

**QUEUEING AND TRAFFIC IN CELLULAR RADIO**

Thesis by  
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## QUEUEING AND TRAFFIC IN CELLULAR RADIO

### ABSTRACT

A Cellular Radio system is analyzed from the communications traffic point of view. A cell within a given system is modeled by a multi-server service facility with or without the possibility of queueing some type of customers. Two types of arrivals are distinguished, corresponding to handoff calls (calls already in progress that enter the cell) and originating calls (calls initiated inside the cell). The queueing system used assumes Poisson distributed arrivals with different rates for the two types of customers. We initially assume, as is usually done for telephone communications, an exponential distribution for the service times of the customers. Due to mobility of the subscribers that can travel through several cells in the system, the channel occupancy time is in general different from the total call duration. Using both a simulation of a cellular system and an analytic model we offer evidence that a memoryless distribution may not be too unrealistic for the channel occupancy time.

We derive some traffic policies that give a higher level of protection to handoff calls, and their influence on the other class of customers as well as on the overall traffic is analyzed. The first policies proposed have the advantage of simplicity and provide an efficient way of reducing the blocking probability of handoff calls while only slightly increasing the blocking probability of originating calls. The price paid is, however, a small decrease in the total carried traffic.

Some more evolved traffic policies are then introduced that still decrease the blocking probability of handoff calls without much penalizing originating calls whose access to the system will only be slightly delayed. These more evolved policies provide the additional advantage of increasing the total carried traffic, while still providing a higher level of protection to handoff calls.

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*To my parents  
and  
to Janne*



## CHAPTER 1

### INTRODUCTION

#### 1.1 Presentation of the Cellular concept.

Cellular Radio aims at making public switched mobile communications available to a very large number of subscribers. While the earlier mobile radio system was using a high-power transmitter positioned at a high point allowing the coverage of the whole service area, a Cellular Radio system ([1]) divides the entire area where service is to be provided into a large number of smaller sub-areas (*cells*), each served by its own low power transmitter (cell site). Each cell will provide service to subscribers located in its own service area. A cell is connected through wireline connections to a system switching center which will in turn provide connection to the Public Switched Telephone Network (PSTN). The system will switch a subscriber over to a new cell as soon as it leaves its original area. This operation is called a *handoff* and ideally goes unnoticed by the customer. We will give details on handoffs later.

In the old system, any customer wishing to receive or initiate a call was served (when possible) by the essentially unique high-power transmitter providing coverage of the entire service area. The problem inherent to this approach was that the total number of customers that could be simultaneously served in the whole service area was simply equal to the number of available frequency channels. Unfortunately, this number was very small (between 10 and 20), which meant that as soon as, say 20 (actually typically 11), calls were present in the system, it would be saturated and all new calls would be blocked. Such a situation is clearly unacceptable if a strong demand for mobile communications exists. The belief that such a need was present, together with important progress in electronics technology, made the implementation of cellular systems both desirable and possible.

In a system based on the cellular concept, the problem of a limited number of channels is overcome through frequency reuse. This principle states that the same frequency channel can be reused in many different cells within the system as long as the mutual interference levels are low enough. This notion has been very commonly used on a larger scale where, for example, television stations in Los Angeles and San Francisco use the same frequencies to transmit their programs. The distance between Los Angeles and San Francisco insures that the two stations will not interfere with each other.

In Cellular Radio, the low level of interference arises from the fact that in an urban environment (not line of sight propagation) the signal power drops rather fast, like the inverse of the 3rd or 4th power of the distance ([2], p. 29, [3]). This implies that the same frequency can be reused many times in the system, increasing the possible number of simultaneous calls on that same frequency by the same factor. Cellular Radio also benefited from a new spectrum allocation in the UHF range with more frequencies available than for the old mobile communications system, but this effect is secondary. The cellular system has over 300 voice channels, but as we shall see each cell only has around 44 channels. However, thousands of calls can be present in the service area at once. The limiting factor in Cellular Radio is not thermal noise, but the self-noise or co-channel interference generated by the other users on the same channel.

It is clear that at this point many problems remain to be solved, such as efficient frequency assignments adapted to the system structure and to the propagation conditions ([4]-[11]), adequate modulation techniques taking into account the special requirements of the transmission channel ([12]-[18]), and optimum cell sites and antenna configuration ([19], [20]). However, even if cellular systems will generally differ from each other in those technical choices, they all rely on this concept of

dividing the whole service area into a large number of cells. Chapter 2 will provide a more detailed exposition of the different choices that have been made for already existing systems, but we shall first present some issues which are of special interest to us.

### **1.2 Problems due to the mobility of the subscribers.**

Let us now consider some interesting problems arising from the mobility of the customers. The first one to arise is the estimation of the distribution of the channel occupancy time. This parameter is of importance for system design and dimensioning. However, even its very definition depends on the cellular concept.

In a classical landbased telephone system, the same telephone line or frequency (cable, radio, or satellite link, etc.) is allocated to a given connection for its entire duration. Under these assumptions, the channel occupancy time is equal to the call duration and usually taken to have an exponential distribution, with departure rate equal to the reciprocal of the average call duration.

In a Cellular Radio system, once a call is initiated, the mobile subscriber might leave the initial cell and be handed off to another cell before call completion. At this point we see that a channel occupancy time will not in general be equal to the call duration. A mobile can move through several cells while involved in a call. In this case, the occupancy time of a given frequency channel will only correspond to the portion of the total call duration time during which the mobile is located in the associated cell. This tells us that even if the call duration itself is still taken to be exponentially distributed as we may expect ([21], p. 9), the channel occupancy time might not be exponentially distributed. We are therefore interested in estimating the actual distribution of the channel occupancy times as well as the influence of the different system parameters on this distribution. This we will do by using both



a computer simulation of a cellular system and by deriving an analytic model for a simplified system.

Another problem arising from the mobility of the customers is the possible blocking of a handoff call. Namely, we can have a call that was initiated in one cell in the system and moves into a new cell where all the frequency channels happen to be already occupied by other calls. This entering call will then have to be cleared from the system involuntarily. Such an event results in a disconnection in the middle of a call, which is highly undesirable. It is therefore necessary to develop traffic policies that help minimize this type of occurrences. On the other hand, the blocking of an originating call merely represents a delay in setting up a communication, which is not as penalizing as an involuntary disconnect.

### **1.3 Possible extensions of a Cellular Radio system.**

We have seen how the introduction of the cellular concept leads to the development of radio systems capable of providing telephone service to a very large number of mobile customers. Let us now try to outline the possible future trends and extensions of cellular systems. We shall do so in the perspective of both new implementations and new offered services.

One of the major trends in Cellular Radio is the introduction of digital technology in the traditionally analog transmission part ([22]-[28]). Several reasons can be found for this tendency. The first one comes from the great progress made in LSI and VLSI technologies, which have made digital devices available at very competitive prices. These techniques are attractive not only because of their low cost, but mainly because they provide feasible solutions to actual needs. For example, digital data encryption provides a perfect answer to the requirement of secure voice and data transmissions. Furthermore, the extension of cellular radio to within-building

communications could require powerful error-correction capability to counteract the extremely hostile propagation environment, thus again requiring digital communications.

The problems encountered with portable units operated inside buildings come from several factors. The first one is that the power level should be kept as low as possible to keep the units light in weight and to maximize the times between battery recharges. On the other hand the propagation conditions inside a building can vary greatly due to the complex nature of the components of the structure of the building itself. This can yield very dramatic variations in the level of the signal power. Due to the previously mentioned power limitation, this cannot be taken care of by sufficiently large power margins, at least not on the uplink (subscriber to base station). A solution to this problem therefore requires some other means (such as coding) to insure the needed reliability of the communication. Finally, we should mention that the possible extension of Cellular Radio to data transmission and other ISDN (Integrated Service Digital Network) type of services makes the trend toward digital technologies even stronger.

Note that in the case of data transmission, codes capable of correcting bursts of errors will be required to counteract the effect of handoffs. For, even if handoffs are imperceptible to telephone users (1/20th of a second), they will represent rather important bursts of errors for data or digital voice users ([23]). For example, if vocoders are used to compress voice in the case of digital voice transmissions, or if some important data file (mobile medical unit waiting for diagnosis, etc.) is being sent, it is either impossible (voice transmission) or extremely undesirable (urgent need of the data) to ask for retransmission of the bits in error after a handoff has taken place. The data should therefore be encoded prior to transmission, and due to

the bursty aspect of error occurrences, burst error correcting codes (Reed-Solomon, Fire, etc.) should be considered.

More precisely, one can note that the burst of errors due to handoffs is rather a burst of erasures. At a bit rate of 9600 bits per second, a handoff of 1/20th of a second represents a sequence of 480 bits in error, or more precisely, missing. Erasing comes from the fact that the mobiles and the cell sites will always know when a handoff occurs (through the use of the control channel, as we will see in Chapter 2). Thus most cyclic code can provide burst erasure fill-in capacity provided that there is only one burst of erasures per codeword. This will typically be the case since handoffs will not usually occur close to each other, and may even be prevented from so doing.

On the other hand, due to the presence of fading, the communication channel itself (between handoffs) will very probably be also a bursty channel ([28], pp. 57-66, pp. 153-163, 185-194), with a different distribution of bursts of errors than the distribution of bursts of erasures due to handoffs. This implies that not only bursts of erasures need to be considered, but regular bursts of errors will also be important. Therefore, designing powerful codes that take the nature of the channel as well as the kind of data sent (digitized voice for example) into account is certainly one of the most necessary future developments for Cellular Radio.

Another possible development in Cellular Radio might come from the use of multi-beam satellites to implement this kind of system in light-traffic areas ([29]-[32]). Such a solution is well adapted to low population density areas, or to regions difficult to access, where the need for mobile communications is present. A good example could be a country like Indonesia which is made up of thousands of small islands with heavy maritime traffic between them. A cellular system covering the

waterways would in this case represent a useful complement to the land telephone network of this country, which is already relying on satellites for long distance. Another practical example is provided by a project that was initiated, and later abandoned, by the Swedish government. This project was meant to provide mobile communications to a fleet of trucks travelling in the northern part of Scandinavia where telecommunication resources are very sparse but badly needed for both safety and dispatching.

Finally, another heavily investigated possible development of Cellular Radio is the use of spread spectrum techniques in place of more classical modulation and frequency assignments ([33]-[37]). The use of spread spectrum techniques offers many advantages from the point of view of interference and multipath protection. On the other hand, it is feared that the use of large spread spectrum systems such as in Cellular Radio could create some nuisance for nearby non-spread spectrum systems. All these questions are under study and involve many regulatory aspects as well as technical ones. This probably means that even if the greater efficiency of spread spectrum over more conventional techniques is established, their actual implementation in Cellular Radio will be rather slow and limited. One of the main reasons for this is that new systems will most certainly have to be compatible with the previous ones, and spread spectrum does not seem to offer such a possibility. Another modulation scheme under study that might offer promising possibilities is Amplitude Comanded Single Sideband (ACSB or ACSSB). This technique offers some of the advantage of the capture effect of classical FM with a seemingly smaller spectral occupancy. However, even if some improved implementation schemes are already available ([13]) the definite advantage of ACSB over classical FM is not yet well established, and appropriate comparisons of the two techniques still need to be performed ([17]).

#### 1.4 Overview and organization.

In our work we analyze a Cellular Radio system from a communications traffic point of view. We start by constructing a model for the distribution of the channel occupancy time. This model is justified and shown to be accurately represented in most practical cases by a negative exponential. This result is obtained by first simulating a cellular system under some very general assumptions. The experimental channel occupancy time distribution obtained from the simulation is then compared to a best-fit exponential model, and excellent agreement is found between the two.

An analytic model for the distribution of the channel occupancy time distribution in a Cellular Radio system is then derived under some simplifying vehicle motion assumptions. In this case the distribution is found equal to a sum of negative exponentials, and is shown to be reasonably close to the previous exponential model. It is to be noted that the assumptions used for the simulation and the ones used for the analytic model are based on two opposite extremes of vehicle behavior. The channel occupancy time distribution of real life cellular systems is likely to be located somewhere in between these two models, so therefore still equal to a negative exponential distribution. One of the main conclusions we can draw from this result is that the times between handoffs can be assumed to be memoryless, that is, exponentially distributed. This result is very useful in analyzing the system traffic behavior.

Using this exponential model for handoffs, we will design efficient traffic policies which give a higher grade of protection to handoff calls. This is required, as we mentioned earlier, because handoff calls represent calls already in progress. The basic idea behind the traffic policies studied is the introduction of **guard channels**. More precisely, for a given cell within a cellular system with a total of  $n$  frequency channels, a guard band of  $g$  ( $0 \leq g < n$ ) guard channels will mean the following:

- As soon as only  $g$  channels remain free in the cell, all newly originating calls are blocked, reserving the use of the guard channels for potential handoff arrivals. When there are again more than  $g$  free channels in the cell, originating calls start being served normally again.

Note that these  $g$  channels are not fixed in advance, but can be any channels depending on the actual traffic and channels in use at the instant at which only  $g$  channels are free.

We establish that this technique yields a noticeable decrease in the blocking of handoff calls without penalizing originating calls too much, resulting in only a slight decrease in carried traffic. We then investigate different queueing policies to further enhance this technique, and we show how it is possible to still greatly decrease the blocking of handoff calls while only delaying a very few originating calls. Closed-form expressions for all system parameters are derived, and it is shown that in addition to higher levels of protection for handoffs, we also increase the total carried traffic. In addition, a simple condition for the ergodicity of the system in the case where infinite queues are considered is obtained. This expression is useful to determine the maximum possible number of guard channels as a function of the different traffic components, and furthermore, it can be used to estimate the corresponding load of the more realistic system with finite queueing capacity.

Let us now outline the general organization of the thesis. Chapter 2 provides the general description of a cellular system, from a structural as well as functional point of view. In Chapter 3 we present the simulation used to establish the experimental channel occupancy time distribution. The assumptions used are introduced, justified, and their domain of validity discussed. A theoretical exponential model for the channel occupancy time distribution is described and compared to the simulated

results. In Chapter 4 we obtain an analytic model describing a simplified cellular system. The chapter concludes with a comparison between the channel occupancy time distribution obtained for the simplified cellular system and the previous exponential model. Chapter 5 presents simple traffic policies to protect handoff calls. A more complex policy is introduced and analyzed in Chapter 6. This policy not only protects handoff calls but also increases the total carried traffic. In Chapter 7 we outline some possible applications of the thesis outside Cellular Radio, and we conclude with a brief summary.

## CHAPTER 2

### DESCRIPTION OF A CELLULAR SYSTEM IN THE U.S.

#### 2.1 Radio parameters.

In the U.S. a given bandwidth has been specially allocated to mobile telephony by the FCC ([38]), but due to the U.S. requirements that there be two servers in every service area—one operated by a “wireline” common carrier, and one by a “non-wireline” or radio common carrier—a single system in the U.S. will never be allocated the whole bandwidth available. We also remark that the cells sites are independently located in the two competing systems. In addition to the duality of system operators for a given service area, there is also a duality between equipment manufacturers. Even if some companies do build both cellular systems (base stations and switching) and mobile equipment, the two manufacturing facilities need not be related. Mobile equipment must be able to operate in any cellular system in the U.S. (they are all compatible), no matter who the service provider is, but the manufacturers can be different.

Cellular voice transmission requires a full duplex-channel (not push-to-talk). Therefore the total bandwidth, currently 40 MHz, has to be divided in two separate sets, one of them for the mobile-to-base link, the other for the base-to-mobile link. Thus, each of the two operators is currently assigned 20 MHz of the 40 MHz. Furthermore, the FCC has reserved a supplementary 20 MHz bandwidth for later allocation, when required by the growth in traffic. This gives an ultimate available bandwidth of 60 MHz for Cellular Radio. The **duplex distance**, or spectral distance between the center frequency of the two spectral components (forward and



reverse channels) of a duplex channel, is equal to 45 MHz in the U.S. The presently available frequency spectrum is given in Table 2.1.

Receiver		Transmitter	
825 → 845 MHz		870 → 890 MHz	
25 → 35 Non-wireline carriers	35 → 45 Wireline carriers	70 → 80 Non-wireline carriers	80 → 90 Wireline carriers

*Table 2.1. Frequency allocation to Cellular Radio.*

The channel spacing is set equal to 30 kHz in the U.S., where by channel spacing we mean the distance between the center frequencies of two adjacent transmission channels. The ultimate total channel allocation for both competing systems in a service area will therefore be 1,000 duplex channels, since 60 MHz will be the final available bandwidth. We shall however see that a typical cell of a single system will only have 44 frequency channels, and therefore only 44 calls can be in progress at once in the cell area. It is frequency reuse that allows the much greater number of calls throughout the service area.

From now on, we restrict our attention to a single cellular system. Most of the channels will be used as traffic channels (TCh's), fully available for speech (or user data) transmission. The others will serve as control channels (CCh's), used for off-air call set up and signaling, therefore avoiding occupancy of the traffic channels with call setup data. As we shall see in Section 2.5, a cell site will be allocated at least one control channel which will be shared by all the mobiles present in the cell. Similarly, one site will be allocated one or usually several channel sets for voice transmission. Depending on the system configuration adopted, the available bandwidth will be divided into seven ([4]) or four ([21], p. 42) channel sets. The

number of control channels associated with each set of traffic channels will therefore depend on the chosen system configuration.

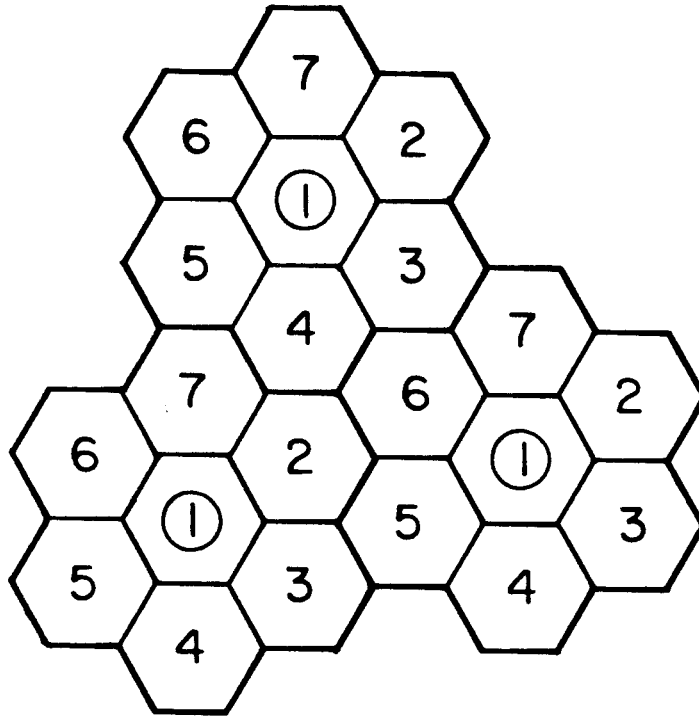
Systems with only four channel sets will usually have *sectorized* cells. More precisely, the coverage area of each cell will be divided into a certain number of sectors, each served by a directional antenna located at the cell site. This is to be compared with systems with seven channel sets where a unique omnidirectional antenna located at the cell site typically provides service to the entire cell area.

The use of sectors with four channel sets is forced by mutual interference problems. With only four channel sets, the distance between two cell sites using the same set of frequencies is smaller than if seven channel sets are used. This increased proximity naturally yields bigger mutual interference problems. Thanks to an increased directivity, the introduction of sectors with directional antennas brings co-channel interference back to an acceptable level. The two configurations with seven or four channel sets are illustrated in Figures 2.1 and 2.2.

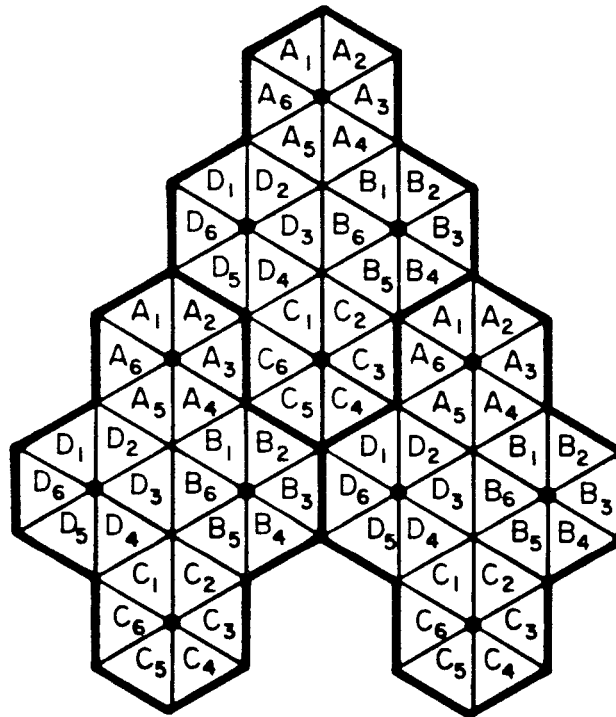
## 2.2 System architecture.

The basic element of a system is as we said the *cell*, also called the *base station coverage area*, roughly representing the zone covered by the transmitter located at the cell site. These sites provide the call processing interface between the subscribers using radio frequency and the trunks using baseband signals. They are connected to the public switched telephone network (PSTN) through wireline connections and a system switch. They are the intelligent relayers of information and provide front-end signal processing facilities to the system.

The system architecture can vary in how these elements are connected and how much independence they are given with respect to higher-level components. We will discuss a typical American system, namely the one developed by AT&T ([4]).



*Fig. 2.1. Seven channel sets.*



*Fig. 2.2. Four channel sets and sectors.*

Some characteristics given here might not be valid for other systems (Motorola, etc.). However, we will try to give a general description rather than stick to a very detailed presentation, and the chosen system is only meant as a support to our explanation.

Depending on the expected traffic, two configurations are possible for the implementation of the higher-level components of the system. Namely, the so-called *Mobile Telephone Switching Office* (MTSO) providing the connections between cells and with the Public Switched Telephone Network (PSTN) can be built in two different ways ([4]):

a. In case of low traffic areas it will be split in two distinct components:

- The *Executive Cellular Processor* (ECP).
- The *Digital Cellular Switch* (DCS).

The ECP provides control of the system. It directs the operation of both the cell sites and the switching offices and it provides control coordination as well as cellular administrative control. In low traffic areas, the ECP will be connected to up to eight remote DCS's with data links to each of them and to each cell site. It is in charge of all call processing and maintenance operations together with billing information.

Conversely, the DCS provides the connections between cell sites and the Public Switched Telephone Network (PSTN). Each DCS is connected to one or more interface points with the network and switches trunks in response to orders from the ECP. It is also in charge of the detection of incoming calls as well as their release later.

b. In the case of high traffic, as the ECP will generally not have any free capacity left to handle several DCS's, it seems logical to centralize the functions of the ECP

and the DCS in a unique component, the MTSO, providing unified switch and control processing.

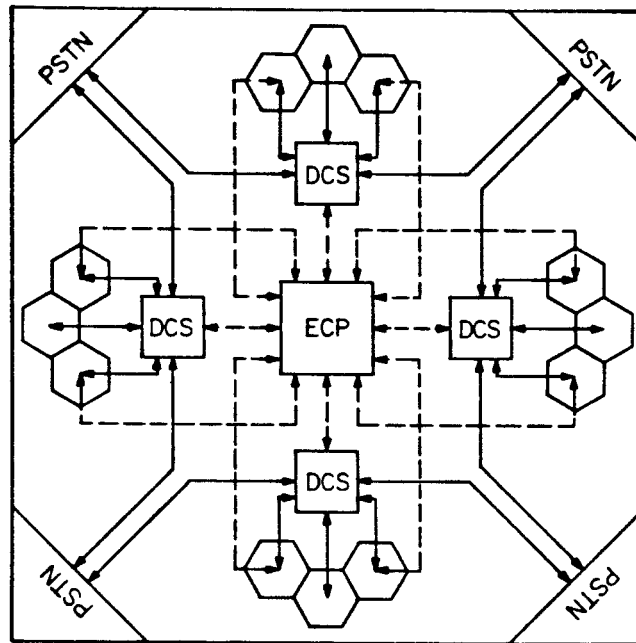
This two different configurations are represented in Figure 2.3.

### **2.3. Functional overview.**

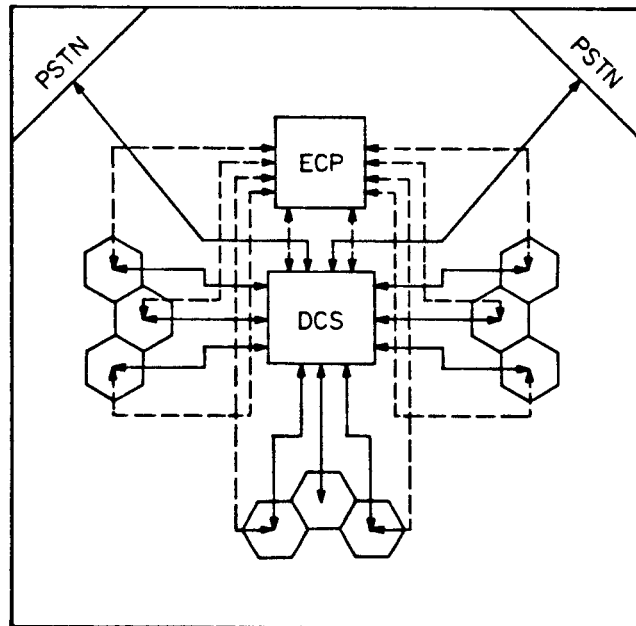
This section will present the procedures involved in a call and highlights the role played by the different system components. We assume the ECP/DCS configuration because the roles are easier to understand.

We first mention that the setup of a call is done through control channels. The information needed to establish a call connection is transmitted between the mobile and the cell site on a control channel using standard procedures which will be outlined later. Once a call is connected, the cell site in charge of the call continuously monitors the link quality. Each site is equipped with a *location radio receiver* ([4], p. 37), operating in a receive mode only, and dedicated to the measure of the received signal strength of all calls in progress. The monitoring of signal quality is made necessary by the propagation characteristics of the channel (fading) and by the mobility of the subscribers.

In case the link quality becomes insufficient the call is either switched to another frequency when possible (case of frequency-dependent fading), or to another antenna if there are more than one at the cell site (case of multipath fading), or finally handed off to another cell site (case of a call moving out of the cell area). The information transmission between the mobile and the cell site necessary to complete the above operations is not done on a control channel. In this case, the system takes advantage of the fact that a channel is already up between the mobile and the cell site. Namely, all the necessary data is transmitted on the voice channel allocated to the call in



----- Data link    ——— Voice link  
ECP: Executive central processor  
DCS: Digital cellular switch  
PSTN: Public switched telephone network



**Fig. 2.3. Mobile telephone switching office.**  
**Top, low traffic areas; bottom, high traffic areas.**

progress using out-of-band signaling. Such signaling is inaudible to the customer since it can be filtered out after reception without any effect on the voice signal.

We now describe some typical sequences of operations that take place between the system and a subscriber. First, when a subscriber set is just turned on it synchronizes itself to the best possible (highest received power) control channel, defining thereby the cell that will initially be in charge of providing service if requested (cell associated to the selected control channel). Note that this process requires no intervention from the system. All power measurements and the synchronization procedure are performed by the mobile unit. Once this step taken, different situations may occur:

- 1- The subscriber does not wish to initiate any call, nor is there a call request, nor does it require any other type of service, and it remains in a steady state. More precisely, the mobile will keep on listening to the initially chosen control channel and regularly rescan the others to determine if it is still locked on the best one. If another control channel turns out to be better, the mobile will simply synchronize itself to the new channel and stop listening to the old one. This procedure is again under the sole control of the mobile and requires no system intervention. Each mobile terminal is provided with an "on-board" chip that directs all the processes that need to be initiated at the mobile level.

- 2- If the subscriber is being called it will be notified through the control channels by the **paging** process. Paging is the process of finding a subscriber unit in the system and informing it that a call is waiting. More precisely, the call request arrives at the DCS level (switching center), which then notifies the ECP via data line connections. As soon as the ECP is notified that a subscriber of the cellular system is being called, it sends a paging order to all cell sites in the system via

ground data lines. Subsequent to the paging order of the ECP, all cell sites will broadcast a paging message that includes the identification number of the called unit. The paging message is broadcast over paging channels, which are forward control channels. We use the term **paging channel** to stand for a forward (base station to mobile) control channel used to page and send orders to mobile stations. The reason for this differentiation between control channels is that some control channels might be used for the transmission of system parameters and status as well as for the transmission of paging and order messages, while other control channels might only be used for paging and orders. Such a choice will depend on the system size and the chosen configuration. There are generally enough control channels for this.

Going back to the paging message, if the called mobile has its unit turned on, it is already synchronized to a paging channel and will therefore receive the paging message and answer it using a standard protocol to access the control channel. For a detailed presentation of this protocol, the reader is referred to [39].

As we mentioned earlier, the response to the paging is sent on the control channel to which the called subscriber is synchronized. This then uniquely determines the cell site that will start serving the subscriber. Once this particular site receives the reply to the page, it notifies the ECP via the data line connection. The ECP then gives the DCS the order to route the calling subscriber from the corresponding interface point with the PSTN (usual case) to the designated cell site, where a free voice channel has in the meantime been allocated. The call is then considered as connected.

3- If the mobile wants to initiate a call it then requires the use of control data transmission. It first has to seize the control channel (reverse channel corresponding



to the forward channel it is synchronized to). After it has been granted access, the request can be sent over the control channel. This request is then sent by the cell site to the ECP over the wire data circuit. The ECP analyzes the dialed digits and notifies the DCS of the validity of the request.

Depending on the final destination of the call initiated by the mobile, several cases can be distinguished (the explanation is helped by thinking of the DCS as a PBX on the PSTN).

If the called party is a landphone (PSTN), the usual case for a call initiated by a mobile subscriber, following the order from the ECP the DCS routes the call to the appropriate network interface point, and the network sets up the connection with the called party. Once this connection is established, the ECP notifies the cell site which completes the connection by allocating a voice channel from among free ones in its channel set. The voice is relayed by voice circuits between the DCS and the cell sites, which are dedicated lines.

If the called party is also a subscriber of the system, not the usual case, we have an intra-system call. The ECP then immediately starts the paging sequence described in 2. Once the called subscriber has been located, frequency channels are assigned by the respective cell sites to both the called and the calling subscriber, and the ECP orders the DCS to complete the connection. The procedure followed is very similar to the one used for intra-PBX calls. The only difference is that the abbreviated dialing adopted in PBX's is not used.

Finally, one need to mention that in the case of a call to another cellular system, the procedure followed is similar to the one used for a PSTN call.

We must now consider the case of a subscriber with a call already in progress moving from one cell to another. It is clear that in order to maintain the link quality

the call must be handed off to the site serving the new cell. Let us look at all the operations involved.

As we said earlier, the cell site control unit continuously monitors the signal strength of calls in progress through its *location radio receiver* which is tunable over the entire received frequency band, and can be ordered to switch to any of the receiving antennas if there are more than one. If a mobile signal strength measured at the serving site becomes inadequate for high-quality service, this cell site orders signal strength measurements on any other antennas of the site (if we assume sectoring or antenna diversity). Note that the whole process is initiated locally, namely at the serving cell site, without any supervision from the ECP. If satisfying results are not obtained, the cell site will then order (via the ECP) the six neighboring sites to start signal strength measurements. If we have a seven channel sets system, six sites are involved. In case of only four channel sets and sectors, the procedure is slightly different ([21], p. 42).

These measurements are made at the frequency of the existing, not the potential new, voice channel of the mobile, since in the case of handoffs, short-term frequency-dependent fast fading will generally not be the main reason for the decrease in signal power. It is however possible to have unnecessary handoffs due to sharp drops in signal power caused by fast fading. One way of avoiding this problem is to monitor the decrease in signal power long enough so as to average out fading. The problem is that this approach can lead to unacceptable link quality if one waits too long. Time thresholds dependent on the local propagation conditions will probably have to be applied. These considerations are proprietary to the manufacturers of the fixed equipment (cell sites, ECP, DCS, etc.).

All instructions and information necessary to the whole handoff process are sent under the control of the ECP. The results are returned to the serving cell site which

establishes a prioritized (ranking after the different received signal powers) list of cells or antennas for handoff. This list is transmitted to the ECP which will choose and notify both the new cell (usually the best available with free channels) as well as the DCS. After a channel has been allocated in the new site, the old site uses out of band signaling on the old voice channel to order the mobile to shift to the new allocated frequency. In the meantime, the DCS is ordered (by the ECP) to switch the call to the landline connected to the new site. All this procedure will normally last around 1/20th of a second and be imperceptible to the talkers on voice calls; data calls may be affected.

Considering the case of motion within the system might not be enough. For example, a subscriber might wish to receive or originate calls when located in another compatible system (and they are all compatible in the U.S.). This feature is called *roaming*. Before describing some solutions investigated to make roaming possible, one should note that even in the case of contiguous distinct cellular systems, a subscriber moving from one system into another while involved in a call would always be disconnected. This is due to the fact that the two independent systems do not have common switching facilities.

The capacity of allowing the subscriber of a given cellular system to roam into other systems throughout the U.S. will depend on how fast agreements can be reached between the service providers in the different American markets. In spite of this, different solutions have already been investigated.

A first one is to make roaming available through two-stage dialing. With this solution a given cellular system would be allocated a *system number* equivalent to a supplementary "area code." This number would be in addition to the 10-digit number of the mobile subscriber. Let us illustrate this principle by working out

a simple example. Suppose we have a subscriber from a cellular system *A* with a 10-digit number *a1* that want to roam into another cellular system *B*. Let  $\beta$  be the supplementary "area code" of system *B*. A call to the subscriber of system *A* would then be placed by first dialing  $\beta$ , upon receipt of a second dial tone the number *a1* would then be dialed. Note that this implies that the calling party has been informed of the intention of the mobile subscriber of system *A* to roam into system *B*, and provided with the number  $\beta$ . The customer from system *A* would be paged in system *B* using his own identification number *a1*. Note that the reason why we need to dial the 10-digit number *a1* after  $\beta$  is in order for the identification number of the customer of system *A* not to coincide with the identification number of a customer of system *B*. This whole process seems rather awkward and furthermore requires that the mobile subscriber has informed all potential callers of the prearranged directory number of the system it intends to roam in.

Another solution investigated is to *temporarily* allocate a new number to a roaming mobile. This method offers the advantage of direct dialing of a subscriber without knowledge of the system it has roamed into, using simply its regular number. Let us illustrate this method with a short example. Suppose again that our mobile from system *A* with number *a1* wants to roam into system *B*. The subscriber must then notify his home system *A* that he wants to roam into system *B*. This will also probably require the specification of some time limits. Provided that some agreements exist between systems *A* and *B*, the subscriber of system *A* is then allocated a temporary number *b1* from system *B*. This number *b1* is stored by the MTSO of system *A* and needs not be dialed by a calling party. Any person wishing to call our mobile will simply dial the usual number *a1*. This number is forwarded to the MTSO of system *A* by the PSTN, and immediately associated with the number *b1* of system *B*. The MTSO of system *A* then connects the calling

party to system *B* and forwards the number *b1*. This typically will involve a long-distance connection. Once in system *B* the call request directed to the customer of system *A* is completed as for a regular subscriber of system *B* using the number *b1*. Note that this process is very similar to the existing call forwarding service, where a customer asks his local phone company to forward any call request directed to his usual number to a new number. This forwarding goes unnoticed from the calling party as does the forwarding of a call directed to a roamer. The customer ordering his call forwarded pays for the forwarding charges.

The above solution requires a supplementary component in the MTSO in order to decide where to route a call directed to a subscriber. However it seems to be more satisfying as it allows a mobile to be reached anywhere using only its home number. Nevertheless, it implies that a mobile informs its home system of its final destination, and that agreements have been reached between different cellular systems and service providers.

On the other hand, calls originated by roamers can be placed identically as in their own home system. In both cases, the billing information is collected at the cell site and transmitted to the MTSO. It can then be forwarded to the appropriate home system or to a predesignated billing center.

We now quickly outline the possible choices available to a cellular system to accommodate an increase in traffic.

#### **2.4. System growth.**

We mentioned in Chapter 1 that the main advantage of Cellular Radio over previous mobile communication systems was that it made mobile communications available to a large number of subscribers. This was achieved by dividing the service area into cells, which allowed the same frequency channel to be reused many times throughout

the system. It is clear that the traffic that can be carried by the system is in direct proportion to the number of times a frequency is reused. In order to handle a given factor increase of the total traffic in the system, one could increase the number of times a frequency is reused in the service area by the same factor.

This can be achieved by either decreasing the coverage area of a cell site or by introducing directional antennas. Generally, directional antennas are introduced to meet a first increase in the existing traffic. This process is called "sectoring" ([2] pp. 31-34, [4], [19]), and usually goes together with a reassignment of different channel sets to the cells ([40], pp. 18-21). The sectors in the cell will generally be either 60° or 120° wide. Sectoring achieves an increase in capacity by decreasing the co-channel interference level, allowing therefore closer reuse of the same frequency ([19], [21], p. 41). The main advantage of this technique is that it allows the operator to meet an initial increase in traffic without having to build any new cell sites, since directional antennas can easily be mounted on the sides of existing sites or buildings. One should mention that cell sites are by large the most expensive component of a cellular system ([2], p. 24).

In a startup configuration, most systems have twelve channel sets instead of seven ([2], p. 33, [40], p. 18). The reason for such a configuration is that it minimizes the number of cell sites needed to cover a given geographical region, and therefore the initial cost of a system. This pattern is certainly not optimum from a frequency reuse point of view, but this will generally not be a problem in a the initial phase of a system. The sectoring of the cells in order to meet a first increase in traffic will then be accompanied by a reduction from twelve to seven in the number of channel sets. At this point, depending on the chosen frequency assignment, a sector can be allocated a whole channel set ([4], p. 44), or a fraction of a channel set ([2], pp. 31-34).

One of the problems due the introduction of directional antennas is a loss in trunking efficiency. Namely, if we have a total of  $N$  channels at a given cell site served by three  $120^\circ$  directional antennas each allocated  $N/3$  channels, we can have calls blocked in one sector if all the  $N/3$  channels are busy, while still having free channels in the other sectors. This would never happen if the cell site was served by a unique omnidirectional antenna that had been allocated all the  $N$  channels, since no call would be blocked unless all channels were busy. However, note that in the case of an omnidirectional cell-site antenna, the number of available channels would be smaller than  $N$ , the sum of the number of channels allocated to the three sectors. This is due to the fact that co-channel interference levels are higher with omnidirectional antennas, forcing a smaller number of channels at each site in order to increase the reuse distance for the same frequency ([2], p.25, [21], p. 41, [40], p. 20). More precisely, a directional antenna delivers the desired signal power only in a given sector, while an omnidirectional antenna has an approximately circular illumination pattern. This implies that all cell sites located outside the main lobe (sector) of the directional antenna will benefit from much lower interference levels, while in the case of an omnidirectional antenna with the same power level, the possibility of high co-channel interferences will not be limited to a certain sector, but will be the same for all directions. In order to keep a desired co-channel interference level, the use of omnidirectional antennas implies therefore a greater distance between two cell sites using the same channel set than with directional antennas.

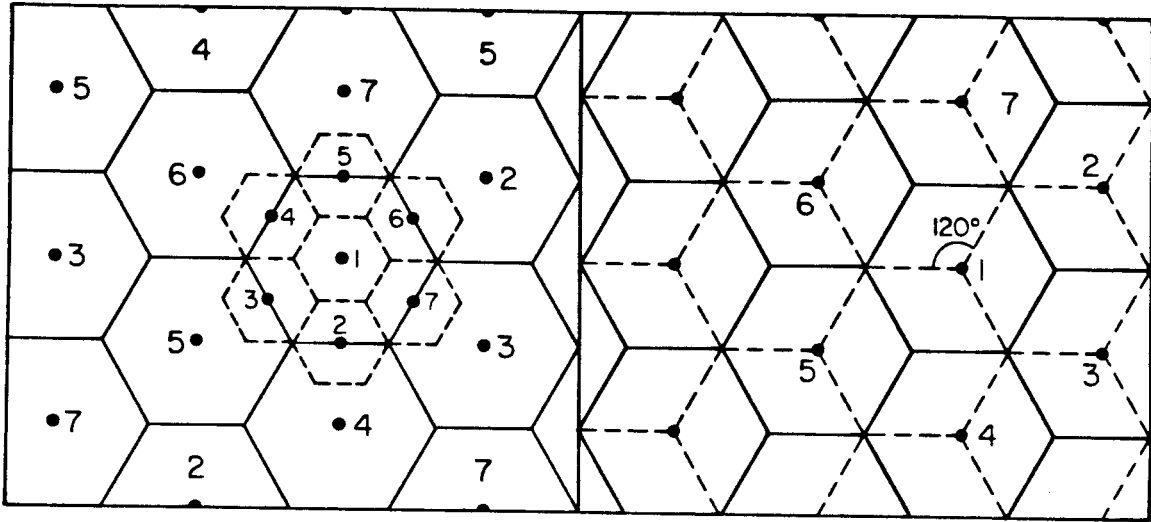
One way to overcome the problem of smaller trunking efficiency with sector antennas is the concept of *sector sharing* ([21], p. 42). Within a cell, two adjacent sectors usually have a significant overlap between them. It is possible to use this overlap to make a given sector keep on serving a mobile moving into another sector

of the same cell, while still keeping adequate signal strength. Conversely, if all channels of a given sector are busy, it is possible to provide service to a mobile located in this sector using channels from adjacent sectors.

Once the additional traffic capacity obtained from sectoring becomes insufficient, the cell size needs to be decreased so that more cells using the same set of frequency channels can coexist in the system, increasing the number of times a given frequency is reused in the service area. This principle is called **cell splitting**, ([2], p. 34) and is usually accomplished while keeping the sector configuration ([40], p. 20). The new cell sites are normally added midway between two existing sites ([40], p. 20), and a lower cell-site transmitter power is chosen to provide adequate signal power in the desired area. In case of ideal propagation conditions with a signal power decreasing like the fourth power of the distance, this would give a forward transmitter power sixteen times smaller for the new sites compared to the transmitter power of the old sites. We assume here new cell sites located at equal distance between two old sites, and corresponding new cells with one-fourth the area of the old cells.

The splitting of cells forces large changes in the system structure. Since all cells will usually not be split at the same time, the simultaneous presence of large and small cells with different transmitter powers can create some problems. Terrain will very probably play a major role in the difficulties that can be encountered with this process. Depending on local propagation conditions, the shape of the cells can greatly vary, making the optimal location of a new site and the proper orientation of its antennas a very complex problem. In any case such an evolution will require careful engineering and system management, which will usually depend as much on theoretical design as on numerous computer simulations and field experiments.





*Fig. 2.4. Cell splitting and sectoring (120°) in Cellular Radio.*

Such a procedure will probably become more accurate in the future as more evolved propagation models based on measured and simulated results are obtained.

However, a theoretical smooth growth pattern can be achieved using the *overlaid-cell* concept ([2], p. 5, [21], p. 10), developed by AT&T Bell Laboratories. It allows the coexistence of cells of different sizes, and has already been successfully applied in several existing systems (Chicago, Houston). Having cells of different size and different transmitter power coexisting in a system makes the task of keeping a minimum distance between cells using the same channel set much more complex. The overlaid-cell concept views a cellular system where cells of different size exist as the superposition of two cellular systems. The first one is a partial cellular system formed by the small cells obtained by splitting a few cells of the initial system. The second system is simply the initial cellular system, where the cells that have been split are still present. This implies that the underlying pattern of large cells will remain present until all the cells in the system have been split. The two concepts of sectoring and cell splitting are illustrated in Figure 2.4.

Based on this double pattern, the channel subset assigned to any sector of an old site will be divided into two subgroups. The first subgroup will be used to keep on providing service to the old large-sector area, while the second subgroup will provide service to the smaller sector area corresponding to the split cell. The actual repartition between the first and the second subgroup will be governed by the interference requirements of the new neighbors of the old site. Frequency channels will be transferred from the first to the second subgroup until all the surrounding cells have also been split. At this point, the old site can be considered as completely integrated into the new smaller cell structure.

### **2.5 Channel efficiency.**

This last section will explain how the use of channels devoted to the transmission of control and system data (as opposed to voice transmission) yield a better efficiency in the way channels are used.

The special channels allocated for the transmission of control information were naturally called *Control Channels*, versus *Traffic Channels* used for voice transmission. Each cell site or sector will be allocated at least one control channel. Control channels will be used to transmit all the data needed to set up a call, initialize a mobile that was just turned on, inform a subscriber that it is being called, etc. Recall that once a call has been connected, and therefore allocated a traffic channel, the latter will be used to transmit the required call supervision data. Control data consists of *access, call setup, paging, synchronization, etc.* All users present in a given cell will share this common control channel and calls can be processed or new requests accepted while others are waiting for an answer, using only one channel.

There exists twenty-one pairs of control channels in a cellular system in the U.S., and there will usually be one control channel per set of traffic channels (seven

channel sets with three  $120^\circ$  sectors per cells gives twenty-one channel subsets, and therefore requires twenty-one control channels). By FCC regulations ([39]), the data rate on control channels is taken equal to 10 kbits/sec with phase modulation. An FCC full control channel includes: a forward control channel (FOCC), transmitting information between land and mobile stations, and a reverse control channel (RECC) between mobile and land stations. For a more detailed presentation of the type of procedures and structures used on control channel, the interested reader is referred to [39], §2.7.1, §2.7.2, §3.7.1, and §3.7.2.

We now provide a heuristic explanation of why the use of control channels increases the system efficiency. Any communication system requires channel capacity to provide access control of users. Namely, a subscriber needs at least to be able to transmit his request for service to the system. This could be done by using voice channels to carry all the necessary procedures. This means that at a given instant we could have several voice channels busy because of some signaling going on with no call in progress. The time during which a voice channel would be kept busy, before and after the effective call connection, includes the message length, the propagation delays, the time spent searching, and all the waiting periods before each answer. This finally results in a rather noticeable loss of capacity. It is therefore more efficient to dedicate a channel to the signaling work. This is part of the rationale for Common Channel Inter-Office Signaling (CCIS) used in in the PSTN ([41], pp. 141-145). In the PSTN, such a common channel handles the signaling of up to 24,000 trunks, avoiding the seizing of a trunk before voice transmission is ready to start. We will now illustrate the control-channel traffic improvement concept with a short example.

Consider a cell (actually sector) which has been allocated fifteen voice channels and one control channel. Suppose that there are on the average five calls per minute

originated inside the cell, and that a call stays an average of two minutes in the cell. For each call, the system spends say twenty seconds, including transmission and waiting times, to set up the connection. Note that this will correspond to less than twenty seconds of use of the control channel, since while a call is waiting for an answer the control channel can be and is used to transmit information concerning other calls. We will assume that a call will never be blocked at the level of the control channel, since due to the structure of the access protocol ([39]), call requests that find the control channel busy simply wait (queue) until they are granted access to the channel.

We can compute the blocking probabilities of calls, first for a system where the setup information (twenty seconds) is directly transmitted on the voice channel, and then for a system that uses a common control channel. For both cases we will assume that the calls arrive to the system according to a Poisson distribution, and in order to simplify the problem we will take the service time to be exponentially distributed for both policies.

We compute the blocking probabilities first for a system with sixteen channels (no control channel) and exponentially distributed calls lasting on the average two minutes and twenty seconds (two minutes for the call itself, and twenty seconds for the setup procedure), and second for a system with only fifteen channels (one control channel), but with exponentially distributed calls that last on the average simply two minutes (only the call itself). Recall that for both cases we have Poisson distributed arrivals with an average of five arrivals per minute. For such systems, the blocking probabilities are easily obtained using Erlang's B formula ([41], pp. 227-228, [42], pp. 105-106). We find a blocking probability  $B_0 \approx 5.3 \times 10^{-2}$  for a system without a control channel, and a lower blocking probability  $B'_0 \approx 3.6 \times 10^{-2}$  for a system with a control channel, as expected.

The actual difference will probably be even larger, since most of the time spent in setting up a call corresponds to waiting periods where the cellular system is expecting an answer. Such periods can typically be much larger than a couple of seconds, especially during the peak hour where circuits in the PSTN are often busy too. Also, note that in Cellular Radio a common control channel would in any case be needed, since the system must be able to reach any subscriber to inform him of an arriving call. Subscribers must therefore be listening to a predetermined channel, which furthermore must be uniquely associated with the given cell that will start providing service. A control channel being a priori required, it is natural to extend its use to the whole call setup procedure, which, as we saw, has the additional advantage of increasing the system efficiency (decrease blocking probability).

This completes our description of a typical cellular system in the U.S., and we now proceed with Chapter 3 on a simulation program to model a general cellular system.

## CHAPTER 3

### MODELING A GENERAL CELLULAR SYSTEM

In this chapter we present the simulation program used to estimate the probability distribution of the channel occupancy time in a Cellular Radio system. We will first give the assumptions used for the system itself as well as for the mobile and traffic behavior. We will then describe the different features of the simulation and its mechanisms, and provide some justifications for the assumptions we made. We also present a theoretical model to which the results of the simulation will be compared. Finally we will state the results we obtained and the conclusions they enabled us to draw.

#### 3.1 Assumptions and method.

In order to simplify the problem, we decided to assume that the cellular system considered was large enough to ignore the fact that some calls could terminate because the mobile would leave the system area altogether. Practically, this means that a channel occupancy time can only end when either a call is terminated by a voluntary disconnect or the mobile moves into another cell. Similarly, the occupation time of a channel can only begin if it is assigned to a newly initiated call, or allocated to a call that has just been handed off.

Unfortunately, the above simplification leads to a new problem. Namely, we now have to consider an infinite potential service area inside which we have to keep track of the vehicle's motions. This requires an enormous memory allocation which makes the whole computational system nearly unfeasible and certainly very costly. To overcome this difficulty, we made use of a further assumption and of a property of a cellular system that enables us to reduce the whole problem to a single cell system.

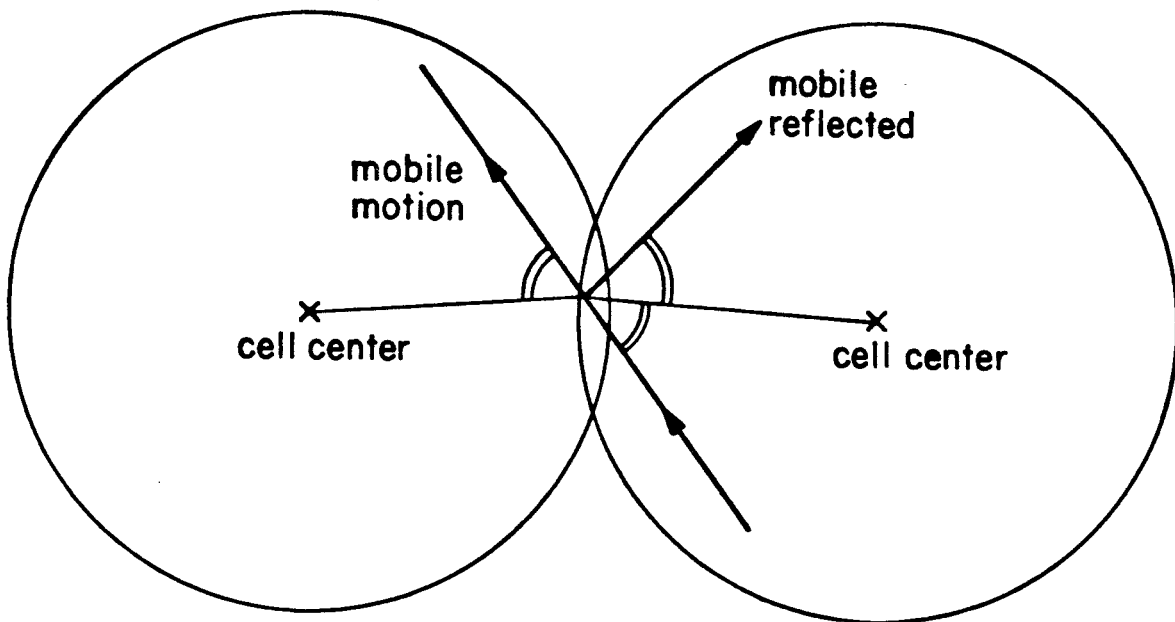
We will assume that all cells in the system generate the same traffic, that is, each cell has the same average rate of call initiation. This will not fit all practical applications ([21], pp. 16, 42, [41]), but it may be able to approximate most conditions by partitioning real-life systems into sub-areas with nearly equivalent traffic characteristics. The complete system could then be approximately reconstructed by superposition of the different subsystems. We expect that this may yield somewhat crude representations. Once the step of assuming equal traffic generation is taken, though, the whole system can be described with only one cell. The arguments leading to this conclusion are the following.

