^{\tau} c_1 dT_{t,1} + c_2 dT_{t,2} \mid p_{0,1}\right] = V^{(a)}[p_{0,1}; (T, \tau)].$$

Thus, $V[p_{0,1}] \geq V^{(a)}[p_{0,1}; (T, \tau)]$.

Since any strategy with $\mathbb{P}\left[\tau < +\infty \mid p_{1,0}\right] < 1$ gives $V^{(a)}[p_{0,1}; (T, \tau)] = -\infty$ and continuing information collection after the state is revealed only decreases the payoff, we have $V[p_{0,1}] \geq V^{(a)}[p_{0,1}; (T, \tau)]$ for all strategies (a^F, T, τ).

The proof concludes by the observation that $V[p_{0,1}] = V^{(a)}[p_{0,1}; (T^*, \tau^*)]$ by definition of the strategy (a^F, T^*, τ^*) and using (A.9). ■

The next step is to guess the optimal a-type strategy (a^F, T^*, τ^*) and check the guess by showing that all conditions listed in Lemma 4 are satisfied by the function equal to the expected payoff from (a^F, T^*, τ^*).[3]

Theorem 1 provides the optimal a-type strategy (a^F, T^*, τ^*) in explicit form. Condition Π is

$$\frac{\pi_k^{\#(a)}}{1-\pi_k^{\#(a)}} \geq \frac{1-R_{3-k}^{(a)}}{R_{3-k}^{(a)}},$$

where $\pi_k^{\#(a)} \in (0, R_k^{(a)}]$ solves

$$1 - \frac{c_k\lambda_{3-k}(1-R_k^{(a)})}{c_{3-k}\lambda_k R_k^{(a)}} + \log\left[\frac{c_k\lambda_{3-k}(1-\pi_k^{\#(a)})}{c_{3-k}\lambda_k\pi_k^{\#(a)}}\right] = 0.$$

When condition Π holds, $\bar{\pi}_k^{(a)} \in \left[\pi_k^{\#(a)}, 1\right)$ is defined as a unique solution to $H_k^{(a)}\left[\bar{\pi}_k^{(a)}\right] = \frac{1-R_{3-k}^{(a)}}{R_{3-k}^{(a)}}$, where

$$H_k^{(a)}[p_k] = \left(\frac{p_k}{1-p_k} + \frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\mathbf{1}\left(p_k > R_k^{(a)}\right)\right)\log\left[\frac{R_k^{(a)}(1-p_k)}{p_k\left(1-R_k^{(a)}\right)}\right]$$
$$+\max\left\{\frac{c_k\lambda_{3-k}\left(p_k - R_k^{(a)}\right)}{c_{3-k}\lambda_k R_k^{(a)}(1-p_k)}, 0\right\} - \frac{p_k}{1-p_k}\left(\frac{c_k\lambda_{3-k}\left(1-R_k^{(a)}\right)}{c_{3-k}\lambda_k R_k^{(a)}} - \log\left[\frac{c_k\lambda_{3-k}\left(1-R_k^{(a)}\right)}{c_{3-k}\lambda_k R_k^{(a)}}\right] - 2\right)$$

is strictly decreasing on $p_k \in \left(\pi_k^{\#(a)}, 1\right)$ from $\frac{\pi_k^{\#(a)}}{1-\pi_k^{\#(a)}}$ to $-\infty$.

Lemma 4 requires to define function $V[p_1]$. Here it is:

Area 1 : $V[p_1] = U^{(a)}[p_1]$,

Area 2.k : $V[p_1] = V_k^{(a)}\left[p_k; R_k^{(a)}\right]$ (recall that $p_2 = 1 - p_1$),

Area 3.k : $V[p_1] = V_k^{(a)}\left[p_k; R_k^{(a)}\right] + \frac{c_k}{\lambda_k}(1-p_k)\Delta_k^{(a)}$,

where

$$V_k^{(a)}\left[p_k; R_k^{(a)}\right] = \frac{1-p_k}{1-R_k^{(a)}}\left(u_k[a]R_k^{(a)} + u_{3-k}[a](1-R_k^{(a)})\right)$$
$$+\frac{p_k - R_k^{(a)}}{1-R_k^{(a)}}\left(u_k[a_k] - \frac{c_k}{\lambda_k}\right) + \frac{c_k}{\lambda_k}(1-p_k)\log\left[\frac{R_k^{(a)}(1-p_k)}{p_k\left(1-R_k^{(a)}\right)}\right].$$

[3] Similar lemma can be formulated for the optimal strategy directly. The only reason to split the solution into two steps is that the optimal a-type strategy is easier to describe, and therefore easier to guess.

It is sufficient to check that function $V[p_1]$ satisfies all conditions listed in Lemma 4:

Step 1 $\mathcal{L}_k[p_1] = 0$ for Areas 2.k and 3.k, $k = 1, 2$,

Step 2 $V[p_1] \in C[0, 1]$,

Step 3 $V[p_1] \in C^1$ everywhere on $(0, 1)$ except for point $\bar{\pi}_k^{(a)}$ in Case 3,

Step 4 $\mathcal{L}_k[p_1] \geq 0$ everywhere on $(0, 1)$, $k = 1, 2$,

Step 5 $V[p_1] \geq U^{(a)}[p_1]$ everywhere on $(0, 1)$.

I omit calculations for Steps 1-3 here since they are straightforward.

Step 4 $\mathcal{L}_1[p_1]\Big|_{\text{Area 1}} = \frac{c_1}{\lambda_1}\left(\frac{1}{p_1} - \frac{1}{R_1^{(a)}}\right) \geq 0$ since $p_1 \leq R_1^{(a)}$ in Area 1.

$$\mathcal{L}_1[p_1]\Big|_{\text{Area 2.2}} = \frac{c_2}{\lambda_2}\log\left[\frac{p_1 R_2^{(a)}}{(1-p_1)(1-R_2^{(a)})}\right] + \frac{c_1}{\lambda_1}\left(\frac{1}{p_1} - \frac{1}{R_1^{(a)}}\right) \equiv g[p_1].$$

$g[1 - R_2^{(a)}] = \frac{c_1}{\lambda_1}\left(\frac{1}{1-R_2^{(a)}} - \frac{1}{R_1^{(a)}}\right) \geq 0$ since $1 - R_2^{(a)} \leq R_1^{(a)}$ in Case 1 and Case 3. Function $g[p_1]$ is decreasing for $p_1 < \frac{c_1\lambda_2}{c_2\lambda_1 + c_1\lambda_2}$ and is increasing for $p_1 > \frac{c_1\lambda_2}{c_2\lambda_1+c_1\lambda_2}$; $g\left[\frac{c_1\lambda_2}{c_2\lambda_1+c_1\lambda_2}\right] = -\frac{c_2}{\lambda_2}\Delta_2^{(a)}$. Thus, $\mathcal{L}_1[p_1]\Big|_{\text{Area 2.2}} \geq 0$ in Case 1. For Case 3 ($k = 1$), $\mathcal{L}_1[p_1]\Big|_{\text{Area 2.2}} \geq 0$ follows immediately from $g[1 - R_2^{(a)}] \geq 0$ and $g'[p_1] < 0$ for $p_1 < 1 - R_2^{(a)}$. For Case 3 ($k = 2$), $\mathcal{L}_1[p_1]\Big|_{\text{Area 2.2}} \geq 0$ follows from $g[1 - \bar{\pi}_2^{(a)}] \geq 0$ (I omit the proof of this fact here).

$\mathcal{L}_1[p_1]\Big|_{\text{Area 3.2}} = \frac{c_2}{\lambda_2}\left(\frac{c_1\lambda_2(1-p_1)}{c_2\lambda_1 p_1} - 1 - \log\left[\frac{c_1\lambda_2(1-p_1)}{c_2\lambda_1 p_1}\right]\right) \geq 0$ since $x - 1 - \log[x] \geq 0$ for all $x \geq 1$ and $\frac{c_1\lambda_2(1-p_1)}{c_2\lambda_1 p_1} \geq 1$ for Area 3.2.

Similar logic applies to $\mathcal{L}_2[p_1]$.

Step 5 Checking $V_k^{(a)}\left[p_k; R_k^{(a)}\right] \geq U^{(a)}[p_1]$ for Area 2.k is straightforward. Since $V[p_1] \geq V_k^{(a)}\left[p_k; R_k^{(a)}\right]$ in Area 3.k, $V[p_1] \geq U^{(a)}[p_1]$ in Area 3.k as well.

Remark 1 *Expression (2.11) can be rewritten as*

$$\Delta_k^{(a)} = \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}} \frac{1-R_1^{(a)}-R_2^{(a)}}{R_{3-k}^{(a)}\left(1-R_k^{(a)}\right)} + \frac{1}{x} + \log[x] - 1\Bigg|_{x=\frac{c_k\lambda_{3-k}\left(1-R_k^{(a)}\right)}{c_{3-k}\lambda_k R_k^{(a)}}}.$$

Since $\frac{1}{x}+\log[x]-1 \geq 0$ *for all* $x \geq 0$, $1-R_2^{(a)} \leq R_1^{(a)}$ *in Case 1. Moreover, when* $1-R_2^{(a)} \geq R_1^{(a)}$, *both* $\Delta_1^{(a)}$ *and* $\Delta_2^{(a)}$ *are nonnegative.*

Remark 2 *Condition (2.9) for* $k=1,2$ *together imply* $\Delta_i^{(a)} \leq 0$ *whenever* $R_i^{(a)} < 1$ *for* $i=1,2$. *Indeed, (2.9) for* $k=1,2$ *imply* $1-R_1^{(a)}-R_2^{(a)} \geq 0$. *Condition* $\frac{\lambda_2 R_2^{(a)}}{c_2} < \frac{\lambda_1 R_1^{(a)}}{c_1}$ *is equivalent to* $\frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1} < \frac{R_1^{(a)}}{R_1^{(a)}+R_2^{(a)}}$. *Whenever* $1-R_1^{(a)}-R_2^{(a)} \geq 0$, *point* $\frac{R_1^{(a)}}{R_1^{(a)}+R_2^{(a)}}$ *lies in between* $1-R_2^{(a)}$ *and* $R_1^{(a)}$. *If* $1-R_2^{(a)} \leq \frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1} \leq R_1^{(a)}$, *then* $\Delta_1^{(a)} \leq 0$ *and* $\Delta_2^{(a)} \leq 0$. *Suppose* $\frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1} \leq 1-R_2^{(a)}$. *Condition (2.9) for* $k=2$ *implies* $\Delta_2^{(a)} \leq 0$. *Thus, it is better to apply Area 2.2 strategy on* $p_1 \leq 1-R_2^{(a)}$. *In particular, it means that at point* $p_1 = \frac{c_1\lambda_2}{c_1\lambda_2+c_2\lambda_1}$ *the agent should use source 2 and not both sources together. Thus, by Markovian property, strategy of Area 3.1 cannot be optimal for any feasible beliefs. Thus,* $\Delta_1^{(a)} \leq 0$.

A.2 Proof of Lemma 1.

Using the Bayes formula, we have $p_{t,1} = \frac{p_{0,1}e^{-\lambda_1 t}}{p_{0,1}e^{-\lambda_1 t}+1-p_{0,1}}\mathbf{1}\left(X_t^{(1)}=0\right) + \mathbf{1}\left(X_t^{(1)}>0\right)$. Thus

$$X_t^{(1)} = 0 \quad \Rightarrow \quad t = \frac{1}{\lambda_1}\log\left[\frac{p_{0,1}\left(1-p_{t,1}\right)}{\left(1-p_{0,1}\right)p_{t,1}}\right].$$

Let τ be the stopping time. It is equal to the "give up" time $\bar{t} = \frac{1}{\lambda_1}\log\left[\frac{p_{0,1}\left(1-\underline{p}_1\right)}{\left(1-p_{0,1}\right)\underline{p}_1}\right]$ if no positive signal is observed. Otherwise, it is equal to the first time the process $X^{(1)}$ becomes positive. The expected stopping time is

$$\mathbb{E}\left[\tau \mid p_{0,1}, p_{0,2}\right] = \left(1-p_{0,1}\right)\bar{t} + p_{0,1}\int_0^{+\infty}\min\{t,\bar{t}\}e^{-\lambda_1 t}\lambda_1 dt\Bigg|_{\bar{t}=\frac{1}{\lambda_1}\log\left[\frac{p_{0,1}\left(1-\underline{p}_1\right)}{\left(1-p_{0,1}\right)\underline{p}_1}\right]} =$$

$$\frac{1-p_{0,1}}{\lambda_1}\log\left[\frac{p_{0,1}\left(1-\underline{p}_1\right)}{\left(1-p_{0,1}\right)\underline{p}_1}\right] + \frac{p_{0,1}-\underline{p}_1}{\lambda_1\left(1-\underline{p}_1\right)}.$$

The probability that the state will not be revealed by the end of the learning process is equal to

$$\mathbb{P}\left[X_\tau^{(1)} = 0 \mid p_{0,1}, p_{0,2}\right] = p_{0,1}e^{-\lambda_1 \bar{t}} + 1 - p_{0,1}\Bigg|_{\bar{t}=\frac{1}{\lambda_1}\log\left[\frac{p_{0,1}\left(1-\underline{p}_1\right)}{\left(1-p_{0,1}\right)\underline{p}_1}\right]} = \frac{1-p_{0,1}}{1-\underline{p}_1}.$$

Thus, the expected payoff is equal to $\mathbb{P}\left[X_\tau^{(1)} = 0 \mid p_{0,1}, p_{0,2}\right]$ times the utility from the default alternative given the beliefs at time $\bar{t}$, $p_{\bar{t},1} = \underline{p}_1$ and $p_{\bar{t},2} = \frac{p_{0,2}}{1-p_{0,1}}\left(1-\underline{p}_1\right)$, plus $\mathbb{P}\left[X_\tau^{(1)} > 0 \mid p_{0,1}, p_{0,2}\right]$ times the utility from the best alternative given state 1, $u_1[a_1]$, minus $c_1\mathbb{E}\left[\tau \mid p_{0,1}, p_{0,2}\right]$. This gives (2.15).

A.3 Optimal Strategy When Only One Source is Available.

Theorem 11 *When only source k, $k = 1, 2$, is available, for any initial beliefs (p_1, p_2), the optimal strategy is:*

- *if for some $i, j \in \{3-k, 3\}$, $i \neq j$, one of the following holds:*
  - $R_k^{(a_i)} < 1 \leq R_k^{(a_j)}$ *and* $R_k^{(a_i)} \leq p_k \leq \min\left\{\bar{\pi}_k^{(i,k)}\left[\frac{p_{3-k}}{1-p_k}\right], \bar{\pi}_k^{(i,j)}\left[\frac{p_{3-k}}{1-p_k}\right]\right\}$,
  - $R_k^{(a_i)} \leq R_k^{(a_j)} < 1$, $\bar{\pi}_k^{(i,k)}\left[\frac{p_{3-k}}{1-p_k}\right] \geq \bar{\pi}_k^{(j,k)}\left[\frac{p_{3-k}}{1-p_k}\right]$ *and* $R_k^{(a_i)} \leq p_k \leq \bar{\pi}_k^{(i,k)}\left[\frac{p_{3-k}}{1-p_k}\right]$,
  - $R_k^{(a_i)} \leq R_k^{(a_j)} < 1$, $\bar{\pi}_k^{(i,k)}\left[\frac{p_{3-k}}{1-p_k}\right] < \bar{\pi}_k^{(j,k)}\left[\frac{p_{3-k}}{1-p_k}\right]$ *and* $R_k^{(a_i)} \leq p_k \leq \min\left\{\bar{\pi}_k^{(i,k)}\left[\frac{p_{3-k}}{1-p_k}\right], \bar{\pi}_k^{(i,j)}\left[\frac{p_{3-k}}{1-p_k}\right]\right\}$,
  - $R_k^{(a_j)} < R_k^{(a_i)} < 1$, $\bar{\pi}_k^{(i,k)}\left[\frac{p_{3-k}}{1-p_k}\right] \geq \bar{\pi}_k^{(j,k)}\left[\frac{p_{3-k}}{1-p_k}\right]$ *and* $R_k^{(a_i)} \leq p_k \leq \bar{\pi}_k^{(i,k)}\left[\frac{p_{3-k}}{1-p_k}\right]$,

  then use source k until $p_k = R_k^{(a_i)}$, with the default action a_i

- *otherwise no learning is optimal and the default action is the one that maximizes the expected payoff:*

$$\max_{a\in\mathcal{A}}\{p_1u_1[a] + p_2u_2[a] + (1-p_1-p_2)u_3[a]\},$$

where $\bar{\pi}_k^{(i,j)}[q_{3-k}] \in \left(0, \min\{1, R_k^{(a_j)}\}\right)$*,* $i, j \in \{1,2,3\}$*, uniquely solves*

$$\frac{1-\frac{\bar{\pi}_k^{(i,j)}[q_{3-k}]}{R_k^{(a_j)}}}{1-\bar{\pi}_k^{(i,j)}[q_{3-k}]} + \log\left[\frac{\bar{\pi}_k^{(i,j)}[q_{3-k}]}{1-\bar{\pi}_k^{(i,j)}[q_{3-k}]}\right] =$$
$$\frac{\lambda_k\left(q_{3-k}\left(u_{3-k}[a_i]-u_{3-k}[a_j]\right)+(1-q_{3-k})\left(u_3[a_i]-u_3[a_j]\right)\right)}{c_k} + \log\left[\frac{R_k^{(a_i)}}{1-R_k^{(a_i)}}\right].$$

Along the curve $p_k = \bar{\pi}_k^{(i,j)}\left[\frac{p_{3-k}}{1-p_k}\right]$, the agent is indifferent between two strategies: (1) no learning, with default alternative a_j, and (2) using source k until $p_k = R_k^{(a_i)}$, with default alternative a_i.

Theorem 11 comes from maximizing $V_k^{(a)}\left[p_1, p_2; R_k^{(a)}\right]$ over $a \in \mathcal{A}$.

A.4 Proof of Lemma 2.

Consider any strategy (a^F, T, τ). Suppose it commands to stop the information collection process if a positive signal is observed (otherwise it is definitely not optimal). Then T is effectively deterministic (the switching strategy conditional on not observing a positive signal). Denote $\bar{\tau}$ the stopping time conditional on not receiving a positive signal. Denote $(\underline{p}_1[p], \underline{p}_2[p])$ the posterior belief at time $\bar{\tau}$ conditional on prior beliefs p and using strategy (a^F, T, τ) with no positive signal observed. Then the expected payoff from strategy (a^F, T, τ) is

$$V[p_1, p_2; (a^F, T, \tau)] \equiv \mathbb{E}\left[u_j[a^F] - \int_0^\tau c_1 dT_{t,1} + c_2 dT_{t,2} \mid p_0 = p\right]$$
$$= \mathbb{P}\left[p_{\tau,1} < 1,\ p_{\tau,2} < 1 \mid p_0 = p\right]\mathbb{E}\left[u_j[a^F] \mid p_1 = \underline{p}_1[p],\ p_2 = \underline{p}_1[p]\right]$$
$$+\sum_{k=1,2}\mathbb{P}\left[p_{\tau,k} = 1 \mid p_0 = p\right]\mathbb{E}\left[u_k[a^F] \mid p_k = 1\right] - \mathbb{E}\left[\int_0^\tau c_1 dT_{t,1} + c_2 dT_{t,2} \mid p_0 = p\right].$$

Since $\mathbb{E}\left[u_k[a^F] \mid p_k = 1\right] \le u_k[a_k]$ and

$$\mathbb{E}\left[u_j[a^F] \mid p_1 = \underline{p}_1[p],\ p_2 = \underline{p}_1[p]\right] \le \max_{a\in\mathcal{A}} \mathbb{E}\left[u_j[a] \mid p_1 = \underline{p}_1[p],\ p_2 = \underline{p}_1[p]\right],$$

for any (T, τ)

$$V[p_1, p_2; (a^F, T, \tau)] \le \max_{a\in\mathcal{A}}\left\{V[p_1, p_2; (a^F, T, \tau)] \colon a^F \text{ is defined by (2.16)}\right\}.$$

Thus,

$$\sup_{a^F} V[p_1, p_2; (a^F, T, \tau)] = \max_{a\in\mathcal{A}} \left\{V[p_1, p_2; (a^F, T, \tau)] \colon a^F \text{ is defined by (2.16)}\right\}.$$

Since $\mathcal{A}$ is finite, we have

$$\begin{aligned} V[p_1, p_2] &= \sup_{(a^F,T,\tau)} V[p_1, p_2; (a^F, T, \tau)] \\ &= \max_{a\in\mathcal{A}} \sup_{(T,\tau)} \left\{V[p_1, p_2; (a^F, T, \tau)] \colon a^F \text{ is defined by (2.16)}\right\} \\ &= \max_{a\in\mathcal{A}} V^{(a)}[p_1, p_2]. \end{aligned}$$

A.5 Proof of Lemma 3.

Consider strategy (a^F, T^*, τ^*). Let $\bar{\tau}^*$ be the maximum time the decision maker spends on learning ("give-up" time). Fix any $t' \in [0, \bar{\tau}^*]$. Let $(a^F, T^*_{-t'}, \tau^* - t')$ be the strategy that describes the agent's behavior starting from the moment t', assuming the agent follows (a^F, T^*, τ^*) and he has not observed a positive signal by time t'.

By definition of the expected payoff from a given strategy, for any $t' \in [0, \bar{\tau}^*]$,

$$\begin{aligned} &V[p_1, p_2; (a^F, T^*, \tau^*)] \\ &\quad= \mathbb{P}\left[\tau^* = \bar{\tau}^* \mid p_0 = p\right] \left(p_{\bar{\tau}^*,1} u_1[a] + p_{\bar{\tau}^*,2} u_2[a] + \left(1 - p_{\bar{\tau}^*,1} - p_{\bar{\tau}^*,2}\right) u_3[a]\right) \\ &\qquad+ \mathbb{P}\left[p_{\tau^*,1} = 1 \mid p_0 = p\right] u_1[a_1] + \mathbb{P}\left[p_{\tau^*,2} = 1 \mid p_0 = p\right] u_2[a_2] \\ &\qquad- \mathbb{E}\left[\int_0^{\tau^*} c_1 dT^*_{t,1} + c_2 dT^*_{t,2} \mid p_0 = p\right] \\ &\quad= \mathbb{P}\left[t' \le \tau^* \mid p_0 = p\right] \mathbb{P}\left[\tau^* = \bar{\tau}^* \mid p_0 = p,\ t' \le \tau^*\right] \times \\ &\qquad \left(p_{\bar{\tau}^*,1} u_1[a] + p_{\bar{\tau}^*,2} u_2[a] + \left(1 - p_{\bar{\tau}^*,1} - p_{\bar{\tau}^*,2}\right) u_3[a]\right) \\ &\qquad+ u_1[a_1] \mathbb{P}\left[t' \le \tau^* \mid p_0 = p\right] \mathbb{P}\left[p_{\tau^*,1} = 1 \mid p_0 = p,\ t' \le \tau^*\right] \\ &\qquad+ u_1[a_1] \mathbb{P}\left[t' > \tau^* \mid p_0 = p\right] \mathbb{P}\left[p_{\tau^*,1} = 1 \mid p_0 = p,\ t' > \tau^*\right] \\ &\qquad+ u_2[a_2] \mathbb{P}\left[t' \le \tau^* \mid p_0 = p\right] \mathbb{P}\left[p_{\tau^*,2} = 1 \mid p_0 = p,\ t' \le \tau^*\right] \\ &\qquad+ u_2[a_2] \mathbb{P}\left[t' > \tau^* \mid p_0 = p\right] \mathbb{P}\left[p_{\tau^*,2} = 1 \mid p_0 = p,\ t' > \tau^*\right] \\ &- \mathbb{P}\left[t' \le \tau^* \mid p_0 = p\right] \left(\int_0^{t'} c_1 dT^*_{t,1} + c_2 dT^*_{t,2} + \mathbb{E}\left[\int_{t'}^{\tau^*} c_1 dT^*_{t,1} + c_2 dT^*_{t,2} \mid p_0 = p,\ t' \le \tau^*\right]\right) \\ &\qquad- \mathbb{P}\left[t' > \tau^* \mid p_0 = p\right] \mathbb{E}\left[\int_0^{\tau^*} c_1 dT^*_{t,1} + c_2 dT^*_{t,2} \mid p_0 = p,\ t' > \tau^*\right] \end{aligned}$$

$$= \mathbb{P}[t' > \tau^* \mid p_0 = p] \times$$
$$\left(u_1[a_1]\mathbb{P}\left[p_{\tau^*,1} = 1 \mid p_0 = p,\ t' > \tau^*\right] + u_2[a_2]\mathbb{P}\left[p_{\tau^*,2} = 1 \mid p_0 = p,\ t' > \tau^*\right]\right)$$
$$-\mathbb{E}\left[\int_0^{\min\{\tau^*,t'\}} c_1 dT^*_{t,1} + c_2 dT^*_{t,2} \mid p_0 = p\right] + \mathbb{P}[t' \leq \tau^* \mid p_0 = p] V\left[p_{t',1}, p_{t',2}; (a^F, T^*_{-t'}, \tau^* - t')\right].$$

Thus, if (a^F, T^*, τ^*) is optimal for the initial beliefs (p_1, p_2), then $(a^F, T^*_{-t'}, \tau^* - t')$ must be optimal for the initial beliefs $(p_{t',1}, p_{t',2})$.

A.6 Proof of Theorem 3.

Lemma 5 is a generalization of Lemma 4. Now, a finite set of points $\mathcal{S}$ transforms into a curve $\mathcal{R}$.[4]

Lemma 5 *If function $V\colon \mathcal{P} \cup \{(1,0)\} \cup \{(0,1)\} \to \mathbb{R}$, where $\mathcal{P} = \{(p_1, p_2) \in [0,1)^2 \colon p_1 + p_2 \leq 1\}$, is such that there exists a curve $\mathcal{R} = \{(\pi_1(r), \pi_2(r)) \in \mathcal{P} \colon 0 \leq r \leq \bar{r}\}$ for some $\bar{r} \geq 0$ such that*

1. *$V[1,0] = u_1[a_1]$, $V[0,1] = u_2[a_2]$,*

2. *V is continuous everywhere on the belief triangle $\mathcal{P}$; moreover, it is continuously differentiable everywhere except on the curve $\mathcal{R}$,*

3. *for all $p \in \mathcal{P} \setminus \mathcal{R}$,*

$$\min\left\{\mathcal{L}_1[p_1, p_2], \mathcal{L}_2[p_1, p_2], V[p_1, p_2] - U^{(a)}[p_1, p_2]\right\} = 0, \tag{A.16}$$

where

$$U^{(a)}[p_1, p_2] = \begin{cases} u_1[a]p_1 + u_2[a]p_2 + u_3[a](1 - p_1 - p_2), & p_1 < 1,\ p_2 < 1, \\ u_1[a_1], & p_1 = 1, \\ u_2[a_2], & p_2 = 1, \end{cases} \tag{A.17}$$

$$\mathcal{L}_1[p_1, p_2] = \frac{c_1}{\lambda_1 p_1} + \frac{\partial V[p_1, p_2]}{\partial p_1}(1 - p_1) - \frac{\partial V[p_1, p_2]}{\partial p_2} p_2 - (u_1[a_1] - V[p_1, p_2]), \tag{A.18}$$

$$\mathcal{L}_2[p_1, p_2] = \frac{c_2}{\lambda_2 p_2} - \frac{\partial V[p_1, p_2]}{\partial p_1} p_1 + \frac{\partial V[p_1, p_2]}{\partial p_2}(1 - p_2) - (u_2[a_2] - V[p_1, p_2]), \tag{A.19}$$

[4] The proof is an adaptation of the ideas of Theorem 7.1, Chapter IV in Fleming and Rishel (2012).

4. *there exists an a-type strategy* (a^F, T^*, τ^*) *such that*

    a) *it is Markovian,*

    b) *the stopping time* τ^* *is almost surely finite and whenever the state is revealed (i.e. the belief process jumps to* $(1,0)$ *or* $(0,1)$*), the agent stops the information collection process,*

    c) *any trajectory of beliefs* (a^F, T^*, τ^*) *can generate (for any initial beliefs and any realization of signals) does not go* along *the curve* $\mathcal{R}$ *at any moment of time (that is, it does not spend any nontrivial time on* $\mathcal{R}$*), unless only one source is used,*

    d) *the following conditions hold:*

$$
\begin{aligned}
dT^*_{t,1} > 0,\ t < \tau^* &\quad\Rightarrow\quad \mathcal{L}_1[p_{t,1}, p_{t,2}] = 0, && \text{(A.20)}\\
dT^*_{t,2} > 0,\ t < \tau^* &\quad\Rightarrow\quad \mathcal{L}_2[p_{t,1}, p_{t,2}] = 0, && \text{(A.21)}\\
V[p_{\tau^*,1}, p_{\tau^*,2}] &\quad=\quad U^{(a)}[p_{\tau^*,1}, p_{\tau^*,2}], && \text{(A.22)}
\end{aligned}
$$

then for any initial beliefs (p_1, p_2)*,*

1. $V[p_1, p_2] = V^{(a)}[p_1, p_2]$*,*

2. (a^F, T^*, τ^*) *is the optimal a-type strategy.*

Proof of Lemma 5 Using the Bayes formula, we have

$$
\begin{aligned}
dp_{t,1} &= -\lambda_1 p_{t,1}(1-p_{t,1})dT_{t,1} + \lambda_2 p_{t,1}p_{t,2}dT_{t,2} + (1-p_{t,1})dX^{(1)}_{T_{t,1}} - p_{t,1}dX^{(2)}_{T_{t,2}},\\
dp_{t,2} &= \lambda_1 p_{t,1}p_{t,2}dT_{t,1} - \lambda_2 p_{t,2}(1-p_{t,2})dT_{t,2} - p_{t,2}dX^{(1)}_{T_{t,1}} + (1-p_{t,2})dX^{(2)}_{T_{t,2}}.
\end{aligned}
$$

Take any a-type strategy (a^F, T, τ) and any initial beliefs $p_0 \in \mathcal{P}$. Let $\{(p_{t,1}, p_{t,2})\}_{t\geq 0}$ be the realization of the belief process, given that the agent follows strategy (a^F, T, τ) and has initial beliefs p_0. Suppose that after observing a positive signal, the agent stops the information collection process: if $t < \tau$, then no positive signals have been observed so far. Let $\tau' \in [0, \tau]$ be any time such that $V \in C^1$ along the belief trajectory so far. The last assumption about the smoothness of V means that

- if only source k is used, V is continuously differentiable along the line $\frac{p_{3-k}}{1-p_k}$, that is, along the line the belief vector is moving,

- if both sources are used, V is continuously differentiable along any direction, that is, both partial derivatives, $\frac{\partial V[p_1,p_2]}{\partial p_1}$ and $\frac{\partial V[p_1,p_2]}{\partial p_2}$, exist and are continuous.

Then Ito's formula gives (García and Griego (1994)):

$$V[p_{\tau',1},p_{\tau',2}] = V[p_{0,1},p_{0,2}]+\lambda_1 \int_0^{\tau'} p_{t,1}\left(\frac{\partial V[p_{t,1},p_{t,2}]}{\partial p_2}p_{t,2} - \frac{\partial V[p_{t,1},p_{t,2}]}{\partial p_1}(1-p_{t,1})\right) dT_{t,1}$$
$$+ \lambda_2 \int_0^{\tau'} p_{t,2}\left(\frac{\partial V[p_{t,1},p_{t,2}]}{\partial p_1}p_{t,1} - \frac{\partial V[p_{t,1},p_{t,2}]}{\partial p_2}(1-p_{t,2})\right) dT_{t,2}$$
$$+ \int_0^{\tau'} \left(u_1[a_1] - V[p_{t,1},p_{t,2}]\right) dX^{(1)}_{T_{t,1}} + \int_0^{\tau'} \left(u_2[a_2] - V[p_{t,1},p_{t,2}]\right) dX^{(2)}_{T_{t,2}}. \quad \text{(A.23)}$$

Let $\tau' = \sup\left\{t \leq \tau :\ V \in C^1 \text{ along } \{p_{t'}\}_{0\leq t' \leq t}\right\}$. If the information collection process always stops eventually (that is, $\mathbb{P}\left[\tau <= \infty \mid p_0\right] = 1$), then τ' is always finite and therefore $V[p_{\tau',1},p_{\tau',2}]$ is well-defined. Taking conditional expectation in (A.23), we have: if $\mathbb{P}\left[\tau < +\infty \mid p_0\right] = 1$, then

$$V[p_{0,1},p_{0,2}] = \mathbb{E}\left[V[p_{\tau',1},p_{\tau',2}] \mid p_0\right]$$
$$- \lambda_1 \mathbb{E}\left[\int_0^{\tau'} p_{t,1}\left(\frac{\partial V[p_{t,1},p_{t,2}]}{\partial p_2}p_{t,2} - \frac{\partial V[p_{t,1},p_{t,2}]}{\partial p_1}(1-p_{t,1})\right) dT_{t,1} \mid p_0\right]$$
$$- \lambda_2 \mathbb{E}\left[\int_0^{\tau'} p_{t,2}\left(\frac{\partial V[p_{t,1},p_{t,2}]}{\partial p_1}p_{t,1} - \frac{\partial V[p_{t,1},p_{t,2}]}{\partial p_2}(1-p_{t,2})\right) dT_{t,2} \mid p_0\right]$$
$$- \lambda_1 \mathbb{E}\left[\int_0^{\tau'} p_{t,1}\left(u_1[a_1] - V[p_{t,1},p_{t,2}]\right) dT_{t,1} \mid p_0\right]$$
$$- \lambda_2 \mathbb{E}\left[\int_0^{\tau'} p_{t,2}\left(u_2[a_2] - V[p_{t,1},p_{t,2}]\right) dT_{t,2} \mid p_0\right],$$

or equivalently,

$$
\begin{aligned}
V[p_{0,1}, p_{0,2}] = \mathbb{E}\left[V[p_{\tau',1}, p_{\tau',2}] \mid p_0\right] \\
+ \lambda_1 \mathbb{E}\left[\int_0^{\tau'} p_{t,1}\mathcal{L}_1[p_{t,1}, p_{t,2}]dT_{t,1} \mid p_0\right] - \mathbb{E}\left[\int_0^{\tau'} c_1 dT_{t,1} \mid p_0\right] \\
+ \lambda_2 \mathbb{E}\left[\int_0^{\tau'} p_{t,2}\mathcal{L}_2[p_{t,1}, p_{t,2}]dT_{t,2} \mid p_0\right] - \mathbb{E}\left[\int_0^{\tau'} c_2 dT_{t,2} \mid p_0\right],
\end{aligned} \tag{A.24}
$$

where $\tau' = \sup\{t \leq \tau \colon V \in C^1$ along the belief trajectory before $t\}$.

Suppose $\tau' < \tau$ is the time when the belief trajectory touches $\mathcal{R}$, but then crosses it, that is $p_{\tau'} \in \mathcal{R}$, $\exists\, \delta > 0$ such that $p_{\tau'+t} \notin \mathcal{R}$ for $t \in (0, \delta)$. Then (A.24) holds for $p_{\tau'}$ as the initial beliefs (recall that V is continuous). Thus,

$$
\begin{aligned}
V[p_{\tau',1}, p_{\tau',2}] = \mathbb{E}\left[V[p_{\tau'',1}, p_{\tau'',2}] \mid p_{\tau'}\right] + \lambda_1 \mathbb{E}\left[\int_{\tau'}^{\tau''} p_{t,1}\mathcal{L}_1[p_{t,1}, p_{t,2}]dT_{t,1} \mid p_{\tau'}\right] - \mathbb{E}\left[\int_{\tau'}^{\tau''} c_1 dT_{t,1} \mid p_{\tau'}\right] \\
+ \lambda_2 \mathbb{E}\left[\int_{\tau'}^{\tau''} p_{t,2}\mathcal{L}_2[p_{t,1}, p_{t,2}]dT_{t,2} \mid p_{\tau'}\right] - \mathbb{E}\left[\int_{\tau'}^{\tau''} c_2 dT_{t,2} \mid p_{\tau'}\right],
\end{aligned} \tag{A.25}
$$

where $\tau'' = \sup\{t \in (\tau', \tau] \colon V \in C^1$ along the belief trajectory on $(\tau', t)\}$. Combining (A.24) and (A.25), I get that (A.24) holds for $\tau' = \sup\{t \leq \tau \colon$ the belief trajectory never goes along the curve $\mathcal{R}$ on $(0, t)$, unless only one source is used$\}$.

(A.16) and (A.24) give

$$
V[p_{0,1}, p_{0,2}] \geq \mathbb{E}\left[V[p_{\tau',1}, p_{\tau',2}] - \int_0^{\tau'} c_1 dT_{t,1} + c_2 dT_{t,2} \mid p_0\right], \tag{A.26}
$$

where $\tau' = \sup\{t \leq \tau \colon \substack{\text{the belief trajectory never goes along the curve } \mathcal{R} \text{ on } (0,t),\\ \text{unless only one source is used}}\}$.

Claim 2 *If* $\mathbb{P}\left[\tau < +\infty \mid p_0\right] = 1$ *and the information collection stops once a positive signal is observed, then*

$$
V[p_{0,1}, p_{0,2}] \geq \mathbb{E}\left[V[p_{\tau,1}, p_{\tau,2}] - \int_0^{\tau} c_1 dT_{t,1} + c_2 dT_{t,2} \mid p_0\right]. \tag{A.27}
$$

Proof: Suppose $\tau' < \tau$ is the time when the belief trajectory touches $\mathcal{R}$ and goes along it for some time: $\exists\, \delta > 0$ such that $p_{\tau'+t} \in \mathcal{R}$ for $t \in [0, \min\{\delta, \tau - \tau'\}]$. It is sufficient to show that

$$V[p_{\tau',1}, p_{\tau',2}] \geq \mathbb{E}\left[V[p_{\min\{\tau'+\delta,\tau\},1}, p_{\min\{\tau'+\delta,\tau\},2}] - \int\limits_{\tau'}^{\min\{\tau'+\delta,\tau\}} c_1 dT_{t,1} + c_2 dT_{t,2} \mid p_{\tau'}\right]. \tag{A.28}$$

Denote $\mathcal{R}[p_{\tau'}, \delta]$ the belief trajectory $p_{\tau'+t}$, $t \in [0, \min\{\delta, \tau - \tau'\}]$. WLOG[5], assume that $\mathcal{R}[p_{\tau'}, \delta]$ can be represented as a function in spheric coordinates: $\rho(\phi)$, $0 \leq \phi \leq \frac{\pi}{4}$ is an angle, $\rho > 0$ is a radius, so that $\pi_1^2 + (1 - \pi_2)^2 = \rho^2$ and $\frac{\pi_1}{1-\pi_2} = \tan[\phi]$. Consider an infinite sequence of strategies $(a^F, T^{(m)}, \tau^{(m)})$, $m = 1, 2, 3, \ldots$, such that strategy $(a^F, T^{(m)}, \tau^{(m)})$ deviates from strategy (a^F, T, τ) in a way that at the moment τ' it chooses only source 2 for $\Delta_m > 0$ amount of time:

- $dT^{(m)}_{\tau'+t,2} = dt$ for $t \in [0, \Delta_m]$ and $dT^{(m)}_{\tau'+\Delta_m+t,k} = dT_{\tau'+t,k}$, $k = 1, 2$,
- $\tau^{(m)} = \begin{cases} \tau' + \Delta_m, & \text{if source 2 produces a positive signal during } t \in [\tau', \tau' + \Delta_m], \\ \tau + \Delta_m, & \text{otherwise.} \end{cases}$

Then it is easy to see that for Δ_m small enough, the belief trajectory of strategy $(a^F, T^{(m)}, \tau^{(m)})$ will not go along $\mathcal{R}$ for $t \in [\tau', \min\{\tau, \tau' + \delta\} + \Delta_m]$. Indeed, conditional on no positive signals, we have: $p^{(m)}_{\tau'+\Delta_m+t,1} = \frac{p_{\tau'+t,1}}{e^{-\lambda_2 \Delta_m} p_{\tau'+t,2} + 1 - p_{\tau'+t,2}}$ and $p^{(m)}_{\tau'+\Delta_m+t,2} = \frac{e^{-\lambda_2 \Delta_m} p_{\tau'+t,2}}{e^{-\lambda_2 \Delta_m} p_{\tau'+t,2} + 1 - p_{\tau'+t,2}}$. Graphically, it means the belief trajectory shifts along the rays $\frac{p_1}{1-p_2} = \text{const}$, and the shift becomes smaller as we decrease Δ_m. Since $\mathcal{R}[p_{\tau'}, \delta]$ can be represented as a function $\rho(\phi)$, $\pi_1^2 + (1 - \pi_2)^2 = \rho^2$ and $\frac{\pi_1}{1-\pi_2} = \tan[\phi]$, the new curve that the belief process of strategy $(a^F, T^{(m)}, \tau^{(m)})$ forms will not coincide with $\mathcal{R}[p_{\tau'}, \delta]$ at any point.

Thus, we can use (A.26):

$$\begin{aligned} &V[p_{\tau',1}, p_{\tau',2}] \geq \mathbb{P}\left[\tau^{(m)} = \tau' + \Delta_m \mid p_{\tau'}\right] \times \\ &\mathbb{E}\left[V[p^{(m)}_{\tau^{(m)},1}, p^{(k)}_{\tau^{(m)},2}] - c_2 \Delta_m \mid \tau^{(m)} = \tau' + \Delta_m,\ p_{\tau'}\right] + \mathbb{P}\left[\tau^{(m)} > \tau' + \Delta_m \mid p_{\tau'}\right] \times \\ &\mathbb{E}\left[V[p^{(m)}_{\min\{\tau'+\delta,\tau\}+\Delta_m,1}, p^{(m)}_{\min\{\tau'+\delta,\tau\}+\Delta_m,2}] - c_2 \Delta_m - \int\limits_{\tau'}^{\min\{\tau'+\delta,\tau\}} c_1 dT_{t,1} + c_2 dT_{t,2} \mid \tau^{(m)} > \tau' + \Delta_m,\ p_{\tau'}\right]. \end{aligned} \tag{A.29}$$

[5] If not, we can always either choose it to be $\pi_2^2 + (1 - \pi_1)^2 = \rho^2$ and $\frac{\pi_2}{1-\pi_1} = \tan[\phi]$ or take a smaller δ.

As $\Delta_m \to 0$ with $m \to +\infty$, the right hand side of (A.29) converges to the right hand side of (A.28). Thus, (A.28) is true.

■

(A.16) and (A.27) together give

$$V[p_{0,1}, p_{0,2}] \geq \mathbb{E}\left[U^{(a)}[p_{\tau,1}, p_{\tau,2}] - \int_0^\tau c_1 dT_{t,1} + c_2 dT_{t,2} \mid p_0\right] \tag{A.30}$$

for any initial belief $p_0 \in \mathcal{P}$ and any strategy (a^F, T, τ) such that $\mathbb{P}[\tau < +\infty \mid p_0] = 1$ and the information collection stops once a positive signal is observed.