In case of traffic-equivalent cells we easily say that since there is no traffic created in the whole system, the average amount of traffic leaving a cell because of calls handed off to other cells must be exactly compensated by the average amount of traffic entering the cell because of calls handed off from other cells. This in effect tells us that we might only need to keep track of calls initiated in one cell. We have reduced the complexity of the problem to be solved in the sense that we do not need to keep track of the calls initiated in the whole infinite service area, but only those initiated in the bounded area formed by a single cell. However, our simplification does not yet suppress the fact that calls initiated in this cell might terminate very far from their initial location. Since we still need to keep track of the successive handoffs of a call in order to characterize the different channel occupancy times, we would still need a large memory allocation. At this point let us give a closer look to the handoff process to simplify further.

Handoff occurrences depend on a geometrical parameter and a time parameter. Namely, a call will be handed off if the mobile crosses the cell boundary before the call terminates. This in turn depends on the mobile location, speed, and direction as well as on the remaining call duration. If we now consider a cellular system

formed of circular cells (most practical cases, ([2], [21], p. 41, [41])), we can make the following remarks.

When a mobile is handed off and leaves the "old" cell area to enter the "new" one, it keeps the same absolute direction. However, its direction relative to the center of the "new" cell is actually its relative direction with respect to the center of the "old" cell, after reflection on the cell boundary following the optical mirror law of reflection. This principle is illustrated in Figure 3.1.



*Fig. 3.1. Reflection principle.*

This artifice allows us to bring the handed-off call back into the initial cell, which means a huge saving in memory allocation since we are now dealing with a closed and small system. The same reflection process is repeated as many times as needed in order to keep a given call inside the initial cell until it terminates. After each handoff occurrences or at call initiation, a new channel occupation time



is started. Each of these time periods terminates at the next handoff, or at the call clearing for the last one.

After presenting the different assumptions and techniques used to construct our Cellular Radio system, let us describe the model used for the vehicle behavior and the (communications) traffic statistics. As far as the traffic statistics are concerned, the assumptions taken for the simulation are very classical. Namely, the total call duration is taken to be exponentially distributed with parameter equal to the reciprocal of the average total call duration. The arrival process has no influence on the simulation itself since time is taken to be arbitrary; however, as we mentioned earlier, we need to assume that it is identical for all cells in the system.

For the mobiles, we assume that they all have the same constant speed. Note that it requires very few modifications to change this to a random constant speed according to a given distribution. The crucial part is, however, the model that directs the mobile's motion.

At a call initiation a mobile is assigned a random initial position from a uniform distribution over the cell area. The call is also assigned a random total duration from an exponential distribution with parameter equal to the reciprocal of the total average call duration. Finally, the mobile is assigned a random initial direction from a uniform distribution over  $[0, 2\pi]$  and a random time until its first change of direction. This random time will be drawn from an exponential distribution with parameter equal to the average number of changes of direction per unit time; thus, changes in direction occur at memoryless instants. The motivation for such a distribution will be emphasized later.

### **3.2 Presentation of the simulation program.**

The mobile provided with all the above parameters is now ready to start its motion through the cell. Two cases have to be distinguished:

**Case 1:**

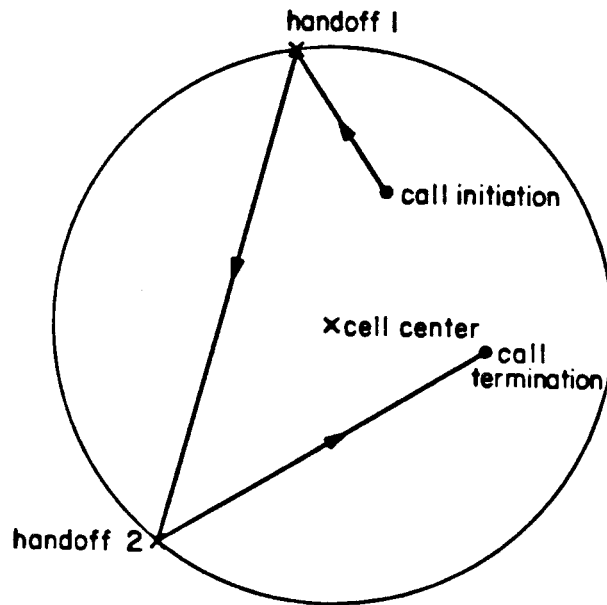
If the time of the first change of direction is **larger** than the total call duration, this parameter will not play any role. The mobile will then move with the same direction for the entire duration of the call, with the possible exceptions of reflections on the cell boundary each time a handoff is needed. The parameters that control the handoff occurrences are the vehicle speed, the cell radius, the vehicle direction and location as well as its remaining service time.

**Case 2:**

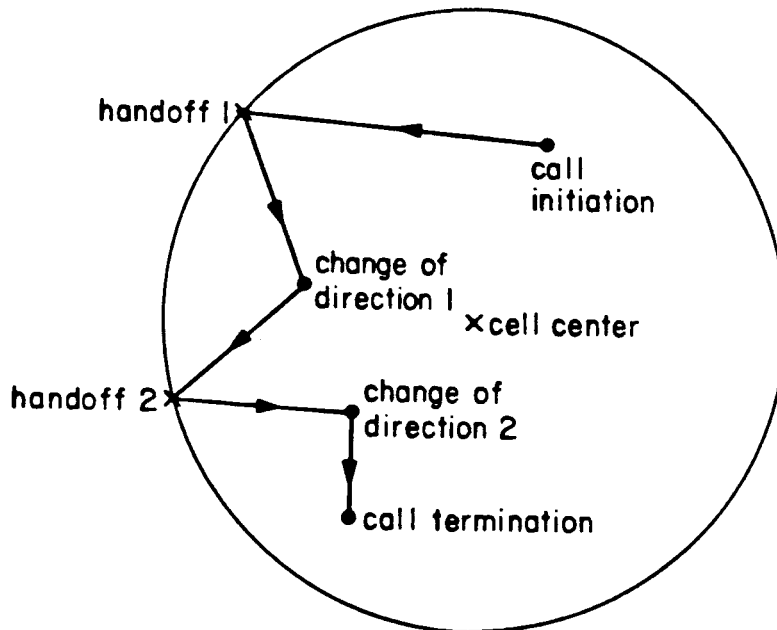
If the time before the first change of direction is **smaller** than the call duration, the mobile will move through the cell, again possibly going through some handoffs, until it reaches the time of change of direction. At this point a new direction is drawn from the uniform distribution over  $[0, 2\pi]$ , as well as a new time until the next change of direction, drawn from the same previously mentioned exponential distribution. The mobile then starts moving in the new direction.

We are now in the same position as at the call initiation, except for the fact that the total call duration is now decreased by the amount of time spent until the change of direction. The whole process is therefore repeated, distinguishing again between Cases 1 and 2. This goes on until the mobile ends up in a situation corresponding to Case 1, after which the call will terminate. The two cases are illustrated in Figures 3.2 and 3.3.

For each call, the times spent on each channel (limited by a handoff or call termination) are stored separately. The process is repeated for a fixed number of calls entered at the beginning of the simulation. After going through all the calls, we end up with a two-dimensional array where each row is associated with a given call and contains all the corresponding information provided by the simulation. Within



*Fig. 3.2. Case 1: Call with no change of direction (2 handoffs).*



*Fig. 3.3. Case 2: Call with 2 changes of direction (2 handoffs).*

a row, the different columns contain the times spent on successive channels. For example, if a call has never been handed off, it only used one frequency channel, on which it spent the entire call duration. In this case, the first column of the corresponding row will contain the total call duration, while the following columns will only contain zeroes. More generally, if a call goes through exactly  $n$  handoffs, the first  $(n + 1)$  columns of the corresponding row will contain the respective times spent on the successive channels, while the coefficients of the following columns will remain equal to zero.

Once the two-dimensional array is completed, the simulation part of the program is finished. We can now use these stored data to obtain estimates for the channel occupation time distribution, the average number of handoffs per call, as well as the distribution of the number of handoffs experienced by the calls. These operations can also be performed simultaneously with the simulation, therefore, greatly reducing the needed memory allocation, since only the call currently in progress needs to be stored while the desired estimates are continuously updated throughout the simulation. This suppresses the need for our large two-dimensional array, but on the other hand, the availability of these data might sometimes be desirable. The accuracy of the resulting estimates will of course depend on the total number of calls that were taken into account into the simulation. The simulation part being rather rapid, however, it is possible to use large samples of calls. For example, the results presented in Section 3.5 are obtained with samples of 10,000 calls (samples an order of magnitude larger could possibly have been used).

In order to summarize the description of the simulation, a flowchart of the program is presented in Figure 3.4. The notation used in the flowchart is listed below.

Simulation variables and parameters:

- R: Cell radius.
- V: Average mobile speed.
- n: Total number of calls in the simulation.
- i: Index characterizing the call in progress.
- $\mu$ : Service rate.
- cd: Number of changes of direction (rate) per unit time.
- mob(1): Radial coordinate of the mobile.
- mob(2): Angular coordinate of the mobile.
- mob(3): Direction of motion of the mobile.
- s: Call duration allocated to a given mobile.
- t(1): Time elapsed since last handoff, or since call initiation in case of no handoff.
- t(2): Time until the next change of direction.
- $\phi$ : New direction of motion at the change of direction.
- dec : Decision parameter that indicates whether a call shall terminate or not.
- j: Index counting the number of frequency channels used by a given call.
- $\theta$ : Angular position of the mobile at a handoff or at a change of direction.
- x: Radial coordinate of the mobile at a time of a change of direction (can be larger than the cell radius, which determines whether a handoff had to take place or not).
- dt: Distance covered by the mobile since the last handoff.
- d: Distance to handoff position.
- th(i, j): Time spent by the  $i^{th}$  call on its  $j^{th}$  frequency channel (if j defined for  $i^{th}$  call).

### 3.3 Justification of the model.

Before presenting the results, we will discuss in more detail the model used for the mobile's motion and more particularly the choice of an exponential distribution for the times between changes of direction.

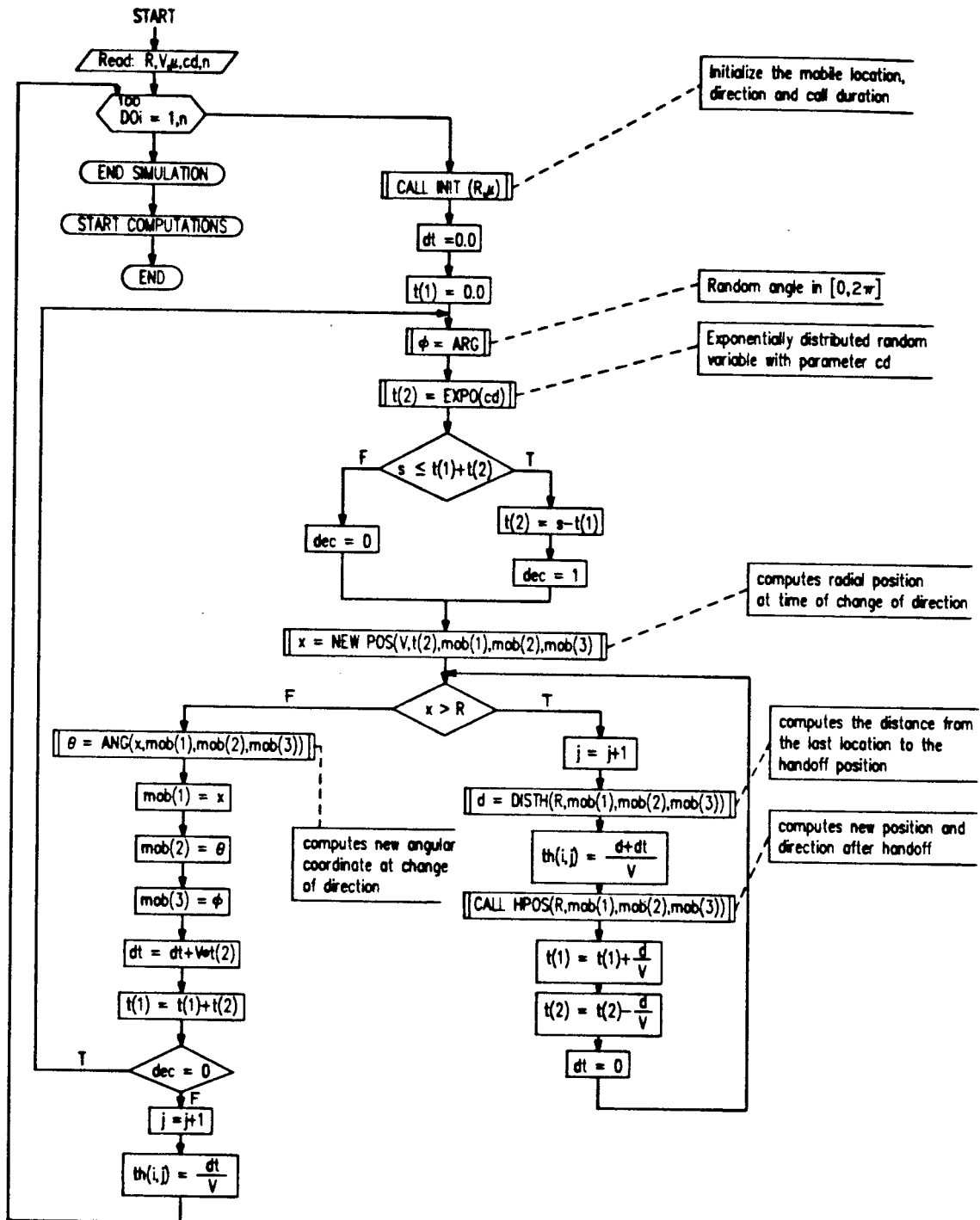


Fig. 3.4. Simulation flowchart.

Let us start with the uniform distribution used for the spatial location of call initiations. It was chosen for two reasons. First, it seems to give a fair representation of reality for cellular systems located in regions with rather homogeneous population density. Furthermore, it is consistent with our initial assumption of traffic-equivalent cells throughout the system, since it forces a uniform spatial traffic distribution in the entire service area. In this sense, we can say that even if it were relatively simple to use a different distribution for the spatial location of call initiations, (e.g., truncated Gaussian centered at the cell center, exponential, etc.), the result might not be extremely meaningful. The reason for this is that the technique used to reduce the whole system to a single cell forces all the cells to be exactly alike in all aspects. A non-uniform distribution would then be identically repeated for all cells in the system. This seems bizarre.

Similarly, the uniform distribution over  $[0, 2\pi]$  for the mobile direction was chosen both for its simplicity and because it seems to provide a rather general and realistic representation of real-life systems. Throughout a cellular system the relative orientation of streets and cells might vary somewhat randomly, giving on the average an approximately uniform distribution of possible directions. This distribution, however, can easily be modified to fit a given particular system.

Finally, the choice of an exponential distribution for the length of time between changes of direction was based on the fact that the time of occurrence of the last change of direction may hardly provide any information on the time of next change of direction. For example, if we are given that a mobile just made a "right" turn, it does not seem to really tell us much about the time at which the next change of direction will occur, nor does it provide any information on what the new direction will be. For example, the driver might be on his way to some rather remote location possibly not involving any new change of direction in the near future, or he might

be looking for a parking space or a close-by address which might then require many changes of direction in the near future. We can therefore assume that the time distribution between two changes of direction is approximately memoryless (the past gives no information on the future). It is well-known, then, that the only continuous time distribution that satisfies this property ([42], pp. 66-67) is the exponential distribution. We now introduce the theoretical model to which we will compare the results of the simulation.

### **3.4 Exponential model for the channel occupancy time distribution.**

From the data provided by the simulation we were able to derive estimates for the distribution of the channel occupancy time and the average number of handoffs experienced by a random call. We now would like to compare these estimates with a theoretical model. Our initial goal was to determine if the distribution of the channel occupancy time could still be approximated by an exponential distribution. At this point it remains to choose how to specify the parameter of this exponential distribution. We decided on the following best-fit approach:

We were originally given an exponentially-distributed total call duration with parameters equal to the reciprocal of the average total call duration. We also can realistically assume that the service time and handoff\* processes are independent. This is because the handoff process depends only on the physical motion of the vehicle, and the motion is taken to have no influence on the total call duration. For example, we ignore here the possibility that people blocked in traffic-jams (small handoff rate) might tend to have longer communication times than the typical customer.

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\* It would be more correct to refer to this as cell boundary crossing, since at this point we only care about the vehicle motion and not the associated service time. We will nevertheless retain the term "handoff."



We would like to be able to get away with memoryless interhandoff times. This choice of an exponential distribution for the interhandoff time distribution was motivated by the above use of another exponential distribution to model the times between the changes of direction of the mobiles. We can hope that the lack of memory between the changes of direction also induces a partial lack of memory between handoff instants, making an exponential distribution not too unrealistic. For example, if we are given that a mobile has just crossed a cell boundary, if it keeps the same direction, the time to the next crossing of a cell boundary will typically be rather large, if it is headed near the cell center. On the other hand, if the mobile changes direction shortly after the first crossing of a cell boundary, this makes the occurrence of a new crossing within a short time more likely.

Since the changes of direction of a mobile are memoryless, we have no information about the situation in which the mobile actually finds itself, so it might be going to have a long journey through a cell or a quick exit. Similarly, if a mobile has not experienced any crossing of a cell boundary for a long time, it can either mean that it is close to a cell boundary and about to cross it, or it can also be that the mobile is still located around the cell center if it went through several successive changes of direction. This would suggest a long time before the next boundary crossing. All these considerations make us feel that an exponential distribution for the interhandoff time may be acceptable. And we do find that for most practical cases, the memoryless handoff model gives a rather good approximation to the simulation results .

Under these assumptions of independent and exponentially distributed service and interhandoff times, we easily obtain, as we will do later, that the distribution of the channel occupancy time is also exponential with parameters equal to the sum of the service rate and the handoff rate. At this point a basic question arises. On

which basis do we compare the exponential model for the channel occupancy time with the results of the simulation?

The answer is, as we mentioned earlier, a best-fit approach. In the case of an exponential distribution there is only one parameter to deal with, namely, the departure rate from the channel. In our model this departure rate is formed by the sum of the service rate and the handoff rate. The service rate will generally be a quantity that depends much more on customer behavior than on system structure. In this sense, it will usually be either given or measurable independently of the cellular system. Therefore, the only parameter really representing the system itself that we can vary in order to adapt the model to the results of the simulation will be the handoff rate. With memoryless (exponential) distributions for the service and interhandoff times we will easily see (to be derived) that the handoff rate can be expressed as the product of the service rate and the average number of handoffs. We will set the memoryless handoff rate so that the memoryless model and the simulation yield the same average number of handoffs for a random call. When we do this, we can give a statistical goodness-of-fit test to the channel occupancy time distribution measured by the simulation and the theoretical (negative) exponential obtained with memoryless handoffs.

Let us now describe how to obtain the distribution of the channel occupancy time as well as the relation between service rate, handoff rate, and the average number of handoffs for a random call, assuming memoryless handoffs. Let  $\mu$  be the reciprocal of the average call duration, or in other words the (overall) service rate. Similarly let  $\eta$  be the average number of handoffs per unit time, or the handoff rate. The density functions associated with the corresponding exponential distributions are then respectively given by:

$$s(t) = \mu \cdot e^{-\mu t}, \quad \text{and} \quad h(t) = \eta \cdot e^{-\eta t}.$$

A channel occupancy time can begin because of a call initiation or a handoff, and end because of a call termination or a handoff. More precisely, assume that we are given that a channel occupancy time just started. We want the probability that it ends before  $t$  time units. Since the service time and interhandoff time are memoryless, and independent of each other, the past history of the call and the associated mobile that initiated the channel occupancy are irrelevant. The channel occupancy time will then be memoryless because the two processes that can terminate it (service termination or handoff) are memoryless and independent. That is, the probability of occupancy ending in  $dt$  can be written as

$$(1 - \mu dt)\eta dt + (1 - \eta dt)\mu dt + \mu dt \eta dt = (\mu + \eta)dt + O(dt^2).$$

So the occupancy termination rate is the constant  $\mu + \eta$ , the sum of the separate rates, independent of what happened before. This suffices to establish that the distribution of the channel occupancy time is memoryless (negative exponential) with rate parameter equal to the sum of the service rate and the handoff rate. The parameter  $\mu$  will generally be given or easily measurable, but, as we said, we will still have to choose  $\eta$  in order to obtain the same average number of handoffs for a random call as in the simulation.

Let us now obtain the relation between the handoff rate, the service rate, and the average number of handoffs for a random call. Let  $\bar{h}$  be the average number of handoffs for a call. We know that a call lasts on the average  $1/\mu$  time units, during which time we have on the average  $\bar{h}$  number of handoffs. The interhandoff time is on the average  $1/\eta$ . Because handoffs occur memorylessly during the overall service time, and independently of service time as usual, the handoff rate  $\eta$  times the average overall service time  $1/\mu$  must be the average number of handoffs during a call:

$$\bar{h} = \eta \cdot \frac{1}{\mu}, \quad \text{or} \quad \eta = \bar{h} \cdot \mu. \quad (3.1)$$

### 3.5 Comparison between the simulation result and the exponential model.

We are now in a position to compare the estimate of the distribution of the channel occupancy time obtained from the simulation with our theoretical model, on the basis of the same average number of handoffs for a random call.

Comparisons will be made for two cellular systems ([21], pp. 22, 31, 45, [43]). The first system (system 1) will have cells one mile in radius and will serve vehicles making on the average one change of direction per minute. This model should fit a dense downtown environment. The second system (system 2) aims to model a more suburban environment, with cells ten miles in radius and vehicles making on the average only one change of direction every two minutes. For both systems the average vehicle speed will be 35 miles per hour and the average total call duration is taken equal to two minutes ([21], p. 16, [40]). In each case, the sample size used for the simulation is 10,000 calls and the time unit will be one minute.

For systems 1 and 2, after running the simulation, we obtain from the accumulated data an estimate for the distribution of the channel occupancy time. At this point we want to determine how good the chances are that this estimate actually corresponds to our memoryless model or to a distribution that can be considered close to it. The evaluation of the agreement between the simulated distribution and the theoretical model is done using the Kolmogorov-Smirnov\* goodness-of-fit test ([44], pp. 177-178, [45], pp. 389-394). This test gives limits for the maximum difference allowable with a given "level of significance" between the simulated distribution and the theoretical model, under the assumption that the samples used in the simulation have been drawn from a population distributed according to the

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\* See Appendix A.

theoretical model. Here, a level of significance  $\alpha$  means that if we really had the distribution assumed, the probability is only  $\alpha$  that a deviation as big as or bigger than the significance limit is attained. If, for example,  $\alpha$  is 20% or 0.2, then we find that the given significance limit is exceeded only 20% of the time. We will call the significance limits as given by the Kolmogorov-Smirnov test the *K-S limits* (at significance  $\alpha$ ).

We are now ready to state the results. The maximum cumulative probability differences between the simulation and the model were found to be acceptably small for practical purposes in all cases. However, the significance is only  $5 \cdot 10^{-6}$  for system 1 (so the negative exponential is extremely unlikely to hold), but is 0.67 for system 2. Here we are **better** than the average significance 0.5 which we would get even if the negative exponential held exactly. Even though the significance for system 1 is too small for accepting theoretical memorylessness, the agreement in cumulative probability between the simulation result and the memoryless model is not too bad in probability difference (around 0.02) for many applications. Moreover, if we increase the rate of change of direction to two changes per minute, as we did in a later simulation, the significance level goes up to 0.19 for the same system 1 as well, so we get good agreement with memorylessness.

The above results, plus the average number of handoffs obtained for systems 1 and 2, are summarized in Table 3.1.

The average numbers of handoffs given are, as we have often said, the ones used to construct the exponential distributions to which the simulated distributions are compared.

Note that the number of samples available in the Kolmogorov-Smirnov test is usually larger than the number  $n$  of calls used in the simulation. This is due to

$n = 10,000$ calls	Average Number of handoffs	K-S limits $\alpha = 0.2$	Maximum Probability Deviation  Model - Simulation
a) System 1	0.7401	$8.1 \cdot 10^{-3}$	$1.93 \cdot 10^{-2}$ ( $\alpha = 5.0 \cdot 10^{-6}$ )
b) System 1	0.7582	$8.07 \cdot 10^{-3}$	$8.18 \cdot 10^{-3}$ ( $\alpha = 0.19$ )
c) System 2	0.0776	$1.03 \cdot 10^{-2}$	$7.21 \cdot 10^{-3}$ ( $\alpha = 0.67$ )

Table 3.1. Comparison simulation and exponential model.

$\alpha$ : significance.

a)  $R = 1$  mi,  
 $v_0 = 35$  mi/hr,  
 $\mu = 0.5 \text{ min}^{-1}$ ,  
 $cd = 1.0$ , (one change of direction per minute).

b)  $R = 1$  mi,  
 $v_0 = 35$  mi/hr,  
 $\mu = 0.5 \text{ min}^{-1}$ ,  
 $cd = 2.0$ , (two changes of direction per minute).

c)  $R = 10$  mi,  
 $v_0 = 35$  mi/hr,  
 $\mu = 0.5 \text{ min}^{-1}$ ,  
 $cd = 0.5$ , (one change of direction every two minutes).

the fact that each call will be made up of possibly several time intervals separated by handoffs, each representing one channel occupation time. It is the total number of channel occupation time intervals that should then be taken as the number of samples used to construct the simulated distribution.

The results are displayed in Figures 3.5 and 3.6 where the dotted curve represents the theoretical model and the plain curve is the simulated distribution.

The good agreement between the two should, however, not lead us to think that we have an exact negative exponential distribution. We will show that the distribution cannot be exactly memoryless with two extreme examples.

Example 1:

We first consider a cellular system identical to the previous system 1, except for the rate of change of direction which is taken to be very small, *e.g.*, one change of direction every 100 minutes. We will show that the resulting distribution for the channel occupation time cannot be exponential.

Suppose for example that we are given that a channel has been occupied by a call for a rather long time, *e.g.*, over 3 minutes. The probability that the channel is about to get freed must be nearly one. This is due to the fact that the probability of the mobile having had, or being about to make, a change of direction is close to zero. The vehicle then has to be close to the cell boundary (about 3.4 minutes to cross the whole cell at 35 miles per hour), and must be about to cross it, which means the termination of the channel occupation time. The call configuration implied in the above situation is depicted in Figure 3.7. This, in turn, contradicts the memoryless property of the exponential distribution, since information provided on the past of the channel nearly completely determines its future behavior. We will however see later that even in this extreme case the resulting simulated distribution does not disagree too much with the exponential model, except around the tail.

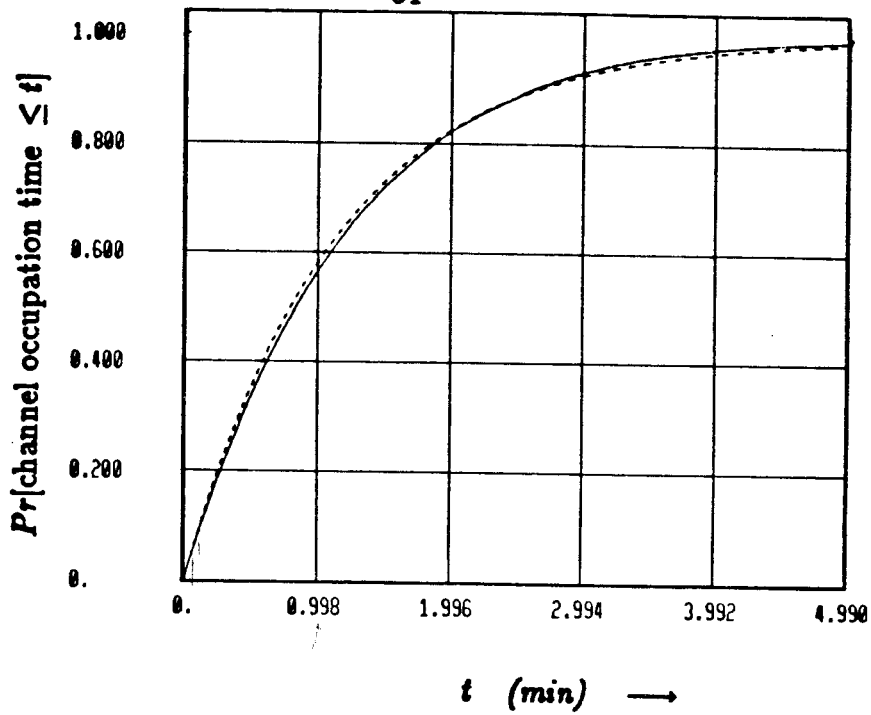


Fig. 9.5. System 1, comparison simulation and exponential model.

Exponential model (---), simulation (—).

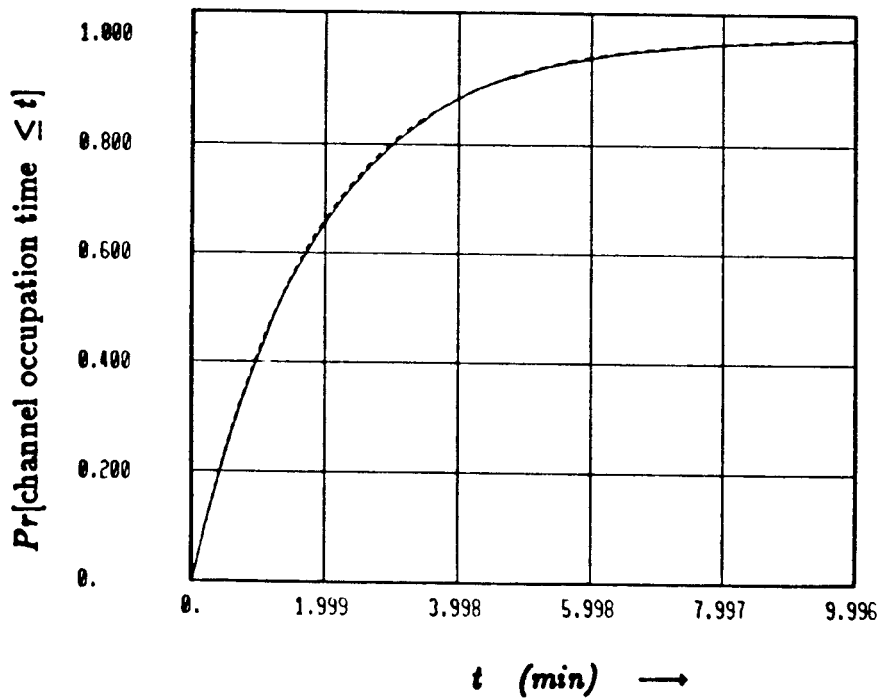
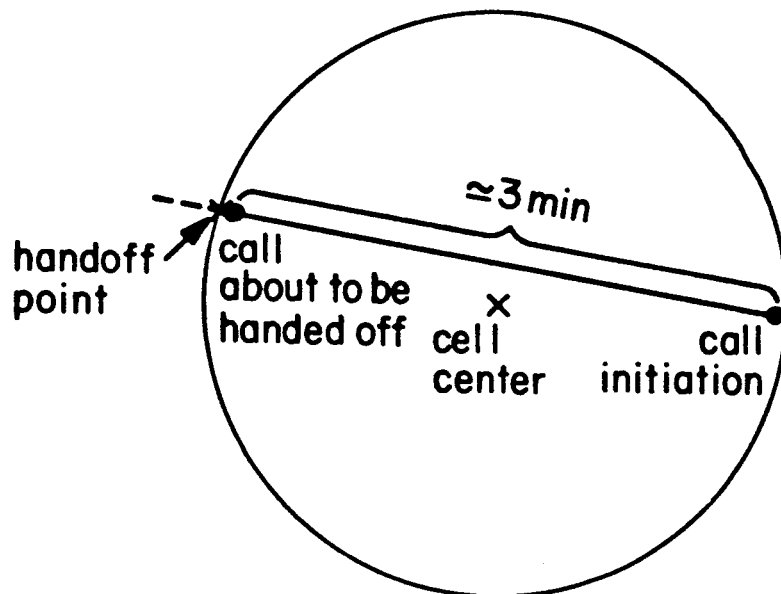


Fig. 9.6. System 2, comparison simulation and exponential model.

Exponential model (---), simulation (—).



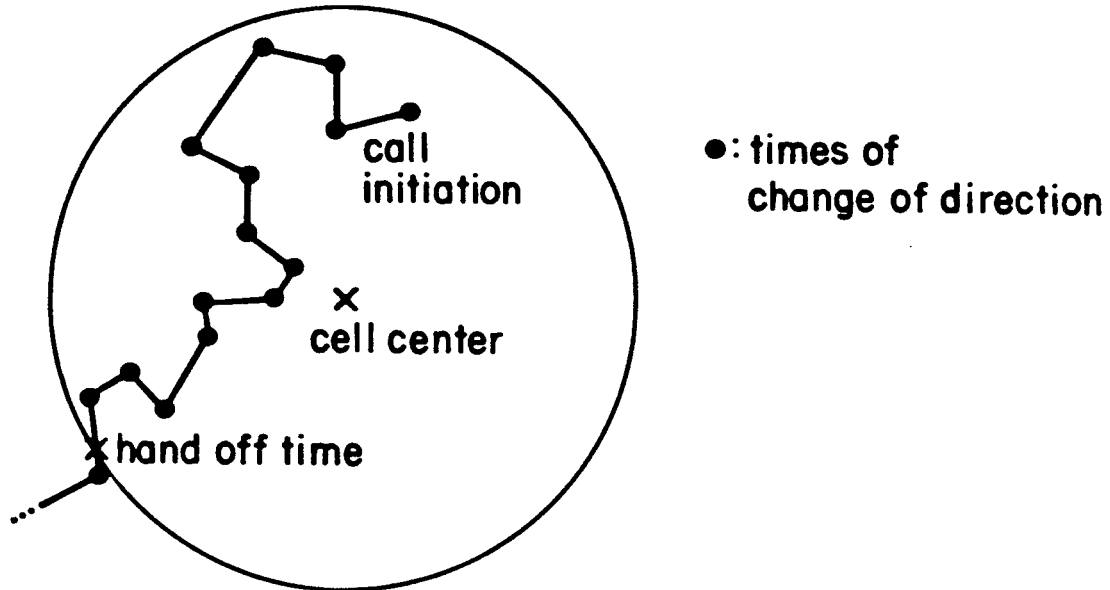


*Fig. 3.7. Example 1: Small rate of change of direction.*

**Example 2:**

The second example relies on the opposite extreme case for the rate of change of direction, which is now taken to be very large. Otherwise keeping the same cellular system as in the first example, we find using the simulation that the average number of handoffs remains nearly the same. The vehicle motion is however different and now crudely approximates a random walk (Brownian motion). If we decide to concentrate on the fraction of calls initiated in the cell that will not terminate inside the cell, that is, that will experience at least one handoff (this fraction is not negligible since the average number of handoffs remains rather high), we can reach several conclusions.

The channel occupation times corresponding to these calls will here be made up of a large number of exponentially-distributed time intervals consisting of times



*Fig. 3.8. Example 2: High rate of change of direction.*

between two changes of direction, except for the last time interval that ends at handoff time. This situation is described in Figure 3.8.

The number of time intervals between changes of direction clearly depends on their respective durations, but the correlation between them is small and decreases when the rate of change of direction increases. The conclusions of the Central Limit Theorem should then be approximately true if we assume a very high rate of change of direction. We can therefore conclude that for the category of calls that we considered, the distribution of the channel occupation time will rather tend to a Gaussian distribution than an exponential one. This completes our showing that the exponential model is a good approximation of the distribution of the channel occupation time, but nevertheless only an approximation.

We now estimate the domain of validity of the exponential approximation in terms of variations of such system parameters as cell radius, average mobile speed, average total call duration, and rate of change of direction.

We first observe that the cell radius, the average mobile speed, and the average total call duration will have the same kind of influence. From the mobile point of view it is equivalent to increase the cell radius or decrease the average mobile speed. Similarly, decreasing or increasing the average total call duration only makes the cell look artificially larger or smaller to the mobile involved in the call. It is therefore sufficient to study the effect of the cell radius variation, for example, in order to understand the influence of the average mobile speed and average total call duration. An evaluation of the influence of the cell radius is provided by the study of two systems with respective radii of 1 and 10 miles (systems 1 and 2). Some additional simulations with other cell radii (not shown) confirm the fact that the exponential distribution remains a good approximation for all practical cases.

At this point the influence of the rate of change of direction still remains to be evaluated. Example 1 showed that in the extreme case of a very small rate of change of direction, the actual distribution could not be exactly memoryless and therefore the result of the simulation would probably disagree with the exponential model. Similarly, example 2 indicated that in the other extreme case of a very high rate of change of direction, the distribution of the channel occupancy time would have an approximately Gaussian component which would again introduce some disagreements between the simulation results and the exponential model.

We would then like to confirm the supposition that, with every thing else held constant, the distribution of the channel occupancy time for low ratio of change of direction first starts by disagreeing with the exponential model, then gets closer and closer to it as the rate of change of direction increases to a realistic practical value, and finally starts drifting away from the exponential distribution as we keep on increasing the rate of change of direction unrealistically.

This verification was made by running the simulation for successive values of the rate of change of direction, each time comparing the simulated distribution with the corresponding exponential one. The cellular system chosen was again the one corresponding to system 1 with the exception of the rate of change of direction which is varied. The results are displayed in Figures 3.9 through 3.16 and give a good validation of the conjectures made on the basis of examples 1 and 2.

Note that the best agreement was obtained for a rate of two changes of direction per minute. In this case the simulated distribution was barely distinguishable from the exponential model. Even for the most extreme cases of either very small or very large rate of change of direction, the difference with the model is not very large.

This terminates the chapter devoted to the simulation program; the next chapter will give an analytic approach.

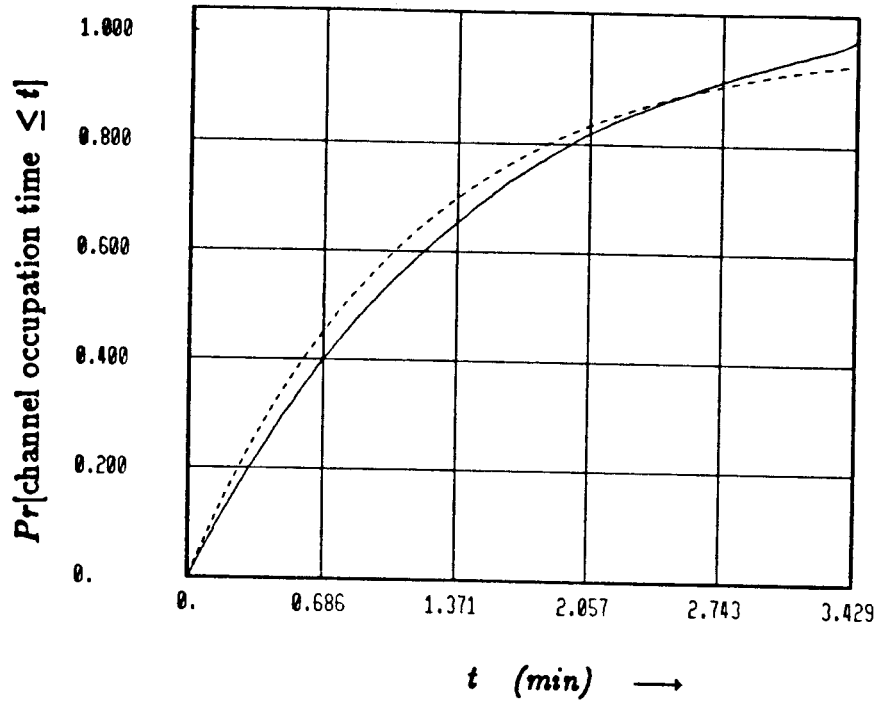


Fig. 3.9. Influence of rate of change of direction,  $cd = 0.0$ .

Exponential model (---), simulation (—).

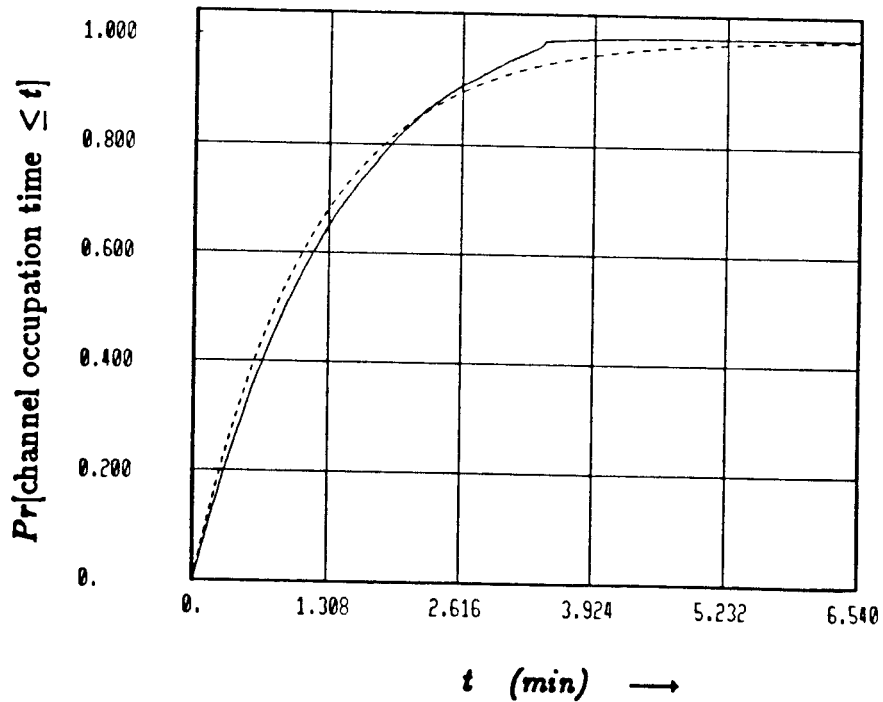


Fig. 3.10. Influence of rate of change of direction,  $cd = 0.1$ .

Exponential model (---), simulation (—).

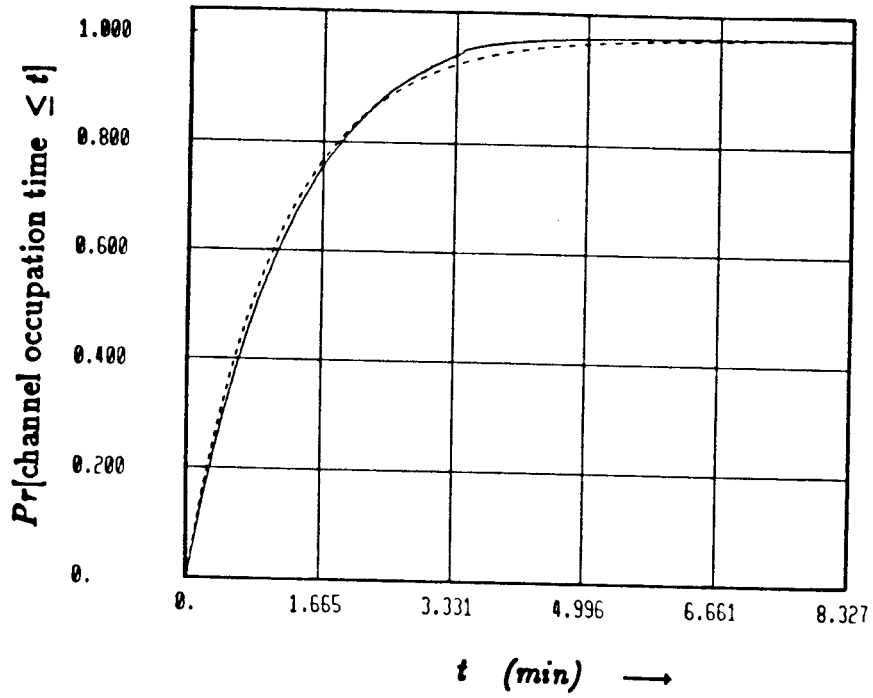


Fig. 3.11. Influence of rate of change of direction,  $cd = 0.5$ .

Exponential model (---), simulation (—).

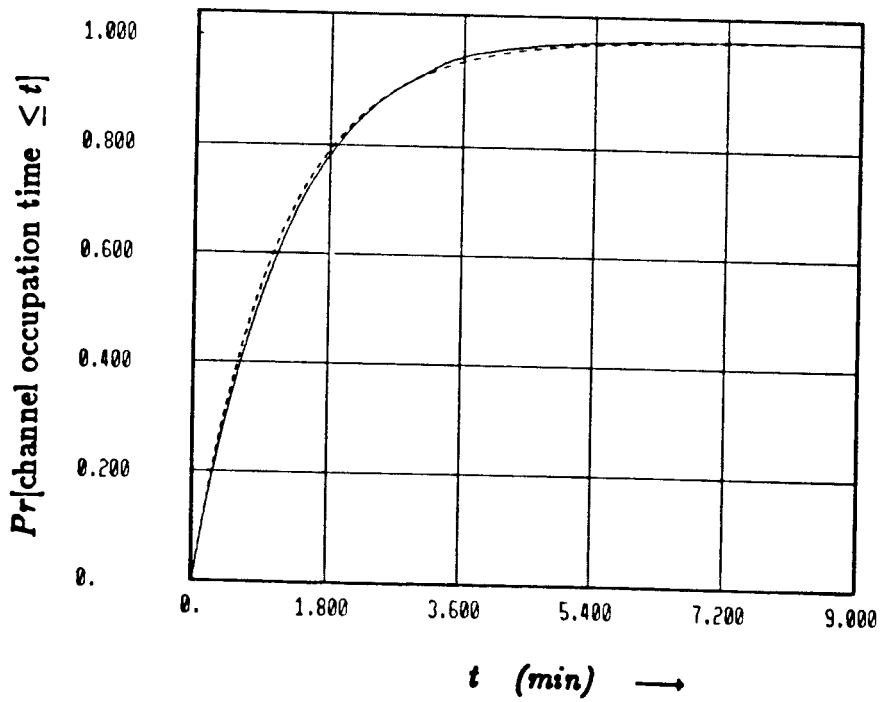


Fig. 3.12. Influence of rate of change of direction,  $cd = 1.0$ .

Exponential model (---), simulation (—).

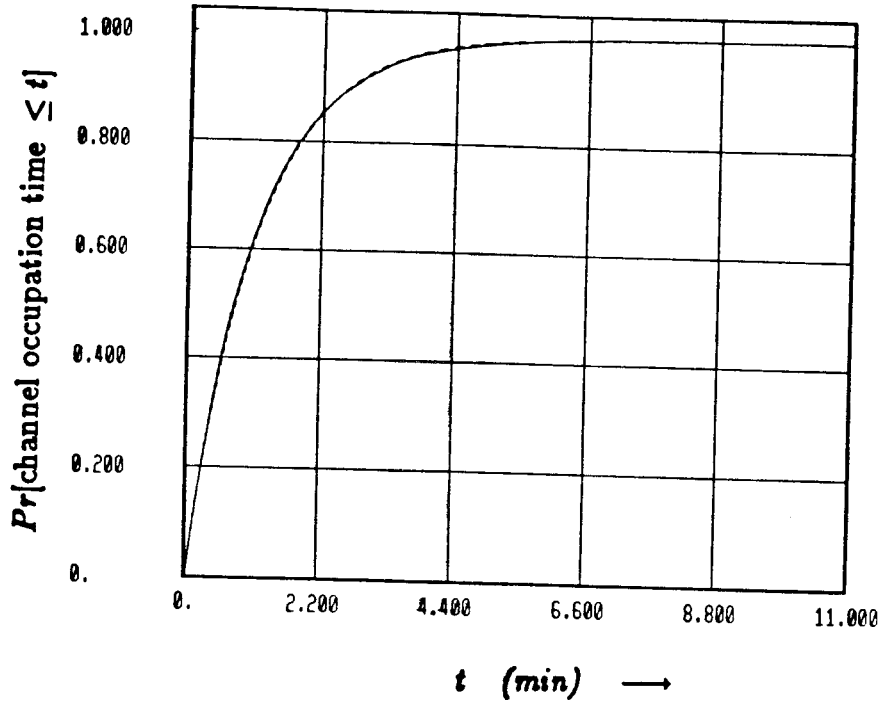


Fig. 3.13. Influence of rate of change of direction,  $cd = 2.0$ .

Exponential model (---), simulation (—).

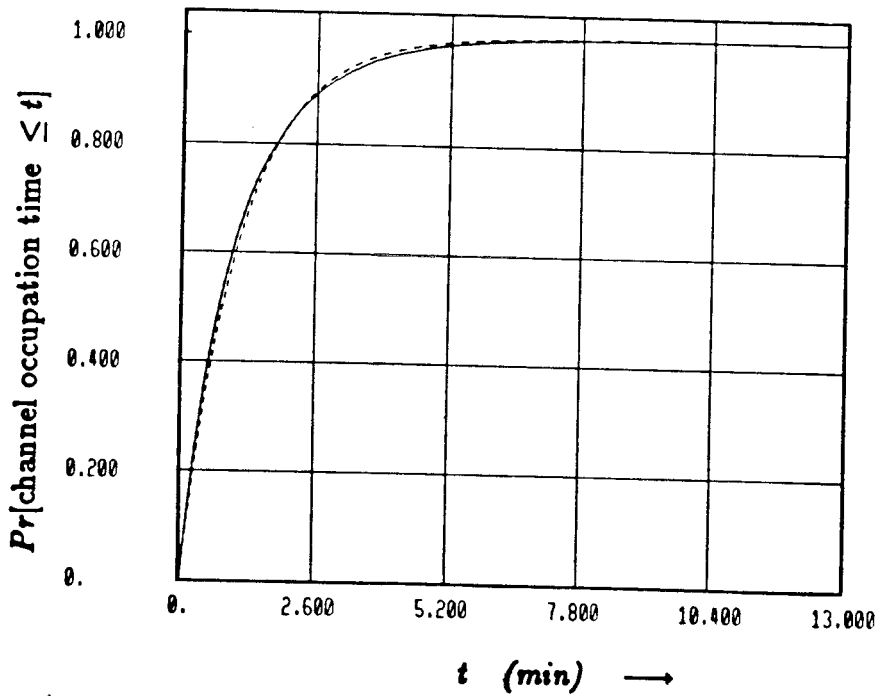


Fig. 3.14. Influence of rate of change of direction,  $cd = 3.0$ .

Exponential model (---), simulation (—).

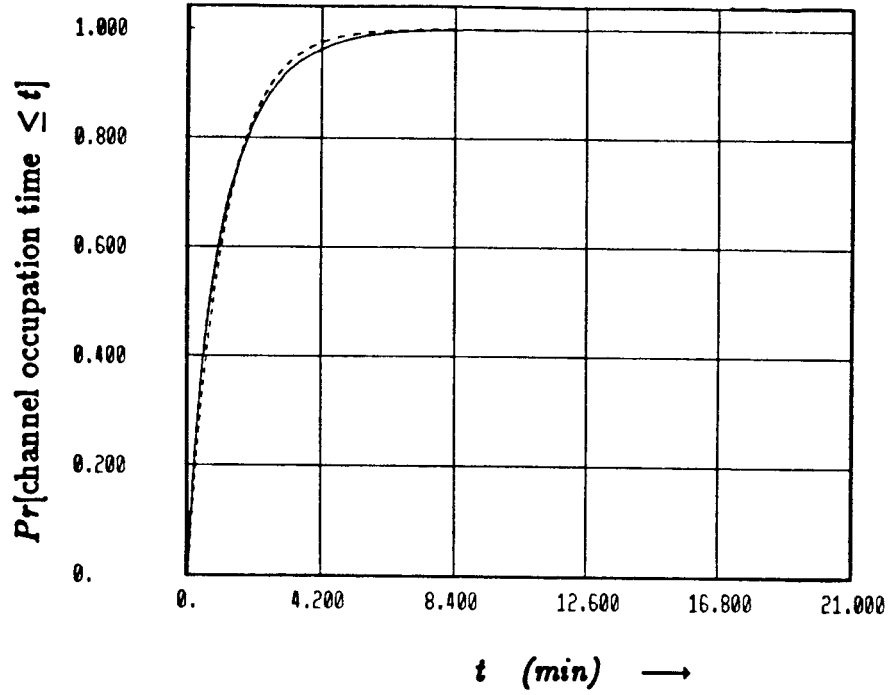


Fig. 9.15. Influence of rate of change of direction,  $cd = 5.0$ .  
Exponential model (---), simulation (—).

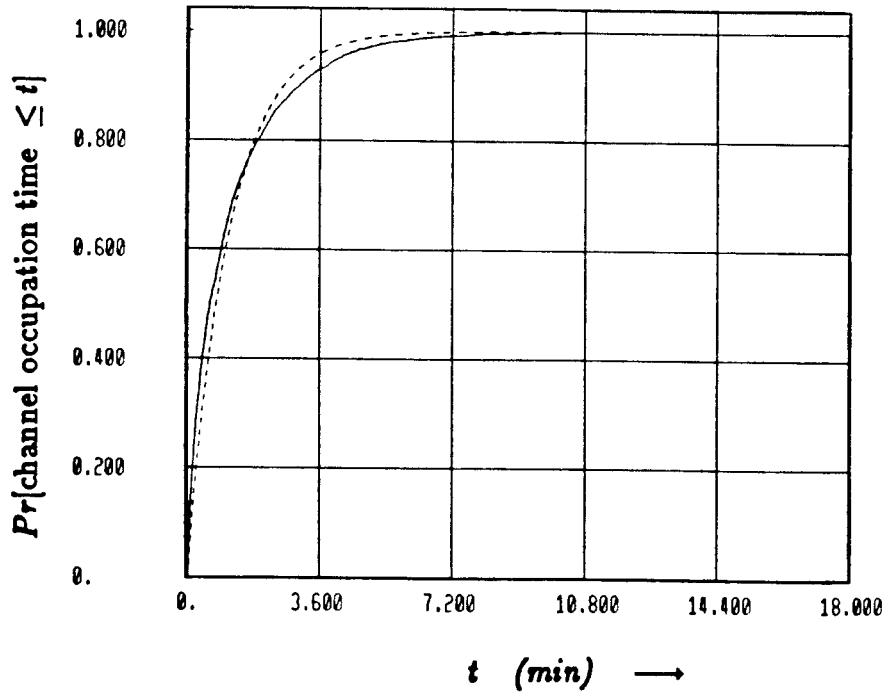


Fig. 9.16. Influence of rate of change of direction,  $cd = 10.0$ .  
Exponential model (---), simulation (—).



## CHAPTER 4

### ANALYTIC MODEL FOR A SIMPLIFIED CELLULAR SYSTEM

#### 4.1 Assumptions and method.

In this Section we present an analytic model for the distribution of the number of handoffs which a random call undergoes. Due to the complexity of the general problem, some simplifying assumptions have to be made. We again assume a constant speed for all the mobiles, which will furthermore keep the same direction for the entire duration of the call. There are only four possible directions of motion here, which are  $90^\circ$  apart and will be assigned randomly with equal probabilities at the beginning of a call. The cellular system will also be assumed to be large enough so that we can consider it as covering the whole plane. In order to cover the plane, we will choose the cells to have a hexagonal shape. This might not be a completely realistic assumption, but it is very classical ([2], pp. 19-22, [21], p. 45, [41]) and more important we will find out that it does not introduce too big a distinction between the possible directions of motion. Finally, we assume a uniform initiation of calls over the whole system area, and an exponentially distributed (memoryless) total call duration.

The coordinate system used in the different calculations is presented in Figure 4.1; we now introduce some useful notation:

$R$ : Cell radius.

$v_0$ : Magnitude of mobile speed.

$d_n$ : Distance to the  $n^{\text{th}}$  cell boundary crossing (direction dependent).

$T_n = d_n/v_0$ : Time before the  $n^{\text{th}}$  cell boundary crossing.

$\mu$ : Service rate, entire call.

$k = R/v_0$ : Ratio of cell radius to mobile speed (system scale).

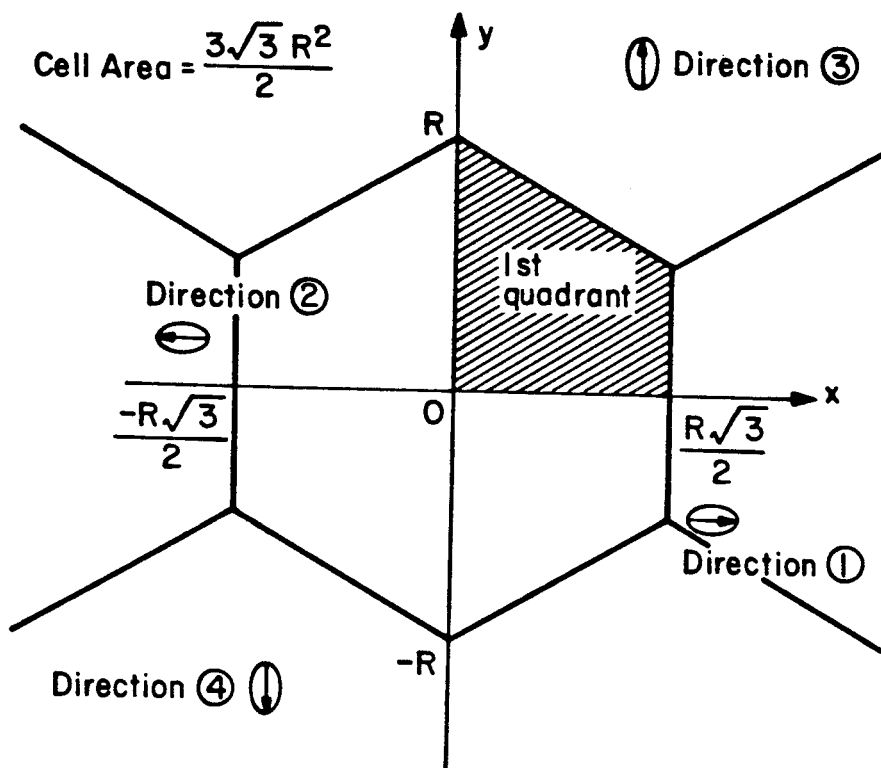


Fig. 4.1. System configuration.

$\alpha = k\mu$ : Product of service rate times system scale.

(1)  $\equiv$  ( $x$ )-direction.

(2)  $\equiv$  ( $-x$ )-direction.

(3)  $\equiv$  ( $y$ )-direction.

(4)  $\equiv$  ( $-y$ )-direction.

Recall that  $Pr[\text{mobile moves in direction } (i)] = 0.25, i = 1, 2, 3, 4.$

We want to obtain the distribution of the number of handoffs. We first find an equivalent quantity, the probability that a random call goes through **at least**  $n$  handoffs for any integer value of  $n$ . The method used will be introduced by means of examples (cases  $n = 1, 2, 3$ ) which will further be used as checks for the general expression.

#### 4.2 Illustration of the method through examples.

Let us first concentrate on the case  $n = 1$ . We will obtain the probability that a call goes through at least one handoff by first computing the four conditional probabilities corresponding to the four possible directions of motion for a vehicle. For each direction we will compute the probability that the mobile crosses a cell boundary before a fixed time  $t$  (independently of call duration). These expressions will then be used to get the overall handoff probability, that is, the probability that a mobile crosses a cell boundary at least once while involved in a call, i.e., before the call terminates.

In order to obtain the above conditional probabilities we will make use of the uniform spatial distribution of call initiations to reduce the problem to the computation of appropriate area ratios. Furthermore, due to the symmetry of the hexagonal cell we can restrict ourself to calls initiated only in the first quadrant of the cell (shaded area in Figure 4.1). This means that we will now have a double condition, the direction of motion and the fact that the call was initiated in the first quadrant. This additional condition will be indicated simply by adding a tilde to the expression of the different probabilities. For example:

$$\tilde{P}_r[T_1 \leq t/(1)]$$

will indicate the probability that the time to the first crossing of a cell boundary is smaller than  $t$ , given that that the vehicle moves in direction (1) and given that it is located inside the first quadrant.

We note that the above probability is equivalent to the following expression:

$$\tilde{P}_r[T_1 \leq t/(1)] = \tilde{P}_r[d_1 \leq v_0 t/(1)].$$

The right hand side of the above equation corresponds to all the possible locations inside the first quadrant that are less than  $v_0 t$  away from a cell boundary when moving in direction (1). The probability sought is then easily found to be the ratio of the shaded area indicated in Figure 4.2(a) over the area of the first quadrant. This then gives

$$\bar{P}_r[T_1 \leq t/(1)] = \begin{cases} -4\left(\frac{t}{3k}\right)^2 + \frac{8\sqrt{3}}{9}\left(\frac{t}{k}\right), & 0 \leq t \leq k\frac{\sqrt{3}}{2}; \\ 1.0, & k\frac{\sqrt{3}}{2} \leq t; \end{cases}$$

recall that  $k = \frac{R}{v_0}$ . The same principle is used for directions (2), (3), (4), with the possible difference that we might obtain different expressions for the same probability depending on the value of  $t$ . This is due to the fact that, as the distance from a cell boundary changes, the hexagonal shape of the cell yields variations in the description of the area associated to this distance. Let us illustrate this by working out direction (2).

Figure 4.2(b) gives the different possibilities from which we easily derive the corresponding probabilities:

$$\bar{P}_r[T_1 \leq t/(2)] = \begin{cases} 2\left(\frac{t}{3k}\right)^2, & 0 \leq t \leq k\frac{\sqrt{3}}{2}; \\ -2\left(\frac{t}{3k}\right)^2 + \frac{8\sqrt{3}}{9}\left(\frac{t}{k}\right) - 1, & k\frac{\sqrt{3}}{2} \leq t \leq k\sqrt{3}; \\ 1.0, & k\sqrt{3} \leq t. \end{cases}$$

Similarly, for direction (3), Figure 4.2(c) gives us the different possibilities from which we derive again the corresponding probabilities:

$$\bar{P}_r[T_1 \leq t/(3)] = \begin{cases} \frac{4}{3}\left(\frac{t}{k}\right), & 0 \leq t \leq k/2; \\ -12\left(\frac{t}{3k}\right)^2 + \frac{8}{3}\left(\frac{t}{k}\right) - \frac{1}{3}, & k/2 \leq t \leq k; \\ 1.0, & k \leq t. \end{cases}$$

Finally, Figure 4.2(d) yields the results in the case of direction (4):

$$\bar{P}_r[T_1 \leq t/(4)] = \begin{cases} 0.0, & 0 \leq t \leq k/2; \\ 12\left(\frac{t}{3k}\right)^2 - \frac{4}{3}\left(\frac{t}{k}\right) + \frac{1}{3}, & k/2 \leq t \leq k; \\ -6\left(\frac{t}{3k}\right)^2 + \frac{8}{3}\left(\frac{t}{k}\right) - \frac{5}{3}, & k \leq t \leq 2k; \\ 1.0, & 2k \leq t. \end{cases}$$

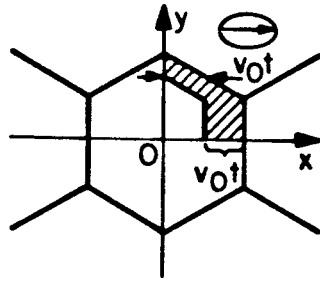


Fig. 4.2(a). Direction (1). Area at distance less than  $t$  away from cell boundary.

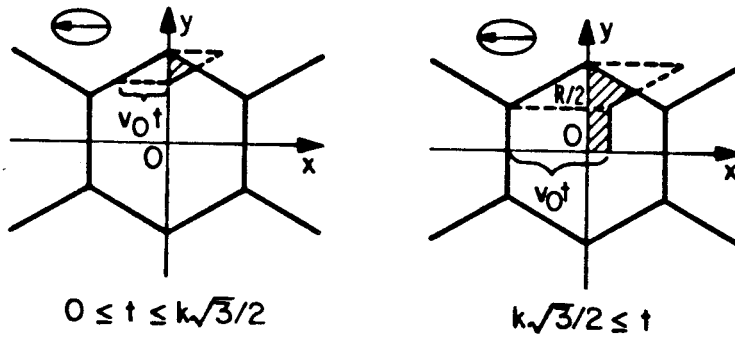


Fig. 4.2(b). Direction (2). Area at distance less than  $t$  away from cell boundary.

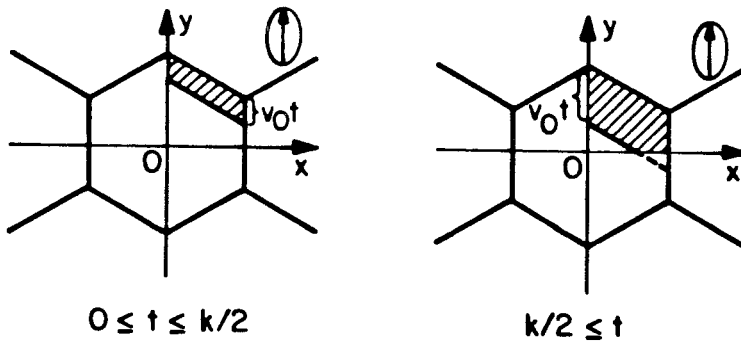


Fig. 4.2(c). Direction (3). Area at distance less than  $t$  away from cell boundary.

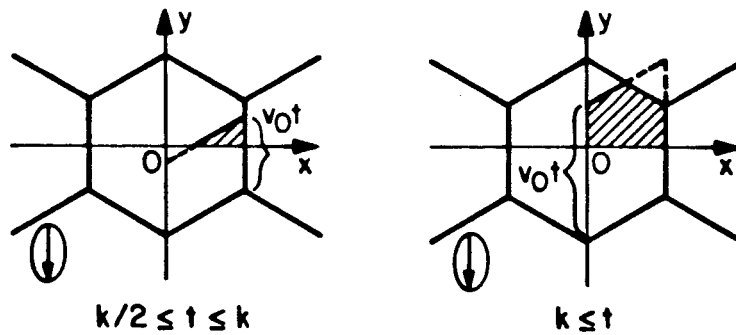


Fig. 4.2(d). Direction (4). Area at distance less than  $t$  away from cell boundary.

We can now compute the unconditioned probability  $P[T_1 \leq t]$ , by averaging all four with equal weight:

$$P[T_1 \leq t] = \frac{1}{4} \left( \bar{P}_r[T_1 \leq t/(1)] + \bar{P}_r[T_1 \leq t/(2)] + \bar{P}_r[T_1 \leq t/(3)] + \bar{P}_r[T_1 \leq t/(4)] \right). \quad (4.1)$$

(The same equation holds of course for  $T_n$  instead of  $T_1$ .) Here we have used the fact that when we consider all possible directions of motion, the initial restriction to the first quadrant for the call initiations becomes irrelevant. So after considering all the possible cases we finally get:

$$P[T_1 \leq t] = \begin{cases} -\frac{1}{2} \left(\frac{t}{3k}\right)^2 + \left(\frac{2\sqrt{3}+3}{9}\right) \left(\frac{t}{k}\right), & 0 \leq t \leq k; \\ -2 \left(\frac{t}{3k}\right)^2 + \left(\frac{2\sqrt{3}+6}{9}\right) \left(\frac{t}{k}\right) - \frac{1}{6}, & k \leq t \leq k\sqrt{3}; \\ -\frac{3}{2} \left(\frac{t}{3k}\right)^2 + \frac{2}{3} \left(\frac{t}{k}\right) + \frac{1}{3}, & k\sqrt{3} \leq t \leq 2k; \\ 1.0, & 2k \leq t. \end{cases}$$

Note that the reason why we didn't have to separately consider the cases:

$$0 \leq t \leq \frac{k}{2}, \quad \frac{k}{2} \leq t \leq \frac{k\sqrt{3}}{2}, \quad \frac{k\sqrt{3}}{2} \leq t \leq k,$$

is merely due to the fact that the corresponding probabilities are incidentally equal; but this yields no generalization, however.

We can now evaluate the probability that a call goes through at least one hand-off, since if we know that the time before the first cell boundary crossing is smaller than  $t$ , we only need to choose the total call duration at least equal to this  $t$  to insure a handoff. More precisely we get,

$$\begin{aligned} P_{\geq 1H} &= \int_0^k \mu e^{-\mu t} \left[ -\frac{t^2}{18k^2} + \left(\frac{2\sqrt{3}+3}{9k}\right) t \right] dt \\ &+ \int_k^{k\sqrt{3}} \mu e^{-\mu t} \left[ -\frac{2}{9k^2} t^2 + \left(\frac{2\sqrt{3}+6}{9k}\right) t - \frac{1}{6} \right] dt \\ &+ \int_{k\sqrt{3}}^{2k} \mu e^{-\mu t} \left[ -\frac{3}{18k^2} t^2 + \frac{2}{3k} t + \frac{1}{3} \right] dt + \int_{2k}^{\infty} \mu e^{-\mu t} dt. \end{aligned}$$

The details of these calculations are provided in Appendix B. After replacing  $k\mu$  by  $\alpha$  we finally get

$$P_{\geq 1H}(\alpha) = -\frac{1}{9\alpha^2} + \frac{2\sqrt{3}+3}{9\alpha} - \frac{1}{3\alpha^2}e^{-\alpha} + \left(\frac{1}{9\alpha^2} - \frac{\sqrt{3}}{9\alpha}\right)e^{-\sqrt{3}\alpha} + \frac{1}{3\alpha^2}e^{-2\alpha}.$$

This can be rewritten in the following form:

$$P_{\geq 1H}(\alpha) = \frac{1}{9\alpha^2} \left[ 3e^{-2\alpha} + e^{-\sqrt{3}\alpha} - 3e^{-\alpha} - 1 \right] + \frac{1}{9\alpha} \left[ 2\sqrt{3} + 3 - \sqrt{3}e^{-\sqrt{3}\alpha} \right]. \quad (4.2)$$

As an easy check we could verify that we still have,

$$\lim_{\alpha \rightarrow \infty} P_{\geq 1H}(\alpha) = 0, \quad \text{and} \quad \lim_{\alpha \rightarrow 0} P_{\geq 1H}(\alpha) = 1.$$

The first case corresponds to a service time going to zero, which should clearly imply that the probability of a call being handed off should also go to zero. This is easily verified since there is factor  $1/\alpha$  in (4.2) and the other factor remains bounded as  $\alpha$  goes to infinity. The second case corresponds to a service time going to infinity which should clearly imply a probability of at least one handoff going to 1. The verification requires here a Taylor series expansion around the origin ( $\alpha = 0$ ) of (4.2); we get:

$$P_{\geq 1H}(\alpha) \approx \frac{1}{9\alpha^2} [3 + 1 - 3 - 1] + \frac{1}{9\alpha} [2\sqrt{3} + 3 - \sqrt{3} - 6 - \sqrt{3} + 3] + \frac{1}{9} [3 + 6 + \frac{3}{2} - \frac{3}{2}] = 1.$$

This provides us with the desired check and completes our study of the case  $n = 1$ .

We will now study the case  $n = 2$ . This uses the exact same method, except for the fact that we are now interested in the distance to the **second** crossing of a cell boundary. Figure 4.3(a) presents the different possible cases for the distance to the second crossing of a cell boundary in the case of a mobile moving in direction (1).

From this we can derive the corresponding conditional probability for the time to the second cell boundary crossing:

$$\bar{P}_r[T_2 \leq t/(1)] = \begin{cases} 2\left(\frac{t}{3k}\right)^2, & 0 \leq t \leq k\frac{\sqrt{3}}{2}; \\ -2\left(\frac{t}{3k}\right)^2 + \frac{4\sqrt{3}}{9}\left(\frac{t}{k}\right) - \frac{1}{3}, & k\frac{\sqrt{3}}{2} \leq t \leq k\sqrt{3}; \\ \frac{4\sqrt{3}}{9}\left(\frac{t}{k}\right) - 1, & k\sqrt{3} \leq t \leq k\frac{3\sqrt{3}}{2}; \\ 1.0, & k\frac{3\sqrt{3}}{2} \leq t. \end{cases}$$

Figures 4.3(b), 4.3(c) and 4.3(d) yield similar interpretations for directions (2), (3) and (4) respectively. For each case we then get the following expressions for the conditional probabilities:

$$\bar{P}_r[T_2 \leq t/(2)] = \begin{cases} 0.0, & 0 \leq t \leq k\frac{\sqrt{3}}{2}; \\ 4\left(\frac{t}{3k}\right)^2 - \frac{4\sqrt{3}}{9}\left(\frac{t}{k}\right) + \frac{1}{3}, & k\frac{\sqrt{3}}{2} \leq t \leq k\sqrt{3}; \\ \frac{1}{3}, & k\sqrt{3} \leq t \leq k\frac{3\sqrt{3}}{2}; \\ \frac{4\sqrt{3}}{9}\left(\frac{t}{k}\right) - \frac{5}{3}, & k\frac{3\sqrt{3}}{2} \leq t \leq k2\sqrt{3}; \\ 1.0, & k2\sqrt{3} \leq t. \end{cases}$$

$$\bar{P}_r[T_2 \leq t/(3)] = \begin{cases} 0.0, & 0 \leq t \leq k; \\ 6\left(\frac{t}{3k}\right)^2 - \frac{4}{3}\left(\frac{t}{k}\right) + \frac{2}{3}, & k \leq t \leq 2k; \\ -12\left(\frac{t}{3k}\right)^2 - \frac{20}{3}\left(\frac{t}{k}\right) - \frac{22}{3}, & 2k \leq t \leq k\frac{5}{2}; \\ 1.0, & k\frac{5}{2} \leq t. \end{cases}$$

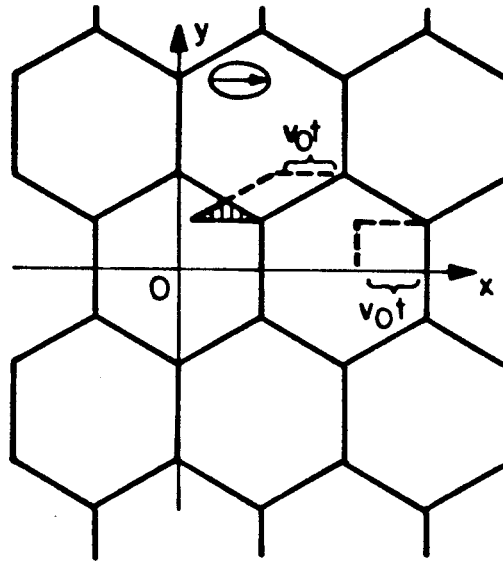
And finally,

$$\bar{P}_r[T_2 \leq t/(4)] = \begin{cases} 0.0, & 0 \leq t \leq 2k; \\ 12\left(\frac{t}{3k}\right)^2 - \frac{16}{3}\left(\frac{t}{k}\right) + \frac{16}{3}, & 2k \leq t \leq k\frac{5}{2}; \\ \frac{4}{3}\left(\frac{t}{k}\right) - 3, & k\frac{5}{2} \leq t \leq 3k; \\ 1.0, & 3k \leq t. \end{cases}$$

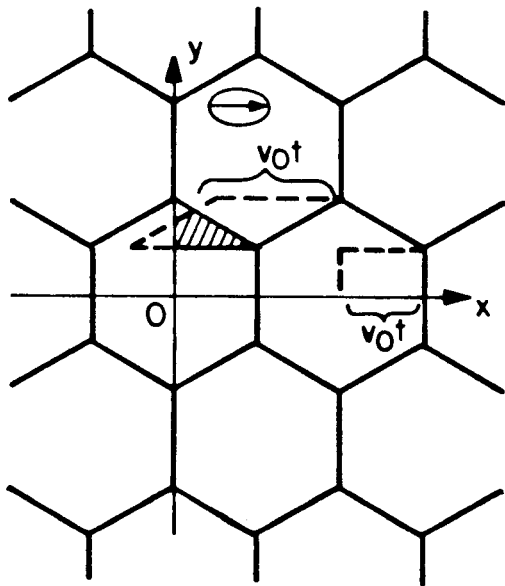
Using Equation (4.1) again with  $n = 2$ , we can derive the overall probability that the time before the second cell boundary crossing is smaller than  $t$ :

$$P[T_2 \leq t] = \begin{cases} \frac{1}{2}\left(\frac{t}{3k}\right)^2, & 0 \leq t \leq k; \\ 2\left(\frac{t}{3k}\right)^2 - \frac{1}{3}\left(\frac{t}{k}\right) + \frac{1}{6}, & k \leq t \leq k\sqrt{3}; \\ \frac{3}{2}\left(\frac{t}{3k}\right)^2 - \left(\frac{3-\sqrt{3}}{9}\right)\left(\frac{t}{k}\right), & k\sqrt{3} \leq t \leq 2k; \\ \left(\frac{3+\sqrt{3}}{9}\right)\left(\frac{t}{k}\right) - \frac{2}{3}, & 2k \leq t \leq 3k; \\ \frac{\sqrt{3}}{9}\left(\frac{t}{k}\right) + \frac{1}{3}, & 3k \leq t \leq 2k\sqrt{3}; \\ 1.0, & 2k\sqrt{3} \leq t. \end{cases}$$

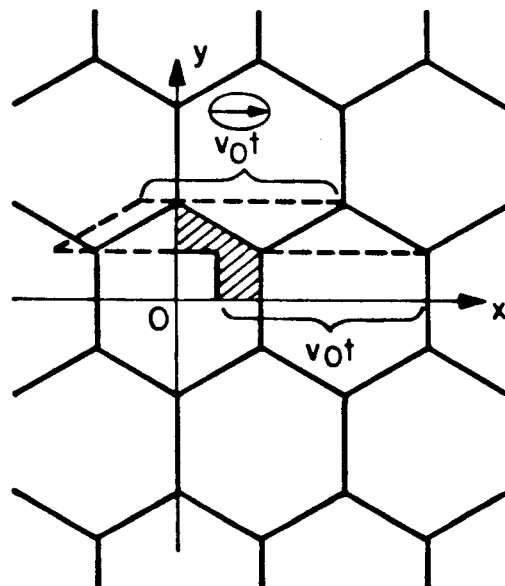




$$0 \leq t \leq k\sqrt{3}/2$$



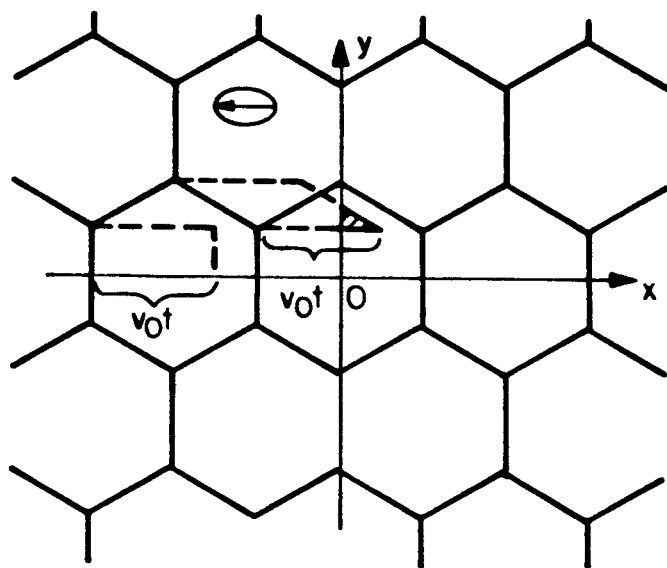
$$k\sqrt{3}/2 \leq t \leq k\sqrt{3}$$



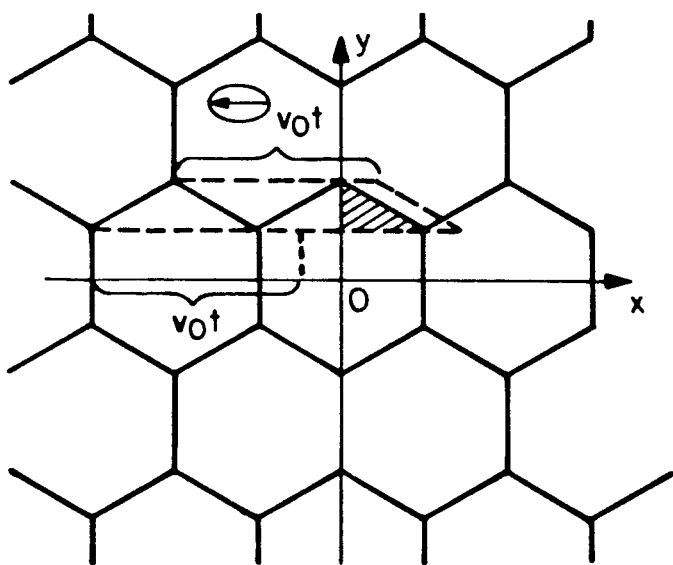
$$k\sqrt{3} \leq t$$

Fig. 4.9(a). Direction (1).

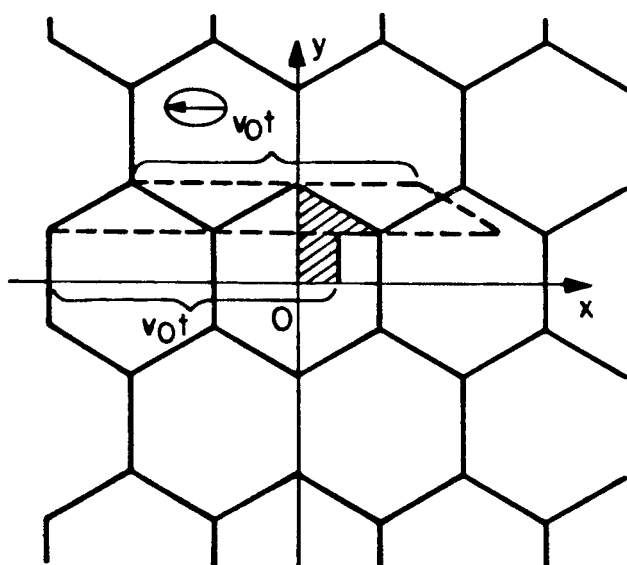
Area at distance less than  $t$  away from second boundary crossing.



$$k\sqrt{3}/2 \leq t \leq k\sqrt{3}$$



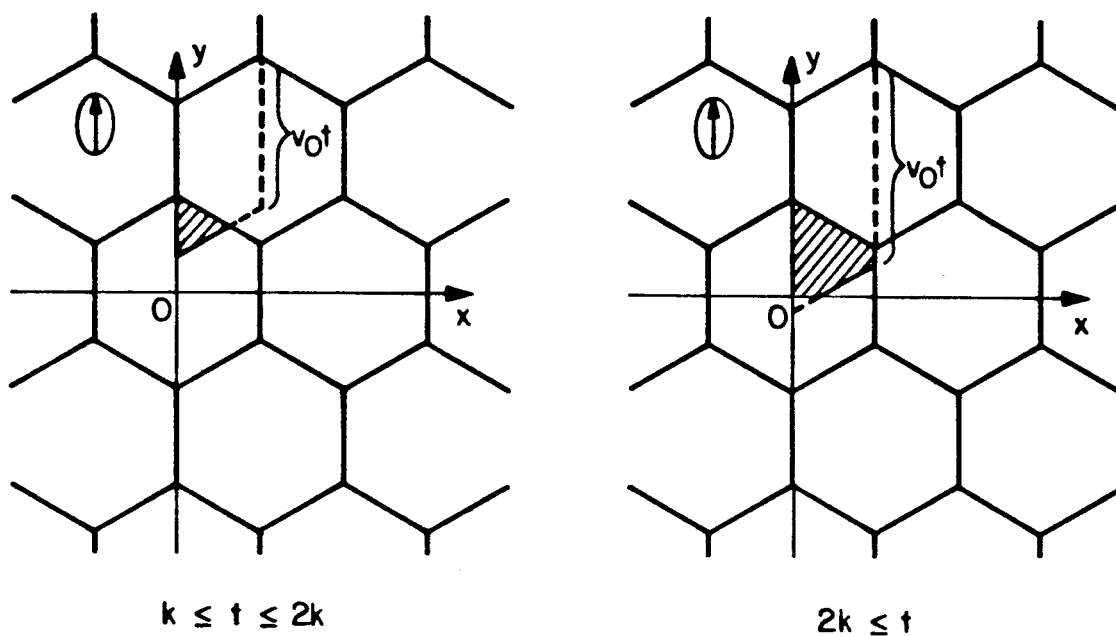
$$k\sqrt{3} \leq t \leq 3k\sqrt{3}/2$$



$$3k\sqrt{3}/2 \leq t$$

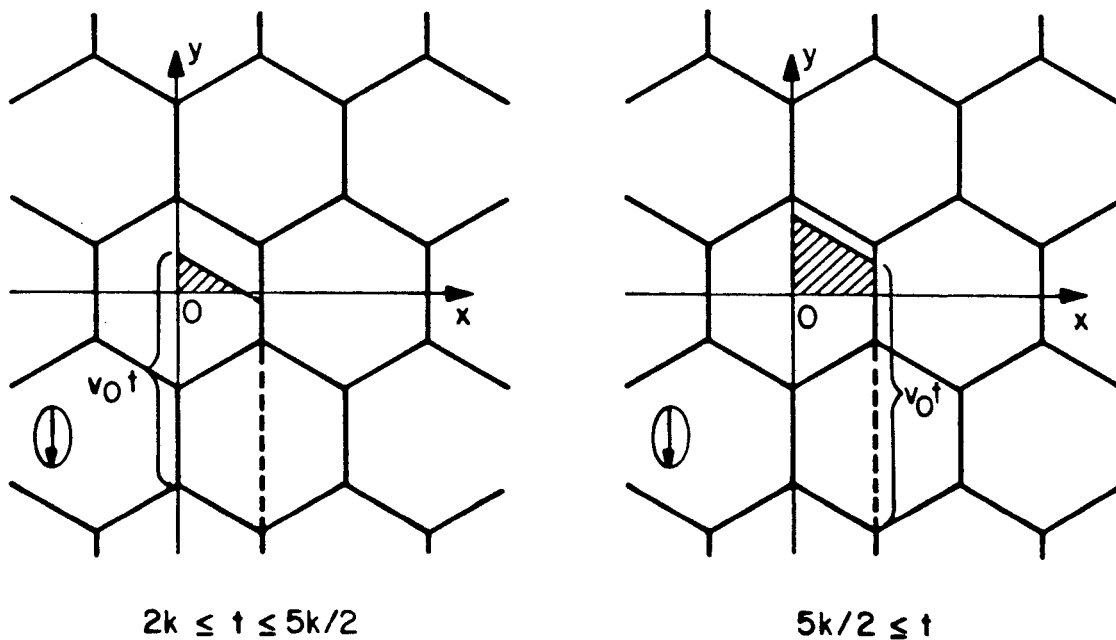
*Fig. 4.3(b). Direction (2).*

*Area at distance less than t away from second boundary crossing.*



*Fig. 4.9(c). Direction (3).*

*Area at distance less than  $t$  away from second boundary crossing.*



*Fig. 4.9(d). Direction (4).*

*Area at distance less than  $t$  away from second boundary crossing.*

As happened for  $n = 1$ , some regions yield the same expression of the probability for the time to the second cell boundary crossing ( $0 \leq t \leq k\frac{\sqrt{3}}{2}$  and  $k\frac{\sqrt{3}}{2} \leq t \leq k$ , for example), making some simplifications possible.

Starting from the above expression we can now derive the probability that a call goes through at least 2 handoffs in a similar fashion as for  $n = 1$  handoff:

$$\begin{aligned}
 P_{\geq 2H} = & \int_0^k \mu e^{-\mu t} \frac{t^2}{18k^2} dt + \int_k^{\sqrt{3}k} \mu e^{-\mu t} \left[ \frac{2}{9k^2} t^2 - \frac{t}{3k} + \frac{1}{6} \right] dt \\
 & + \int_{\sqrt{3}k}^{2k} \mu e^{-\mu t} \left[ \frac{t^2}{6k^2} - \left( \frac{3-\sqrt{3}}{9k} \right) t \right] dt + \int_{2k}^{3k} \mu e^{-\mu t} \left[ \left( \frac{3+\sqrt{3}}{9k} \right) t - \frac{2}{3} \right] dt \\
 & \int_{3k}^{2\sqrt{3}k} \mu e^{-\mu t} \left[ \frac{\sqrt{3}}{9k} t + \frac{1}{3} \right] dt + \int_{2\sqrt{3}k}^{\infty} \mu e^{-\mu t} dt
 \end{aligned}$$

A detailed calculation is again provided in Appendix B and we shall only state the result after replacing  $k\mu$  by  $\alpha$ :

$$P_{\geq 2H}(\alpha) = \frac{1}{9\alpha^2} \left[ 1 + 3e^{-\alpha} - e^{-\sqrt{3}\alpha} - 3e^{-2\alpha} \right] - \frac{1}{9\alpha} \left[ 3e^{-3\alpha} + \sqrt{3}e^{-2\sqrt{3}\alpha} \right]. \quad (4.3)$$

This result can again be checked for the limit cases  $\alpha \rightarrow \infty$  and  $\alpha \rightarrow 0$ . We easily find that we still have

$$\lim_{\alpha \rightarrow \infty} P_{\geq 2H}(\alpha) = 0.$$

If we make a Taylor series expansion of (4.3) around the origin, we get:

$$\begin{aligned}
 P_{\geq 2H}(\alpha) \approx & \frac{1}{9\alpha^2} [1 + 3 - 1 - 3] + \frac{1}{9\alpha} [-3 + \sqrt{3} + 6 - 3 - \sqrt{3}] \\
 & + \frac{1}{9} \left[ \frac{3}{2} - \frac{3}{2} - 6 + 9 + 6 \right] = 1,
 \end{aligned}$$

which gives us the desired result

$$\lim_{\alpha \rightarrow 0} P_{\geq 2H}(\alpha) = 1.$$

We now proceed with the case  $n = 3$ , which will again use the same method. This is the last special case we will work. The purpose of these three examples is not only to illustrate the use of area ratios to compute the different probabilities, but also to provide some checks for the general expression of the probability of  $n$  handoffs.

The different areas that need to be taken into account to compute the different distances to the third boundary crossing are illustrated in Figures 4.4(a), 4.4(b), 4.4(c), 4.4(d), for directions (1), (2), (3), (4) respectively. This again allows us to derive the expressions of the probability distribution of the time to the third cell boundary crossing. In the case of direction (1) we get:

$$\bar{P}_r[T_3 \leq t/(1)] = \begin{cases} 0.0, & 0 \leq t \leq k\sqrt{3}; \\ -4\left(\frac{t}{3k}\right)^2 + \frac{4\sqrt{3}}{3}\left(\frac{t}{k}\right) - \frac{8}{3}, & k\sqrt{3} \leq t \leq k\frac{3\sqrt{3}}{2}; \\ \frac{1}{3}, & k\frac{3\sqrt{3}}{2} \leq t \leq 2\sqrt{3}k; \\ \frac{4\sqrt{3}}{9}\left(\frac{t}{k}\right) - \frac{7}{3}, & 2\sqrt{3}k \leq t \leq k\frac{5\sqrt{3}}{2}; \\ 1.0, & k\frac{5\sqrt{3}}{2} \leq t. \end{cases}$$

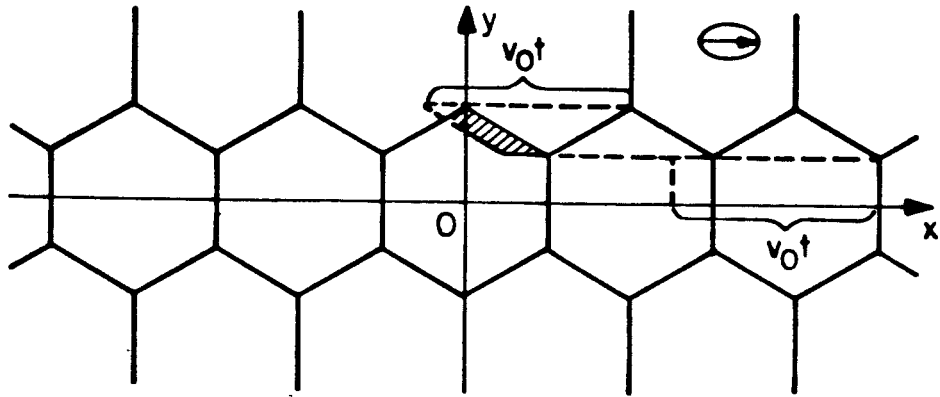
Similarly, for directions (2), (3), and (4) we get:

$$\bar{P}_r[T_3 \leq t/(2)] = \begin{cases} 0.0, & 0 \leq t \leq k\sqrt{3}; \\ 2\left(\frac{t}{3k}\right)^2 - \frac{4\sqrt{3}}{9}\left(\frac{t}{k}\right) + \frac{2}{3}, & k\sqrt{3} \leq t \leq k\frac{3\sqrt{3}}{2}; \\ -2\left(\frac{t}{3k}\right)^2 + \frac{8\sqrt{3}}{9}\left(\frac{t}{k}\right) - \frac{7}{3}, & k\frac{3\sqrt{3}}{2} \leq t \leq 2\sqrt{3}k; \\ \frac{1}{3}, & 2\sqrt{3}k \leq t \leq k\frac{5\sqrt{3}}{2}; \\ \frac{4\sqrt{3}}{9}\left(\frac{t}{k}\right) - 3, & k\frac{5\sqrt{3}}{2} \leq t \leq 3\sqrt{3}k; \\ 1.0, & 3\sqrt{3}k \leq t. \end{cases}$$

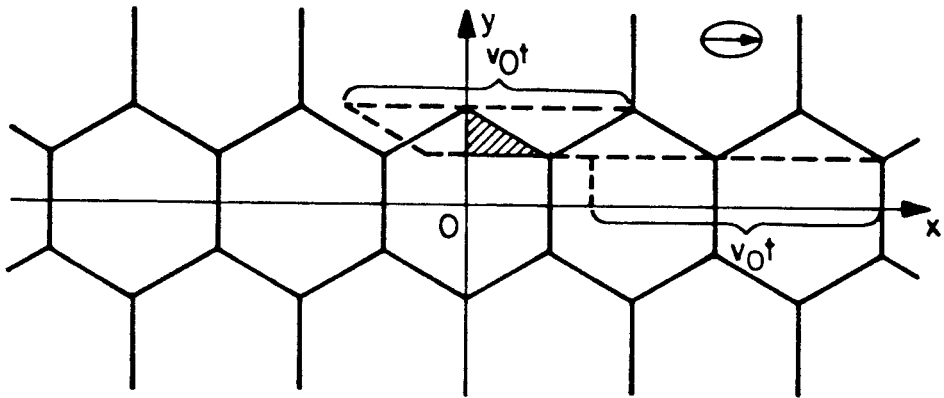
$$\bar{P}_r[T_3 \leq t/(3)] = \begin{cases} 0.0, & 0 \leq t \leq 3k; \\ \frac{4}{3}\left(\frac{t}{k}\right) - 4, & 3k \leq t \leq \frac{7}{2}k; \\ -12\left(\frac{t}{3k}\right)^2 + \frac{32}{3}\left(\frac{t}{k}\right) - \frac{61}{3}, & \frac{7}{2}k \leq t \leq 4k; \\ 1.0, & 4k \leq t. \end{cases}$$

And finally,

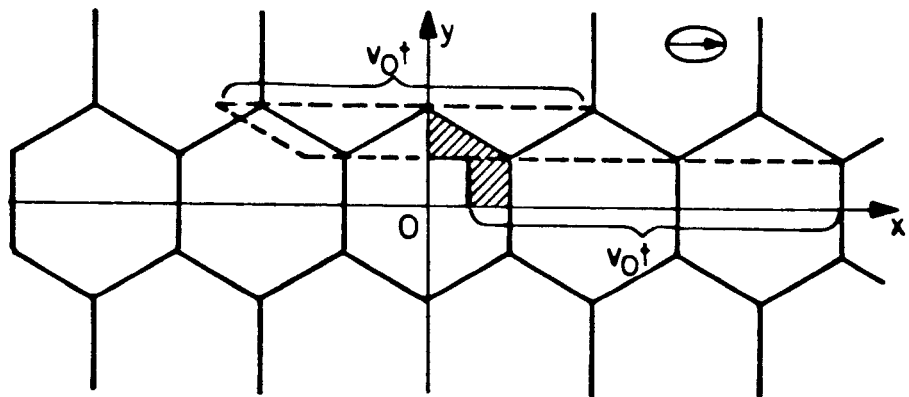
$$\bar{P}_r[T_3 \leq t/(4)] = \begin{cases} 0.0, & 0 \leq t \leq \frac{7}{2}k; \\ 12\left(\frac{t}{3k}\right)^2 - \frac{28}{3}\left(\frac{t}{k}\right) + \frac{49}{3}, & \frac{7}{2}k \leq t \leq 4k; \\ -6\left(\frac{t}{3k}\right)^2 + \frac{20}{3}\left(\frac{t}{k}\right) + \frac{49}{3}, & 4k \leq t \leq 5k; \\ 1.0, & 5k \leq t. \end{cases}$$



$$k\sqrt{3} \leq t \leq 3k\sqrt{3}/2$$



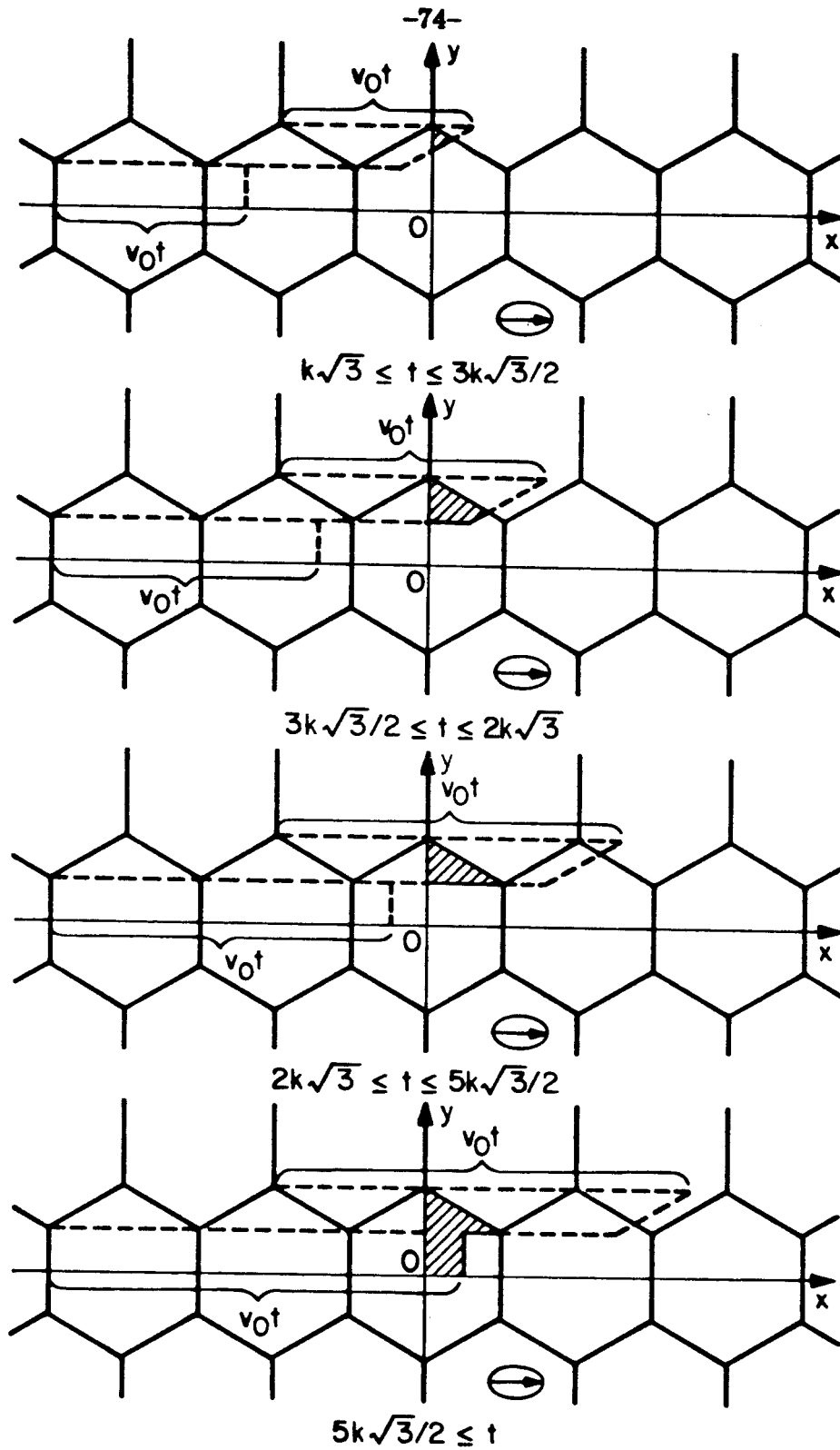
$$3k\sqrt{3}/2 \leq t \leq 2k\sqrt{3}$$



$$2k\sqrt{3} \leq t$$

Fig. 4.4(a). Direction (1).

Area at distance less than  $t$  away from third boundary crossing.



*Fig. 4.4(b). Direction (2).*

*Area at distance less than  $t$  away from third boundary crossing.*

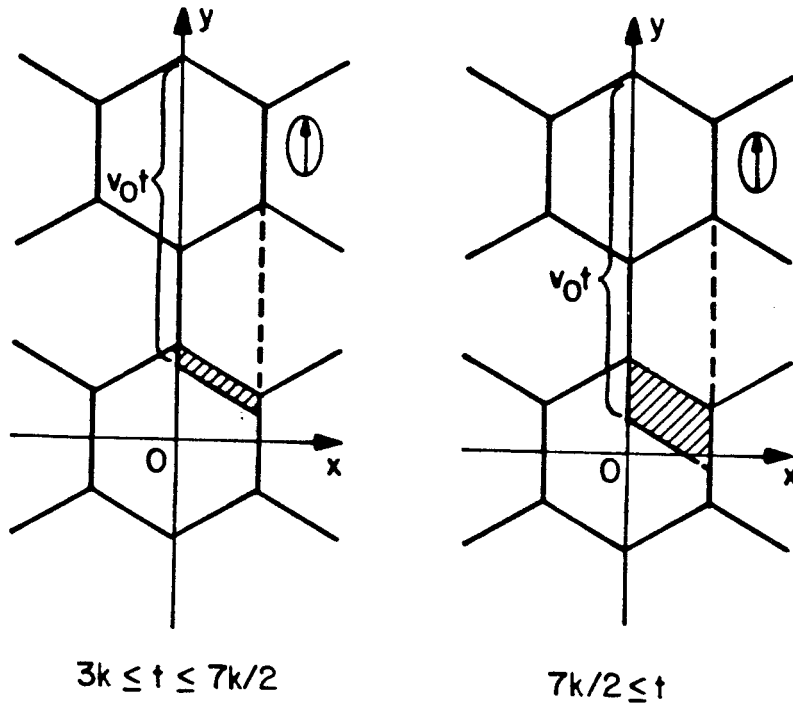


Fig. 4.4(c). Direction (3).

Area at distance less than  $t$  away from third boundary crossing.

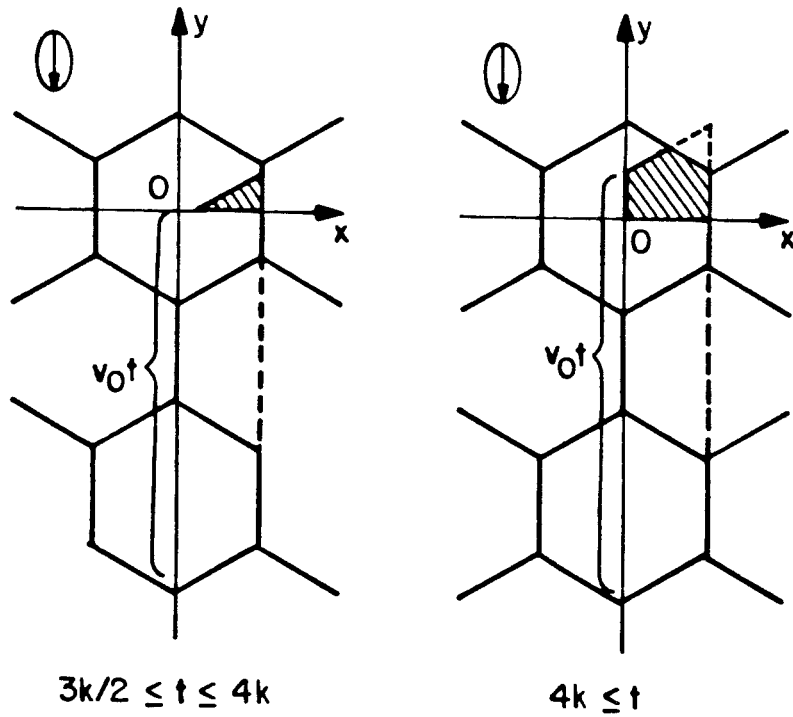


Fig. 4.4(d). Direction (4).

Area at distance less than  $t$  away from third boundary crossing.