By definition, the right hand side of (A.30) is the expected payoff from the strategy (a^F, T, τ):

$$\mathbb{E}\left[U^{(a)}[p_{\tau,1}, p_{\tau,2}] - \int_0^\tau c_1 dT_{t,1} + c_2 dT_{t,2} \mid p_0\right] = V^{(a)}[p_{0,1}, p_{0,2}; (T, \tau)].$$

Thus, $V[p_{0,1}, p_{0,2}] \geq V^{(a)}[p_{0,1}, p_{0,2}; (T, \tau)]$.

Since any strategy with $\mathbb{P}[\tau < +\infty \mid p_0] < 1$ has $V^{(a)}[p_{0,1}, p_{0,2}; (T, \tau)] = -\infty$ and continuing information collection after the state is revealed only decreases the payoff, we have $V[p_{0,1}, p_{0,2}] \geq V^{(a)}[p_{0,1}, p_{0,2}; (T, \tau)]$ for all strategies (a^F, T, τ).

The proof concludes by the observation that $V[p_{0,1}, p_{0,2}] = V^{(a)}[p_{0,1}, p_{0,2}; (T^*, \tau^*)]$ by definition of the strategy (a^F, T^*, τ^*) and using (A.24).

■

The next step is to guess the optimal a-type strategy (a^F, T^*, τ^*) and check the guess by showing that all conditions listed in Lemma 5 are satisfied by the function equal to the expected payoff from (a^F, T^*, τ^*).

Theorem 3 provides the optimal a-type strategy (a^F, T^*, τ^*) in explicit form. $\bar{\pi}^{(a)}_{3-k}\left[\frac{p_k}{1-p_{3-k}}\right]$ is defined as follows:

$$\bar{\pi}^{(a)}_{3-k}[q_k] = \begin{cases} 1, \quad \text{if } R^{(a)}_{3-k} < 1,\ B^{(a)}_k\left[R^{(a)}_{3-k}, 1\right] < \frac{1-R^{(a)}_{3-k}}{R^{(a)}_{3-k}} \text{ and } q_k \geq q^{**(a)}_k, \\ \in \left[\bar{\bar{p}}^{(a)}_{3-k}[q_k], \min\left\{1, R^{(a)}_{3-k}\right\}\right] : \quad B^{(a)}_k\left[\bar{\pi}^{(a)}_{3-k}[q_k], q_k\right] = \frac{1-R^{(a)}_{3-k}}{R^{(a)}_{3-k}}, \quad \text{otherwise.} \end{cases}$$

$q_k^{**(a)}$ solves $B_k^{(a)}\left[R_{3-k}^{(a)}, q_k^{**(a)}\right] = \frac{1-R_{3-k}^{(a)}}{R_{3-k}^{(a)}}$ and

$$
\begin{aligned}
B_k^{(a)}[p_{3-k}, q_k] = {} & \left(\frac{p_k}{1-p_k}\frac{1}{q_k} + \frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\mathbf{1}\left(p_k > R_k^{(a)}\right)\right)\log\left[\frac{R_k^{(a)}(1-p_k)}{p_k\left(1-R_k^{(a)}\right)}\right] \\
& + \frac{c_k\lambda_{3-k}\left(p_k - R_k^{(a)}\right)\left(q_{3-k} - \mathbf{1}\left(p_k \le R_k^{(a)}\right)\right)}{c_{3-k}\lambda_k R_k^{(a)}(1-p_k)} \\
& + \frac{p_k}{1-p_k}\left(2 - \frac{c_k\lambda_{3-k}\left(1-R_k^{(a)}\right)q_{3-k}}{c_{3-k}\lambda_k R_k^{(a)}} + \frac{1}{q_k}\log\left[\frac{c_k\lambda_{3-k}\left(1-R_k^{(a)}\right)q_{3-k}}{c_{3-k}\lambda_k R_k^{(a)}}\right]\right) \\
& - \frac{p_k}{1-p_k}\left(1 - \frac{c_k\lambda_{3-k}\underline{\tilde{p}}_{3-k}^{(a)}[q_k]}{c_{3-k}\lambda_k\bar{p}_k^{(a)}\left[\underline{\tilde{p}}_{3-k}^{(a)}[q_k]\right]} + \frac{1}{q_k}\log\left[\frac{c_k\lambda_{3-k}\underline{\tilde{p}}_{3-k}^{(a)}[q_k]}{c_{3-k}\lambda_k\bar{p}_k^{(a)}\left[\underline{\tilde{p}}_{3-k}^{(a)}[q_k]\right]}\right]\right)\mathbf{1}\left(q_k \le q_k^*\right) \\
& + (1-q_{3-k})\left(\left(\left(1 + \frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\right)\log\left[\frac{(1-q_k^*)q_k}{(1-q_k)q_k^*}\right] + \frac{1}{p_{3-k}^{*(a)}}\right)\mathbf{1}\left(q_k > q_k^*\right) + \right. \\
& \left.\left.\frac{1}{\underline{\tilde{p}}_{3-k}^{(a)}[q_k]}\mathbf{1}\left(q_k \le q_k^*\right)\right)\right|_{\substack{p_k=(1-p_{3-k})q_k, \\ q_{3-k}=\frac{p_{3-k}}{1-(1-p_{3-k})q_k}, \\ q_k^*=\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}}} .
\end{aligned}
$$

The expected payoff function that corresponds to the optimal a-type strategy is

Area 1 : $V[p_1, p_2] = U^{(a)}[p_1, p_2]$;

Area 2.k : $V[p_1, p_2] = V_k^{(a)}\left[p_1, p_2; R_k^{(a)}\right]$ defined in Lemma 1;

Area 3.k.1 : $V[p_1, p_2] = V_k^{(a)}\left[p_1, p_2; R_k^{(a)}\right] + \frac{c_k}{\lambda_k}(1-p_k)\Delta_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$; for beliefs in this area, it is always true that $\frac{p_{3-k}}{1-p_k} \ge \frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}\mathbf{1}\left(\frac{\lambda_k R_k^{(a)}}{c_k} \le \frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}}\right) + \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}\mathbf{1}\left(\frac{\lambda_k R_k^{(a)}}{c_k} > \frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}}\right)$; here, $\Delta_k^{(a)}[q_{3-k}]$ is defined as

$$\Delta_k^{(a)}[q_{3-k}] = \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\left(\frac{1}{R_{3-k}^{(a)}} - 1\right) + \log\left[\frac{c_k\lambda_{3-k}\left(1 - R_k^{(a)}\right)q_{3-k}}{c_{3-k}\lambda_k R_k^{(a)}}\right] - q_{3-k}$$

$$-(1-q_{3-k})\times\left(\begin{cases}\left(1 + \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\right)\log\left[\frac{\frac{1}{p_{3-k}^{*(a)}} - \frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k} - 1}{\frac{1}{q_{3-k}} - 1}\right] + \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}p_{3-k}^{*(a)}}, & \frac{\lambda_k R_k^{(a)}}{c_k} \leq \frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \\ \left(1 + \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\right)\log\left[\frac{\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\left(\frac{1}{p_k^{*(a)}} - 1\right) - 1}{\frac{1}{q_{3-k}} - 1}\right] + \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}R_{3-k}^{(a)}} - \frac{1}{R_k^{(a)}} + \frac{1}{p_k^{*(a)}}, & \frac{\lambda_k R_k^{(a)}}{c_k} \geq \frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}}\end{cases}\right);$$

Area 3.k.2 : $V[p_1, p_2] = V_k^{(a)}\left[p_1, p_2; R_k^{(a)}\right] + \frac{c_k}{\lambda_k}(1 - p_k)\Delta_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$; this area can only appear if $\frac{\lambda_k R_k^{(a)}}{c_k} \geq \frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}}$; for beliefs in this area, it is always true that $\frac{p_{3-k}}{1-p_k} \leq \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}$; here, $\Delta_k^{(a)}[q_{3-k}]$ is defined as

$$\Delta_k^{(a)}[q_{3-k}] = \frac{c_{3-k}\lambda_k q_{3-k}}{c_k\lambda_{3-k}}\left(\frac{1}{R_{3-k}^{(a)}} - \frac{1}{\bar{p}_{3-k}^{(a)}\left[\tilde{\underline{p}}_k[q_{3-k}]\right]}\right) + (1-q_{3-k})\left(\frac{1}{R_k^{(a)}} - \frac{1}{\tilde{\underline{p}}_k[q_{3-k}]}\right)$$
$$+ \log\left[\frac{\left(1 - R_k^{(a)}\right)\tilde{\underline{p}}_k[q_{3-k}]}{R_k^{(a)}\left(1 - \tilde{\underline{p}}_k[q_{3-k}]\right)}\right].$$

Note that $\Delta_k^{(a)}[1]$ is equal to $\Delta_k^{(a)}$ in Theorem 1.

Although the description of the optimal a-type strategy in Theorem 3 is complete, it is not very convenient for the purpose of the proof. I start the proof with more detailed description of the optimal a-type strategy.

The way to present the optimal a-type strategy is on the belief triangle $\mathcal{P} \cup \{(1, 0)\} \cup \{(0, 1)\}$. Each point corresponds to the best action: use source 1, use source 2, or stop. For illustration, black points correspond to source 1, gray points refer to source 2, and white points imply stopping. There are several cases possible, depending on the parameters of the model.

Case 1: Information from both sources is too costly. The whole belief triangle is white, that is, no matter what the beliefs are, the optimal a-type strategy is always stop the information collection process and get utility $U^{(a)}[p_1, p_2]$.

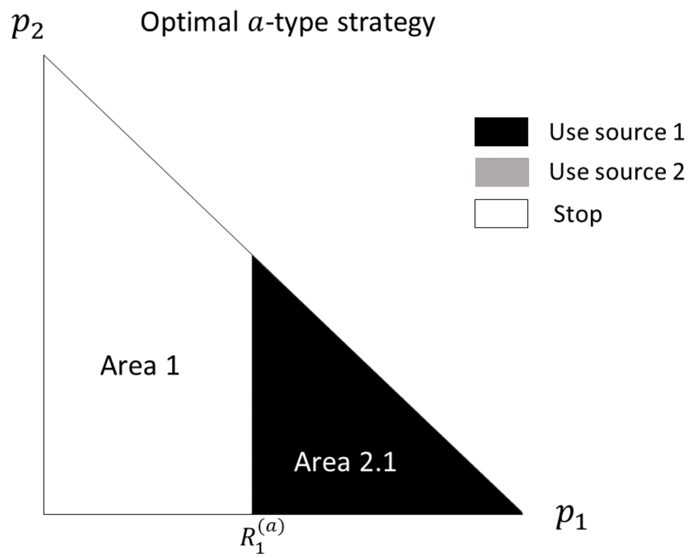

Figure A.1: Optimal a-type strategy for $R_1^{(a)} = 0.4$, $R_2^{(a)} = 10$, $\frac{c_1}{\lambda_1} = 0.1$, $\frac{c_2}{\lambda_2} = 10$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). White region means it is optimal to stop. Black region means the first information source should be used. Gray region means the second information source should be used. Case 2: $R_k^{(a)} < 1$, $R_{3-k}^{(a)} \geq 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} < \frac{\lambda_k R_k^{(a)}}{c_k}$, $k = 1$.

This is optimal when the marginal cost of getting information (of any type) is too high compared to the marginal benefit from this information. Formally, the condition is $R_1^{(a)} \geq 1$ and $R_2^{(a)} \geq 1$.

Case 2: Only one source might be used according to the optimal a-type strategy. Suppose $R_k^{(a)} < 1$ and $R_{3-k}^{(a)} \geq 1$ for some $k = 1, 2$. That means the cost of getting information of type k is less than the benefit from this information, as long as state k is likely enough ($p_k > R_k^{(a)}$). Suppose also that the benefit of getting alternative a_{3-k} correct is greater than getting alternative a_k correct. Formally, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} < \frac{\lambda_k R_k^{(a)}}{c_k}$, which is equivalent to $u_k[a_k] - u_k[a] < u_{3-k}[a_{3-k}] - u_{3-k}[a]$. Then the optimal a-type strategy is to use source k if the current belief about state k is greater than the cost-benefit ratio of source k, that is $p_k > R_k^{(a)}$, and to choose the default alternative a right away otherwise. See Figure A.1.

Denote Area 1 the area where it is optimal to stop right away. Denote Area 2.k the area where source k is used until $p_k = R_k^{(a)}$ (and then stop).

Case 3: Both sources might be used but only separately, that is, no switching can be optimal. Suppose it is not Case 1 or Case 2. That means either $R_k^{(a)} < 1$, $R_{3-k}^{(a)} \geq 1$ and

$\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$ for some k, or $R_1^{(a)} < 1$ and $R_2^{(a)} < 1$. Equivalently, there exists k such that $R_k^{(a)} < 1$ and $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$.

Case 3 corresponds to the optimal a-type strategy such that if $p_j > R_j^{(a)}$, then source j is used. Otherwise, it is optimal to stop. See Figure A.2.

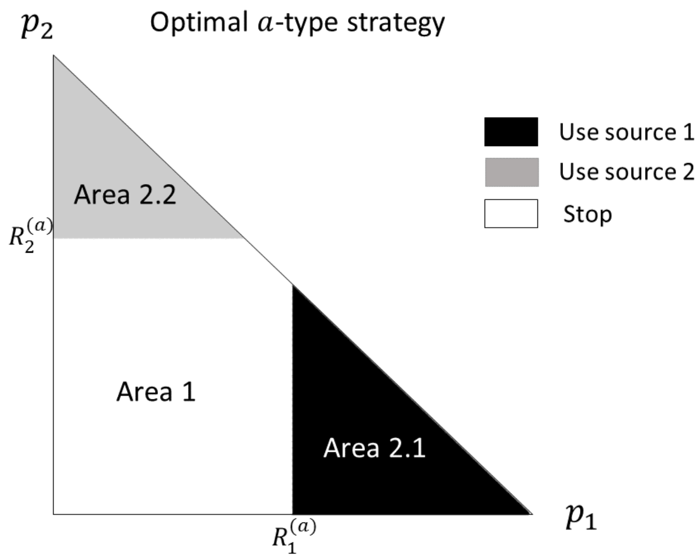

Figure A.2: Optimal a-type strategy for $R_1^{(a)} = 0.5$, $R_2^{(a)} = 0.6$, $\frac{c_1}{\lambda_1} = 1.5$, $\frac{c_2}{\lambda_2} = 1$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). White region means it is optimal to stop. Black region means the first information source should be used. Gray region means the second information source should be used. Case 3: $R_k^{(a)} < 1$, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] \geq 1$, $k = 1$.

It is easy to see that we need an additional condition here. Indeed, if $R_1^{(a)} + R_2^{(a)} < 1$ (the case that has not been ruled out so far), then the above description of the strategy has a contradiction: what source shall we use when both beliefs are above its thresholds, that is $p_1 > R_1^{(a)}$ and $p_2 > R_2^{(a)}$? This additional condition is $\Delta_k^{(a)}[1] \leq 0$. This conditional is actually even stronger than $R_1^{(a)} + R_2^{(a)} \geq 1$. To explain where this condition came from, I need to introduce function $\bar{p}_k^{(a)}[p_{3-k}]$ and point $p_{3-k}^{*(a)}$.

Properties of $\bar{p}_k^{(a)}[p_{3-k}]$ and $p_{3-k}^{*(a)}$.

For $k = 1, 2$, define $\bar{p}_k^{(a)}[p_{3-k}]$ by (2.17). As I explained in Section 2.5, the curve $p_k = \bar{p}_k^{(a)}[p_{3-k}]$ is an indifference curve between two sources.

For $k = 1, 2$, when $R_k^{(a)} < 1$ and $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, define $p_{3-k}^{*(a)}$ by (2.18). As I explained in Section 2.5, point $p_{3-k} = p_{3-k}^{*(a)}$, $p_k = \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]$ is such that at this point the agent is indifferent between simultaneous use of two sources and using source k from now on.

The solution to (2.18) always exists and is unique. Indeed, function $\bar{p}_k^{(a)}[p_{3-k}]$ is decreasing, $\bar{p}_k^{(a)}[0] = 1$ and $\bar{p}_k^{(a)}\left[R_{3-k}^{(a)}\right] = R_k^{(a)}$:

Lemma 6 *For $k = 1, 2$, when $R_k^{(a)} < 1$, function $\bar{p}_k^{(a)}[p_{3-k}]$ is strictly decreasing for $p_{3-k} \in (0, +\infty)$ from 1.*

<u>*Proof:*</u>

$$\bar{p}_k^{(a)\prime}[p_{3-k}] = -\frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}} \frac{R_k^{(a)}(1-R_k^{(a)})e^{\frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\left(\frac{1}{R_{3-k}^{(a)}}-\frac{1}{p_{3-k}}\right)}}{p_{3-k}^2\left(R_k^{(a)}+(1-R_k^{(a)})e^{\frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\left(\frac{1}{R_{3-k}^{(a)}}-\frac{1}{p_{3-k}}\right)}\right)^2} < 0.$$

■

Intuitively, the higher p_{3-k}, the more valuable source $3-k$ is, the lower the indirect benefit of using this source should be to satisfy condition (2.21) of cost equal to the total benefit. Condition $\bar{p}_k^{(a)}\left[R_{3-k}^{(a)}\right] = R_k^{(a)}$ also has a natural meaning: when $p_{3-k} = R_{3-k}^{(a)}$, the direct benefit is equal to the cost, which pushes the indirect benefit to zero.

Condition (2.9) is equivalent to the requirement that when $R_k^{(a)} < 1$ and $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, the curve $p_k = \bar{p}_k^{(a)}[p_{3-k}]$ lies outside of the belief triangle. To see it, first note that

Lemma 7 *For $k = 1, 2$, when $R_k^{(a)} < 1$ and $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, condition $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] \geq 1$ is equivalent to $\Delta_k^{(a)}[1] \leq 0$.*

<u>*Proof:*</u> Using (2.21) and (2.18), we can rewrite condition $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] \geq 1$ as follows: $p_1 + p_2 \geq 1$ where $p_1 \in (0,1)$ and $p_2 \in (0,1)$ uniquely solve

$$\begin{cases} c_{3-k} = \lambda_{3-k} p_{3-k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] + \frac{c_k}{\lambda_k}\log\left[\frac{p_k\left(1-R_k^{(a)}\right)}{R_k^{(a)}(1-p_k)}\right]\right), \\ \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}} = \frac{\lambda_k p_k}{c_k}, \quad p_{3-k} \leq R_{3-k}^{(a)}. \end{cases}$$

Substituting p_{3-k} from the last equation, we get: $p^{*(a)}_{3-k} + \bar{p}^{(a)}_k \left[p^{*(a)}_{3-k}\right] \geq 1$ is equivalent to

$$p_k + \frac{c_{3-k}\lambda_k p_k}{c_k \lambda_{3-k}} \geq 1,$$

where p_k solves

$$c_k = \lambda_k p_k \left(u_{3-k}[a_{3-k}] - u_{3-k}[a] + \frac{c_k}{\lambda_k} \log\left[\frac{p_k\left(1-R^{(a)}_k\right)}{R^{(a)}_k(1-p_k)}\right]\right), \quad \frac{c_{3-k}\lambda_k p_k}{c_k\lambda_{3-k}} \leq R^{(a)}_{3-k}.$$

Inequality $p_k + \frac{c_{3-k}\lambda_k p_k}{c_k\lambda_{3-k}} \geq 1$ is equivalent to

$$\frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}} \geq \frac{1-p_k}{p_k}.$$

Equation $c_k = \lambda_k p_k \left(u_{3-k}[a_{3-k}] - u_{3-k}[a] + \frac{c_k}{\lambda_k} \log\left[\frac{p_k\left(1-R^{(a)}_k\right)}{R^{(a)}_k(1-p_k)}\right]\right)$ is equivalent to

$$\frac{1-p_k}{p_k} + \log\left[\frac{1-p_k}{p_k}\right] = \frac{\lambda_k\left(u_{3-k}[a_{3-k}] - u_{3-k}[a]\right)}{c_k} - 1 + \log\left[\frac{\left(1-R^{(a)}_k\right)}{R^{(a)}_k}\right].$$

Inequality $\frac{c_{3-k}\lambda_k p_k}{c_k\lambda_{3-k}} \leq R^{(a)}_{3-k}$ is equivalent to

$$\frac{\lambda_k\left(u_{3-k}[a_{3-k}] - u_{3-k}[a]\right)}{c_k} - 1 \leq \frac{1-p_k}{p_k}.$$

Note that function $x + \log[x]$ is an increasing function of x. Thus, $p^{*(a)}_{3-k} + \bar{p}^{(a)}_k \left[p^{*(a)}_{3-k}\right] \geq 1$ is equivalent to

$$\frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}} + \log\left[\frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\right] \geq \frac{\lambda_k\left(u_{3-k}[a_{3-k}] - u_{3-k}[a]\right)}{c_k} - 1 + \log\left[\frac{\left(1-R^{(a)}_k\right)}{R^{(a)}_k}\right] \geq$$
$$\frac{\lambda_k\left(u_{3-k}[a_{3-k}] - u_{3-k}[a]\right)}{c_k} - 1 + \log\left[\frac{\lambda_k\left(u_{3-k}[a_{3-k}] - u_{3-k}[a]\right)}{c_k} - 1\right]$$

$\Leftrightarrow$

$$\begin{cases} \frac{\lambda_k}{c_k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] - \frac{c_{3-k}}{\lambda_{3-k}}\right) + \log\left[\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k] - u_k[a] - \frac{c_k}{\lambda_k}\right)\right] \leq 1, \\ \frac{\lambda_{3-k}R^{(a)}_{3-k}}{c_{3-k}} \geq \frac{\lambda_k R^{(a)}_k}{c_k} \end{cases}$$

$\Leftrightarrow \Delta^{(a)}_k \leq 0$ and $\frac{\lambda_{3-k}R^{(a)}_{3-k}}{c_{3-k}} \geq \frac{\lambda_k R^{(a)}_k}{c_k}$. ■

Moreover,

Lemma 8 *For $k = 1, 2$, if $R_k^{(a)} < 1$ and $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, then for all $p_{3-k} > 0$*

$$\frac{p_{3-k}}{1 - \bar{p}_k^{(a)}[p_{3-k}]} \geq \frac{p_{3-k}^{*(a)}}{1 - \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}.$$

Proof: Consider a function $\frac{p_{3-k}}{1-\bar{p}_k^{(a)}[p_{3-k}]}$.

$$\frac{d\left(\frac{p_{3-k}}{1-\bar{p}_k^{(a)}[p_{3-k}]}\right)}{dp_{3-k}} = \frac{\frac{\lambda_{3-k} p_{3-k}}{c_{3-k}} - \frac{\lambda_k \bar{p}_k^{(a)}[p_{3-k}]}{c_k}}{\frac{\lambda_{3-k} p_{3-k}}{c_{3-k}}\left(1 - \bar{p}_k^{(a)}[p_{3-k}]\right)}.$$

Thus, when $R_k^{(a)} < 1$ and $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, we have

- $\frac{p_{3-k}}{1-\bar{p}_k^{(a)}[p_{3-k}]}$ is increasing for $p_{3-k} \in \left(p_{3-k}^{*(a)}, +\infty\right)$,
- $\frac{p_{3-k}}{1-\bar{p}_k^{(a)}[p_{3-k}]}$ is decreasing for $p_{3-k} \in \left(0, p_{3-k}^{*(a)}\right)$.

■

Thus, if $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] \geq 1$, then $p_{3-k} + \bar{p}_k^{(a)}[p_{3-k}] \geq 1$ for all $p_{3-k} > 0$. In sum, when $R_k^{(a)} < 1$ and $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, requirement $\Delta_k^{(a)}[1] \leq 0$ guarantees that the curve $p_k = \bar{p}_k^{(a)}[p_{3-k}]$ lies outside of the belief triangle.

Note that when condition (2.10) holds, the belief triangle does not contain Area 3.k.2.[6] Since the switching in Area 3.3-k.2 strategy can only occur at the curve $p_k = \bar{p}_k^{(a)}[p_{3-k}]$, if the whole curve $p_k = \bar{p}_k^{(a)}[p_{3-k}]$ lies outside of the belief triangle, such strategy is not feasible.

Finally, notice that

$$p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] \geq 1 \quad \Rightarrow \quad R_1^{(a)} + R_2^{(a)} \geq 1. \tag{A.31}$$

Indeed, by Lemma 8, $R_{3-k}^{(a)} + \bar{p}_k^{(a)}\left[R_{3-k}^{(a)}\right] \geq 1$. Statement (A.31) follows from $\bar{p}_k^{(a)}\left[R_{3-k}^{(a)}\right] = R_k^{(a)}$.

In sum, the conditions for Case 3 are:

1. $\frac{c_k}{\lambda_k} < u_k[a_k] - u_k[a]$,

[6] $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} = \frac{\lambda_k R_k^{(a)}}{c_k}$, the belief triangle does not have either Area 3.1.2 or Area 3.2.2.

2. $u_k[a_k] - u_k[a] \geq u_{3-k}[a_{3-k}] - u_{3-k}[a]$,
3. $\frac{\lambda_k}{c_k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] - \frac{c_{3-k}}{\lambda_{3-k}}\right) + \log\left[\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k] - u_k[a] - \frac{c_k}{\lambda_k}\right)\right] \leq 1$,

with the interpretation that, given that it is optimal to use at least one source sometimes ($\frac{c_k}{\lambda_k} < u_k[a_k] - u_k[a]$), the cost of the sources are still large enough. In some sense, here we have that high costs lead to "myopic" behavior being dynamically optimal. Indeed, no switching means at most one source is used until the final decision is made. So, the decision maker chooses the source that is optimal if he would be restricted by having only a short period of time for learning, that is, his behavior is "myopic".

Case 4: Both sources might be used, switching is sometimes optimal; all indifference curves between two sources are switching curves. Suppose it is not Case 1, or Case 2, or Case 3. That means there exists k such that $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$ and $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$.

Case 4 corresponds to the optimal a-type strategy such that

- when $p_k > \bar{p}_k[p_{3-k}]$ and $\frac{\lambda_j p_j}{c_j} > \frac{\lambda_{3-j} p_{3-j}}{c_{3-j}}$, then use source j,
- when $p_k < \bar{p}_k[p_{3-k}]$ and $p_j > R_j^{(a)}$, then use source j,
- otherwise it is optimal to stop.

See Figure A.3.

Recall that the curve $p_k = \bar{p}_k[p_{3-k}]$ goes through the point $(R_1^{(a)}, R_2^{(a)})$ which means the above description has no contradictions. Moreover, the stopping region can be described as the set of points (p_1, p_2) such that $p_1 < R_1^{(a)}$ and $p_2 < R_2^{(a)}$.

However, the described a-type strategy is not necessarily optimal unless we add one more condition. To understand the nature of this condition, let's discuss how beliefs are moving according to this strategy. When $p_k > R_k^{(a)}$ and $\frac{p_{3-k}}{1-p_k} < \frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$, source k should be used until the belief about state k becomes equal to the threshold $R_k^{(a)}$. This is Area 2.k. When $R_k^{(a)} < p_k < \min\left\{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right], \bar{p}_k^{(a)}\left[p_{3-k}\right]\right\}$ and $\frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]} < \frac{p_{3-k}}{1-p_k} < \frac{R_{3-k}^{(a)}}{1-R_k^{(a)}}$, source k again should be used until $p_k = R_k^{(a)}$. Thus, this is again Area 2.k. When $\frac{\lambda_k p_k}{c_k} > \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}}$ and $\frac{p_{3-k}}{1-p_k} > \frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$, source k is used until the belief

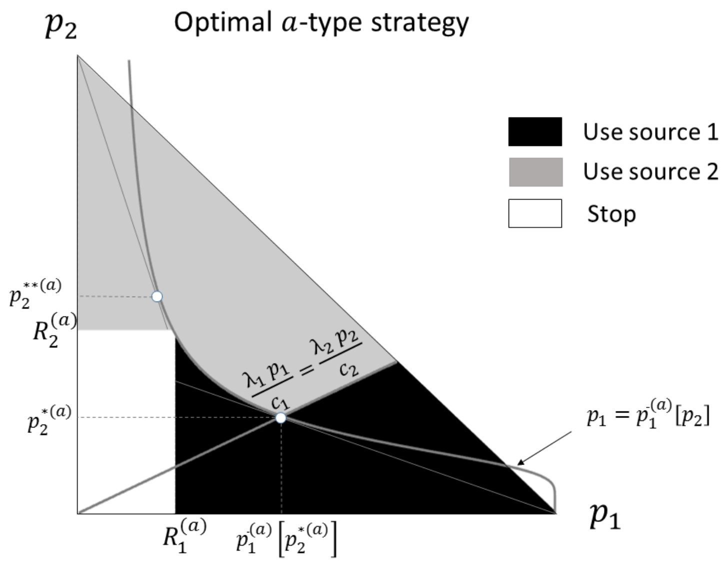

Figure A.3: Optimal a-type strategy for $R_1^{(a)} = 0.2$, $R_2^{(a)} = 0.4$, $\frac{c_1}{\lambda_1} = 2$, $\frac{c_2}{\lambda_2} = 1$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). White region means it is optimal to stop. Black region means the first information source should be used. Gray region means the second information source should be used. Case 4: $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} \geq R_{3-k}^{(a)}$, $k = 1$.

vector hits the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$, along which beliefs start crawling until they reach $p_k = \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]$, $p_{3-k} = p_{3-k}^{*(a)}$, then source k is used until $p_k = R_k^{(a)}$. Denote Area 3.k.1 the area where source k is used until $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$. When $p_{3-k} > R_{3-k}^{(a)}$ and $\frac{p_k}{1-p_{3-k}} < \frac{R_k^{(a)}}{1-R_{3-k}^{(a)}}$, source 3-k is used until $p_{3-k} = R_{3-k}^{(a)}$, which means (p_1, p_2) belongs to Area 2.3-k.

Up until now, the partition of the belief triangle into areas went without any contradictions, no additional conditions needed. However, to finish the description of the partition, I need to impose a new condition, $p_{3-k}^{**(a)} \geq R_{3-k}^{(a)}$. As it was announced in the heading of Case 4, whenever the agent is indifferent between two sources, he must switch sources. While this is always true for the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$, it might not be the case with the curve $p_k = \bar{p}_k[p_{3-k}]$. Indeed, consider the set of points such that $p_{3-k} < R_{3-k}^{(a)}$ and $\frac{p_k}{1-p_{3-k}} < \frac{R_k^{(a)}}{1-R_{3-k}^{(a)}}$. If, according to the described tactic, source 3-k might be used there, there should exist a ray $\frac{p_k}{1-p_{3-k}} = \text{const} < \frac{R_k^{(a)}}{1-R_{3-k}^{(a)}}$ that crosses the curve $p_k = \bar{p}_k[p_{3-k}]$ twice on the interval $p_{3-k} \in \left(p_{3-k}^{*(a)}, R_{3-k}^{(a)}\right)$. The crossing point that has higher

p_{3-k} would be an indifference point but not a switching point: at this point the decision maker is indifferent between two sources but no belief trajectory crosses this point unless it starts there. The additional condition $p_{3-k}^{**(a)} \geq R_{3-k}^{(a)}$ is needed precisely to exclude that kind of situation.

Before discussing this condition, let's finish the description of the areas (assuming this condition holds). Suppose $\frac{R_k^{(a)}}{1-R_{3-k}^{(a)}} < \frac{p_k}{1-p_{3-k}} < \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$. Then whenever p_{3-k} is greater than the smallest solution to $\frac{p_k}{1-p_{3-k}} = \frac{\bar{p}_k^{(a)}[p_{3-k}]}{1-p_{3-k}}$, namely $\underline{\tilde{p}}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$, source 3-k is used. Once leaving this area (that is, hitting $p_{3-k} = \underline{\tilde{p}}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$), the agent permanently switches to source k, which is used until $p_k = R_k^{(a)}$. Denote Area 3.3-k.2 the area where source 3-k is used until $p_{3-k} = \underline{\tilde{p}}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$. Finally, when $\frac{p_k}{1-p_{3-k}} > \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$ and $\frac{\lambda_k p_k}{c_k} < \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}}$, source 3-k is used and therefore (p_1, p_2) belongs to Area 3.3-k.1.

See Figure A.4.

Lemma 9 *For* $k = 1, 2$, *when* $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$ *and* $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$,

1. $p_{3-k}^{**(a)}$ *is well-defined by (2.19);*
2. *function* $\frac{\bar{p}_k^{(a)}[p_{3-k}]}{1-p_{3-k}}$ *is decreasing for* $p_{3-k} \in \left(p_{3-k}^{*(a)}, p_{3-k}^{**(a)}\right)$ *and it is increasing for* $p_{3-k} \in \left(p_{3-k}^{**(a)}, 1\right)$*;*
3. $\frac{\bar{p}_k^{(a)}[p_{3-k}]}{1-p_{3-k}}$ *is decreasing when* $p_{3-k} < p_{3-k}^{*(a)}$ *and* $p_{3-k} + \bar{p}_k^{(a)}[p_{3-k}] \leq 1$.

Proof: Consider the second derivative of $\bar{p}_k^{(a)}[p_{3-k}]$:

$$\bar{p}_k^{(a)''}[p_{3-k}] = \frac{c_{3-k}^2 \lambda_k \left(1 - \bar{p}_k^{(a)}[p_{3-k}]\right) \bar{p}_k^{(a)}[p_{3-k}]}{c_k \lambda_{3-k}^2 p_{3-k}^4} \left(\frac{\lambda_k}{c_k} + 2\left(\frac{p_{3-k}\lambda_{3-k}}{c_{3-k}} - \frac{\lambda_k \bar{p}_k^{(a)}[p_{3-k}]}{c_k}\right)\right).$$

Therefore, if $R_k^{(a)} < 1$ and $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, then $\bar{p}_k^{(a)}[p_{3-k}]$ is convex for $p_{3-k} > p_{3-k}^{*(a)}$. Hence, if $p_{3-k}^{*(a)} < 1$, then the function $\frac{\bar{p}_k^{(a)}[p_{3-k}]}{1-p_{3-k}}$ has at most one

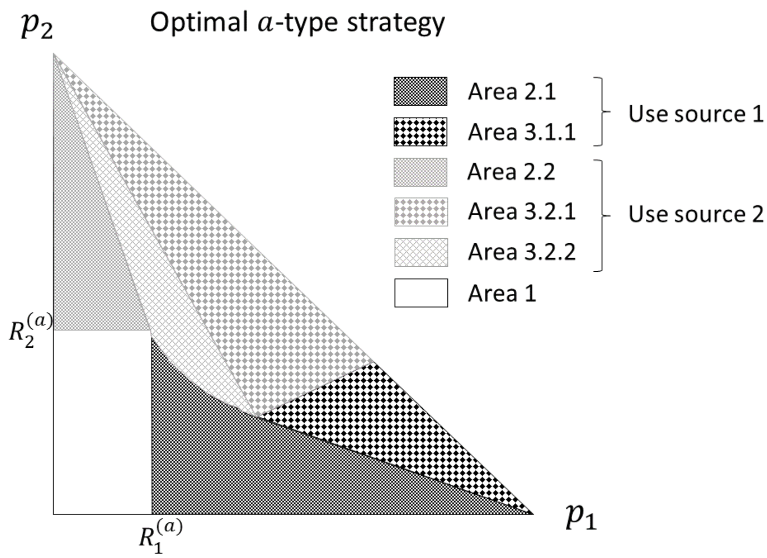

Figure A.4: Optimal a-type strategy for $R_1^{(a)} = 0.2$, $R_2^{(a)} = 0.4$, $\frac{c_1}{\lambda_1} = 2$, $\frac{c_2}{\lambda_2} = 1$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). Area 1 means it is optimal to stop. Areas 2.1 and 3.1.1 mean the first information source should be used. Areas 2.2, 3.2.1 and 3.2.2 mean the second information source should be used. Areas 2.1 and 2.2 correspond to the payoff optimal phase. Areas 3.1.1, 3.2.1, 3.2.2 correspond to the informatively optimal phase. Case 4: $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} \geq R_{3-k}^{(a)}$, $k = 1$.

local minimum on $p_{3-k} \in \left(p_{3-k}^{*(a)}, 1\right)$.

$$\frac{d\left(\frac{\bar{p}_k^{(a)}[p_{3-k}]}{1-p_{3-k}}\right)}{dp_{3-k}} = \frac{\lambda_k \bar{p}_k^{(a)}[p_{3-k}]}{c_k(1-p_{3-k})p_{3-k}^2}\left(\frac{c_k p_{3-k}^2}{(1-p_{3-k})\,\lambda_k} - \frac{c_{3-k}\left(1-\bar{p}_k^{(a)}[p_{3-k}]\right)}{\lambda_{3-k}}\right). \tag{A.32}$$

Note that $\lim\limits_{p_{3-k}\to 1-0} \frac{d\left(\frac{\bar{p}_k^{(a)}[p_{3-k}]}{1-p_{3-k}}\right)}{dp_{3-k}} = +\infty$ and $\lim\limits_{p_{3-k}\to p_{3-k}^{*(a)}} \frac{d\left(\frac{\bar{p}_k^{(a)}[p_{3-k}]}{1-p_{3-k}}\right)}{dp_{3-k}} = -\frac{1-p_{3-k}^{*(a)}-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{p_{3-k}^{*(a)}\left(1-p_{3-k}^{*(a)}\right)^2}$.

Moreover, from (A.32), we know that $\frac{d\left(\frac{\bar{p}_k^{(a)}[p_{3-k}]}{1-p_{3-k}}\right)}{dp_{3-k}} < 0$ if and only if

$$\frac{c_k p_{3-k}^2}{(1-p_{3-k})\,\lambda_k} < \frac{c_{3-k}\left(1-\bar{p}_k^{(a)}[p_{3-k}]\right)}{\lambda_{3-k}},$$

which is equivalent to

$$\frac{\lambda_{3-k}p_{3-k}}{c_{3-k}} - \frac{\lambda_k \bar{p}_k^{(a)}[p_{3-k}]}{c_k} < \frac{\lambda_k \left(1 - p_{3-k} - \bar{p}_k^{(a)}[p_{3-k}]\right)}{c_k p_{3-k}}.$$

The left hand side is negative when $p_{3-k} < p_{3-k}^{*(a)}$, the right hand side is nonnegative when $p_{3-k} + \bar{p}_k^{(a)}[p_{3-k}] \leq 1$. ■

Lemma 10 *When* $R_k^{(a)} < 1$, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$ *and* $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, *condition* $p_{3-k}^{**(a)} \geq R_{3-k}^{(a)}$ *is equivalent to* $R_{3-k}^{(a)} < 1$ *and*

$$\frac{u_k[a_k] - u_k[a]}{u_{3-k}[a_{3-k}] - u_{3-k}[a]} \leq \frac{\lambda_k \lambda_{3-k}}{c_{3-k}c_k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] - \frac{c_{3-k}}{\lambda_{3-k}}\right)\left(u_k[a_k] - u_k[a] - \frac{c_k}{\lambda_k}\right). \tag{A.33}$$

Proof: We know that function $\frac{c_k p_{3-k}^2}{(1-p_{3-k})\lambda_k} - \frac{c_{3-k}\left(1-\bar{p}_k^{(a)}[p_{3-k}]\right)}{\lambda_{3-k}}$ is negative for $p_{3-k} \in \left[p_{3-k}^{*(a)}, p_{3-k}^{**(a)}\right)$ and is positive for $p_{3-k} \in \left(p_{3-k}^{**(a)}, 1\right)$. Given $p_{3-k}^{*(a)} \leq R_{3-k}^{(a)}$, we conclude that $p_{3-k}^{**(a)} \geq R_{3-k}^{(a)}$ is equivalent to $R_{3-k}^{(a)} < 1$ and

$$\frac{c_k\left(R_{3-k}^{(a)}\right)^2}{\left(1 - R_{3-k}^{(a)}\right)\lambda_k} - \frac{c_{3-k}\left(1 - \bar{p}_k^{(a)}\left[R_{3-k}^{(a)}\right]\right)}{\lambda_{3-k}} \leq 0.$$

The last inequality is equivalent to (A.33) ■

In sum, the conditions for Case 4 are:

1. $\frac{c_k}{\lambda_k} < u_k[a_k] - u_k[a]$, $\frac{c_{3-k}}{\lambda_{3-k}} < u_{3-k}[a_{3-k}] - u_{3-k}[a]$,
2. $u_k[a_k] - u_k[a] \geq u_{3-k}[a_{3-k}] - u_{3-k}[a]$,
3. $\frac{\lambda_k}{c_k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] - \frac{c_{3-k}}{\lambda_{3-k}}\right) + \log\left[\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k] - u_k[a] - \frac{c_k}{\lambda_k}\right)\right] > 1$,
4. $\frac{1}{u_{3-k}[a_{3-k}] - u_{3-k}[a]} \leq \frac{\lambda_k\lambda_{3-k}}{c_{3-k}c_k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] - \frac{c_{3-k}}{\lambda_{3-k}}\right)\left(1 - \frac{c_k}{\lambda_k(u_k[a_k] - u_k[a])}\right)$.

It is easy to see that all these conditions (except $\frac{c_{3-k}}{\lambda_{3-k}} < u_{3-k}[a_{3-k}] - u_{3-k}[a]$) are in the form of $u_k[a_k] - u_k[a]$ greater than something. Thus, Case 4 describes the optimal a-type strategy when the benefit from the correct choice of the alternative for one of the states is large enough ($u_k[a_k] - u_k[a]$ is large enough), while such benefit for the other state ($u_{3-k}[a_{3-k}] - u_{3-k}[a]$) is not too low compared to the cost for the corresponding source ($\frac{c_{3-k}}{\lambda_{3-k}}$).

Case 5: Both sources might be used, switching is sometimes optimal; if the agent is indifferent between the informatively superior source and something else, this something else is always the other source. Suppose it is not the first four cases. That means there exists k such that $R_k^{(a)} < 1$, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$ and $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$.

To describe the optimal a-type strategy in Case 5, I start by recalling the description of the optimal a-type strategy in Case 4:

- when $p_k > \bar{p}_k[p_{3-k}]$ and $\frac{\lambda_j p_j}{c_j} > \frac{\lambda_{3-j}p_{3-j}}{c_{3-j}}$, then use source j,
- when $p_k < \bar{p}_k[p_{3-k}]$ and $p_j > R_j^{(a)}$, then use source j,
- otherwise it is optimal to stop.

This description has no contradiction if $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$ instead of $p_{3-k}^{**(a)} \geq R_{3-k}^{(a)}$. However, this would not be the optimal a-type strategy. Why? Recall that the curve $p_k = \bar{p}_k[p_{3-k}]$ is the set of points such that the cost of using source $3-k$ is equal to a sum of the direct benefit of using that source $(\lambda_{3-k}p_{3-k}\,(u_{3-k}[a_{3-k}] - u_{3-k}[a]))$ and the indirect benefit $\left(\lambda_{3-k}p_{3-k}\left(\frac{c_k}{\lambda_k}\log\left[\frac{\bar{p}_k^{(a)}[p_{3-k}]\left(1-R_k^{(a)}\right)}{R_k^{(a)}\left(1-\bar{p}_k^{(a)}[p_{3-k}]\right)}\right]\right)\right)$. Thus, these are points where the agent is indifferent between continuing using source 3-k and switching to source k. So, all such points must be switching points. If $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, this is not always true. Thus, the part of the curve $p_k = \bar{p}_k[p_{3-k}]$ that does not correspond to the switching behavior must be changed. How?