Equation (4.1) with  $n = 3$  is now used to obtain the overall probability that the time before the third cell boundary crossing is smaller than  $t$ :

$$Pr[T_3 \leq t] = \begin{cases} 0.0, & 0 \leq t \leq k\sqrt{3}; \\ -\frac{1}{2}\left(\frac{t}{3k}\right)^2 + \frac{2\sqrt{3}}{9}\left(\frac{t}{k}\right) - \frac{1}{2}, & k\sqrt{3} \leq t \leq 3k; \\ -\frac{1}{2}\left(\frac{t}{3k}\right)^2 + \left(\frac{3+2\sqrt{3}}{9}\right)\left(\frac{t}{k}\right) - \frac{3}{2}, & 3k \leq t \leq 2\sqrt{3}k; \\ \left(\frac{3+\sqrt{3}}{9}\right)\left(\frac{t}{k}\right) - \frac{3}{2}, & 2\sqrt{3}k \leq t \leq 4k; \\ -\frac{3}{2}\left(\frac{t}{3k}\right)^2 + \left(\frac{15+\sqrt{3}}{9}\right)\left(\frac{t}{k}\right) - \frac{25}{6}, & 4k \leq t \leq 5k; \\ \frac{\sqrt{3}}{9}\left(\frac{t}{k}\right), & 5k \leq t \leq 3\sqrt{3}k; \\ 1.0, & 3\sqrt{3}k \leq t. \end{cases}$$

Using the same techniques as for the cases  $n = 1$  and  $n = 2$ , we derive the probability that a call goes through at least 3 handoffs:

$$\begin{aligned} P_{\geq 3H} &= \int_{\sqrt{3}k}^{3k} \mu e^{-\mu t} \left[ -\frac{t^2}{18k^2} + \frac{2\sqrt{3}}{9k}t - \frac{1}{2} \right] dt \\ &+ \int_{3k}^{2\sqrt{3}k} \mu e^{-\mu t} \left[ -\frac{t^2}{18k^2} + \left( \frac{3+2\sqrt{3}}{9k} \right) t - \frac{3}{2} \right] dt \\ &+ \int_{2\sqrt{3}k}^{4k} \mu e^{-\mu t} \left[ \left( \frac{3+\sqrt{3}}{9k} \right) t - \frac{3}{2} \right] dt \\ &+ \int_{4k}^{5k} \mu e^{-\mu t} \left[ -\frac{t^2}{6k^2} + \left( \frac{15+\sqrt{3}}{9k} \right) t - \frac{25}{6} \right] dt \\ &+ \int_{5k}^{3\sqrt{3}k} \mu e^{-\mu t} \left[ \frac{\sqrt{3}}{9k}t \right] dt + \int_{3\sqrt{3}k}^{\infty} \mu e^{-\mu t} dt \end{aligned}$$

After some calculations which are also described in Appendix B we get

$$\begin{aligned} P_{\geq 3H}(\alpha) &= \frac{1}{9\alpha^2} \left[ -e^{-\sqrt{3}\alpha} + e^{-2\sqrt{3}\alpha} - 3e^{-4\alpha} + 3e^{-5\alpha} \right] \\ &+ \frac{1}{9\alpha} \left[ \sqrt{3}e^{-\sqrt{3}\alpha} + 3e^{-3\alpha} + \sqrt{3}e^{-2\sqrt{3}\alpha} - \sqrt{3}e^{-3\sqrt{3}\alpha} \right]. \end{aligned} \quad (4.4)$$

We can immediately check that we still have

$$\lim_{\alpha \rightarrow \infty} P_{\geq 3H}(\alpha) = 0.$$

We can also make a Taylor series expansion around the origin to verify

$$\lim_{\alpha \rightarrow 0} P_{\geq 3H}(\alpha) = 1 :$$

$$\begin{aligned}
 P_{\geq 3H}(\alpha) &\approx \frac{1}{9\alpha^2} [-1 + 1 - 3 + 3] \\
 &+ \frac{1}{9\alpha} [\sqrt{3} - 2\sqrt{3} + 12 - 15 + \sqrt{3} + 3 + \sqrt{3} - \sqrt{3}] \\
 &+ \frac{1}{9} \left[ -\frac{3}{9} + 6 - 24 + \frac{75}{2} - 3 - 9 - 6 + 9 \right] = 1.
 \end{aligned}$$

We are now ready to treat the general case of the probability that a call goes through at least  $n$  handoffs ( $n \geq 2$ ). The method used will be slightly different from the one used in the previous examples, in the sense that we will compute the conditional probability of going through  $n$  handoffs for each direction, before deriving the general expression.

### 4.3 General handoff probability.

For each direction we will first obtain the distribution of the time until the  $n^{\text{th}}$  cell boundary crossing. While computing these quantities we will have to distinguish between  $n$  even and  $n$  odd. Once we have obtained this distribution we will immediately compute the probability that a call goes through at least  $n$  handoffs given that it moves in a given direction. After obtaining these four conditional probabilities we will finally get to the unconditioned probability that a call goes through at least  $n$  handoffs. Here again we will have to distinguish between the cases  $n$  even and  $n$  odd.

It is worth noting that since we are restricting the calls to be located in the first quadrant, we will obtain different expression for directions (1) and (2) (similarly for (3) and (4)). This comes from the fact that the restriction to the first quadrant breaks the symmetry of the cell. This symmetry will be reestablished when the general handoff probability is derived. This general expression will be checked using the examples  $n = 2$  and  $n = 3$ .

We are now ready to start with direction (1) using the same technique of area ratios to compute the desired conditional probabilities for the time distribution to the  $n^{\text{th}}$  cell boundary crossing.

**Direction (1):**

$n$  odd:

$$\bar{P}_r[T_n \leq t/(1)] = \begin{cases} 0.0, & 0 \leq t \leq \frac{(n-1)\sqrt{3}}{2}k; \\ -4\left(\frac{t}{3k}\right)^2 + \frac{4n\sqrt{3}}{9}\left(\frac{t}{k}\right) - \left(\frac{n^2-1}{3}\right), & \frac{(n-1)\sqrt{3}}{2}k \leq t \leq \frac{n\sqrt{3}}{2}k; \\ \frac{1}{3}, & \frac{n\sqrt{3}}{2}k \leq t \leq (n-1)\sqrt{3}k; \\ \frac{4\sqrt{3}}{9}\left(\frac{t}{k}\right) - \left(\frac{4n-5}{3}\right), & (n-1)\sqrt{3}k \leq t \leq \frac{(2n-1)\sqrt{3}}{2}k; \\ 1.0, & \frac{(2n-1)\sqrt{3}}{2}k \leq t. \end{cases}$$

$n$  even:

$$\bar{P}_r[T_n \leq t/(1)] = \begin{cases} 0.0, & 0 \leq t \leq \frac{(n-2)\sqrt{3}}{2}k; \\ 2\left(\frac{t}{3k}\right)^2 - \frac{2\sqrt{3}(n-2)}{9}\left(\frac{t}{k}\right) + \frac{(n-2)^2}{6}, & \frac{(n-2)\sqrt{3}}{2}k \leq t \leq \frac{(n-1)\sqrt{3}}{2}k; \\ -2\left(\frac{t}{3k}\right)^2 + \frac{2n\sqrt{3}}{9}\left(\frac{t}{k}\right) - \left(\frac{n^2-2}{6}\right), & \frac{(n-1)\sqrt{3}}{2}k \leq t \leq \frac{n\sqrt{3}}{2}k; \\ \frac{1}{3}, & \frac{n\sqrt{3}}{2}k \leq t \leq (n-1)\sqrt{3}k; \\ \frac{4\sqrt{3}}{9}\left(\frac{t}{k}\right) - \left(\frac{4n-5}{3}\right), & (n-1)\sqrt{3}k \leq t \leq \frac{(2n-1)\sqrt{3}}{2}k; \\ 1.0, & \frac{(2n-1)\sqrt{3}}{2}k \leq t. \end{cases}$$

At this point we diverge from the method used in the examples, in the sense that we immediately compute the probability that a call goes through at least  $n$  handoffs, given that it moves in direction (1) and that it is located in the first quadrant. The technique is, however, exactly the same as in the case of the computation of the overall handoff probability. We simply use the following relation:

$$\bar{P}_{\geq nH/(1)} = \int_0^\infty \bar{P}_r[T_n \leq t/(1)]s(t)dt,$$

where  $s(t)$  is the service time density function. This gives for  $n$  odd:

$$\begin{aligned} \bar{P}_{\geq nH/(1)} &= \int_{(n-1)\frac{\sqrt{3}}{2}k}^{n\frac{\sqrt{3}}{2}k} \mu e^{-\mu t} \left[ -\frac{4}{9k^2}t^2 + \frac{4\sqrt{3}n}{9k}t - \left(\frac{n^2-1}{3}\right) \right] dt \\ &+ \frac{1}{3} \int_{n\frac{\sqrt{3}}{2}k}^{(n-1)\sqrt{3}k} \mu e^{-\mu t} dt + \int_{(n-1)\sqrt{3}k}^{(2n-1)\frac{\sqrt{3}}{2}k} \mu e^{-\mu t} \left[ \frac{4\sqrt{3}}{9k}t - \left(\frac{4n-5}{3}\right) \right] dt \\ &+ \int_{(2n-1)\frac{\sqrt{3}}{2}k}^\infty \mu e^{-\mu t} dt. \end{aligned}$$

The details of the calculations are again provided in Appendix B, and we finally obtain:

$$\begin{aligned}\bar{P}_{\geq nH/(1)}(\alpha) &= \frac{8}{9\alpha^2} \left[ e^{-n\frac{\sqrt{3}}{2}\alpha} - e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} \right] \\ &+ \frac{4\sqrt{3}}{9\alpha} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-(n-1)\sqrt{3}\alpha} - e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \right] \quad (4.5), \quad n \text{ odd}\end{aligned}$$

We can easily check that we have

$$\lim_{\alpha \rightarrow \infty} \bar{P}_{\geq nH/(1)}(\alpha) = 0,$$

as must be. Similarly, for the case where  $\alpha \rightarrow 0$  we make a Taylor series expansion at the origin of (5) and obtain

$$\begin{aligned}\bar{P}_{\geq nH/(1)} &\approx \frac{8}{9\alpha^2} [1 - 1] + \frac{4\sqrt{3}}{9\alpha} [1 + 1 - 1 - n + (n - 1)] \\ &+ \left[ \frac{n^2}{3} - \frac{(n-1)^2}{3} - \frac{2(n-1)}{3} - \frac{4(n-1)}{3} + \frac{2(2n-1)}{3} \right] \\ &= \frac{1}{3} [n^2 - n^2 + 2n - 1 - 2n + 2 - 4n + 4n - 2] = 1,\end{aligned}$$

as again must be.

Similar calculations can be made for the case  $n$  even (see Appendix B), and we obtain

$$\begin{aligned}\bar{P}_{\geq nH/(1)}(\alpha) &= \frac{4}{9\alpha^2} \left[ e^{-n\frac{\sqrt{3}}{2}\alpha} - 2e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-(n-2)\frac{\sqrt{3}}{2}\alpha} \right] \\ &+ \frac{4\sqrt{3}}{9\alpha} \left[ e^{-(n-1)\sqrt{3}\alpha} - e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \right]. \quad (4.6), \quad n \text{ even}\end{aligned}$$

This result again gives

$$\lim_{\alpha \rightarrow \infty} \bar{P}_{\geq nH/(1)}(\alpha) = 0,$$

as must be. For the case  $\alpha \rightarrow 0$ , a Taylor series expansion gives

$$\begin{aligned}\bar{P}_{\geq nH/(1)} &\approx \frac{4}{9\alpha^2} [1 - 2 + 1] + \frac{4\sqrt{3}}{9\alpha} \left[ \frac{(n-2)}{2} + (n-1) - \frac{n}{2} \right] \\ &+ \frac{1}{6} [(n-2)^2 - 2(n-1)^2 + n^2 - 8(n-1) + 4(2n-1)] \\ &= \frac{1}{6} [n^2 - 4n + 4 - 2n^2 + 4n - 2 + n^2 - 8n + 8 + 8n - 4] = 1,\end{aligned}$$

which provides, as before, the desired result.

We are now ready to start with direction (2).

**Direction (2):**

$n$  odd:

$$\tilde{P}_r[T_n \leq t/(2)] = \begin{cases} 0.0, & 0 \leq t \leq \frac{(n-1)\sqrt{3}}{2}k; \\ 2\left(\frac{t}{3k}\right)^2 - \frac{2\sqrt{3}(n-1)}{9}\left(\frac{t}{k}\right) + \frac{(n-1)^2}{6}, & \frac{(n-1)\sqrt{3}}{2}k \leq t \leq \frac{n\sqrt{3}}{2}k; \\ -2\left(\frac{t}{3k}\right)^2 + \frac{2\sqrt{3}(n+1)}{9}\left(\frac{t}{k}\right) - \frac{(n+1)^2-2}{6}, & \frac{n\sqrt{3}}{2}k \leq t \leq \frac{(n+1)\sqrt{3}}{2}k; \\ \frac{1}{3}, & \frac{(n+1)\sqrt{3}}{2}k \leq t \leq \frac{(2n-1)\sqrt{3}}{2}k; \\ \frac{4\sqrt{3}}{9}\left(\frac{t}{k}\right) - \frac{(4n-3)}{3}, & \frac{(2n-1)\sqrt{3}}{2}k \leq t \leq n\sqrt{3}k; \\ 1.0, & n\sqrt{3}k \leq t. \end{cases}$$

$n$  even:

$$\tilde{P}_r[T_n \leq t/(2)] = \begin{cases} 0.0, & 0 \leq t \leq \frac{(n-1)\sqrt{3}}{2}k; \\ 4\left(\frac{t}{3k}\right)^2 - \frac{4\sqrt{3}(n-1)}{9}\left(\frac{t}{k}\right) + \frac{(n-1)^2}{3}, & \frac{(n-1)\sqrt{3}}{2}k \leq t \leq \frac{n\sqrt{3}}{2}k; \\ \frac{1}{3}, & \frac{n\sqrt{3}}{2}k \leq t \leq \frac{(2n-1)\sqrt{3}}{2}k; \\ \frac{4\sqrt{3}}{9}\left(\frac{t}{k}\right) - \frac{(4n-3)}{3}, & \frac{(2n-1)\sqrt{3}}{2}k \leq t \leq n\sqrt{3}k; \\ 1.0, & n\sqrt{3}k \leq t. \end{cases}$$

Using a similar procedure as for direction (1) (also described in Appendix B), we obtain, in the case  $n$  odd, the following expression for the probability that a call goes through at least  $n$  handoffs given that it moves in direction (2) and is located in the first quadrant:

$$\begin{aligned} \tilde{P}_{\geq nH/(2)}(\alpha) &= \frac{4}{9\alpha^2} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} - 2e^{-n\frac{\sqrt{3}}{2}\alpha} + e^{-(n+1)\frac{\sqrt{3}}{2}\alpha} \right] \\ &\quad + \frac{4\sqrt{3}}{9\alpha} \left[ e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} - e^{-n\sqrt{3}\alpha} \right]. \end{aligned} \quad (4.7), \text{ } n \text{ odd}$$

This result can again be checked for the limit values  $\alpha \rightarrow \infty$  and  $\alpha \rightarrow 0$ . We immediately find 0 as the limit when  $\alpha \rightarrow \infty$ ; for  $\alpha \rightarrow 0$  we need, as before, a Taylor

series expansion around the origin:

$$\begin{aligned}\bar{P}_{\geq nH/(2)}(\alpha) &= \frac{4}{9\alpha^2}[1 - 2 + 1] + \frac{4\sqrt{3}}{9\alpha} \left[ -\frac{(n-1)}{2} + n - \frac{(n-1)}{2} + 1 - 1 \right] \\ &+ \frac{1}{6} \left[ (n-1)^2 - 2n^2 + (n+1)^2 - 4(2n-1) + 8n \right] \\ &= \frac{1}{6} \left[ n^2 - 2n + 1 - 2n^2 + n^2 + 2n + 1 - 8n + 4 + 8n \right] = 1,\end{aligned}$$

which yields the desired result.

Similarly, for the case  $n$  even (in Appendix B), we get the following expression for the probability of at least  $n$  handoffs given that the mobile moves in direction (2) and is located in the first quadrant:

$$\begin{aligned}\bar{P}_{\geq nH/(2)}(\alpha) &= \frac{8}{9\alpha^2} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} - e^{-n\frac{\sqrt{3}}{2}\alpha} \right] \\ &- \frac{4\sqrt{3}}{9\alpha} \left[ e^{-n\frac{\sqrt{3}}{2}\alpha} - e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-n\sqrt{3}\alpha} \right]. \quad (4.8), \quad n \text{ even}\end{aligned}$$

We again easily check that we have

$$\lim_{\alpha \rightarrow \infty} \bar{P}_{\geq nH/(2)}(\alpha) = 0,$$

while a Taylor series at the origin gives

$$\begin{aligned}\bar{P}_{\geq nH/(2)}(\alpha) &= \frac{8}{9\alpha^2}[1 - 1] + \frac{4\sqrt{3}}{9\alpha} [-(n-1) + n - 1 + 1 - 1] \\ &+ \frac{1}{3} \left[ (n-1)^2 - n^2 + 2n - 2(2n-1) + 4n \right] \\ &= \frac{1}{3} \left[ n^2 - 2n + 1 - n^2 + 2n - 4n + 2 + 4n \right] = 1,\end{aligned}$$

the correct limit when  $\alpha \rightarrow 0$ .

**Direction (3):**

The case of direction (3) is treated in complete parallelism with the two previous ones, and we get:

$n$  odd:

$$\bar{P}_r[T_n \leq t/(3)] = \begin{cases} 0.0, & 0 \leq t \leq \frac{(3n-3)}{2}k; \\ \frac{4}{3}\left(\frac{t}{k}\right) - 2(n-1), & \frac{(3n-3)}{2}k \leq t \leq \frac{(3n-2)}{2}k; \\ -12\left(\frac{t}{3k}\right)^2 + \frac{4(3n-1)}{3}\left(\frac{t}{k}\right) - \frac{(3n-1)^2}{3} + 1, & \frac{(3n-2)}{2}k \leq t \leq \frac{(3n-1)}{2}k; \\ 1.0, & \frac{(3n-1)}{2}k \leq t. \end{cases}$$

$n$  even:

$$\bar{P}_r[T_n \leq t/(3)] = \begin{cases} 0.0, & 0 \leq t \leq \frac{(3n-4)}{2}k; \\ 6\left(\frac{t}{3k}\right)^2 - \frac{2(3n-4)}{3}\left(\frac{t}{k}\right) + \frac{(3n-4)^2}{6}, & \frac{(3n-4)}{2}k \leq t \leq \frac{(3n-2)}{2}k; \\ -12\left(\frac{t}{3k}\right)^2 + \frac{4(3n-1)}{3}\left(\frac{t}{k}\right) - \frac{(3n-1)^2}{3} + 1, & \frac{(3n-2)}{2}k \leq t \leq \frac{(3n-1)}{2}k; \\ 1.0, & \frac{(3n-1)}{2}k \leq t. \end{cases}$$

From the above expressions we can again compute (in Appendix B) the probability that a call goes through  $n$  handoffs given that the associated mobile moves in direction (3) and that it is located in the first quadrant. Let us start with the case  $n$  odd:

$$\bar{P}_{\geq nH/(3)}(\alpha) = \frac{8}{3\alpha^2} \left[ e^{-\frac{(3n-1)}{2}\alpha} - e^{-\frac{(3n-2)}{2}\alpha} \right] + \frac{4}{3\alpha} \left[ e^{-\frac{(3n-3)}{2}\alpha} \right]. \quad (4.9), \text{ } n \text{ odd}$$

The same checks as before are again used and we find that

$$\lim_{\alpha \rightarrow \infty} \bar{P}_{\geq nH/(3)}(\alpha) = 0,$$

as must be; if  $\alpha \rightarrow 0$ , we get,

$$\begin{aligned} \bar{P}_{\geq nH/(3)}(\alpha) &\approx \frac{8}{3\alpha^2} [1 - 1] + \frac{4}{3\alpha} [-(3n-1) + (3n-2) + 1] \\ &\quad + \frac{1}{3} [(3n-1)^2 - (3n-2)^2 - 2(3n-3)] \\ &= \frac{1}{3} [9n^2 - 6n + 1 - 9n^2 + 12n - 4 - 6n + 6] = 1, \end{aligned}$$

which gives the proper limit.

The case  $n$  even yields some very similar expressions:

$$\tilde{P}_{\geq nH/(3)}(\alpha) = \frac{4}{3\alpha^2} \left[ e^{-\frac{(3n-4)}{2}\alpha} - 3e^{-\frac{(3n-2)}{2}\alpha} + 2e^{-\frac{(3n-1)}{2}\alpha} \right]. \quad (4.10), \quad n \text{ even}$$

The above probability also goes to 0 when  $\alpha \rightarrow \infty$ ; for  $\alpha \rightarrow 0$  we get:

$$\begin{aligned} \tilde{P}_{\geq nH/(3)}(\alpha) &\approx \frac{4}{3\alpha^2} [1 - 3 + 2] + \frac{2}{3\alpha} [-(3n-4) + 3(3n-2) - 2(3n-1)] \\ &\quad + \frac{1}{6} [(3n-4)^2 - 3(3n-2)^2 + 2(3n-1)^2] \\ &= \frac{1}{6} [9n^2 - 24n + 16 - 27n^2 + 36n - 12 + 18n^2 - 12n + 2] = 1, \end{aligned}$$

which is again the desired result.

#### Direction (4):

Finally we treat the case of direction (4), obtaining the probability distribution for the time to the  $n^{\text{th}}$  cell boundary crossing for  $n$  odd and even:

$n$  odd:

$$\tilde{P}_r[T_n \leq t/(4)] = \begin{cases} 0.0, & 0 \leq t \leq \frac{(3n-2)}{2}k; \\ 12\left(\frac{t}{3k}\right)^2 - \frac{4(3n-2)}{3}\left(\frac{t}{k}\right) + \frac{(3n-2)^2}{3}, & \frac{(3n-2)}{2}k \leq t \leq \frac{(3n-1)}{2}k; \\ -6\left(\frac{t}{3k}\right)^2 + \frac{2(3n+1)}{3}\left(\frac{t}{k}\right) - \frac{(3n+1)^2}{6} + 1, & \frac{(3n-1)}{2}k \leq t \leq \frac{(3n+1)}{2}k; \\ 1.0, & \frac{(3n+1)}{2}k \leq t. \end{cases}$$

$n$  even:

$$\tilde{P}_r[T_n \leq t/(4)] = \begin{cases} 0.0, & 0 \leq t \leq \frac{(3n-2)}{2}k; \\ 12\left(\frac{t}{3k}\right)^2 - \frac{4(3n-2)}{3}\left(\frac{t}{k}\right) + \frac{(3n-2)^2}{3}, & \frac{(3n-2)}{2}k \leq t \leq \frac{(3n-1)}{2}k; \\ \frac{4}{3}\left(\frac{t}{k}\right) - (2n-1), & \frac{(3n-1)}{2}k \leq t \leq \frac{3n}{2}k; \\ 1.0, & \frac{3n}{2}k \leq t. \end{cases}$$

From this distribution we again derive (in Appendix B) the probability that a call goes through at least  $n$  handoffs given that the associated mobile moves in direction (4) and is located in the first quadrant. For  $n$  odd we get:

$$\tilde{P}_{\geq nH/(4)}(\alpha) = \frac{4}{3\alpha^2} \left[ 2e^{-\frac{(3n-2)}{2}\alpha} - 3e^{-\frac{(3n-1)}{2}\alpha} + e^{-\frac{(3n+1)}{2}\alpha} \right]. \quad (4.11), \quad n \text{ odd}$$



As usual the above probability also goes to 0 when  $\alpha \rightarrow \infty$ ; and for  $\alpha \rightarrow 0$  we get:

$$\begin{aligned}\tilde{P}_{\geq nH/(4)}(\alpha) &\approx \frac{4}{3\alpha^2}[2 - 3 + 1] + \frac{2}{3\alpha}[-2(3n - 2) + 3(3n - 1) - (3n + 1)] \\ &+ \frac{1}{6}[2(3n - 2)^2 - 3(3n - 1)^2 + 2(3n + 1)^2] \\ &= \frac{1}{6}[18n^2 - 24n + 8 - 27n^2 + 18n - 3 + 9n^2 + 6n + 1] = 1,\end{aligned}$$

which is again correct.

The same process is repeated for  $n$  even and we also obtain the probability that a call goes through at least  $n$  handoffs when the associated mobile is in the first quadrant and moves in direction (4):

$$\tilde{P}_{\geq nH/(4)}(\alpha) = \frac{8}{3\alpha^2} \left[ e^{-\frac{(3n-2)}{2}\alpha} - e^{-\frac{(3n-1)}{2}\alpha} \right] - \frac{4}{3\alpha} \left[ e^{-\frac{3n}{2}\alpha} \right]. \quad (4.12), \text{ } n \text{ even}$$

As before we again find that

$$\lim_{\alpha \rightarrow \infty} \tilde{P}_{\geq nH/(4)}(\alpha) = 0,$$

as must be. If  $\alpha \rightarrow 0$ , we get

$$\begin{aligned}\tilde{P}_{\geq nH/(4)}(\alpha) &= \frac{8}{3\alpha^2}[1 - 1] + \frac{4}{3\alpha}[-(3n - 2) + (3n - 1) - 1] \\ &+ \frac{1}{3}[(3n - 2)^2 - (3n - 1)^2 + 6n] \\ &= \frac{1}{3}[9n^2 - 12n + 4 - 9n^2 + 6n - 1 + 6n] = 1,\end{aligned}$$

which is the desired limit.

We are now in a position to compute the overall probability for a call to be handed off at least  $n$  times, with a condition neither on its location nor on its direction. This probability will be computed using the arithmetic mean:

$$\begin{aligned}P_{\geq nH}(\alpha) &= \frac{1}{4} \left( \tilde{P}_{\geq nH/(1)}(\alpha) + \tilde{P}_{\geq nH/(2)}(\alpha) \right. \\ &\left. + \tilde{P}_{\geq nH/(3)}(\alpha) + \tilde{P}_{\geq nH/(4)}(\alpha) \right). \quad (4.13)\end{aligned}$$

Let us start with the case  $n$  odd by replacing the different conditional probabilities in (4.13) by the expressions we have just evaluated:

$$\begin{aligned}
 P_{\geq nH}(\alpha) &= \frac{2}{9\alpha^2} \left[ e^{-n\frac{\sqrt{3}}{2}\alpha} - e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} \right] \\
 &+ \frac{\sqrt{3}}{9\alpha} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-(n-1)\sqrt{3}\alpha} - e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \right] \\
 &+ \frac{1}{9\alpha^2} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} - 2e^{-n\frac{\sqrt{3}}{2}\alpha} + e^{-(n+1)\frac{\sqrt{3}}{2}\alpha} \right] \\
 &+ \frac{\sqrt{3}}{9\alpha} \left[ e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} - e^{-n\sqrt{3}\alpha} \right] + \frac{2}{3\alpha^2} \left[ e^{-\frac{(3n-1)}{2}\alpha} - e^{-\frac{(3n-2)}{2}\alpha} \right] \\
 &+ \frac{1}{3\alpha} \left[ e^{-\frac{(3n-3)}{2}\alpha} \right] + \frac{1}{3\alpha^2} \left[ 2e^{-\frac{(3n-2)}{2}\alpha} - 3e^{-\frac{(3n-1)}{2}\alpha} + e^{-\frac{(3n+1)}{2}\alpha} \right]
 \end{aligned}$$

This finally gives

$$\begin{aligned}
 P_{\geq nH}(\alpha) &= \frac{1}{9\alpha^2} \left[ e^{-(n+1)\frac{\sqrt{3}}{2}\alpha} - e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + 3e^{-\frac{(3n+1)}{2}\alpha} - 3e^{-\frac{(3n-1)}{2}\alpha} \right] \\
 &+ \frac{\sqrt{3}}{9\alpha} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-(n-1)\sqrt{3}\alpha} - e^{-n\sqrt{3}\alpha} + \sqrt{3}e^{-\frac{(3n-3)}{2}\alpha} \right].
 \end{aligned}$$

(4.14),  $n$  odd

If, as a check, we evaluate the above expression for  $n = 3$ , we get:

$$\begin{aligned}
 P_{\geq 3H}(\alpha) &= \frac{1}{9\alpha^2} \left[ e^{-2\sqrt{3}\alpha} - e^{-\sqrt{3}\alpha} + 3e^{-5\alpha} - 3e^{-4\alpha} \right] \\
 &+ \frac{\sqrt{3}}{9\alpha} \left[ e^{-\sqrt{3}\alpha} + e^{-2\sqrt{3}\alpha} - e^{-3\sqrt{3}\alpha} + \sqrt{3}e^{-3\alpha} \right],
 \end{aligned}$$

which agrees with Equation (4.4).

Note that if we evaluate the case  $n = 1$ , the expression we get is identical to Equation (4.2). Therefore, even though some intermediate expressions are only defined for  $n$  greater than 3, the final expression turns out to be valid for all odd  $n$ . This is due to the fact that all the terms that disappear when we set  $n = 1$  are already equal to 0 in the general formulas.

We can now similarly treat the case  $n$  even ( $n > 0$ ):

$$\begin{aligned}
 P_{\geq nH}(\alpha) &= \frac{1}{9\alpha^2} \left[ e^{-(n-2)\frac{\sqrt{3}}{2}\alpha} - 2e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-n\frac{\sqrt{3}}{2}\alpha} \right] \\
 &+ \frac{\sqrt{3}}{9\alpha} \left[ e^{-(n-1)\sqrt{3}\alpha} - e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \right] + \frac{2}{9\alpha^2} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} - 2e^{-n\frac{\sqrt{3}}{2}\alpha} \right] \\
 &- \frac{\sqrt{3}}{9\alpha} \left[ e^{-n\frac{\sqrt{3}}{2}\alpha} - e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} - e^{-n\sqrt{3}\alpha} \right] \\
 &+ \frac{1}{3\alpha^2} \left[ e^{-\frac{(3n-4)}{2}\alpha} - 3e^{-\frac{(3n-2)}{2}\alpha} + 2e^{-\frac{(3n-1)}{2}\alpha} \right] \\
 &+ \frac{2}{3\alpha^2} \left[ e^{-\frac{(3n-2)}{2}\alpha} - e^{-\frac{(3n-1)}{2}\alpha} \right] - \frac{1}{3\alpha} \left[ e^{-\frac{3n}{2}\alpha} \right]
 \end{aligned}$$

This gives:

$$\begin{aligned}
 P_{\geq nH}(\alpha) &= \frac{1}{9\alpha^2} \left[ e^{-(n-2)\frac{\sqrt{3}}{2}\alpha} - e^{-n\frac{\sqrt{3}}{2}\alpha} + 3e^{-\frac{(3n-4)}{2}\alpha} - 3e^{-\frac{(3n-2)}{2}\alpha} \right]; \\
 &+ \frac{\sqrt{3}}{9\alpha} \left[ e^{-(n-1)\sqrt{3}\alpha} - e^{-n\frac{\sqrt{3}}{2}\alpha} - e^{-n\sqrt{3}\alpha} - \sqrt{3}e^{-\frac{3n}{2}\alpha} \right]. \quad (4.15), \quad n \text{ even}
 \end{aligned}$$

Similarly, if we set  $n = 2$  in the above expression, we get:

$$P_{\geq 2H}(\alpha) = \frac{1}{9\alpha^2} \left[ 1 - e^{-\sqrt{3}\alpha} + 3e^{-\alpha} - 3e^{-2\alpha} \right] - \frac{\sqrt{3}}{9\alpha} \left[ e^{-2\sqrt{3}\alpha} + \sqrt{3}e^{-3\alpha} \right],$$

which checks with the expression given by Equation (4.3).

At this point let us summarize the results we have obtained so far. We have been able to derive in closed form the probability that a call goes through at least  $n$  handoffs for any value of  $n$ . This was achieved by computing some intermediate conditional probabilities of handoff, the probabilities that a call goes through at least  $n$  handoffs given that the associated mobile was moving in a given direction and was located in the first quadrant. For completeness, the conditional probabilities that a call goes through at least  $n$  handoffs, given only the direction of motion of the associated vehicle without any restriction on its location, are presented in Appendix B.

We now derive the probability that a call goes through exactly  $n$  handoffs. From this, or another but simpler method, we find the average number of handoffs for a random call. For both  $n$  odd and  $n$  even the following relation holds true:

$$P_{nH}(\alpha) = P_{\geq nH}(\alpha) - P_{\geq (n+1)H}(\alpha)$$

In the case  $n$  odd,  $(n + 1)$  will be even, so by using Equations (4.14) and (4.15) in the above expression we get:

$$\begin{aligned} P_{nH}(\alpha) = & \frac{1}{9\alpha^2} \left[ e^{-(n+1)\frac{\sqrt{3}}{2}\alpha} - e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + 3e^{-\frac{(3n+1)}{2}\alpha} - 3e^{-\frac{(3n-1)}{2}\alpha} \right. \\ & \left. + e^{-(n+1)\frac{\sqrt{3}}{2}\alpha} - e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + 3e^{-\frac{(3n+1)}{2}\alpha} - 3e^{-\frac{(3n-1)}{2}\alpha} \right] \\ & + \frac{\sqrt{3}}{9\alpha} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-(n-1)\sqrt{3}\alpha} - e^{-n\sqrt{3}\alpha} + \sqrt{3}e^{-\frac{(3n-3)}{2}\alpha} \right. \\ & \left. + e^{-(n+1)\frac{\sqrt{3}}{2}\alpha} + e^{-(n+1)\sqrt{3}\alpha} - e^{-n\sqrt{3}\alpha} + \sqrt{3}e^{-\frac{(3n+3)}{2}\alpha} \right]. \end{aligned}$$

If we now let  $n = 2m + 1$ , we get:

$$\begin{aligned} P_{(2m+1)H}(\alpha) = & \frac{2}{9\alpha^2} \left[ e^{-\sqrt{3}m\alpha} \left( e^{-\sqrt{3}\alpha} - 1 \right) + 3e^{-3m\alpha} \left( e^{-2\alpha} - e^{-\alpha} \right) \right] \\ & + \frac{\sqrt{3}}{9\alpha} \left[ e^{-\sqrt{3}m\alpha} \left( 1 + e^{-\sqrt{3}\alpha} \right) + e^{-2\sqrt{3}m\alpha} \left( 1 + e^{-2\sqrt{3}\alpha} - 2e^{-\sqrt{3}\alpha} \right) \right. \\ & \left. + \sqrt{3}e^{-3m\alpha} \left( 1 + e^{-3\alpha} \right) \right] \end{aligned}$$

This finally gives:

$$\begin{aligned} P_{(2m+1)H}(\alpha) = & e^{-\sqrt{3}m\alpha} \left[ -\frac{2}{9\alpha^2} \left( 1 - e^{-\sqrt{3}\alpha} \right) + \frac{\sqrt{3}}{9\alpha} \left( 1 + e^{-\sqrt{3}\alpha} \right) \right] \quad (4.16) \\ & + e^{-2\sqrt{3}m\alpha} \left[ \frac{\sqrt{3}}{9\alpha} \left( 1 - e^{-\sqrt{3}\alpha} \right)^2 \right] + e^{-3m\alpha} \left[ -\frac{2}{3\alpha^2} \left( e^{-\alpha} - e^{-2\alpha} \right) \right]. \end{aligned}$$

In the case of  $n$  even,  $(n + 1)$  is now odd and we must reverse the order in which

we use Equations (4.14) and (4.15), which yields:

$$\begin{aligned}
 P_{nH}(\alpha) = & \frac{1}{9\alpha^2} \left[ e^{-(n-2)\frac{\sqrt{3}}{2}\alpha} - e^{-n\frac{\sqrt{3}}{2}\alpha} + 3e^{-\frac{(3n-4)}{2}\alpha} - 3e^{-\frac{(3n-2)}{2}\alpha} \right. \\
 & \left. - e^{-(n+2)\frac{\sqrt{3}}{2}\alpha} + e^{-n\frac{\sqrt{3}}{2}\alpha} - 3e^{-\frac{(3n+4)}{2}\alpha} + 3e^{-\frac{(3n+2)}{2}\alpha} \right] \\
 & + \frac{\sqrt{3}}{9\alpha} \left[ -e^{-n\frac{\sqrt{3}}{2}\alpha} + e^{-(n-1)\sqrt{3}\alpha} - e^{-n\sqrt{3}\alpha} - \sqrt{3}e^{-\frac{3n}{2}\alpha} \right. \\
 & \left. - e^{-n\frac{\sqrt{3}}{2}\alpha} + e^{-(n+1)\sqrt{3}\alpha} - e^{-n\sqrt{3}\alpha} - \sqrt{3}e^{-\frac{3n}{2}\alpha} \right].
 \end{aligned}$$

After letting  $n = 2m$  ( $m > 0$ ), we get:

$$\begin{aligned}
 P_{2mH}(\alpha) = & \frac{1}{9\alpha^2} \left[ e^{-\sqrt{3}m\alpha} \left( e^{\sqrt{3}\alpha} - e^{-\sqrt{3}\alpha} \right) + 3e^{-3m\alpha} \left( e^{2\alpha} - e^{-2\alpha} - e^{\alpha} + e^{-\alpha} \right) \right] \\
 & + \frac{\sqrt{3}}{9\alpha} \left[ -2e^{-\sqrt{3}m\alpha} + e^{-2\sqrt{3}m\alpha} \left( e^{\sqrt{3}\alpha} + e^{-\sqrt{3}\alpha} \right) \right. \\
 & \left. - 2e^{-2\sqrt{3}m\alpha} - 2\sqrt{3}e^{-3m\alpha} \right].
 \end{aligned}$$

This can be rewritten as:

$$\begin{aligned}
 P_{2mH}(\alpha) = & e^{-\sqrt{3}m\alpha} \left[ \frac{1}{9\alpha^2} \left( e^{\sqrt{3}\alpha} - e^{-\sqrt{3}\alpha} \right) - \frac{2\sqrt{3}}{9\alpha} \right] \\
 & + e^{-2\sqrt{3}m\alpha} \left[ \frac{\sqrt{3}}{9\alpha} \left( e^{\sqrt{3}\alpha} + e^{-\sqrt{3}\alpha} - 2 \right) \right] \\
 & + e^{-3m\alpha} \left[ \frac{1}{3\alpha^2} \left( e^{2\alpha} - e^{-2\alpha} - e^{\alpha} + e^{-\alpha} \right) - \frac{2}{3\alpha} \right]. \quad (4.17)
 \end{aligned}$$

We have derived the exact distribution of the number of handoffs.

Note, as already pointed out in Chapter 3, that the parameters  $R$ ,  $v_0$ , and  $\mu$  have a related effect on the calls present in the system. In all the expressions we have obtained so far, the three parameters are present only through the parameter  $\alpha$ , which, as we recall, is given by:

$$\alpha = \frac{R\mu}{v_0}.$$

We now present some results on the probability that a call goes through a certain number of handoffs. Figures 4.7 and 4.8 show the probability of at least  $n$  handoffs for the cases  $n = 1$  to 6 as a function of  $\alpha$ . Note that the model becomes rather unrealistic for large values of  $n$  since a large number of handoffs will, in a real-life system, presumably imply large service times. Under our assumptions, however, this implies a constant direction of motion for the mobile during an unrealistically large time interval. It is therefore reassuring to find that even for small values of  $\alpha$ , that is, as we shall see, a large average number of handoffs, the probability of at least  $n$  handoffs still decreases rapidly when  $n$  increases. This means that the part where the model is most questionable (large service times) does not have too important a contribution to the overall system behavior.

Figure 4.9 illustrates the fact that the assumed hexagonal shape for the cells (not completely realistic) does not introduce too big a distinction between the handoff probabilities given that the mobile moves in direction  $x$  or  $y$ , except for extremely small values of the handoff probability. This tells that the handoff probability is not too sensitive to the cell shape and hence makes the hexagonal cell assumption used in the model acceptable in the real world of irregular cells.

#### 4.4 Average number of handoffs.

We could now compute the average number  $\bar{h}$  of handoffs experienced by a random call from the above distribution, and we will do so. But before doing this, let us derive  $\bar{h}$  in a simpler way. What kind of expression might we obtain for  $\bar{h}$  in terms of  $\alpha$ ?

Recall that the cell boundary crossings and the call duration are independent; therefore, the average number of handoffs is equal to the average call duration times the average number of cell boundary crossings experienced per unit time by a

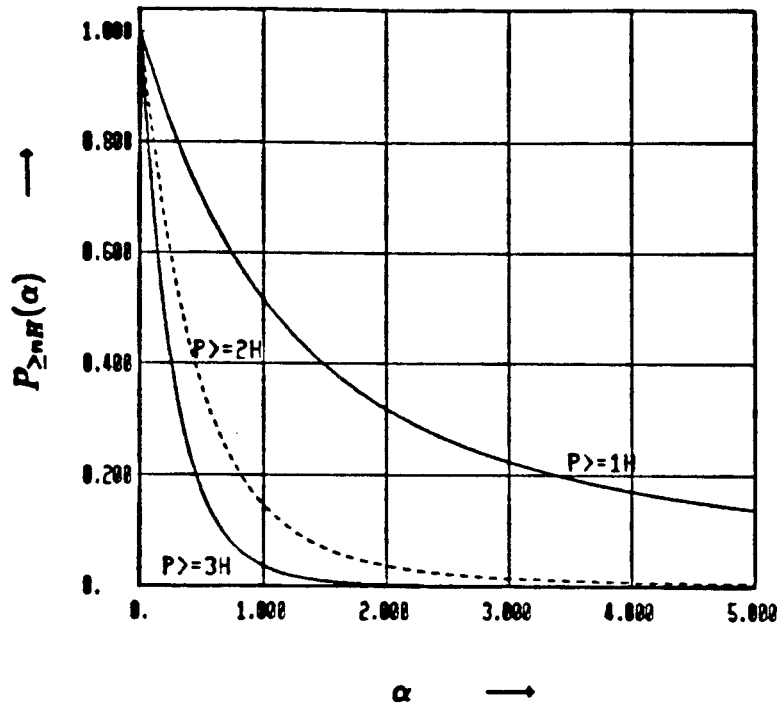


Fig. 4.5. Probability of  $n = 1, 2, 3$ , handoffs as a function of  $\alpha$ .

$\alpha = R\mu/v_0$ ,  $R$ : cell radius,  $v_0$ : average mobile speed,  $\mu$ : service rate.

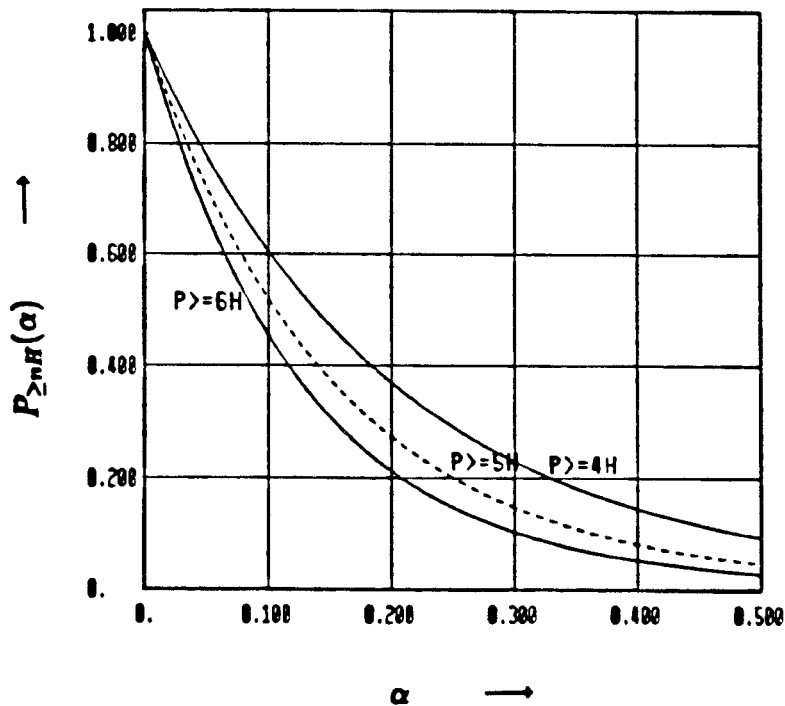


Fig. 4.6. Probability of  $n = 4, 5, 6$ , handoffs as a function of  $\alpha$ .

$\alpha = R\mu/v_0$ ,  $R$ : cell radius,  $v_0$ : average mobile speed,  $\mu$ : service rate.

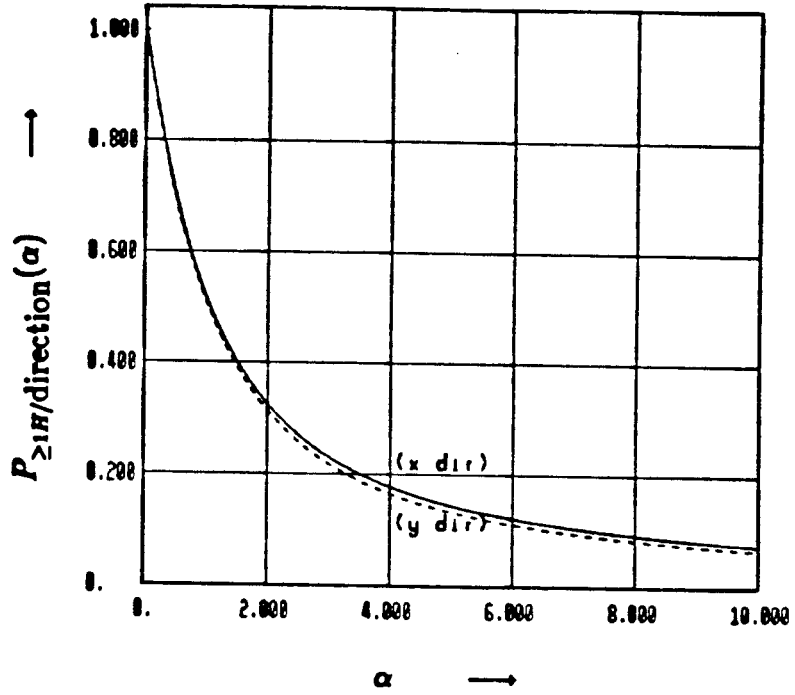


Fig. 4.7. Probability of  $n \geq 1$  handoff, given the direction of motion.



mobile. The mobile moves at constant speed and all the cell boundaries are straight lines; this makes the average number of cell boundary crossings equal to a weighted sum (to take the specific geometry of the cell packing into account) of linear terms in  $R/v_0$ , which represents the scale factor of the system. Let us specify these weight coefficients.

We need to consider the possible directions of motion for the calls. Due to the symmetry of the hexagonal cell, directions (1) and (2) will be equivalent as will directions (3) and (4). Without loss of generality, we now assume the scale factor of the system to be equal to unity ( $k = 1$ , or  $R = 1$  mi,  $v_0 = 1$  mi/min, for example).

In the case of direction (1) and (2), two distinct regions have to be distinguished for the possible location of the mobile, as illustrated in Figure 4.8.

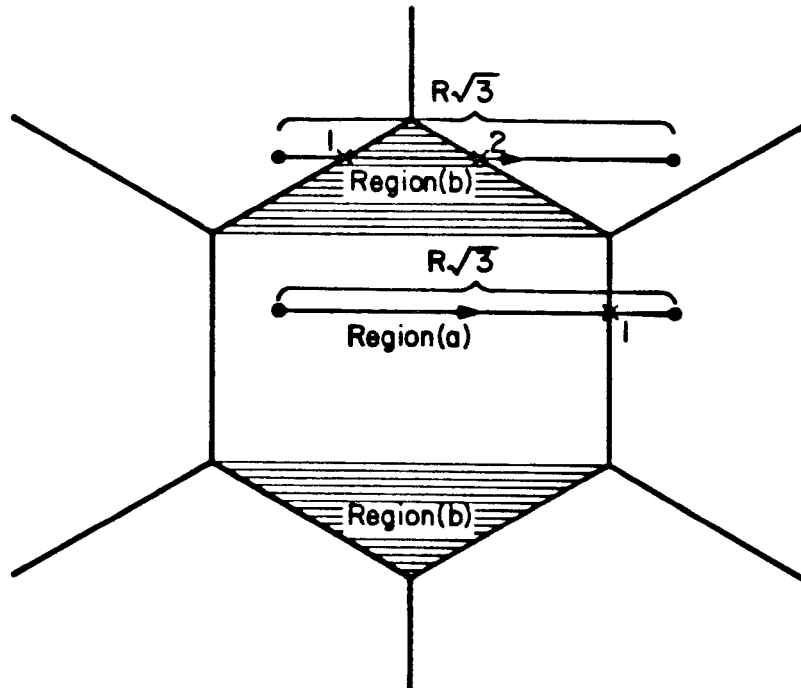


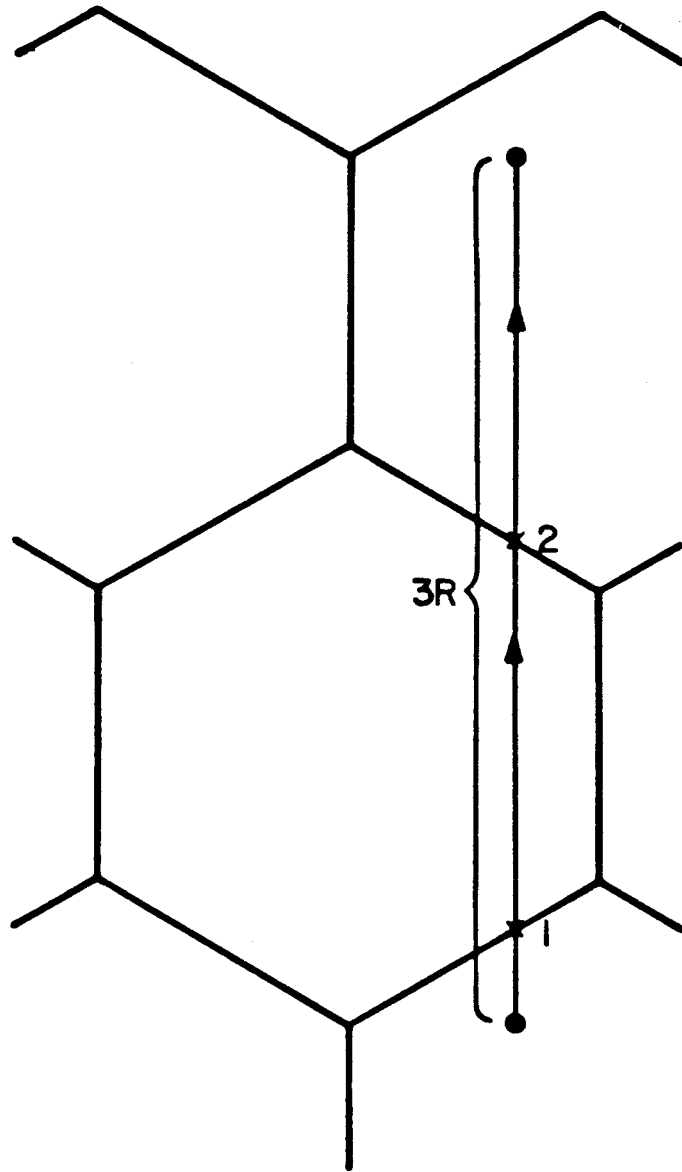
Fig. 4.8. Cell boundary crossings, directions (1) or (2).

Region (a) corresponds to the rectangular center part of the cell of width  $R\sqrt{3}$  and height  $R$ . Region (a) represents  $2/3$  of the total cell area. Region (b) (shaded area in Figure 4.8) is made up of the triangular upper and lower parts of the hexagonal cell of width  $R\sqrt{3}$  and height  $R/2$ . Region (b) represents the remaining  $1/3$  of the total cell area. If we consider a mobile moving in either direction (1) or (2) and located in Region(a), it takes  $\sqrt{3}$  time units to cross one cell and therefore experiences on the average  $1/\sqrt{3}$  cell boundary crossing every time unit. A mobile located in Region (b) and moving in either direction (1) or (2) will, no matter what its exact position is, take  $\sqrt{3}$  time units to go through two consecutive cells, crossing in the meantime two cell boundaries. This gives an average of  $2/\sqrt{3}$  cell boundary crossings every time unit.

In the case of directions (3) and (4), no sub-regions have to be distinguished, and any mobile moving in one of these two directions will take 3 time units to cross two consecutive cells (see Figure 4.9), which gives an average of  $2/3$  cell boundary crossings every time unit.

Using the fact that half the mobiles move in directions (1) or (2) and the other half moves in direction (3) or (4), we immediately obtain the average number of cell boundary crossings  $\bar{\tau}_1$  for the case  $k = 1$ . Namely, among the half of the calls that moves in directions (1) or (2),  $2/3$  of them will experience an average of  $1/\sqrt{3}$  cell boundary crossings per unit time, while the remaining  $1/3$  goes through  $2/\sqrt{3}$  cell boundary crossings on the average per unit time. The other half of the calls moves in directions (3) or (4) and experiences  $2/3$  cell boundary crossings per unit time. This finally gives:

$$\begin{aligned}\bar{\tau}_1 &= \frac{1}{2} \left[ \frac{2}{3} \left( \frac{1}{\sqrt{3}} \right) + \frac{1}{3} \left( \frac{2}{\sqrt{3}} \right) \right] + \frac{1}{2} \left[ \frac{2}{3} \right] \\ &= \frac{1}{3\sqrt{3}} + \frac{1}{3\sqrt{3}} + \frac{1}{3} \\ &= \frac{3 + 2\sqrt{3}}{9}.\end{aligned}$$



*Fig. 4.9. Cell boundary crossings, directions (3) or (4).*

Since the average call duration is  $1/\mu$ , the expression for  $\bar{h}$  with an arbitrary value of the scale factor  $k$  is given by:

$$\begin{aligned}\bar{h} &= \frac{3 + 2\sqrt{3}}{9} \cdot \frac{1}{k} \cdot \frac{1}{\mu} \\ \Rightarrow \bar{h} &= \frac{3 + 2\sqrt{3}}{9\alpha}.\end{aligned}\tag{4.18}$$

This expression can also be derived analytically as a check of our expressions for the handoffs probabilities. We have:

$$\begin{aligned}\bar{h} &= 2 \sum_{m=1}^{\infty} m P_{2mH}(\alpha) + 2 \sum_{m=0}^{\infty} m P_{(2m+1)H}(\alpha) + \sum_{m=0}^{\infty} P_{(2m+1)H}(\alpha) \\ &= 2 \sum_{m=0}^{\infty} m \left( P_{2mH}(\alpha) + P_{(2m+1)H}(\alpha) \right) + \sum_{m=0}^{\infty} P_{(2m+1)H}(\alpha).\end{aligned}$$

Let us first compute the quantity in parenthesis in the above expression, using Equations (4.16) and (4.17):

$$\begin{aligned}P_{2mH}(\alpha) + P_{(2m+1)H}(\alpha) &= \\ &e^{-\sqrt{3}m\alpha} \left[ \frac{1}{9\alpha^2} \left( e^{\sqrt{3}\alpha} + e^{-\sqrt{3}\alpha} - 2 \right) - \frac{\sqrt{3}}{9\alpha} \left( 1 - e^{-\sqrt{3}\alpha} \right) \right] \\ &+ e^{-2\sqrt{3}m\alpha} \left[ \frac{\sqrt{3}}{9\alpha} \left( e^{\sqrt{3}\alpha} + e^{-2\sqrt{3}\alpha} - e^{-\sqrt{3}\alpha} - 1 \right) \right] \\ &+ e^{-3m\alpha} \left[ \frac{1}{3\alpha^2} \left( e^{2\alpha} + e^{-2\alpha} - e^{\alpha} - e^{-\alpha} \right) - \frac{1}{3\alpha} \left( 1 - e^{-3\alpha} \right) \right]\end{aligned}$$

We will require two useful and well-known summations:

$$\begin{aligned}\sum_{m=0}^{\infty} m \left( e^{-rx} \right)^m &= \frac{e^{-rx}}{\left( 1 - e^{-rx} \right)^2}, \\ \sum_{m=0}^{\infty} \left( e^{-rx} \right)^m &= \frac{1}{1 - e^{-rx}}.\end{aligned}$$

Using these results we get the following expression for  $\bar{h}$ :

$$\bar{h} = 2 \left[ \frac{e^{-\sqrt{3}\alpha}}{\left( 1 - e^{-\sqrt{3}\alpha} \right)^2} \right] \cdot \left[ \frac{1}{9\alpha^2} \left( e^{\sqrt{3}\alpha} + e^{-\sqrt{3}\alpha} - 2 \right) - \frac{\sqrt{3}}{9\alpha} \left( 1 - e^{-\sqrt{3}\alpha} \right) \right]$$

$$\begin{aligned}
 &+ 2 \left[ \frac{e^{-2\sqrt{3}\alpha}}{(1 - e^{-2\sqrt{3}\alpha})^2} \right] \cdot \left[ \frac{\sqrt{3}}{9\alpha} (e^{\sqrt{3}\alpha} + e^{-2\sqrt{3}\alpha} - e^{-\sqrt{3}\alpha} - 1) \right] \\
 &+ 2 \left[ \frac{e^{-3\alpha}}{(1 - e^{-3\alpha})^2} \right] \cdot \left[ \frac{1}{3\alpha^2} (e^{2\alpha} + e^{-2\alpha} - e^\alpha - e^{-\alpha}) - \frac{1}{3\alpha} (1 - e^{-3\alpha}) \right] \\
 &+ \left[ \frac{1}{1 - e^{-\sqrt{3}\alpha}} \right] \cdot \left[ -\frac{2}{9\alpha^2} (1 - e^{-\sqrt{3}\alpha}) + \frac{\sqrt{3}}{9\alpha} (1 + e^{-\sqrt{3}\alpha}) \right] \\
 &+ \left[ \frac{1}{1 - e^{-2\sqrt{3}\alpha}} \right] \cdot \left[ \frac{\sqrt{3}}{9\alpha} (1 - e^{-\sqrt{3}\alpha})^2 \right] \\
 &+ \left[ \frac{1}{1 - e^{-3\alpha}} \right] \cdot \left[ -\frac{2}{3\alpha^2} (e^{-\alpha} - e^{-2\alpha}) + \frac{1}{3\alpha} (1 + e^{-3\alpha}) \right].
 \end{aligned}$$

After further tedious calculations which are detailed in Appendix C, we finally obtain the same expression as before:

$$\bar{h} = \frac{3 + 2\sqrt{3}}{9\alpha} \approx \frac{0.7182}{\alpha}.$$

This gives us a great deal of confidence in our derivation of the probability distribution for the number of handoffs.

Moreover, there is good agreement with the average number of handoffs found in the simulation of Chapter 3 for the same cellular system. Recall that the simulation was based on the same assumptions as for the analytic model, except for the fact that the mobiles were going through changes of direction while involved in a call. The times between changes of direction were exponentially distributed and their average was an input of the simulation. The good agreement between the analytic model and the simulation results for the average number of handoffs tells us that the constant direction assumption is not too penalizing with respect to the number of handoffs. For example, in the case of a system with cell radius  $R = 1$  mi, average mobile speed  $v_0 = 35$  mi/h, and average service rate  $\mu = 0.5 \text{ min}^{-1}$ , the simulation gives  $\bar{h} \approx 0.75$ , while Equation (4.18) gives  $\bar{h} = 0.84$ , within 10% of the simulated

value. If we increase the cell radius to  $R = 10$  mi, the simulation now gives  $\bar{h} \approx 0.078$  while Equation (4.18) yields  $\bar{h} = 0.084$ , which is within 10% of the simulation result. This tells us that if we want a quick estimate of the channel occupation time, we might, since we know  $\alpha$ , use Equation (4.18) to approximate the average number of handoffs and use this somewhat crude value to obtain the handoff parameter of the memoryless distribution of Chapter 3.

This completes the sections devoted to handoff probabilities. We will now derive the channel occupancy time distribution. This quantity is not to be confused with the  $T_n(t)$ 's which represent the times to a cell boundary crossing independently of whether the mobile has a call in progress or not.

#### 4.5 Channel occupancy time distribution.

In this Section we derive the distribution of the channel occupancy time as induced by the model of Section 4.1. We will assume a call initiation rate of  $\lambda$  calls per unit time and per cell, all cells being taken identical. In order to compute the channel occupancy time distribution we will consider two cases. In the first case, the mobile associated with the call occupying the channel moves in directions (1) or (2), and in the second case it moves in directions (3) or (4). For these two cases, we will distinguish between calls that were originated in the cell and calls that were handed off to the cell. We already know that we have on the average  $\lambda$  calls originated per unit time inside the cell, and therefore on the average  $\lambda/2$  calls originated inside the cell are moving in directions (1) or (2), and  $\lambda/2$  calls originated inside the cell are moving in directions (3) or (4). However, it remains to determine similar quantities for the calls handed off to the cell. We now proceed with this computation.

We will compute for the two cases of directions of motion ((1), (2), and (3), (4)), the average rate of calls entering the cell. Because of the memorylessness of

the exponential distribution of the total service time, we will not need to remember the time spent by the call from its start until it entered the cell considered. This is due to the fact that the distribution of the residual service time of a call entering the cell considered will be identical to the exponential distribution of the total service time, because of memorylessness.

Let us first consider a call moving in direction (1). Due to our assumption of constant direction for the entire call duration, for such a call to ultimately enter our cell, it must have originated between the dotted lines of Figure 4.10 ( $-R \leq y \leq R$ ).

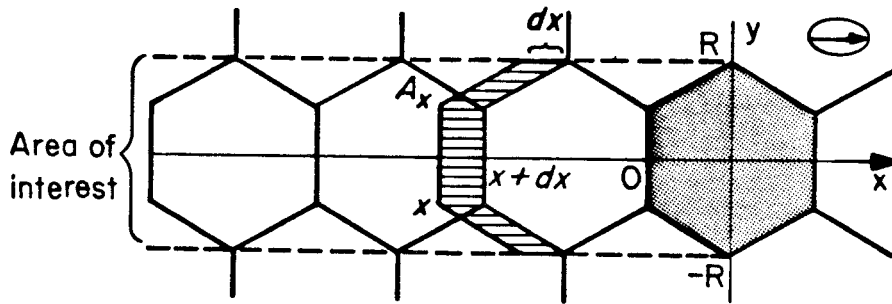


Fig. 4.10. Handoff arrivals from directions (1) or (2).

If we now take a call that originated in the elementary shaded area  $A_x$  of Figure 4.10, the probability that it reaches the boundary of our cell is given by:

$$\begin{aligned} Pr\{\text{call originated in } A_x \text{ enters the cell}\} &= Pr\{\text{its service time } t \geq \left\lfloor \frac{x}{v_0} \right\rfloor\} \\ &= \int_{|x/v_0|}^{\infty} \mu e^{-\mu t} dt \\ &= e^{-\mu|x/v_0|}. \end{aligned}$$

We know that 1/4 of the mobiles move in direction (1) and that we have  $\lambda$  arrivals per unit time per cell (cell area =  $3\sqrt{3}/2R^2$ ) while the elementary area  $A_x$  has area

$2Rdx$ . This gives a total origination rate for calls moving in direction (1) in the elementary area  $A_x$  equal to:

$$\begin{aligned}\lambda_x &= \frac{\lambda}{4} \cdot \frac{2Rdx}{3\sqrt{3}/2R^2} \\ &= \frac{\lambda dx}{3\sqrt{3}R}.\end{aligned}$$

Since we have computed the probability that a call originated in  $A_x$  enters our cell, we obtain the average arrival rate into the cell due to calls originated in  $A_x$  by multiplying the two, as we can by independence:

$$\eta_x = \lambda_x \cdot e^{-\mu|x/v_0|}.$$

If we now want the total (handoff) arrival rate in the cell due to calls moving in direction (1), we need to sum the elementary rate  $\eta_x$  over all possible values of  $x$ . In the case of direction (1), which corresponds to  $x \leq 0$ , we therefore get:

$$\begin{aligned}\eta_{(1)} &= \frac{\lambda}{3\sqrt{3}R} \int_{-\infty}^0 e^{-\mu|x/v_0|} dx \\ &= \frac{\sqrt{3}}{9} \frac{\lambda}{k\mu},\end{aligned}$$

where we recall that  $k = R/v_0$ .

The case of direction (2) is identical, with the only difference being that it is now the region  $x \geq 0$  that needs to be taken into account. The resulting rate is, however, the same because of the symmetry of the hexagonal cell with respect to the  $y$ -axis, and we get a total (handoff) arrival rate into the cell due to calls moving in direction (1) or (2) equal to:

$$\eta_{(1)\&(2)} = \frac{2\sqrt{3}}{9} \frac{\lambda}{k\mu}.$$



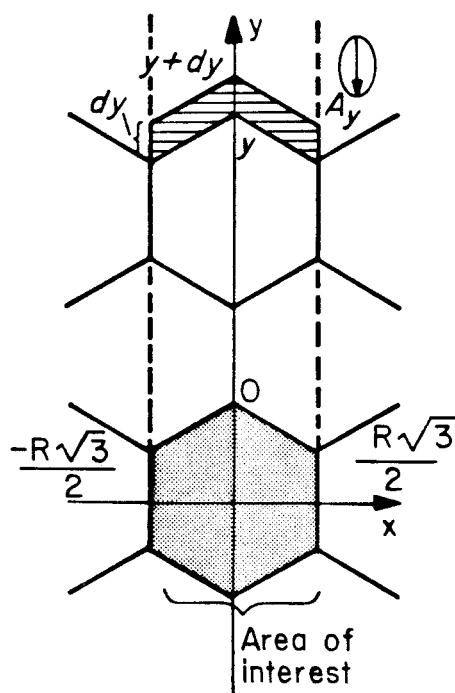


Fig. 4.11. Handoff arrivals from directions (3) or (4).

Calls moving in directions (3) or (4) are treated in a similar fashion as illustrated by Figure 4.11.

We consider a call that originated in the elementary area  $A_y$  and is moving in direction (4). The probability that it enters the cell considered is given by:

$$\begin{aligned} Pr\{\text{call originated in } A_y \text{ enters the cell}\} &= Pr\{\text{its service time } t \geq \left| \frac{y}{v_0} \right|\} \\ &= \int_{|y/v_0|}^{\infty} \mu e^{-\mu t} dt \\ &= e^{-\mu|y/v_0|}. \end{aligned}$$

Since  $A_y = \sqrt{3}Rdy$ , we have in complete parallelism with the case of direction (1):

$$\begin{aligned} \lambda_y &= \frac{\lambda}{4} \cdot \frac{\sqrt{3}Rdy}{3\sqrt{3}/2R^2} \\ &= \frac{\lambda dy}{6R} \\ \Rightarrow \eta_y &= \lambda_y e^{-\mu|y/v_0|}. \end{aligned}$$

After summing the above for all values of  $y \geq 0$ , we derive:

$$\eta_{(4)} = \frac{1}{6} \frac{\lambda}{k\mu}.$$

The case of direction (3) being entirely symmetric, we get the overall average handoff arrival rate into the cell due to calls moving in directions (3) or (4):

$$\eta_{(3)\&(4)} = \frac{1}{3} \frac{\lambda}{k\mu}.$$

This is, as we would expect from the geometry, somewhat less than  $\eta_{(1)\&(2)}$ , since handoff calls moving in directions (3) or (4) must have originated in the band intercepted by the hexagonal cell of width  $R\sqrt{3}$  and centered on the  $y$ -axis, while for the case of directions (1) or (2), the corresponding band centered on the  $x$ -axis has larger width  $2R$ . Call originations being in proportion to the area they arise from, this explains the difference in rates for the two cases.

All this gives the total arrival rate of calls inside the cell, originating and handoff, equal to:

$$\begin{aligned} \eta &= \lambda + \eta_{(1)\&(2)} + \eta_{(3)\&(4)} \\ &= \lambda + \frac{2\sqrt{3}}{9} \frac{\lambda}{k\mu} + \frac{1}{3} \frac{\lambda}{k\mu} \\ \Rightarrow \eta &= \frac{\lambda(9k\mu + 2\sqrt{3} + 3)}{9k\mu}. \end{aligned} \tag{4.19}$$

We are now almost ready to start the computation of the distribution of the channel occupancy time. But first note that if we are given that a channel occupancy time has just started, it will end before time  $t$  if either the service time of the call is smaller than  $t$  or its service time is larger than  $t$  but the call leaves the cell before  $t$ . These two aspects will have to be taken into account.

**Directions (1) and (2):**

a) Calls originated inside the cell.

Let us first compute the probability that an arrival belongs to the category of calls moving in directions (1) or (2) having originated inside the cell. This probability is equal to the ratio of the arrival rate of calls originated inside the cell and moving in directions (1) or (2) to the total arrival rate:

$$P_{(1)\&(2)}^a = \frac{\lambda/2}{\eta} = \frac{9k\mu}{2(9k\mu + 2\sqrt{3} + 3)}$$

Let  $S_c$  be the channel occupancy time. We want the probability that  $S_c$  is smaller than  $t$ . This quantity will depend of course on the value of  $t$ . For example, if  $t \geq k\sqrt{3}$ , the probability is equal to 1, since  $k\sqrt{3}$  represents the maximum possible time spent in the cell by a mobile moving in directions (1) or (2) at speed  $v_0$ . However, the case  $0 \leq t \leq k\sqrt{3}$  requires the use of the previous remark on how a channel occupancy time might terminate. Namely, if the call has a service time already smaller than  $t$ , it insures that the channel occupancy time will be smaller than  $t$ . However, even if the service time is greater than  $t$  but the call is at a travel time less than  $t$  from a cell boundary, the channel occupation time will also terminate before  $t$ . This possibility corresponds to the shaded area of Figure 4.12(a). We can then write that the probability that the channel occupancy time is smaller than  $t$  as the sum of the probability that the service time is smaller than  $t$  and the probability that the service time is greater than  $t$  with the call located in the shaded area of Figure 4.12(a):

$$P_{(1)\&(2)}^a[S_c \leq t] = \int_0^t \mu e^{-\mu s} ds + \left[ \frac{4\sqrt{3}}{9} \left( \frac{t}{k} \right) - \left( \frac{t}{3k} \right)^2 \right] \cdot \int_t^\infty \mu e^{-\mu s} ds.$$

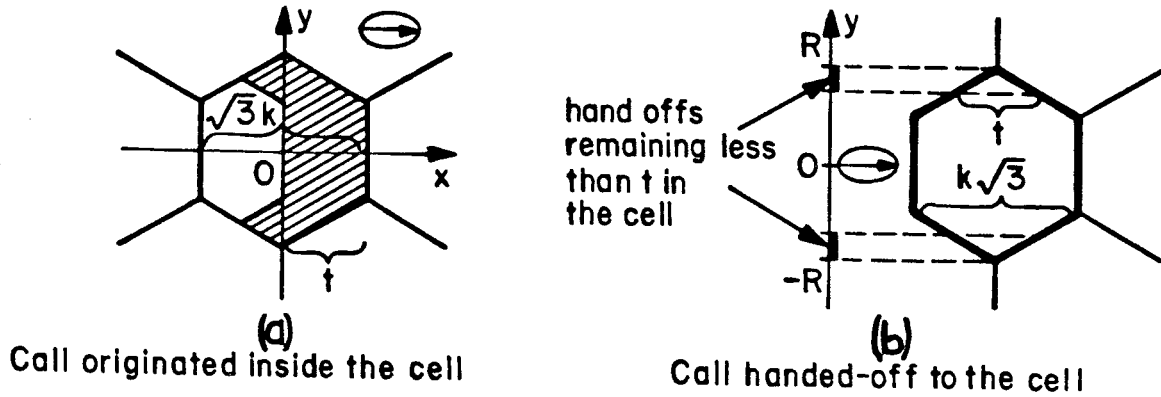


Fig. 4.12. Time spent in the cell by a call.

Case of directions (1) or (2).

This gives:

$$P_{(1)\&(2)}^a [S_c \leq t] = \begin{cases} 1.0, & k\sqrt{3} \leq t; \\ 1 - e^{-\mu t} \left[ 1 - \frac{4\sqrt{3}}{9} \left(\frac{t}{k}\right) + \left(\frac{t}{3k}\right)^2 \right], & 0 \leq t \leq k\sqrt{3}. \end{cases}$$

b) Calls handed off to the cell.

A given arrival will correspond to this case with probability equal to the ratio of the corresponding arrival rate to the total arrival rate. Namely, we get:

$$\begin{aligned} P_{(1)\&(2)}^b &= \frac{\eta_{(1)\&(2)}}{\eta} \\ &= \frac{2\sqrt{3}}{9k\mu + 2\sqrt{3} + 3}. \end{aligned}$$

As before, if  $t \geq k\sqrt{3}$ , the probability that the service time ends before  $t$  is equal to 1, since any mobile moving in directions (1) or (2) can remain in the cell at most  $k\sqrt{3}$  time units. For  $0 \leq t \leq k\sqrt{3}$ , however, we again need to take into account the calls with a service time larger than  $t$ , but that stay less than  $t$  in the cell. The difference is that we know that the calls are located on the cell boundary (handed

off to the cell). Depending on the mobile ordinate ( $y$ ), we will have different possible times for the mobile to cross the cell.

For example, the handed-off mobiles with ordinates in the range  $-R/2 \leq y \leq R/2$  will all take  $k\sqrt{3}$  time units to cross the cell. They will therefore not contribute to  $P_{(1)\&(2)}^b[S_c \leq t]$  if  $0 \leq t < k\sqrt{3}$ , but they will introduce a discontinuity in the expression of  $P_{(1)\&(2)}^b[S_c \leq t]$  at  $t = k\sqrt{3}$  since at this point a non-zero fraction of the arrivals is either taken into account if  $t \geq k\sqrt{3}$  or discarded if  $0 \leq t < k\sqrt{3}$ . If the handed-off mobile has its ordinate in the range  $R/2 \leq |y| \leq R$ , then the call will take  $2\sqrt{3}(k - |y/v_0|)$  to cross the cell. So if the associated call has a service time greater than  $t$ , but we want it to correspond to a channel occupancy time smaller than  $t$ , this forces the following:

$$2\sqrt{3} \left( k - \left| \frac{y}{v_0} \right| \right) \leq t, \quad 0 \leq t \leq k\sqrt{3}$$

$$\Rightarrow \left| \frac{y}{v_0} \right| \geq k - \frac{\sqrt{3}}{6}t.$$

Since  $y$  is in the range  $-R \leq y \leq R$  for calls moving in directions (1) or (2), and since the handoff arrivals are uniform with respect to  $y$  (the same elementary contribution for each  $y$  in this range), the fraction of handoff calls that will satisfy the above requirement is given by:

$$h_t = \frac{2 \left( k - \left( -\frac{\sqrt{3}}{6}t + k \right) \right)}{2k} = \frac{\sqrt{3}t}{6k}.$$

We conclude that for the case  $0 \leq t \leq k\sqrt{3}$ :

$$P_{(1)\&(2)}^b[S_c \leq t] = \int_0^t \mu e^{-\mu s} ds + \frac{\sqrt{3}t}{6k} \int_t^\infty \mu e^{-\mu s} ds + \frac{e^{-\sqrt{3}\mu k}}{2} u(t - \sqrt{3}k).$$

Here  $u(t - \sqrt{3}k)$ , a unit step function, takes into account the fact that half the handoff arrivals ( $-R/2 \leq y \leq R/2$ ) with service time at least  $k\sqrt{3}$  will correspond to a channel occupancy time exactly equal to  $k\sqrt{3}$ .

More precisely, we know that the probability that the channel occupancy time is less than or equal to  $k\sqrt{3}$  is equal to 1, since no matter what the service time of the call is, it will never remain more than  $k\sqrt{3}$  time units in the cell. However, if we now want the probability that the channel occupancy time is strictly smaller than  $k\sqrt{3}$ , we do not obtain probability 1 anymore. This discontinuity is due to the presence in our model of a non-zero fraction of calls with channel occupancy time exactly equal to  $k\sqrt{3}$ . Specifically, all the handoff arrivals with service time greater or equal to  $k\sqrt{3}$  and ordinates in the range  $-R/2 \leq y \leq R/2$  will take exactly  $k\sqrt{3}$  time units to cross the cell, and will therefore correspond to a channel occupancy time exactly equal to  $k\sqrt{3}$ . In the expression of the distribution of the channel occupancy time, this discontinuity is taken into account by the unit step function starting at  $t = k\sqrt{3}$ , where the multiplicative factor  $e^{-\sqrt{3}\mu k}/2$  represents all the handoff arrivals moving in directions (1) or (2), with ordinates in the range  $-R/2 \leq y \leq R/2$ , and with service time greater than  $k\sqrt{3}$ . All these different configurations are illustrated in Figure 4.12(b), and we finally obtain:

$$P_{(1)\&(2)}^b[S_c \leq t] = \begin{cases} 1.0, & k\sqrt{3} \leq t; \\ 1 - e^{-\mu t} \left[ -\frac{\sqrt{3}}{6} \left( \frac{t}{k} \right) + 1 \right] \\ \quad + e^{-\sqrt{3}\mu k}/2 \cdot u(t - \sqrt{3}k), & 0 \leq t \leq k\sqrt{3}. \end{cases}$$

**Directions (3) and (4):**

a) Calls originated inside the cell.

The probability that a channel occupancy starts with a call moving in directions (3) or (4) and was originated inside the cell is given by the ratio of the arrival rate of such calls to the overall arrival rate. The expression of this probability is the same as for directions (1) and (2):

$$P_{(3)\&(4)}^a = \frac{9k\mu}{2(9k\mu + 2\sqrt{3} + 3)}$$

Similarly to the case of directions (1) and (2), we will have to distinguish between different domains for the value of  $t$  when computing the probability that the channel occupancy time is smaller than  $t$ . This is again due to the fact that calls with a service time larger than  $t$  can still correspond to a channel occupancy time smaller than  $t$  if the associated mobile stays less than  $t$  in the cell. The different cases and the corresponding areas are described in Figures 4.13(a<sub>1</sub>) and (a<sub>2</sub>) for a mobile moving in direction (3).

We will have to consider three regions for  $t$ . If  $t \geq 2k$ , then since  $2k$  is the maximum time that a mobile moving in directions (3) or (4) can spend in the cell, we will have  $P_{(3)\&(4)}^a[S_c \leq t] = 1.0$ . If  $k \leq t \leq 2k$ , the channel occupancy time will be smaller than  $t$  either because the service time is smaller than  $t$ , or because the service time is greater than  $t$  but the mobile is located in the shaded region (probability =  $-1/3 - 4/3(t/k) + 3(t/3k)^2$ ) of Figure 4.13(a<sub>1</sub>). This gives a probability of channel occupancy time smaller than  $t$  equal to:

$$P_{(3)\&(4)}^a[S_c \leq t] = \int_0^t \mu e^{-\mu s} ds + \left[ -\frac{1}{3} + \frac{4}{3} \left( \frac{t}{k} \right) - 3 \left( \frac{t}{3k} \right)^2 \right] \int_t^\infty \mu e^{-\mu s} ds.$$

Finally if  $0 \leq t \leq k$ , the channel occupancy time will be smaller than  $t$  again for a service time smaller than  $t$ , but also for a service time greater than  $t$  with the associated mobile located in the shaded region (probability =  $2/3(t/k)$ ) of Figure 4.13(a<sub>2</sub>). The probability of a channel occupation time smaller than  $t$  is then given by:

$$P_{(3)\&(4)}^a[S_c \leq t] = \int_0^t \mu e^{-\mu s} ds + \frac{2}{3} \left( \frac{t}{k} \right) \int_t^\infty \mu e^{-\mu s} ds.$$

So the distribution of the channel occupancy time when the call was originated inside the cell and is associated to a mobile moving in directions (3) or (4) is given by:

$$P_{(3)\&(4)}^a[S_c \leq t] = \begin{cases} 1.0, & 2k \leq t; \\ 1 - e^{-\mu t} \left[ \frac{4}{3} - \frac{4}{3} \left( \frac{t}{k} \right) + 3 \left( \frac{t}{3k} \right)^2 \right], & k \leq t \leq 2k; \\ 1 - e^{-\mu t} \left[ 1 - \frac{2}{3} \left( \frac{t}{k} \right) \right], & 0 \leq t \leq k. \end{cases}$$

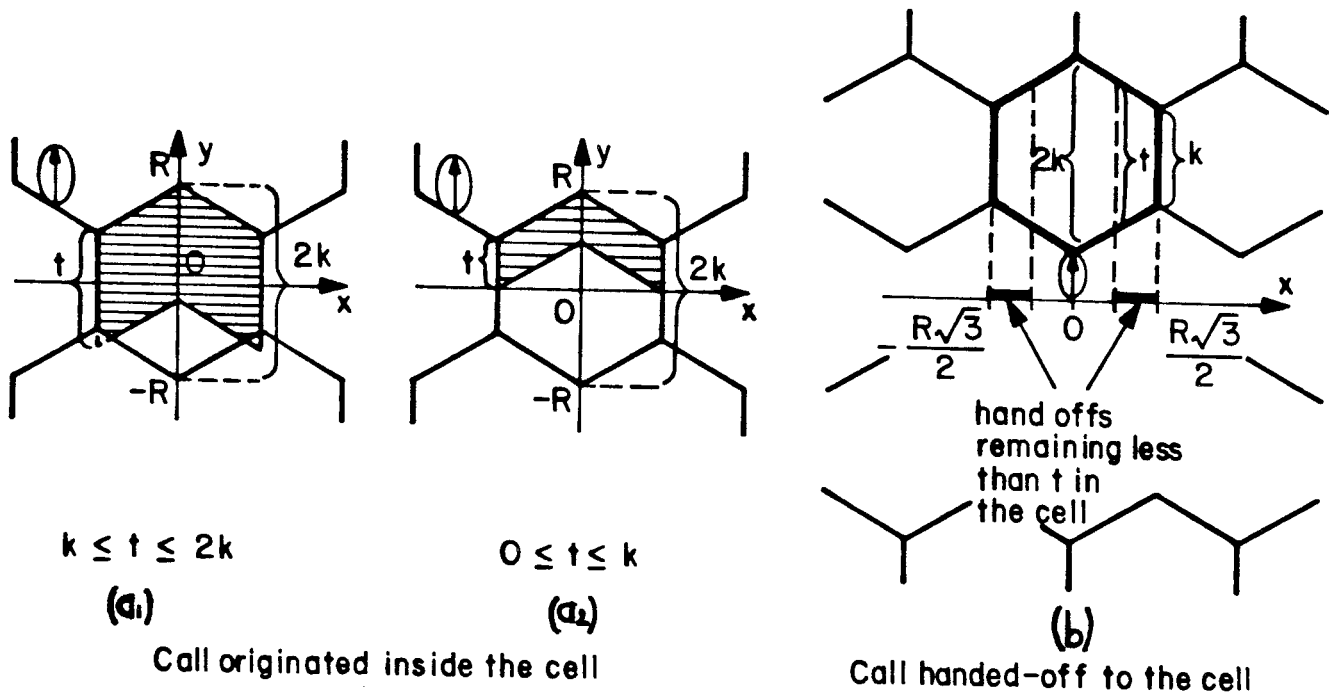


Fig. 4.13. Time spent in the cell by a call.

Case of directions (3) or (4).

b) Calls handed off to the cell.

A channel occupancy time will start because of a call handed off to the cell, and associated to a mobile moving in directions (3) or (4), with probability equal to the ratio of the arrival rate of this type of call to the total arrival rate. This ratio is equal to:

$$\begin{aligned}
 P_{(3)\&(4)}^b &= \frac{\eta_{(3)\&(4)}}{\eta} \\
 &= \frac{3}{9k\mu + 2\sqrt{3} + 3}
 \end{aligned}$$

We know that at the beginning of the channel occupancy time the associated mobile is located on the cell boundary, and the arrivals are uniformly distributed with respect to the abscissa ( $x$ ) (same elementary contribution for all  $x$ 's). For a given value of  $x$ , a mobile located on the cell boundary and moving in directions (3) or



(4) will take  $2k - (2\sqrt{3}/3)|x/v_0|$  to cross the cell (see Figure 4.13(b)). Note that this quantity is never smaller than  $k$  since we have  $0 \leq |x/v_0| \leq k\sqrt{3}/2$ .

So the probability that the channel occupancy time is smaller than  $t$  is 1.0 if  $2k \leq t$ , since  $2k$  is the maximum time spent in the cell by a mobile moving in directions (3) or (4). If  $k \leq t \leq 2k$ , we need to join to the set of calls with service time smaller than  $t$  the calls with service time greater than  $t$  but that cross the cell in time less than  $t$ . This corresponds to the case:

$$2k - 2\sqrt{3}/3 \left| \frac{x}{v_0} \right| \leq t, \quad k \leq t \leq 2k$$

$$\Rightarrow (2k - t) \frac{\sqrt{3}}{2} \leq \left| \frac{x}{v_0} \right| \leq \frac{\sqrt{3}}{2} k.$$

Since  $x$  is in the range  $-R\sqrt{3}/2 \leq x \leq R\sqrt{3}/2$ , the fraction of handoff arrivals that will satisfy this requirement is given by:

$$h'_t = \frac{2 \left( \frac{\sqrt{3}}{2} k - \frac{\sqrt{3}}{2} (2k - t) \right)}{\sqrt{3} k} = \left( \frac{t}{k} \right) - 1.$$

So the distribution of the channel occupancy time  $P_{(3)\&(4)}^b[S_c \leq t]$  for this case is given by:

$$P_{(3)\&(4)}^b[S_c \leq t] = \int_0^t \mu e^{-\mu s} ds + \left[ \left( \frac{t}{k} \right) - 1 \right] \int_t^\infty \mu e^{-\mu s} ds, \quad k \leq t \leq 2k.$$

To derive the result for the case  $0 \leq t \leq k$ , we only need to consider calls with service time smaller than  $t$ , since any mobile moving in directions (3) or (4) and handed off to the cell will spend more than  $k$  time units in the cell. Therefore, the probability that the channel occupancy time is smaller than  $t$  is simply equal to the probability that the service time is smaller than  $t$ , that is, equals  $1 - e^{-\mu t}$ . So we

finally obtain the distribution of the channel occupancy time corresponding to calls handed off to the cell and moving in directions (3) or (4):

$$P_{(3)\&(4)}^b[S_c \leq t] = \begin{cases} 1.0, & 2k \leq t; \\ 1 - e^{-\mu t} \left[ 2 - \left( \frac{t}{k} \right) \right], & k \leq t \leq 2k; \\ 1 - e^{-\mu t}, & 0 \leq t \leq k. \end{cases}$$

After studying all the different cases we are now in a position to compute the general channel occupancy time distribution by taking a weighted average of the previous expressions. Thus, we have:

$$P[S_c \leq t] = P_{(1)\&(2)}^a \cdot P_{(1)\&(2)}^a[S_c \leq t] + P_{(1)\&(2)}^b \cdot P_{(1)\&(2)}^b[S_c \leq t] \\ + P_{(3)\&(4)}^a \cdot P_{(3)\&(4)}^a[S_c \leq t] + P_{(3)\&(4)}^b \cdot P_{(3)\&(4)}^b[S_c \leq t].$$

This directly gives after a little algebra:

$$P[S_c \leq t] = \begin{cases} 1.0, & 2k \leq t; \\ 1.0 - \rho^{-1} e^{-\mu t} \left[ 6(k\mu + 1) - 3t \left( 2\mu + \frac{1}{k} \right) + \frac{3\mu}{2k} t^2 \right], & k\sqrt{3} \leq t \leq 2k; \\ 1.0 - \rho^{-1} e^{-\mu t} \left[ \frac{21k\mu}{2} + 2\sqrt{3} + 6 - t \left( (2\sqrt{3} + 6)\mu + \frac{4}{k} \right) + \frac{2\mu}{k} t^2 \right], \\ \quad + \rho^{-1} \sqrt{3} e^{-k\sqrt{3}\mu} u(t - \sqrt{3}k) & k \leq t \leq k\sqrt{3}; \\ 1.0 - \rho^{-1} e^{-\mu t} \left[ (9k\mu + 2\sqrt{3} + 3) - t \left( (2\sqrt{3} + 3)\mu + \frac{1}{k} \right) + \frac{\mu}{2k} t^2 \right], & 0 \leq t \leq k, \end{cases} \quad (4.20)$$

where we have used the following notation:

$$\rho = 9k\mu + 2\sqrt{3} + 3.$$

#### 4.6 Comparison with the exponential model.

Figures 4.14 and 4.15 illustrate the comparison between the above distribution of the channel occupancy time and a memoryless distribution that would give the same average number of handoffs. This is done for two systems, both with  $v_0 = 35$  mi/h

and  $\mu = 0.5 \text{ min}^{-1}$ , but one with  $R = 1 \text{ mi}$  and the other with  $R = 10 \text{ mi}$ . We see that the two distributions are rather close in the case  $R = 10 \text{ mi}$ , but decreasing  $R$  down to  $1 \text{ mi}$  causes a noticeable difference between them.

It is possible to obtain a heuristic interpretation of this difference. In the model, the assumption of constant direction plays a major role in the derivation of the handoff distribution. Furthermore, we note that forcing a constant direction for the mobile motion completely determines its past and its future, which is the complete opposite of the memorylessness assumption. We indeed expect some disagreements between the handoff distributions under the assumption of constant direction and under the memorylessness assumption. If we now decrease the cell radius from  $10 \text{ mi}$  to  $1 \text{ mi}$ , then, at the same time we must increase the average number of handoffs. This increases the importance of the contribution of handoff calls to the overall channel occupancy time distribution. It then seems rather natural to have an increased difference between the distributions of channel occupancy time under the constant direction assumption and under the memorylessness assumption, when the cell radius decreases.

It is important to realize that we do not compare the channel occupancy time distribution obtained under the assumption of constant direction with what a real life system would give, but merely with another model based on an exponential distribution for the times between handoffs. As a matter of fact, in real life, mobiles keep a constant direction for a certain time on the average. If for a fixed average service time the cell radius is decreased until the time taken to cross a cell becomes very small compared to the time spent in travelling in a given direction, this then makes the constant-direction model probably more realistic than the memorylessness assumption. But in the constant-direction case we certainly have memory in the handoff process. A call that has not been handed off for a while is much more

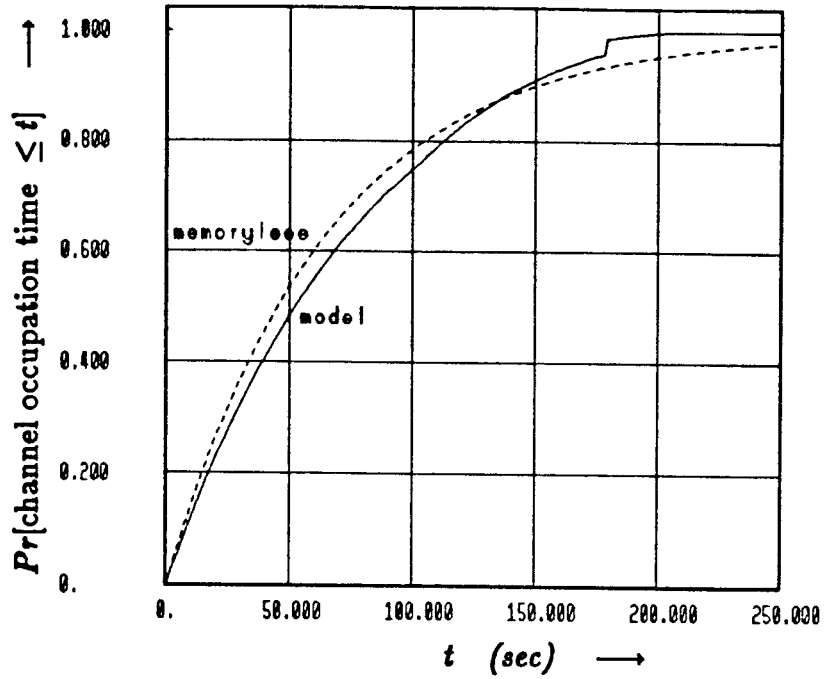


Fig. 4.14. Probability distribution of the channel occupancy time.

$$R = 1 \text{ mi}, v_0 = 35 \text{ mi/h}, \mu = 0.5 \text{ min}^{-1}$$

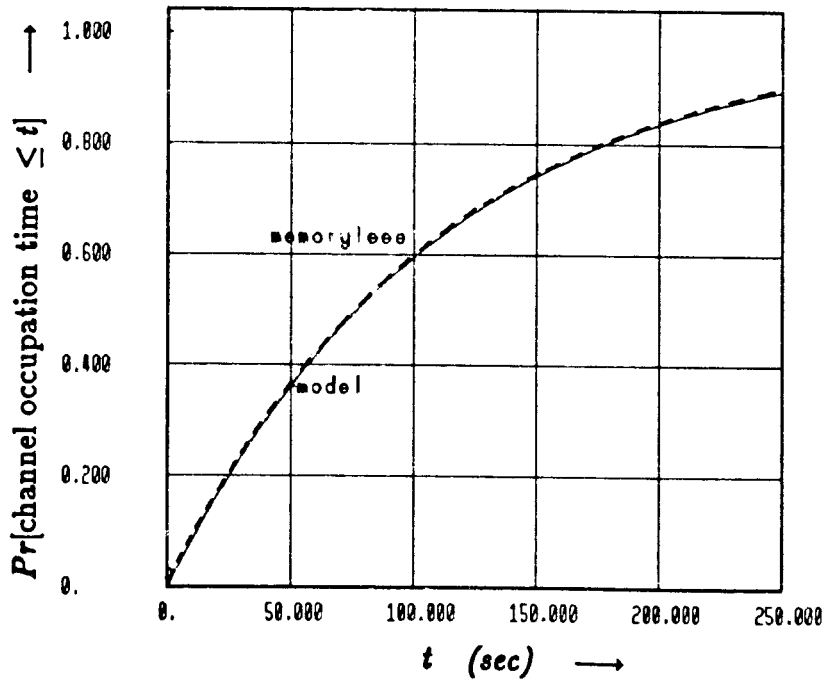


Fig. 4.15. Probability distribution of the channel occupancy time.

$$R = 10 \text{ mi}, v_0 = 35 \text{ mi/h}, \mu = 0.5 \text{ min}^{-1}$$

likely to cross a cell boundary (this means a handoff) than a call that just went through a handoff.

This kind of situation might not fit most real life systems, but it can be a good approximation for systems where the assumption of no change of direction is not too unrealistic. A possible application could be a cellular marine system implemented by satellite, since ships are more likely to maintain a constant direction for a long time than most land vehicles, with the possible exception of trucks driving on interstate freeways. In any case, the constant-direction assumption and the memoryless assumption represent the two possible extremes to model the behavior of the mobiles. Most real life systems will probably be located somewhere in between, and the two methods presented in this chapter (simulation and analytic model) can provide us with tools to bound and analyze a system.

In closing this section, let us note that from a (communications) traffic point of view, each cell will still handle a total traffic equal to  $\lambda/\mu$ . This is due to the fact that handoffs represent multiple accesses to the system but not to traffic creation. This forces the total traffic handled by the system to remain constant. Since all cells are taken to be equivalent, this means that they still carry a traffic equal to  $\lambda/\mu$ . In Appendix D, we verify directly as a check that the traffic handled by a cell remains equal to  $\lambda/\mu$ , and we derive the traffic contribution of each category of calls. That is, we obtain, among other things, expressions for the traffic contributions of handoff calls and originating calls. We will see in Chapter 5 that these quantities can be of some help to dimension the arrival process to the system.

Note that here we have neglected the fact that calls might be blocked. Taking the blocking into account would modify the traffic, since the carried traffic would no longer equal the offered traffic. Furthermore, if an originating call is blocked,

this results in a net loss for the system, but if a handoff call is blocked it only represents a partial loss, since the call was already provided with part of its service time. We shall not try to account for this effect; the blocking it is hoped is small. This terminates the study of the channel occupancy time in a Cellular Radio system.

## CHAPTER 5

### SIMPLE TRAFFIC POLICIES THAT PROTECT HANDOFFS

#### 5.1 Assumptions and presentation of the guard channel method.

In this chapter we derive some simple traffic policies that can help to significantly decrease the blocking probability of handoff calls, while not penalizing the originating calls too much. We recall that handoff calls are calls already in progress when they enter a cell as a result of their motion through the cellular system. On the other hand, originating calls correspond to calls initiated by mobiles (as calling or called parties) located inside the service area of the cell.

A cellular system is made up of cells which can all be considered as equivalent. Cells will most certainly differ from each other in the amount of traffic they carry, their coverage area, and perhaps even the number of frequency channels they have been allocated. However, in general, any cell can be regarded as a service facility with a certain number of servers (the frequency channels) and some customers that require service (the calls). The service facility can be a queueing system if queueing is provided, or simply a blocking system where customers that find all channels busy are cleared from the facility.

In order to be able to use this model to derive some efficient traffic policies, we need to give more detailed information on how customers arrive into the system, and what kind of service they require. This means that we need precise formulations of the arrival process and of the service process. Since our final goal is to give a higher level of protection to handoff calls, this requires that we distinguish between the arrival in the cell of a handoff call and an originating call. This in turn implies two distinct arrival processes for these two types of calls.

In most communication systems, arrivals are assumed to occur at Poisson-distributed times ([41], Chapter 5, p. 225), which means that the interarrival time

can be taken to be exponentially distributed or memoryless. This allows us to assume an exponential distribution for interarrival time of **originating** calls. The distribution of the interarrival time of handoff calls can also be taken as exponential. This is due to the large number of independent customers present in the system with random position and direction. We can therefore assume that the fact that a handoff arrival just occurred gives no information on the arrival time of the next handoff. This kind of assumption is classical and very similar to the one that leads to an exponential distribution for the interarrival time distribution of originating calls.

The arrival rate of handoff calls can be estimated in terms of the arrival rate of originating calls. This can be done by using the expressions for handoff probabilities derived in Chapter 4 (they can also be estimated from the simulation of Chapter 3) to find the fraction of calls initiated in surrounding cells that will be handed off to our cell. It will usually be sufficient to consider calls that are handed off only once or twice. Another possibility is to use the expressions derived in Appendix D that give the traffic served by the cell corresponding to handoff as well as the traffic due to originating calls. The arrival rate of handoff calls is then simply equal to the product of the arrival rate of originating calls times the ratio of the traffic due to handoffs over the traffic due to originating calls. Both methods yield approximately the same answer, and in most practical cases the arrival rate of handoff calls will be much smaller than the arrival rate of originating calls, typically by a factor of 5 to 10.

We now need to adopt a distribution for the service time provided by the cell. Namely, we need the distribution of the channel occupancy time. At this point we make use of the results established in Chapters 3 and 4. We start by assuming that the distribution of the total service time required by the customers (communication



duration) is exponential. This is again a classical assumption for voice communication systems, which, as we mentioned in Chapter 3, has furthermore been found to be realistic in Cellular Radio ([21], p. 9). The service rate is merely the inverse of the average call duration, and is a quantity that can usually be estimated or obtained rather easily.

Using the results of Chapters 3 and 4, we know that the handoff process is also well approximated by an exponential distribution. Recall that the handoff rate can be obtained either from the simulation of Chapter 3 or estimated using Equations (3.1) and (4.18). Under these assumptions, we saw that the distribution of the channel occupancy time is also an exponential distribution formed by the product of the exponential service time distribution and the exponential interhandoff time distribution. Calls depart from the channels memorylessly, either because of a handoff or a call completion.

Provided with all the above data, we finally have the following model of a cell within a cellular system:

- We regard a cell as a service facility with, say,  $n$  servers (frequency channels);  $n$  is often a number like 44. This service facility provides service to two initially different kind of customers, the handoff calls and the originating calls.
- Originating calls have an exponential interarrival time distribution with parameter  $\lambda$ .
- Handoff calls have also an exponential interarrival time distribution, but with different parameter  $\gamma$ , usually smaller than  $\lambda$  (more time between handoff arrivals).
- Both type of calls are taken to have the same total service time distribution. This distribution is exponential with parameter  $\mu$ , where  $1/\mu$  is the average total call duration in the cellular system.

- The interhandoff time is also taken to be exponentially distributed with parameter  $\eta$ .

Note, as we already mentioned in Chapter 4, that it is possible to choose the same distribution for the total service time of both originating and handoff calls because of the memoryless property of the exponential distribution. This tells us that even if we know that handoff calls must have already been in service for a certain time, the distribution of their remaining service time when they enter the cell is the same exponential as the overall service time distribution.

Let us now describe the policies we will study in order to improve the protection of handoff calls.

- The first investigated is the blocking of newly-originating calls as soon as only  $g$  channels among the initial  $n$  remain free. This allows us to build a "guard band" before handoff calls will be blocked, which will happen only when all  $n$  channels are occupied. Note that the  $g$  guard channels are not fixed in advance, but will be created only when required by the traffic conditions. Clearly, the influence of having guard channels on the blocking probability of originating calls will depend on the value of  $g$ , with the blocking probability of originating calls increasing as  $g$  increases. We would like to keep the blocking probabilities of originating calls relatively low while obtaining a substantial decrease in the blocking probability of handoff calls.

- The second possibility is to allow handoff calls to be queued, hoping for a short average delay which would not make the service interruption too perceptible and cause voluntary but unwanted disconnects.

- The third policy combines the two above solutions, keeping a certain number of guard channels for the handoff calls, while also allowing them to be queued if no

channel is available. We will treat the two cases where the allowed queue size is taken either infinite or finite and equal to some number  $L$ .

In what follows we start with the simple first policy in order to illustrate the usefulness of the guard channel concept. We then directly study the third and more general policy which combines queueing and guard channels. (The second policy outlined above merely corresponds to the special case  $g = 0$  of this more general policy.) Note that the first possibility is also a special case of the third one corresponding to the case where the queue length  $L = 0$ .

For each method, we will first obtain the state probabilities of the cell. Here a state is the number of customers in service or waiting for service in the cell. We will then derive the expression of meaningful parameters such as blocking probabilities, probability of delay, average delay, etc.

## 5.2. Illustration of the method for a simple system without queueing.

First we define some useful notation:

$$\begin{aligned} \theta &= \mu + \eta, & a &= \frac{\alpha}{\theta}, \\ \alpha &= \lambda + \gamma, & c &= \frac{\gamma}{\theta}. \end{aligned}$$

We recall that  $\mu$  is the total service rate,  $\eta$  is the handoff rate,  $\lambda$  is the arrival rate of originating calls and  $\gamma$  is the arrival rate of handoff calls. So  $a$  can be considered as the total traffic offered to the cell, and  $c$  is the handoff traffic offered to the cell. We call  $g$  the number of guard channels, and we want to determine an appropriate value of  $g$  in order to protect handoff calls from high blocking without increasing the blocking of originating calls too much. Our system is a simple blocking system with memoryless arrivals and memoryless service time, and state-dependent arrival rate. Namely, the arrival rate into the cell is equal to  $(\lambda + \gamma)$  if we have fewer than

$n - g$  customers present in the system, and simply equal to  $\gamma$  if we have  $n - g$  or more customers in the system. Basic queueing theory can then be shown to give ([42], Chapter 3, Section 1):

State probabilities:

$$\begin{aligned} 0 \leq i \leq n - g & \quad P(i) = \frac{a^i}{i!} P(0), \\ n - g \leq i \leq n & \quad P(i) = \frac{a^{n-g} c^{i-(n-g)}}{i!} P(0), \end{aligned}$$

where

$$P(0) = 1 / \left[ \sum_{i=0}^{n-g-1} \frac{a^i}{i!} + a^{n-g} \sum_{i=n-g}^n \frac{c^{i-(n-g)}}{i!} \right]$$

is merely a normalizing factor so that the state probabilities add up to 1. The blocking probability  $B_O$  of originating calls is then given by:

$$\begin{aligned} B_O &= P(i \geq n - g) \\ \Rightarrow B_O &= \left[ a^{n-g} \sum_{i=n-g}^n \frac{c^{i-(n-g)}}{i!} \right] P(0). \end{aligned} \quad (5.1)$$

Similarly for handoff calls we have the blocking probability  $B_H$  given by:

$$\begin{aligned} B_H &= P(i \geq n) \\ \Rightarrow B_H &= \frac{a^{n-g} c^g}{n!} P(0). \end{aligned} \quad (5.2)$$

We now evaluate the two blocking probabilities given by Equations (5.1) and (5.2) for a typical cellular system. We choose a cell with  $n = 44$  frequency channels.\* We first choose a total offered traffic  $a = 40$  Erlangs (heavy traffic), and

---

\* With the initial bandwidth of 40 MHz and a channel spacing of 30 kHz, we have 333 channels. If we choose a configuration with 7 channel sets, taking into account the 21 control channels, we have 312 channels left for calls, and  $7 \times 44 = 308$ .

a handoff traffic  $c = 8$  Erlangs. Figure 5.1 illustrates the effect of increasing the value of  $g$  for the blocking probabilities of the two types of calls.

The first thing we can notice is the difference in slope between the two curves. It implies that the decrease in blocking probability of handoff calls with increasing  $g$  will be much steeper than the corresponding increase in the blocking probability of originating calls. This is what makes the guard channel method so useful in Cellular Radio.

We can now estimate the behavior of the guard channel method when the total offered traffic varies. If we compute the ratio of the blocking probability of handoffs over the blocking probability of originating calls we simply get:

$$\frac{B_H}{B_O} = \frac{c^n/n!}{\sum_{i=n-g}^n c^i/i!}$$

For a fixed value of  $g$  (other than 0, for which the ratio is 1), this expression is easily shown to decrease if  $c$  is decreased, as one would probably guess:

$$\begin{aligned} \frac{d}{dc} \left( \frac{B_H}{B_O} \right) &= \left[ \frac{c^{n-1}}{(n-1)!} \left( \sum_{i=n-g}^n \frac{c^i}{i!} \right) - \frac{c^n}{n!} \left( \sum_{i=n-g}^n \frac{c^{i-1}}{(i-1)!} \right) \right] \Sigma^{-2} \\ &= \frac{c^{n-1}}{(n-1)!} \left[ \sum_{i=n-g}^n \frac{c^i}{i!} - \frac{c}{n} \sum_{i=n-g}^n \frac{c^{i-1}}{(i-1)!} \right] \Sigma^{-2} \\ &= \frac{c^{n-1}}{(n-1)!} \left[ \sum_{i=n-g}^n \frac{c^i}{i!} - \sum_{i=n-g}^n \frac{c^i}{n(i-1)!} \right] \Sigma^{-2} \\ &\geq \frac{c^{n-1}}{(n-1)!} \left[ \sum_{i=n-g}^n \frac{c^i}{i!} - \sum_{i=n-g}^n \frac{c^i}{i!} \right] \Sigma^{-2} \end{aligned}$$

(because  $n(i-1)! \geq i!$  in the range of  $i$ )

$$\geq 0,$$

where  $\Sigma = \sum_{i=n-g}^n c^i/i!$ .