Figure A.5 shows the optimal a-type strategy for Case 5.

The part of the curve $p_k = \bar{p}_k[p_{3-k}]$ that the belief trajectory never crosses is changed to $p_{3-k} = \bar{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$. Where does it come from and how does this curve look like in algebraic form? To answer these questions, I need to partition the belief triangle into the areas as before. The set of beliefs such that $p_k > R_k^{(a)}$ and $\frac{p_{3-k}}{1-p_k} < \frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$ belongs to Area 2.k. When $\frac{\lambda_k p_k}{c_k} > \frac{\lambda_{3-k}p_{3-k}}{c_{3-k}}$ and $\frac{p_{3-k}}{1-p_k} > \frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$, then we're in Area 3.k.1. If $p_{3-k} > R_{3-k}^{(a)}$, then (p_1, p_2) belongs to Area 2.3-k (this part is void in Figure A.5 since $R_{3-k}^{(a)} > 1$ given the values of the parameters the picture is drawn for). If $p_1 < R_1^{(a)}$ and $p_2 < R_2^{(a)}$, then (p_1, p_2) belongs to Area 1. So far, nothing is new, we've already seen the same in previous cases.

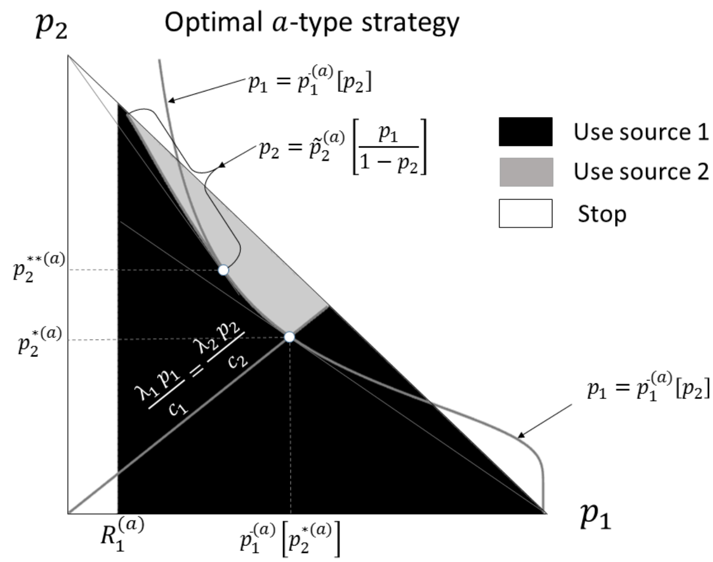

Figure A.5: Optimal a-type strategy for $R_1^{(a)} = 0.1$, $R_2^{(a)} = 10$, $\frac{c_1}{\lambda_1} = 1.2$, $\frac{c_2}{\lambda_2} = 1$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). White region means it is optimal to stop. Black region means the first information source should be used. Gray region means the second information source should be used. Case 5: $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} \geq 0$, $k = 1$.

Case 5 describes the situation when, if the agent is indifferent between using source 3-k and then switching to source k, and something else, this something else is always source k. Graphically, the curve $p_{3-k} = \tilde{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ always divides black and gray regions, not touching the white one. Thus, this curve divides Area 2.k and Areas 3.3-k.1 and 3.3-k.2 (see Figure A.6).

Lemma 11 *For $k = 1, 2$, when $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $\frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}} < q_k < 1$,*

1. *$\underline{\tilde{p}}_{3-k}^{(a)}[q_k]$ and $\bar{\tilde{p}}_{3-k}^{(a)}[q_k]$ are well-defined by (2.20);*
2. *$\underline{\tilde{p}}_{3-k}^{(a)\prime}[q_k] < 0$ and $\bar{\tilde{p}}_{3-k}^{(a)\prime}[q_k] > 0$; intuitively, $\underline{\tilde{p}}_{3-k}^{(a)}[q_k]$ decreases since the higher the relative probability of state k (i.e., $q_k = \frac{p_k}{1-p_{3-k}}$, which stays constant while we use source 3-k), the more useful source k is, and the quicker the switch will occur;*

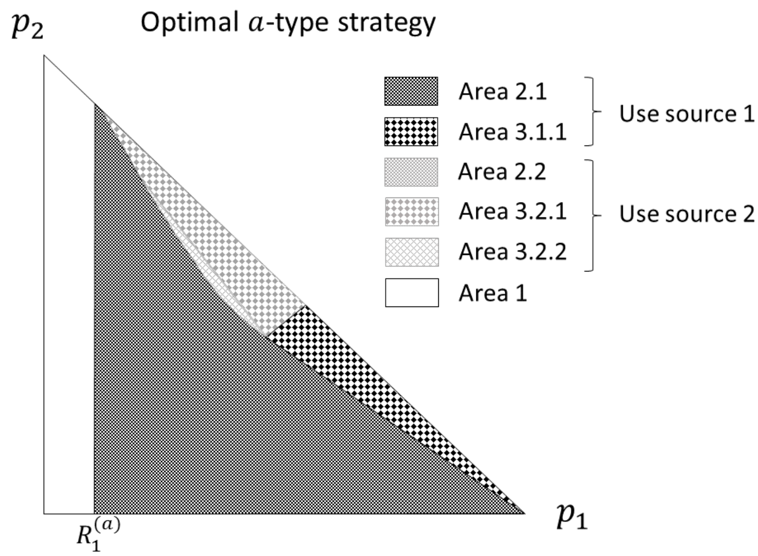

Figure A.6: Optimal a-type strategy for $R_1^{(a)} = 0.1$, $R_2^{(a)} = 10$, $\frac{c_1}{\lambda_1} = 1.2$, $\frac{c_2}{\lambda_2} = 1$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). Area 1 means it is optimal to stop. Areas 2.1 and 3.1.1 mean the first information source should be used. Areas 2.2, 3.2.1 and 3.2.2 mean the second information source should be used. Areas 2.1 and 2.2 correspond to the payoff optimal phase. Areas 3.1.1, 3.2.1, 3.2.2 correspond to the informatively optimal phase. Case 5: $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} \geq 0$, $k = 1$.

3. *line* $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ *lies entirely on one side of the curve* $p_{3-k} = \bar{\tilde{p}}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$*:*

$$\frac{\lambda_k \bar{p}_k^{(a)}\left[\bar{\tilde{p}}_{3-k}^{(a)}[q_k]\right]}{c_k} < \frac{\lambda_{3-k}\bar{\tilde{p}}_{3-k}^{(a)}[q_k]}{c_{3-k}}; \tag{A.34}$$

4. $\lim\limits_{q_k \to \frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}}} \underline{\tilde{p}}_{3-k}^{(a)}[q_k] = \lim\limits_{q_k \to \frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}}} \bar{\tilde{p}}_{3-k}^{(a)}[q_k] = p_{3-k}^{**(a)}$; $\underline{\tilde{p}}_{3-k}^{(a)}\left[\frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}\right] = p_{3-k}^{*(a)}$.

<u>*Proof:*</u>

1. Recall that $\bar{p}_k^{(a)}[p_{3-k}]$ is convex for $p_{3-k} > p_{3-k}^{*(a)}$. Moreover, function $\frac{\bar{p}_k^{(a)}[p_{3-k}]}{1-p_{3-k}}$ is decreasing for $p_{3-k} \in \left(p_{3-k}^{*(a)}, p_{3-k}^{**(a)}\right)$ and is increasing for $p_{3-k} \in \left(p_{3-k}^{**(a)}, 1\right)$; it is decreasing when $p_{3-k} < p_{3-k}^{*(a)}$ and $\frac{\bar{p}_k^{(a)}[p_{3-k}]}{1-p_{3-k}} < 1$.

Thus, equation $\frac{\bar{p}_k^{(a)}[p_{3-k}]}{1-p_{3-k}} = q_k$ has two solutions when $q_k > \frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}}$. These solutions are $\underline{\tilde{p}}_{3-k}^{(a)}[q_k]$ and $\bar{\tilde{p}}_{3-k}^{(a)}[q_k]$.

2.

$$\frac{\bar{p}_k^{(a)}\left[p_{3-k}[q_k]\right]}{1-p_{3-k}[q_k]} = q_k \quad \Rightarrow \quad p'_{3-k}[q_k] = \frac{c_k\lambda_{3-k}p_{3-k}^2[q_k]}{q_k\left(\frac{c_k\lambda_{3-k}p_{3-k}^2[q_k]}{1-p_{3-k}[q_k]} - c_{3-k}\lambda_k\left(1-\bar{p}_k^{(a)}\left[p_{3-k}[q_k]\right]\right)\right)}. \tag{A.35}$$

3. Since $\bar{p}_k^{(a)}[p_{3-k}]$ is decreasing and $\bar{\tilde{p}}_{3-k}^{(a)}[q_k] > p_{3-k}^{*(a)}$.

■

It is easy to see that if $p_{3-k} = \underline{\tilde{p}}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$, then $p_k = \bar{p}_k^{(a)}[p_{3-k}]$. Similarly, if $p_{3-k} = \bar{\tilde{p}}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$, then $p_k = \bar{p}_k^{(a)}[p_{3-k}]$. Thus, when $q_k < \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}$, the curve $p_{3-k} = \underline{\tilde{p}}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ is the switching curve from source 3-k to source k, permanently (when $q_k > \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}$, the switching line is $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ and not $p_k = \bar{p}_k^{(a)}[p_{3-k}]$). The curve $p_{3-k} = \bar{\tilde{p}}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ is the part of the curve $p_k = \bar{p}_k^{(a)}[p_{3-k}]$ that needs to be changed.

To define the curve $p_{3-k} = \tilde{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ formally, I first need to introduce functions $W_k^{(a)}[p_k, q_{3-k}]$ and $F_k^{(a)}[x, q_{3-k}]$.

Definition and properties of $W_k^{(a)}[p_k, q_{3-k}]$**.**

For $k = 1, 2$, when $R_{3-k}^{(a)} < 1$, $p_k \in (0,1)$, $q_{3-k} \in (0,1)$, denote

$$W_k^{(a)}[p_k, q_{3-k}] = \left(1 - \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}} - \frac{1}{R_k^{(a)}}\right)\frac{1}{1-p_k} + \log\left[\frac{p_k}{1-p_k}\right] + \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\left(\frac{1}{1-p_k} - q_{3-k}\right)\log\left[\frac{R_{3-k}^{(a)}\left(\frac{1}{1-p_k} - q_{3-k}\right)}{\left(1-R_{3-k}^{(a)}\right)q_{3-k}}\right]. \tag{A.36}$$

To understand the meaning of this function, note that

$$\left.\frac{\partial W_{3-k}\left[p_{3-k}, q_k\right]}{\partial p_{3-k}}\right|_{q_k = \frac{p_{3-k}}{1-p_k}} = \frac{c_{3-k} - \lambda_{3-k}p_{3-k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] + \frac{c_k}{\lambda_k}\log\left[\frac{p_k\left(1-R_k^{(a)}\right)}{R_k^{(a)}(1-p_k)}\right]\right)}{c_{3-k}p_{3-k}(1-p_{3-k})^2}. \tag{A.37}$$

Thus, the marginal change of $W_{3-k}\left[p_{3-k}, \frac{p_{3-k}}{1-p_k}\right]$ as p_{3-k} decreases while $\frac{p_{3-k}}{1-p_k}$ stays constant (this is the direction the belief vector would move if source 3-k is used and no positive signal is observed) is the adjusted difference between the marginal benefit of using source 3-k (direct and indirect) and its marginal cost. Thus, this function has the meaning of the adjusted expected benefit of using source 3-k before switching to source k.

Properties of $W^{(a)}_{3-k}[p_{3-k}, q_k]$:

1. $\lim_{p_{3-k}\to 1} W^{(a)}_{3-k}[p_{3-k}, q_k] = +\infty$. Intuitively, when the agent is almost certain it is state 3-k, he should definitely use source 3-k to avoid the default alternative a and choose a_{3-k} after observing a positive signal (which will eventually appear almost certainly).

2. $W^{(a)}_{3-k}[p_{3-k}, q_k]$ is increasing in p_{3-k} when $\frac{\bar{p}^{(a)}_k[p_{3-k}]}{1-p_{3-k}} > q_k$ and is decreasing in p_{3-k} when $\frac{\bar{p}^{(a)}_k[p_{3-k}]}{1-p_{3-k}} < q_k$. Essentially, it says that the curve $p_k = \bar{p}^{(a)}_k[p_{3-k}]$ is the set of extremal points for $W^{(a)}_{3-k}[p_{3-k}, q_k]$. Recall that these points are the points where marginal cost of using source 3-k is equal to its marginal benefit. Given (A.37), these are the points when $\left.\frac{\partial W_{3-k}[p_{3-k},q_k]}{\partial p_{3-k}}\right|_{q_k=\frac{p_{3-k}}{1-p_k}} = 0$. <u>*Proof:*</u>

$$\frac{\partial W^{(a)}_{3-k}[p_{3-k}, q_k]}{\partial p_{3-k}} = \frac{c_k \lambda_{3-k}}{c_{3-k}\lambda_k (1-p_{3-k})^2} \log\left[\frac{\frac{1}{(1-p_{3-k})q_k} - 1}{\frac{1}{\bar{p}^{(a)}_k[p_{3-k}]} - 1}\right].$$

■

3. Assume $R^{(a)}_k < 1$, $\frac{\lambda_{3-k} R^{(a)}_{3-k}}{c_{3-k}} \geq \frac{\lambda_k R^{(a)}_k}{c_k}$, $p^{*(a)}_{3-k} + \bar{p}^{(a)}_k\left[p^{*(a)}_{3-k}\right] < 1$. Given the previous point, the following are true.

   a) When $q_k < \frac{\bar{p}^{(a)}_k[p^{**(a)}_{3-k}]}{1-p^{**(a)}_{3-k}} = \inf_{p_{3-k}\in(0,1)} \frac{\bar{p}^{(a)}_k[p_{3-k}]}{1-p_{3-k}}$, $W^{(a)}_{3-k}[p_{3-k}, q_k]$ is increasing in p_{3-k} for $p_{3-k} \in (0, 1)$.

   b) When $q_k > \frac{\bar{p}^{(a)}_k[p^{**(a)}_{3-k}]}{1-p^{**(a)}_{3-k}}$:

   - $W^{(a)}_{3-k}[p_{3-k}, q_k]$ is increasing in p_{3-k} for $p_{3-k} \in \left(0, \underline{\tilde{p}}^{(a)}_{3-k}[q_k]\right)$ and for $p_{3-k} \in \left(\bar{\tilde{p}}^{(a)}_{3-k}[q_k], 1\right)$,
   - $W^{(a)}_{3-k}[p_{3-k}, q_k]$ is decreasing in p_{3-k} for $p_{3-k} \in \left(\underline{\tilde{p}}^{(a)}_{3-k}[q_k], \bar{\tilde{p}}^{(a)}_{3-k}[q_k]\right)$.

Definition and properties of $F^{(a)}_k[x, q_{3-k}]$.

For $k = 1, 2$, when $x \in \left(0, \frac{1}{\frac{c_1}{\lambda_1}+\frac{c_2}{\lambda_2}}\right)$, $q_{3-k} \in (0,1)$, denote

$$F_k^{(a)}[x, q_{3-k}] = -q_{3-k}\left(1+\frac{1}{R_k^{(a)}}\right) - \frac{\lambda_k}{c_k}\left(\frac{c_{3-k}}{\lambda_{3-k}} + \frac{1-q_{3-k}}{x}\right) + \log\left[\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}q_{3-k}\right]$$
$$-(1-q_{3-k})\left(1+\frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\right)\log\left[\frac{\frac{\lambda_{3-k}}{c_{3-k}}\left(\frac{1}{x}-\frac{c_1}{\lambda_1}-\frac{c_2}{\lambda_2}\right)}{\frac{1}{q_{3-k}}-1}\right].$$

Suppose that $R_k^{(a)} < 1$, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$. Note that from (2.18) we get $\frac{\lambda_{3-k}p_{3-k}^{*(a)}}{c_{3-k}} < \frac{1}{\frac{c_1}{\lambda_1}+\frac{c_2}{\lambda_2}}$. Thus, $\frac{\lambda_{3-k}p_{3-k}^{*(a)}}{c_{3-k}}$ falls in the range of admissible values for the argument x in function $F_{3-k}^{(a)}[x, q_k]$.

Suppose the current belief vector lies on the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ so that $p_{3-k} > p_{3-k}^{*(a)}$ (this part of the line corresponds to using both sources simultaneously in such a way that the belief vector moves along this line). Function $W_{3-k}^{(a)}\left[p_{3-k}, \frac{p_k}{1-p_{3-k}}\right] - F_{3-k}^{(a)}\left[\frac{\lambda_{3-k}p_{3-k}^{*(a)}}{c_{3-k}}, \frac{p_k}{1-p_{3-k}}\right]$ has the meaning of the adjusted expected benefit from using both sources until $p_{3-k} = p_{3-k}^{*(a)}$. This follows from two observations. First, $F_{3-k}^{(a)}\left[\frac{\lambda_{3-k}p_{3-k}^{*(a)}}{c_{3-k}}, \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}\right] = W_{3-k}^{(a)}\left[p_{3-k}^{*(a)}, \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}\right]$ so that once $p_{3-k} = p_{3-k}^{*(a)}$ is reached, function $W_{3-k}^{(a)}\left[p_{3-k}, \frac{p_k}{1-p_{3-k}}\right] - F_{3-k}^{(a)}\left[\frac{\lambda_{3-k}p_{3-k}^{*(a)}}{c_{3-k}}, \frac{p_k}{1-p_{3-k}}\right]$ is equal to zero. Second, note that along the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ we have $p_{3-k} = \frac{1}{1+\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1-p_{3-k}}{p_k}}$. The derivative of this function,

$$\frac{\partial\left(W_{3-k}^{(a)}\left[\frac{1}{1+\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1}{q_k}}, q_k\right] - F_{3-k}^{(a)}\left[\frac{\lambda_{3-k}p_{3-k}^{*(a)}}{c_{3-k}}, q_k\right]\right)}{\partial q_k}$$
$$= -\left(\frac{1}{\xi} - \frac{1}{q_k} + 2\log\left[\frac{\frac{1}{\xi}-1}{\frac{1}{q_k}-1}\right]\right)$$
$$+\left(1-\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\right)\left(\frac{1}{\xi} + \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\left(\frac{R_{3-k}^{(a)}-1}{R_{3-k}^{(a)}}\right) + \log\left[\frac{R_k^{(a)}\left(\frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}+\frac{1}{q_k}-1\right)\left(\frac{1}{\xi}-1\right)}{\left(1-R_k^{(a)}\right)\left(\frac{1}{q_k}-1\right)}\right]\right)\Bigg|_{\xi=\frac{\bar{p}_k\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}},$$

consists of two parts. The first part, $-\left(\frac{1}{\xi} - \frac{1}{q_k} + 2\log\left[\frac{\frac{1}{\xi}-1}{\frac{1}{q_k}-1}\right]\right)$, is the decreasing function of q_k, it is equal to 0 when $p_{3-k} = p_{3-k}^{*(a)}$, and therefore it has the interpretation of the total net marginal cost of using both sources along the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ until $p_{3-k} = p_{3-k}^{*(a)}$. The second part is the adjustment for the asymmetry of the sources; it is equal to 0 when $\frac{c_1}{\lambda_1} = \frac{c_2}{\lambda_2}$. Thus, as the belief vector is moving down along the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$, function $W_{3-k}^{(a)}\left[p_{3-k}, \frac{p_k}{1-p_{3-k}}\right] - F_{3-k}^{(a)}\left[\frac{\lambda_{3-k} p_{3-k}^{*(a)}}{c_{3-k}}, \frac{p_k}{1-p_{3-k}}\right]$ increases at the rate of the total net marginal benefit. To sum up, function $W_{3-k}^{(a)}\left[p_{3-k}, \frac{p_k}{1-p_{3-k}}\right] - F_{3-k}^{(a)}\left[\frac{\lambda_{3-k} p_{3-k}^{*(a)}}{c_{3-k}}, \frac{p_k}{1-p_{3-k}}\right]$ is the total net benefit of using both sources along the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ until $p_{3-k} = p_{3-k}^{*(a)}$.

Lemma 12 *Suppose that* $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$. *Then* $F_{3-k}^{(a)}\left[\frac{\lambda_{3-k} p_{3-k}^{*(a)}}{c_{3-k}}, q_k\right] > W_{3-k}^{(a)}\left[\bar{\bar{p}}_{3-k}^{(a)}[q_k], q_k\right]$ *for* $q_k \geq \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$.

Proof: From (2.18) we get

$$F_{3-k}^{(a)}\left[\frac{\lambda_{3-k} p_{3-k}^{*(a)}}{c_{3-k}}, \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}\right] = W_{3-k}^{(a)}\left[p_{3-k}^{*(a)}, \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}\right] > W_{3-k}^{(a)}\left[\bar{\bar{p}}_{3-k}^{(a)}\left[\frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}\right], \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}\right],$$

$$\frac{d\left(F_{3-k}^{(a)}\left[\frac{\lambda_{3-k} p_{3-k}^{*(a)}}{c_{3-k}}, q_k\right] - W_{3-k}^{(a)}\left[\bar{\bar{p}}_{3-k}^{(a)}[q_k], q_k\right]\right)}{dq_k} =$$

$$\frac{c_k \lambda_{3-k} \bar{\bar{p}}_{3-k}^{(a)}[q_k]}{c_{3-k}\lambda_k \bar{p}_k^{(a)}\left[\bar{\bar{p}}_{3-k}^{(a)}[q_k]\right]} - \frac{1}{\bar{\bar{p}}_{3-k}^{(a)}[q_k]} - 1 + \frac{1}{p_{3-k}^{*(a)}} + \left(1 + \frac{c_k \lambda_{3-k}}{c_{3-k}\lambda_k}\right)\log\left[\frac{\frac{c_{3-k}\lambda_k\left(1-p_{3-k}^{*(a)}\right)}{c_k \lambda_{3-k} p_{3-k}^{*(a)}} - 1}{\frac{1}{q_k} - 1}\right] > 0$$

for $q_k \geq \frac{c_k \lambda_{3-k} p_{3-k}^{*(a)}}{c_{3-k}\lambda_k\left(1-p_{3-k}^{*(a)}\right)} = \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$ (the last inequality holds because (A.34) and $\bar{\bar{p}}_{3-k}^{(a)}[q_k] > p_{3-k}^{*(a)}$). ■

I'll use this lemma when I define $\tilde{p}_{3-k}^{(a)}[q_k]$.

Definition and properties of $\tilde{p}_{3-k}^{(a)}[q_k]$ ***and*** $q_k^{*(a)}$.

For $k = 1, 2$, when $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, denote

$$L_{3-k}^{(a)}[q_k] = \begin{cases} W_{3-k}^{(a)}\left[\underline{\tilde{p}}_{3-k}^{(a)}[q_k], q_k\right], & \frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}} < q_k \leq \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}, \\ F_{3-k}^{(a)}\left[\frac{\lambda_{3-k} p_{3-k}^{*(a)}}{c_{3-k}}, q_k\right], & \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}} < q_k < 1, \end{cases} \tag{A.38}$$

$$\tilde{p}^{(a)}_{3-k}[q_k] \in \left(\bar{\bar{p}}^{(a)}_{3-k}[q_k], 1\right): \quad W^{(a)}_{3-k}\left[\tilde{p}^{(a)}_{3-k}[q_k], q_k\right] = L^{(a)}_{3-k}[q_k] \tag{A.39}$$

and

$$\tilde{q}^{(a)}_{3-k}[q_k] = \frac{\tilde{p}^{(a)}_{3-k}[q_k]}{1 - q_k\left(1 - \tilde{p}^{(a)}_{3-k}[q_k]\right)}. \tag{A.40}$$

Properties:

1. Definition (A.39) has no contradiction, that is, the solution p_{3-k} to $W^{(a)}_{3-k}[p_{3-k}, q_k] = L^{(a)}_{3-k}[q_k]$ always exists and is unique on the interval $\left(\bar{\bar{p}}^{(a)}_{3-k}[q_k], 1\right)$.

   Proof: When $\frac{\bar{p}^{(a)}_k[p^{**(a)}_{3-k}]}{1-p^{**(a)}_{3-k}} < q_k \leq \frac{\bar{p}^{(a)}_k[p^{*(a)}_{3-k}]}{1-p^{*(a)}_{3-k}}$, this follows from the properties 1 and 3b of function $W^{(a)}_{3-k}[p_{3-k}, q_k]$.

   When $\frac{\bar{p}^{(a)}_k[p^{*(a)}_{3-k}]}{1-p^{*(a)}_{3-k}} < q_k < 1$, this follows from Lemma 12 and properties 1 and 3c of function $W^{(a)}_{3-k}[p_{3-k}, q_k]$.

   So, since $W^{(a)}_{3-k}[p_{3-k}, q_k]$ is increasing in p_{3-k} for $p_{3-k} \in \left(\bar{\bar{p}}^{(a)}_{3-k}[q_k], 1\right)$ from $W^{(a)}_{3-k}\left[\bar{\bar{p}}^{(a)}_{3-k}[q_k], q_k\right] < L^{(a)}_{3-k}[q_k]$ to $+\infty$, the solution to (A.39) exists and is unique. ■

2. $L^{(a)}_{3-k}[q_k]$ and $\tilde{p}^{(a)}_{3-k}[q_k]$ are continuous for $q_k \in \left(\frac{\bar{p}^{(a)}_k[p^{**(a)}_{3-k}]}{1-p^{**(a)}_{3-k}}, 1\right)$.

   Proof: Follows from $F^{(a)}_{3-k}\left[\frac{\lambda_{3-k} p^{*(a)}_{3-k}}{c_{3-k}}, \frac{\bar{p}^{(a)}_k[p^{*(a)}_{3-k}]}{1-p^{*(a)}_{3-k}}\right] = W^{(a)}_{3-k}\left[p^{*(a)}_{3-k}, \frac{\bar{p}^{(a)}_k[p^{*(a)}_{3-k}]}{1-p^{*(a)}_{3-k}}\right]$. ■

3. $\tilde{p}^{(a)}_{3-k}[q_k]$ and $\tilde{q}^{(a)}_{3-k}[q_k]$ are strictly increasing. Intuitively, the higher the relative probability of state k (q_k), the more useful source k becomes, the higher the threshold p_{3-k} would be to make the decision maker indifferent between the sources.

   Proof: If $\frac{\bar{p}^{(a)}_k[p^{**(a)}_{3-k}]}{1-p^{**(a)}_{3-k}} < q_k < \frac{\bar{p}^{(a)}_k[p^{*(a)}_{3-k}]}{1-p^{*(a)}_{3-k}}$, then

$$\tilde{p}^{(a)\prime}_{3-k}[q_k] = \frac{c_k \lambda_{3-k}}{c_{3-k}\lambda_k} \frac{\frac{1}{q_k}\left(\frac{1}{1-\tilde{p}^{(a)}_{3-k}[q_k]} - \frac{1}{1-\underline{\tilde{p}}^{(a)}_{3-k}[q_k]}\right) + \log\left[\frac{\frac{1}{q_k\left(1-\tilde{p}^{(a)}_{3-k}[q_k]\right)}-1}{\frac{1}{q_k\left(1-\underline{\tilde{p}}^{(a)}_{3-k}[q_k]\right)}-1}\right]}{\frac{\partial W^{(a)}_{3-k}\left[\tilde{p}^{(a)}_{3-k}[q_k], q_k\right]}{\partial p_{3-k}}}.$$

If $\frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}} < q_k < 1$, then

$$\tilde{p}_{3-k}^{(a)\prime}[q_k] = \frac{c_k \lambda_{3-k}}{c_{3-k}\lambda_k} \frac{\frac{1}{q_k}\left(\frac{1}{1-\tilde{p}_{3-k}^{(a)}[q_k]} - \frac{1}{1-\bar{\tilde{p}}_{3-k}^{(a)}[q_k]}\right) + \log\left[\frac{\frac{1}{q_k\left(1-\tilde{p}_{3-k}^{(a)}[q_k]\right)}-1}{\frac{1}{q_k\left(1-\bar{\tilde{p}}_{3-k}^{(a)}[q_k]\right)}-1}\right]}{\frac{\partial W_{3-k}^{(a)}\left[\tilde{p}_{3-k}^{(a)}[q_k],q_k\right]}{\partial p_{3-k}}} + \frac{\frac{d\left(F_{3-k}^{(a)}\left[\frac{\lambda_{3-k}p_{3-k}^{*(a)}}{c_{3-k}},q_k\right]-W_{3-k}^{(a)}\left[\bar{\tilde{p}}_{3-k}^{(a)}[q_k],q_k\right]\right)}{dq_k}}{\frac{\partial W_{3-k}^{(a)}\left[\tilde{p}_{3-k}^{(a)}[q_k],q_k\right]}{\partial p_{3-k}}}.$$

In any case, $\tilde{p}_{3-k}^{(a)\prime}[q_k] > 0$. Therefore,

$$\tilde{q}_{3-k}^{(a)\prime}[q_k] = \frac{\tilde{p}_{3-k}^{(a)}[q_k]\left(1-\tilde{p}_{3-k}^{(a)}[q_k]\right) + (1-q_k)\tilde{p}_{3-k}^{(a)\prime}[q_k]}{\left(1-q_k\left(1-\tilde{p}_{3-k}^{(a)}[q_k]\right)\right)^2} > 0.$$

■

The curve $p_{3-k} = \tilde{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ is the indifference curve in Case 5. It consists of two parts, the part that divides Area 2.k and Area 3.3-k.2 and the part that divides Area 2.k and Area 3.3-k.1.

- When $\frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}} < \frac{p_k}{1-p_{3-k}} \leq \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}$, the curve $p_{3-k} = \tilde{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ is the set of points when the decision maker is indifferent between using source k only and using source 3-k until $p_{3-k} = \underline{\tilde{p}}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ and then switching to source k permanently. In that case, condition (A.39) is equivalent to

$$\int_{\underline{\tilde{p}}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]}^{\tilde{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]} \frac{\partial W_{3-k}\left[p_{3-k}', \frac{p_k}{1-p_{3-k}}\right]}{\partial p_{3-k}'} dp_{3-k}' = 0,$$

where $\frac{\partial W_{3-k}\left[p_{3-k}', \frac{p_k}{1-p_{3-k}}\right]}{\partial p_{3-k}}$ is defined by (A.37). Thus, $p_{3-k} = \tilde{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ sets the net benefit of using source 3-k before permanently switching to source k to zero.

- When $\frac{p_k}{1-p_{3-k}} > \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}$, the curve $p_{3-k} = \tilde{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ is the set of points when the decision maker is indifferent between using source k only and using source 3-k until the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ (that is until $p_{3-k} = \frac{1}{1+\frac{c_k \lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1-p_{3-k}}{p_k}}$), then using both sources along this line until $p_{3-k} = p^{*(a)}_{3-k}$ and only then switching to source k permanently. In that case, condition (A.39) is equivalent to

$$\int\limits_{\frac{1}{1+\frac{c_k \lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1-p_{3-k}}{p_k}}}^{\tilde{p}^{(a)}_{3-k}\left[\frac{p_k}{1-p_{3-k}}\right]} \frac{\partial W_{3-k}\left[p'_{3-k}, \frac{p_k}{1-p_{3-k}}\right]}{\partial p'_{3-k}} dp'_{3-k}$$

$$+ \int\limits_{\frac{\tilde{p}_k\left[p^{*(a)}_{3-k}\right]}{1-p^{*(a)}_{3-k}}}^{\frac{p_k}{1-p_{3-k}}} \frac{\partial \left(W^{(a)}_{3-k}\left[\frac{1}{1+\frac{c_k \lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1}{q_k}}, q_k\right] - F^{(a)}_{3-k}\left[\frac{\lambda_{3-k} p^{*(a)}_{3-k}}{c_{3-k}}, q_k\right]\right)}{\partial q_k} dq_k = 0.$$

The first part is equal to $W_{3-k}\left[\tilde{p}^{(a)}_{3-k}\left[\frac{p_k}{1-p_{3-k}}\right], \frac{p_k}{1-p_{3-k}}\right] - W_{3-k}\left[\frac{1}{1+\frac{c_k \lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1-p_{3-k}}{p_k}}, \frac{p_k}{1-p_{3-k}}\right]$ and it is the net benefit of using source 3-k until the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$. The second part is equal to $W_{3-k}\left[\frac{1}{1+\frac{c_k \lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1-p_{3-k}}{p_k}}, \frac{p_k}{1-p_{3-k}}\right] - F^{(a)}_{3-k}\left[\frac{\lambda_{3-k} p^{*(a)}_{3-k}}{c_{3-k}}, \frac{p_k}{1-p_{3-k}}\right]$ and it is the net benefit of using both sources along the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ until $p_{3-k} = p^{*(a)}_{3-k}$. Thus, $p_{3-k} = \underline{\tilde{p}}^{(a)}_{3-k}\left[\frac{p_k}{1-p_{3-k}}\right]$ sets the net benefit of using source 3-k and then using both sources before permanently switching to source k to zero.

In sum, $p_{3-k} = \underline{\tilde{p}}^{(a)}_{3-k}\left[\frac{p_k}{1-p_{3-k}}\right]$ makes the net benefit of using source 3-k equal to zero.

To understand what function $\tilde{q}^{(a)}_{3-k}[q_k]$ means, note that wherever $p_{3-k} = \tilde{p}^{(a)}_{3-k}\left[\frac{p_k}{1-p_{3-k}}\right]$, we have $\tilde{q}^{(a)}_{3-k}\left[\frac{p_k}{1-p_{3-k}}\right] = \frac{p_{3-k}}{1-p_k}$. This is just a useful notation for a description of the optimal a-type strategy.

Recall that Case 5 requires the curve $p_{3-k} = \tilde{p}^{(a)}_{3-k}\left[\frac{p_k}{1-p_{3-k}}\right]$ to be outside the stopping region, that is, if $p_{3-k} = \tilde{p}^{(a)}_{3-k}\left[\frac{p_k}{1-p_{3-k}}\right]$, then $p_k \geq R^{(a)}_k$. In other words, this condition is equivalent to $\tilde{p}^{(a)}_{3-k}[q_k] \leq 1 - \frac{R^{(a)}_k}{q_k}$ for all $q_k \in \left[\frac{\tilde{p}^{(a)}_k[p^{**(a)}_{3-k}]}{1-p^{**(a)}_{3-k}}, 1\right)$:

$$\tilde{p}^{(a)}_{3-k}\left[\frac{p_k}{1-p_{3-k}}\right] \leq 1 - \frac{R^{(a)}_k}{\frac{p_k}{1-p_{3-k}}} \quad \Leftrightarrow \quad \frac{p_k}{1-p_{3-k}}\left(1 - \tilde{p}^{(a)}_{3-k}\left[\frac{p_k}{1-p_{3-k}}\right]\right) \geq R^{(a)}_k.$$

Denote

$$G_k^{(a)I} = 1 - \frac{c_k \lambda_{3-k}\left(1 - R_k^{(a)}\right)}{c_{3-k}\lambda_k R_k^{(a)}} + \log\left[\frac{c_k \lambda_{3-k}\left(1 - R_k^{(a)}\right)}{c_{3-k}\lambda_k R_k^{(a)}}\right] - \frac{1}{R_k^{(a)}}\left(\frac{1 - R_k^{(a)}}{R_{3-k}^{(a)}} - 1\right).$$

For $k = 1, 2$, when $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, denote

$$q_k^{*(a)} \in \left(\frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1 - p_{3-k}^{**(a)}}, 1\right): \quad W_{3-k}^{(a)}\left[1 - \frac{R_k^{(a)}}{q_k^{*(a)}}, q_k^{*(a)}\right] = L_{3-k}^{(a)}\left[q_k^{*(a)}\right]. \quad \text{(A.41)}$$

$q_k^{*(a)}$ has the meaning of the threshold above which condition $\tilde{p}_{3-k}^{(a)}[q_k] \leq 1 - \frac{R_k^{(a)}}{q_k}$ does not hold. When $G_k^{(a)I} \geq 0$, there is no such threshold and condition $\tilde{p}_{3-k}^{(a)}[q_k] \leq 1 - \frac{R_k^{(a)}}{q_k}$ holds all the time. Formally:

Lemma 13 *When $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, we have*

- $p_{3-k}^{**(a)} < 1 - \frac{R_k^{(a)}}{q_k} < 1$ *whenever* $\frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1 - p_{3-k}^{**(a)}} < q_k < 1$, *so that* $1 - \frac{R_k^{(a)}}{q_k}$ *is a valid first argument for function* $W_{3-k}[p_{3-k}, q_k]$*;*
- *if* $G_k^{(a)I} < 0$, *then the solution to (A.41) is unique,*
  - $W_{3-k}^{(a)}\left[1 - \frac{R_k^{(a)}}{q_k}, q_k\right] > L_{3-k}^{(a)}[q_k]$ *for* $q_k \in \left(\frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1 - p_{3-k}^{**(a)}}, q_k^{*(a)}\right)$, *and therefore* $\tilde{p}_{3-k}^{(a)}[q_k] < 1 - \frac{R_k^{(a)}}{q_k}$,
  - $W_{3-k}^{(a)}\left[1 - \frac{R_k^{(a)}}{q_k}, q_k\right] < L_{3-k}^{(a)}[q_k]$ *for* $q_k \in \left(q_k^{*(a)}, 1\right)$, *and therefore* $\tilde{p}_{3-k}^{(a)}[q_k] > 1 - \frac{R_k^{(a)}}{q_k}$*;*
- *if* $G_k^{(a)I} \geq 0$, *then* $\tilde{p}_{3-k}^{(a)}[q_k] < 1 - \frac{R_k^{(a)}}{q_k}$ *for* $q_k \in \left(\frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1 - p_{3-k}^{**(a)}}, 1\right)$.

Proof: When $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $\frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1 - p_{3-k}^{**(a)}} < q_k < 1$, we have

$$p_{3-k}^{**(a)} = 1 - \frac{R_k^{(a)}}{\frac{R_k^{(a)}}{1 - p_{3-k}^{**(a)}}} < 1 - \frac{R_k^{(a)}}{\frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1 - p_{3-k}^{**(a)}}} < 1 - \frac{R_k^{(a)}}{q_k} < 1 - R_k^{(a)} < 1.$$

When $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, consider a function $W_{3-k}^{(a)}\left[1-\frac{R_k^{(a)}}{q_k}, q_k\right] - L_{3-k}^{(a)}[q_k]$ for $\frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}} < q_k < 1$. It is continuous, its derivative is continuous,

$$\left.\frac{d\left(W_{3-k}^{(a)}\left[1-\frac{R_k^{(a)}}{q_k}, q_k\right] - W_{3-k}^{(a)}\left[\underline{\tilde{p}}_{3-k}^{(a)}[q_k], q_k\right]\right)}{dq_k}\right|_{q_k=\frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}}$$

$$= \left.\frac{d\left(W_{3-k}^{(a)}\left[1-\frac{R_k^{(a)}}{q_k}, q_k\right] - F_{3-k}^{(a)}\left[\frac{\lambda_{3-k} p_{3-k}^{*(a)}}{c_{3-k}}, q_k\right]\right)}{dq_k}\right|_{q_k=\frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}}$$

$$= \frac{\frac{1}{1-\frac{c_{3-k}\lambda_k R_k^{(a)}\left(1-p_{3-k}^{*(a)}\right)}{c_k \lambda_{3-k} p_{3-k}^{*(a)}}} - \frac{1-R_k^{(a)}}{R_{3-k}^{(a)}} - \frac{c_k \lambda_{3-k}}{c_{3-k}\lambda_k}}{R_k^{(a)}},$$

and it is concave:

$$\frac{d^2\left(W_{3-k}^{(a)}\left[1-\frac{R_k^{(a)}}{q_k}, q_k\right] - L_{3-k}^{(a)}[q_k]\right)}{dq_k^2} =$$

$$\begin{cases} -\left(\frac{1}{\left(q_k-R_k^{(a)}\right)^2} + \frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k q_k^2\left(1-\underline{\tilde{p}}_{3-k}^{(a)}[q_k]\right)\left(1-\bar{p}_k^{(a)}\left[\underline{\tilde{p}}_{3-k}^{(a)}[q_k]\right]\right)}\right), & \frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}} < q_k < \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}} \\ -\left(\frac{1}{\left(q_k-R_k^{(a)}\right)^2} + \frac{1}{(1-q_k)q_k}\left(1+\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k q_k}\right)\right), & \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}} < q_k < 1 \end{cases} < 0.$$

It is positive near $q_k = \frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}}$:

$$\lim_{q_k \to \frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}}} \left(W_{3-k}^{(a)}\left[1-\frac{R_k^{(a)}}{q_k}, q_k\right] - W_{3-k}^{(a)}\left[\underline{\tilde{p}}_{3-k}^{(a)}[q_k], q_k\right]\right) > 0$$

since $W_{3-k}^{(a)}\left[p_{3-k}, \frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}}\right]$ is increasing p_{3-k} for $p_{3-k} > p_{3-k}^{*(a)}$, $\lim_{q_k \to \frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}}} \underline{\tilde{p}}_{3-k}^{(a)}[q_k] = p_{3-k}^{**(a)}$ and $1 - \frac{R_k^{(a)}}{\frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}}} \in \left(p_{3-k}^{**}, 1\right)$.

The statement follows from

$$\lim_{q_k\to 1}\left(W^{(a)}_{3-k}\left[1-\frac{R^{(a)}_k}{q_k},q_k\right]-F^{(a)}_{3-k}\left[\frac{\lambda_{3-k}p^{*(a)}_{3-k}}{c_{3-k}},q_k\right]\right)=G^{(a)I}_k.$$

■

In sum, the conditions for Case 5 are $R^{(a)}_k<1$, $\frac{\lambda_{3-k}R^{(a)}_{3-k}}{c_{3-k}}\geq\frac{\lambda_k R^{(a)}_k}{c_k}$, $p^{*(a)}_{3-k}+\bar{p}^{(a)}_k\left[p^{*(a)}_{3-k}\right]<1$, $p^{**(a)}_{3-k}<R^{(a)}_{3-k}$, $G^{(a)I}_k\geq 0$, or equivalently

1. $\frac{c_k}{\lambda_k}<u_k[a_k]-u_k[a]$,
2. $u_k[a_k]-u_k[a]\geq u_{3-k}[a_{3-k}]-u_{3-k}[a]$,
3. $\frac{\lambda_k}{c_k}\left(u_{3-k}[a_{3-k}]-u_{3-k}[a]-\frac{c_{3-k}}{\lambda_{3-k}}\right)+\log\left[\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k]-u_k[a]-\frac{c_k}{\lambda_k}\right)\right]>1$,
4. $\frac{u_k[a_k]-u_k[a]}{u_k[a_k]-u_k[a]-\frac{c_k}{\lambda_k}}>\frac{\lambda_k\lambda_{3-k}}{c_{3-k}c_k}\left(u_{3-k}[a_{3-k}]-u_{3-k}[a]-\frac{c_{3-k}}{\lambda_{3-k}}\right)(u_{3-k}[a_{3-k}]-u_{3-k}[a])$,
5. $\frac{2+\log\left[\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k]-u_k[a]-\frac{c_k}{\lambda_k}\right)\right]}{\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k]-u_k[a]-\frac{c_k}{\lambda_k}\right)}\geq\frac{\lambda_k}{c_k}\left(u_{3-k}[a_{3-k}]-u_{3-k}[a]-\frac{c_{3-k}}{\lambda_{3-k}}\right)+1$,

with the interpretation that, given it is sometimes optimal to use source 3-k before switching to source k, the direct benefit from source 3-k is small enough (conditions 2, 4 and 5 limit $u_{3-k}[a_{3-k}]-u_{3-k}[a]$ from above).