Therefore, if we keep the fraction of the total traffic due to handoffs constant, and decrease the total traffic, the efficiency of the guard channel method should

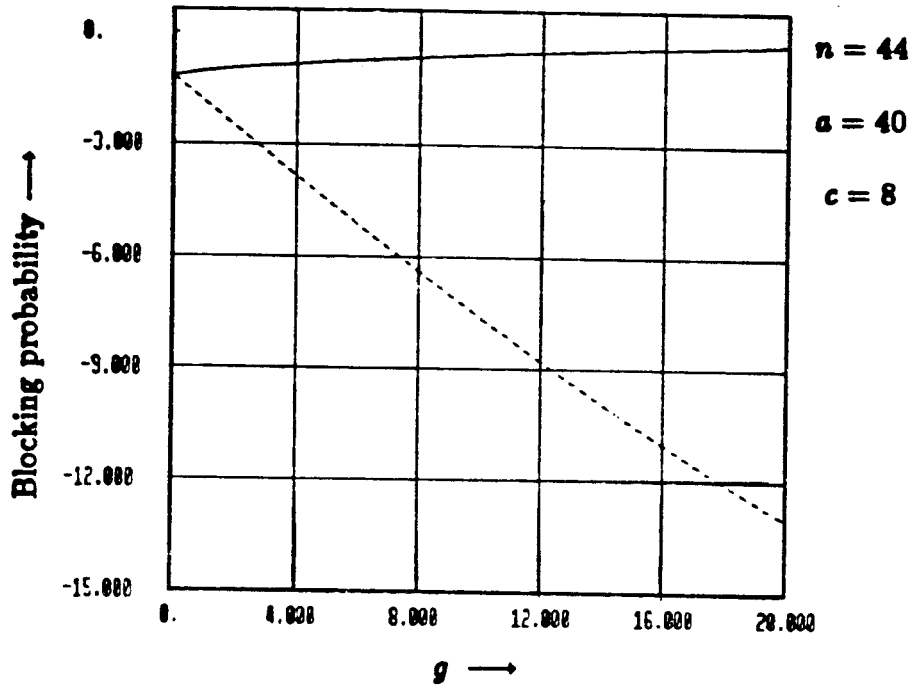


Fig. 5.1.  $\text{Log}_{10}(B_H)$  (---) and  $\text{Log}_{10}(B_0)$  (—) as functions of the number of guard channels  $g$ .

decrease too. This is easily verified by plotting blocking probabilities of the handoff and originating calls as functions of the number of guard channels for different values of the total traffic  $a$ . Figures 5.2 and 5.3 illustrate this for  $a = 30$ ,  $c = 6$  Erlangs, and  $a = 20$ ,  $c = 4$  Erlangs respectively.

This decrease in efficiency is, however, not too penalizing, since in the case of low traffic we do not really need to worry about the possible blocking of handoff calls, because of an already very low blocking probability. This tells us that the guard channel method is most efficient precisely in the region where improvement of the blocking of handoff calls is most desirable. Note that our assumption of keeping the fraction of the total offered traffic due to handoff calls constant is not too unrealistic, since the percentage of calls that are handed off should not really depend on how many calls enter the system.

However, we must realize that the introduction of guard channels does not have only advantages. A price paid for the decrease of the blocking probability of handoff calls is a slight decrease in total carried traffic, where the carried traffic is the traffic that actually goes through the system. Namely, it is the total offered traffic  $a$  decreased by the blocked calls. The carried traffic is of importance for most communication systems because it is in direct proportion to revenues (blocked calls are not charged, even though the system has done some work trying to establish a connection). The reason for the slight decrease in carried traffic when guard channels are introduced is due to the fact that the greater number of originating calls blocked is not completely compensated by the additional number of handoff calls accepted by the system. This mainly comes from the fact that the fraction of the total offered traffic  $a$  due to originating calls is greater than the one due to handoff calls. This decrease in traffic as a function of the number of guard channels is illustrated in Figure 5.4 for the case of  $a = 40$  Erlangs and  $c = 8$  Erlangs.

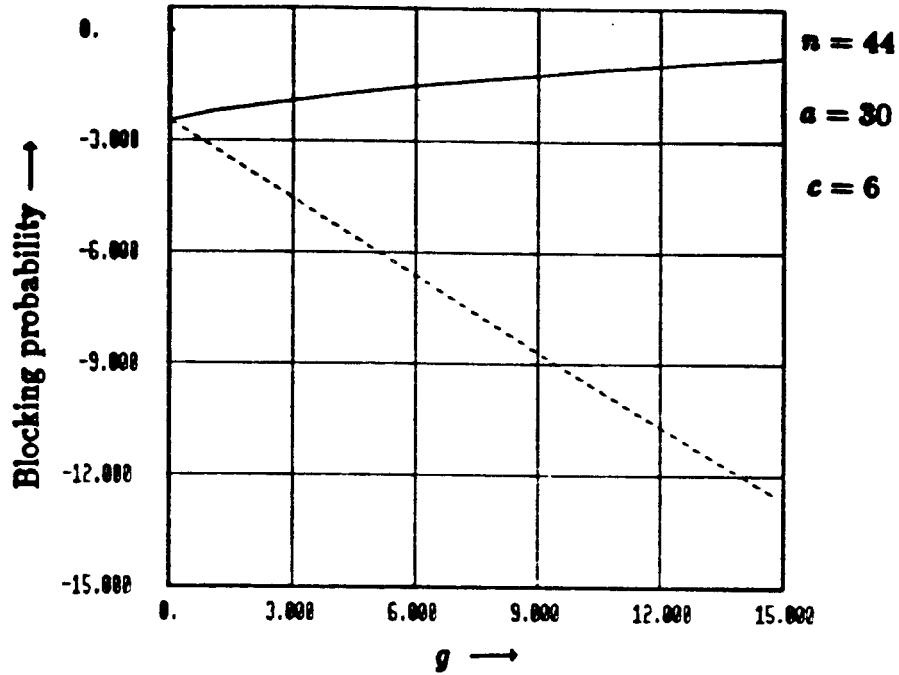


Fig. 5.2.  $\text{Log}_{10}(B_H)$  (---) and  $\text{Log}_{10}(B_O)$  (—) in medium traffic conditions.

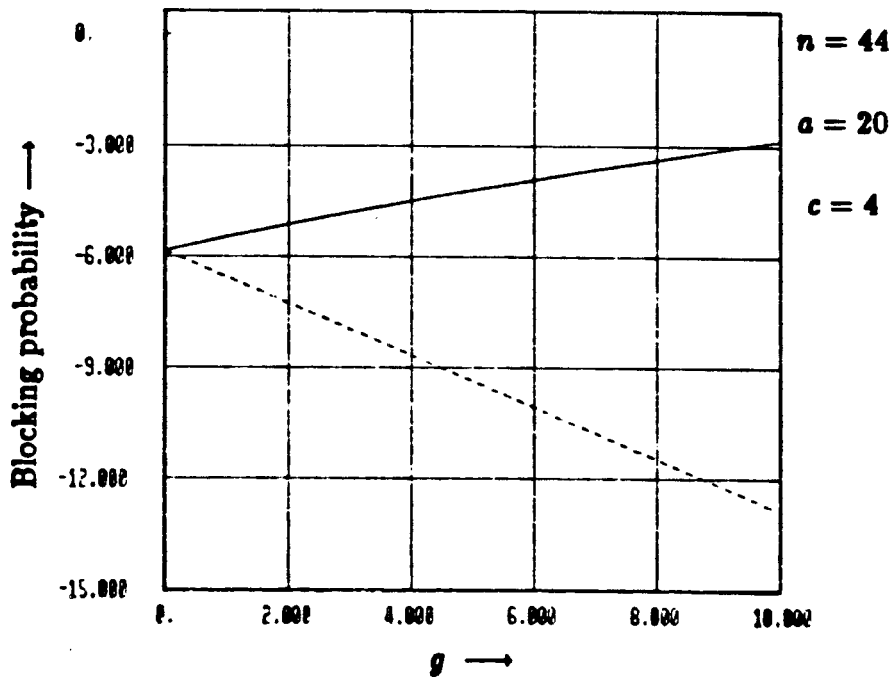
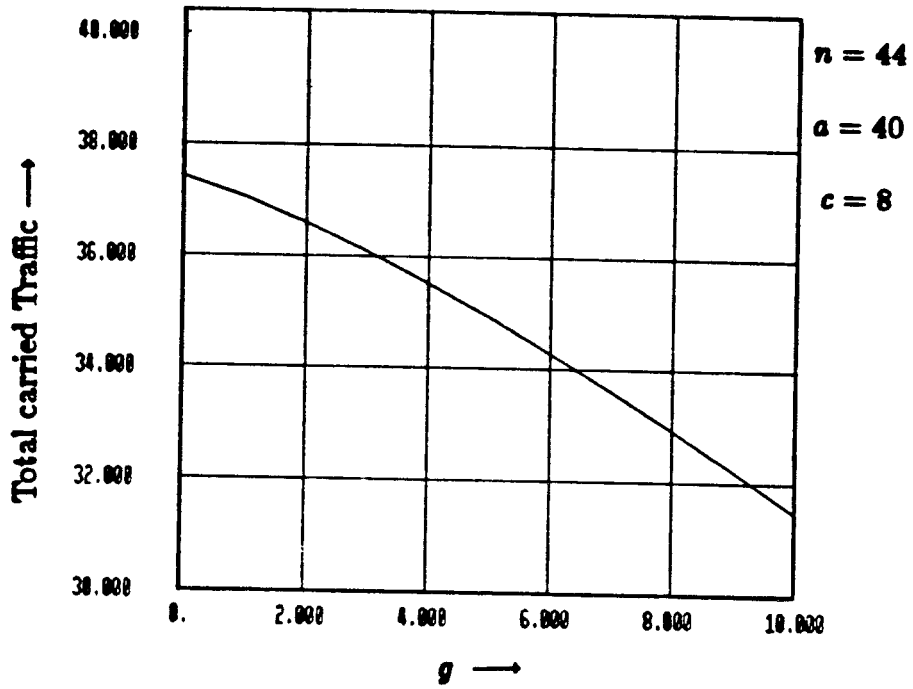


Fig. 5.3.  $\text{Log}_{10}(B_H)$  (---) and  $\text{Log}_{10}(B_O)$  (—) in light traffic conditions.





*Fig. 5.4. Total carried traffic as a function of the number of guard channels  $g$ .*

We now treat the more complicated case where guard channels are combined with the possibility of queueing handoff calls.

### 5.3 General case with guard channels and queueing of handoffs.

We first assume that infinite queues are allowed for handoff calls. The state balance equations can then be easily derived using the state diagram representation of the system. Three distinct cases have to be distinguished, corresponding to the three possible behaviors of the system. Either all customers are served without any distinction of type, or originating calls are blocked and handoff calls are served, or finally, originating calls are blocked and handoff calls are queued. More precisely, we have the following relations:

$$1. 0 \leq i \leq n - g - 1$$

$$P(i + 1) = \frac{a}{i + 1} P(i),$$

$$2. n - g \leq i \leq n - 1$$

$$P(i + 1) = \frac{c}{i + 1} P(i),$$

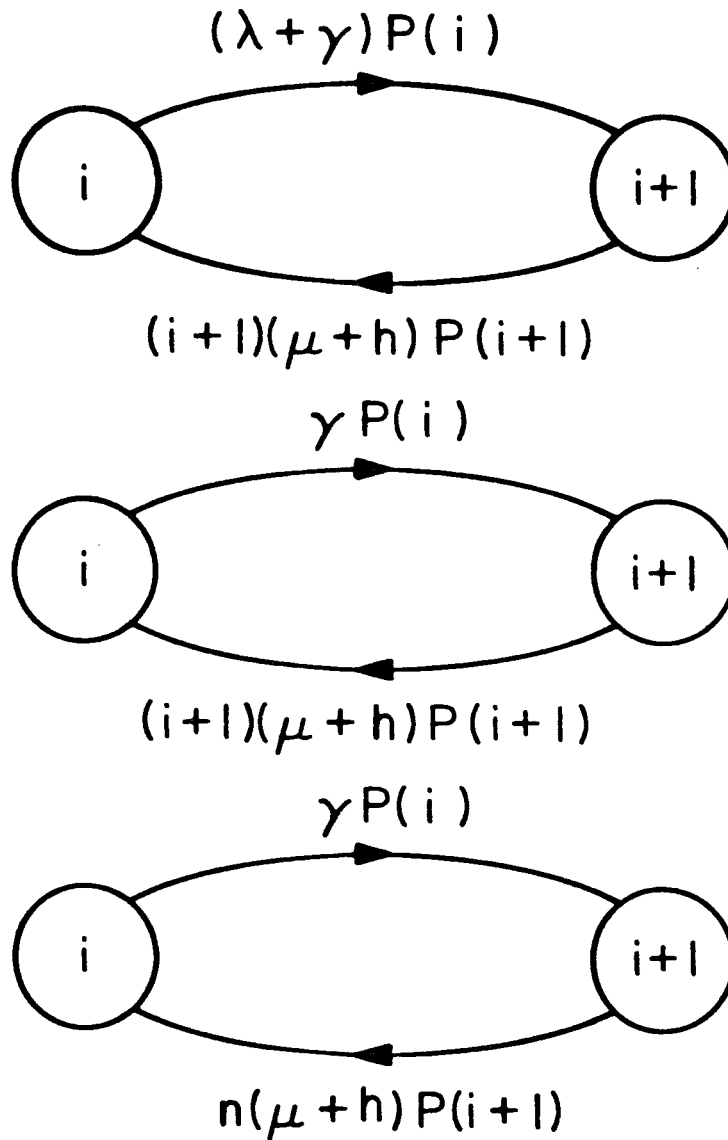
$$3. n \leq i$$

$$P(i + 1) = \frac{c}{n} P(i).$$

The associated state diagram is Figure 5.5.

From the above relations between the state probabilities obtained from the state transitions we easily derive the exact expressions of the state probabilities. Namely, we get:

$$\begin{cases} 0 \leq i \leq n - g, & P(i) = \frac{a^i}{i!} P(0); \\ n - g \leq i \leq n, & P(i) = \frac{a^{n-g} c^{i-(n-g)}}{i!} P(0); \\ n \leq i, & P(i) = \frac{a^{n-g} c^{i-(n-g)}}{n! n^{i-n}} P(0), \end{cases} \quad (5.3)$$



*Fig. 5.5. State diagram for the infinite queue case.*

where  $P(0)$  is given by:

$$P(0) = 1 / \left[ \sum_{i=0}^{n-g-1} \frac{a^i}{i!} + a^{n-g} \sum_{i=n-g}^{n-1} \frac{c^{i-(n-g)}}{i!} + \frac{a^{n-g} c^g}{(n-1)!(n-c)} \right]. \quad (5.4)$$

Note that  $P(0)$  is again merely a normalization factor so that the state probabilities add up to 1.

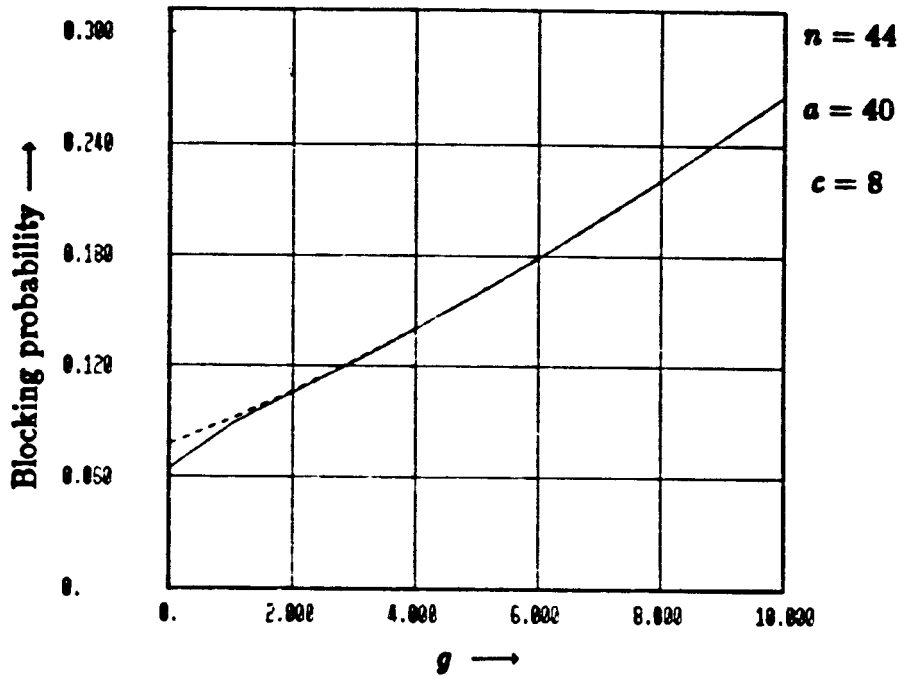
Being now provided with the state probabilities as given by (5.3), we can compute the blocking probability  $B'_O$  for originating calls:

$$\begin{aligned} B'_O &= P\{i \geq n-g\} \\ \Rightarrow B'_O &= \left[ \left(\frac{a}{c}\right)^{n-g} \sum_{i=n-g}^{n-1} \frac{c^i}{i!} + \frac{a^{n-g} c^g}{(n-1)!(n-c)} \right] P(0). \end{aligned} \quad (5.5)$$

We remark that due to the fact that handoff calls are allowed to queue,  $B'_O$  is slightly greater than the previous  $B_O$ , even in the case  $g = 0$ , because all handoff calls present in the queue will be served prior to any originating call. This fact is illustrated in Figure 5.6, which gives the blocking probability of originating calls as a function of the number of guard channels for the case where handoff calls are blocked if all channels are busy, and the case where handoff calls are allowed to queue when all channels are busy.

Based on Figure 5.6 we note that the difference between the two cases is not really significant. This means that allowing handoff calls to be queued does not bring in additional penalty for originating calls. The reason for this is that when guard channels are introduced handoff calls nearly always find a free channel upon their arrival, and therefore hardly ever have to queue. The only noticeable difference for originating calls occurs for  $g = 0$  where we have:

$$B_O \simeq 6.46 \cdot 10^{-2} \quad \text{versus} \quad B'_O \simeq 7.79 \cdot 10^{-2} \quad (a = 40, c = 8, n = 44)$$



**Fig. 5.6. Blocking probability of originating calls with (—) and without (---) queuing handoff calls.**

Furthermore, the fact that we assume infinite queueing capacity for handoff calls implies that no handoff call will ever be blocked, even if the price paid is a long waiting time. We will treat the more realistic case of finite queue length later, and we will also estimate the average waiting time experienced by handoff calls.

Since, as we have said, handoff calls are queued, there is no blocking probability for them. However, we can compute an equivalent expression which is the probability of being delayed. It is easily obtained from the state probabilities given in system (5.3):

$$\begin{aligned}
 P \{ \text{handoff call is delayed} \} &= P(i \geq n) = P_H(> 0) \\
 \Rightarrow P_H(> 0) &= \frac{a^{n-g} c^g}{(n-1)!(n-c)} \cdot P(0). \quad (5.6)
 \end{aligned}$$

We can also obtain the probability of a handoff being delayed more than a certain time  $t$ . This probability is the sum over all possible values  $i \geq n$  of finding  $i$  customers in the system, times the probability that there are fewer than  $n - i$  departures from the system in time  $t$ . The departure process from the system when all servers are busy is a Poisson process with parameter  $n\theta$ .

We start the calculation by computing the density function of the waiting time of a handoff call. Namely, a handoff call will have a waiting time between  $t$  and  $t + dt$  if it finds  $i \geq n$  customers in the system upon its arrival, if  $(i - n)$  depart within time  $t$ , and if another one departs between  $t$  and  $t + dt$ . For a given value of  $i \geq n$  this probability is given by:

$$P\{w \in [t, t + dt) / i\} = e^{-\theta n t} \frac{(\theta n)^{i-n}}{(i-n)!} \theta n dt.$$

The expression for the density function  $w(t)$  of the waiting time of a handoff call is then obtained by summing for all values of  $i$  the above expression multiplied by the probability of finding  $i \geq n$  customers in the system:

$$\begin{aligned}
 w(t)dt &= \sum_{i=n}^{\infty} P(i) P\{w \in [t, t+dt] / i\} \\
 &= \frac{a^{n-g} c^g}{n!} e^{-\theta n t} \theta n dt P(0) \sum_{i=n}^{\infty} \left(\frac{c}{n}\right)^{i-n} \frac{(\theta n t)^{i-n}}{(i-n)!} \\
 &= \frac{a^{n-g} c^g}{n!} e^{-\theta n t} \theta n dt P(0) \sum_{i=n}^{\infty} \frac{(c\theta t)^{i-n}}{(i-n)!} \\
 &= \frac{a^{n-g} c^g}{n!} e^{-\theta n t} \theta n dt P(0) \sum_{i=0}^{\infty} \frac{(c\theta t)^i}{i!} \\
 &= \frac{a^{n-g} c^g}{n!} e^{-\theta n t} \theta n dt P(0) e^{c\theta t} \\
 \Rightarrow w(t) &= \frac{a^{n-g} c^g}{(n-1)!} \theta e^{-t\theta(n-c)} P(0). \tag{5.7}
 \end{aligned}$$

After integrating, we obtain the probability that a handoff call waits more than a certain time  $t$ . Namely, we get:

$$P_H(> t) = \frac{a^{n-g} c^g}{(n-1)!(n-c)} e^{-t\theta(n-c)} P(0). \tag{5.8}$$

It is interesting to note that the waiting time turns out to be conditionally memoryless, given that the system is busy. This tells us that the probability that a handoff call waits more than a certain time  $t$  is simply the probability of finding a busy system, times the probability of no departure from the system in time  $t$ , divided by the probability of no arrival in time  $t$ . Equation (5.7) also gives us the expression of the average delay  $W_H$  of handoff calls. We have:

$$W_H = \frac{a^{n-g} c^g}{(n-1)!(n-c)^2 \theta} \cdot P(0) = \frac{1}{\theta(n-c)} \cdot P(> 0). \tag{5.9}$$

We recall that we decided to introduce the possibility of queueing handoff calls, in order to further decrease their blocking probability, but we were also hoping

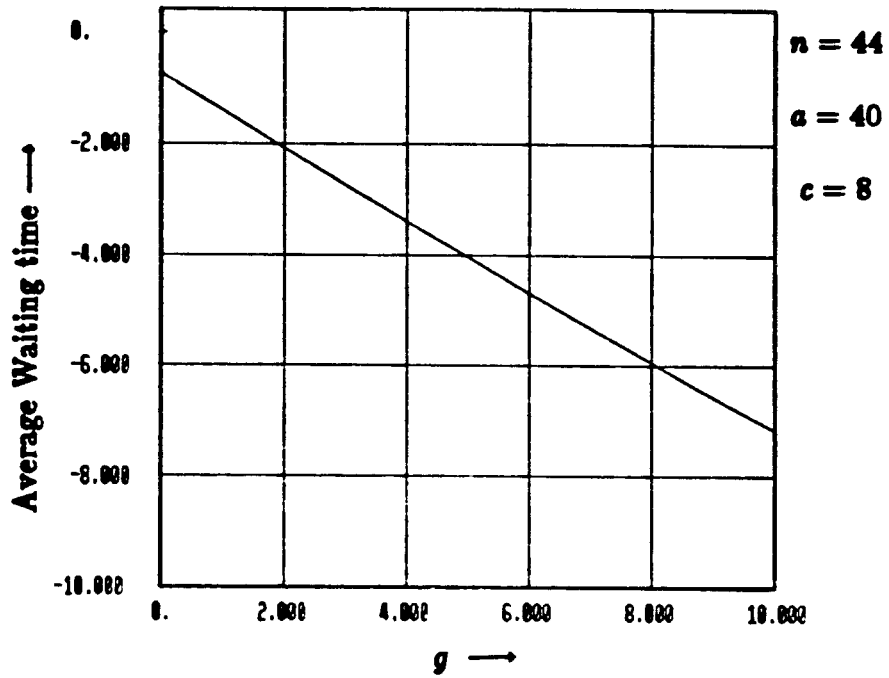
for a small average waiting time. The need for a small waiting time comes from the fact that we want the queueing process to go nearly unnoticed by a customer involved in a call. An example of the behavior of the average waiting time of handoff calls is provided in Figure 5.7 for  $a = 40$  Erlangs,  $c = 8$  Erlangs, an average total service time of 2 min, and on the average one handoff every 5 min ( $\theta \simeq 0.012 / \text{sec}$ ). Figure 5.7 gives the logarithm base 10 of the average waiting time of handoff calls expressed in seconds. For example, we have an average waiting time of approximately 0.18 sec for handoff calls with no guard channels, but this is decreased to  $10^{-2}$  sec if we have  $g = 2$  guard channels, and further decreased to  $10^{-4}$  sec for  $g = 5$  guard channels.

Figure 5.7 tells us that the decrease in the average waiting time of handoff calls is rather steep when the number of guard channels is increased. We could therefore conclude that guard channels are efficient in decreasing the average waiting time of handoff calls, making the possibility of queueing handoff calls useful. However, what we are really interested in is the average delay of handoff calls that actually do experience a delay. That is, we want the average delay given that the call is delayed. This conditional delay is simply obtained by dividing the overall average delay  $W_H$  given in Equation (5.9) by the probability of being delayed  $P(> 0)$  given in Equation (5.6). This quantity turns out to be independent of the number of guard channels  $g$ , and is easily found equal to:

$$D_H = \frac{1}{\theta(n - c)} \quad (5.10)$$

This tells us that increasing the number of guard channels will only decrease the number of handoff calls that have to be queued, without reducing their average delay. A rigorous explanation for this can be obtained by remembering that once a call is in progress, it is impossible to determine its previous type (handoff or





*Fig. 5.7.  $\text{Log}_{10}(W_H)$  (average waiting time) for handoff calls as a function of the number of guard channels  $g$ .*

originating). Therefore, once we are in the situation where all servers (channels) are busy, we have no information on the possible value of  $g$ , and our waiting time will depend only on the residual service times of the customers in service. These times are clearly independent of  $g$  because of the memoryless distribution of the channel occupancy time.

If we take again:  $a = 40$ ,  $c = 8$ ,  $n = 44$  and choose  $\theta \simeq 0.012/\text{sec}$  ( $\mu \simeq 0.5/\text{min}$ ,  $\eta \simeq 0.2/\text{min}$ ), we find:

$$D_H \simeq 2.31 \text{ sec.}$$

On the average the delay introduced in the handoff process by queueing, using the above parameters, is greater than 2 seconds and therefore perceptible when experienced. It may be too large for adequate service quality.

We now investigate the case where queueing is again allowed, but the queue size is more realistically taken to be finite, equal to some integer  $L$ . We are hoping that the queue size limitation might help decrease the average waiting time of calls that are delayed, since long delays come from customers that find a large queue ahead of them upon their arrival. However, limiting the queue size implies that a blocking probability for handoff calls will again appear.

The only change in the state probabilities is that they are now equal to zero for  $i$  strictly bigger than  $n + L$ . On the other hand, the state probabilities for  $0 \leq i \leq n + L$  are still given by Equation (5.3), and the only modification needed is in the expression of the normalizing factor  $P(0)$ , which can be obtained without recalculation by truncation.

What we mean by **truncation** is a simple technique that allows one to obtain the state probabilities for a finite-storage birth and death system, such as ours,

directly from the state probabilities of the corresponding infinite-storage-capacity system. This technique does not seem to be published in the literature, and therefore we will justify it. We first note that all the customers that find our finite storage system in a non-blocking situation (queue not full) have no means of distinguishing it from an infinite-capacity system (memoryless arrivals and service times). This is illustrated by the fact that the state diagram remains the same as given in Figure 5.5, until the system is saturated (queue is full). This tells us that the state probabilities should have the same literal expressions in terms of  $P(0)$  as the ones given by Equation (5.3). The only difference with the infinite storage capacity system is that the state probabilities are equal to zero for all states exceeding the system capacity. Furthermore, since the state probabilities must add up to one, the only modification we need is to choose the normalization factor  $P(0)$  such as to yield this result. Namely, we have:

$$P_L(0) = 1 / \left[ \sum_{i=0}^{n-g-1} \frac{a^i}{i!} + \left(\frac{a}{c}\right)^{n-g} \sum_{i=n-g}^{n-1} \frac{c^i}{i!} + \frac{a^{n-g}c^g}{(n-1)!} \frac{1 - (\frac{c}{n})^{L+1}}{n-c} \right]. \quad (5.11)$$

The other state probabilities are, as we said, obtained simply by replacing  $P(0)$  by  $P_L(0)$ . This gives a blocking probability for originating calls  $B'_{LO}$  equal to:

$$B'_{LO} = \left[ \left(\frac{a}{c}\right)^{n-g} \sum_{i=n-g}^{n-1} \frac{c^i}{i!} + \frac{a^{n-g}c^g}{(n-1)!} \frac{1 - (\frac{c}{n})^{L+1}}{n-c} \right] P_L(0). \quad (5.12)$$

As a check, Equation (5.12) reduces to Equation (5.5) if we let  $L \rightarrow \infty$  and to Equation (5.1) if  $L = 0$ .

Similarly, for handoff calls we can compute the blocking probability  $B'_{LH}$  introduced by the finite queue length  $L$ . We have:

$$\begin{aligned} B'_{LH} &= P\{i = n + L\} \\ \Rightarrow B'_{LH} &= \frac{a^{n-g}c^{L+g}}{n! n^L} P(0). \end{aligned} \quad (5.13)$$

Again we can check that Equation (5.13) goes to 0 if  $L \rightarrow \infty$  and reduces to Equation (5.2) if  $L = 0$ .

We can also derive the probability for a handoff call to be delayed:

$$\begin{aligned} P\{\text{handoff call is delayed}\} &= P[n \leq i < n + L] = P_{LH}(> 0) \\ \Rightarrow P_{LH}(> 0) &= \frac{a^{n-g}c^g}{(n-1)!} \frac{1 - (\frac{c}{n})^L}{n-c} P(0). \end{aligned} \quad (5.14)$$

Equation (5.14) gives 0 for  $L = 0$  and reduces to Equation (5.6) if we let  $L \rightarrow \infty$ , as must be.

Finally, we want the probability that a handoff call is delayed more than  $t$ . As in the case of infinite queue length, we start by deriving the density function of the waiting time of handoff calls. The computations are completely parallel to the ones that led to Equation (5.7) except for the fact we now only consider the cases  $n \leq i < L$ , and we get:

$$w_L(t) = P_L(0) \frac{a^{n-g}c^g}{n!} \theta n e^{-\theta n t} \sum_{i=0}^{L-1} \frac{(c\theta t)^i}{i!}. \quad (5.15)$$

After integration, this gives the desired expression for the probability that a handoff call be delayed more than  $t$ :

$$\begin{aligned} P_{LH}(> t) &= P_L(0) \frac{a^{n-g}c^g}{n!} \sum_{i=0}^{L-1} \int_t^{\infty} \theta n e^{-\theta n x} \frac{(c\theta x)^i}{i!} dx \\ P_{LH}(> t) &= P_L(0) \frac{a^{n-g}c^g}{n!} e^{-\theta n t} \sum_{i=0}^{L-1} \left( \sum_{j=0}^i \frac{(c\theta t)^{i-j}}{(i-j)!} \left(\frac{c}{n}\right)^j \right). \end{aligned} \quad (5.16)$$

Equation (5.16) can be checked to reduce to Equation (5.8) when we let  $L \rightarrow \infty$ . This is best seen by realizing that for  $L \rightarrow \infty$  the expression of  $w_L(t)$  gives the same expression as Equation (5.7), since we have:

$$\lim_{L \rightarrow \infty} \left( \sum_{i=0}^{L-1} \frac{(c\theta t)^i}{i!} \right) = e^{c\theta t}.$$

This immediately yields the desired result.

From Equation (5.15), we can also deduce the expression of the average waiting time  $W_{LH}$  of handoff calls. Namely, we have:

$$\begin{aligned}
 W_{LH} &= \int_0^{\infty} t w_L(t) dt \\
 &= P_L(0) \frac{a^{n-g} c^g}{n!} \sum_{i=0}^{L-1} \int_0^{\infty} \theta n e^{-\theta n t} \frac{(c\theta t)^i}{i!} dt \\
 &= P_L(0) \frac{a^{n-g} c^g}{n!} \sum_{i=0}^{L-1} \frac{i+1}{c\theta} \int_0^{\infty} \theta n e^{-\theta n t} \frac{(c\theta t)^{i+1}}{(i+1)!} dt \\
 &= P_L(0) \frac{a^{n-g} c^g}{n!} \frac{1}{c\theta} \sum_{i=0}^{L-1} (i+1) \left(\frac{c}{n}\right)^{i+1}.
 \end{aligned}$$

Here we can simplify:

$$\begin{aligned}
 \sum_{i=0}^{L-1} (i+1) \left(\frac{c}{n}\right)^{i+1} &= \frac{c}{n} \sum_{i=1}^L i \left(\frac{c}{n}\right)^{i-1} \\
 &= cn \cdot \frac{1 - (L+1) \left(\frac{c}{n}\right)^L + L \left(\frac{c}{n}\right)^{L+1}}{(n-c)^2}.
 \end{aligned}$$

So we finally get:

$$W_{LH} = P_L(0) \frac{a^{n-g} c^g}{(n-1)!(n-c)^2 \theta} \left( 1 - (L+1) \left(\frac{c}{n}\right)^L + L \left(\frac{c}{n}\right)^{L+1} \right). \quad (5.17)$$

Equation (5.17) indeed does give  $W_{LH} = 0$  for  $L = 0$ , as must be, and it also reduces to Equation (5.9) when we let  $L \rightarrow \infty$ .

We can now obtain the average delay given delayed,  $D_{LH}$ , simply by dividing the above expression for  $W_{LH}$  by  $P_{LH}(> 0)$  as given in Equation (5.14). We find that it is again independent of the number of guard channels  $g$ , as must be, and equal to:

$$D_{LH} = \frac{1}{\theta(n-c)} \frac{1 - (L+1) \left(\frac{c}{n}\right)^L + L \left(\frac{c}{n}\right)^{L+1}}{1 - \left(\frac{c}{n}\right)^L}. \quad (5.18)$$

This quantity is to be compared with the average delay of handoff calls  $D_H$  (Equation (5.10)), when infinite queue size is allowed. In particular, we find that for  $L = 1$ , which should give the smallest possible delay when queueing is allowed, we get:

$$D_{LH} = \frac{1}{\theta n}, \quad \text{versus} \quad D_H = \frac{1}{\theta(n-c)}.$$

With our previous values of  $c = 8$  Erlangs,  $n = 44$ , and  $\theta = 0.012 \text{ sec}^{-1}$ , this gives:

$$D_{LH} = 1.89 \text{ sec} \quad \text{versus} \quad D_H = 2.31 \text{ sec}.$$

We therefore have a slight decrease in the average delay of handoff calls that have to wait when we limit the queue size, but it is not really significant. This decrease is independent of the number of guard channels chosen.

Based on all the results we have obtained so far, we will now evaluate the method of guard channels for protecting handoff calls in a cellular system.

#### 5.4 Evaluation of the method.

In this chapter we introduced some simple policies to decrease the blocking probability of handoff calls without penalizing originating calls too much. These policies rely on two basic concepts. The first one consists in using guard channels to increase the protection level of handoffs, the second offers the possibility of queueing handoff calls if all channels are busy upon their arrival.

Section 5.2 describes the effect of guard channels, and we show that this method is successful in decreasing the blocking of handoff calls, while not increasing too much the blocking of originating calls. The method is also found to be most efficient in the heavy traffic case which is the one where improvement is most needed for handoff calls. Furthermore, this technique is rather simple to implement in the case of Cellular Radio, since the system can easily distinguish between handoffs

and originating calls through the use of the control channels, and always monitors the number of busy channels in each cell. The guard channel policy can therefore be implemented in a centralized way (at the MTSO or ECP level), and is rather flexible, since the number of guard channels needed to achieve a given blocking probability for handoff calls can be fixed independently for each cell.

In Section 5.3, we treated the more general case where, in addition to guard channels, handoff calls were allowed to queue. The queueing facility was introduced with the hope that the average delay experienced by handoff calls that would find all channels busy would stay small. In the case where infinite queueing capacity was considered, the average delay of customers that have to queue was found independent of the number of guard channels, and in typical cases would yield noticeable communications interruptions of several seconds for the handed-off customers (see Equation (5.10)).

In the more realistic case of finite queue length, we found that the average delay experienced by queued handoff calls was again independent of the number of guard channels and only slightly smaller than in the case of infinite queue (Equation (5.18)). The main conclusion is that for both cases of infinite or finite queue, the average delay of queued handoff calls is approximately inversely proportional to the total number of channels in the cell. This tells us that the introduction of queueing will be useful mainly in the case of cells with a large number of frequency channels, probably more than the 44 we actually have. For example, with one waiting space available for the queueing of handoff calls, one would need a cell with 167 channels in order to have an average waiting time of 0.5 sec for the queued handoff calls ( $\theta = 0.012 \text{ sec}^{-1}$  as before).

However, in real system queueing might be applicable even for cells with small number of frequency channels. This is due to the fact that handoffs usually do

not need to be performed instantaneously, and there is usually a small time margin during which the received signal power is decreasing but still acceptable. It is therefore possible to use this time margin to decrease the waiting time of queued handoff calls to a more appropriate level. In other words this means that a system might be willing to accept a short time with rather bad signal quality in order to decrease the time (queueing time) during which no transmissions at all take place. Recall that this is done in order to avoid a potential blocking of a handoff call.

To summarize, guard channels and queueing of handoff calls represent simple and useful methods to reduce the blocking of handoff calls in order to improve the perceived quality of cellular service. They require no modification of the mobiles. This type of improvement might be rather critical in a start-up market, where customers do not necessarily have any particular level of confidence in the cellular concept. Moreover, the two methods are based on techniques familiar to classic telephony, and can therefore be implemented at a rather low investment cost. Finally, let us mention that the price paid in both cases to improve the protection of handoff calls is a slight decrease in total carried traffic. In the next chapter we will study traffic policies that not only decrease the blocking probability of handoff calls, but also increase the total carried traffic.



**CHAPTER 6**  
**TRAFFIC POLICIES THAT PROTECT HANDOFFS**  
**AND**  
**INCREASE THE CARRIED TRAFFIC**

**6.1 Assumptions and system description.**

In this chapter we present some traffic policies that not only help decrease the blocking probability of handoff calls, but also increase the total traffic carried by the system.

In Chapter 5 we introduced the method of guard channels and we combined it with the queueing of handoff calls. We found that guard channels were very efficient in decreasing the blocking probability of handoff calls, but queueing handoff calls did not always offer such a dramatic improvement. This is due to the fact that the average waiting time of queued handoff calls (conditioned on waiting) is independent of the number of guard channels and usually corresponds to perceptible service interruptions except for cells with a large number of frequency channels (around 160 channels to get an average delay of 0.5 sec). Furthermore, we recall that there was a small decrease in the total traffic carried by the system when using these methods, since even though fewer handoff calls were blocked, the other type of calls (originating calls) were blocked more often than before.

This naturally leads us to consider methods where we still have guard channels to protect handoff calls, but where the queueing of handoff calls is no longer allowed, since it did not prove to be really useful in most cases. Also, in order to compensate for the guard channels that give higher priority to handoff calls, we allow originating calls to be queued. Introducing the possibility of queueing for originating calls has the additional advantage of increasing the total traffic carried by the system. This

is due to the fact that, as before, thanks to the guard channels, more handoff calls go through the system, but now this is no longer paid by less originating calls going through the system as it was the case in Chapter 5. With the possibility of queueing originating calls the decrease in the blocking probability of handoff calls only causes a small delay for originating calls. The important point is that originating calls are delayed and not blocked, so they will end up going through the system; therefore contributing to the total carried traffic. All this finally yields a larger total carried traffic.

These concepts will be more precisely detailed in what follows, but the purpose of this section is simply to motivate and introduce the chosen policies. We now describe the queueing system associated to the above method.

We consider a system with two types of customers having access to a service facility, which is formed by a set of  $n$  servers. The two types of customers (handoff calls and originating calls) have the same service rate  $\mu$  but distinct arrival rates  $\gamma$  and  $\lambda$ , respectively. In addition to this, the following features are present: Customers of type I (handoff,  $\gamma$ ) have access to all channels with no restriction but will be blocked if all servers are busy. On the other hand, customers of type II (originating,  $\lambda$ ) have free access to the servers as long as there are more than  $g$  servers free. In case only  $g$  or fewer servers remain free, customers of type II will be queued and will receive service in a first come, first served basis as soon as more than  $g$  servers become idle.

We will now describe the two-dimensional state diagram representing the above system. The pair  $(i_1, i_2)$  will denote the state of the system, with the following notation:  $i_1$  will be the number of customers in the queue, and  $i_2$  will be the number of customers in service. The state diagram then has two distinct parts.

The first one describes the cases where fewer than  $(n - g)$  servers are busy. This part forms the lower part of the first column, and is entirely similar to what we would have with a purely blocking system. The second part is the one that really represents the special features of the system, and it is only at this point that the second dimension is introduced. We will however see later that even this part can be reduced to a one-dimensional diagram, for some purposes. A complete description of the entire diagram is given in Figure 6.1. Also, when a stationary solution exists (ergodic system), the states are related to each other through the balance equations, whose expressions are given in equations  $B_1, B_2, B_3$ .

$$\underline{B_1.} \quad (i_1 = 0, \quad 0 \leq i_2 \leq n - g)$$

$$[b + c]P(0, 0) = P(0, 1)$$

$$\vdots$$

$$[b + c]P(0, n - g - 1) = [n - g]P(0, n - g)$$

$$\underline{B_2.} \quad (i_1 = 0, \quad n - g \leq i_2 \leq n)$$

$$[(n - g) + (b + c)]P(0, n - g) = [b + c]P(0, n - g - 1)$$

$$+ [n - g + 1]P(0, n - g + 1)$$

$$+ [n - g]P(1, n - g)$$

$$[(n - g + 1) + (b + c)]P(0, n - g + 1) = cP(0, n - g) + [n - g + 2]P(0, n - g + 2)$$

$$\vdots$$

$$[(n - 1) + (b + c)]P(0, n - 1) = cP(0, n - 2) + nP(0, n)$$

$$[n + b]P(0, n) = cP(0, n - 1)$$

$$\underline{B_3.} \quad (i_1 \geq 1)$$

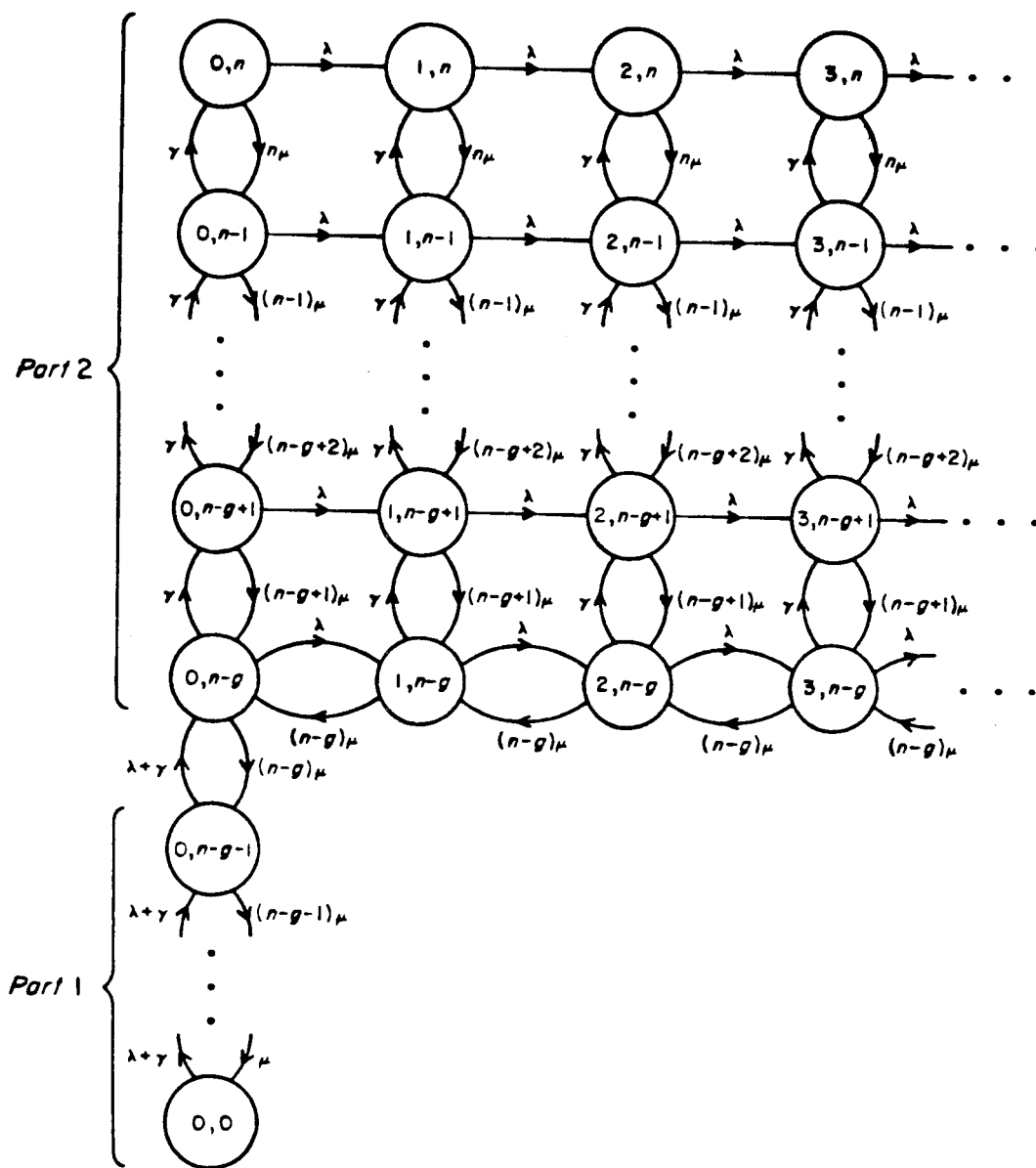


Fig. 6.1. Complete State Diagram.

$$\begin{aligned}
 [(n-g) + (b+c)]P(i_1, n-g) &= bP(i_1-1, n-g) \\
 &+ [n-g+1]P(i_1, n-g+1) \\
 &+ [n-g]P(i_1+1, n-g) \\
 [(n-g+1) + (b+c)]P(i_1, n-g+1) &= bP(i_1-1, n-g+1) \\
 &+ [n-g+2]P(i_1, n-g+2) + cP(i_1, n-g) \\
 &\vdots \\
 [(n-1) + (b+c)]P(i_1, n-1) &= bP(i_1-1, n-1) + nP(i_1, n) \\
 &+ cP(i_1, n-2) \\
 [n+b]P(i_1, n) &= bP(i_1-1, n) + cP(i_1, n-1).
 \end{aligned}$$

Here the following notation has been used:

$$b = \frac{\lambda}{\mu}, \quad c = \frac{\gamma}{\mu}.$$

A solution for the part ( $i_1 = 0, 0 \leq i_2 \leq n-g$ ) is obtained in a straightforward manner from basic queueing theory, ([42], p. 105):

$$P(0, i_2) = \frac{a^{i_2}}{i_2!} P(0, 0), \quad 0 \leq i_2 \leq n-g \quad \text{and} \quad a = \frac{\lambda + \gamma}{\mu}.$$

We will therefore restrict ourself to the subsystem ( $i_1 \geq 0, n-g \leq i_2 \leq n$ ). The state diagram then has the form given in Figure 6.2:

If we now consider the modified system, we can get a general balance equation, valid for any state of the stationary system:

$$\begin{aligned}
 [b + c\bar{\delta}_{i_2, n} + (1 - \delta_{i_1, n-g}\delta_{i_2, n-g})i_2] P(i_1, i_2) &= (i_2 + 1)\bar{\delta}_{i_2, n}P(i_1, i_2 + 1) \\
 &+ c\bar{\delta}_{i_2, n-g}P(i_1, i_2 - 1) \\
 &+ b\bar{\delta}_{i_1, n-g}P(i_1 - 1, i_2) \\
 &+ (n-g)\delta_{i_2, n-g}P(i_1 + 1, i_2). \quad (Bg)
 \end{aligned}$$

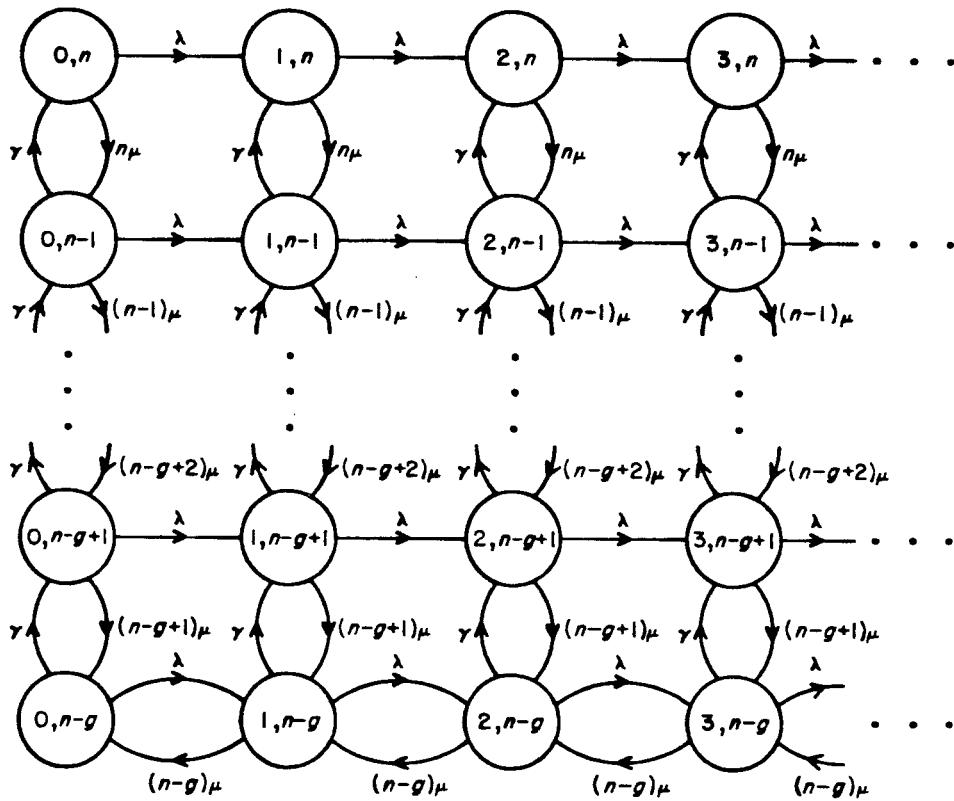


Fig. 6.2. Modified State Diagram.

where we have:

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j; \end{cases}$$

$$\bar{\delta}_{i,j} = 1 - \delta_{i,j}.$$

The two-dimensional Z-Transform describing the modified system can then be obtained from this expression (see Appendix E). Unfortunately, the transform equation is actually a  $g^{th}$  order differential equation and therefore may not be extremely useful from a practical point of view. We will now try another method in order to solve our problem.

This method is mainly based on the fact that the state diagram is sequential with respect to its columns ( $i_1 = \text{constant}$ ). A natural idea is then to solve the problem for the first column ( $i_1 = 0$ ), and, once we have these solutions, go to the

second column and obtain a new set of solutions, and so on. Ultimately, we want a recursive pattern that will enable us to derive the general solutions.

### 6.2 Derivation of the Z-Transform for the first column of the system.

We will initially concentrate on the first column ( $i_1 = 0$ ). Since there is only one index varying ( $i_2$ ), we will use the more convenient notation:

$$P(0, n - k) = q_k, \quad 0 \leq k \leq g.$$

The basic idea is then to try to express all the  $q_k$ 's in terms of  $q_0$ , and use the fact that  $q_g = P(0, n - g)$  is already known as a function of  $P(0, 0)$ , namely:

$$q_g = \frac{a^{(n-g)}}{(n-g)!} P(0, 0).$$

to express  $P(0, i_2)$ , ( $0 \leq i_2 \leq n$ ) in terms of  $P(0, 0)$ . The passage to the second column is obtained by using the existing balance equation involving  $P(0, n - g)$  and  $P(1, n - g)$ .