The following lemma is useful for assuring a non-contradicting description of the optimal a-type strategy. Roughly speaking, it says that if point $(R^{(a)}_1,R^{(a)}_2)$ is inside the belief triangle, using source 3-k is never a bad idea at this point (we saw it in Case 4 and we will see it in Case 7).

Lemma 14 *When* $R^{(a)}_1<1$, $R^{(a)}_2<1$, $\frac{\lambda_{3-k}R^{(a)}_{3-k}}{c_{3-k}}\geq\frac{\lambda_k R^{(a)}_k}{c_k}$, $p^{*(a)}_{3-k}+\bar{p}^{(a)}_k\left[p^{*(a)}_{3-k}\right]<1$, $p^{**(a)}_{3-k}<R^{(a)}_{3-k}$, *we have:*

- $G^{(a)I}_k\geq 0$ *only if* $R^{(a)}_1+R^{(a)}_2\geq 1$. *Thus, point* $(R^{(a)}_1,R^{(a)}_2)$ *always lies outside the belief triangle.*
- *if* $G^{(a)I}_k<0$, *then* $q^{*(a)}_k<\frac{R^{(a)}_k}{1-R^{(a)}_{3-k}}$.

<u>*Proof:*</u> When $G^{(a)I}_k\geq 0$ we have $R^{(a)}_1+R^{(a)}_2\geq 1$ since $1-x+\log[x]\leq 0$ for all x. If $G^{(a)I}_k<0$, we have $\frac{R^{(a)}_k}{1-R^{(a)}_{3-k}}>\frac{\bar{p}^{(a)}_k\left[p^{**(a)}_{3-k}\right]}{1-p^{**(a)}_{3-k}}$, $\bar{\bar{p}}_{3-k}\left[\frac{R^{(a)}_k}{1-R^{(a)}_{3-k}}\right]=R^{(a)}_{3-k}$ and therefore

$$W^{(a)}_{3-k}\left[1-\frac{R^{(a)}_k}{q_k},q_k\right]-L^{(a)}_{3-k}[q_k]\Bigg|_{q_k=\frac{R^{(a)}_k}{1-R^{(a)}_{3-k}}}=W^{(a)}_{3-k}\left[\bar{\bar{p}}_{3-k}[q_k],q_k\right]-L^{(a)}_{3-k}[q_k]\Bigg|_{q_k=\frac{R^{(a)}_k}{1-R^{(a)}_{3-k}}}<0.$$

■

Case 6: The agent might be indifferent between the informatively superior source and stopping when his belief about state 3 is close to zero. Suppose it is not the first five cases. That means there exists k such that $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$. As we know from Lemma 13, this is the case when $\tilde{p}_{3-k}^{(a)}[q_k] < 1 - \frac{R_k^{(a)}}{q_k}$ for $q_k < q_k^{*(a)}$ and $\tilde{p}_{3-k}^{(a)}[q_k] > 1 - \frac{R_k^{(a)}}{q_k}$ for $q_k > q_k^{*(a)}$. Once the curve $p_{3-k} = \tilde{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ enters the stopping region, it changes its supposed interpretation. Now, we want the indifference curve to equalize the utility of stopping and the utility of using source $3-k$ before switching to source k. Thus, we need to change the equation for the indifference curve when $\frac{p_k}{1-p_{3-k}} > q_k^{*(a)}$. Let's denote this new curve as $p_{3-k} = \hat{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$.

Case 6 covers the range of parameters when, in addition to what has been mentioned just above, the indifference curve $p_{3-k} = \hat{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ goes all the way to the line $p_1 + p_2 = 1$, not touching $p_{3-k} = R_{3-k}^{*(a)}$ along the way.

Figures A.7, A.8 and A.9 show the optimal a-type strategy for Case 6.

Figures A.10, A.11 and A.12 show the partition of the belief triangle into Areas according to the type of the a-type strategy optimal for given beliefs.

To define the curve $p_{3-k} = \hat{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ formally, I need to introduce function $J_k^{(a)}[p_k, q_{3-k}]$ first.

Definition and properties of $J_k^{(a)}[p_k, q_{3-k}]$.

For $k = 1, 2$, when $p_k \in (0,1)$, denote

$$J_k^{(a)}[p_k, q_{3-k}] = \log\left[\frac{p_k}{1-p_k}\right] + \frac{1 - \frac{1}{R_k^{(a)}}}{1-p_k} - \frac{c_{3-k}\lambda_k q_{3-k}}{c_k \lambda_{3-k} R_{3-k}^{(a)}}.$$

To understand the meaning of this function, note that

$$\left.\frac{\partial\left(W_{3-k}^{(a)}[p_{3-k}, q_k] - J_{3-k}^{(a)}[p_{3-k}, q_k]\right)}{\partial p_{3-k}}\right|_{q_k = \frac{p_k}{1-p_{3-k}}} = \frac{c_k \lambda_{3-k}}{c_{3-k}\lambda_k} \frac{\log\left[\frac{R_k^{(a)}}{1-R_k^{(a)}}\right] - \log\left[\frac{p_k}{1-p_k}\right]}{(1-p_{3-k})^2},$$

so that the function $W_{3-k}^{(a)}\left[p_{3-k}, \frac{p_k}{1-p_{3-k}}\right] - J_{3-k}^{(a)}\left[p_{3-k}, \frac{p_k}{1-p_{3-k}}\right]$ is the total net benefit of using source k until $p_k = R_k^{(a)}$ after switching from source 3-k.

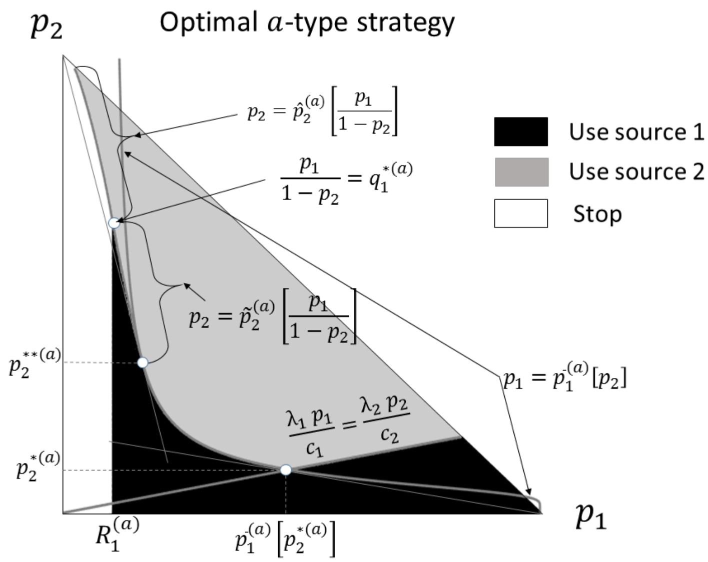

Figure A.7: Optimal a-type strategy for $R_1^{(a)} = 0.1$, $R_2^{(a)} = 10$, $\frac{c_1}{\lambda_1} = 5$, $\frac{c_2}{\lambda_2} = 1$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). White region means it is optimal to stop. Black region means the first information source should be used. Gray region means the second information source should be used. Case 6: $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, $R_{3-k}^{(a)} \geq 1$ or $G_k^{(a)II} \geq 0$, $k = 1$.

Coupled with the interpretation of $W_{3-k}^{(a)}\left[p_{3-k}, \frac{p_k}{1-p_{3-k}}\right]$ as the benefit of the a-type strategy of using source 3-k and then switching to source k permanently, we conclude that $J_{3-k}^{(a)}\left[p_{3-k}, \frac{p_k}{1-p_{3-k}}\right]$ is the benefit of using of source 3-k, conditional on switching to source k afterwards.

Properties:

1. $J_{3-k}^{(a)}[p_{3-k}, q_k]$ is increasing in p_{3-k} when $p_{3-k} < R_{3-k}^{(a)}$ and it is decreasing in p_{3-k} when $p_{3-k} > R_{3-k}^{(a)}$. Intuitively, $R_{3-k}^{(a)}$ is the threshold above which using source 3-k until $p_{3-k} = R_{3-k}^{(a)}$ is preferred to stopping. Hence, once we pass this threshold, the use of source 3-k, conditional on switching to source k afterwards, becomes less desirable.

   Proof: Follows from

$$\frac{\partial J_{3-k}^{(a)}[p_{3-k}, q_k]}{\partial p_{3-k}} = \frac{R_{3-k}^{(a)} - p_{3-k}}{(1-p_{3-k})^2 p_{3-k} R_{3-k}^{(a)}}.$$

■

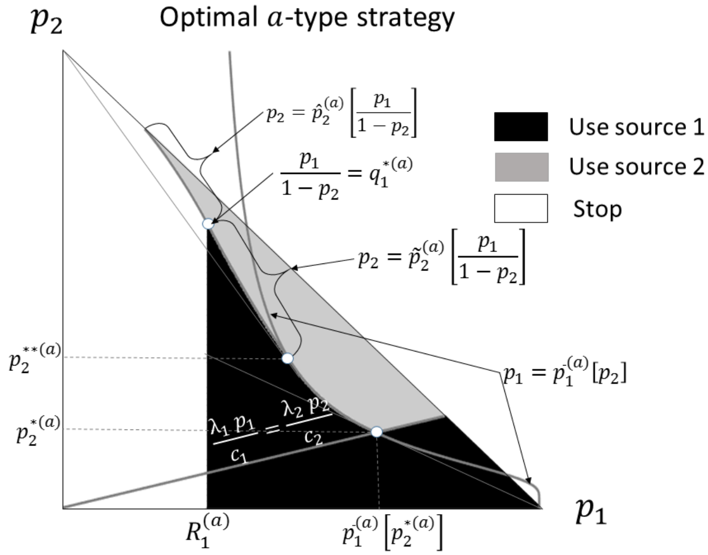

Figure A.8: Optimal a-type strategy for $R_1^{(a)} = 0.3$, $R_2^{(a)} = 10$, $\frac{c_1}{\lambda_1} = 4$, $\frac{c_2}{\lambda_2} = 1$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). White region means it is optimal to stop. Black region means the first information source should be used. Gray region means the second information source should be used. Case 6: $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, $R_{3-k}^{(a)} \geq 1$ or $G_k^{(a)II} \geq 0$, $k = 1$.

2. $\lim_{p_{3-k} \to 0} J_{3-k}^{(a)}[p_{3-k}, q_k] = -\infty$. Indeed, when the probability of state 3-k is zero, source 3-k is completely useless.

3. When $R_{3-k}^{(a)} < 1$, $\lim_{p_{3-k} \to 1} J_{3-k}^{(a)}[p_{3-k}, q_k] = -\infty$.

4. When $R_{3-k}^{(a)} \geq 1$, $\lim_{p_{3-k} \to 1} J_{3-k}^{(a)}[p_{3-k}, q_k] = +\infty$.

5. For $k = 1, 2$, when $R_k^{(a)} < 1$, $p_{3-k} \in (0, 1)$, $q_k \in (0, 1)$, we have $J_{3-k}^{(a)}[p_{3-k}, q_k] \leq W_{3-k}^{(a)}[p_{3-k}, q_k]$, with equality if and only if $p_{3-k} = 1 - \frac{R_k^{(a)}}{q_k}$. This echoes the interpretation of $W_{3-k}^{(a)}\left[p_{3-k}, \frac{p_k}{1-p_{3-k}}\right] - J_{3-k}^{(a)}\left[p_{3-k}, \frac{p_k}{1-p_{3-k}}\right]$ as the benefit of using source k after switching. When $p_{3-k} = 1 - \frac{R_k^{(a)}}{\frac{p_k}{1-p_{3-k}}}$ (equivalent to $p_k = R_k^{(a)}$), this benefit is zero since according to the optimal a-type strategy the agent is indifferent between stopping and using source k. When $p_{3-k} \neq 1 - \frac{R_k^{(a)}}{\frac{p_k}{1-p_{3-k}}}$ (equivalent to $p_k \neq R_k^{(a)}$), the farther p_k from the threshold, the higher the benefit from the opportunity to use source k after switching from source 3-k in the course of applying the

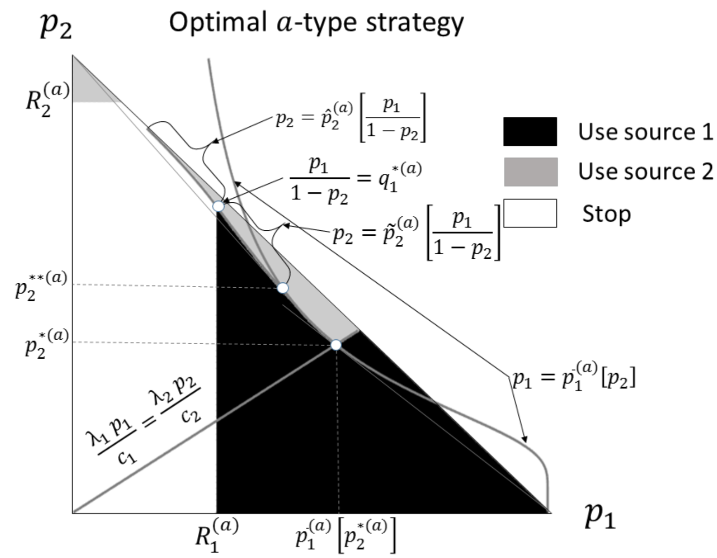

Figure A.9: Optimal a-type strategy for $R_1^{(a)} = 0.3$, $R_2^{(a)} = 0.9$, $\frac{c_1}{\lambda_1} = 1.5$, $\frac{c_2}{\lambda_2} = 1$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). White region means it is optimal to stop. Black region means the first information source should be used. Gray region means the second information source should be used. Case 6: $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, $R_{3-k}^{(a)} \geq 1$ or $G_k^{(a)II} \geq 0$, $k = 1$.

optimal a-type strategy.

Proof:

$$J_{3-k}^{(a)}[p_{3-k}, q_k] - W_{3-k}^{(a)}[p_{3-k}, q_k] =$$
$$\frac{c_k \lambda_{3-k}}{c_{3-k}\lambda_k} q_k \left(\frac{1}{R_k^{(a)}} - 1\right)(x - 1 - x\log[x])\Bigg|_{x=\frac{\frac{1}{q_k(1-p_{3-k})}-1}{\frac{1}{R_k^{(a)}}-1}} \leq 0.$$

■

Definition and properties of $\hat{p}_{3-k}^{(a)}[q_k]$ and $q_k^{**(a)}$.

Without specifying the details, I say that $\hat{p}_{3-k}^{(a)}[q_k]$, $q_k \geq q_k^{*(a)}$, is the solution to $J_{3-k}^{(a)}\left[\hat{p}_{3-k}^{(a)}[q_k], q_k\right] = L_{3-k}^{(a)}[q_k]$. Before introducing the full definition, let's discuss the intuition behind it.

- When $\frac{p_k}{1-p_{3-k}} < \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}$, the curve $p_{3-k} = \hat{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$ is the set of points when the agent is indifferent between stopping, and using

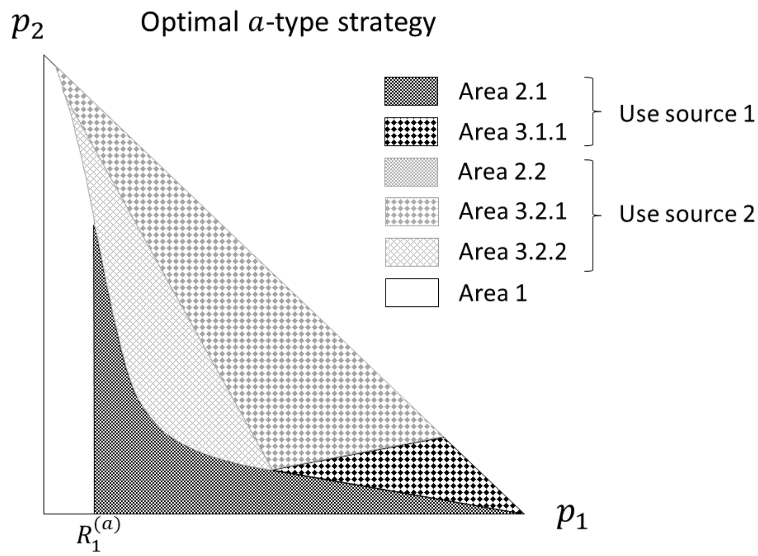

Figure A.10: Optimal a-type strategy for $R_1^{(a)} = 0.1$, $R_2^{(a)} = 10$, $\frac{c_1}{\lambda_1} = 5$, $\frac{c_2}{\lambda_2} = 1$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). Area 1 means it is optimal to stop. Areas 2.1 and 3.1.1 mean the first information source should be used. Areas 2.2, 3.2.1 and 3.2.2 mean the second information source should be used. Areas 2.1 and 2.2 correspond to the payoff optimal phase. Areas 3.1.1, 3.2.1, 3.2.2 correspond to the informatively optimal phase. Case 6: $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)} \left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, $R_{3-k}^{(a)} \geq 1$ or $G_k^{(a)II} \geq 0$, $k = 1$.

source 3-k until $p_{3-k} = \underaccent{\tilde}{p}_{3-k}^{(a)} \left[\frac{p_k}{1-p_{3-k}}\right]$ and then switching to source k permanently:

$$L_{3-k}^{(a)} \left[\frac{p_k}{1-p_{3-k}}\right] - J_{3-k}^{(a)} \left[\hat{p}_{3-k}^{(a)} \left[\frac{p_k}{1-p_{3-k}}\right], \frac{p_k}{1-p_{3-k}}\right] =$$
$$\left(W_{3-k}^{(a)} \left[\underaccent{\tilde}{p}_{3-k}^{(a)} \left[\frac{p_k}{1-p_{3-k}}\right], \frac{p_k}{1-p_{3-k}}\right] - W_{3-k}^{(a)} \left[\hat{p}_{3-k}^{(a)} \left[\frac{p_k}{1-p_{3-k}}\right], \frac{p_k}{1-p_{3-k}}\right]\right) +$$
$$\left(W_{3-k}^{(a)} \left[\hat{p}_{3-k}^{(a)} \left[\frac{p_k}{1-p_{3-k}}\right], \frac{p_k}{1-p_{3-k}}\right] - J_{3-k}^{(a)} \left[\hat{p}_{3-k}^{(a)} \left[\frac{p_k}{1-p_{3-k}}\right], \frac{p_k}{1-p_{3-k}}\right]\right) = 0.$$

The first part reflects the benefit of using source 3-k until $p_{3-k} = \underaccent{\tilde}{p}_{3-k}^{(a)} \left[\frac{p_k}{1-p_{3-k}}\right]$. The second part shows the benefit of using source k afterwards.

- When $\frac{p_k}{1-p_{3-k}} > \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}$, the curve $p_{3-k} = \hat{p}_{3-k}^{(a)} \left[\frac{p_k}{1-p_{3-k}}\right]$ is the set of points when the agent is indifferent between stopping, and using source 3-k until the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ (that is, until $p_{3-k} = \frac{1}{1+\frac{c_k \lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1-p_{3-k}}{p_k}}$), then

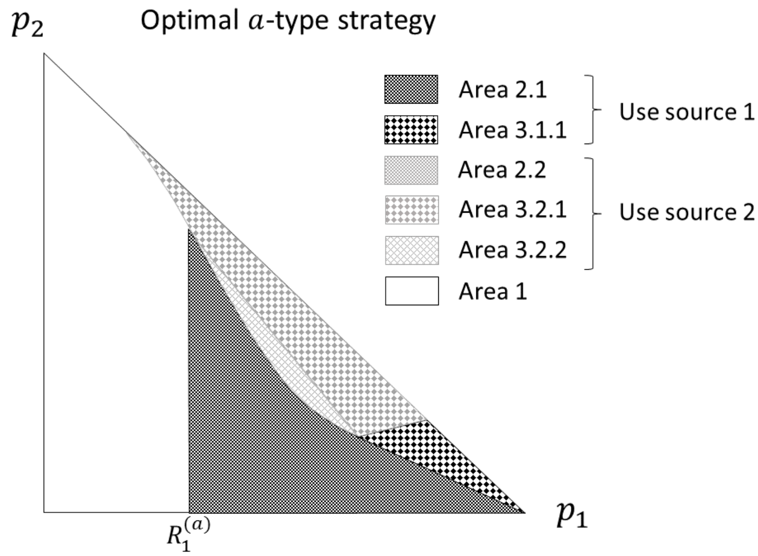

Figure A.11: Optimal a-type strategy for $R_1^{(a)} = 0.3$, $R_2^{(a)} = 10$, $\frac{c_1}{\lambda_1} = 4$, $\frac{c_2}{\lambda_2} = 1$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). Area 1 means it is optimal to stop. Areas 2.1 and 3.1.1 mean the first information source should be used. Areas 2.2, 3.2.1 and 3.2.2 mean the second information source should be used. Areas 2.1 and 2.2 correspond to the payoff optimal phase. Areas 3.1.1, 3.2.1, 3.2.2 correspond to the informatively optimal phase. Case 6: $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, $R_{3-k}^{(a)} \geq 1$ or $G_k^{(a)II} \geq 0$, $k = 1$.

using both sources along this line until $p_{3-k} = p_{3-k}^{*(a)}$ and only then switching to source k permanently:

$$L_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right] - J_{3-k}^{(a)}\left[\hat{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right], \frac{p_k}{1-p_{3-k}}\right] =$$

$$\left(W_{3-k}\left[\frac{1}{1+\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1-p_{3-k}}{p_k}}, \frac{p_k}{1-p_{3-k}}\right] - W_{3-k}^{(a)}\left[\hat{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right], \frac{p_k}{1-p_{3-k}}\right]\right) +$$

$$\left(F_{3-k}^{(a)}\left[\frac{\lambda_{3-k} p_{3-k}^{*(a)}}{c_{3-k}}, \frac{p_k}{1-p_{3-k}}\right] - W_{3-k}\left[\frac{1}{1+\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1-p_{3-k}}{p_k}}, \frac{p_k}{1-p_{3-k}}\right]\right) +$$

$$\left(W_{3-k}^{(a)}\left[\hat{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right], \frac{p_k}{1-p_{3-k}}\right] - J_{3-k}^{(a)}\left[\hat{p}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right], \frac{p_k}{1-p_{3-k}}\right]\right) = 0.$$

The first part is equal to the benefit of using source 3-k until the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$. The second part is equal to the benefit of using both sources

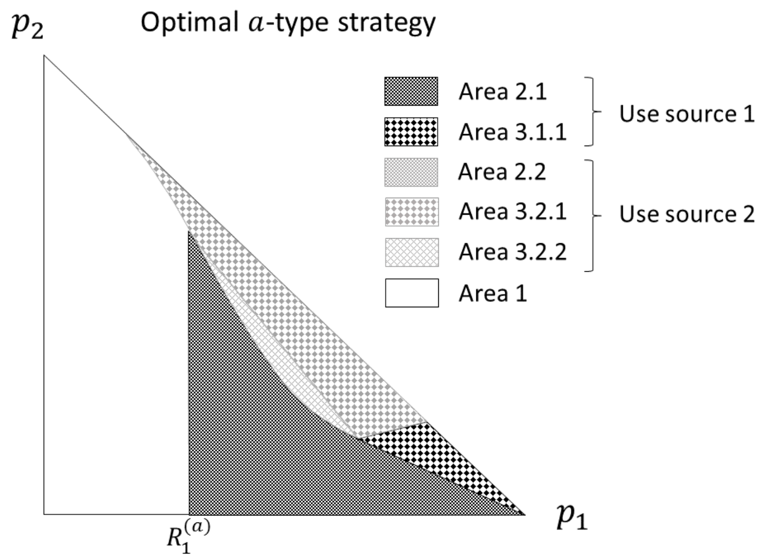

Figure A.12: Optimal a-type strategy for $R_1^{(a)} = 0.3$, $R_2^{(a)} = 0.9$, $\frac{c_1}{\lambda_1} = 1.5$, $\frac{c_2}{\lambda_2} = 1$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). Area 1 means it is optimal to stop. Areas 2.1 and 3.1.1 mean the first information source should be used. Areas 2.2, 3.2.1 and 3.2.2 mean the second information source should be used. Areas 2.1 and 2.2 correspond to the payoff optimal phase. Areas 3.1.1, 3.2.1, 3.2.2 correspond to the informatively optimal phase. Case 6: $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)} \left[p_{3-k}^{*(a)} \right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, $R_{3-k}^{(a)} \geq 1$ or $G_k^{(a)II} \geq 0$, $k = 1$.

along the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ until $p_{3-k} = p_{3-k}^{*(a)}$. The last part is equal to the benefit of using source k afterwards.

So, now, in contrast to function $\tilde{p}_k^{(a)}[q_k]$, I need to add the benefit of the last part of the trajectory, which corresponds to using source k until $p_k = R_k^{(a)}$.

Now I proceed to the full definition. It consists of three scenarios: when $R_{3-k}^{(a)} \geq 1$, when $R_{3-k}^{(a)} < 1$ and the equation $J_{3-k}^{(a)} \left[\hat{p}_{3-k}^{(a)}[q_k], q_k \right] = L_{3-k}^{(a)}[q_k]$ has the solution for all $q_k^{*(a)} < q_k < 1$, and when $R_{3-k}^{(a)} < 1$ and I can define $\hat{p}_k^{(a)}[q_k]$ only up to some threshold $q_k^{**(a)}$. The first two scenarios correspond to Case 6, the last one is relevant for Case 7.

For $k = 1, 2$, when $R_k^{(a)} < 1$, $R_{3-k}^{(a)} \geq 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)} \left[p_{3-k}^{*(a)} \right] < 1$, denote

$$\hat{p}_{3-k}^{(a)}[q_k] \in (0, 1) : \quad J_{3-k}^{(a)} \left[\hat{p}_{3-k}^{(a)}[q_k], q_k \right] = L_{3-k}^{(a)}[q_k]. \tag{A.42}$$

Note that since $J_{3-k}^{(a)}[p_{3-k}, q_k]$ is increasing in p_{3-k} from $-\infty$ to $+\infty$, the solution to (A.42) always exists and is unique.

When $R_{3-k}^{(a)} < 1$, function $J_{3-k}^{(a)}[p_{3-k}, q_k]$ is increasing when $p_{3-k} < R_{3-k}^{(a)}$ and is decreasing to $-\infty$ afterwards. So, it might be the case that equation $J_{3-k}^{(a)}[p_{3-k}, q_k] = L_{3-k}^{(a)}[q_k]$ has no solution (more precisely, it happens when $J_{3-k}^{(a)}\left[R_{3-k}^{(a)}, q_k\right] - L_{3-k}^{(a)}[q_k] < 0$). Moreover, when it does, it might have two solutions. The following lemma describes the behavior of function $J_{3-k}^{(a)}\left[R_{3-k}^{(a)}, q_k\right] - L_{3-k}^{(a)}[q_k] = \sup\limits_{p_{3-k}} J_{3-k}^{(a)}[p_{3-k}, q_k] - L_{3-k}^{(a)}[q_k]$.

Denote

$$G_k^{(a)II} = 1 - \frac{c_k\lambda_{3-k}\left(1 - R_k^{(a)}\right)}{c_{3-k}\lambda_k R_k^{(a)}} + \log\left[\frac{c_k\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}\lambda_k\left(1 - R_{3-k}^{(a)}\right)}\right].$$

Lemma 15 *For $k = 1, 2$, when $R_1^{(a)} < 1$, $R_2^{(a)} < 1$, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, function $J_{3-k}^{(a)}\left[R_{3-k}^{(a)}, q_k\right] - L_{3-k}^{(a)}[q_k]$ is decreasing for $\frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}} < q_k < 1$ to $G_k^{(a)II}$.*

<u>*Proof:*</u>

- $\frac{\bar{p}_k^{(a)}[p_{3-k}^{**(a)}]}{1-p_{3-k}^{**(a)}} < q_k < \frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}$:

$$\begin{aligned}&\frac{d\left(J_{3-k}^{(a)}\left[R_{3-k}^{(a)}, q_k\right] - L_{3-k}^{(a)}[q_k]\right)}{dq_k}\\ &\quad= \frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\left(\log\left[\frac{\frac{1}{\bar{p}_k^{(a)}\left[\underline{\tilde{p}}_{3-k}^{(a)}[q_k]\right]} - 1}{\frac{1}{R_k^{(a)}} - 1}\right] + \frac{1}{\bar{p}_k^{(a)}\left[\underline{\tilde{p}}_{3-k}^{(a)}[q_k]\right]} - \frac{1}{R_k^{(a)}}\right) < 0\end{aligned}$$

since $\underline{\tilde{p}}_{3-k}^{(a)}[q_k] < p_{3-k}^{**(a)} < R_{3-k}^{(a)} \Rightarrow \bar{p}_k^{(a)}\left[\underline{\tilde{p}}_{3-k}^{(a)}[q_k]\right] > \bar{p}_k^{(a)}\left[R_{3-k}^{(a)}\right] = R_k^{(a)}$.

- $\left.\frac{d\left(J_{3-k}^{(a)}\left[R_{3-k}^{(a)},q_k\right]-L_{3-k}^{(a)}[q_k]\right)}{dq_k}\right|_{q_k=\frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}-0} = \left.\frac{d\left(J_{3-k}^{(a)}\left[R_{3-k}^{(a)},q_k\right]-L_{3-k}^{(a)}[q_k]\right)}{dq_k}\right|_{q_k=\frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}}+0} = \frac{1}{R_{3-k}^{(a)}} - \frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k R_k^{(a)}}$.

- $\frac{\bar{p}_k^{(a)}[p_{3-k}^{*(a)}]}{1-p_{3-k}^{*(a)}} < q_k < 1$:

$$\frac{d^2\left(J_{3-k}^{(a)}\left[R_{3-k}^{(a)}, q_k\right] - L_{3-k}^{(a)}[q_k]\right)}{dq_k^2} = -\frac{q_k + \frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}}{(1-q_k)q_k^2} < 0.$$

- $\lim_{q_k\to 1}\left(J_{3-k}^{(a)}\left[R_{3-k}^{(a)}, q_k\right] - L_{3-k}^{(a)}[q_k]\right) = G_k^{(a)II}$.

∎

Now, I can define $\hat{p}_k^{(a)}[q_k]$ for the second and the third scenario.

For $k = 1,2$, when $R_1^{(a)} < 1$, $R_2^{(a)} < 1$, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, $G_k^{(a)II} \geq 0$, $q_k \geq q_k^{*(a)}$, denote

$$\hat{p}_{3-k}^{(a)}[q_k] \in \left(0, R_{3-k}^{(a)}\right): \quad J_{3-k}^{(a)}\left[\hat{p}_{3-k}^{(a)}[q_k], q_k\right] = L_{3-k}^{(a)}[q_k]. \tag{A.43}$$

Note that since $J_{3-k}^{(a)}[p_{3-k}, q_k]$ is increasing in p_{3-k} from $-\infty$ to $J_{3-k}^{(a)}[R_{3-k}, q_k] \geq L_{3-k}^{(a)}[q_k]$, the solution to (A.43) always exists and is unique.

For $k = 1,2$, when $R_1^{(a)} < 1$, $R_2^{(a)} < 1$, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, $G_k^{(a)II} < 0$, denote

$$q_k^{**(a)} \in \left[q_k^{*(a)}, 1\right): \quad J_{3-k}^{(a)}\left[R_{3-k}^{(a)}, q_k^{**(a)}\right] = L_{3-k}^{(a)}\left[q_k^{**(a)}\right]. \tag{A.44}$$

Note that since $J_{3-k}^{(a)}\left[R_{3-k}^{(a)}, q_k\right] - L_{3-k}^{(a)}[q_k]$ is decreasing on $q_k \in \left(q_k^{*(a)}, 1\right)$ from $J_{3-k}^{(a)}\left[R_{3-k}^{(a)}, q_k^{*(a)}\right] - J_{3-k}^{(a)}\left[1 - \frac{R_k^{(a)}}{q_k^{*(a)}}, q_k^{*(a)}\right] \geq 0$ to $G_k^{(a)II} < 0$, the solution to (A.44) always exists and is unique.

For $k = 1,2$, when $R_1^{(a)} < 1$, $R_2^{(a)} < 1$, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, $G_k^{(a)II} < 0$, $q_k^{*(a)} \leq q_k \leq q_k^{**(a)}$, define $\hat{p}_{3-k}^{(a)}[q_k]$ by (A.43). Note that the solution to (A.43) always exists and is unique since $q_k^{**(a)}$ is defined in such a way that $J_{3-k}^{(a)}[R_{3-k}, q_k] \geq L_{3-k}^{(a)}[q_k]$ for $q_k^{*(a)} \leq q_k \leq q_k^{**(a)}$.

Note that since $J_{3-k}^{(a)}\left[R_{3-k}^{(a)}, q_k\right] - L_{3-k}^{(a)}[q_k]$ is decreasing and since $J_{3-k}^{(a)}[p_k, q_k] - J_{3-k}^{(a)}\left[R_{3-k}^{(a)}, q_k\right]$ does not depend on q_k, function $\hat{p}_k^{(a)}[q_k]$ is increasing in q_k.

The following lemma is the analog of Lemma 13 for function $\hat{p}_{3-k}^{(a)}[q_k]$. It says that the indifference curve is continuous and crosses the line $p_k = R_k^{(a)}$ only when $\frac{p_k}{1-p_{3-k}} = q_k^{*(a)}$.

Lemma 16 *When* $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)} \left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, *we have*

- $\tilde{p}_{3-k}^{(a)}[q_k^{*(a)}] = \hat{p}_{3-k}^{(a)}[q_k^{*(a)}] = 1 - \frac{R_k^{(a)}}{q_k^{*(a)}}$,
- *when* $R_{3-k}^{(a)} \geq 1$, *we have* $\hat{p}_{3-k}^{(a)}[q_k] > 1 - \frac{R_k^{(a)}}{q_k}$ *for* $q_k \in \left(q_k^{*(a)}, 1\right)$,
- *when* $R_{3-k}^{(a)} < 1$, $G_k^{(a)II} \geq 0$, *we have* $\hat{p}_{3-k}^{(a)}[q_k] > 1 - \frac{R_k^{(a)}}{q_k}$ *for* $q_k \in \left(q_k^{*(a)}, \min\left\{1, \frac{R_k^{(a)}}{1-R_{3-k}^{(a)}}\right\}\right)$,
- *when* $R_{3-k}^{(a)} < 1$, $G_k^{(a)II} < 0$, *we have* $\hat{p}_{3-k}^{(a)}[q_k] > 1 - \frac{R_k^{(a)}}{q_k}$ *for* $q_k \in \left(q_k^{*(a)}, \min\left\{q_k^{**(a)}, \frac{R_k^{(a)}}{1-R_{3-k}^{(a)}}\right\}\right)$.

<u>*Proof:*</u> Since $J_{3-k}^{(a)}\left[1 - \frac{R_k^{(a)}}{q_k}, q_k\right] = W_{3-k}^{(a)}\left[1 - \frac{R_k^{(a)}}{q_k}, q_k\right]$, we have $J_{3-k}^{(a)}\left[1 - \frac{R_k^{(a)}}{q_k}, q_k\right] - L_{3-k}^{(a)}[q_k] = W_{3-k}^{(a)}\left[1 - \frac{R_k^{(a)}}{q_k}, q_k\right] - L_{3-k}^{(a)}[q_k]$. The properties of function $W_{3-k}^{(a)}\left[1 - \frac{R_k^{(a)}}{q_k}, q_k\right] - L_{3-k}^{(a)}[q_k]$ has been studied in Lemma 13. ■

Case 6 corresponds to $G_k^{(a)I} \geq 0$. The analog of Lemma 14 holds here as well.

Lemma 17 *When* $R_1^{(a)} < 1$, $R_2^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$ *we have* $G_k^{(a)II} \geq 0$ *only if* $R_1^{(a)} + R_2^{(a)} \geq 1$. *Thus, point* $(R_1^{(a)}, R_2^{(a)})$ *always lies outside the belief triangle.*

<u>*Proof:*</u> Since $1 - x + \log[x] \leq 0$ for all x. ■

In sum, the conditions for Case 6 are $R_k^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, $R_{3-k}^{(a)} \geq 1$ or $G_k^{(a)II} \geq 0$, or equivalently

1. $\frac{c_k}{\lambda_k} < u_k[a_k] - u_k[a]$,
2. $u_k[a_k] - u_k[a] \geq u_{3-k}[a_{3-k}] - u_{3-k}[a]$,
3. $\frac{\lambda_k}{c_k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] - \frac{c_{3-k}}{\lambda_{3-k}}\right) + \log\left[\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k] - u_k[a] - \frac{c_k}{\lambda_k}\right)\right] > 1$,
4. $\frac{u_k[a_k]-u_k[a]}{u_k[a_k]-u_k[a]-\frac{c_k}{\lambda_k}} > \frac{\lambda_k \lambda_{3-k}}{c_{3-k} c_k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] - \frac{c_{3-k}}{\lambda_{3-k}}\right)(u_{3-k}[a_{3-k}] - u_{3-k}[a])$,
5. $\frac{2+\log\left[\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k]-u_k[a]-\frac{c_k}{\lambda_k}\right)\right]}{\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k]-u_k[a]-\frac{c_k}{\lambda_k}\right)} < \frac{\lambda_k}{c_k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] - \frac{c_{3-k}}{\lambda_{3-k}}\right) + 1$,

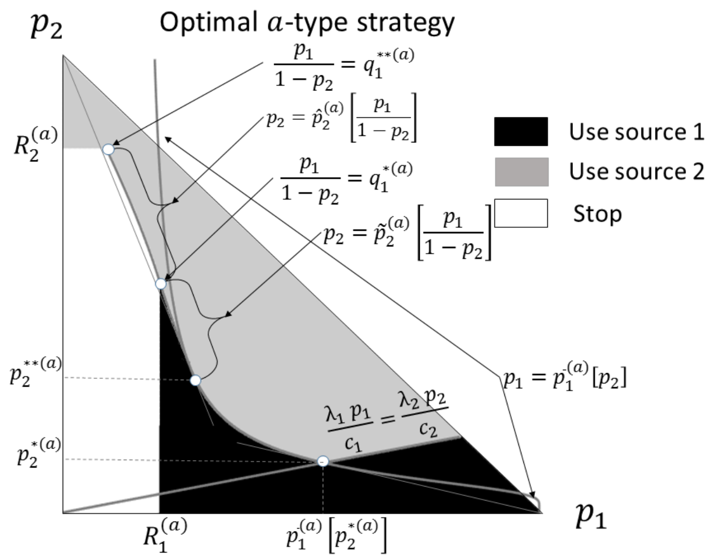

Figure A.13: Optimal a-type strategy for $R_1^{(a)} = 0.2$, $R_2^{(a)} = 0.8$, $\frac{c_1}{\lambda_1} = 5$, $\frac{c_2}{\lambda_2} = 1$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). White region means it is optimal to stop. Black region means the first information source should be used. Gray region means the second information source should be used. Case 7: $R_1^{(a)} < 1$, $R_2^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, $G_k^{(a)II} < 0$, $k = 1$.

6. if $\frac{c_{3-k}}{\lambda_{3-k}} < u_{3-k}[a_{3-k}] - u_{3-k}[a]$, then $\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k] - u_k[a] - \frac{c_k}{\lambda_k}\right) + \log\left[\frac{\lambda_k}{c_k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] - \frac{c_{3-k}}{\lambda_{3-k}}\right)\right] \leq 1$.

Case 7. This is the last case.

Figure A.13 shows the optimal a-type strategy for Case 7.

Figure A.14 shows the partition of the belief triangle into Areas according to the type of the a-type strategy optimal for given beliefs.

Conditions for Case 7 are $R_1^{(a)} < 1$, $R_2^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, $G_k^{(a)II} < 0$, or equivalently

1. $\frac{c_k}{\lambda_k} < u_k[a_k] - u_k[a]$, $\frac{c_{3-k}}{\lambda_{3-k}} < u_{3-k}[a_{3-k}] - u_{3-k}[a]$,
2. $u_k[a_k] - u_k[a] \geq u_{3-k}[a_{3-k}] - u_{3-k}[a]$,
3. $\frac{\lambda_k}{c_k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] - \frac{c_{3-k}}{\lambda_{3-k}}\right) + \log\left[\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k] - u_k[a] - \frac{c_k}{\lambda_k}\right)\right] > 1$,
4. $\frac{u_k[a_k] - u_k[a]}{u_k[a_k] - u_k[a] - \frac{c_k}{\lambda_k}} > \frac{\lambda_k \lambda_{3-k}}{c_{3-k} c_k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] - \frac{c_{3-k}}{\lambda_{3-k}}\right)(u_{3-k}[a_{3-k}] - u_{3-k}[a])$,

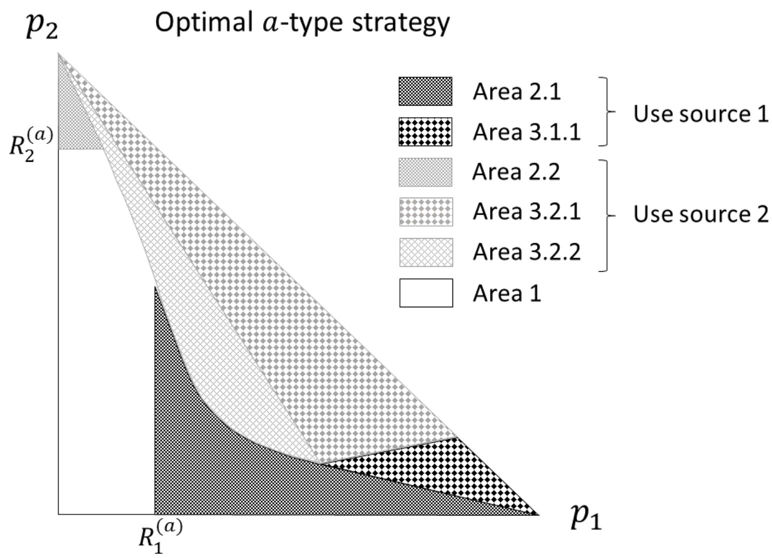

Figure A.14: Optimal a-type strategy for $R_1^{(a)} = 0.2$, $R_2^{(a)} = 0.8$, $\frac{c_1}{\lambda_1} = 5$, $\frac{c_2}{\lambda_2} = 1$. Each point (p_1, p_2) corresponds to what to do if the current beliefs are (p_1, p_2). Area 1 means it is optimal to stop. Areas 2.1 and 3.1.1 mean the first information source should be used. Areas 2.2, 3.2.1 and 3.2.2 mean the second information source should be used. Areas 2.1 and 2.2 correspond to the payoff optimal phase. Areas 3.1.1, 3.2.1, 3.2.2 correspond to the informatively optimal phase. Case 7: $R_1^{(a)} < 1$, $R_2^{(a)} < 1$, $\frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)} \left[p_{3-k}^{*(a)} \right] < 1$, $p_{3-k}^{**(a)} < R_{3-k}^{(a)}$, $G_k^{(a)I} < 0$, $G_k^{(a)II} < 0$, $k = 1$.

5. $\frac{2+\log\left[\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k]-u_k[a]-\frac{c_k}{\lambda_k}\right)\right]}{\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k]-u_k[a]-\frac{c_k}{\lambda_k}\right)} < \frac{\lambda_k}{c_k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] - \frac{c_{3-k}}{\lambda_{3-k}}\right) + 1,$

6. $\frac{\lambda_{3-k}}{c_{3-k}}\left(u_k[a_k] - u_k[a] - \frac{c_k}{\lambda_k}\right) + \log\left[\frac{\lambda_k}{c_k}\left(u_{3-k}[a_{3-k}] - u_{3-k}[a] - \frac{c_{3-k}}{\lambda_{3-k}}\right)\right] > 1.$

To understand how these cases partition the parameter space, see Figure A.15.

Theorem 3 has three cases in the optimal a-type strategy description. Case 1 there corresponds to Cases 1-3 here, Case 2 there corresponds to Case 4 here, and Case 3 there corresponds to Cases 5-7 here. Notation $\bar{\pi}_{3-k}^{(a)}[q_k]$ combines $\tilde{p}_{3-k}^{(a)}[q_k]$ and $\hat{p}_{3-k}^{(a)}[q_k]$:

Case 5 : $\bar{\pi}_{3-k}^{(a)}[q_k] = \tilde{p}_{3-k}^{(a)}[q_k]$ for all $q_k \in \left(\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{**(a)}\right]}{1-p_{3-k}^{**(a)}}, 1\right)$,

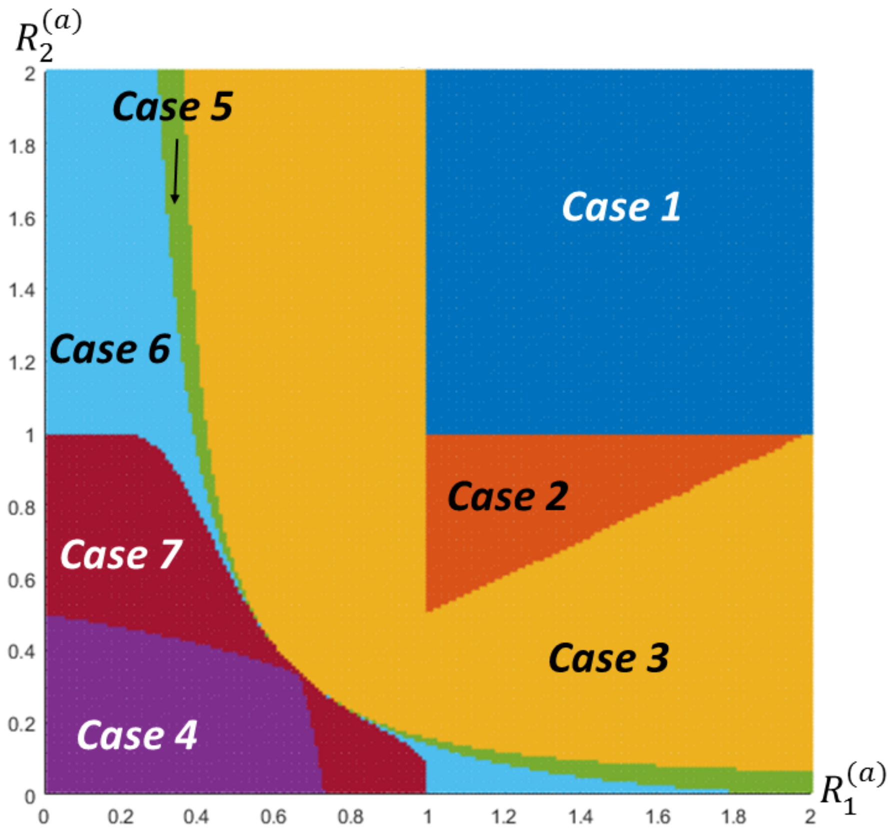

Figure A.15: The partition of the space (R_1, R_2) into Cases 1-7 when $\frac{c_1\lambda_2}{c_2\lambda_1} = 2$.

Case 6 :

$$\bar{\pi}_{3-k}^{(a)}[q_k] = \begin{cases} \tilde{p}_{3-k}^{(a)}[q_k], & q_k \in \left(\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{**(a)}\right]}{1-p_{3-k}^{**(a)}}, q_k^{*(a)} \right], \\ \hat{p}_{3-k}^{(a)}[q_k], & q_k \in \left[q_k^{*(a)}, 1 \right), \end{cases}$$

Case 7 :

$$\bar{\pi}_{3-k}^{(a)}[q_k] = \begin{cases} \tilde{p}_{3-k}^{(a)}[q_k], & q_k \in \left(\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{**(a)}\right]}{1-p_{3-k}^{**(a)}}, q_k^{*(a)} \right], \\ \hat{p}_{3-k}^{(a)}[q_k], & q_k \in \left[q_k^{*(a)}, q_k^{**(a)} \right], \\ 1, & q_k \in \left[q_k^{**(a)}, 1 \right). \end{cases}$$

Function $B_k^{(a)}[p_{3-k}, q_k]$ solves

$$W_{3-k}^{(a)}[p_{3-k}, q_k]\mathbf{1}\left((1-p_{3-k})q_k > R_k^{(a)}\right)+J_{3-k}^{(a)}[p_{3-k}, q_k]\mathbf{1}\left((1-p_{3-k})q_k \leq R_k^{(a)}\right)-L_{3-k}^{(a)}[q_k]$$
$$= \frac{1-(1-p_k)q_k}{1-p_{3-k}}\left(B_k^{(a)}[p_{3-k}, q_k] - \frac{1-R_{3-k}^{(a)}}{R_{3-k}^{(a)}}\right).$$

Proof of Theorem 3

Given Lemma 5, it is sufficient to check that function $V[p_1, p_2]$ satisfies the following five conditions:

Step 1 $\mathcal{L}_k[p_1, p_2] = 0$ for Areas 2.k, 3.k.1 and 3.k.2, $k = 1, 2$,

Step 2 $\mathcal{L}_{3-k}[p_1, p_2] \geq 0$ for Areas 1, 2.k, 3.k.1 and 3.k.2,

Step 3 $V[p_1, p_2] \in C$ everywhere,

Step 4 $V[p_1, p_2] \in C^1$ along the line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ that separates Area 3.1.1 and Area 3.2.1,

Step 5 $U^{(a)}[p_1, p_2] \leq V[p_1, p_2]$ for Area 2.k, 3.k.1 and 3.k.2.

Step 1 Step 1 can be checked directly.

Step 2 For Step 2 we have:

$$\mathcal{L}_{3-k}[p_1, p_2]\bigg|_{\text{Area 1}} = \frac{c_{3-k}}{\lambda_{3-k}}\left(\frac{1}{p_{3-k}} - \frac{1}{R_{3-k}^{(a)}}\right).$$

For Area 1, we always have $p_1 \leq R_1^{(a)}$ and $p_2 \leq R_2^{(a)}$. Thus, $\mathcal{L}_{3-k}[p_1, p_2]\Big|_{\text{Area 1}} \geq 0$.

$$\mathcal{L}_{3-k}[p_1, p_2]\bigg|_{\text{Area 2.k}} = \frac{c_k}{\lambda_k}\log\left[\frac{(1-p_k)\bar{p}_k^{(a)}[p_{3-k}]}{p_k\left(1-\bar{p}_k^{(a)}[p_{3-k}]\right)}\right].$$

For Area 2.k, we always have $p_k \leq \bar{p}_k^{(a)}[p_{3-k}]$. Thus, $\mathcal{L}_{3-k}[p_1, p_2]\Big|_{\text{Area 2.k}} \geq 0$.

$$\mathcal{L}_{3-k}[p_1, p_2]\bigg|_{\text{Area 3.k.1}} = \frac{c_k}{\lambda_k}\left(\frac{c_{3-k}\lambda_k p_k}{c_k\lambda_{3-k}p_{3-k}} - 1 - \log\left[\frac{c_{3-k}\lambda_k p_k}{c_k\lambda_{3-k}p_{3-k}}\right]\right).$$

For Area 3.k.1, we always have $\frac{\lambda_k p_k}{c_k} \geq \frac{\lambda_{3-k} p_{3-k}}{c_{3-k}}$. Since $x - 1 - \log[x] \geq 0$ for all $x \geq 1$, we have $\mathcal{L}_{3-k}[p_1, p_2]\Big|_{\text{Area 3.k.1}} \geq 0$.

$$\mathcal{L}_{3-k}[p_1, p_2]\Big|_{\text{Area 3.k.2}} = \frac{c_k}{\lambda_k}\left(\left(\frac{c_{3-k}\lambda_k p_k}{c_k \lambda_{3-k} p_{3-k}} - \log\left[\frac{c_{3-k}\lambda_k p_k}{c_k \lambda_{3-k} p_{3-k}}\right]\right)\right.$$
$$\left.-\left(\frac{c_{3-k}\lambda_k(1-p_k)\underline{\tilde{p}}_k\left[\frac{p_{3-k}}{1-p_k}\right]}{c_k\lambda_{3-k}p_{3-k}\left(1-\underline{\tilde{p}}_k\left[\frac{p_{3-k}}{1-p_k}\right]\right)} - \log\left[\frac{c_{3-k}\lambda_k(1-p_k)\underline{\tilde{p}}_k\left[\frac{p_{3-k}}{1-p_k}\right]}{c_k\lambda_{3-k}p_{3-k}\left(1-\underline{\tilde{p}}_k\left[\frac{p_{3-k}}{1-p_k}\right]\right)}\right]\right)\right).$$

For Area 3.k.2, we always have $\underline{\tilde{p}}_k\left[\frac{p_{3-k}}{1-p_k}\right] \leq p_k$ and $\frac{\lambda_k \underline{\tilde{p}}_k\left[\frac{p_{3-k}}{1-p_k}\right]}{c_k} \geq \frac{\lambda_{3-k}\bar{p}^{(a)}_{3-k}\left[\underline{\tilde{p}}_k\left[\frac{p_{3-k}}{1-p_k}\right]\right]}{c_{3-k}}$, which is equivalent to $1 \leq \frac{c_{3-k}\lambda_k(1-p_k)\underline{\tilde{p}}_k\left[\frac{p_{3-k}}{1-p_k}\right]}{c_k\lambda_{3-k}p_{3-k}\left(1-\underline{\tilde{p}}_k\left[\frac{p_{3-k}}{1-p_k}\right]\right)} \leq \frac{c_{3-k}\lambda_k p_k}{c_k\lambda_{3-k}p_{3-k}}$. Since $x - \log[x]$ is increasing for all $x \geq 1$, we have $\mathcal{L}_{3-k}[p_1, p_2]\Big|_{\text{Area 3.k.2}} \geq 0$.

Step 3 One can check directly that

$$V[p_1, p_2]\Big|_{LHS} = V[p_1, p_2]\Big|_{RHS},$$

where LHS and RHS are defined in Table A.1.

Step 4 One can check directly that

$$\frac{\partial V[p_1, p_2]}{\partial p_j}\Bigg|_{\text{Area 3.1.1, } \frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}} = \frac{\partial V[p_1, p_2]}{\partial p_j}\Bigg|_{\text{Area 3.2.1, } \frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}}, \quad j = 1, 2.$$

Step 5 For Step 5, we differentiate along the line $\frac{p_{3-k}}{1-p_k} = \text{const}$

$$\frac{\partial\left(\frac{V[p_1,p_2]-U^{(a)}[p_1,p_2]}{1-p_k}\Big|_{p_{3-k}=q_{3-k}(1-p_k)}\right)}{\partial p_k}\Bigg|_{\text{Area 2.k, Area 3.k.1, Area 3.k.2}} = \frac{c_k(p_k - R_k^{(a)})}{\lambda_k(1-p_k)^2 p_k R_k^{(a)}}.$$

Since for Area 2.k we always have $p_k \geq R_k^{(a)}$ and $V[p_1, p_2] = U^{(a)}[p_1, p_2]$ when $p_k = R_k^{(a)}$, we have $V[p_1, p_2] - U^{(a)}[p_1, p_2]\Big|_{\text{Area 2.k}} \geq 0$.

LHS	RHS
Area 2.k, $p_k = R_k^{(a)}$	Area 1, $p_k = R_k^{(a)}$
Area 2.k, $\frac{p_{3-k}}{1-p_k} = \frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$	Area 3.k.1, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $\frac{p_{3-k}}{1-p_k} = \frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$
Area 3.1.1, $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$	Area 3.2.1, $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$
Area 3.k.1, $\frac{\lambda_k R_k^{(a)}}{c_k} > \frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}}$, $\frac{p_{3-k}}{1-p_k} = \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}$	Area 3.k.2, $\frac{p_{3-k}}{1-p_k} = \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}$
Area 2.k, $\frac{p_{3-k}}{1-p_k} = \frac{R_{3-k}^{(a)}}{1-R_k^{(a)}}$	Area 3.k.2, $\frac{p_{3-k}}{1-p_k} = \frac{R_{3-k}^{(a)}}{1-R_k^{(a)}}$, $\underline{\tilde{p}}_k^{(a)}\left[\frac{R_{3-k}^{(a)}}{1-R_k^{(a)}}\right] = R_k^{(a)}$
Area 2.k, $\frac{p_{3-k}}{1-p_k} = q_{3-k}^{**(a)}$	Area 3.k.2, $\frac{p_{3-k}}{1-p_k} = q_{3-k}^{**(a)} \leq \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}$
Area 2.k, $\frac{p_{3-k}}{1-p_k} = q_{3-k}^{**(a)}$	Area 3.k.1, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \leq \frac{\lambda_k R_k^{(a)}}{c_k}$, $\frac{p_{3-k}}{1-p_k} = q_{3-k}^{**(a)} \geq \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}$
Area 3.k.2, $p_k = \underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$	Area 2.3-k, $p_k = \underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$
Area 3.k.2, $p_k = \tilde{p}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$, $\frac{p_{3-k}}{1-p_k} \leq \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}$	Area 2.3-k, $p_k = \tilde{p}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$, $\frac{p_{3-k}}{1-p_k} \leq \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}$
Area 3.k.1, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \leq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_k=\tilde{p}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$, $\frac{p_{3-k}}{1-p_k} \geq \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}$	Area 2.3-k, $p_k=\tilde{p}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$, $\frac{p_{3-k}}{1-p_k} \geq \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}$
Area 3.k.2, $p_k = \hat{p}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$, $\frac{p_{3-k}}{1-p_k} \leq \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}$	Area 1, $p_k = \hat{p}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$, $\frac{p_{3-k}}{1-p_k} \leq \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}$
Area 3.k.1, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \leq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_k=\hat{p}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$, $\frac{p_{3-k}}{1-p_k} \geq \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}$	Area 1, $p_k=\hat{p}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]$, $\frac{p_{3-k}}{1-p_k} \geq \frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}$

Table A.1: Proof of Theorem 3. Step 3.

For Area 3.k.2, we need to make sure that if $p_k = R_k^{(a)}$ lies inside this area, we still have $V[p_1, p_2] \geq U^{(a)}[p_1, p_2]$ (the rest follows from continuity of V).

$$\left.\frac{\partial\left(\left.\frac{V[p_1,p_2]-U^{(a)}[p_1,p_2]}{1-p_k}\right|_{\substack{p_{3-k}=q_{3-k}(1-R_k^{(a)}),\\ p_k=R_k^{(a)}}}\right)}{\partial q_{3-k}}\right|_{\text{Area 3.k.2}} =$$

$$\frac{c_k}{\lambda_k}\left(\frac{1}{\underline{\tilde{p}}_k^{(a)}[q_{3-k}]} - \frac{1}{R_k^{(a)}}\right) + \frac{c_{3-k}}{\lambda_{3-k}}\left(\frac{1}{R_{3-k}^{(a)}} - \frac{1}{\bar{p}_{3-k}^{(a)}\left[\underline{\tilde{p}}_k^{(a)}[q_{3-k}]\right]}\right).$$

Area 3.k.2 is present only when $\frac{\lambda_k R_k^{(a)}}{c_k} > \frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}}$. We also know that

$\underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right] < R_k^{(a)}$ and $\bar{p}_k^{(a)}\left[\underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]\right] > R_{3-k}^{(a)}$. Thus, $V[p_1,p_2]-U^{(a)}[p_1,p_2]\Big|_{\text{Area 3.k.2}} \geq 0$ because $V[p_1,p_2]$ is continuous and $V[p_1,p_2]-U^{(a)}[p_1,p_2]\Big|_{\text{Area 2.k}} \geq 0$.

For Area 3.k.1, $\frac{\lambda_k R_k^{(a)}}{c_k} \leq \frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}}$, we have $p_k \geq R_k^{(a)}$. So, $\frac{V[p_1,p_2]-U^{(a)}[p_1,p_2]}{1-p_k}$ is increasing in p_k along the trajectory of beliefs movement. Thus, it is sufficient to show that $V[p_1,p_2] \geq U^{(a)}[p_1,p_2]$ for the switching line $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$. Consider function

$$g_1[q_{3-k}] = \left(\left(c_k\lambda_{3-k}q_{3-k} + c_{3-k}\lambda_k\right)\left(V[p_1,p_2]-U^{(a)}[p_1,p_2]\right)\Big|_{p_k=\frac{\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}q_{3-k}}{1+\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}q_{3-k}}}\right)\Bigg|_{\text{Area 3.k.1, } \frac{\lambda_k R_k^{(a)}}{c_k} \leq \frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}}}$$

for $q_{3-k} \geq \frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$. It is convex,

$$g_1''[q_{3-k}] = \frac{c_{3-k}(c_k\lambda_{3-k} + c_{3-k}\lambda_k)}{(1-q_{3-k})q_{3-k}^2\lambda_{3-k}} > 0,$$

and it is non-decreasing at $q_{3-k} = \frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$:

$$g_1'\left[\frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}\right] = \frac{c_k^2\lambda_{3-k}\left(\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]-R_k^{(a)}\right)}{R_k^{(a)}\lambda_k\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]} \geq 0.$$

Given the continuity of V and the fact that we already showed $V[p_1,p_2] \geq U^{(a)}[p_1,p_2]$ for Area 2.k, we have $g_1[q_{3-k}] \geq 0$ for $q_{3-k} \geq \frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$. Thus,

$$V[p_1,p_2]-U^{(a)}[p_1,p_2]\Big|_{\text{Area 3.k.1, } \frac{\lambda_k R_k^{(a)}}{c_k} \leq \frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}}} \geq 0.$$

For Area 3.k.1, $\frac{\lambda_k R_k^{(a)}}{c_k} > \frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}}$, we define

$$g_2[q_{3-k}] = \left(\frac{V[p_1,p_2]-U^{(a)}[p_1,p_2]}{1-p_k}\Bigg|_{\substack{p_{3-k}=q_{3-k}(1-R_k^{(a)}),\\ p_k=R_k^{(a)}}}\right)\Bigg|_{\text{Area 3.k.1, } \frac{\lambda_k R_k^{(a)}}{c_k} > \frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}}}.$$

$$g_2''[q_{3-k}] = \frac{c_k\lambda_{3-k}q_{3-k} + c_{3-k}\lambda_k}{(1-q_{3-k})q_{3-k}^2\lambda_k\lambda_{3-k}} > 0, \quad g_2'\left[\frac{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-p_k^{*(a)}}\right] = \frac{c_{3-k}}{\lambda_{3-k}R_{3-k}^{(a)}} - \frac{c_k}{\lambda_k R_k^{(a)}} > 0.$$

Thus, $V[p_1,p_2] \geq U^{(a)}[p_1,p_2]$ for Area 3.k.1, $\frac{\lambda_k R_k^{(a)}}{c_k} > \frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}}$.

■

Remark 3 *At the curve* $p_{3-k} = \bar{\pi}_{3-k}^{(a)}\left[\frac{p_k}{1-p_{3-k}}\right]$, *function* $V[p_1,p_2]$ *is non-differentiable. This is the curve* $\mathcal{R}$ *in Lemma 5.*

A.7 Proof of Theorem 4.

Lemma 18 gives explicit expressions for the expected time the agent spends using each of the sources.

Lemma 18 *Let* (p_1,p_1) *be the initial beliefs. Then the expected time* $T_j[p_1,p_2]$ *of using source* j, $j=1,2$, *can be expressed as follows:*

1. *If for some* $k=1,2$, $p_k > R_k^{(a)}$ *and the optimal strategy is to use source* k *until* $p_k = R_k^{(a)}$, *with the default alternative* a, *then* $T_{3-k}[p_1,p_2]=0$ *and*

$$T_k[p_1,p_2] = \frac{p_k - R_k^{(a)}}{1-R_k^{(a)}}\frac{1}{\lambda_k} + \frac{1-p_k}{\lambda_k}\log\left[\frac{(1-R_k^{(a)})p_k}{R_k^{(a)}(1-p_k)}\right]. \tag{A.45}$$

2. *If the optimal strategy is* 3.*a-strategy and*

   - *if* $a=a_k$ *for some* $k\in\{1,2\}$, *then (2.24) holds,*
   - *if* $a=a_3$ *and* $k\in\{1,2\}$ *is such that (2.25) holds, then (2.28) holds,*

   then

$$T_k[p_1,p_2] = \frac{p_k - \underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]}{1-\underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]}\frac{1}{\lambda_k} + \frac{1-p_k}{\lambda_k}\log\left[\frac{\left(1-\underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]\right)p_k}{\underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right](1-p_k)}\right], \tag{A.46}$$

$$T_{3-k}[p_1,p_2] = \frac{p_{3-k}}{\lambda_{3-k}}\left(\frac{1-R_{3-k}^{(a)}x}{1-R_{3-k}^{(a)}} + (x-1)\log\left[\frac{\frac{1}{R_{3-k}^{(a)}}-1}{x-1}\right]\right)\Bigg|_{x=\frac{1}{\frac{p_{3-k}}{1-p_k}\left(1-\underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]\right)}}. \tag{A.47}$$

3. *If the optimal strategy is* 3.*a-strategy and*

   - *if* $a=a_k$ *for some* $k\in\{1,2\}$, *then either (2.22) or (2.23) holds,*

- *if $a = a_3$ and $k \in \{1,2\}$ is such that (2.25) holds, then either (2.26) or (2.27) holds,*

then

$$T_k[p_1,p_2] = \frac{(1-p_1-p_2)\log\left[\frac{\frac{1-p_k^{*(a)}}{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}-1}{\frac{1-p_k}{p_{3-k}}-1}\right] + (1-p_k)\log\left[\frac{p_k\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{p_{3-k}p_k^{*(a)}}\right] + 1 - \frac{1-p_1-p_2}{1-p_k^{*(a)}-\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}}{\lambda_k}, \tag{A.48}$$

$$\begin{aligned} T_{3-k}[p_1,p_2] &= \frac{(1-p_1-p_2)\log\left[\frac{\frac{1-p_k^{*(a)}}{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}-1}{\frac{1-p_k}{p_{3-k}}-1}\right] + p_k^{*(a)}\left(\frac{p_{3-k}}{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]} - \frac{p_k}{p_k^{*(a)}}\right)}{\lambda_{3-k}} \\ &+ \frac{1 - \frac{1-p_1-p_2}{1-p_k^{*(a)}-\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}\left(\frac{1-\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}\left(\frac{p_k^{*(a)}}{R_k^{(a)}}-1\right) + \frac{1-\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]}{1-R_{3-k}^{(a)}}\right)}{\lambda_{3-k}}. \end{aligned} \tag{A.49}$$

Proof of Lemma 18

Step 1 It is straightforward to show that

1. $T_1[p_1,p_2]$ and $T_2[p_1,p_2]$ are continuously differentiable along $\frac{\lambda_1 p_1}{c_1} = \frac{\lambda_2 p_2}{c_2}$ when both sources are used simultaneously for a non-zero period of time, and
2. $T_1[p_1,p_2]$ and $T_2[p_1,p_2]$ are continuous along any trajectory of beliefs (so that they are continuous at switching curves).

Step 2 $T_j[p_1,p_2] = \mathbb{E}\left[T_{\tau,j} \mid p_0 = p\right]$ *for* $j = 1,2$

Proof: Let (a^F, T, τ) be the optimal strategy given the initial beliefs p_1, p_2.

By Step 1 of the proof, $T_j[p_1,p_2] \in C^1$ along the belief trajectory (except maybe one switching point). Thus, I can use Ito's formula in a similar way to how I used it in Lemma 5 and get the analog of the expression (A.24):

$$T_j[p_{0,1},p_{0,2}] = \mathbb{E}\left[T_j[p_{\tau,1},p_{\tau,2}] \mid p_0\right] + \lambda_1 \mathbb{E}\left[\int_0^\tau p_{t,1}\mathcal{L}_1^{T_j}[p_{t,1},p_{t,2}]dT_{t,1} \mid p_0\right]$$
$$-\mathbb{E}\left[\int_0^\tau c_1 dT_{t,1} \mid p_0\right] + \lambda_2 \mathbb{E}\left[\int_0^\tau p_{t,2}\mathcal{L}_2^{T_j}[p_{t,1},p_{t,2}]dT_{t,2} \mid p_0\right] - \mathbb{E}\left[\int_0^\tau c_2 dT_{t,2} \mid p_0\right], \tag{A.50}$$

where

$$\mathcal{L}_i^{T_j}[p_1,p_2] = \frac{c_i}{\lambda_i p_i} + \frac{\partial T_j[p_1,p_2]}{\partial p_i}(1-p_i) - \frac{\partial T_j[p_1,p_2]}{\partial p_{3-i}} p_{3-i} - \left(0 - T_j[p_1,p_2]\right). \tag{A.51}$$

(A.50) is equivalent to

$$T_j[p_{0,1},p_{0,2}] = \mathbb{E}\left[T_j[p_{\tau,1},p_{\tau,2}] \mid p_0\right] + \lambda_1 \mathbb{E}\left[\int_0^\tau p_{t,1}\mathcal{M}_{1,j}[p_{t,1},p_{t,2}]dT_{t,1} \mid p_0\right]$$
$$+ \mathbb{E}\left[T_{\tau,j} \mid p_0\right] + \lambda_2 \mathbb{E}\left[\int_0^\tau p_{t,2}\mathcal{M}_{2,j}[p_{t,1},p_{t,2}]dT_{t,2} \mid p_0\right], \tag{A.52}$$

where

$$\mathcal{M}_{i,j}[p_1,p_2] = -\frac{\mathbf{1}(i=j)}{\lambda_j p_j} + \frac{\partial T_j[p_1,p_2]}{\partial p_i}(1-p_i) - \frac{\partial T_j[p_1,p_2]}{\partial p_{3-i}} p_{3-i} + T_j[p_1,p_2]. \tag{A.53}$$

It is straightforward to show that $\mathcal{M}_{i,j}[p_1,p_2] = 0$ whenever source i is used and $T_j[p_1,p_2] = 0$ for all stopping points. Thus, $T_j[p_1,p_2] = \mathbb{E}\left[T_{\tau,j} \mid p_0 = p\right]$. ■

■

Let a be the optimal default alternative.

For $k = 1,2$, when $R_k^{(a)} < 1$, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$ and $q_k \in \left(\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{**(a)}\right]}{1-p_{3-k}^{**(a)}}, 1\right)$, denote

$$H_k^{(a)}[q_k] = \begin{cases} \frac{1-p_{3-k}^{*(a)}-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{\left(1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)(1-q_k)}\left(\frac{1-p_{3-k}^{*(a)}}{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}q_k - 1\right) - \frac{1}{1-R_k^{(a)}} + \log\left[\frac{\frac{1}{R_k^{(a)}}-1}{\frac{1}{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}-1}\right], & q_k > \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}, \\ -\frac{1}{1-R_k^{(a)}} + \log\left[\frac{\frac{1}{R_k^{(a)}}-1}{\frac{1}{\bar{p}_k^{(a)}\left[\underline{\tilde{p}}_{3-k}^{(a)}[q_k]\right]}-1}\right], & q_k \leq \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}. \end{cases} \tag{A.54}$$

Note that this function is continuous at point $q_k = \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$.

The reason to introduce this function is revealed when I calculate the derivatives $\frac{\partial T_j[c_1,c_2]}{\partial c_{3-j}}$:

1. If initial beliefs are in Area 3.k.1 of the optimal a-type strategy and $\frac{\lambda_k R_k^{(a)}}{c_k} \leq \frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}}$, then

$$\frac{\partial T_j\left[c_1,c_2\right]}{\partial c_{3-j}} = \frac{p_{3-k}^{*(a)}(1-p_1-p_2)\left(1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)H_k^{(a)}\left[\frac{1}{1+\frac{c_{3-k}\lambda_k(1-p_1-p_2)}{c_k\lambda_{3-k}p_{3-k}}}\right]}{c_{3-k}\lambda_k\left(1-p_{3-k}^{*(a)}-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)^2}, \quad j=1,2. \tag{A.55}$$

2. If initial beliefs are in Area 3.k.1 of the optimal a-type strategy and $\frac{\lambda_k R_k^{(a)}}{c_k} > \frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}}$, then

$$\frac{\partial T_j\left[c_1,c_2\right]}{\partial c_{3-j}} = \frac{p_k^{*(a)}(1-p_1-p_2)\left(1-\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]\right)H_{3-k}^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]}{c_k\lambda_{3-k}\left(1-p_k^{*(a)}-\bar{p}_{3-k}^{(a)}\left[p_k^{*(a)}\right]\right)^2}, \quad j=1,2. \tag{A.56}$$

3. If initial beliefs are in Area 3.k.2 of the optimal a-type strategy, then $\frac{\lambda_k R_k^{(a)}}{c_k} > \frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}}$ and

$$\frac{\partial T_j\left[c_1,c_2\right]}{\partial c_{3-j}} = \frac{p_{3-k}\left(1-\bar{p}_{3-k}^{(a)}\left[\underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]\right]\right)\left(-\underline{\tilde{p}}_k^{(a)\prime}\left[\frac{p_{3-k}}{1-p_k}\right]\right)H_{3-k}^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]}{c_{3-k}\lambda_k\underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]\left(1-\underline{\tilde{p}}_k^{(a)}\left[\frac{p_{3-k}}{1-p_k}\right]\right)^2}, \quad j=1,2. \tag{A.57}$$

Thus, the sign of function $H_k^{(a)}$ at a certain point coincides with the sign of $\frac{\partial T_j[c_1,c_2]}{\partial c_{3-j}}$, where $k \in \{1,2\}$ is such that $\frac{\lambda_k R_k^{(a)}}{c_k} \leq \frac{\lambda_{3-k} R_{3-k}^{(a)}}{c_{3-k}}$.

Properties of $H_k^{(a)}\left[q_k\right]$: $\lim_{q_k \to 1} H_k^{(a)}[q_k] = +\infty$, $\lim_{q_k \to \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{**(a)}\right]}{1-p_{3-k}^{**(a)}}} H_k^{(a)}[q_k] = \frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\left(\frac{1}{p_{3-k}^{**(a)}} - \frac{1}{R_{3-k}^{(a)}}\right) - \frac{1}{1-R_k^{(a)}}$, $H_k^{(a)}[q_k]$ is increasing in q_k.

For $k=1,2$, when $R_k^{(a)}<1$, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}}\geq\frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)}+\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]<1$ and $p_{3-k}^{**(a)}>\frac{R_{3-k}^{(a)}}{\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{R_{3-k}^{(a)}}{1-R_k^{(a)}}+1}$, denote

$$\hat{q}_k^{(a)}\in\left(\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{**(a)}\right]}{1-p_{3-k}^{**(a)}},1\right):\quad H_k^{(a)}\left[\hat{q}_k^{(a)}\right]=0. \tag{A.58}$$

Note that the condition $p_{3-k}^{**(a)}>\frac{R_{3-k}^{(a)}}{\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{R_{3-k}^{(a)}}{1-R_k^{(a)}}+1}$ is equivalent to $\lim_{q_k\to\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{**(a)}\right]}{1-p_{3-k}^{**(a)}}}H_k^{(a)}[q_k]<0$, so that the solution to (A.58) exists and is unique.

$\hat{q}_k^{(a)}$ serves as a threshold, so that when $\frac{p_k}{1-p_{3-k}}>\hat{q}_k^{(a)}$, the sources are substitutes and when $\frac{p_k}{1-p_{3-k}}<\hat{q}_k^{(a)}$, they are complements.

Lemma 19 *For $k=1,2$,*

1. *when $R_1^{(a)}+R_2^{(a)}<1$ (which corresponds to Cases 4 and 7), we have $\hat{q}_k^{(a)}>\frac{R_k^{(a)}}{1-R_{3-k}^{(a)}}$;*
2. *$\hat{q}_k^{(a)}<\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$ if and only if $p_{3-k}^{*(a)}<\frac{R_{3-k}^{(a)}}{\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{R_{3-k}^{(a)}}{1-R_k^{(a)}}+1}$ if and only if $H_k^{(a)I}\equiv e^{-\frac{1}{1-R_k^{(a)}}}\frac{1-R_k^{(a)}}{R_k^{(a)}}-\frac{R_k^{(a)}}{1-R_k^{(a)}}-\frac{\lambda_k c_{3-k}}{c_k\lambda_{3-k}}\frac{1}{R_{3-k}^{(a)}}>0$;*
3. *when $R_k^{(a)}<1$, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}}\geq\frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)}+\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]<1$, we have*

$$p_{3-k}^{**(a)}>\frac{R_{3-k}^{(a)}}{\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{R_{3-k}^{(a)}}{1-R_k^{(a)}}+1}\quad\Leftrightarrow\quad \textit{if } H_k^{(a)I}>0, \textit{ then } H_k^{(a)II}<0,$$

where

$$H_k^{(a)II}\equiv e^{\frac{1}{1-R_k^{(a)}}}\frac{R_k^{(a)}}{1-R_k^{(a)}}+1-\left(\frac{1}{R_{3-k}^{(a)}}+\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1}{1-R_k^{(a)}}\right)\left(\frac{c_{3-k}\lambda_k}{c_k\lambda_{3-k}}\frac{1-R_{3-k}^{(a)}}{R_{3-k}^{(a)}}+\frac{1}{1-R_k^{(a)}}\right).$$

Proof:

1. It follows from $\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{**(a)}\right]}{1-p_{3-k}^{**(a)}} \leq \frac{R_k^{(a)}}{1-R_{3-k}^{(a)}} \leq \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$ and $H_k^{(a)}\left[\frac{R_k^{(a)}}{1-R_{3-k}^{(a)}}\right] = -\frac{1}{1-R_k^{(a)}} < 0$.

2. $H_k^{(a)}\left[\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}\right] = -\frac{1}{1-R_k^{(a)}} + \log\left[\frac{\frac{1}{R_k^{(a)}}-1}{\frac{1}{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}-1}\right]$.

   $-\frac{1}{1-R_k^{(a)}} + \log\left[\frac{\frac{1}{R_k^{(a)}}-1}{\frac{1}{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}-1}\right] > 0$ is equivalent to $p_{3-k}^{*(a)} < \frac{R_{3-k}^{(a)}}{\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{R_{3-k}^{(a)}}{1-R_k^{(a)}}+1}$, which is equivalent to $e^{-\frac{1}{1-R_k^{(a)}}}\frac{1-R_k^{(a)}}{R_k^{(a)}} - \frac{R_k^{(a)}}{1-R_k^{(a)}} > \frac{\lambda_k c_{3-k}}{c_k\lambda_{3-k}}\frac{1}{R_{3-k}^{(a)}}$.

   Indeed, $-\frac{1}{1-R_k^{(a)}} + \log\left[\frac{\frac{1}{R_k^{(a)}}-1}{\frac{1}{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}-1}\right] > 0$ is equivalent to $1 + e^{-\frac{1}{1-R_k^{(a)}}}\frac{1-R_k^{(a)}}{R_k^{(a)}} > \frac{1}{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$. Recall that by definitions (2.17) and (2.18), we have $\log\left[\frac{1}{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]} - 1\right] + \frac{1}{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]} = \frac{\lambda_k c_{3-k}}{c_k\lambda_{3-k}}\frac{1}{R_{3-k}^{(a)}} + \log\left[\frac{1-R_k^{(a)}}{R_k^{(a)}}\right]$. Thus, $1 + e^{-\frac{1}{1-R_k^{(a)}}}\frac{1-R_k^{(a)}}{R_k^{(a)}} > \frac{1}{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$ is equivalent to

   $$\log\left[\left(1 + e^{-\frac{1}{1-R_k^{(a)}}}\frac{1-R_k^{(a)}}{R_k^{(a)}}\right) - 1\right] + \left(1 + e^{-\frac{1}{1-R_k^{(a)}}}\frac{1-R_k^{(a)}}{R_k^{(a)}}\right) > \frac{\lambda_k c_{3-k}}{c_k\lambda_{3-k}}\frac{1}{R_{3-k}^{(a)}} + \log\left[\frac{1-R_k^{(a)}}{R_k^{(a)}}\right].$$

   After some simplification, this becomes $e^{-\frac{1}{1-R_k^{(a)}}}\frac{1-R_k^{(a)}}{R_k^{(a)}} - \frac{R_k^{(a)}}{1-R_k^{(a)}} > \frac{\lambda_k c_{3-k}}{c_k\lambda_{3-k}}\frac{1}{R_{3-k}^{(a)}}$.

   On the other hand, $1 + e^{-\frac{1}{1-R_k^{(a)}}}\frac{1-R_k^{(a)}}{R_k^{(a)}} > \frac{1}{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}$ can be rewritten as $p_{3-k}^{*(a)} < \frac{R_{3-k}^{(a)}}{\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{R_{3-k}^{(a)}}{1-R_k^{(a)}}+1}$.

3. Recall that function $\frac{c_k p_{3-k}^2}{(1-p_{3-k})\lambda_k} - \frac{c_{3-k}\left(1-\bar{p}_k^{(a)}[p_{3-k}]\right)}{\lambda_{3-k}}$ is negative for $p_{3-k} \in \left[p_{3-k}^{*(a)}, p_{3-k}^{**(a)}\right)$ and is positive for $p_{3-k} \in \left(p_{3-k}^{**(a)}, 1\right)$. Thus, $p_{3-k}^{**(a)} > \frac{R_{3-k}^{(a)}}{\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{R_{3-k}^{(a)}}{1-R_k^{(a)}}+1} \Leftrightarrow$

   $$\text{if } p_{3-k}^{*(a)} < \frac{R_{3-k}^{(a)}}{\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{R_{3-k}^{(a)}}{1-R_k^{(a)}} + 1}, \text{ then } \left.\frac{c_k p_{3-k}^2}{(1-p_{3-k})\lambda_k} - \frac{c_{3-k}\left(1-\bar{p}_k^{(a)}[p_{3-k}]\right)}{\lambda_{3-k}}\right|_{p_{3-k}=\frac{R_{3-k}^{(a)}}{\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{R_{3-k}^{(a)}}{1-R_k^{(a)}}+1}} < 0$$

   $\Leftrightarrow$ if $H_k^{(a)I} > 0$, then $H_k^{(a)II} < 0$.

■

In sum,

- if $H_k^{(a)I} \leq 0$, then $\hat{q}_k^{(a)} \geq \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$;
- if $H_k^{(a)I} > 0$ and $H_k^{(a)II} < 0$, then $\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{**(a)}\right]}{1-p_{3-k}^{**(a)}} < \hat{q}_k^{(a)} < \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$;
- if $H_k^{(a)I} > 0$ and $H_k^{(a)II} \geq 0$, then $H_k^{(a)}[q_k] > 0$ for all $q_k > \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{**(a)}\right]}{1-p_{3-k}^{**(a)}}$.

Cases 1-3 : The sources are independent for all possible beliefs.

Cases 4-7, $H_k^{(a)I} \leq 0$: In Area 3.3-k.1, the sources are substitutes when $\frac{p_k}{1-p_{3-k}} > \hat{q}_k^{(a)}$ and they are complements when $\frac{p_k}{1-p_{3-k}} < \hat{q}_k^{(a)}$. In Area 3.k.1, the sources are substitutes when $\frac{p_{3-k}}{1-p_k} > \frac{1}{1+\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1-\hat{q}_k^{(a)}}{\hat{q}_k^{(a)}}}$ and they are complements when $\frac{p_{3-k}}{1-p_k} < \frac{1}{1+\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1-\hat{q}_k^{(a)}}{\hat{q}_k^{(a)}}}$. The sources are complements in Area 3.3-k.2. Otherwise, the sources are independent.

Cases 4 and 7, $H_k^{(a)I} > 0$: The sources are substitutes in Areas 3.1.1 and 3.2.1. In Area 3.3-k.2, the sources are substitutes when $\frac{p_k}{1-p_{3-k}} > \hat{q}_k^{(a)}$ and they are complements when $\frac{p_k}{1-p_{3-k}} < \hat{q}_k^{(a)}$. Otherwise, the sources are independent.

Cases 5 and 6, $H_k^{(a)I} > 0$ **and** $H_k^{(a)II} < 0$: The sources are substitutes in Areas 3.1.1 and 3.2.1. In Area 3.3-k.2, the sources are substitutes when $\frac{p_k}{1-p_{3-k}} > \hat{q}_k^{(a)}$ and they are complements when $\frac{p_k}{1-p_{3-k}} < \hat{q}_k^{(a)}$. Otherwise, the sources are independent.

Cases 5 and 6, $H_k^{(a)I} > 0$ **and** $H_k^{(a)II} \geq 0$: The sources are substitutes in Areas 3.1.1, 3.2.1 and 3.3-k.2. Otherwise, the sources are independent.

The next lemma shows the comparative statics for the threshold $\hat{q}_k^{(a)}$ that separates the regions of substitutes and complements. Under the condition that this threshold exists (that is, **Cases 4-7,** $H_k^{(a)I} \leq 0$, or **Cases 4 and 7,** $H_k^{(a)I} > 0$, or **Cases 5 and 6,** $H_k^{(a)I} > 0$ **and** $H_k^{(a)II} < 0$), it increases with the cost of information ($\frac{c_1}{\lambda_1}$ and / or $\frac{c_2}{\lambda_2}$) and decreases with the benefit of information ($u_1[a_1]-u_1[a]$ and / or $u_2[a_2]-u_2[a]$). For **Cases 4-7,** $H_k^{(a)I} \leq 0$, complements and substitutes regions are separated by two lines, $\frac{p_k}{1-p_{3-k}} = \hat{q}_k^{(a)}$ and $\frac{p_{3-k}}{1-p_k} = \frac{1}{1+\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1-\hat{q}_k^{(a)}}{\hat{q}_k^{(a)}}}$. Lemma 20 states that both lines move towards (away from) $p_1 + p_2 = 1$ as the cost (benefit) of information increases.