We have the following relations for the  $q_k$ 's:

$$\begin{cases} q_0 = q_0, \\ q_1 = (nc^{-1} + t)q_0, \\ q_k = (\bar{\alpha} - k\bar{\delta})q_{k-1} - (\bar{\beta} - k\bar{\delta})q_{k-2}, \end{cases} \quad 2 \leq k \leq g. \quad (6.1)$$

Here we have defined:

$$\bar{\alpha} = (n + 1)c^{-1} + (t + 1), \quad \bar{\beta} = (n + 2)c^{-1},$$

$$\bar{\delta} = c^{-1}, \quad t = \frac{\lambda}{\gamma} = \frac{b}{c}.$$

Equation (6.1) defines the sequence of the  $q_k$ 's valid for  $2 \leq k \leq g$ ; we will, however, pretend it is valid for all  $k \geq 2$  and compute the corresponding Z-Transform equation. Once we have obtained a function satisfying this equation, we will need

to get an asymptotic power expansion at the origin of rank at least  $g$ , since the corresponding coefficients are the ones that make sense in the context of our problem. A rigorous justification of this procedure is given in Appendix F.

We will now proceed with the computation of the Z-Transform equation, starting with equation (6.1). Summing both sides of equation (6.1) for all values of the index  $k$  ranging from  $k = 0$  to  $k = \infty$ , we reach the following equality:

$$\begin{aligned} \sum_{k=2}^{\infty} q_k Z^k &= \bar{\alpha} \left[ \sum_{k=2}^{\infty} q_{k-1} Z^k \right] - \bar{\delta} \left[ \sum_{k=2}^{\infty} k q_{k-1} Z^k \right] \\ &\quad - \bar{\beta} \left[ \sum_{k=2}^{\infty} q_{k-2} Z^k \right] + \bar{\delta} \left[ \sum_{k=2}^{\infty} k q_{k-2} Z^k \right] \\ \Rightarrow Q(Z) - q_1 Z - q_0 &= \bar{\alpha} Z [Q(Z) - q_0] - \bar{\delta} \left[ \sum_{k=1}^{\infty} (k+1) q_k Z^{k+1} \right] \\ &\quad - \bar{\beta} Z^2 Q(Z) + \bar{\delta} \left[ \sum_{k=0}^{\infty} (k+2) q_k Z^{k+2} \right]. \end{aligned}$$

Where we have assumed:

$$\begin{aligned} Q(Z) &= \sum_{k=0}^{\infty} q_k Z^k \\ \Rightarrow Q(Z) - q_1 Z - q_0 &= Q(Z)[\bar{\alpha} Z - \bar{Z}^2] - \bar{\alpha} Z q_0 \\ &\quad - \bar{\delta} \left[ \sum_{k=1}^{\infty} k q_k Z^{k+1} + \sum_{k=1}^{\infty} q_k Z^{k+1} \right] \\ &\quad + \bar{\delta} \left[ \sum_{k=0}^{\infty} k q_k Z^{k+2} + 2 \sum_{k=0}^{\infty} q_k Z^{k+2} \right] \\ \Rightarrow Q(Z) - q_1 Z - q_0 &= Q(Z)[\bar{\alpha} Z - \bar{\beta} Z^2] - \bar{\alpha} Z q_0 - \bar{\delta} Z^2 \dot{Q}(Z) \\ &\quad - \bar{\delta} Z [Q(Z) - q_0] + \bar{\delta} Z^3 \dot{Q}(Z) + 2\bar{\delta} Z^2 Q(Z). \end{aligned}$$

Here we have taken:

$$\dot{Q}(Z) = \frac{\partial Q}{\partial Z}.$$

$$\Rightarrow Q(Z)[1 - \bar{\alpha} Z + \bar{\beta} Z^2 + \bar{\delta} Z - 2\bar{\delta} Z^2] + \dot{Q}(Z)\bar{\delta}[Z^2 - Z^3] = q_0[1 - \bar{\alpha} Z] + Z q_1 + \bar{\delta} q_0$$



$$\Rightarrow Q(Z)[1 + (\bar{\delta} - \bar{\alpha})Z + (\bar{\beta} - 2\bar{\delta})Z^2] + \dot{Q}(Z)\bar{\delta}Z^2[1 - Z] = q_0[1 + (\bar{\delta} - \bar{\alpha})Z] + Zq_1.$$

But recall that we have:

$$\begin{aligned}\bar{\alpha} - \bar{\delta} &= nc^{-1} + (t + 1) = \alpha, \\ \bar{\beta} - 2\bar{\delta} &= nc^{-1} = \beta, \\ \bar{\delta} &= c^{-1} = \delta.\end{aligned}$$

With this new notation and using the fact that  $q_1 = [nc^{-1} + t]q_0$ , we get:

$$\begin{aligned}Q(Z)[1 - \alpha Z + \beta Z^2] + \dot{Q}(Z)\delta Z^2[1 - Z] &= q_0[1 - \alpha Z] + Z[nc^{-1} + t]q_0, \\ &= q_0[1 + Z(nc^{-1} + t - nc^{-1} - (r + 1))], \\ &= q_0(1 - Z),\end{aligned}$$

So we finally have the following transform equation:

$$Q(Z)[1 - \alpha Z + \beta Z^2] + \dot{Q}(Z)\delta Z^2[1 - Z] = q_0[1 - Z]. \quad (6.2)$$

where we have taken:

$$\begin{aligned}\alpha &= nc^{-1} + (t + 1) \\ \beta &= nc^{-1} \\ \delta &= c^{-1}\end{aligned}$$

We can now rewrite (6.2) in a more traditional way:

$$\dot{Q}(Z) + Q(Z)\frac{[\beta Z^2 - \alpha Z + 1]}{\delta Z^2[1 - Z]} = \frac{q_0}{\delta Z^2}. \quad (6.3)$$

This is of the form:

$$\dot{p} + rp = q.$$

Let us set:

$$r = \frac{\beta Z^2 - \alpha Z + 1}{\delta Z^2[1 - Z]} \quad \text{and} \quad q = \frac{q_0}{\delta Z^2} = \frac{q_0 c}{Z^2},$$

and take:

$$A(Z) = \int_{\epsilon}^Z r(x) dx, \quad \text{where } \epsilon \text{ is an arbitrary constant.}$$

The general form of the solution then is given ([47], pp. 60-65) by:

$$Q(Z) = Q(\epsilon)e^{-A(Z)} + e^{-A(Z)} \int_{\epsilon}^Z e^{A(x)} q(x) dx.$$

We will now try to obtain a solution continuous in the region  $|Z| < 1$ . The first step towards this solution is to obtain a partial fraction expansion for  $r(Z)$ , which is given by:

$$r(Z) = \frac{b}{Z-1} - \frac{(n+b)}{Z} + \frac{c}{Z^2}.$$

This gives:

$$A(Z) = bLn|1-Z| - (n+b)Ln|Z| - \frac{c}{Z} - bLn|1-\epsilon| + (n+b)Ln|\epsilon| + \frac{c}{\epsilon},$$

and

$$e^{A(Z)} = (1-Z)^b Z^{-(n+b)} e^{-cZ^{-1}} (1-\epsilon)^{-b} \epsilon^{(n+b)} e^{c\epsilon^{-1}},$$

$$e^{-A(Z)} = (1-Z)^{-b} Z^{(n+b)} e^{cZ^{-1}} (1-\epsilon)^b \epsilon^{-(n+b)} e^{-c\epsilon^{-1}}.$$

Here we limit ourself to the region:  $(0 < \epsilon < Z < 1)$ . We can then write:

$$Q(Z) = Q(\epsilon)e^{-A(Z)} + e^{-A(Z)} c \int_{\epsilon}^Z e^{A(x)} \frac{q_0}{x^2} dx$$

$$\Rightarrow Q(Z) = Q(\epsilon) \left[ (1-Z)^{-b} Z^{(n+b)} e^{cZ^{-1}} (1-\epsilon)^b \epsilon^{-(n+b)} e^{-c\epsilon^{-1}} \right] \quad (i)$$

$$+ q_0 (1-Z)^{-b} Z^{(n+b)} e^{cZ^{-1}} c \int_{\epsilon}^Z (1-x)^b x^{-(n+b)} e^{-cx^{-1}} \frac{dx}{x^2}. \quad (ii)$$

Let us first look at (ii) and make the following change of variable:

$$u = \frac{1}{x}; \quad du = -\frac{dx}{x^2}.$$

$$\begin{aligned} \Rightarrow (ii) &= -q_0(1-Z)^{-b} Z^{(n+b)} e^{cZ^{-1}} c \int_{\epsilon^{-1}}^{Z^{-1}} \left(\frac{u-1}{u}\right)^b u^{(n+b)} e^{-cu} du \\ \Rightarrow (ii) &= -q_0 \frac{Z^{(n+b)}}{(1-Z)^b} e^{cZ^{-1}} c \int_{\epsilon^{-1}}^{Z^{-1}} u^n (u-1)^b e^{-cu} du \\ \Rightarrow (ii) &= -q_0 \frac{Z^{(n+b)}}{(1-Z)^b} e^{cZ^{-1}} c \int_{\epsilon^{-1}}^{Z^{-1}} ([u-1]+1)^n (u-1)^b e^{-cu} du \\ \Rightarrow (ii) &= -q_0 \frac{Z^{(n+b)}}{(1-Z)^b} e^{cZ^{-1}} c \left[ \sum_{i=0}^n \binom{n}{i} \int_{\epsilon^{-1}}^{Z^{-1}} (u-1)^{(b+i)} e^{-cu} du \right]. \end{aligned}$$

Now let  $v = u - 1$ :

$$\Rightarrow (ii) = -q_0 \frac{Z^{(n+b)}}{(1-Z)^b} e^{cZ^{-1}} \left[ \sum_{i=0}^n \binom{n}{i} e^{-c} \int_{[\epsilon^{-1}-1]}^{[Z^{-1}-1]} v^{(b+i)} e^{-cv} d(cv) \right].$$

Finally, let  $y = cv$ :

$$\begin{aligned} \Rightarrow (ii) &= -q_0 \frac{Z^{(n+b)}}{(1-Z)^b} e^{cZ^{-1}} e^{-c} \left[ \sum_{i=0}^n \binom{n}{i} c^{-(b+i)} \int_{c[\epsilon^{-1}-1]}^{c[Z^{-1}-1]} y^{(b+i)} e^{-y} dy \right] \\ \Rightarrow (ii) &= q_0 \frac{Z^{(n+b)}}{(1-Z)^b} e^{c(Z^{-1}-1)} c^{-b} \left[ \sum_{i=0}^n \binom{n}{i} c^{-i} \int_{c[Z^{-1}-1]}^{c[\epsilon^{-1}-1]} y^{(b+i)} e^{-y} dy \right]. \end{aligned}$$

We want  $Q(Z)$  to be defined and continuous in the region  $|Z| < 1$  around  $Z = 0$ .

This then implies:

$$\lim_{Z \rightarrow 0} Q(Z) = Q(0) = q_0.$$

If we then decide to let  $\epsilon \rightarrow 0$  we note that (i)  $\rightarrow 0$ , except perhaps at  $Z = \epsilon$ . But failure to approach 0 at  $Z = \epsilon$  would introduce a discontinuity at the origin. This argument leads us to ignore (i) when  $\epsilon \rightarrow 0$  and take:

$$Q(Z) = \lim_{\epsilon \rightarrow 0} (ii). \quad (*)$$

We will verify later that (\*) defines a continuous function of  $Z$  and that we still have  $Q(Z) \rightarrow q_0$  for  $Z \rightarrow 0$  when we define  $Q(Z)$  by (\*). But let us first prove that

(i) can be ignored when we let  $\varepsilon \rightarrow 0$ . We have:

$$\begin{aligned} \lim_{Z \rightarrow 0} Q(Z) &= \lim_{Z \rightarrow 0} [(i) + (ii)], \quad \forall \varepsilon \quad 0 < \varepsilon < 1, \\ &= \lim_{Z \rightarrow 0} \left[ \lim_{\varepsilon \rightarrow 0} (i) + \lim_{\varepsilon \rightarrow 0} (ii) \right], \\ &= \lim_{Z \rightarrow 0} \left[ \lim_{\varepsilon \rightarrow 0} (i) \right] + \lim_{Z \rightarrow 0} \left[ \lim_{\varepsilon \rightarrow 0} (ii) \right], \quad (\text{since both terms are finite}), \\ \Rightarrow q_0 &= \lim_{Z \rightarrow 0} \left[ \lim_{\varepsilon \rightarrow 0} (i) \right] + q_0, \quad (\text{assume } \lim_{Z \rightarrow 0} \left[ \lim_{\varepsilon \rightarrow 0} (ii) \right] = q_0 \text{ already proved}), \\ \Rightarrow 0 &= \lim_{Z \rightarrow 0} \left[ \lim_{\varepsilon \rightarrow 0} (i) \right]. \end{aligned}$$

This proves that we have:

$$\lim_{\varepsilon \rightarrow 0} (i) = 0, \quad \forall Z, \quad 0 < Z < 1.$$

We will use the above result and the fact that the factor  $e^{-c\varepsilon^{-1}}$  in the expression of (i) gives 0 as the limit for  $Z \neq \varepsilon$  and  $\varepsilon \rightarrow 0$ . Going back to (ii) we have:

$$\lim_{\varepsilon \rightarrow 0} (ii) = q_0 \frac{Z^{(n+b)}}{(1-Z)^b} e^{c[Z^{-1}-1]} c^{-b} \left[ \sum_{i=0}^n \binom{n}{i} \int_{c[Z^{-1}-1]}^{\infty} y^{(b+i)} e^{-y} dy \right].$$

Note that we have  $c(Z^{-1}-1) > 0$  since  $0 < Z < 1$ . Then ([48], p. 940, Eq. 8.350-2) gives us:

$$\begin{aligned} \int_{c[Z^{-1}-1]}^{\infty} y^{(b+i)} e^{-y} dy &= \Gamma(b+i+1, c[Z^{-1}-1]) \\ \Rightarrow Q(Z) &= q_0 \frac{Z^{(n+b)}}{(1-Z)^b} e^{c(Z^{-1}-1)} c^{-b} \left[ \sum_{i=0}^n \binom{n}{i} c^{-i} \Gamma(b+i+1, c[Z^{-1}-1]) \right]. \end{aligned} \tag{6.4}$$

In order to rewrite  $Q(Z)$  in a more convenient way, note (ibid, p. 942, Eq. 8.356-2) that:

$$\begin{aligned} \Gamma(\alpha+1, x) &= \alpha \Gamma(\alpha, x) + x^\alpha e^{-x} \\ \Rightarrow \Gamma(\alpha+2, x) &= (\alpha+1) \Gamma(\alpha+1, x) + x^{(\alpha+1)} e^{-x}, \\ &= \alpha(\alpha+1) \Gamma(\alpha, x) + x^\alpha e^{-x} [(\alpha+1) + x]. \end{aligned}$$

Similarly:

$$\begin{aligned}\Gamma(\alpha + 3, x) &= (\alpha + 2)\Gamma(\alpha + 2, x) + x^{(\alpha+2)}e^{-x}, \\ &= \alpha(\alpha + 1)(\alpha + 2)\Gamma(\alpha, x) \\ &\quad + x^\alpha e^{-x} [(\alpha + 1)(\alpha + 2) + (\alpha + 2)x + x^2].\end{aligned}$$

We can then easily prove the following formula by induction ([48] p. 942, Eq. 8.356-5):

$$\Gamma(\alpha + k, x) = (\alpha)_k \Gamma(\alpha, x) + x^\alpha e^{-x} \sum_{j=0}^{k-1} (\alpha + j + 1)_{k-j-1} x^j, \quad k > 0,$$

where  $(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ . This then implies:

$$\begin{aligned}\Gamma(b + i + 1, c[Z^{-1} - 1]) &= (b)_{i+1} \Gamma(b, c[Z^{-1} - 1]) \\ &\quad + (c[Z^{-1} - 1])^b e^{-c[Z^{-1}-1]} \left[ \sum_{j=0}^i (b + j + 1)_{i-j} (c[Z^{-1} - 1])^j \right].\end{aligned}$$

More generally:

$$\begin{aligned}\sum_{i=0}^n \binom{n}{i} c^{-i} \Gamma(b + i + 1, c[Z^{-1} - 1]) &= \\ \Gamma(b, c[Z^{-1} - 1]) \left[ \sum_{i=0}^n \binom{n}{i} c^{-i} (b)_{i+1} \right] \\ + (c[Z^{-1} - 1])^b e^{-c[Z^{-1}-1]} \left[ \sum_{i=0}^n \binom{n}{i} c^{-i} \left( \sum_{j=0}^i (b + j + 1)_{i-j} (c[Z^{-1}-1])^j \right) \right].\end{aligned}$$

This finally gives:

$$\begin{aligned}Q(Z) &= q_0 \frac{Z^{(n+b)}}{(1-Z)^b} e^{c[Z^{-1}-1]} c^{-b} \Gamma(b, c[Z^{-1} - 1]) \left[ \sum_{i=0}^n \binom{n}{i} c^{-i} (b)_{i+1} \right] \\ &\quad + q_0 Z^n \left[ \sum_{i=0}^n \binom{n}{i} c^{-i} \left( \sum_{j=0}^i (b + j + 1)_{i-j} (c[Z^{-1} - 1])^j \right) \right].\end{aligned} \quad (6.5)$$

We are now interested in verifying that we still have:

$$\lim_{Z \rightarrow 0} Q(Z) = Q(0) = q_0.$$

Note that since equation (6.5) defines a continuous function of  $Z$  for  $0 < Z < 1$  this will also prove that  $Q(Z)$  is a continuous function of  $Z$  in a non-empty region around the origin.

Letting  $Z \rightarrow 0$  is equivalent to letting  $c[Z^{-1} - 1] \rightarrow \infty$  (assuming  $c > 0$ ). We will therefore use the following asymptotic expansion of  $\Gamma(\alpha, x)$  for  $|x| \rightarrow \infty$  and  $M > 0$ . ([49], p. 135, Eq. 6):

$$\Gamma(\alpha, x) \asymp x^{(\alpha-1)} e^{-x} \left[ \sum_{m=0}^{M-1} \frac{(1-\alpha)_m}{(-x)^m} + O(|x|^{-M}) \right].$$

Before proceeding any further, let us state more precisely what is meant by asymptotic expansion. A function  $F(x)$  is said to have an **asymptotic expansion**  $\tilde{F}(x)$  when  $x \rightarrow x_0$ , if the following conditions are met:

$$\forall \epsilon > 0, \quad \exists \eta_\epsilon > 0: \quad \forall x, |x - x_0| < \eta_\epsilon \quad \Rightarrow \quad \left| \frac{F(x)}{\tilde{F}(x)} - 1 \right| < \epsilon.$$

In the case where  $x_0 = \infty$ , the above condition is replaced by:

$$\forall \epsilon > 0, \quad \exists X_\epsilon: \quad \forall x, |x| > |X_\epsilon| \quad \Rightarrow \quad \left| \frac{F(x)}{\tilde{F}(x)} - 1 \right| < \epsilon.$$

Note that in the case where  $x_0 = 0$ , if  $\tilde{F}(x) = \tilde{F}_M(x)$  is of the form:

$$\tilde{F}_M(x) = \sum_{n=0}^{M-1} a_n x^n + O(x^M),$$

it does not mean that  $F(x)$  has a power series expansion at the origin, since the values of  $\eta_\epsilon$  depends on the chosen value of  $M$ . Therefore, the power series associated with the  $a_n$ 's can have an empty region of convergence, since the asymptotic expansion only deals with a finite number of terms.

Returning to the incomplete  $\Gamma$ -function, we have:

$$\begin{aligned} \Gamma(b, c[Z^{-1} - 1]) &\asymp (c[Z^{-1} - 1])^{b-1} e^{-c[Z^{-1}-1]} \\ &\quad \left[ \sum_{m=0}^{M-1} \frac{(1-b)_m}{(-c[Z^{-1} - 1])^m} + O\left((c[Z^{-1} - 1])^{-M}\right) \right] \\ \Rightarrow \Gamma(b, c[Z^{-1} - 1]) &\asymp c^{(b-1)} \frac{(1-Z)^{(b-1)}}{Z^{(b-1)}} e^{-c[Z^{-1}-1]} \\ &\quad \left[ \sum_{m=0}^{M-1} \frac{(1-b)_m Z^m}{(-c)^m (1-Z)^m} + O(Z^M) \right]. \end{aligned}$$

This in turn allows us to write for  $Z \rightarrow 0$ :

$$\begin{aligned} Q(Z) &\asymp q_0 \frac{Z^{(n+1)}}{c(1-Z)} \left[ \sum_{i=0}^n \binom{n}{i} c^{-i(b)_{i+1}} \right] \left[ \sum_{m=0}^{M-1} \frac{(1-b)_m Z^m}{(-c)^m (1-Z)^m} + O(Z^M) \right] \\ &\quad + q_0 Z^n \left[ \sum_{i=0}^n \binom{n}{i} c^{-i} \left( \sum_{j=0}^i (b+j+1)_{i-j} c^j \frac{(1-Z)^j}{Z^j} \right) \right]. \end{aligned} \quad (6.6)$$

So  $Q(Z)$  is of the form  $Q(Z) \asymp q_0(a) + q_0(b)$  where:

$$\begin{aligned} (a) &= \frac{Z^{(n+1)}}{c(1-Z)} \left[ \sum_{i=0}^n \binom{n}{i} c^{-i(b)_{i+1}} \right] \left[ \sum_{m=0}^{M-1} \frac{(1-b)_m Z^m}{(-c)^m (1-Z)^m} + O(Z^M) \right], \\ (b) &= Z^n \left[ \sum_{i=0}^n \binom{n}{i} c^{-i} \left( \sum_{j=0}^i (b+j+1)_{i-j} c^j \frac{(1-Z)^j}{Z^j} \right) \right]. \end{aligned}$$

We will now let  $Z \rightarrow 0$ . Since (a) is equal to  $Z^{n+1}$  time a sum of rational fractions in  $Z$ , none having  $Z = 0$  as a pole, we can immediately conclude:

$$\lim_{Z \rightarrow 0} (a) = 0.$$

On the other hand (b) reduces to a sum of polynomials (since  $n \geq j$ ). And the only one that does not have  $Z = 0$  as a zero is the one corresponding to the indices  $i = j = n$ , with the following coefficient:

$$q_0 Z^n \binom{n}{n} c^{-n(b+n)_0} c^n \frac{(1-Z)^n}{Z^n} = q_0 (1-Z)^n \rightarrow q_0, \quad \text{as } Z \rightarrow 0.$$

So we have proved that we have:

$$\begin{aligned}\lim_{Z \rightarrow 0} Q(Z) &= \lim_{Z \rightarrow 0} (a) + \lim_{Z \rightarrow 0} (b), \\ &= 0 + q_0 = q_0.\end{aligned}$$

This completes the proof we gave when we neglected term (i) for  $\epsilon \rightarrow 0$  in our earlier expression of  $Q(Z)$ . It also insures the continuity of  $Q(Z)$  as given by (6.4) in a non-empty region around the origin.

### 6.3 State probabilities for the first column of the system.

We can now go back to our initial task which was to determine the  $q_i$ 's in terms of  $q_0$ . We know (see Appendix F) that  $q_i$  is the coefficient of  $Z^i$  in the asymptotic power expansion of  $Q(Z)$ , where, due to the initial structure of our problem, we are only interested in indices  $i$  in the range  $0 \leq i \leq g$ . We can now note that in equation (6.6), the term corresponding to (a) will contribute to powers of  $Z$  only greater than or equal to  $(n+1)$ . This allows us to say that the relevant part of the power expansion of  $Q(Z)$  will be given by part (b). Namely we get:

$$(b) = q_0 Z^n \left[ \sum_{i=0}^n \binom{n}{i} c^{-i} \left( \sum_{j=0}^i (b+j+1)_{i-j} c^j \frac{(1-Z)^j}{Z^j} \right) \right],$$

or,

$$(b) = q_0 \left[ \sum_{i=0}^n \binom{n}{i} c^{-i} \left( \sum_{j=0}^i (b+j+1)_{i-j} Z^{n-j} (1-Z)^j \right) \right],$$

or again,

$$(b) = q_0 \left[ \sum_{j=0}^n c^j Z^{n-j} (1-Z)^j \left( \sum_{i=j}^n (b+j+1)_{i-j} \binom{n}{i} c^{-i} \right) \right].$$

From the above expression, we can easily derive the different coefficients of the powers of  $Z$ . Namely, if we call  $q_{n-k}$  the coefficient of  $Z^{n-k}$  for  $k = 1, \dots, n$ , we get:

$$q_{n-k} = q_0 \sum_{j=k}^n c^j \binom{j}{j-k} (-1)^{j-k} \left[ \sum_{i=j}^n (b+j+1)_{i-j} \binom{n}{i} c^{-i} \right]. \quad (6.7)$$



So, since

$$q_0 = P(0, n),$$

$$q_g = P(0, n - g),$$

and we already know:

$$P(0, n - g) = \frac{a^{n-g}}{(n-g)!} P(0, 0),$$

we find that (6.7) gives:

$$P(0, n - g) = P(0, n) \sum_{j=n-g}^n \left( c^j \binom{j}{j-(n-g)} (-1)^{j-(n-g)} \left[ \sum_{i=j}^n (b+j+1)_{i-j} \binom{n}{i} c^{-i} \right] \right).$$

This allows us to express  $P(0, n)$  in terms of  $P(0, 0)$ , and then express all the  $P(0, k)$  in terms of  $P(0, 0)$ , by using equation (6.7). However for notational simplicity we will express them in terms of  $P(0, n - g)$ . So if in equation (6.7) we make the following changes of indices:

$$i_2 = k \quad \text{and} \quad l = j - i_2.$$

we obtain:

$$P(0, i_2) = \frac{c^{i_2}}{i_2!} \lambda(n, i_2) P(0, n), \quad (6.8)$$

where:

$$\lambda(n, i_2) = \sum_{l=0}^{n-i_2} c^l \frac{(l+i_2)!}{l!} (-1)^l \left[ \sum_{i=l+i_2}^n (b+i_2+l+1)_{i-l-i_2} \binom{n}{i} c^{-i} \right]. \quad (6.9)$$

This implies that  $P(0, i_2)$  can be written in terms of  $\lambda(n, i_2)$  and  $\lambda(n, n - g)$  as:

$$P(0, i_2) = \frac{(n-g)! c^{i_2}}{c^{(n-g)} i_2!} \frac{\lambda(n, i_2)}{\lambda(n, n-g)} P(0, n-g). \quad (6.10)$$

Let us now pause to summarize the result we have obtained so far, and express all the state probabilities of the first column in term of  $P(0, 0)$ . Due to the fact that

the balance equation of the first non-trivial state of our system:  $(0, n-g)$  involved a term from the second column of the state diagram, we were unable to use standard inductive methods ([42], Chap. 3 Section 1) to solve for the state probabilities of the first column in terms of  $P(0,0)$ . We therefore decided to start from the top state of the first column:  $(0, n)$ . Then, using Z-Transform methods, we were able to obtain the probabilities of the states  $(0, n-g)$  to  $(0, n)$  in term of  $P(0, n)$ . Inverting this relation for state  $(0, n-g)$  and using the fact that  $P(0, n-g)$  was already known as a function of  $P(0,0)$  gave us an expression for  $P(0, n)$  in term of  $P(0,0)$ . This then yielded expressions for all the state probabilities of the first column in term of  $P(0,0)$ . The state probabilities of the first column as functions of  $P(0,0)$  are now given by  $(0 \leq g < n)$ :

$$P(0, i_2) = \begin{cases} \frac{a^{i_2}}{i_2!} P(0,0), & \text{for } 0 \leq i_2 \leq n-g; \\ \left(\frac{a}{c}\right)^{(n-g)} \frac{c^{i_2}}{i_2!} \frac{\lambda(n, i_2)}{\lambda(n, n-g)} P(0,0), & \text{for } n-g \leq i_2 \leq n. \end{cases} \quad (6.11)$$

#### 6.4 Verification in the case of a single traffic stream.

We now pause for a rapid check of Equation (6.11). Namely, we will take the traffic stream of customers of type (II) equal to 0, i.e.,  $b = 0$ . In this very special case, we are in the simple situation of a classical blocking system with  $n$  servers and input traffic equal to  $c = a$ . The state probabilities are then given by:

$$P(0, i_2) = \frac{c^{i_2}}{i_2!} P(0,0) \quad 0 \leq i_2 \leq n.$$

We therefore want to verify that Equation (6.11) reduces to this simple expression when we set  $b = 0$ . This verification will have to be done only for the cases  $n-g \leq i_2 \leq n$ , since Equation (6.11) agrees with the above formula for  $0 \leq i_2 \leq n-g$ . Now recall that for  $n-g \leq i_2 \leq n$ , we had:

$$P(0, i_2) = \left(\frac{a}{c}\right)^{(n-g)} \frac{c^{i_2}}{i_2!} \frac{\lambda(n, i_2)}{\lambda(n, n-g)} P(0,0).$$

For the case  $b = 0$  we have  $a = 0 + c = c$ , so the above formula becomes:

$$P(0, i_2) = \frac{c^{i_2}}{i_2!} P(0, 0) \left( \frac{\lambda(n, i_2)}{\lambda(n, n-g)} \right).$$

So, in order to complete our verification we must prove that we have:

$$\lambda(n, i_2) = \lambda(n, n-g), \quad \forall i_2, \quad n-g \leq i_2 \leq n \quad \text{and} \quad b = 0.$$

Now, when  $b = 0$ , and  $n-g \leq i_2 \leq n$ , we have the following expression for  $\lambda(n, i_2)$ :

$$\begin{aligned} \lambda(n, i_2) &= \sum_{l=0}^{n-i_2} c^l (-1)^l \frac{(l+i_2)!}{l!} \left[ \sum_{i=l+i_2}^n \frac{i!}{(l+i_2)! (n-i)! i!} c^{-i} \right], \\ &= \sum_{l=0}^{n-i_2} \sum_{i=l+i_2}^n \frac{n!}{l!(n-i)!} (-1)^l x^{i-l}, \quad (x = c^{-1}). \end{aligned}$$

Make the change of index  $m = i - l$ . This gives:

$$\lambda(n, i_2) = \sum_{l=0}^{n-i_2} \sum_{m=i_2}^{n-l} \frac{n!}{l!(n-(l+m))!} (-1)^l x^m.$$

We then reverse the order of summation to get:

$$\lambda(n, i_2) = n! \sum_{m=i_2}^n \left[ \sum_{l=0}^{n-m} \frac{(-1)^l}{l!(n-(l+m))!} \right] x^m$$

We will now consider two distinct cases:

\* (1)  $m = n$ ; the inner summation then gives:

$$\sum_{l=0}^0 \frac{(-1)^l}{l!(n-(l+m))!} = \frac{(-1)^0}{0!0!} = 1$$

\* (2)  $m \neq n$ ; then let  $t = n - m > 0$  and multiply the inner summation  $\Sigma_l$  by  $t!$ :

$$\begin{aligned} \Sigma_l t! &= \sum_{l=0}^t (-1)^l \frac{t!}{l!(t-l)!} \\ &= \sum_{l=0}^t \binom{t}{l} (-1)^l \\ &= (1-1)^t \\ &= 0, \quad \text{since } t > 0. \end{aligned}$$

This proves that the inner summation is equal to 0 if  $m \neq n$ . Therefore in the case where  $b = 0$ , the expression of  $\lambda(n, i_2)$  reduces to:

$$\lambda(n, i_2) = n!c^{-n}. \quad (6.12)$$

This indeed does hold for all values of  $i_2$  in the range  $n-g \leq i_2 \leq n$ . This completes the proof that Equation (6.11) with  $b = 0$  agrees with the expression obtained for a purely blocking system, as must be.

At this point, we comment on the simplicity of expression (6.12) for  $\lambda(n, i_2)$  when  $b = 0$ . Unfortunately, it cannot be extended to the general case where  $b \neq 0$ . This is due to the fact that the coefficient  $(b + i_2 + l + 1)_{i-l-i_2}$  is, for arbitrary  $b$ , a ratio of  $\Gamma$ -functions and not of factorials. Even if by chance  $b$  happens to be an integer, the ratio of  $\Gamma$ -functions reduces to a ratio of factorials, but this still does not enable us to perform the necessary simplifications that led to the above result. Namely, the cancellation of a factorial term from the inner summation by one from the outer summation does not occur anymore.

Before trying to extend the expressions obtained for the first column of our system to an arbitrary state, let us pause to derive some interesting consequences of what we have done, consequences that will further motivate our attempt to generalize the previous results.

### 6.5 Some important system parameters.

Recall that in Section 6.3 we obtained the state probabilities of the first column of our state diagram in terms of  $P(0,0)$ . We realize, however, that at this point,  $P(0,0)$  still remains unknown. Furthermore, since we consider a system with infinite queue capacity, the derivation of  $P(0,0)$  might turn out to be rather painful. This leads us to try to find a short cut that would enable us to directly obtain  $P(0,0)$ , without having to know all the state probabilities.

Let us first define some larger states " $i_2$ ", corresponding to the situation where  $i_2$  servers are busy. Then in the cases where  $i_2 \geq n - g$ , we would have:

$$P_{i_2} = \sum_{i_1=0}^{\infty} P(i_1, i_2).$$

A state diagram involving only these larger states would then have the following condensed representation (for  $n - g \leq i_2 \leq n$ ):

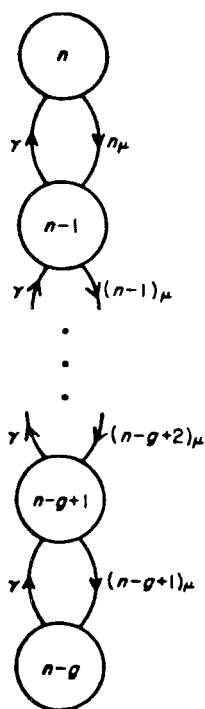


Fig. 6.3. Condensed State Diagram.

This representation is made possible by the fact that, as can be seen from Figure 6.2, all the states lying on the same row have the same incoming and outgoing flux. For example, if we choose row  $(n - 1)$ , all states have an outgoing flux of rate  $\gamma$  to the upper row and an outgoing flux of rate  $(n - 1)\theta$  to the lower row. Similarly all the incoming fluxes from the upper row are equal to  $n\theta$ , and equal to  $\gamma$  for the lower row. This implies that the larger state " $i_2$ ", which represents the whole row

$i_2$ , can be uniquely related to the larger states " $i_2 - 1$ " and " $i_2 + 1$ ", using these outgoing and incoming flux parameters.

This diagram is typical of a classical purely blocking system, and the "state probabilities" are easily expressed in terms of  $P_{n-g}$ ; namely, we get:

$$P_{i_2} = c^{i_2-(n-g)} \frac{(n-g)!}{i_2!} P_{n-g}.$$

This in turn gives:

$$\sum_{i_1=0}^{\infty} P(i_1, i_2) = c^{i_2-(n-g)} \frac{(n-g)!}{i_2!} \sum_{i_1=0}^{\infty} P(i_1, n-g), \quad n-g \leq i_2 \leq n. \quad (\alpha)$$

This gives us some useful information on the rows of our initial state diagram. We will now try to get some similar information for the columns of the system. It can be acquired by realizing that the outgoing flux (to the right) of column  $i_1$  is balanced only by the incoming flux from state  $(i_1 + 1, n - g)$ . We can therefore write:

$$P(i_1 + 1, n - g) = \frac{b}{(n-g)} \sum_{i_2=n-g}^n P(i_1, i_2), \quad i_1 \geq 0. \quad (\beta)$$

Starting from this point, we would like to relate the information we obtained from  $(\alpha)$  and  $(\beta)$ .

As a first attempt, sum  $(\beta)$  for all values of  $i_1 \geq 0$ .

$$\begin{aligned} \sum_{i_1=0}^{\infty} P(i_1 + 1, n - g) &= \frac{b}{(n-g)} \sum_{i_1=0}^{\infty} \left( \sum_{i_2=n-g}^n P(i_1, i_2) \right) \\ \Rightarrow \sum_{i_1=1}^{\infty} P(i_1, n - g) &= \frac{b}{(n-g)} \sum_{i_2=n-g}^n \left( \sum_{i_1=0}^{\infty} P(i_1, i_2) \right) \\ \Rightarrow \sum_{i_1=0}^{\infty} P(i_1, n - g) &= P(0, n - g) + \frac{b}{(n-g)} \sum_{l=n-g}^n \left( \sum_{i_1=0}^{\infty} P(i_1, l) \right). \end{aligned}$$

Here we have replaced the index  $i_2$  by  $l$ . We can substitute this new expression for  $\sum_{i_1=0}^{\infty} P(i_1, n-g)$  into  $(\alpha)$ . We then get:

$$\sum_{i_1=0}^{\infty} P(i_1, i_2) = c^{i_2-(n-g)} \frac{(n-g)!}{i_2!} \left[ P(0, n-g) + \frac{b}{(n-g)} \sum_{l=n-g}^n \left( \sum_{i_1=0}^{\infty} P(i_1, l) \right) \right].$$

But the double summation in the previous equation represents the sum of all state probabilities except for the cases where fewer than  $(n-g)$  servers are busy. These missing probabilities are, however, already known as functions of  $P(0,0)$  (see Equation (6.11)).

Since for an ergodic system, the state probabilities must add up to 1, we can write:

$$\sum_{l=n-g}^n \left( \sum_{i_1=0}^{\infty} P(i_1, l) \right) = 1 - \sum_{l=0}^{n-g-1} P(0, l).$$

This allows us to write:

$$\begin{aligned} P_{i_2} &= \sum_{i_1=0}^{\infty} P(i_1, i_2), \quad \text{for } n-g \leq i_2 \leq n. \\ &= c^{i_2-(n-g)} \frac{(n-g)!}{i_2!} \left[ P(0, n-g) + \frac{b}{(n-g)} \left( 1 - \sum_{l=0}^{n-g-1} P(0, l) \right) \right] \\ \Rightarrow P_{i_2} &= c^{i_2-(n-g)} \frac{(n-g)!}{i_2!} \left[ \frac{a^{(n-g)}}{(n-g)!} P(0,0) + \frac{b}{(n-g)} \left( 1 - P(0,0) \sum_{l=0}^{n-g-1} \frac{a^l}{l!} \right) \right]. \end{aligned}$$

Since we have:  $P(0, l) = \frac{a^l}{l!} P(0,0)$  for  $0 \leq l \leq n-g$ , we finally get for  $P_{i_2}$ :

$$\begin{aligned} P_{i_2} &= P(0,0) \left[ \left( \frac{a}{c} \right)^{(n-g)} \frac{c^{i_2}}{i_2!} - \frac{b}{(n-g)} \frac{(n-g)!}{c^{(n-g)}} \frac{c^{i_2}}{i_2!} \left( \sum_{l=0}^{n-g-1} \frac{a^l}{l!} \right) \right] \\ &\quad + \frac{b}{(n-g)} \frac{(n-g)!}{c^{(n-g)}} \frac{c^{i_2}}{i_2!}. \end{aligned} \tag{6.13}$$

But we also know in the case of an ergodic system that the following relation holds:

$$\sum_{i_2=0}^{n-g-1} P(0, i_2) + \sum_{i_2=n-g}^n P_{i_2} = 1.$$

We therefore can at last deduce an equation with  $P(0,0)$  as the only unknown:

$$\begin{aligned} 1 &= P(0,0) \sum_{i_2=0}^{n-g-1} \frac{a^{i_2}}{i_2!} + P(0,0) \left(\frac{a}{c}\right)^{(n-g)} \sum_{i_2=n-g}^n \frac{c^{i_2}}{i_2!} \\ &\quad - P(0,0) \frac{b}{(n-g)} \frac{(n-g)!}{c^{(n-g)}} \left( \sum_{i_2=n-g}^n \frac{c^{i_2}}{i_2!} \right) \left( \sum_{l=0}^{n-g-1} \frac{a^l}{l!} \right) \\ &\quad + \frac{b}{(n-g)} \frac{(n-g)!}{c^{(n-g)}} \left( \sum_{i_2=n-g}^n \frac{c^{i_2}}{i_2!} \right). \end{aligned}$$

This finally gives us the expression of  $P(0,0)$  we were looking for:

$$P(0,0) = \left( 1 - \frac{b}{(n-g)} \frac{(n-g)!}{c^{(n-g)}} \sum_{i_2=n-g}^n \frac{c^{i_2}}{i_2!} \right) \cdot \Delta^{-1}. \quad (6.14),$$

where:

$$\Delta = \left[ \left( \sum_{i_2=0}^{n-g-1} \frac{a^{i_2}}{i_2!} \right) \left( 1 - \frac{(n-g)!}{c^{(n-g)}} \frac{b}{(n-g)} \sum_{i_2=n-g}^n \frac{c^{i_2}}{i_2!} \right) + \left(\frac{a}{c}\right)^{(n-g)} \sum_{i_2=n-g}^n \frac{c^{i_2}}{i_2!} \right].$$

We are now in a position to derive several interesting system parameters.

Namely, we can get:

- \* The blocking probability for customers of type I:  $P_{B_I}$ ,
- \* The probability of delay for customers of type II:  $P_{D_{II}}$ ,
- \* The probability of having  $i_2$  servers busy.

Let us start with  $P_{B_I}$ :

$$\begin{aligned} P_{B_I} &= P_n \\ \Rightarrow P_{B_I} &= P(0,0) \left[ \left(\frac{a}{c}\right)^{(n-g)} \frac{c^n}{n!} - \frac{b}{(n-g)} \frac{(n-g)!}{c^{(n-g)}} \frac{c^n}{n!} \left( \sum_{l=0}^{n-g-1} \frac{a^l}{l!} \right) \right] \\ &\quad + \frac{b}{(n-g)} \frac{(n-g)!}{c^{(n-g)}} \frac{c^n}{n!}. \end{aligned} \quad (6.15)$$

For  $P_{D_{II}}$ , we have:

$$P_{D_{II}} = 1 - \left( \sum_{l=0}^{n-g-1} \frac{a^l}{l!} \right) P(0,0). \quad (6.16)$$



Finally,

$$P(i_2 \text{ servers busy}) = \begin{cases} \frac{a^{i_2}}{i_2!} P(0,0), & \text{if } 0 \leq i_2 < n - g; \\ P_{i_2}, & \text{if } n - g \leq i_2 \leq n. \end{cases} \quad (6.17)$$

Another interesting parameter of our system is a condition for its stability or, in other words, ergodicity. Since the system can be represented by a Markov chain, a necessary and sufficient condition for its ergodicity is that all the states are non-transient ([50], Chap. VIII, Section 7; Chap. XI, Section 8; Chap. XIV, Section 9) ([51], pp. 102-104). Furthermore, the Markov chain defined by our state diagram is obviously irreducible and aperiodic. This is due to the fact that any state can be reached from any other state. Since the probability of any state can be expressed in terms of the probability of the empty state (as we will see later), we can say that the system will be ergodic **if and only if** the empty state is non-transient, or in other words, if and only if  $P(0,0)$  is non-zero. So we have:

$$\text{Queueing-Blocking system ergodic} \iff P(0,0) > 0.$$

But recall that equation (6.14) gave us the expression for  $P(0,0)$ . We can therefore conclude:

$$\text{System is ergodic} \iff \frac{b}{(n-g)} \frac{(n-g)!}{c^{(n-g)}} \left( \sum_{i_2=n-g}^n \frac{c^{i_2}}{i_2!} \right) < 1. \quad (6.18)$$

Several comments can be made about this condition. First consider the extreme case  $g = 0$ , or in other words, a system with no guard channels. The above expression then reduces to:

$$\text{System ergodic} \iff \frac{b}{n} < 1.$$

This result is well known, since for the above system, ergodicity means that the queue length does not start diverging. This in turn can only happen if, when a queue is present, the departure rate from the system is always smaller than the corresponding arrival rate. Now, with no guard channel, the departure rate from the system is equal to  $n\theta$  when a queue is present. Similarly, since customers of type I are blocked when the  $n$  servers are busy, the arrival rate to the system, when a queue is present, is equal to  $\lambda$  only. The condition for ergodicity would then be:

$$\lambda < n\theta \iff \frac{b}{n} < 1.$$

This agrees with the expression given by equation (6.18) for the case  $g = 0$ .

In the general case, we can give the following partial interpretation of the condition for ergodicity. Equation (6.18) is formed by the product of two terms:

$$\frac{b}{n-g} \quad \text{and} \quad \frac{(n-g)!}{c^{n-g}} \left( \sum_{i_2=n-g}^n \frac{c^{i_2}}{i_2!} \right).$$

The first term  $\left(\frac{b}{n-g}\right)$  represent the ergodicity condition for a queueing-blocking system with  $(n-g)$  servers, no guard channels, and the same traffic characteristics (this is actually the "serving" system seen by customers of type II). Unfortunately, in our case, we do not depart from the "queueing" situation with a rate  $(n-g)\theta$  since we still have arrivals from customers of type I until all servers are busy. These arrivals slow down our departing rate from the "queueing state". The second term represents this "slowing-down" factor, since it is equal to the ratio of the probability of being in a state that allows us to leave the queueing state ( $P_{n-g}$ ) over the sum of the probabilities of being in any of the states where queueing is present ( $\sum_{i_2=n-g}^n P_{i_2}$ ). The ergodicity condition for the whole system is then the product of the classical ergodicity factor for the smaller system with only  $(n-g)$

servers and the "slowing-down" factor, which decreases the departing rate from the states where customers of type II have to be queued.

The limiting value for the traffic  $b$  of customers of type II is plotted as a function of the number of guard channels for several values of the total number of channels  $n$  and the traffic  $c$  of customers of type II in Figures 6.4 and 6.5;  $P_{B_I}$  and  $P_{D_{II}}$  are displayed in Figures 6.6 and 6.7 as functions of the number of guard channels. Figure 6.8 illustrates the slight increase in the blocking probability of handoff calls due to the queueing of originating calls, by comparing it with the blocking probability of handoff calls for the same number of guard channels, but without queueing of originating calls. Finally, Figure 6.9 presents the increase in carried traffic obtained when originating calls are queued, versus a system where no queueing is allowed.

### 6.6 Recursive formula for the state probabilities.

The purpose of this section is to extend the results derived in Section 6.3 to any state of our system. Equation (6.11) allowed us to express all  $P(0, i_2)$  for  $0 \leq i_2 \leq n$  as functions of  $P(0, 0)$ . Since  $P(0, 0)$  could be directly computed using equation (6.14), this method provided us with a direct computation of the state probabilities for the first column of the system. Unfortunately, this work is not directly applicable to the second column, since for each of the states of this column, there is an incoming flux from the neighbor state in the first column. However, several similarities do exist, which make possible a partial use of the previous results. We now emphasize these similarities, and we will apply them insofar as possible to the derivation of a recursive formula relating the state probabilities of a given column to those of the previous column.

We can first note, as we did in Section 6.3, that for both the second and first columns, it is impossible to use a method starting from the bottom state  $(1, n -$

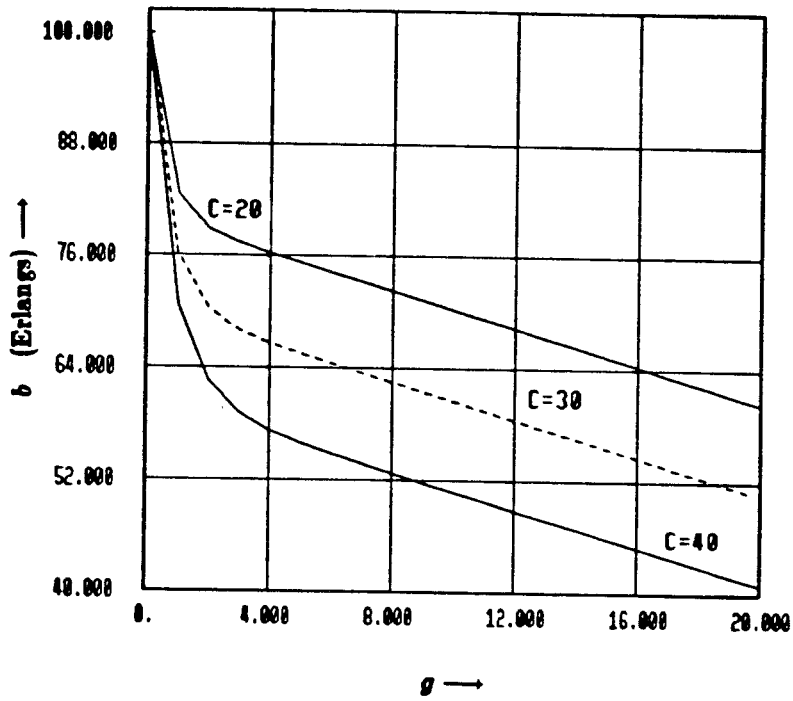


Fig. 6.4. Maximum traffic  $b$  for  $n=100$

( $c$ : Traffic of customers I;  $g$ : No of guard channels).

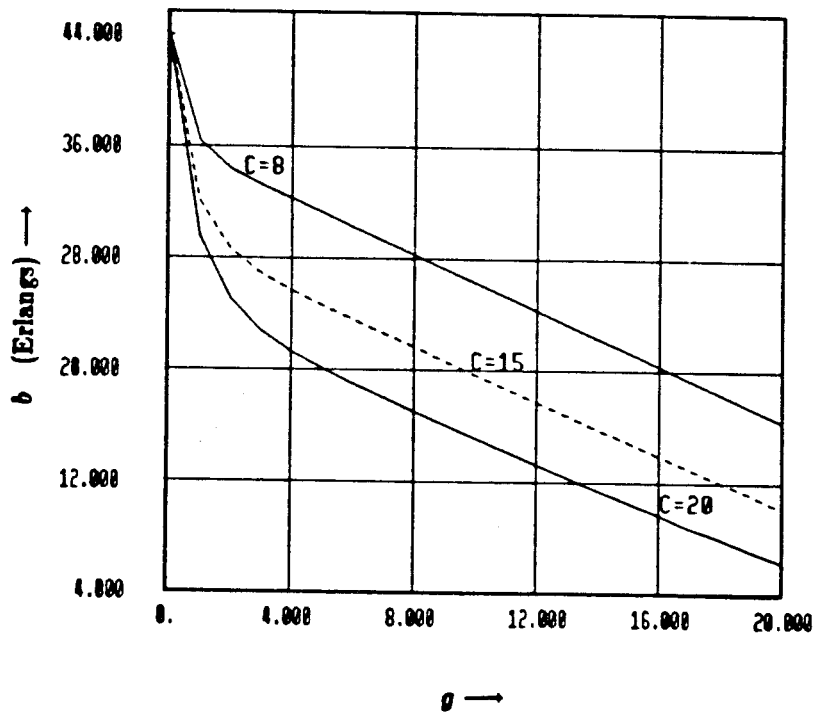


Fig. 6.5. Maximum traffic  $b$  for  $n=44$

( $c$ : Traffic of customers I;  $g$ : No of guard channels).

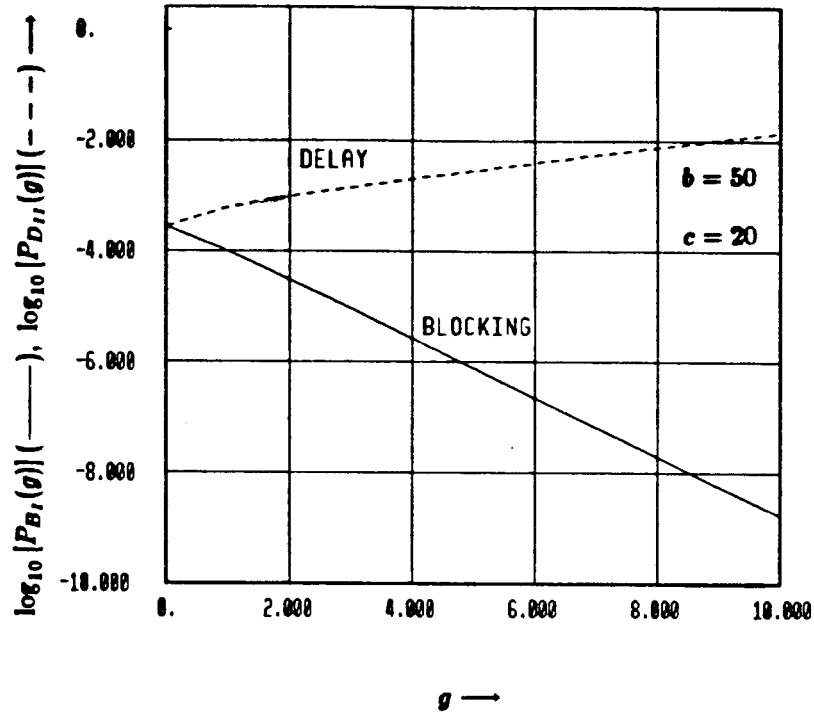


Fig. 6.6.  $P_{B_I}$  and  $P_{D_{II}}$  for  $n=100$

( $b, c$ : Traffic of customers II, I;  $g$ : No of guard channels).