Lemma 20 *For $k = 1, 2$, when $R_k^{(a)} < 1$, $\frac{\lambda_{3-k}R_{3-k}^{(a)}}{c_{3-k}} \geq \frac{\lambda_k R_k^{(a)}}{c_k}$, $p_{3-k}^{*(a)} + \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right] < 1$ and $p_{3-k}^{**(a)} > \frac{R_{3-k}^{(a)}}{\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{R_{3-k}^{(a)}}{1-R_k^{(a)}}+1}$,*

- *$\hat{q}_k^{(a)}$ is a well-defined function of $\frac{c_k}{\lambda_k}$, $\frac{c_{3-k}}{\lambda_{3-k}}$, $u_k[a_k] - u_k[a]$ and $u_{3-k}[a_{3-k}] - u_{3-k}[a]$ and we have*

$$\frac{\partial \hat{q}_k^{(a)}}{\partial \frac{c_i}{\lambda_i}} > 0, \quad \frac{\partial \hat{q}_k^{(a)}}{\partial\left(u_i[a_i] - u_i[a]\right)} < 0, \quad i = 1, 2;$$

- *when in addition $H_k^{(a)I} < 0$, then $\hat{Q}_{3-k}^{(a)} \equiv \frac{1}{1+\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\frac{1-\hat{q}_k^{(a)}}{\hat{q}_k^{(a)}}}$ is a well-defined function of $\frac{c_k}{\lambda_k}$, $\frac{c_{3-k}}{\lambda_{3-k}}$, $u_k[a_k] - u_k[a]$ and $u_{3-k}[a_{3-k}] - u_{3-k}[a]$ and we have*

$$\frac{\partial \hat{Q}_{3-k}^{(a)}}{\partial \frac{c_i}{\lambda_i}} > 0, \quad \frac{\partial \hat{Q}_{3-k}^{(a)}}{\partial\left(u_i[a_i] - u_i[a]\right)} < 0, \quad i = 1, 2.$$

<u>*Proof:*</u> When $\hat{q}_k^{(a)} > \frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$, we have

$$\frac{c_k}{\lambda_k}\frac{\partial \hat{q}_k^{(a)}}{\partial \frac{c_k}{\lambda_k}} = \frac{\left(1 + p_{3-k}^{*(a)} - \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)\left(\left(1 - p_{3-k}^{*(a)}\right)\hat{q}_k^{(a)} - \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)^2}{\left(1 - \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)\left(1 - p_{3-k}^{*(a)} - \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)}$$
$$+ \frac{\left(2 - R_k^{(a)}\right)R_k^{(a)}\left(1 - \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\left(1 - \hat{q}_k^{(a)}\right)^2}{\left(1 - R_k^{(a)}\right)^2\left(1 - p_{3-k}^{*(a)} - \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)^2} + \left(1 - \hat{q}_k^{(a)}\right)\hat{q}_k^{(a)} > 0,$$

$$\frac{c_{3-k}}{\lambda_{3-k}}\frac{\partial \hat{q}_k^{(a)}}{\partial \frac{c_{3-k}}{\lambda_{3-k}}} = \frac{p_{3-k}^{*(a)}\left(1 - \hat{q}_k^{(a)}\right)\left(\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\left(\frac{1-p_{3-k}^{*(a)}}{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}\hat{q}_k^{(a)} - 1\right) + \left(1 - p_{3-k}^{*(a)} - \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)\hat{q}_k^{(a)}\right)}{\left(1 - p_{3-k}^{*(a)} - \bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)^2} > 0,$$

$$\frac{c_k}{\lambda_k}\frac{\partial \hat{q}_k^{(a)}}{\partial\left(u_k[a_k]-u_k[a]\right)}=-\left(\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}\right)^2\frac{R_k^{(a)}\left(1-\hat{q}_k^{(a)}\right)}{\left(1-R_k^{(a)}\right)^2}$$

$$\times\left\{\frac{\left(1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)^3}{\hat{Q}_{3-k}^{(a)}\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}\left(\hat{Q}_{3-k}^{(a)}-\left(\frac{p_{3-k}^{*(a)}}{1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}\right)^2+\frac{\left(\left(p_{3-k}^{*(a)}\right)^2+\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\hat{Q}_{3-k}^{(a)}\right)\left(1-\hat{q}_k^{(a)}\right)}{\left(1-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)^2}\right)\right.$$

$$\left.+\left(\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]-R_k^{(a)}\right)\left(1+p_{3-k}^{*(a)}-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)\left(\frac{1-p_{3-k}^{*(a)}}{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}\hat{q}_k^{(a)}-1\right)\right\}<0,$$

$$\frac{c_{3-k}}{\lambda_{3-k}}\frac{\partial \hat{q}_k^{(a)}}{\partial\left(u_{3-k}[a_{3-k}]-u_{3-k}[a]\right)}$$
$$=-\frac{p_{3-k}^{*(a)}\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\left(1+p_{3-k}^{*(a)}-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)\left(1-\hat{q}_k^{(a)}\right)}{\left(1-p_{3-k}^{*(a)}-\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]\right)^2}\left(\frac{1-p_{3-k}^{*(a)}}{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}\hat{q}_k^{(a)}-1\right)<0,$$

$$\frac{c_k}{\lambda_k}\frac{\partial \hat{Q}_{3-k}^{(a)}}{\partial\frac{c_k}{\lambda_k}}=\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\left(\frac{\hat{Q}_{3-k}^{(a)}}{\hat{q}_k^{(a)}}\right)^2\left(\frac{c_k}{\lambda_k}\frac{\partial \hat{q}_k^{(a)}}{\partial\frac{c_k}{\lambda_k}}-\left(1-\hat{q}_k^{(a)}\right)\hat{q}_k^{(a)}\right)>0,$$

$$\frac{c_{3-k}}{\lambda_{3-k}}\frac{\partial \hat{Q}_{3-k}^{(a)}}{\partial\frac{c_{3-k}}{\lambda_{3-k}}}=\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\left(\frac{\hat{Q}_{3-k}^{(a)}}{\hat{q}_k^{(a)}}\right)^2\left(\frac{c_{3-k}}{\lambda_{3-k}}\frac{\partial \hat{q}_k^{(a)}}{\partial\frac{c_{3-k}}{\lambda_{3-k}}}+\left(1-\hat{q}_k^{(a)}\right)\hat{q}_k^{(a)}\right)>0,$$

$$\frac{\partial \hat{Q}_{3-k}^{(a)}}{\partial\left(u_i[a_i]-u_i[a]\right)}=\frac{c_k\lambda_{3-k}}{c_{3-k}\lambda_k}\left(\frac{\hat{Q}_{3-k}^{(a)}}{\hat{q}_k^{(a)}}\right)^2\frac{\partial \hat{q}_k^{(a)}}{\partial\left(u_i[a_i]-u_i[a]\right)}<0,\quad i=1,2.$$

When $\hat{q}_k^{(a)}<\frac{\bar{p}_k^{(a)}\left[p_{3-k}^{*(a)}\right]}{1-p_{3-k}^{*(a)}}$, we have

$$\frac{c_{3-k}}{\lambda_{3-k}}\frac{\partial \hat{q}_k^{(a)}}{\partial\frac{c_k}{\lambda_k}}=\frac{-1}{\underline{\tilde{p}}_{3-k}^{(a)\prime}\left[\hat{q}_k^{(a)}\right]}\left(\frac{\underline{\tilde{p}}_{3-k}^{(a)}\left[\hat{q}_k^{(a)}\right]}{1-R_k^{(a)}}\right)^2>0,\quad \frac{c_{3-k}}{\lambda_{3-k}}\frac{\partial \hat{q}_k^{(a)}}{\partial\frac{c_{3-k}}{\lambda_{3-k}}}=\frac{\hat{q}_k^{(a)}\underline{\tilde{p}}_{3-k}^{(a)}\left[\hat{q}_k^{(a)}\right]}{1-\underline{\tilde{p}}_{3-k}^{(a)}\left[\hat{q}_k^{(a)}\right]}>0,$$

$$\frac{c_{3-k}}{\lambda_{3-k}}\frac{\partial \hat{q}_k^{(a)}}{\partial\left(u_k[a_k]-u_k[a]\right)}=-R_k^{(a)}\left(\frac{\underline{\tilde{p}}_{3-k}^{(a)}\left[\hat{q}_k^{(a)}\right]}{1-R_k^{(a)}}\right)^2\left(\frac{\left(1-R_k^{(a)}\right)\hat{q}_k^{(a)}}{1-\underline{\tilde{p}}_{3-k}^{(a)}\left[\hat{q}_k^{(a)}\right]}-\frac{1}{\underline{\tilde{p}}_{3-k}^{(a)\prime}\left[\hat{q}_k^{(a)}\right]}\right)<0,$$

$$\frac{c_{3-k}}{\lambda_{3-k}}\frac{\partial \hat{q}_k^{(a)}}{\partial\left(u_{3-k}[a_{3-k}]-u_{3-k}[a]\right)}=-\frac{\hat{q}_k^{(a)}\left(\underline{\tilde{p}}_{3-k}^{(a)}\left[\hat{q}_k^{(a)}\right]\right)^2}{1-\underline{\tilde{p}}_{3-k}^{(a)}\left[\hat{q}_k^{(a)}\right]}<0.$$

■

APPENDIX FOR CHAPTER 3

B.1 Proof for Theorem 5

The proof is based on a well-known property of normal distribution:

$$x|\theta \sim \mathcal{N}(\theta, \sigma^2), \quad \theta \sim \mathcal{N}(\theta_0, \sigma_0^2) \quad \Rightarrow \quad \theta|x \sim \mathcal{N}\left(\frac{\sigma_0^2}{\sigma^2+\sigma_0^2}x + \frac{\sigma^2}{\sigma^2+\sigma_0^2}\theta_0, \left(\frac{1}{\sigma_0^2}+\frac{1}{\sigma^2}\right)^{-1}\right). \tag{B.1}$$

Obviously, $x_i^1 = s_i$ since $\theta|s_i \sim \mathcal{N}\left(s_i, \sigma^2\right)$. Given that, in the second stage agent i knows not only their own private signal, but also their neighbor's private signal: $y_i^1|\theta \sim \mathcal{N}\left(\theta, \sigma^2\right)$.

$$\begin{matrix} \theta|s_i \sim \mathcal{N}\left(s_i, \sigma^2\right) \\ y_i^1|\theta \sim \mathcal{N}\left(\theta, \sigma^2\right) \end{matrix} \quad \overset{(B.1)}{\Rightarrow} \quad \theta|s_i, y_i^1 \sim \mathcal{N}\left(\frac{s_i+y_i^1}{2}, \frac{\sigma^2}{2}\right) \quad \Rightarrow \quad x_i^2 = \frac{s_i+y_i^1}{2}.$$

In the third stage, y_i^2 gives another signal, as it is an average of y_i^1 and the neighbor's neighbor's signal.

$$\begin{matrix} \theta|s_i, y_i^1 \sim \mathcal{N}\left(\frac{s_i+y_i^1}{2}, \frac{\sigma^2}{2}\right) \\ 2y_i^2 - y_i^1|\theta \sim \mathcal{N}\left(\theta, \sigma^2\right) \end{matrix} \quad \overset{(B.1)}{\Rightarrow} \quad \theta|s_i, y_i^1, y_i^2 \sim \mathcal{N}\left(\frac{2}{3}y_i^2 + \frac{1}{3}s_i, \frac{\sigma^2}{3}\right) \quad \Rightarrow \quad x_i^3 = \frac{2}{3}y_i^2 + \frac{1}{3}s_i.$$

By induction, for $3 \le t \le n$,

$$\begin{matrix} \theta|s_i, y_i^1, \dots, y_i^{t-2} \sim \mathcal{N}\left(\frac{t-2}{t-1}y_i^{t-2} + \frac{1}{t-1}s_i, \frac{\sigma^2}{t-1}\right) \\ (t-1)y_i^{t-1} - (t-2)y_i^{t-2}|\theta \sim \mathcal{N}\left(\theta, \sigma^2\right) \end{matrix} \quad \overset{(B.1)}{\Rightarrow} \quad \theta|s_i, y_i^1, \dots, y_i^{t-1} \sim \mathcal{N}\left(\frac{t-1}{t}y_i^{t-1} + \frac{1}{t}s_i, \frac{\sigma^2}{t}\right).$$

Thus, $x_i^t = \frac{t-1}{t}y_i^{t-1} + \frac{1}{t}s_i$.

In any further stage, when $t > n$, new guesses from the neighbor do not bring any new information to agent i. Thus, $x_i^t = x_i^n$ for all $t > n$.

B.2 Proof for Theorem 6

Necessity Let's take a close look at the assumption (3.4) in light of the belief updating rule (3.3):

$$\left.\begin{matrix} x_i^t = \lambda_i^t s_i + \sum_{\tau=1}^{t-1} \mu_i^{\tau,t} \cdot y_i^\tau \\ x_i^{t+1} = c_i^{t+1,t}\left(\lambda_i^t s_i + \sum_{\tau=1}^{t-1} \mu_i^{\tau,t} \cdot y_i^\tau\right) + \mu_i^{t,t+1} \cdot y_i^t \end{matrix}\right\} \quad \Rightarrow \quad x_i^{t+1} = w_{ii}^t x_i^t + w_{i,\mathcal{N}(i)}^t y_i^t,$$

where $w^t_{i,\mathcal{N}(i)} = \mu_i^{t,t+1}$ and $w^t_{ii} = c_i^{t+1,t}$.

This gives us (3.1).

Sufficiency For any $i \in \mathcal{N}$ and for any $t \geq 2$

$$\begin{aligned} x_i^t &= w_{ii}^{t-1} x_i^{t-1} + w_{i,\mathcal{N}(i)}^{t-1} y_i^{t-1} = w_{ii}^{t-1}\left(w_{ii}^{t-2} x_i^{t-2} + w_{i,\mathcal{N}(i)}^{t-2} y_i^{t-2}\right) + w_{i,\mathcal{N}(i)}^{t-1} y_i^{t-1} = \ldots \\ &= \left(\prod_{\tau=1}^{t-1} w_{ii}^{\tau}\right) x_i^1 + w_{i,\mathcal{N}(i)}^{t-1} y_i^{t-1} + \sum_{\tau=1}^{t-2}\left(\prod_{\tau'=\tau+1}^{t-1} w_{ii}^{\tau'}\right) w_{i,\mathcal{N}(i)}^{\tau} y_i^{\tau}. \end{aligned}$$

On the other hand, by Assumption 2,

$$x_i^t = \lambda_i^t s_i + \sum_{\tau=1}^{t-1} \mu_i^{\tau,t} \cdot y_i^{\tau}.$$

So, given Assumption 1, we must have

$$\lambda_i^t = \prod_{\tau=1}^{t-1} w_{ii}^{\tau}, \quad \mu_i^{t-1,t} = w_{i,\mathcal{N}(i)}^{t-1}, \quad \mu_i^{\tau,t} = \left(\prod_{\tau'=\tau+1}^{t-1} w_{ii}^{\tau'}\right) w_{i,\mathcal{N}(i)}^{\tau}, \quad 1 \leq \tau \leq t-2.$$

Therefore, for any $t_1 > t_2$ there exists a constant

$$c_i^{t_1,t_2} = \prod_{\tau=t_2}^{t_1-1} w_{ii}^{\tau}$$

such that (3.4) holds.

B.3 Instructions

At the beginning of this experiment, you will be divided into groups. Each group consists of four members: A, B, C, and D. These letters correspond to the members' IDs. This experiment consists of several periods. Your group and your ID are fixed across all periods. Your ID will appear on the top right corner of your screen.

There are 1001 jars enumerating from 0 to 1000. Each jar contains 53 balls. Each ball has one and only one integer on it. In jar m ($m = 0, 1, \ldots, 1000$) there are

- 5 balls with integer m
- 5 balls with integer $m + 1$
- 5 balls with integer $m - 1$
- 5 balls with integer $m + 2$

- 5 balls with integer $m - 2$
- 4 balls with integer $m + 3$
- 4 balls with integer $m - 3$
- 4 balls with integer $m + 4$
- 4 balls with integer $m - 4$
- 3 balls with integer $m + 5$
- 3 balls with integer $m - 5$
- 3 balls with integer $m + 6$
- 3 balls with integer $m - 6$

The above distribution of balls with different integers is graphically displayed on top left of the screen.

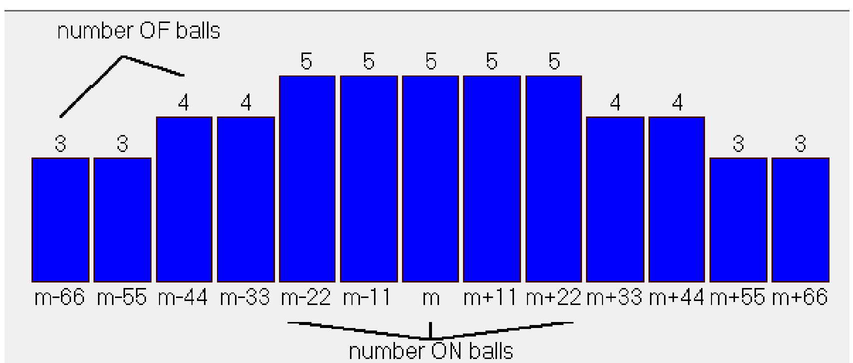

Figure B.1: The distribution of the balls.

In each period, for each group, the experimenter picks a jar randomly from 1001 jars. Your task in each period is to guess which jar the experimenter picks for your group, i.e. to guess the value of m. Your will make guesses in 5 stages and your payment depends on your performance in all stages. Note that m is also the average of integers on all balls in jar m.

Besides the distribution of balls with different numbers, you will get some additional information to helps you make guesses:

- Your draws

  At the beginning of each period, each member of your group (including yourself) will draw one ball independently *with replacement* from the jar. The number on the ball that you have drawn can be seen only by you and this information is located on top right of your screen.

- Guesses of your group members

  You will make guesses in 5 stages. From stage 2 and on, you will be able to see the guesses that one member in your group has made in previous stages. If you are A, you can see D's guesses; if you are B, you can see A's guesses; if you are C, you can see B's guesses; if you are D, you can see C's guesses. These relations are graphically described by the picture on top right of the screen. The history of guesses is recorded at bottom right of the screen.

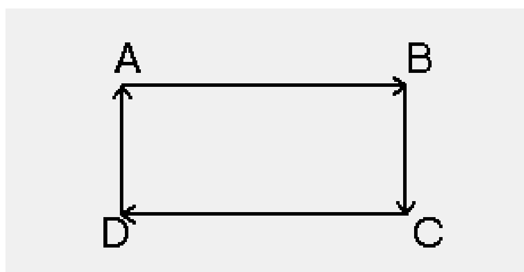

Figure B.2: The network.

To sum up, the procedure of the experiment is as the following:

1. In the beginning of this experiment, you are divided into groups of four and assigned an ID.

2. In the beginning of each period, a jar is picked randomly from the 1001 jars for each group.

3. Each member in your group makes one draw from the jar.

4. You input your guess about the number of the jar. You need to confirm your guess by pressing the button "OK". Then you are at the waiting stage. When all your group members submit their guesses, stage 2 starts.

5. In stage 2 you are able to see the guess made by one of your group members in stage 1, as well as your previous guess. You need to input a new guess and submit it by pressing the button “OK".

6. Similar procedure continues until the last stage.

7. The experiment will last 5 stages for each period. Your payment is based on your guesses in **all** stages: the closer your guesses are to the actual value of m, the higher payoff you will get.

   Here is how your payoff in each stage is calculated:

$$\text{Profit} = \max\left(10 - \frac{1}{11}|\text{Your Guess} - m|, 0\right)$$

   You will earn 4 cents for every point.[1]

   Adding up your profit in all stages gives your final payoff.

A calculator is available if you click the button on the bottom right corner.

The clock on the top right corner counting down from 60 seconds incentivizes you to make decisions within a reasonable length of time. You will still be able to input your guesses after the time expires.

[1] The conversion rate is 1 ruble for 1 point for all sessions in Russia.

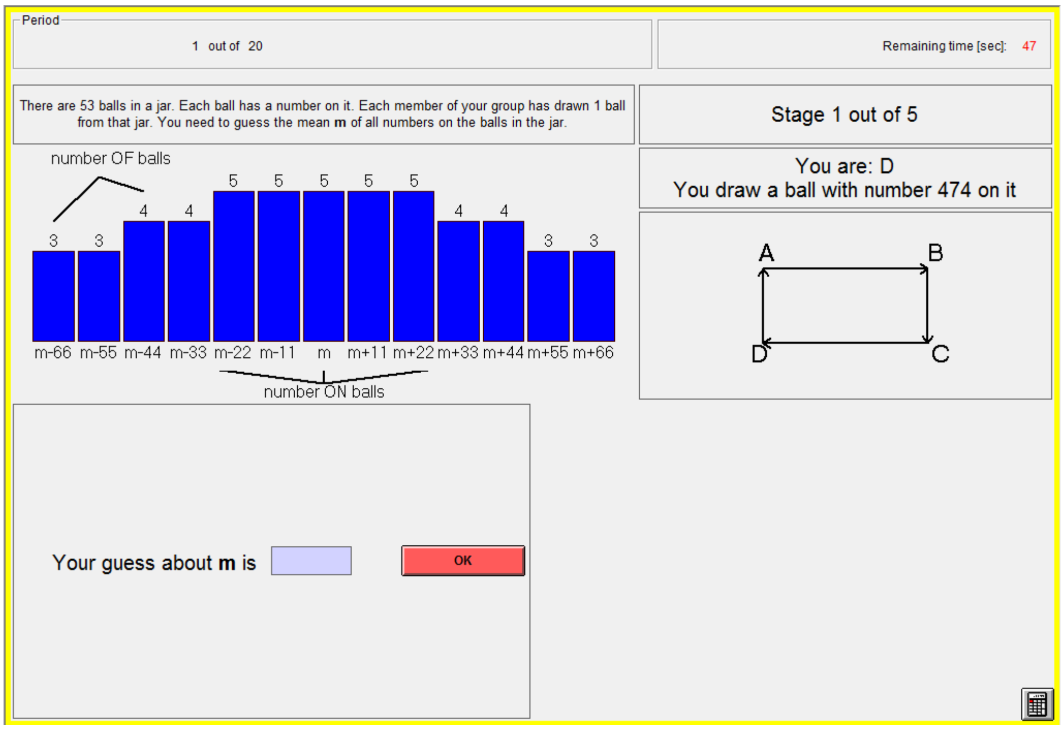

Figure B.3: Screenshot.

B.4 Test (Questions Asked Before the Experiment Starts)

Please answer the following questions:

- Before observing any draws, which of the following do you think is more likely:
  - The jar is No.0.
  - The jar is No.500.
- Which of the following statements is true:
  - In stage 3, B can observe C's guesses in stage 1 and 2.
  - In stage 4, A can observe D's guesses in stage 1, 2, and 3.
  - In stage 2, A can observe B and C's guesses in stage 1.
  - In stage 5, C can observe B's guesses in stage 1 only.

Suppose you see the following screen during the experiment:

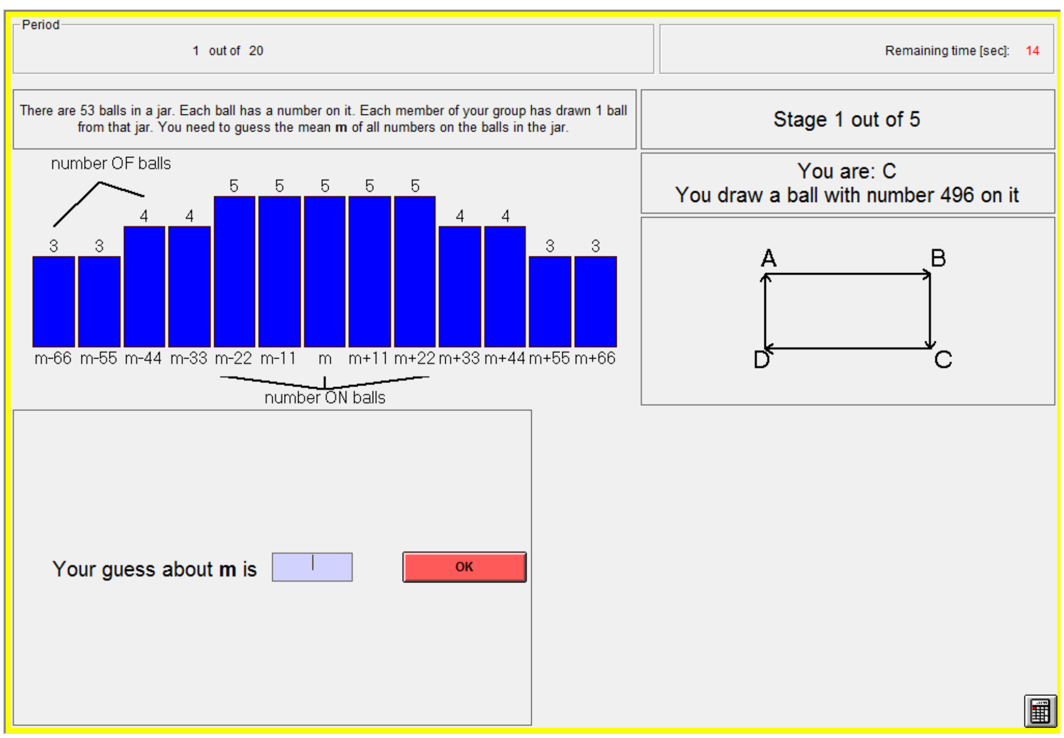

Figure B.4: Screenshot.

- Your ID is ____.
- There are ____ (how many) balls in the jar.
- **In this period**, will you or your group members make any additional draws after stage 1?
- You would like to input No. ____ as your guess about the jar number.

Suppose you see the following screen during the experiment:

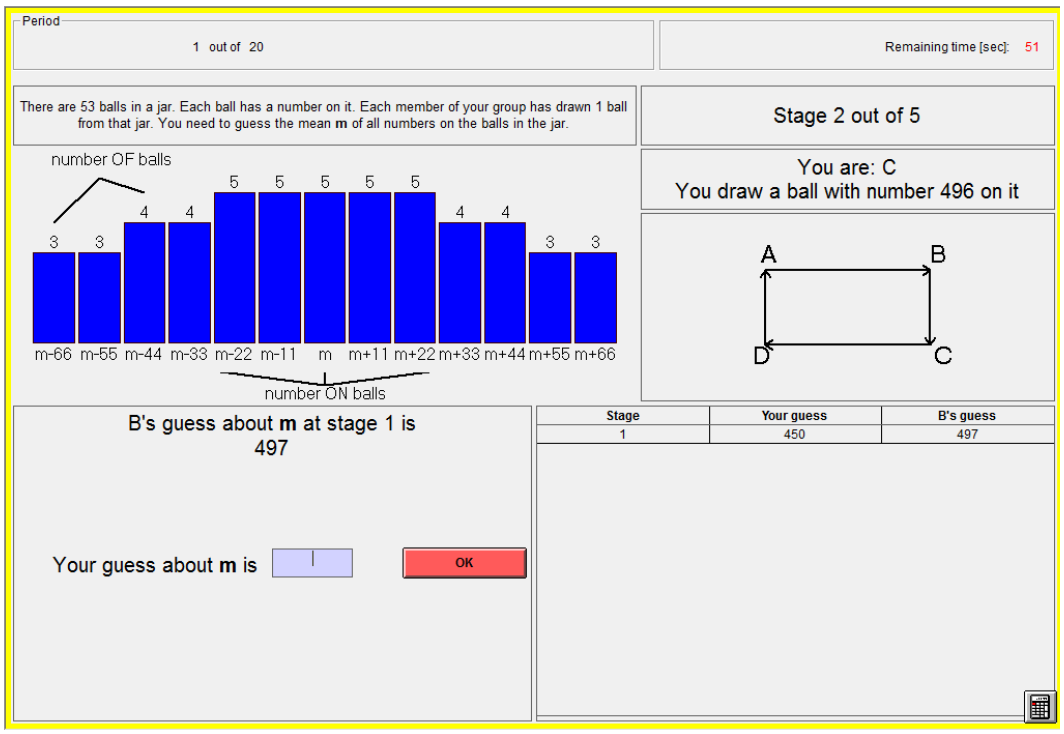

Figure B.5: Screenshot.

- ____ (how many) stages have passed and there are ____ stages left including the current stage.
- Your guess in the first stage was ____ (It is already shown on the screen.)
- B's guess in the first stage was ____.

B.5 Robustness Check for Section 3.7: Maximum Likelihood Comparison Across Different Specifications

Consider the following specifications:

GM-S2345 $x_i^t = \left(\lambda^2 s_i + \mu^{1,2} y_i^1\right) \cdot \mathbf{1}(t=2) + \left(\lambda^3 s_i + \mu^{1,3} y_i^1 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(\lambda^4 s_i + \mu^{1,4} y_i^1 + \mu^{2,4} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4) + \left(\lambda^5 s_i + \mu^{1,5} y_i^1 + \mu^{2,5} y_i^2 + \mu^{3,5} y_i^3 + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t=5)$

BUH-S2345 $x_i^t = \left(\lambda^2 s_i + \mu^{1,2} y_i^1\right) \cdot \mathbf{1}(t=2) + \left(\lambda^3 s_i + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(\lambda^4 s_i + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4) + \left(\lambda^5 s_i + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t=5)$

PBH-S2345 $x_i^t = \left(\lambda^2 s_i + \mu^{1,2} y_i^1\right) \cdot \mathbf{1}(t=2) + \left(c^3 \lambda^2 s_i + c^3 \mu^{1,2} y_i^1 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(c^4 c^3 \lambda^2 s_i + c^4 c^3 \mu^{1,2} y_i^1 + c^4 \mu^{2,3} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4) + \left(c^5 c^4 c^3 \lambda^2 s_i + c^5 c^4 c^3 \mu^{1,2} y_i^1 + c^5 c^4 \mu^{2,3} y_i^2 + c^5 \mu^{3,4} y_i^3 + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t=5)$

GM-S234 $x_i^t = \left(\lambda^2 s_i + \mu^{1,2} y_i^1\right) \cdot \mathbf{1}(t=2) + \left(\lambda^3 s_i + \mu^{1,3} y_i^1 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(\lambda^4 s_i + \mu^{1,4} y_i^1 + \mu^{2,4} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4)$

BUH-S234 $x_i^t = \left(\lambda^2 s_i + \mu^{1,2} y_i^1\right) \cdot \mathbf{1}(t=2) + \left(\lambda^3 s_i + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(\lambda^4 s_i + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4)$

PBH-S234 $x_i^t = \left(\lambda^2 s_i + \mu^{1,2} y_i^1\right) \cdot \mathbf{1}(t=2) + \left(c^3 \lambda^2 s_i + c^3 \mu^{1,2} y_i^1 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(c^4 c^3 \lambda^2 s_i + c^4 c^3 \mu^{1,2} y_i^1 + c^4 \mu^{2,3} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4)$

GM-S23 $x_i^t = \left(\lambda^2 s_i + \mu^{1,2} y_i^1\right) \cdot \mathbf{1}(t=2) + \left(\lambda^3 s_i + \mu^{1,3} y_i^1 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3)$

BUH-S23 $x_i^t = \left(\lambda^2 s_i + \mu^{1,2} y_i^1\right) \cdot \mathbf{1}(t=2) + \left(\lambda^3 s_i + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3)$

PBH-S23 $x_i^t = \left(\lambda^2 s_i + \mu^{1,2} y_i^1\right) \cdot \mathbf{1}(t=2) + \left(c^3 \lambda^2 s_i + c^3 \mu^{1,2} y_i^1 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3)$

GM-S345 $x_i^t = \left(\lambda^3 s_i + c^3 x_i^2 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(\lambda^4 s_i + c^4 x_i^2 + \mu^{2,4} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4) + \left(\lambda^5 s_i + c^5 x_i^2 + \mu^{2,5} y_i^2 + \mu^{3,5} y_i^3 + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t=5)$

BUH-S345 $x_i^t = \left(\lambda^3 s_i + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(\lambda^4 s_i + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4) + \left(\lambda^5 s_i + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t=5)$

PBH-S345 $x_i^t = \left(c^3 x_i^2 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(c^4 c^3 x_i^2 + c^4 \mu^{2,3} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4) + \left(c^5 c^4 c^3 x_i^2 + c^5 c^4 \mu^{2,3} y_i^2 + c^5 \mu^{3,4} y_i^3 + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t=5)$

GM-S34 $x_i^t = \left(\lambda^3 s_i + c^3 x_i^2 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(\lambda^4 s_i + c^4 x_i^2 + \mu^{2,4} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4)$

BUH-S34 $x_i^t = \left(\lambda^3 s_i + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t = 3) + \left(\lambda^4 s_i + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t = 4)$

PBH-S34 $x_i^t = \left(c^3 x_i^2 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t = 3) + \left(c^4 c^3 x_i^2 + c^4 \mu^{2,3} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t = 4)$

GM-S45 $x_i^t = \left(\lambda^4 s_i + c^4 x_i^3 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t = 4) + \left(\lambda^5 s_i + c^5 x_i^3 + \mu^{3,5} y_i^3 + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t = 5)$

BUH-S45 $x_i^t = \left(\lambda^4 s_i + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t = 4) + \left(\lambda^5 s_i + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t = 5)$

PBH-S45 $x_i^t = \left(c^4 x_i^3 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t = 4) + \left(c^5 c^4 x_i^3 + c^5 \mu^{3,4} y_i^3 + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t = 5)$

GM-S3 $x_i^3 = \lambda^3 s_i + c^3 x_i^2 + \mu^{2,3} y_i^2$

BUH-S3 $x_i^3 = \lambda^3 s_i + \mu^{2,3} y_i^2$

PBH-S3 $x_i^3 = c^3 x_i^2 + \mu^{2,3} y_i^2$

GM-S4 $x_i^4 = \lambda^4 s_i + c^4 x_i^3 + \mu^{3,4} y_i^3$

BUH-S4 $x_i^4 = \lambda^4 s_i + \mu^{3,4} y_i^3$

PBH-S4 $x_i^4 = c^4 x_i^3 + \mu^{3,4} y_i^3$

GM-S5 $x_i^5 = \lambda^5 s_i + c^5 x_i^4 + \mu^{4,5} y_i^4$

BUH-S5 $x_i^5 = \lambda^5 s_i + \mu^{4,5} y_i^4$

PBH-S5 $x_i^5 = c^5 x_i^4 + \mu^{4,5} y_i^4$

GM-S345x $x_i^t = \left(\lambda^3 s_i + c^3 x_i^2 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t = 3) + \left(\lambda^4 s_i + c^4 x_i^3 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t = 4) + \left(\lambda^5 s_i + c^5 x_i^4 + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t = 5)$

BUH-S345x Equivalent to BUH-S345

PBH-S345x $x_i^t = \left(c^3 x_i^2 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t = 3) + \left(c^4 x_i^3 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t = 4) + \left(c^5 x_i^4 + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t = 5)$

GM-S34x $x_i^t = \left(\lambda^3 s_i + c^3 x_i^2 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t = 3) + \left(\lambda^4 s_i + c^4 x_i^3 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t = 4)$

BUH-S34x Equivalent to BUH-S34

PBH-S34x $x_i^t = \left(c^3 x_i^2 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t = 3) + \left(c^4 x_i^3 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t = 4)$

GM-S345i $x_i^t = \left(\lambda^3 s_i + \mu^{1,3} y_i^1 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t = 3) + \left(\lambda^4 s_i + \mu^{1,4} y_i^1 + \mu^{2,4} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t = 4) + \left(\lambda^5 s_i + \mu^{1,5} y_i^1 + \mu^{2,5} y_i^2 + \mu^{3,5} y_i^3 + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t = 5)$

BUH-S345i $x_i^t = \left(\lambda^3 s_i + \mu^{1,3} y_i^1 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(\lambda^4 s_i + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4) + \left(\lambda^5 s_i + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t=5)$

PBH-S345i $x_i^t = \left(\lambda^3 s_i + \mu^{1,3} y_i^1 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(c^4 \lambda^3 s_i + c^4 \mu^{1,3} y_i^1 + c^4 \mu^{2,3} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4) + \left(c^5 c^4 \lambda^3 s_i + c^5 c^4 \mu^{1,3} y_i^1 + c^5 c^4 \mu^{2,3} y_i^2 + c^5 \mu^{3,4} y_i^3 + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t=5)$

GM-S34i $x_i^t = \left(\lambda^3 s_i + \mu^{1,3} y_i^1 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(\lambda^4 s_i + \mu^{1,4} y_i^1 + \mu^{2,4} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4)$

BUH-S34i $x_i^t = \left(\lambda^3 s_i + \mu^{1,3} y_i^1 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(\lambda^4 s_i + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4)$

PBH-S34i $x_i^t = \left(\lambda^3 s_i + \mu^{1,3} y_i^1 + \mu^{2,3} y_i^2\right) \cdot \mathbf{1}(t=3) + \left(c^4 \lambda^3 s_i + c^4 \mu^{1,3} y_i^1 + c^4 \mu^{2,3} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4)$

GM-S45i $x_i^t = \left(\lambda^4 s_i + \mu^{1,4} y_i^1 + \mu^{2,4} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4) + \left(\lambda^5 s_i + \mu^{1,5} y_i^1 + \mu^{2,5} y_i^2 + \mu^{3,5} y_i^3 + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t=5)$

BUH-S45i $x_i^t = \left(\lambda^4 s_i + \mu^{1,4} y_i^1 + \mu^{2,4} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4) + \left(\lambda^5 s_i + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t=5)$

PBH-S45i $x_i^t = \left(\lambda^4 s_i + \mu^{1,4} y_i^1 + \mu^{2,4} y_i^2 + \mu^{3,4} y_i^3\right) \cdot \mathbf{1}(t=4) + \left(c^5 \lambda^4 s_i + c^5 \mu^{1,4} y_i^1 + c^5 \mu^{2,4} y_i^2 + c^5 \mu^{3,4} y_i^3 + \mu^{4,5} y_i^4\right) \cdot \mathbf{1}(t=5)$

Table B.1 below shows comparison between different specifications based on maximum likelihood and information criteria (AIC and BIC). Since PBH and BUH type specifications have the same number of parameters, they can be safely compared using the log-likelihood function. In contrast, GM specifications have more parameters, so AIC or BIC should be used to account for the change in degrees of freedom.

The first row in Table B.1 compares GM-S2345, PBH-S2345 and BUH-S2345. These specifications are the same as models (3.7), (3.8) and (3.9) and we include them here for completeness. They include stages 2,3,4 and 5 and have only "external" information on the left side of a regression (private signal and a neighbor's guesses). $LL_{\text{PBH-BUH}}$ is positive, which means PBH-S2345 fits better than BUH-S2345. Both $AIC_{\text{GM-BUH}}$ and $BIC_{\text{GM-BUH}}$ are negative, which means GM-S2345 fits better than BUH-S2345. However, the comparison between GM-S2345 and PBH-S2345 is ambiguous: GM-S2345 is better according to AIC, while PBH-S2345 is better according to BIC.

The second row in Table B.1 compares GM-S234, PBH-S234 and BUH-S234. These specifications exclude stage 5 from consideration. The conclusion is exactly

	$LL_{\text{PBH-BUH}}$	$AIC_{\text{GM-BUH}}$	$BIC_{\text{GM-BUH}}$	$AIC_{\text{GM-PBH}}$	$BIC_{\text{GM-PBH}}$
S2345	96.4 [95.5,97.3]	-199 [-201,-197]	-168 [-169,-166]	-6.42 [-7.39,-5.44]	25.2 [24.2,26.2]
S234	61.7 [61.0,62.4]	-126 [-127,-125]	-111 [-112,-110]	-2.48 [-3.18,-1.78]	12.5 [11.8,13.2]
S23	42.7 [42.1,43.2]	-83.5 [-84.6,-82.5]	-79.0 [-80.0,-78.0]	1.84 [1.49,2.19]	6.42 [6.06,6.77]
S345	-46.1 [-49.9,-42.2]	-282 [-285,-279]	-252 [-255,-249]	-374 [-380,-368]	-344 [-350,-338]
S34	-50.0 [-52.7,-47.3]	-174 [-175,-172]	-160 [-162,-159]	-274 [-278,-270]	-260 [-265,-256]
S45	147 [146,148]	-333 [-335,-331]	-319 [-322,-317]	-39.3 [-40.0,-38.6]	-25.6 [-26.2,-24.9]
S3	-15.6 [-17.4,-13.9]	-111 [-113,-110]	-107 [-108.74,-106]	-143 [-145,-140]	-139 [-141,-136]
S4	79.5 [78.7,80.3]	-179 [-180,-177]	-175 [-176,-173]	-19.5 [-19.9,-19.1]	-15.6 [-16.0,-15.2]
S5	137 [135,138]	-280 [-282,-277]	-276 [-278,-273]	-6.05 [-6.27,-5.84]	-2.17 [-2.38,-1.95]
S345x	95.4 [90.7,100.2]	-486 [-489,-482]	-471 [-474,-467]	-295 [-301,-288]	-280 [-286,-273]
S34x	12.7 [9.7,15.8]	-258 [-260,-256]	-249 [-251,-247]	-232 [-237,-228]	-223 [-228,-219]
S45x	218 [216,219]	-461 [-464,-459]	-452 [-455,-450]	-26.2 [-26.7,-25.7]	-17.0 [-17.5,-16.5]
S345i	62.7 [61.5,63.9]	-151 [-153,-149]	-126 [-129,-124]	-26.0 [-26.6,-25.4]	-1.05 [-1.58,-0.51]
S34i	11.3 [10.9,11.7]	-36.9 [-37.5,-36.2]	-27.7 [-28.4,-27.0]	-14.2 [-14.5,-13.9]	-5.03 [-5.33,-4.73]
S45i	72.1 [71.1,73.2]	-140 [-142,-138]	-127 [-129,-125]	3.82 [3.68,3.96]	17.6 [17.4,17.7]

Table B.1: $LL_{\text{PBH-BUH}} = \ln(L_{\text{PBH}}) - \ln(L_{\text{BUH}})$ is the difference between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model; $AIC_{\text{GM-BUH}} = AIC_{\text{GM}} - AIC_{\text{BUH}}$ is the difference between AIC for the GM model and the BUH model; $BIC_{\text{GM-BUH}} = BIC_{\text{GM}} - BIC_{\text{BUH}}$ is the difference between BIC for the GM model and the BUH model; $AIC_{\text{GM-PBH}}$ ($BIC_{\text{GM-PBH}}$) is the difference between AIC (BIC) for the GM model and the PBH model. The first column shows the type of specification used. All subjects are pooled. 99% confidence intervals are found by bootstrapping by uniformly sampling by sessions and by periods within each session (only the last 15 original periods are used). Bootstrap sample size is 10,000.

the same: PBH-S234 fits better than BUH-S234, GM-S234 fits better than BUH-S234, GM-S234 is better than PBH-S234 according to AIC, while PBH-S234 is better than GM-S234 according to BIC.

The next row in Table B.1 compares GM-S23, PBH-S23 and BUH-S23. These specifications only consider stages 2 and 3. The conclusion is almost the same. Both GM-S23 and PBH-S23 are better than BUH-S23. However, now both AIC and BIC indicate that PBH-S23 is better than GM-S23.

The next row in Table B.1 compares GM-S345, PBH-S345 and BUH-S345. In contrast to the S2345 specifications, these specifications exclude stage 2 and have x_i^2 instead of y_i^1 on the left side of a regression. Specifically, the persuasion bias hypothesis (3.5) connects the weights in the general model (3.2)-(3.3) for two subsequent stages. Thus, if we want to restrict weights in stage 3, we need to estimate weights in stage 2 as well. However, excluding stage 2 data makes it impossible. Luckily, there is another way to impose persuasion bias restriction for stage 3, namely to include x_i^2 as a proxy for $\lambda^2 s_i + \mu^{1,2} y_i^1$. In the world where the PBH model holds exactly, this is an equivalent transformation. However, the presence of an error term means that including x_i^2 as a proxy for $\lambda^2 s_i + \mu^{1,2} y_i^1$ changes the error correlation structure. To match this transformation, we also substitute y_i^1 with x_i^2 in the general model (GM-S345). Table B.1 shows that BUH-S345 is better than PBH-S345, while GM-S345 is better than both of them.

Excluding stage 5 does not change this conclusion: BUH-S34 is better than PBH-S34, while GM-S34 is better than both of them.

The next row in Table B.1 compares GM-S45, PBH-S45 and BUH-S45. These specifications only consider stages 4 and 5, using x_i^3 as a proxy for $\lambda^3 s_i + \mu^{1,3} y_i^1 + \mu^{2,3} y_i^2$ in the PBH model and substituting y_i^1 and y_i^2 with x_i^3 in the general model. $\mathrm{LL}_{\text{PBH-BUH}}$ is positive and relatively large, which means PBH-S45 fits much better than BUH-S45. However, the general model again is better than the other two.

The next three rows in Table B.1 compare three models by stages (3,4 and 5). In every stage, the general model is better, through GM-S5 is only slightly better than PBH-S5. The PBH model is better than the BUH model in stages 4 and 5 but it is worse in stage 3.

The next row in Table B.1 compares GM-S345x, PBH-S345x and BUH-S345x. The S345x specification is a union of S3, S4 and S5 specifications. In other words, here we use the previous stage's guess as a proxy for a corresponding linear combination of a private signal and the neighbor's guesses in each stage. Table B.1 shows that PBH-S345x is better than BUH-S345x, while GM-S345x is better than both of them. Excluding the last stage does not change this conclusion (see the S34x specification), neither does considering only stages 4 and 5 (see the S45x specification).

Specifications GM-S345i, PBH-S345i and BUH-S345i use data from stages 3, 4 and 5 but all three models are identical in stage 3. So, the S234i specification tests stages 4 and 5 together, using only "external" information on the left side of a regression. Table B.1 shows that PBH-S345i is better than BUH-S345i, while GM-S345i is better than both of them. Excluding the last stage does not change this conclusion (see S34i specification). Finally, specification S45i tests stage 5 solely and it favors PBH model over BUH and even GM.

In sum, Table B.1 shows:

1. The general model is much better than the BUH model ($\mathrm{AIC}_{\text{GM-BUH}}$ and $\mathrm{BIC}_{\text{GM-BUH}}$ are all negative).
2. The general model is slightly better than the PBH model.
3. When only "external" information on the left side of a regression is used, the PBH model is better than the BUH model (see S2345, S234, S23, S345i, S34i, S45i).
4. Using x_i^2 as a proxy for $\lambda^2 s_i + \mu^{1,2} y_i^1$ makes the PBH model worse (compare S2345, S234, S23 with S345, S34, S3).
5. Using x_i^3 as a proxy for $\lambda^3 s_i + \mu^{1,3} y_i^1 + \mu^{2,3} y_i^2$ makes the PBH model better (compare S34 with S34x).
6. Using x_i^4 as a proxy for $\lambda^4 s_i + \mu^{1,4} y_i^1 + \mu^{2,4} y_i^2 + \mu^{3,4} y_i^3$ makes the PBH model better (compare S45 with S45x).
7. The PBH model is arguably worse than the BUH model in stage 3 (compare S345, S34, S345x, S34x, S345, S34 with S45, S4, S45x, S4, S345i, S34i and see S3; S23 gives the opposite result). More precisely, using x_i^2 makes the PBH model worse than the BUH model, while using $\lambda^2 s_i + \mu^{1,2} y_i^1$ reverses the comparison.
8. The PBH model is arguably better than the BUH model in stage 4 (compare S234, S34x, S45x with S23, S3, S5 and see S4 and S34i; S34 vs S3 gives the opposite result).

9. The PBH model is better than the BUH model in stage 5 (compare S2345, S345, S45, S345x, S45x, S345i with S234, S34, S4, S34x, S4, S34i and see S5 and S45i).

B.6 Equilibrium for Experimental Setup

Model	PBH	BUH	GM
	x_i^t	x_i^t	x_i^t
$s_i \cdot \mathbf{1}(t=2)\ (\lambda^2)$	0.566*** (0.070)	0.498*** (0.005)	0.498*** (0.004)
$y_i^1 \cdot \mathbf{1}(t=2)\ (\mu^{1,2})$	0.433*** (0.070)	0.501*** (0.005)	0.501*** (0.004)
$\left(\lambda^2 s_i + \mu^{1,2} y_i^1\right) \cdot \mathbf{1}(t=3)\ (c^3)$	0.571*** (0.108)		
$s_i \cdot \mathbf{1}(t=3)\ (\lambda^3 = c^3\lambda^2)$	*0.323*	0.334*** (0.008)	0.335*** (0.008)
$y_i^1 \cdot \mathbf{1}(t=3)\ (\mu^{1,3} = c^3\mu^{1,2})$	*0.247*		0.021 (0.020)
$y_i^2 \cdot \mathbf{1}(t=3)\ (\mu^{2,3})$	0.429*** (0.108)	0.665*** (0.007)	0.644*** (0.020)
$\left(\lambda^3 s_i + \sum_{\tau=1,2} \mu^{\tau,3} y_i^\tau\right) \cdot \mathbf{1}(t=4)\ (c^4)$	0.429*** (0.046)		
$s_i \cdot \mathbf{1}(t=4)\ (\lambda^4 = c^4\lambda^3)$	*0.139*	0.242*** (0.008)	0.243*** (0.009)
$y_i^1 \cdot \mathbf{1}(t=4)\ (\mu^{1,4} = c^4\mu^{1,3})$	*0.106*		0.0125 (0.015)
$y_i^2 \cdot \mathbf{1}(t=4)\ (\mu^{2,4} = c^4\mu^{2,3})$	*0.184*		-0.002 (0.023)
$y_i^3 \cdot \mathbf{1}(t=4)\ (\mu^{3,4})$	0.569*** (0.046)	0.755*** (0.008)	0.744*** (0.013)
AIC	7620.819	6784.275	6784.261
BIC	7655.713	6819.168	6834.108
$\ln(L)$	-3803.41	-3385.138	-3382.13

Signif. codes: *** 1%, ** 5%, * 10%

Table B.2: Analog of Table 3.1 from Section 3.7 without stage 5 for equilibrium data (specification S234). All subjects are pooled, standard errors are clustered by subjects, periods and stages. Only the last 15 periods are used. *Italic entries* are derived from the estimated coefficients.

The equilibrium in the experimental setup requires a long description, so we omit it. Instead, we calculate it for every period in our experiment and then fit the linear model:

$$\begin{cases} x_i^2 &= \lambda_i^2 s_i + \mu_i^{1,2} y_i^1, \\ x_i^3 &= \lambda_i^3 s_i + \mu_i^{1,3} y_i^1 + \mu_i^{2,3} y_i^2, \\ x_i^4 &= \lambda_i^4 s_i + \mu_i^{1,4} y_i^1 + \mu_i^{2,4} y_i^2 + \mu_i^{3,4} y_i^3. \end{cases} \tag{B.2}$$

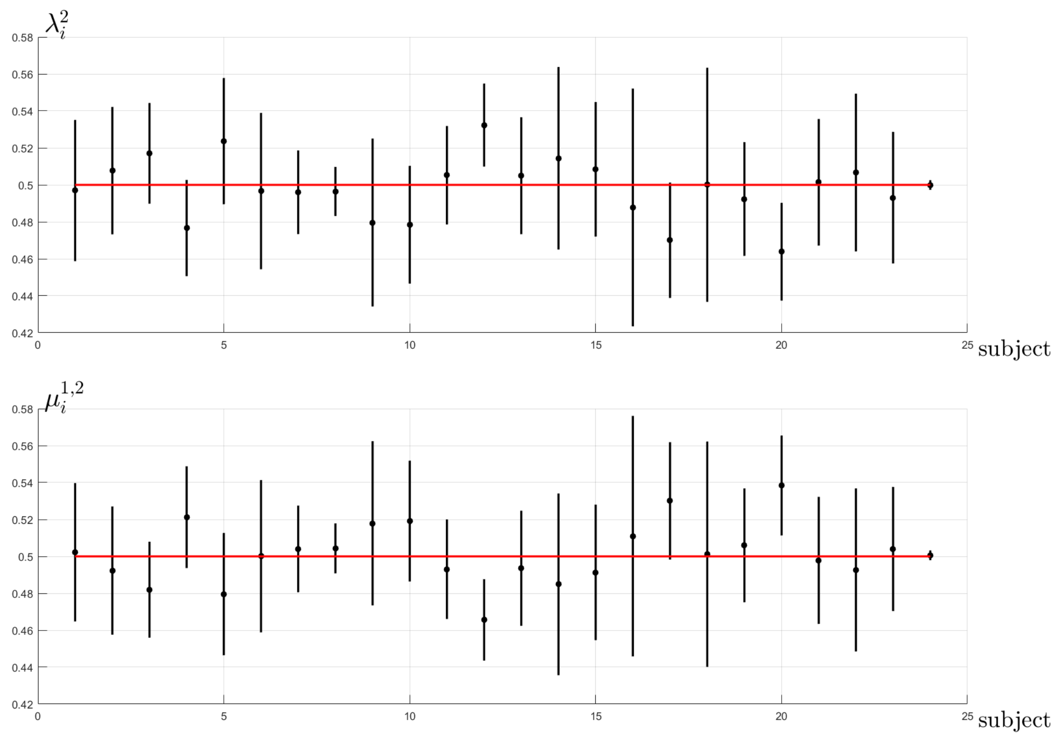

Figure B.6: Model (B.2), stage 2, equilibrium data. Only the last 15 periods are used for each subject. 95% confidence interval is used. Red line indicates Theorem 5's prediction.

Note that all guesses converge at the last stage, so that $x_i^5 = x_i^4$.

Table B.2, last column shows the fitted model (B.2). To match the actual data analysis, we exclude the first 5 periods and use only the last 15 periods. Comparing this general model with the PBH model and the BUH model based on the S234 specification, we see that

1) the BUH model has an advantage over the general model based on BIC and the direct comparison of the coefficients (all additional terms have nonsignificant effect);

2) the BUH model is definitely better than the PBH model based on the log-likelihood functions and the direct comparison of the coefficients for the PBH model and the general model.

What is more, the weights from the BUH model are very close to the equilibrium

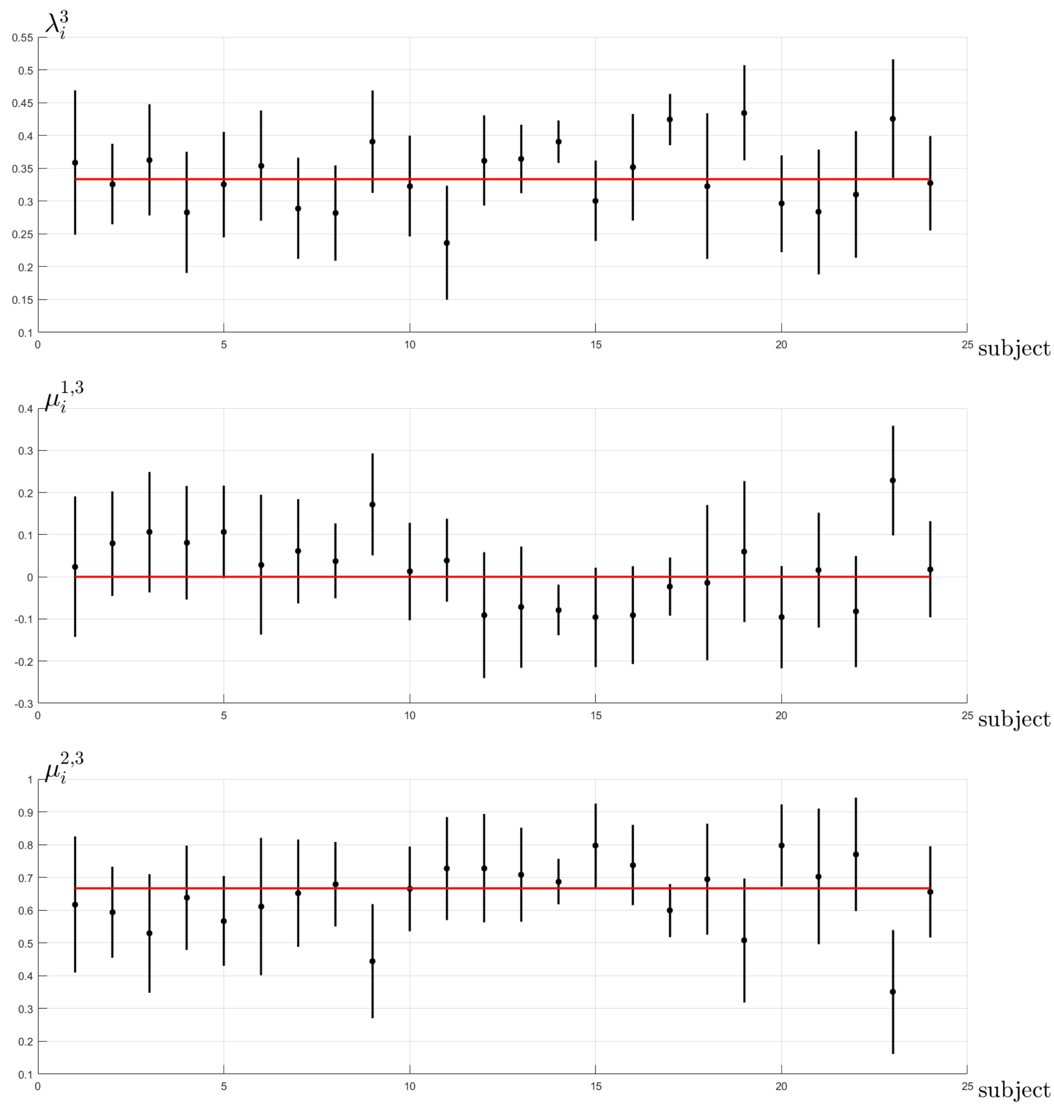

Figure B.7: Model (B.2), stage 3, equilibrium data. Only the last 15 periods are used for each subject. 95% confidence interval is used. Red line indicates Theorem 5's prediction.

weights from Theorem 5:

$$\begin{cases} x_i^2 &= 0.5 \cdot s_i + 0.5 \cdot y_i^1, \\ x_i^3 &= 0.333 \cdot s_i + 0 \cdot y_i^1 + 0.666 \cdot y_i^2, \\ x_i^4 &= 0.25 \cdot s_i + 0 \cdot y_i^1 + 0 \cdot y_i^2 + 0.75 \cdot y_i^3. \end{cases} \tag{B.3}$$

Figures B.6, B.7 and B.8 show model (B.2) fit for *each* subject separately. Again, all weights are very close to the equilibrium weights from Theorem 5.

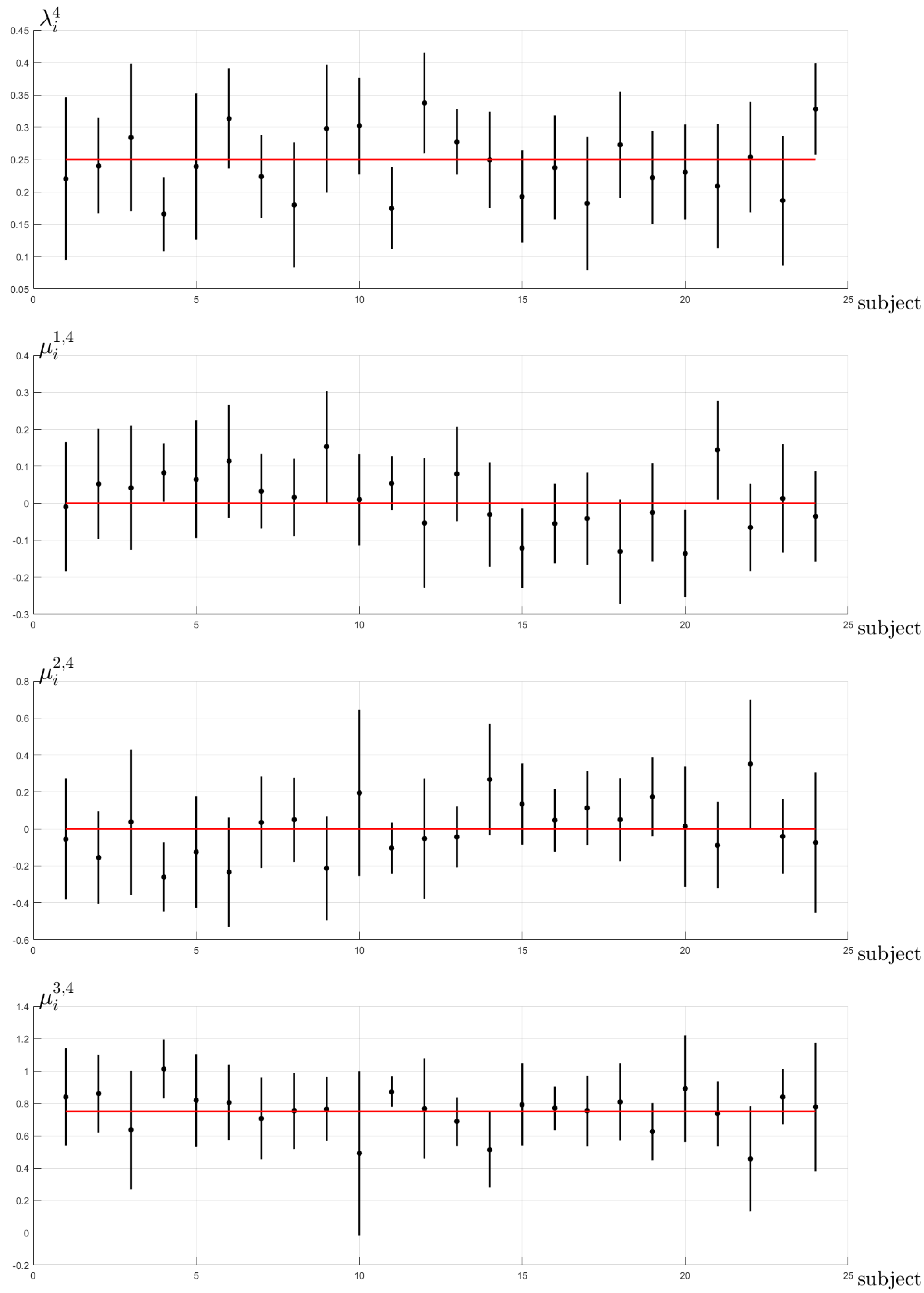

Figure B.8: Model (B.2), stage 4, equilibrium data. Only the last 15 periods are used for each subject. 95% confidence interval is used. Red line indicates Theorem 5's prediction.

B.7 Maximum Likelihood Comparison Across Different Specifications For Each Subject

	A	B	C	D
S2345	3.66 [3.31,4.01]	-4.29 [-4.47,-4.11]	2.57 [2.34,2.79]	17.03 [16.61,17.45]
S234	4.24 [3.97,4.5]	-7.83 [-7.98,-7.68]	0.61 [0.45,0.77]	1.28 [1.17,1.39]
S23	4.83 [4.68,4.98]	-0.16 [-0.33,0.01]	-0.06 [-0.16,0.03]	0.42 [0.37,0.47]
S345	-4.44 [-4.77,-4.11]	-4.79 [-4.95,-4.64]	18.42 [18.14,18.71]	29.61 [29.25,29.96]
S34	-0.79 [-1.03,-0.55]	-6.82 [-6.94,-6.7]	14.66 [14.4,14.91]	7.9 [7.75,8.05]
S45	-3.92 [-4.13,-3.71]	-1.15 [-1.26,-1.03]	3.05 [2.86,3.24]	24.64 [24.32,24.97]
S3	3.78 [3.64,3.92]	-1.73 [-1.86,-1.6]	9.35 [9.15,9.55]	5.78 [5.66,5.9]
S4	-2.83 [-2.94,-2.73]	-3.29 [-3.36,-3.22]	0.21 [0,0.42]	4.99 [4.89,5.08]
S5	6.9 [6.79,7.01]	13.36 [12.04,14.68]	5.7 [5.56,5.84]	18.48 [18.26,18.7]
S345x	7.62 [7.34,7.9]	0.13 [-0.02,0.28]	20.78 [20.49,21.07]	37.52 [37.13,37.9]
S34x	3.36 [3.14,3.57]	-5.5 [-5.61,-5.38]	14.56 [14.33,14.79]	10.83 [10.69,10.97]
S45x	1.94 [1.76,2.12]	1.42 [1.3,1.54]	6.71 [6.52,6.9]	29.15 [28.79,29.51]
S345i	-8.28 [-8.51,-8.06]	-2.2 [-2.32,-2.09]	3.93 [3.76,4.11]	25.85 [25.43,26.26]
S34i	-6.06 [-6.18,-5.94]	-5.35 [-5.44,-5.27]	1.03 [0.91,1.15]	1.93 [1.85,2.02]
S45i	0.93 [0.8,1.06]	4.16 [4.08,4.24]	9.29 [9.09,9.49]	26.71 [26.31,27.11]

Table B.3: Session 1. For each subject and each specification this table shows $LL_{\text{PBH-BUH}} = \ln(L_{\text{PBH}}) - \ln(L_{\text{BUH}})$, which is the difference between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model. 99% confidence intervals are found by bootstrapping by uniformly sampling by periods (only the last 15 original periods are used). Bootstrap sample size is 10,000.

	A	B	C	D
S2345	14.81 [14.36,15.27]	-6.36 [-6.51,-6.21]	27.65 [27.24,28.06]	-0.08 [-0.35,0.2]
S234	15.34 [14.09,16.6]	-2.48 [-2.64,-2.31]	23.51 [23.09,23.93]	-0.06 [-0.29,0.17]
S23	11.99 [11.16,12.82]	-2.31 [-2.62,-1.99]	19.38 [19.06,19.7]	-0.05 [-0.17,0.08]
S345	8.59 [8.09,9.09]	-4.77 [-4.89,-4.64]	19.12 [18.75,19.49]	-7.28 [-7.65,-6.91]
S34	8.42 [6.98,9.86]	-1.54 [-1.66,-1.41]	14.44 [14.09,14.79]	-6.34 [-6.62,-6.06]
S45	0.06 [-0.14,0.26]	-2.14 [-2.24,-2.05]	19.25 [19.13,19.38]	11.76 [11.55,11.97]
S3	5.8 [5.05,6.55]	-1.11 [-1.19,-1.03]	9.12 [8.91,9.33]	-5.98 [-6.15,-5.82]
S4	-45.1 [-45.61,-44.59]	1.92 [1.85,1.99]	20.49 [20.33,20.66]	7.31 [7.17,7.45]
S5	1.45 [1.06,1.83]	-2.79 [-2.89,-2.7]	4.07 [3.99,4.14]	5.08 [4.51,5.65]
S345x	11.46 [11.01,11.91]	-3.41 [-3.56,-3.27]	27.97 [27.65,28.29]	-1.34 [-1.69,-0.99]
S34x	10.22 [8.81,11.62]	0.03 [-0.09,0.15]	21.29 [20.95,21.63]	-3.48 [-3.74,-3.22]
S45x	1.09 [0.93,1.26]	-2.31 [-2.43,-2.19]	19.45 [19.33,19.58]	12.63 [12.42,12.85]
S345i	0.49 [0.32,0.67]	-2.54 [-2.64,-2.44]	6.42 [6.21,6.64]	10.43 [10.19,10.68]
S34i	-2.04 [-3.01,-1.07]	1.54 [1.46,1.62]	3.69 [3.5,3.89]	7.98 [7.81,8.16]
S45i	3.28 [2.65,3.92]	-2.51 [-2.62,-2.39]	3.06 [2.95,3.16]	3.96 [3.83,4.08]

Table B.4: Session 2. For each subject and each specification this table shows $\mathrm{LL}_{\text{PBH-BUH}} = \ln(L_{\text{PBH}}) - \ln(L_{\text{BUH}})$, which is the difference between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model. 99% confidence intervals are found by bootstrapping by uniformly sampling by periods (only the last 15 original periods are used). Bootstrap sample size is 10,000.

	A	B	C	D
S2345	16.42 [16.14,16.7]	-2.39 [-2.55,-2.23]	-0.22 [-0.42,-0.02]	-16.52 [-16.72,-16.31]
S234	17.41 [17.19,17.64]	0.07 [-0.07,0.21]	-2.34 [-2.54,-2.14]	-12.75 [-12.91,-12.58]
S23	22.79 [22.56,23.02]	-0.96 [-1.05,-0.87]	1.73 [1.54,1.92]	-5.33 [-5.45,-5.21]
S345	12.02 [11.81,12.23]	-2.31 [-2.5,-2.12]	-0.86 [-1.01,-0.71]	-16.43 [-16.68,-16.19]
S34	11.41 [11.27,11.56]	0.4 [0.26,0.55]	-2.33 [-2.46,-2.21]	-11.48 [-11.67,-11.3]
S45	2.04 [1.89,2.2]	3.84 [3.72,3.96]	-0.52 [-0.64,-0.39]	-6.41 [-6.64,-6.18]
S3	11.5 [11.29,11.7]	-0.9 [-1,-0.8]	0.29 [0.19,0.4]	-3.67 [-3.78,-3.57]
S4	2.56 [2.46,2.65]	3.35 [3.27,3.44]	-1.95 [-2.04,-1.86]	-3.29 [-3.44,-3.14]
S5	3.27 [3.2,3.35]	0.43 [-0.16,1.02]	2.63 [2.56,2.69]	0.18 [-0.06,0.41]
S345x	18.06 [17.86,18.25]	2.95 [2.77,3.13]	-0.08 [-0.23,0.07]	-9.68 [-9.94,-9.42]
S34x	13.06 [12.91,13.21]	2.5 [2.36,2.63]	-2.05 [-2.18,-1.92]	-7.07 [-7.26,-6.88]
S45x	5.49 [5.33,5.64]	3.84 [3.67,4.02]	-0.32 [-0.45,-0.2]	-5.21 [-5.44,-4.98]
S345i	1.35 [1.14,1.57]	1.03 [0.92,1.13]	-0.12 [-0.25,0.01]	-7.48 [-7.62,-7.35]
S34i	2.79 [2.64,2.94]	2.09 [2.01,2.17]	-1.87 [-1.98,-1.77]	-4.95 [-5.06,-4.84]
S45i	0.53 [0.42,0.64]	-0.88 [-0.93,-0.82]	2.61 [2.53,2.69]	-0.7 [-0.75,-0.66]

Table B.5: Session 3. For each subject and each specification this table shows $\mathrm{LL}_{\text{PBH-BUH}} = \ln(L_{\text{PBH}}) - \ln(L_{\text{BUH}})$, which is the difference between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model. 99% confidence intervals are found by bootstrapping by uniformly sampling by periods (only the last 15 original periods are used). Bootstrap sample size is 10,000.

	A	B	C	D
S2345	22.43 [22.26,22.6]	33.12 [32.72,33.53]	-0.52 [-0.68,-0.37]	0.9 [0.76,1.05]
S234	14.32 [14.19,14.46]	25.58 [25.19,25.96]	-0.31 [-0.43,-0.19]	0.05 [-0.07,0.17]
S23	7.84 [7.75,7.94]	17.66 [17.41,17.9]	0.01 [-0.07,0.08]	-0.28 [-0.35,-0.22]
S345	14.38 [14.16,14.61]	23.71 [23.39,24.04]	16.79 [16.52,17.06]	5.13 [4.82,5.45]
S34	8.49 [8.32,8.67]	16.32 [16.05,16.59]	16.07 [15.88,16.25]	1.68 [1.42,1.94]
S45	13.24 [13.09,13.39]	20.64 [20.47,20.8]	5.59 [5.41,5.76]	9.37 [9.23,9.51]
S3	2.48 [2.37,2.59]	8.46 [8.34,8.59]	10.85 [10.74,10.96]	-0.89 [-1.13,-0.64]
S4	5.57 [5.48,5.65]	12.05 [11.87,12.23]	9.89 [9.78,10]	4.16 [4.06,4.25]
S5	10.45 [10.36,10.55]	12.59 [12.51,12.66]	2.31 [2.21,2.41]	7.19 [7.12,7.26]
S345x	16.87 [16.65,17.08]	34.22 [34.03,34.41]	27.71 [27.46,27.95]	7.76 [7.45,8.06]
S34x	8.78 [8.61,8.95]	20.47 [20.25,20.69]	21.34 [21.16,21.53]	3.25 [3.01,3.49]
S45x	13.98 [13.82,14.13]	24.82 [24.68,24.95]	11.86 [11.68,12.03]	10.11 [9.96,10.26]
S345i	19.87 [19.69,20.05]	23.47 [23.23,23.72]	-0.94 [-1.03,-0.85]	5.02 [4.89,5.15]
S34i	10.98 [10.86,11.09]	12.87 [12.66,13.09]	-0.63 [-0.69,-0.57]	1.81 [1.71,1.91]
S45i	7.96 [7.74,8.18]	14.39 [14.23,14.55]	-1.1 [-1.19,-1.02]	3.17 [3.11,3.24]

Table B.6: Session 4. For each subject and each specification this table shows $LL_{\text{PBH-BUH}} = \ln(L_{\text{PBH}}) - \ln(L_{\text{BUH}})$, which is the difference between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model. 99% confidence intervals are found by bootstrapping by uniformly sampling by periods (only the last 15 original periods are used). Bootstrap sample size is 10,000.

	A	B	C	D
S2345	22.57 [22.26,22.87]	-5.12 [-5.3,-4.94]	9.23 [8.84,9.62]	168.37 [168.08,168.66]
S234	11.28 [11.07,11.49]	-4.01 [-4.2,-3.81]	-3.21 [-3.46,-2.96]	126.03 [125.82,126.23]
S23	3.46 [3.31,3.6]	1.27 [1.16,1.38]	-10.09 [-10.28,-9.9]	80.43 [80.26,80.59]
S345	16.37 [16.01,16.73]	-9.03 [-9.25,-8.82]	5.18 [4.87,5.49]	130.85 [130.6,131.11]
S34	6.76 [6.53,6.99]	-7.61 [-7.83,-7.4]	-3.93 [-4.12,-3.74]	88.13 [87.94,88.32]
S45	20.63 [20.4,20.86]	-0.17 [-0.26,-0.08]	16.66 [16.41,16.91]	94.04 [93.85,94.24]
S3	0.78 [0.64,0.92]	-1.61 [-1.74,-1.48]	-7.12 [-7.24,-6.99]	43.26 [42.97,43.55]
S4	7.62 [7.49,7.76]	0.37 [0.27,0.47]	5.82 [5.32,6.32]	50.49 [50.37,50.6]
S5	13.06 [12.99,13.13]	-0.48 [-0.54,-0.42]	18.04 [17.46,18.62]	47.35 [47.23,47.47]
S345x	20.63 [20.35,20.92]	-1.71 [-1.9,-1.52]	16.68 [16.44,16.91]	138.25 [138.03,138.48]
S34x	9.16 [8.94,9.38]	-1.23 [-1.42,-1.04]	0.37 [0.18,0.56]	91.3 [91.13,91.46]
S45x	19.62 [19.43,19.8]	-0.09 [-0.19,0.01]	22.25 [22.05,22.46]	97.72 [97.54,97.9]
S345i	22.79 [22.6,22.97]	-2.62 [-2.71,-2.52]	13.69 [13.29,14.09]	120.35 [120.06,120.65]
S34i	11.24 [11.11,11.36]	-2.55 [-2.66,-2.45]	3.37 [3.2,3.55]	77.09 [76.88,77.29]
S45i	14.1 [13.95,14.25]	0.89 [0.82,0.96]	9.43 [9.13,9.73]	78.34 [78.1,78.58]

Table B.7: Session 5. For each subject and each specification this table shows $LL_{\text{PBH-BUH}} = \ln(L_{\text{PBH}}) - \ln(L_{\text{BUH}})$, which is the difference between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model. 99% confidence intervals are found by bootstrapping by uniformly sampling by periods (only the last 15 original periods are used). Bootstrap sample size is 10,000.

	A	B	C	D
S2345	0.28 [0.22,0.33]	4.74 [4.57,4.91]	119.66 [119.07,120.26]	5.68 [5.58,5.77]
S234	0.25 [0.2,0.29]	0.98 [0.85,1.11]	86.17 [85.71,86.62]	5.06 [4.98,5.14]
S23	0.27 [0.26,0.28]	0.9 [0.83,0.97]	73.08 [72.92,73.24]	0.97 [0.94,1.01]
S345	4.24 [4.14,4.33]	-0.27 [-0.45,-0.09]	90.96 [90.46,91.45]	2.42 [2.35,2.5]
S34	4.53 [4.44,4.62]	-2.87 [-3,-2.74]	58.44 [58.09,58.79]	2.94 [2.87,3]
S45	1.24 [1.19,1.3]	8.24 [8.1,8.38]	58.85 [58.46,59.24]	0.46 [0.39,0.53]
S3	4.92 [4.84,4.99]	-1.43 [-1.5,-1.36]	42.08 [41.93,42.23]	3.39 [3.35,3.42]
S4	1.21 [1.16,1.27]	4.69 [4.59,4.79]	27.31 [27.07,27.54]	0.12 [0.04,0.2]
S5	0.38 [0.34,0.41]	7.15 [7.02,7.27]	45.72 [45.56,45.89]	1.44 [1.35,1.52]
S345x	5.87 [5.78,5.96]	8.18 [7.98,8.38]	97.75 [97.26,98.23]	3.53 [3.4,3.65]
S34x	5.9 [5.81,6]	1.8 [1.68,1.93]	59.45 [59.08,59.82]	1.69 [1.58,1.81]
S45x	1.51 [1.46,1.57]	11.95 [11.8,12.11]	62.61 [62.23,63]	1.87 [1.77,1.97]
S345i	0.49 [0.41,0.56]	3.89 [3.75,4.02]	85.34 [84.84,85.85]	7.58 [7.47,7.69]
S34i	0.34 [0.28,0.41]	0.44 [0.36,0.51]	52.36 [52,52.71]	7.31 [7.21,7.42]
S45i	0.28 [0.19,0.37]	3.17 [3.06,3.28]	46.4 [45.97,46.83]	3.68 [3.55,3.8]

Table B.8: Session 6. For each subject and each specification this table shows $LL_{\text{PBH-BUH}} = \ln(L_{\text{PBH}}) - \ln(L_{\text{BUH}})$, which is the difference between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model. 99% confidence intervals are found by bootstrapping by uniformly sampling by periods (only the last 15 original periods are used). Bootstrap sample size is 10,000.

B.8 Performance vs PBH-Fit

For all tables in this section: *Only the last 15 periods are used. One data point corresponds to one subject. Two subjects are dropped: subject D in session 5 and subject C in session 6.* So, the number of observations is 22 for every regression in this section.

We use the following notations:

$$\mathrm{LL}^{+}_{\text{PBH-BUH}} = \mathrm{LL}_{\text{PBH-BUH}} \cdot \mathbf{1}\left(\mathrm{LL}_{\text{PBH-BUH}} > 0\right),$$

$$\mathrm{LL}^{-}_{\text{PBH-BUH}} = \mathrm{LL}_{\text{PBH-BUH}} \cdot \mathbf{1}\left(\mathrm{LL}_{\text{PBH-BUH}} < 0\right).$$

	Specification S34				Specification S34x			
	Profit	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*
Intercept		-137 (292)	-134 (285)			-136 (294)	-144 (288)	
$\mathrm{LL}_{\text{PBH-BUH}}$			-5.06 (5.32)	-5.63 (5.08)			-3.96 (5.23)	-4.57 (4.99)
$\mathrm{LL}^{+}_{\text{PBH-BUH}}$	-3.09 (8.49)	-2.41 (8.79)			-2.28 (6.44)	-1.80 (6.65)		
$\mathrm{LL}^{-}_{\text{PBH-BUH}}$	-11.5 (16.5)	-11.2 (16.9)			-18.9 (25.2)	-17.7 (25.9)		
IdealProfit	1.17*** (0.06)	1.06*** (0.23)	1.04*** (0.23)	1.15*** (0.04)	1.16*** (0.06)	1.06*** (0.24)	1.03*** (0.23)	1.14*** (0.04)
Adjusted R^2	0.9815	0.4575	0.4818	0.9823	0.9813	0.4527	0.4730	0.9819

Signif. codes: *** 1%, ** 5%, * 10%

Table B.9: *Profit* sums the actual profit from stages 1,2,3 and 4. *IdealProfit* sums the profit based on *Ideal Guesses* from stages 1,2,3 and 4.

	Specification S2345				Specification S345				Specification S345x			
	Profit	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*
Intercept		-406	-304			-194	-224			-238	-231	
		(321)	(337)			(338)	(326)			(340)	(331)	
$LL_{PBH\text{-}BUH}$			3.35	2.48			-4.01	-4.19			-2.41	-2.58
			(4.21)	(4.08)			(4.31)	(4.25)			(4.04)	(3.98)
$LL^{+}_{PBH\text{-}BUH}$	7.17	8.94*			-0.94	-1.29			-2.05	-1.74		
	(4.82)	(4.94)			(6.68)	(6.83)			(4.73)	(4.81)		
$LL^{-}_{PBH\text{-}BUH}$	-19.3	-20.9			-13.4	-11.8			-8.26	-9.54		
	(13.6)	(13.5)			(15.0)	(15.5)			(26.01)	(26.43)		
IdealProfit	1.23***	0.97***	0.98***	1.18***	1.19***	1.05***	1.00***	1.15***	1.16***	1.00***	0.99***	1.15***
	(0.05)	(0.21)	(0.23)	(0.04)	(0.06)	(0.25)	(0.22)	(0.04)	(0.06)	(0.23)	(0.23)	(0.05)
Adjusted R^2	0.983	0.534	0.4716	0.9815	0.9815	0.4569	0.4778	0.9820	0.9806	0.4366	0.4641	0.9815

Signif. codes: *** 1%, ** 5%, * 10%

Table B.10: *Profit* sums the actual profit from stages 1,2,3,4 and 5. *IdealProfit* sums the profit based on *Ideal Guesses* from stages 1,2,3,4 and 5.

	Specification S2345				Specification S345				Specification S345x			
	Profit	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*
Intercept		-405^{**}	-300			-314^{*}	-270			-326	-277	
		(182)	(178)			(174)	(177)			(188)	(179)	
$LL_{\text{PBH-BUH}}$			2.03	1.10			-1.26	-1.60			0.17	-0.46
			(2.85)	(2.92)			(2.98)	(3.07)			(2.76)	(2.83)
$LL^{+}_{\text{PBH-BUH}}$	2.81	5.57			2.13	3.51			0.09	1.60		
	(3.55)	(3.46)			(4.48)	(4.30)			(3.26)	(3.22)		
$LL^{-}_{\text{PBH-BUH}}$	-7.07	-13.0			-12.5	-14.9			-6.92	-15.3		
	(9.99)	(9.47)			(10.0)	(9.6)			(18.13)	(17.9)		
IdealProfit	1.27^{***}	0.72^{**}	0.80^{***}	1.23^{***}	1.28^{***}	0.85^{***}	0.81^{***}	1.20^{***}	1.23^{***}	0.79^{***}	0.82^{***}	1.21^{***}
	(0.07)	(0.25)	(0.26)	(0.06)	(0.09)	(0.25)	(0.26)	(0.06)	(0.09)	(0.27)	(0.26)	(0.07)
Adjusted R^2	0.9591	0.3474	0.2872	0.9597	0.9605	0.3191	0.2748	0.9599	0.9576	0.2589	0.2682	0.9594

Signif. codes: *** 1%, ** 5%, * 10%

Table B.11: *Profit* sums the actual profit from stages 3,4 and 5. *IdealProfit* sums the profit based on *Ideal Guesses* from stages 3,4 and 5.

	Specification S234				Specification S34				Specification S34x			
	Profit	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*
Intercept		-371*** (124)	-286** (130)			-305** (134)	-237* (129)			-277** (131)	-244* (132)	
$LL_{\text{PBH-BUH}}$			2.29 (2.59)	0.57 (2.69)			-1.64 (3.06)	-2.83 (3.17)			-0.66 (3.03)	-2.21 (3.08)
$LL^{+}_{\text{PBH-BUH}}$	2.70 (3.21)	6.35** (2.95)			-0.63 (4.95)	3.84 (4.90)			-0.15 (3.80)	2.41 (3.70)		
$LL^{-}_{\text{PBH-BUH}}$	-9.39 (8.75)	-13.8* (7.5)			-8.13 (9.59)	-13.7 (9.0)			-15.9 (15.0)	-19.4 (13.9)		
IdealProfit	1.27*** (0.08)	0.54** (0.25)	0.63** (0.28)	1.22*** (0.06)	1.24*** (0.10)	0.67** (0.27)	0.70** (0.27)	1.20*** (0.06)	1.25*** (0.09)	0.70** (0.27)	0.69** (0.28)	1.19*** (0.07)
Adjusted R^2	0.9544	0.3433	0.2006	0.9535	0.9536	0.2211	0.1802	0.9551	0.9542	0.2074	0.1698	0.9545

Signif. codes: *** 1%, ** 5%, * 10%

Table B.12: *Profit* sums the actual profit from stages 3 and 4. *IdealProfit* sums the profit based on *Ideal Guesses* from stages 3 and 4.

	Specification S23				Specification S3			
	Profit	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*
Intercept		-219*** (65)	-166** (64)			-148** (64)	-127** (57)	
$LL_{PBH\text{-}BUH}$			0.90 (1.65)	-0.97 (1.70)			-3.24 (2.25)	-4.70* (2.35)
$LL^{+}_{PBH\text{-}BUH}$	-0.55 (1.93)	2.98 (1.88)			-5.17 (3.38)	-1.21 (3.50)		
$LL^{-}_{PBH\text{-}BUH}$	-3.78 (6.02)	-8.45 (5.04)			-3.31 (7.32)	-8.24 (6.94)		
IdealProfit	1.18*** (0.07)	0.40 (0.24)	0.55** (0.24)	1.17*** (0.06)	1.13*** (0.08)	0.63** (0.23)	0.66*** (0.22)	1.14*** (0.05)
Adjusted R^2	0.9539	0.3010	0.1978	0.9556	0.9605	0.249	0.2654	0.9624

Signif. codes: *** 1%, ** 5%, * 10%

Table B.13: *Profit* is the actual profit from stage 3. *IdealProfit* is the profit based on *Ideal Guesses* from stage 3.

	Profit	*Profit*	*Profit*	*Profit*
Intercept		-232 (240)	-71 (238)	
$LL_{PBH\text{-}BUH}$			4.08 (4.70)	3.58 (4.30)
$LL^{+}_{PBH\text{-}BUH}$	7.09 (4.67)	9.53* (5.31)		
$LL^{-}_{PBH\text{-}BUH}$	-18.5 (14.3)	-22.0 (14.8)		
IdealProfit	1.18*** (0.04)	0.98*** (0.22)	1.09*** (0.22)	1.15*** (0.03)
Adjusted R^2	0.9835	0.6173	0.5681	0.9822

Signif. codes: *** 1%, ** 5%, * 10%

Table B.14: Specification S23. *Profit* sums the actual profit from stages 1,2 and 3. *IdealProfit* sums the profit based on *Ideal Guesses* from stages 1,2 and 3.

	Specification S45				Specification S45x				Specification S345i			
	Profit	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*
Intercept		-197	-187			-205	-189			-188	-187	
		(129)	(127)			(136)	(127)			(128)	(127)	
$\text{LL}_{\text{PBH-BUH}}$			-0.44	-0.97			0.12	-0.34			-0.30	-0.85
			(3.00)	(3.07)			(2.85)	(2.92)			(2.75)	(2.80)
$\text{LL}^{+}_{\text{PBH-BUH}}$	0.30	1.14			-0.39	0.67			0.95	1.53		
	(3.63)	(3.55)			(3.24)	(3.21)			(3.44)	(3.37)		
$\text{LL}^{-}_{\text{PBH-BUH}}$	-13.8	-16.0			0.54	-9.94			-12.5	-12.1		
	(19.2)	(18.6)			(24.77)	(24.94)			(13.1)	(12.7)		
IdealProfit	1.26^{***}	0.82^{**}	0.79^{**}	1.21^{***}	1.22^{***}	0.78^{**}	0.79^{**}	1.22^{***}	1.28^{***}	0.85^{**}	0.79^{**}	1.22^{***}
	(0.10)	(0.30)	(0.30)	(0.08)	(0.10)	(0.31)	(0.30)	(0.09)	(0.10)	(0.31)	(0.30)	(0.07)
Adjusted R^2	0.9408	0.1838	0.1958	0.9425	0.9392	0.1579	0.1949	0.9422	0.9419	0.1912	0.1953	0.9424

Signif. codes: *** 1%, ** 5%, * 10%

Table B.15: *Profit* sums the actual profit from stages 4 and 5. *IdealProfit* sums the profit based on *Ideal Guesses* from stages 4 and 5.

	Specification S4				Specification S34i			
	Profit	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*
Intercept		-134* (73)	-127* (73)			-153** (68)	-135* (75)	
$LL_{\text{PBH-BUH}}$			-0.34 (1.32)	-0.81 (1.35)			0.45 (3.00)	-1.28 (2.99)
$LL^{+}_{\text{PBH-BUH}}$	1.59 (3.24)	2.58 (3.10)			3.78 (3.92)	6.39 (3.75)		
$LL^{-}_{\text{PBH-BUH}}$	-1.73 (1.77)	-1.42 (1.68)			-17.0* (8.9)	-17.1** (8.1)		
IdealProfit	1.30*** (0.10)	0.70* (0.34)	0.66* (0.34)	1.24*** (0.07)	1.39*** (0.11)	0.72** (0.32)	0.63* (0.35)	1.24*** (0.08)
Adjusted R^2	0.9259	0.07933	0.0758	0.9271	0.9346	0.2434	0.07376	0.9265

Signif. codes: *** 1%, ** 5%, * 10%

Table B.16: *Profit* is the actual profit from stage 4. *IdealProfit* is the profit based on *Ideal Guesses* from stage 4.

	Specification S5				Specification S45i			
	Profit	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*	*Profit*
Intercept		-59.8 (62.6)	-57.1 (60.6)			-58.3 (63.2)	-58.8 (61.4)	
$LL_{\text{PBH-BUH}}$			1.08 (2.23)	1.12 (2.22)			-0.28 (2.04)	-0.04 (2.02)
$LL^{+}_{\text{PBH-BUH}}$	0.90 (2.50)	0.73 (2.52)			-0.22 (2.28)	-0.40 (2.30)		
$LL^{-}_{\text{PBH-BUH}}$	6.16 (23.8)	9.03 (24.01)			4.44 (23.69)	2.80 (23.85)		
IdealProfit	1.23*** (0.10)	0.95*** (0.31)	0.98*** (0.30)	1.24*** (0.09)	1.20*** (0.09)	0.93*** (0.31)	0.93*** (0.30)	1.21*** (0.08)
Adjusted R^2	0.9457	0.2685	0.3027	0.9483	0.9449	0.2564	0.2949	0.9476

Signif. codes: *** 1%, ** 5%, * 10%

Table B.17: *Profit* is the actual profit from stage 5. *IdealProfit* is the profit based on *Ideal Guesses* from stage 5.

B.9 Maximum Likelihood Comparison for Section 3.9.2

	All subjects	BUH-subjects	PBH-subjects	Unclear	PBH3BUH4	BUH3PBH4
S2345	66.58	-168.35	24.14	4.02	-11.25	73.97
S234	42.69	-234.85	14.76	4.71	-11.24	25.31
S23	60.8	-187.47	8.67	6.53	0.54	-56.52
S345	151.55	-153.50	20.96	52.64	-12.06	42.48
S34	105.68	-174.45	13.12	38.50	-9.66	7.76
S45	123.89	-33.57	21.13	23.62	-4.43	84.28
S3	97.3	-103.59	10.96	26.18	-0.56	-34.12
S4	38.27	-53.84	8.51	14.27	-7.10	38.80
S5	11812	4241	997	1973	954	3280
S345x	382.36	-49.71	48.75	78.35	7.47	134.85
S34x	155.91	-147.68	19.51	46.64	-4.92	26.83
S45x	340.7	38.58	39.99	51.89	14.27	152.33
S345i	6.87	-28.95	26.03	-5.65	-6.66	78.37
S34i	-18.93	-76.20	12.24	-2.91	-7.31	39.38
S45i	57.52	63.66	15.44	-0.87	13.99	40.69

Table B.18: $LL_{\text{PBH-BUH}}$ for different subsets of subjects when the ideal guesses are used instead of the actual guesses. *BUH-subjects* are B and C in session 1, A and C in session 3, C in session 4 and all subjects in session 5. *PBH-subjects* are A in session 1 and C in session 2. *Unclear* group are D in session 1, A in session 2, B in session 3, D in session 4. *PBH3BUH4* group are B in session 2 and A in session 6. *BUH3PBH4* group are D in session 2, D in session 3, A and B in session 4, and B, C and D in session 6.

	A	B	C	D
S2345	9.52 [9.31,9.72]	0.79 [0.46,1.11]	-18.7 [-18.92,-18.47]	1.86 [1.51,2.21]
S234	5.33 [5.18,5.47]	-19.44 [-19.66,-19.22]	-32.44 [-32.62,-32.25]	0.43 [0.03,0.84]
S23	3.27 [3.11,3.44]	-13.51 [-13.73,-13.29]	-39.59 [-39.76,-39.41]	0.9 [0.37,1.43]
S345	3.37 [3.15,3.58]	-2.31 [-2.58,-2.04]	-17.66 [-17.82,-17.49]	0.44 [0.15,0.73]
S34	0.65 [0.5,0.8]	-14.65 [-14.81,-14.5]	-24.19 [-24.31,-24.07]	-0.32 [-0.61,-0.02]
S45	6.39 [6.25,6.52]	9.96 [9.79,10.14]	1.14 [0.98,1.29]	12.08 [11.79,12.38]
S3	0.78 [0.55,1.01]	-7.52 [-7.64,-7.41]	-21.12 [-21.21,-21.04]	0.1 [-0.17,0.37]
S4	2.08 [2.01,2.16]	-4.32 [-4.42,-4.22]	-4.71 [-4.8,-4.62]	7.22 [7.02,7.42]
S5	491.49 [491.14,491.84]	484.41 [484.09,484.73]	482.96 [482.64,483.27]	481.47 [481.16,481.79]
S345x	15.49 [15.27,15.72]	6.05 [5.8,6.31]	-7.59 [-7.78,-7.39]	11.01 [10.68,11.34]
S34x	2.93 [2.78,3.07]	-12.69 [-12.83,-12.55]	-20.84 [-20.95,-20.73]	5.65 [5.3,6]
S45x	15.77 [15.61,15.93]	17.96 [17.79,18.14]	7.92 [7.74,8.11]	17.68 [17.41,17.95]
S345i	12.94 [12.71,13.17]	13.76 [13.58,13.93]	2.14 [1.95,2.34]	0.68 [0.5,0.86]
S34i	5.89 [5.73,6.05]	-3.86 [-3.95,-3.76]	-8.15 [-8.3,-8]	-0.25 [-0.4,-0.1]
S45i	7.77 [7.61,7.94]	20.27 [20.07,20.48]	9.58 [9.44,9.72]	1.66 [1.45,1.87]

Table B.19: Session 1. For each subject and each specification this table shows $\mathrm{LL}_{\text{PBH-BUH}} = \ln(L_{\text{PBH}}) - \ln(L_{\text{BUH}})$, which is the difference between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model. Every real guess x_i^t is changed to *Ideal Guess*. 99% confidence intervals are found by bootstrapping by uniformly sampling by periods (only the last 15 original periods are used). Bootstrap sample size is 10,000.

	A	B	C	D
S2345	32.74 [32.15,33.33]	-1.01 [-1.68,-0.33]	54.74 [54.39,55.1]	-6.3 [-6.92,-5.68]
S234	24.49 [23.94,25.05]	-2.72 [-3.88,-1.56]	47.83 [47.54,48.12]	-14.96 [-15.46,-14.46]
S23	16.93 [16.11,17.75]	7.87 [6.34,9.4]	76.97 [75.44,78.5]	-43.06 [-43.7,-42.43]
S345	24.13 [23.67,24.58]	-3.94 [-4.46,-3.41]	41.05 [40.78,41.32]	-9.7 [-10.16,-9.23]
S34	16.03 [15.65,16.41]	-3.75 [-4.53,-2.96]	31.88 [31.68,32.07]	-12.93 [-13.26,-12.59]
S45	0.33 [0.21,0.45]	-9.26 [-9.44,-9.09]	15.01 [14.86,15.16]	7.63 [7.31,7.94]
S3	8.32 [7.9,8.74]	3.16 [2.39,3.94]	38.15 [37.36,38.94]	-22.72 [-23,-22.43]
S4	-1.31 [-1.38,-1.25]	-21.56 [-21.65,-21.46]	6.39 [6.31,6.47]	3.06 [2.88,3.24]
S5	466.12 [465.8,466.44]	462.69 [462.37,463.02]	473.24 [472.92,473.56]	470.73 [470.38,471.07]
S345x	22.53 [22.12,22.95]	4.09 [3.57,4.61]	54.3 [54.05,54.56]	2.18 [1.74,2.62]
S34x	14.04 [13.67,14.4]	-2.14 [-2.91,-1.37]	32.63 [32.46,32.8]	-9.9 [-10.22,-9.58]
S45x	7.27 [7.13,7.41]	-3.37 [-3.55,-3.19]	23.56 [23.38,23.74]	17.99 [17.68,18.31]
S345i	4.71 [4.6,4.81]	-5.25 [-5.46,-5.04]	21.34 [21.07,21.6]	14.73 [14.28,15.18]
S34i	1.56 [1.48,1.65]	-9.81 [-10.22,-9.4]	11.66 [11.49,11.84]	7.46 [7.13,7.78]
S45i	9 [8.86,9.14]	37.3 [37,37.61]	10.46 [10.16,10.77]	8.88 [8.72,9.05]

Table B.20: Session 2. For each subject and each specification this table shows $LL_{\text{PBH-BUH}} = \ln(L_{\text{PBH}}) - \ln(L_{\text{BUH}})$, which is the difference between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model. Every real guess x_i^t is changed to *Ideal Guess*. 99% confidence intervals are found by bootstrapping by uniformly sampling by periods (only the last 15 original periods are used). Bootstrap sample size is 10,000.

	A	B	C	D
S2345	-30.02 [-30.31,-29.72]	-7.33 [-7.64,-7.02]	-3.96 [-4.29,-3.64]	7.72 [7.46,7.98]
S234	-34.59 [-34.83,-34.36]	-11.58 [-11.8,-11.36]	-8.11 [-8.39,-7.83]	-1.46 [-1.67,-1.24]
S23	-32.11 [-32.32,-31.89]	-67.49 [-68.1,-66.88]	-2.2 [-2.54,-1.85]	-25.5 [-26.16,-24.83]
S345	-27.45 [-27.69,-27.2]	-8.9 [-9.14,-8.65]	-5.41 [-5.67,-5.15]	3.91 [3.72,4.11]
S34	-27 [-27.17,-26.83]	-10.06 [-10.22,-9.91]	-6.8 [-6.99,-6.6]	-2.46 [-2.6,-2.32]
S45	-3.29 [-3.46,-3.13]	-0.88 [-0.98,-0.78]	-0.61 [-0.77,-0.46]	6.6 [6.45,6.75]
S3	-18.76 [-18.87,-18.65]	-34.89 [-35.2,-34.59]	-1.75 [-1.93,-1.57]	-13.85 [-14.18,-13.52]
S4	-5.1 [-5.2,-5]	-0.15 [-0.2,-0.09]	-2.9 [-2.98,-2.83]	2 [1.91,2.09]
S5	491.85 [491.53,492.17]	493.11 [492.79,493.43]	493.93 [493.61,494.25]	493.04 [492.73,493.36]
S345x	-14.68 [-14.91,-14.45]	-2.65 [-2.77,-2.52]	4.94 [4.71,5.17]	6.82 [6.65,6.99]
S34x	-22.19 [-22.34,-22.04]	-10.95 [-11.04,-10.87]	-4.73 [-4.92,-4.55]	-3.92 [-4.04,-3.79]
S45x	4.61 [4.43,4.79]	8.97 [8.86,9.07]	7.13 [6.97,7.29]	13.14 [12.97,13.3]
S345i	-3.99 [-4.2,-3.79]	3.26 [3.09,3.43]	0.13 [-0.07,0.32]	12.02 [11.77,12.27]
S34i	-8.2 [-8.33,-8.07]	1.98 [1.86,2.1]	-3.71 [-3.82,-3.61]	5.27 [5.09,5.45]
S45i	5.45 [5.27,5.63]	5.6 [5.28,5.92]	8.27 [8.1,8.45]	7.25 [7.06,7.44]

Table B.21: Session 3. For each subject and each specification this table shows $\mathrm{LL}_{\text{PBH-BUH}} = \ln(L_{\text{PBH}}) - \ln(L_{\text{BUH}})$, which is the difference between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model. Every real guess x_i^t is changed to *Ideal Guess*. 99% confidence intervals are found by bootstrapping by uniformly sampling by periods (only the last 15 original periods are used). Bootstrap sample size is 10,000.

	A	B	C	D
S2345	25.75 [25.21,26.29]	-5.44 [-5.66,-5.22]	-24.12 [-24.43,-23.81]	-2.5 [-2.68,-2.31]
S234	14.49 [14.05,14.93]	-7.06 [-7.26,-6.87]	-24.69 [-24.93,-24.45]	-1.13 [-1.27,-0.99]
S23	-6.88 [-7.17,-6.58]	-7.01 [-7.18,-6.84]	-98.54 [-98.7,-98.37]	-0.21 [-0.31,-0.12]
S345	17.75 [17.34,18.16]	-7.46 [-7.65,-7.26]	-22.24 [-22.47,-22.02]	2.12 [1.91,2.33]
S34	8.63 [8.33,8.92]	-6.72 [-6.86,-6.57]	-18.89 [-19.06,-18.73]	2.69 [2.53,2.85]
S45	11.43 [11.16,11.71]	7.45 [7.35,7.55]	-4.99 [-5.2,-4.78]	7.81 [7.6,8.01]
S3	-4 [-4.15,-3.84]	-4.63 [-4.72,-4.54]	-50.42 [-50.5,-50.34]	3.18 [3.07,3.29]
S4	4.99 [4.84,5.15]	2.98 [2.93,3.04]	-2.78 [-2.88,-2.68]	4.96 [4.85,5.07]
S5	487.47 [487.09,487.85]	485.07 [484.69,485.45]	474.47 [474.11,474.82]	480.13 [479.75,480.51]
S345x	27.47 [27.07,27.87]	14.94 [14.78,15.11]	-7.75 [-7.97,-7.53]	14.86 [14.6,15.11]
S34x	7.45 [7.17,7.73]	0.38 [0.27,0.48]	-14.76 [-14.91,-14.61]	7.72 [7.55,7.89]
S45x	21.5 [21.22,21.78]	17.82 [17.7,17.93]	4.05 [3.83,4.26]	14.78 [14.55,15.01]
S345i	23.8 [23.41,24.2]	2.16 [2.04,2.27]	-3.58 [-3.89,-3.26]	0.65 [0.48,0.81]
S34i	13.77 [13.49,14.05]	-0.09 [-0.18,0.01]	-3.84 [-4.03,-3.64]	1.39 [1.28,1.5]
S45i	11.57 [11.35,11.79]	2.29 [2.2,2.39]	2.98 [2.79,3.18]	0.7 [0.54,0.86]

Table B.22: Session 4. For each subject and each specification this table shows $\mathrm{LL}_{\text{PBH-BUH}} = \ln(L_{\text{PBH}}) - \ln(L_{\text{BUH}})$, which is the difference between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model. Every real guess x_i^t is changed to *Ideal Guess*. 99% confidence intervals are found by bootstrapping by uniformly sampling by periods (only the last 15 original periods are used). Bootstrap sample size is 10,000.

	A	B	C	D
S2345	-28.6 [-28.92,-28.27]	-26.56 [-26.78,-26.33]	-16.82 [-17.12,-16.51]	-29.64 [-29.82,-29.45]
S234	-40 [-40.25,-39.75]	-20.33 [-20.51,-20.16]	-34.25 [-34.49,-34.01]	-39.5 [-39.77,-39.24]
S23	-108.27 [-108.44,-108.1]	-10.76 [-10.93,-10.59]	-20.89 [-21.03,-20.75]	-42.32 [-42.59,-42.05]
S345	-24.19 [-24.42,-23.97]	-22.4 [-22.59,-22.22]	-15 [-15.23,-14.78]	-25.06 [-25.2,-24.91]
S34	-28.49 [-28.65,-28.33]	-14.94 [-15.05,-14.82]	-24.35 [-24.51,-24.2]	-28.26 [-28.45,-28.07]
S45	-11.48 [-11.69,-11.26]	-11.16 [-11.29,-11.02]	-4.33 [-4.54,-4.12]	-3.77 [-3.88,-3.66]
S3	-55.33 [-55.41,-55.24]	-6.03 [-6.12,-5.94]	-11.44 [-11.51,-11.37]	-22.17 [-22.31,-22.04]
S4	-10.4 [-10.51,-10.29]	-5.92 [-5.99,-5.86]	-11.01 [-11.15,-10.87]	-6.49 [-6.58,-6.39]
S5	476.13 [475.81,476.46]	474.25 [473.93,474.57]	478.71 [478.38,479.03]	476.74 [476.41,477.06]
S345x	-16.07 [-16.29,-15.84]	-6.73 [-6.89,-6.56]	-6.97 [-7.18,-6.76]	-9.89 [-10.02,-9.76]
S34x	-27.12 [-27.27,-26.97]	-11.92 [-12.03,-11.81]	-22.37 [-22.52,-22.22]	-22.53 [-22.71,-22.35]
S45x	-4.37 [-4.6,-4.14]	-1.68 [-1.82,-1.55]	1.36 [1.12,1.6]	4.31 [4.21,4.41]
S345i	-11.99 [-12.29,-11.69]	-10.79 [-10.94,-10.65]	-2.31 [-2.57,-2.05]	-5.28 [-5.44,-5.13]
S34i	-17.99 [-18.19,-17.79]	-7.12 [-7.21,-7.04]	-15.05 [-15.25,-14.85]	-12.07 [-12.21,-11.93]
S45i	43.83 [43.48,44.19]	2.45 [2.33,2.57]	13.95 [13.75,14.16]	8.05 [7.8,8.3]

Table B.23: Session 5. For each subject and each specification this table shows $LL_{\text{PBH-BUH}} = \ln(L_{\text{PBH}}) - \ln(L_{\text{BUH}})$, which is the difference between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model. Every real guess x_i^t is changed to *Ideal Guess*. 99% confidence intervals are found by bootstrapping by uniformly sampling by periods (only the last 15 original periods are used). Bootstrap sample size is 10,000.

	A	B	C	D
S2345	-0.24 [-0.38,-0.09]	24.58 [24.29,24.86]	2.85 [2.59,3.12]	12.26 [12.02,12.49]
S234	-3.03 [-3.14,-2.92]	12.6 [12.42,12.78]	2.27 [2.04,2.49]	-8.94 [-9.16,-8.72]
S23	0.46 [0.35,0.57]	-1.33 [-1.53,-1.12]	-10.71 [-10.93,-10.49]	-105.62 [-105.75,-105.49]
S345	-0.18 [-0.29,-0.08]	18.22 [18.01,18.43]	-1.39 [-1.61,-1.17]	5.94 [5.75,6.12]
S34	-2.02 [-2.1,-1.95]	7.98 [7.87,8.1]	-0.61 [-0.78,-0.44]	-8.32 [-8.48,-8.17]
S45	-0.03 [-0.13,0.08]	23.15 [23.02,23.29]	5.84 [5.7,5.98]	7.63 [7.5,7.75]
S3	0.21 [0.16,0.27]	-1.29 [-1.41,-1.18]	-6.91 [-7.02,-6.8]	-54.13 [-54.19,-54.06]
S4	-1.54 [-1.59,-1.49]	11.26 [11.19,11.32]	3.47 [3.4,3.55]	0.84 [0.8,0.88]
S5	498.73 [498.39,499.07]	503.75 [503.42,504.09]	497.43 [497.1,497.76]	499.24 [498.9,499.58]
S345x	9.22 [9.1,9.34]	24.93 [24.75,25.1]	10.53 [10.36,10.7]	8.68 [8.53,8.83]
S34x	-1.21 [-1.28,-1.14]	9.5 [9.41,9.59]	1.41 [1.27,1.55]	-10.21 [-10.32,-10.1]
S45x	10.33 [10.22,10.43]	32.08 [31.93,32.23]	13.36 [13.23,13.5]	17.68 [17.54,17.81]
S345i	-0.58 [-0.69,-0.47]	18.09 [17.83,18.34]	5.08 [4.91,5.25]	17.68 [17.51,17.86]
S34i	-2.46 [-2.51,-2.4]	8.38 [8.22,8.54]	3.99 [3.86,4.11]	4.17 [4.08,4.27]
S45i	7.36 [7.15,7.58]	16.24 [15.98,16.51]	1.08 [0.93,1.22]	24.25 [23.97,24.54]

Table B.24: Session 6. For each subject and each specification this table shows $LL_{\text{PBH-BUH}} = \ln(L_{\text{PBH}}) - \ln(L_{\text{BUH}})$, which is the difference between the log-likelihood functions at the estimated parameters' values for the PBH model and the BUH model. Every real guess x_i^t is changed to *Ideal Guess*. 99% confidence intervals are found by bootstrapping by uniformly sampling by periods (only the last 15 original periods are used). Bootstrap sample size is 10,000.

B.10 Robustness Check for Table 3.5

We use the following notations:

$$\mathrm{LL}^{\text{ideal+}}_{\text{PBH-BUH}} = \mathrm{LL}^{\text{ideal}}_{\text{PBH-BUH}} \cdot \mathbf{1}\left(\mathrm{LL}^{\text{ideal}}_{\text{PBH-BUH}} > 0\right),$$

$$\mathrm{LL}^{\text{ideal}-}_{\text{PBH-BUH}} = \mathrm{LL}^{\text{ideal}}_{\text{PBH-BUH}} \cdot \mathbf{1}\left(\mathrm{LL}^{\text{ideal}}_{\text{PBH-BUH}} < 0\right).$$

	Intercept	$\mathrm{LL}^{\text{ideal}}_{\text{PBH-BUH}}$	$\mathrm{LL}^{\text{ideal+}}_{\text{PBH-BUH}}$	$\mathrm{LL}^{\text{ideal}-}_{\text{PBH-BUH}}$	Adjusted R^2
S2345	6.63**	0.144			0.014
	(2.62)	(0.127)			
	0.34		0.468**	-0.341	0.173
	(3.73)		(0.187)	(0.249)	
S234	5.12**	0.161			0.072
	(2.15)	(0.099)			
	-0.35		0.542***	-0.143	0.273
	(2.86)		(0.173)	(0.148)	
S23	4.66**	0.059			0.058
	(1.86)	(0.039)			
	2.62		0.225**	0.013	0.152
	(2.10)		(0.099)	(0.045)	
S345	5.77**	0.022			-0.049
	(2.61)	(0.157)			
	0.83		0.371	-0.385	0.026
	(4.00)		(0.266)	(0.297)	
S34	3.51*	0.049			-0.042
	(1.97)	(0.126)			
	-0.72		0.492*	-0.256	0.099
	(2.77)		(0.248)	(0.191)	
S45	5.91***	0.233			0.004
	(2.06)	(0.224)			
	2.02		0.645*	-0.584	0.066
	(3.24)		(0.346)	(0.577)	
S3	2.46*	0.008			-0.049
	(1.33)	(0.055)			
	1.17		0.214	-0.048	0.016
	(1.54)		(0.146)	(0.064)	

Signif. codes: *** 1%, ** 5%, * 10%

Table B.25: The dependent variable is $\mathrm{LL}^{\text{real}}_{\text{PBH-BUH}}$. Only the last 15 periods are used. One data point corresponds to one subject. Two subjects are dropped: subject D in session 5 and subject C in session 6. So, the total number of observations is 22 for each regression.

	Intercept	$LL^{ideal}_{PBH\text{-}BUH}$	$LL^{ideal+}_{PBH\text{-}BUH}$	$LL^{ideal-}_{PBH\text{-}BUH}$	Adjusted R^2
S4	1.85	0.189			-0.036
	(2.60)	(0.366)			
	-3.01		1.524	-0.450	0.038
	(3.95)		(0.908)	(0.534)	
S345x	10.56***	0.118			-0.027
	(3.08)	(0.178)			
	2.07		0.529**	-1.376**	0.224
	(4.10)		(0.216)	(0.568)	
S34x	6.42***	0.125			-0.004
	(1.89)	(0.132)			
	1.81		0.583**	-0.244	0.138
	(2.83)		(0.252)	(0.216)	
S45x	5.97*	0.240			0.007
	(3.28)	(0.224)			
	3.83		0.366	-1.254	-0.018
	(4.47)		(0.288)	(2.100)	
S345i	5.66**	-0.022			-0.049
	(2.41)	(0.216)			
	4.78		0.049	-0.253	-0.098
	(3.59)		(0.304)	(0.719)	
S34i	2.10*	0.011			-0.050
	(1.16)	(0.144)			
	-0.45		0.477	-0.329	0.038
	(1.88)		(0.309)	(0.245)	

Signif. codes: *** 1%, ** 5%, * 10%

Table B.26: The dependent variable is $LL^{real}_{PBH\text{-}BUH}$. Only the last 15 periods are used. One data point corresponds to one subject. Two subjects are dropped: subject D in session 5 and subject C in session 6. So, the total number of observations is 22 for each regression.

Appendix C

APPENDIX FOR CHAPTER 4

C.1 Proof for Theorem 7

The Hamilton-Jacobi-Bellman equation

$$\min\left\{\mathcal{L}(p), V(p) - \max_{v\in\{G,I\}} U(v,p)\right\} = 0, \tag{C.1}$$

where

$$\mathcal{L}(p) = \kappa - \frac{2}{\sigma^2}p^2(1-p)^2V''(p),$$

gives the sufficient condition for a continuously differentiable function $V\colon [0,1] \to \mathbb{R}$ to be the value function

$$V(p) = \sup_{(\tau,v)} \mathbb{E}\left[U(v,p_\tau) - \kappa\tau \mid p_0 = p\right]. \tag{C.2}$$

Differential equation $\mathcal{L}(p) = 0$ has the following solution:

$$V(p) = C_1 + pC_2 + \kappa\sigma^2\left(p - \frac{1}{2}\right)\log\left(\frac{p}{1-p}\right),$$

where C_1 and C_2 are some constants.

Consider the following class of functions defined on $p \in [0,1]$ parameterized with $\lambda \in (0, 0.5]$:

$$V_\lambda(p) = \begin{cases} pQ + R, & p \geq 1-\lambda, \\ (1-\lambda)Q + R + \kappa\sigma^2\left(p - \frac{1}{2}\right)\log\left(\frac{p}{1-p}\right) - \kappa\sigma^2\left(\lambda - \frac{1}{2}\right)\log\left(\frac{\lambda}{1-\lambda}\right), & \lambda < p < 1-\lambda, \\ (1-p)Q + R, & p \leq \lambda. \end{cases}$$

Note that these functions are continuous, symmetric around $p = 0.5$, that is $V_\lambda(p) = V_\lambda(1-p)$, and satisfy $\mathcal{L}(p) = 0$ for $\lambda < p < 1-\lambda$. Moreover, function $V_\lambda(p)$ is continuously differentiable if and only if

$$\lim_{p\to\lambda+0} V'_\lambda(p) = -Q,$$

which is equivalent to (4.3). Note that the left hand side of (4.3) is a decreasing function of $\lambda \in (0, 0.5]$ from $+\infty$ to 0. Thus, the solution to (4.3) always exists and unique.

Finally, note that

1. $V_\lambda(p) \geq \max\limits_{v\in\{G,I\}} U(v,p)$ for $\lambda < p < 1-\lambda$ since $V_\lambda(p)$ is convex for $\lambda < p < 1-\lambda$,
2. $\mathcal{L}(p) \geq 0$ for $p \geq 1-\lambda$ and $p \leq \lambda$ since the utility function is linear over the belief.

Thus, $V_\lambda(p)$ is the value function and therefore the strategy (4.2) is the unique optimal one.

C.2 Proof for Theorem 8

First, we calculate $\lambda'\left(\sigma^2\right)$ from (4.3) holding κ, Q, R fixed:

$$\lambda'\left(\sigma^2\right) = \frac{(1-\lambda)\lambda}{\sigma^2}\left(1-2\lambda-2(1-\lambda)\lambda\log\left(\frac{\lambda}{1-\lambda}\right)\right). \tag{C.3}$$

Substituting (C.3) into

$$\mathcal{X}'\left(\sigma^2\right) = \frac{1}{2}\left(\log\left(\frac{1-\lambda}{\lambda}\right) - \frac{\sigma^2}{(1-\lambda)\lambda}\lambda'\left(\sigma^2\right)\right), \tag{C.4}$$

we get $\mathcal{X}'\left(\sigma^2\right) = g\left(\lambda\left(\sigma^2\right)\right)$, where for any $\lambda \in (0,0.5)$ function $g(\lambda)$ is defined as

$$g(\lambda) = \lambda - \frac{1}{2} + \left(\frac{1}{2} - (1-\lambda)\lambda\right)\log\left(\frac{1-\lambda}{\lambda}\right).$$

Function $g(\lambda)$ is decreasing in $\lambda \in (0,0.5)$ from $+\infty$ to 0. Thus, it is always positive and therefore $\mathcal{X}'\left(\sigma^2\right) > 0$.

C.3 Proof for Theorem 9

Consider function $g(\eta) = \eta c(\eta) - \hat{c}$. $c'(\eta) > 0$ if $g(\eta) < 0$ and $c'(\eta) < 0$ if $g(\eta) > 0$. Since $g'(\eta)|_{g(\eta)=0} = c(\eta) > 0$, function $g(\eta)$ can cross 0 only once and only from below. From (4.14), if the solution to $g(\eta^*) = 0$ exists, it is equal to $\eta^* = \frac{1}{Qh'(\hat{c})}$. Substituting $\eta^* = \frac{1}{Qh'(\hat{c})}$ to (4.14), we get $c(\eta^*) = \frac{\hat{c}}{\eta^*}$ and thus $g(\eta^*) = 0$.

C.4 Proof for Theorem 10

Differentiating (4.17), we get $c'(\eta) = \frac{cQh'(\eta c)}{1-\eta h'(\eta c)Q} > 0$.

C.5 Dynamic Model, $\delta > 0$

The decision maker faces the following optimization problem:[1]

$$\sup_{(\tau,v)} \mathbb{E}\left[e^{-\delta\tau}U(v,p_\tau) - \kappa\int_0^\tau e^{-\delta t}dt\right]. \tag{C.5}$$

[1] A more general strategy space includes the opportunity to allocate partial attention to the information flow. If the agent allocates $\Delta \in [0,1]$ amount of attention at time t, then he pays $\kappa\Delta dt$

Before presenting the optimal strategy, we need to make one more assumption:

Assumption 7 $-\kappa < (Q + R)\delta$.

This assumption says that if $R < -Q$, then δ should be small enough. It guarantees that stopping is optimal if the true state is known. Indeed, if $R < -Q$, then the decision maker gets negative utility when he makes the decision. Basically, the final utility payment acts as a cost. If the discount factor is large, he would want to postpone the payment of this cost forever. If Assumption 7 does not hold, the optimal strategy does not exist.

Theorem 12 *The optimal strategy exists and is given by (4.2), where threshold* $\lambda \in (0, 0.5)$ *is uniquely defined by*

$$\frac{2\lambda(1-\lambda)}{\left(1 - \frac{2}{1+\left(\frac{1-\lambda}{\lambda}\right)^{\sqrt{1+2\delta\sigma^2}}}\right)\sqrt{1+2\delta\sigma^2} - 1 + 2\lambda} - (1-\lambda) = \frac{R + \frac{\kappa}{\delta}}{Q}. \tag{C.6}$$

Proof:

The Hamilton-Jacobi-Bellman equation (C.1), where

$$\mathcal{L}(p) = \kappa - \frac{2}{\sigma^2} p^2 (1-p)^2 V''(p) + \delta V(p),$$

gives the sufficient condition for a continuously differentiable function $V\colon [0,1] \to \mathbb{R}$ to be the value function

$$V(p) = \sup_{(\tau,v)} \mathbb{E}\left[e^{-\delta\tau} U(v, p_\tau) - \kappa \int_0^\tau e^{-\delta t} dt \mid p_0 = p \right]. \tag{C.7}$$

Differential equation $\mathcal{L}(p) = 0$ has the following solution:

$$V(p) = -\frac{\kappa}{\delta} + C_1 p^{\frac{1-\sqrt{1+2\delta\sigma^2}}{2}} (1-p)^{\frac{1+\sqrt{1+2\delta\sigma^2}}{2}} + C_2 p^{\frac{1+\sqrt{1+2\delta\sigma^2}}{2}} (1-p)^{\frac{1-\sqrt{1+2\delta\sigma^2}}{2}},$$

where C_1 and C_2 are some constants.

---

and observes $X_{t+\Delta dt} - X_t$. It turns out that it is never optimal to allocate partial attention when the discount factor is positive. Suppose $\Delta^* > 0$ is optimal. Then $\mathcal{L}(p_t, \Delta^*) = \min_{\Delta \in [0,1]} \mathcal{L}(p_t, \Delta) = 0$, where $\mathcal{L}(p, \Delta) = \Delta\left(\kappa - \frac{2}{\sigma^2} p^2 (1-p)^2 V''(p)\right) + \delta V(p)$, $V(p)$ is the value function. Thus, whenever $\delta V(p) \neq 0$, we must have $\Delta^* = 1$.

Consider the following class of functions defined on $p \in [0,1]$ parameterized with $\lambda \in (0, 0.5]$:

$$V_\lambda(p) = \begin{cases} pQ + R, & p \geq 1-\lambda, \\ \frac{\left(\frac{1-p}{p}\right)^{\frac{\sqrt{1+2\delta\sigma^2}}{2}} + \left(\frac{p}{1-p}\right)^{\frac{\sqrt{1+2\delta\sigma^2}}{2}}}{\left(\frac{1-\lambda}{\lambda}\right)^{\frac{\sqrt{1+2\delta\sigma^2}}{2}} + \left(\frac{\lambda}{1-\lambda}\right)^{\frac{\sqrt{1+2\delta\sigma^2}}{2}}} \sqrt{\frac{(1-p)p}{(1-\lambda)\lambda}} \left((1-\lambda)Q + R + \frac{c}{\delta}\right) - \frac{c}{\delta}, & \lambda < p < 1-\lambda, \\ (1-p)Q + R, & p \leq \lambda. \end{cases}$$

Note that these functions are continuous, symmetric around $p = 0.5$, that is $V_\lambda(p) = V_\lambda(1-p)$, and satisfy $\mathcal{L}(p, 1) = 0$ for $\lambda < p < 1-\lambda$. Moreover, function $V_\lambda(p)$ is continuously differentiable if and only if

$$\lim_{p \to \lambda+0} V'_\lambda(p) = -Q,$$

which is equivalent to (C.6). Note that the left hand side of (C.6) is an increasing function of $\lambda \in (0, 0.5]$ from -1 to $+\infty$. Thus,

- if $\frac{R+\frac{\kappa}{\delta}}{Q} > -1$, then the solution to (C.6) always exists and unique,
- if $\frac{R+\frac{\kappa}{\delta}}{Q} \leq -1$, then there is no $\lambda \in (0, 0.5]$ such that $V_\lambda(p)$ is continuously differentiable.

Note that (C.6) implies that $(1-\lambda)Q + R + \frac{\kappa}{\delta} > 0$.

Finally, note that

1. $V_\lambda(p) \geq \max\limits_{v \in \{G,I\}} U(v,p)$ for $\lambda < p < 1-\lambda$ since $V_\lambda(p)$ is convex for $\lambda < p < 1-\lambda$ as long as $(1-\lambda)Q + R + \frac{\kappa}{\delta} > 0$,
2. $\mathcal{L}(p) \geq 0$ for $p \geq 1-\lambda$ and $p \leq \lambda$ as long as $(1-\lambda)Q + R + \frac{\kappa}{\delta} \geq 0$.

Thus, $V_\lambda(p)$ is the value function and therefore the strategy (4.2) is the unique optimal one. ∎

Note that when Assumption 7 just holds, that is when $(Q+R)\delta + \kappa$ is very small, the optimal λ is very close to 0, which corresponds to long learning.

Theorem 13 *$\mathcal{X}\left(\sigma^2\right)$ defined by (4.10) and (C.6) is increasing in σ^2.*

Proof: First, we calculate $\lambda'\left(\sigma^2\right)$ from (C.6) holding κ, Q, R, δ fixed:

$$\lambda'\left(\sigma^2\right) = \frac{(1-\lambda)\lambda\left(\left(\frac{1-\lambda}{\lambda}\right)^{2\sqrt{1+2\delta\sigma^2}} + 2\sqrt{1+2\delta\sigma^2}\left(\frac{1-\lambda}{\lambda}\right)^{\sqrt{1+2\delta\sigma^2}}\log\left(\frac{1-\lambda}{\lambda}\right) - 1\right)}{\left(1+\left(\frac{1-\lambda}{\lambda}\right)^{\sqrt{1+2\delta\sigma^2}}\right)^2 \sigma^2\sqrt{1+2\delta\sigma^2}}. \tag{C.8}$$

Substituting (C.8) into (C.4), we get $\mathcal{X}'\left(\sigma^2\right) = g\left(\lambda\left(\sigma^2\right), \sqrt{1+2\delta\sigma^2}\right)$, where for any $\lambda \in (0, 0.5)$ and $\alpha > 1$ function $g(\lambda, \alpha)$ is defined as

$$g(\lambda, \alpha) = \frac{1+\alpha\left(1+\left(\frac{1-\lambda}{\lambda}\right)^{2\alpha}\right)\log\left(\frac{1-\lambda}{\lambda}\right) - \left(\frac{1-\lambda}{\lambda}\right)^{2\alpha}}{2\alpha\left(1+\left(\frac{1-\lambda}{\lambda}\right)^{\alpha}\right)^2}.$$

For any fixed $\alpha > 1$, function $g(\lambda, \alpha)$ is decreasing in $\lambda \in (0, 0.5)$ from $+\infty$ to 0. Thus, it is always positive and therefore $\mathcal{X}'\left(\sigma^2\right) > 0$. ■

C.6 Dynamic Asymmetric Model

Consider a general form of the utility function:

$$U(v, p) = pu(v, I) + (-p)u(v, G).$$

Assumption 8 $u(A, I) > \max\{u(A, G), u(C, I)\}$, $u(C, G) > \max\{u(C, I), u(A, G)\}$.

Denote

$$p^* = \frac{u(C, G) - u(A, G)}{u(C, G) - u(A, G) + u(A, I) - u(C, I)}.$$

Then, the judge wants to acquit when $p > p^*$ and he wants to convict when $p < p^*$.

Without loss of generality, assume that the judge is weakly biased towards convicting

Assumption 9 $p^* \geq 0.5$.

Theorem 14 *The optimal strategy exists and is given by*

$$\tau = \inf\left\{t \geq 0\colon p_t \notin (\lambda, \mu)\right\}, \quad v = \begin{cases} A, & p_\tau \geq \mu, \\ C, & p_\tau \leq \lambda, \end{cases} \tag{C.9}$$

where p_t is the belief that the true state is I at time t. Thresholds $0 < \lambda < p^* < \mu < 1$ are uniquely defined by

$$f(\mu) - f(\lambda) = \frac{G}{2}, \quad \mu = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4(1-\lambda)\lambda}{1 + (2p^* - 1)G(1-\lambda)\lambda}}\right), \tag{C.10}$$

where $G = \frac{2(u(A,I)-u(A,G)+u(C,G)-u(C,I))}{\kappa\sigma^2}$, $f(x) = \log\left(\frac{x}{1-x}\right) - \frac{1-2x}{2(1-x)x}$.

The proof is similar to Theorem 7.

In asymmetric case, the choice of the welfare function is not so obvious. In symmetric case, it is natural to take the probability of the correct decision as the welfare criterion. When there is bias in prior belief and / or in preferences $u(v, z)$, there are many different options one can take as the welfare criterion. We are not going to consider them all and just focus on how the strategy changes with the overconfidence level.

Theorem 15 confirms the conclusion of Theorem 8 for the upper threshold:

Theorem 15 $\frac{\sigma^2}{2}\log\left(\frac{\mu(\sigma^2)}{1-\mu(\sigma^2)}\right)$ *is increasing in σ^2.*

However, this conclusion is no longer true for the lower threshold:

Theorem 16 *When $p^* > \frac{1}{2}$, there exists $\Sigma^2 > 0$ such that $\frac{\sigma^2}{2}\log\left(\frac{\lambda(\sigma^2)}{1-\lambda(\sigma^2)}\right)$ is decreasing for $\sigma^2 < \Sigma^2$ and it is increasing for $\sigma^2 > \Sigma^2$.*

Theorem 16 states that there is a unique level of overconfidence $\eta = \frac{\sigma^2}{\Sigma^2}$ that minimizes the lower threshold for X_t. Intuitively, there is a trade-off between the preference bias and the overall precision of the decision. When there is a lot of noise in information, the bias is more prominent since the decision is not precise anyway. As information becomes more precise, the trade-off optimal resolution moves towards the decision precision, which means that the thresholds become more symmetric.