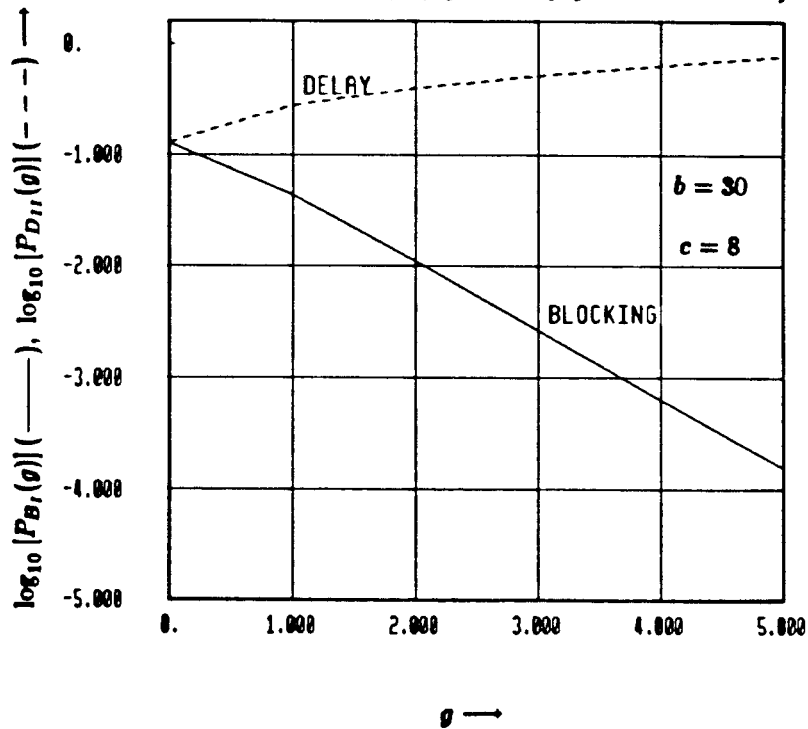


Fig. 6.7.  $P_{B_I}$  and  $P_{D_{II}}$  for  $n=44$

( $b, c$ : Traffic of customers II, I;  $g$ : No of guard channels).

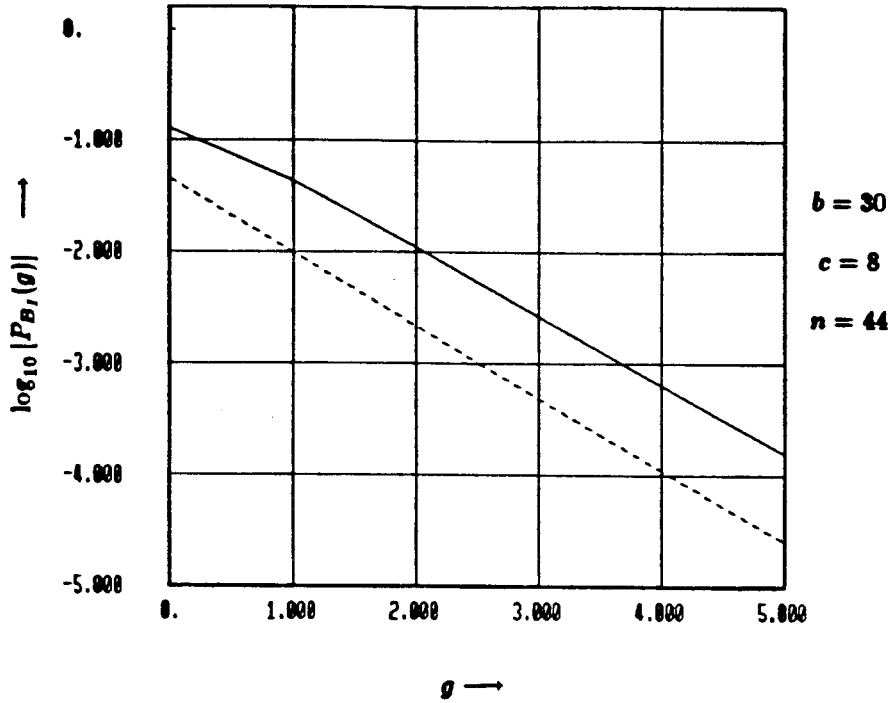


Fig 6.8. Blocking probability of customers I with (—) and without (---) queueing customers II.

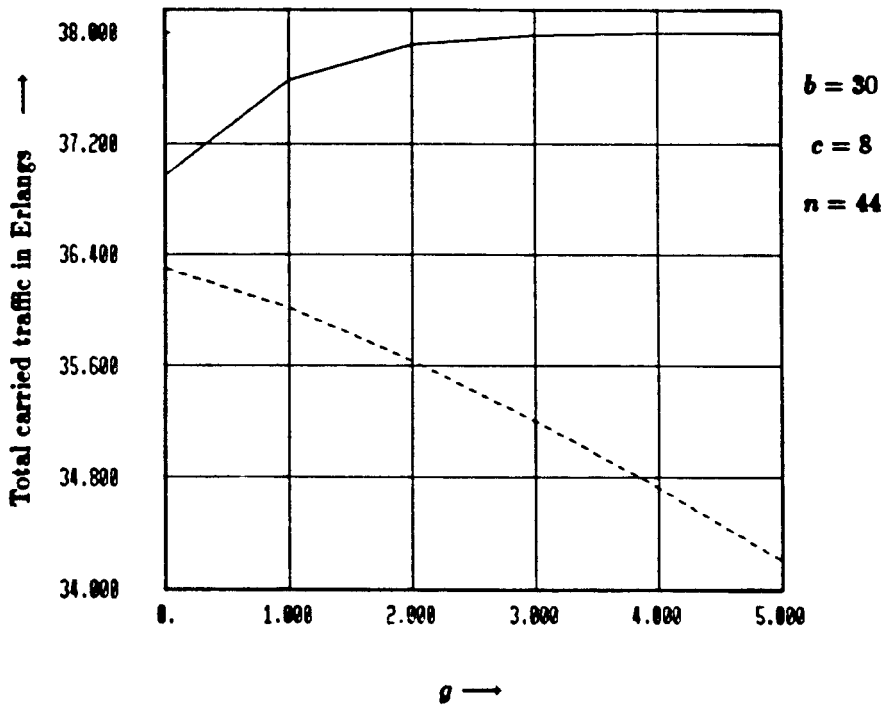


Fig 6.9. Total carried traffic with (—) and without (---) queueing customers II. ( $b, c$ : Traffic of customers II, I;  $g$ : No of guard channels).

g). This is because solving its balance equation would require the knowledge of  $P(2, n - g)$ , which at this point remains unknown. However we can note that the value of  $P(1, n - g)$  can be determined using the fact that this state must balance the entire outgoing flux from the first column. Namely:

$$P(1, n - g) = \frac{b}{(n - g)} \sum_{i_2=n-g}^n P(0, i_2).$$

We are therefore in a very similar position as we were at the beginning of Section 6.3. We have a known probability for the bottom state of the column, but are forced to start from the top state. The basic difference between the two systems is that the probabilities  $P(1, i_2)$  cannot be expressed in terms of  $P(1, n)$  alone. Such an expression will also involve the probabilities of the first column, due to that incoming flux from the left. We thus have the following general relation:

$$\begin{array}{l} P(1, i_2) = \frac{(b+c+i_2+1)}{c} P(1, i_2 + 1) - \frac{(i_2+2)}{c} P(1, i_2 + 2) \quad | \quad -\frac{b}{c} P(0, i_2 + 1) \\ \text{for } 0 \leq i_2 \leq n - 2 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad | \\ P(1, n - 1) = \frac{(b+n)}{c} P(1, n) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad | \quad -\frac{b}{c} P(0, n) \end{array} \quad (6.19)$$

(For the first column, the system collapses to the left side of the dashed line.)

In order to rewrite (6.19) in a simpler form, we adopt the following notation:

$$r = \frac{b}{c}, \quad \beta_{i_2} = \begin{cases} \frac{b+i_2+c}{c}, & \text{for } i_2 < n; \\ \frac{b+n}{c}; & \text{for } i_2 = n. \end{cases} \quad \gamma_{i_2} = -\frac{i_2}{c}.$$

We can then rewrite (6.19) in the following form:

$$\begin{array}{l} y_{i_2} = \beta_{i_2+1} y_{i_2+1} + \gamma_{i_2+2} y_{i_2+2} - r x_{i_2+1}, \quad n - g \leq i_2 \leq n - 2; \\ y_{n-1} = \beta_n y_n - r x_n. \end{array} \quad (6.19')$$

Here  $y_{i_2}$  stands for  $P(1, i_2)$  and  $x_{i_2}$  for  $P(0, i_2)$ .

We would now like to make use of the results derived for the first column to solve for the state probabilities in the second column. This is motivated by the strong

similarities that the two systems bear. A direct approach, however, turns out to be somewhat confusing, and the underlying similarities between the structures of the second and first columns do not appear as obvious as desired.

In order to gain greater insight in the system's behavior, we will use a slightly different approach. We will consider equation (6.19) as the description of a **signal flow graph** where the different coefficients  $r, \beta_{i_2}, \gamma_{i_2}$ , represent the **gains** of the corresponding branches in the graph. This approach is only formally different from the preceding one, but it will enable us to make use of Mason's formula ([52]) to compute the relationships or "network gains" between the outputs:  $y_{n-1}, \dots, y_{n-g}$ , and the inputs:  $y_n, x_n, x_{n-1}, \dots, x_{n-g}$ . But before going through all the computations, we review Mason's formula.

Let us start with some general notions. A signal flow graph is an oriented graph, where the **vertices** represent the inputs and outputs (variables). Two vertices are connected together by **branches** if the two variables they represent are involved in a common linear equation. A **gain** is then associated with each branch and is taken equal to the proportionality coefficient obtained from the linear equation involving the two variables (vertices). In the graph, a **forward path** is defined to be a continuous path that starts at a vertex associated with an input and ends at a vertex associated with an output. Furthermore, in a forward path no vertex can be traversed more than once. The gain associated with a forward path is then the product of the gains of all its branches. Finally, a **loop** is a closed path where all nodes except the originating/terminating node are traversed at most once. The gain of a loop is the product of the gains of its branches, and two loops are said to be **nontouching** if they have no vertex in common.

We are now ready to state Mason's formula. Mason's formula, the general gain formula for signal flow graphs, tells us that for a given signal flow graph, the



relationship of any input-output pair can be determined by mere inspection, when using the following formula:

$$M = \frac{y_{out}}{y_{in}} = \sum_{k=1}^N \frac{M_k \Delta_k}{\Delta}.$$

$M$  = gain between  $y_{in}$  and  $y_{out}$ .

$N$  = total number of forward paths.

$M_k$  = gain of the  $k^{th}$  forward path.

$P_{mr}$  = gain product of the  $m^{th}$  combination of  $r$  nontouching loops.

$$\Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2} - \sum_m P_{m3} + \dots$$

$\Delta_k$  = equivalent of  $\Delta$  for the subgraph nontouching the  $k^{th}$  forward path.

For a more detailed statement of Mason's formula, the reader is referred to ([52], Chap. 3, Sections 5 to 10).

We can note that the above formula greatly simplifies when the number of loops is small. This will turn out to be the case with our system. In order to compute the global value of a given output, we will make use of the superposition principle. This principle tells us that the value of an output can be computed by summing all values of the partial outputs obtained by setting one input non-zero, and all the others to zero. We present the representation of system (6.19) by a flow graph in Figure 6.10.

Since the graph of Figure 6.10 is rather regular, we can derive the general

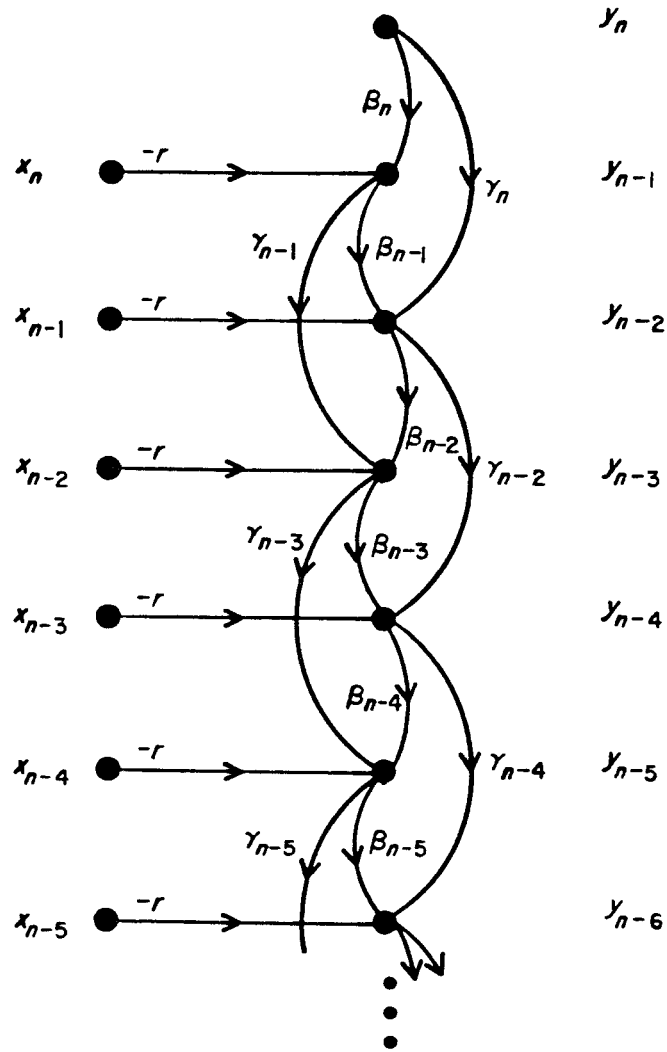


Fig. 6.10. Flow graph representation of Equation (6.19).

expression for the outputs by working out some simple examples:

$$\begin{aligned}
 y_{n-1} &= \beta_n y_n - r x_n \\
 y_{n-2} &= [\beta_n \beta_{n-1} + \gamma_n] y_n - r[\beta_{n-1}] x_n - r x_{n-1} \\
 y_{n-3} &= [\beta_n \beta_{n-1} \beta_{n-2} + \beta_n \gamma_{n-1} + \gamma_n \beta_{n-2}] y_n \\
 &\quad - r[\beta_{n-1} \beta_{n-2} + \gamma_{n-1}] x_n - r[\beta_{n-2}] x_{n-1} - r x_{n-2} \\
 y_{n-4} &= [\beta_n \beta_{n-1} \beta_{n-2} \beta_{n-3} + \beta_n \beta_{n-3} \gamma_{n-2} + \beta_n \gamma_{n-1} \beta_{n-3} \\
 &\quad + \gamma_n \beta_{n-2} \beta_{n-3} + \gamma_n \gamma_{n-2}] y_n \\
 &\quad - r[\beta_{n-1} \beta_{n-2} \beta_{n-3} + \beta_{n-1} \gamma_{n-2} + \gamma_{n-1} \beta_{n-3}] x_n \\
 &\quad - r[\beta_{n-2} \beta_{n-3} + \gamma_{n-2}] x_{n-1} - r[\beta_{n-3}] x_{n-2} - r x_{n-3}
 \end{aligned}$$

We can immediately remark that the coefficients of the inputs from the first column ( $x_k$ 's) go through the same type of behavior as the coefficients of the input  $y_n$ , except for a shift in the index value. For example, the coefficient of  $x_n$  for output  $y_{n-4}$  is the coefficient of  $y_n$  for output  $y_{n-3}$  with  $n$  replaced by  $(n-1)$ . Similarly, the coefficient of  $x_{n-1}$  for the same output  $y_{n-4}$  is the coefficient of  $y_n$  for output  $y_{n-2}$  with  $n$  replaced by  $(n-2)$ , and so on.

In general, let  $\bar{\lambda}(n, n-k)$  be the coefficient of the input  $y_n$  with respect to the output  $y_{n-k}$ . We then have:

$$y_{n-k} = \bar{\lambda}(n, n-k) y_n - r \sum_{i=1}^k \bar{\lambda}(n-i, n-k) x_{n-i+1}, \quad 0 \leq k \leq g.$$

Note that this equation will characterize the behavior of any column of the state diagram, simply by replacing the  $x_i$ 's by the state probabilities of the preceding column. This then gives the general formula:

$$P(i_1, i_2) = \bar{\lambda}(n, i_2) P(i_1, n) - \bar{\delta}_{i_1, 0} r \sum_{i=i_2+1}^n \bar{\lambda}(i-1, i_2) P(i_1-1, i). \quad (6.20)$$

This expression is valid for all values of  $i_1 \geq 0$ , and for  $n - g \leq i_2 \leq n$ , where the summation is taken equal to zero if  $i_2 = n$  (also recall that  $r = b/c$ ). The factor  $\delta_{i_1,0}$  is equal to 0 if  $i_1 = 0$  and to 1 otherwise. It arises to take into account the fact that the first column has no incoming flux from the left. The generality of Equation (6.20) enables us to identify the coefficients  $\bar{\lambda}(n, i_2)$  since they satisfy the same relation for all values of  $i_1$ , and are already known in the case  $i_1 = 0$ . Namely, we have, after identifying Equations (6.8) and (6.20):

$$\bar{\lambda}(n, i_2) = \frac{c^{i_2}}{i_2!} \lambda(n, i_2), \quad n - g \leq i_2 \leq n. \quad (6.21)$$

Here  $\lambda(n, i_2)$  is given by Equation (6.9). Unfortunately, recall that in our flow graph we had the following notation:

$$\beta_{i_2} = \begin{cases} \frac{b+i_2+c}{c}, & \text{for } n - g \leq i_2 < n; \\ \frac{b+n}{c}. & \end{cases}$$

We can immediately realize that the two expressions do not agree for  $i_2 = n$ . Then, since  $\beta_p$  gives the initial condition of the sequence  $\bar{\lambda}(p, i_2)$ ,  $i_2 \leq p$ , we will clearly have differences between the general expression for  $\bar{\lambda}(p, i_2)$  with  $p \neq n$  and for  $\bar{\lambda}(n, i_2)$ . This means that we cannot simply use Equation (6.21) for  $\bar{\lambda}(n, i_2)$  and replace  $n$  by  $p$  to obtain the expression of  $\bar{\lambda}(p, i_2)$ . In order to obtain the general formula for the  $\bar{\lambda}(p, i_2)$ ,  $p \neq n$ , some additional calculations will be needed. Fortunately, it turns out that the change of initial condition does not induce drastic modifications, and there is a nearly complete parallelism with Section 6.2. We will now proceed with this calculation.

For simplicity, let  $q_k = \bar{\lambda}(p, p - k)q_0$  ( $q_0 = P(i_1, n)$ ). Now, by analogy with Equation (6.1), we know that the  $q_k$ 's must satisfy:

$$q_k = [(p+1)c^{-1} + (r+1) - kc^{-1}]q_{k-1} - [(p+2)c^{-1} - kc^{-1}]q_{k-2}.$$

Using similar notation, we immediately get the Z-Transform equation:

$$Q(Z)[1 - \alpha Z + \beta Z^2] + \dot{Q}(Z)\delta Z^2(1 - Z) = q_0(1 - \alpha Z) + Zq_1.$$

Here we have  $\alpha = (p + b + c)/c$ ,  $\beta = p/c$ ,  $\delta = c^{-1}$ . Note that so far, there is no difference between the new expression and the one we obtained for the first column. This is due to the fact that we still have not made use of our initial condition. But, if we now decide to replace  $q_1$  by its expression in terms of  $q_0$ , the resulting Z-Transform equation will differ from the one we obtained in Section 6.2 (Equation (6.2)). Using the above notation we know that  $q_1 = \alpha q_0$  (instead of  $q_1 = (\alpha - 1)q_0$  which held before). This then gives us the following differential equation:

$$Q(Z)[1 - \alpha Z + \beta Z^2] + \dot{Q}(Z)\delta Z^2(1 - Z) = q_0.$$

Or, in other terms,

$$\dot{Q}(Z) + Q(Z) \frac{[\beta Z^2 - \alpha Z + 1]}{\delta Z^2(1 - Z)} = \frac{q_0}{\delta Z^2(1 - Z)}. \quad (6.22)$$

Using the same techniques as before, the solution can be written in the form:

$$Q(Z) = Q(\epsilon)e^{-A(Z)} + e^{-A(Z)} \int_{\epsilon}^Z e^{A(x)} \frac{cq_0}{(1-x)x^2} dx = (i') + (ii'),$$

$$\text{where } A(Z) = \int_{\epsilon}^Z \frac{\beta x^2 - \alpha x + 1}{\delta x^2(1-x)} dx.$$

As in Section 6.2 we want a solution continuous in the region  $|Z| < 1$ , and we again have two distinct terms:

- (i') Represents the contribution of the homogeneous equation associated with Equation (6.22), with the same initial conditions. Note that (i') has the same expression as the term (i) of Section 6.2. This is due to the fact that (i') does not depend on the inhomogeneous part of Equation (6.22) and the homogeneous

parts of Equations (6.22) and (6.2) of Section 6.2 are identical. This will allow us to use for (i') all the results we obtained for (i) in Section 6.2.

(ii') Is a particular solution of the inhomogeneous Equation (6.22).

As before, let us first look at (ii'); since  $A(Z)$  has the same expression as in Section 6.2, we can immediately write the following expression for (ii'):

$$(ii') = q_0(1-Z)^{-b} Z^{(p+b)} e^{cZ^{-1}} c \int_{\epsilon}^Z (1-x)^{(b-1)} x^{-(p+b)} e^{-cx^{-1}} \frac{dx}{x^2}.$$

Now let  $u = 1/x$ ,  $du = -dx/x^2$  to find

$$\begin{aligned} (ii') &= -q_0 \frac{Z^{(p+b)}}{(1-Z)^b} e^{cZ^{-1}} c \int_{\epsilon^{-1}}^{Z^{-1}} u^{(p+b)} \left(\frac{u-1}{u}\right)^{(b-1)} e^{-cu} du \\ \Rightarrow (ii') &= -q_0 \frac{Z^{(p+b)}}{(1-Z)^b} e^{cZ^{-1}} c \int_{\epsilon^{-1}}^{Z^{-1}} u^{(p+1)} (u-1)^{(b-1)} e^{-cu} du \\ \Rightarrow (ii') &= -q_0 \frac{Z^{(p+b)}}{(1-Z)^b} e^{cZ^{-1}} c \int_{\epsilon^{-1}}^{Z^{-1}} [(u-1)+1]^{(p+1)} (u-1)^{(b-1)} e^{-cu} du \\ \Rightarrow (ii') &= -q_0 \frac{Z^{(p+b)}}{(1-Z)^b} e^{cZ^{-1}} c \left[ \sum_{i=0}^{p+1} \binom{p+1}{i} \int_{\epsilon^{-1}}^{Z^{-1}} (u-1)^{(b+i-1)} e^{-cu} du \right]. \end{aligned}$$

Then let  $v = u - 1$  to obtain

$$(ii') = -q_0 \frac{Z^{(p+b)}}{(1-Z)^b} e^{cZ^{-1}} c \left[ \sum_{i=0}^{p+1} e^{-c} \binom{p+1}{i} \int_{\epsilon^{-1}-1}^{Z^{-1}-1} v^{(b+i-1)} e^{-cv} dv \right].$$

Finally, let  $y = cv$ ,  $dy = cdv$ , which implies

$$(ii') = -q_0 \frac{Z^{(p+b)}}{(1-Z)^b} e^{c[Z^{-1}-1]} \left[ \sum_{i=0}^{p+1} \binom{p+1}{i} \int_{c[\epsilon^{-1}-1]}^{c[Z^{-1}-1]} c^{-(b+i-1)} y^{(b+i-1)} e^{-y} dy \right].$$

We now are in a position to make the same assumptions and apply the same manipulations as in Section 6.2. Namely, under the assumption of continuity at the origin of our solution, we have  $Q(Z) \rightarrow Q(0) = q_0$  when  $Z \rightarrow 0$ . We can then let  $\epsilon \rightarrow 0$ , and assume that we have:

$$Q(Z) = \lim_{\epsilon \rightarrow 0} (ii') \quad (**)$$

This is again justified by the fact that with  $Z$  fixed,  $(i') \rightarrow 0$  as  $\epsilon \rightarrow 0$  for all values of  $Z \neq 0$ . We here make use of the fact that  $(i')$  and  $(i)$  of Section 6.2 are identical. And as in Section 6.2 we will prove that  $Q(Z)$  can be defined by (\*\*) by establishing its continuity. Namely, we will be able to verify that we then still have  $Q(Z) \rightarrow Q(0) = q_0$  when  $Z \rightarrow 0$ . To accomplish this, we first write:

$$Q(Z) = q_0 \frac{Z^{(p+b)}}{(1-Z)^b} e^{c[Z^{-1}-1]} c^{-(b-1)} \left[ \sum_{i=0}^{p+1} \binom{p+1}{i} c^{-i} \int_{c[Z^{-1}-1]}^{\infty} y^{(b+i-1)} e^{-y} dy \right]$$

$$\Rightarrow Q(Z) = q_0 \frac{Z^{(p+b)}}{(1-Z)^b} e^{c[Z^{-1}-1]} c^{-(b-1)} \left[ \sum_{i=0}^{p+1} \binom{p+1}{i} c^{-i} \Gamma(b+i, c[Z^{-1}-1]) \right].$$

Again using the same relation as in Section 6.2 for the incomplete  $\Gamma$ -function ([48], p. 942, Eq. 8.356-5), we can write for  $i > 0$ :

$$\Gamma(b+i, c[Z^{-1}-1]) = (b)_i \Gamma(b, c[Z^{-1}-1])$$

$$+ (c[Z^{-1}-1])^b e^{-c[Z^{-1}-1]} \left[ \sum_{j=0}^{i-1} (b+j+1)_{i-(j+1)} (c[Z^{-1}-1])^j \right]$$

$$\Rightarrow \Gamma(b+i, c[Z^{-1}-1]) = (b)_i \Gamma(b, c[Z^{-1}-1])$$

$$+ (c[Z^{-1}-1])^{(b-1)} e^{-c[Z^{-1}-1]} \left[ \sum_{j=1}^i (b+j)_{i-j} (c[Z^{-1}-1])^j \right].$$

This gives us:

$$\sum_{i=0}^{p+1} \binom{p+1}{i} c^{-i} \Gamma(b+i, c[Z^{-1}-1]) = \Gamma(b, c[Z^{-1}-1]) \left[ \sum_{i=0}^{p+1} \binom{p+1}{i} c^{-i} (b)_i \right]$$

$$+ (c[Z^{-1}-1])^{(b-1)} e^{-c[Z^{-1}-1]} \left[ \sum_{i=1}^{p+1} \binom{p+1}{i} c^{-i} \left( \sum_{j=1}^i (b+j)_{i-j} (c[Z^{-1}-1])^j \right) \right].$$

So, we finally get:

$$Q(Z) = q_0 \frac{Z^{(p+b)}}{(1-Z)^b} e^{c[Z^{-1}-1]} c^{-(b-1)} \Gamma(b, c[Z^{-1}-1]) \left[ \sum_{i=0}^{p+1} c^{-i} (b)_i \right]$$

$$+ q_0 \frac{Z^{(p+1)}}{(1-Z)} \left[ \sum_{i=1}^{p+1} \binom{p+1}{i} c^{-i} \left( \sum_{j=1}^i (b+j)_{i-j} (c[Z^{-1}-1])^j \right) \right]$$

$$\begin{aligned} \Rightarrow Q(Z) &= q_0 \frac{Z^{(p+b)}}{(1-Z)^b} e^{c[Z^{-1}-1]} c^{-(b-1)} \Gamma(b, c[Z^{-1}-1]) \left[ \sum_{i=0}^{p+1} \binom{p+1}{i} c^{-i(b)_i} \right] \\ &\quad + q_0 Z^p \left[ \sum_{i=0}^p \binom{p+1}{i+1} c^{-i} c^{-1} \left( \sum_{j=0}^i c(b+j+1)_{i-j} \frac{(1-Z)}{Z} (c[Z^{-1}-1])^j \right) \right] \\ \Rightarrow Q(Z) &= q_0 \frac{Z^{(p+b)}}{(1-Z)^b} e^{c[Z^{-1}-1]} c^{-(b-1)} \Gamma(b, c[Z^{-1}-1]) \left[ \sum_{i=0}^{p+1} \binom{p+1}{i} c^{-i(b)_i} \right] \\ &\quad + q_0 Z^p \left[ \sum_{i=0}^p \binom{p+1}{i+1} c^{-i} \left( \sum_{j=0}^i (b+j+1)_{i-j} (c[Z^{-1}-1])^j \right) \right]. \quad (6.23) \end{aligned}$$

We now want to verify that we still have:

$$\lim_{Z \rightarrow 0} Q(Z) = Q(0) = q_0.$$

We will use the same expansion of the incomplete  $\Gamma$ -function as in Section 6.2, valid for small values of  $Z$ . We then have for  $Z \rightarrow 0$ :

$$\begin{aligned} Q(Z) &= q_0 \frac{Z^{(p+1)}}{(1-Z)} \left[ \sum_{i=0}^{p+1} \binom{p+1}{i} c^{-i(b)_i} \right] \left[ \sum_{m=0}^{M-1} \frac{(1-b)_m Z^m}{(-c)^m (1-Z)^m} + O(Z^M) \right] \\ &\quad + q_0 Z^p \left[ \sum_{i=0}^p \binom{p+1}{i+1} c^{-i} \left( \sum_{j=0}^i (b+j+1)_{i-j} c^j \frac{(1-Z)^j}{Z^j} \right) \right] = (a') + (b'). \end{aligned}$$

If we now let  $Z \rightarrow 0$ , we again find that the only non-zero term is the one corresponding to indices  $i = j = p$  in the second component of the above equation. The corresponding coefficient is:

$$q_0 \left[ \binom{p+1}{p+1} c^{-p} c^p (1-Z)^p \right] \longrightarrow q_0 \quad \text{as } Z \rightarrow 0.$$

This, as in Section 6.2 (also see Appendix F), justifies the step we took in assuming:

$$Q(Z) = \lim_{\epsilon \rightarrow 0} (i\epsilon').$$



We can now go back to our initial task, that of obtaining an expression for  $\bar{\lambda}(p, i_2)$  with  $p \neq n$ . We are therefore interested in evaluating the coefficients of powers of  $Z$  smaller or equal to  $p$ . This in turn tells us that (a') ((a) in Section 6.2) can be ignored in the expression of  $Q(Z)$ . We can now concentrate on (b') only.

$$(b') = q_0 Z^p \left[ \sum_{i=0}^p \binom{p+1}{i+1} c^{-i} \left( \sum_{j=0}^i (b+j+1)_{i-j} c^j \frac{(1-Z)^j}{Z^j} \right) \right],$$

or

$$(b') = q_0 \left[ \sum_{i=0}^p \binom{p+1}{i+1} c^{-i} \left( \sum_{j=0}^i (b+j+1)_{i-j} c^j Z^{p-j} (1-Z)^j \right) \right],$$

or again

$$(b') = q_0 \left[ \sum_{j=0}^p c^j Z^{p-j} (1-Z)^j \left( \sum_{i=j}^p (b+j+1)_{i-j} \binom{p+1}{i+1} c^{-i} \right) \right].$$

We can now obtain the desired expression for  $\bar{\lambda}(p, p-k)$ :

$$\begin{aligned} q_k &= \bar{\lambda}(p, p-k) q_0 \\ \Rightarrow q_k &= q_0 \sum_{j=p-k}^p c^j \binom{j}{j-(p-k)} (-1)^{j-(p-k)} \left( \sum_{i=j}^p (b+j+1)_{i-j} \binom{p+1}{i+1} c^{-i} \right). \end{aligned}$$

If this is rewritten with indices in agreement with Equations (6.20) and (6.21), we find:

$$\bar{\lambda}(p, i_2) = \frac{c^{i_2}}{i_2!} \sum_{l=0}^{p-i_2} c^l (-1)^l \frac{(l+i_2)!}{l!} \left( \sum_{i=l+i_2}^p (b+l+i_2+1)_{i-l-i_2} \binom{p+1}{i+1} c^{-i} \right). \quad (6.24)$$

We note that the expression of  $\bar{\lambda}(p, i_2)$  for  $p \neq n$  is the same as for  $p = n$ , except that the coefficient  $\binom{p+1}{i+1}$  present in Equation (6.24) replaces the former coefficient  $\binom{n}{i}$ .

Equation (6.20) tells us that the state probabilities in any column are determined by the state probabilities of the previous column, and the probability of the

top state of that given column. Recall that we do not know the probability of the top state. However, the probability of the bottom state can be obtained from the state probabilities of the previous column by using Equation ( $\beta$ ) of Section 6.5. Therefore, if we use Equation (6.20) for  $i_2 = n - g$  to express  $P(i_1, n)$  in terms of  $P(i_1, n - g)$ , we will be able to write any state probability in a given column in terms of state probabilities of the previous column.

We now proceed with exactly this computation, first by specializing Equation (6.20) to the case  $i_2 = n - g$ :

$$P(i_1, n - g) = \bar{\lambda}(n, n - g)P(i_1, n) - \bar{\delta}_{i_1, 0} r \sum_{i=n-g+1}^n \bar{\lambda}(i - 1, n - g)P(i_1 - 1, i).$$

Since Equation ( $\beta$ ) of Section 6.5 gives us

$$P(i_1, n - g) = \frac{b}{(n - g)} \sum_{i=n-g}^n P(i_1 - 1, i),$$

we can write:

$$\begin{aligned} \bar{\lambda}(n, n - g)P(i_1, n) &= \frac{b}{(n - g)} \sum_{i=n-g}^n P(i_1 - 1, i) \\ &\quad + \bar{\delta}_{i_1, 0} r \sum_{i=n-g+1}^n \bar{\lambda}(i - 1, n - g)P(i_1 - 1, i). \end{aligned}$$

If we define, in agreement with Equations (6.21) and (6.24),

$$\bar{\lambda}(p, i_2) = \frac{c^{i_2}}{i_2!} \lambda(p, i_2), \quad \forall p \quad 0 \leq p \leq n, \quad (6.25)$$

we will have:

$$\begin{aligned} P(i_1, n) &= \frac{(n - g)!}{c^{(n-g)}} \frac{1}{\lambda(n, n - g)} \left[ \frac{b}{(n - g)} \left( \sum_{i=n-g}^n P(i_1 - 1, i) \right) \right. \\ &\quad \left. + \bar{\delta}_{i_1, 0} \frac{b}{c} \frac{c^{n-g}}{(n - g)!} \left( \sum_{i=n-g+1}^n \lambda(i - 1, n - g)P(i_1 - 1, i) \right) \right]. \end{aligned}$$

If we now use this new expression for  $P(i_1, n)$  in Equation (6.20), we can express all the state probabilities of column  $i_1$  in terms of the state probabilities of column  $i_1 - 1$ . For notational simplicity, and since we already derived the state probabilities for column  $i_1 = 0$  in Section 6.3, we will limit ourselves to the cases  $i_1 > 0$ . We then get:

$$\begin{aligned}
 P(i_1, i_2) = \frac{(n-g)! c^{i_2}}{c^{(n-g)} i_2!} & \left[ \frac{\lambda(n, i_2)}{\lambda(n, n-g)} \frac{b}{(n-g)} \left( \sum_{i=n-g}^n P(i_1 - 1, i) \right) \right. \\
 & + \frac{\lambda(n, i_2)}{\lambda(n, n-g)} \frac{b c^{(n-g)}}{c (n-g)!} \left( \sum_{i=n-g+1}^n \lambda(i-1, n-g) P(i_1 - 1, i) \right) \\
 & \left. - \frac{b c^{(n-g)}}{c (n-g)!} \left( \sum_{i=i_2+1}^n \lambda(i-1, i_2) P(i_1 - 1, i) \right) \right]. \quad (6.26)
 \end{aligned}$$

Equation (6.26) is valid for  $i_1 > 0$  and  $n-g \leq i_2 \leq n$ . If we consider the special case  $i_1 = 1$ , we get the state probabilities of the second column in terms of the state probabilities of the first column. The latter probabilities are known from Equation (6.11) of Section 6.3. Unfortunately, even though we can indeed obtain the desired result, the expression does not seem very useful, and is not easily generalizable. The next section will be devoted to establishing a simple general expression for the state probabilities of any column in terms of the state probabilities of the first column that will be much more useful than the formula we could obtain here in closed form.

### 6.7 General expression for the state probabilities.

In order to obtain a practical formulation for the state probabilities, we will express Equation (6.26) in a matrix form. We are naturally led to this type of representation by the recursive form of the expression relating each column of the state diagram. Furthermore, the column form itself calls for a corresponding vector representation. We therefore define the following column probability vector of dimension  $(g+1)$ :

$$\vec{P}_{i_1} = \begin{pmatrix} P(i_1, n-g) \\ P(i_1, n-g+1) \\ \vdots \\ P(i_1, n-1) \\ P(i_1, n) \end{pmatrix}.$$

We now want to obtain the  $(g+1) \times (g+1)$  matrix  $T$  such that:

$$\vec{P}_{i_1} = T\vec{P}_{i_1-1}, \quad i_1 > 0.$$

Let  $t_{ij}$ ,  $i, j = 1, \dots, g+1$  be the coefficients of the matrix  $T$ . They can be determined after identification with Equation (6.26):

- $t_{1j} = \frac{b}{(n-g)}, \quad \text{for } j = 1, \dots, g+1;$
- $t_{i1} = \frac{(n-g)!}{(i+n-g-1)!} c^{(i-1)} \frac{b}{(n-g)} \frac{\lambda(n, i+n-g-1)}{\lambda(n, n-g)}, \quad \text{for } i = 1, \dots, g+1;$
- $t_{ij} = \frac{(n-g)!}{(i+n-g-1)!} c^{(i-1)} \frac{\lambda(n, i+n-g-1)}{\lambda(n, n-g)} \left[ \frac{b}{(n-g)} + \frac{b}{c} \frac{c^{(n-g)}}{(n-g)!} \lambda(j+n-g-2, n-g) \right],$   
for  $\begin{cases} i = 2, \dots, g+1, \\ 2 \leq j \leq i; \end{cases}$
- $t_{ij} = \frac{(n-g)!}{(i+n-g-1)!} c^{(i-1)} \left[ \frac{\lambda(n, i+n-g-1)}{\lambda(n, n-g)} \left( \frac{b}{(n-g)} + \frac{b}{c} \frac{c^{(n-g)}}{(n-g)!} \lambda(j+n-g-2, n-g) \right) - \frac{b}{c} \frac{c^{(n-g)}}{(n-g)!} \lambda(j+n-g-2, i+n-g-1) \right],$   
for  $\begin{cases} i = 1, \dots, g, \\ j > i. \end{cases}$

With the matrix  $T$  defined as above, the state probability vector for column  $i_1$  is easily expressable in terms of  $\vec{P}_0$ . We have the following relation:

$$\vec{P}_{i_1} = T^{i_1} \vec{P}_0, \quad i_1 \geq 0. \tag{6.27}$$

where:

$$\vec{P}_0 = \begin{pmatrix} P(0, n - g) \\ P(0, n - g + 1) \\ \vdots \\ P(0, n - 1) \\ P(0, n) \end{pmatrix}.$$

And the state probabilities  $P(0, i_2)$  for  $n - g \leq i_2 \leq n$  are determined by Equation (6.11). Note that Equation (6.27) also holds for  $i_1 = 0$ .

We note that Equation (6.27) provides a computationally efficient way of computing the state probabilities, since the size of the matrix  $T$  only depends on the number of guard channels. This number being typically small compared with the total number of channels  $n$ , the matrix can usually be easily diagonalized, which makes the computation of its powers easier, or we can also compute the state probabilities in an iterative fashion. This method is to be compared with the classical way of dealing with this problem ([53]), by numerically solving the system:

$$A\vec{P} = \vec{0}. \tag{R}$$

where  $A$  is the  $N \times N$  matrix of transition rates between states,  $N$  is the total number of states in the system, and  $\vec{P}$  is the state vector of dimension  $N$ .

Note that we are now restricted to the case of finite queue length, while Equation (6.27) treats the more general case of infinite queue length. We will later describe the modifications needed to adapt Equation (6.27) to the finite queue length case, but one can already say that the computational complexity remains nearly exactly the same. In any case, solving system (R) can be extremely costly for our problem, even with an efficient algorithm.

Let us illustrate the solution method by an example. Consider a system with  $n = 100$  channels,  $g = 10$  guard channels, and a waiting space of maximum size

$w = 30$ . The total number of states is then  $N = 419$ . Among these, 89 correspond to the cases with more than 10 free servers. The remaining 330 are formed by the possible queue of length 30 for each of the 11 cases where the number of busy servers is between 90 and 100. System (R) then involves finding the normalized eigenvector of the  $419 \times 419$  matrix  $A$ , corresponding to the eigenvalue  $\lambda = 0$ . On the other hand, Equation (6.27) only requires the computation of powers of the  $11 \times 11$  matrix  $T$ , which, for example, after diagonalization is relatively simple. Or Equation (6.27) can also be used to obtain the state probabilities iteratively. Furthermore, we might not be interested in computing all the state probabilities. Then, Equation (6.27) allows us to stop at any desired level, while system (R) leaves us no choice. Finally, the complexity of (R) increases with the number of waiting spaces, while this parameter has no influence on Equation (6.27).

Before going into the modifications needed to adapt Equation (6.27) to the case of finite queue length, we will derive some interesting system parameters such as average queue length and average waiting time.

### 6.8 Average queue length and average waiting time.

The matrix form of Equation (6.27) will allow us to obtain some rather simple expressions for the average queue length as well as for the average waiting time. But first let us define the vector  $\vec{U}$  as the column vector of dimension  $(g+1)$  with all its coefficients equal to 1. Since along a given column the queue length remains the same, and since column  $i_1$  corresponds to a queue of length  $i_1$ , the average queue length is given by:

$$L_q = \sum_{i_1=0}^{\infty} i_1 \vec{U}^T \vec{P}_{i_1}$$

$$\Rightarrow L_q = \vec{U}^T \left[ \sum_{i_1=0}^{\infty} i_1 T^{i_1} \right] \vec{P}_0.$$

Here  $L_q$  stands for the average queue length.

The above matrix series must be defined, since we assumed an ergodic system. We therefore have:

$$L_q = \vec{U}^T (I - T)^{-2} \vec{P}_0. \quad (6.28)$$

The average queue length is plotted as a function of the number of guard channels for two values of the total number of channels  $n$  in Figures 6.11 ( $n = 100$ ) and 6.12 ( $n = 44$ ).

If we now want the average waiting time  $W$  for customers of type (II), we can simply apply Little's formula ([42], p. 17) to our queueing system. This yields:

$$\begin{aligned} L_q &= \lambda W \\ \Rightarrow W &= \frac{1}{\lambda} [\vec{U}^T (I - T)^{-2} \vec{P}_0]. \end{aligned} \quad (6.29)$$

Similarly, if we are interested in  $W^*$ , the average waiting time of customers that actually experience some delay, we have the following expression:

$$W^* = \frac{W}{P_{D_{II}}}. \quad (6.30)$$

Here  $P_{D_{II}}$  is given by Equation (6.16). This ends Section 6.8, and we will now proceed with the modifications needed to adapt our result to a system with finite queue length.

### 6.9 System with finite queue length.

We now consider a queueing-blocking system identical to the previous one, except for the fact that we now limit the queue length to a finite number  $L$ . The first remark we can make is that Equations (6.26) and  $(\beta)$  are still valid, for those indices  $i_1$  ranging from 0 to  $(L - 1)$ . For the last column, however, there is no outgoing flux to the right, and therefore Equation  $(\beta)$  is not defined; furthermore that column

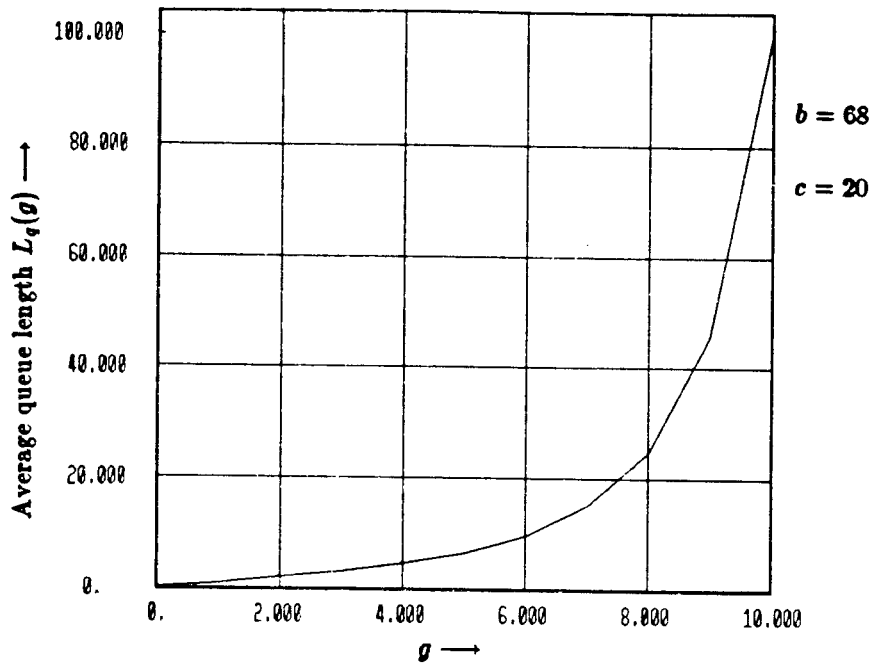


Fig 6.11. Average queue length for customers II,  $n = 100$ .

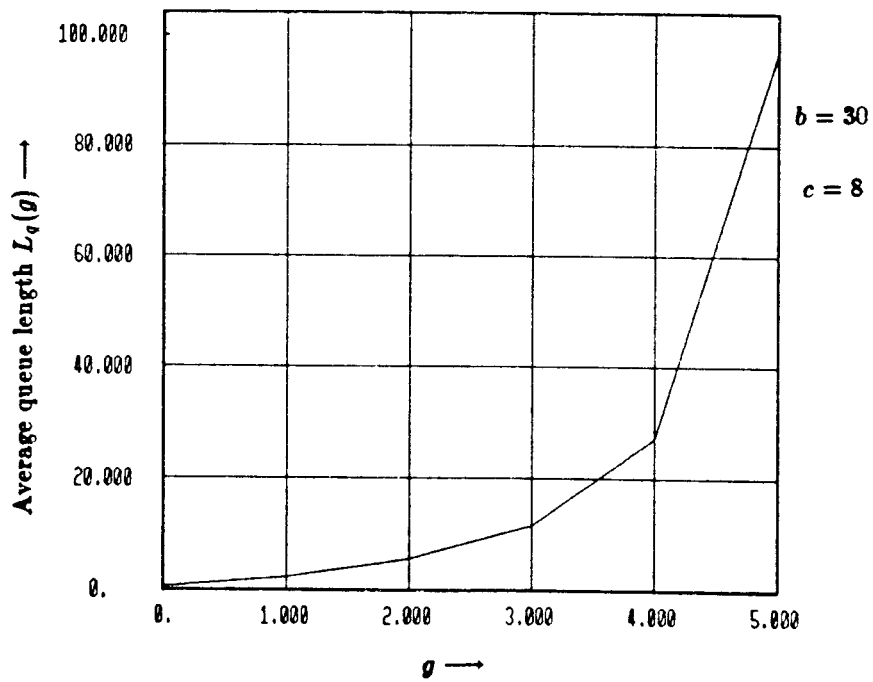


Fig 6.12. Average queue length for customers II,  $n = 44$ .



will satisfy a relation different from Equation (6.26). We will now proceed with the computation of this new expression.

The absence of outflow to the right in this case makes a direct derivation possible. We illustrate this by first working out some examples before obtaining the general formula. If we limit ourselves to a maximum queue length of  $L$ , the last column of the state diagram will then correspond to the index  $i_1 = L$ . We can start by writing the balance equation for the top state of this column ( $i_1 = L, i_2 = n$ ), and then proceed down the column for a few values of the index  $i_2$ :

$$\bullet \quad nP(L, n) = cP(L, n-1) + bP(L-1, n)$$

$$\Rightarrow P(L, n-1) = \frac{n}{c}P(L, n) - \frac{b}{c}P(L-1, n),$$

$$\bullet \quad (n-1+c)P(L, n-1) = nP(L, n) + cP(L, n-2) + bP(L-1, n-1)$$

$$\Rightarrow (n-1)P(L, n-1) = cP(L, n-2) + b(P(L-1, n) + P(L-1, n-1))$$

$$\Rightarrow P(L, n-2) = \frac{n(n-1)}{c^2}P(L, n) - \frac{b}{c} \left[ \frac{(n-1)}{c} + 1 \right] P(L-1, n) - \frac{b}{c}P(L-1, n-1),$$

$$\bullet \quad (n-2+c)P(L, n-2) = (n-1)P(L, n-1) + cP(L, n-3) + bP(L-1, n-2)$$

$$\Rightarrow (n-2)P(L, n-2) = cP(L, n-3) + b[P(L-1, n) + P(L-1, n-1) + P(L-1, n-2)]$$

$$\Rightarrow P(L, n-3) = \frac{n(n-1)(n-2)}{c^3}P(L, n) - \frac{b}{c} \left[ \frac{(n-1)(n-2)}{c^2} + \frac{(n-2)}{c} + 1 \right] P(L-1, n) - \frac{b}{c} \left[ \frac{(n-2)}{c} + 1 \right] P(L-1, n-1) - \frac{b}{c}P(L-1, n-2).$$

The general formula can then be extended from the above examples. We get:

$$P(L, n - k) = \frac{n!}{(n - k)!} c^{-k} P(L, n) - \frac{b}{c} \left[ \sum_{p=1}^k \left( \sum_{l=p}^k \frac{(n - l)!}{(n - k)!} c^{(l-k)} \right) P(L - 1, n - p + 1) \right].$$

If we specialize this expression to the case  $k = g$ , we get:

$$P(L, n - g) = \frac{n!}{(n - g)!} c^{-g} P(L, n) - \frac{b}{c} \left[ \sum_{p=1}^g \left( \sum_{l=p}^g \frac{(n - l)!}{(n - g)!} c^{(l-g)} \right) P(L - 1, n - p + 1) \right].$$

We also have from Equation ( $\beta$ ):

$$P(L, n - g) = \frac{b}{(n - g)} \sum_{p=0}^g P(L - 1, n - p).$$

So combining these two equations gives:

$$\frac{b}{(n - g)} \sum_{p=0}^g P(L - 1, n - p) = \frac{n!}{(n - g)!} c^{-g} P(L, n) - \frac{b}{c} \left[ \sum_{p=0}^{g-1} \left( \sum_{l=p+1}^g \frac{(n - l)!}{(n - g)!} c^{(l-g)} \right) P(L - 1, n - p) \right].$$

We can now express  $P(L, n)$  in terms of the state probabilities of column ( $L - 1$ ):

$$P(L, n) = \frac{(n - g)!}{n!} c^g \left[ \frac{b}{(n - g)} \sum_{p=0}^g P(L - 1, n - p) + \frac{b}{c} \left( \sum_{p=0}^{g-1} \left[ \sum_{l=p+1}^g \frac{(n - l)!}{(n - g)!} c^{(l-g)} \right] P(L - 1, n - p) \right) \right].$$

The above will enable us to obtain all state probabilities of column  $L$  in terms of state probabilities of column ( $L - 1$ ):

$$P(L, n - k) = \frac{(n - g)!}{c^{(n-g)}} \frac{c^{(n-k)}}{(n - k)!} \left[ \frac{b}{(n - g)} \sum_{p=0}^g P(L - 1, n - p) + \frac{b}{c} \left( \sum_{p=0}^{g-1} \left[ \sum_{l=p+1}^g \frac{(n - l)!}{(n - g)!} c^{(l-g)} \right] P(L - 1, n - p) \right) - \frac{b}{c} \left( \sum_{p=0}^{k-1} \left[ \sum_{l=p+1}^k \frac{(n - l)!}{(n - g)!} c^{(l-g)} \right] P(L - 1, n - p) \right) \right] \quad (6.31)$$

The above equation is valid for  $0 \leq k \leq g$ , where the last summation is as usual equal to zero for  $k = 0$ . We can, as before, express Equation (6.31) in the form of a matrix-vector multiplication. With the same notation as in Section 6.7, we get:

$$\vec{P}_L = S\vec{P}_{L-1}.$$

Here  $S$  is a  $(g+1) \times (g+1)$  matrix whose entries are defined in the following way:

- $s_{1j} = \frac{b}{(n-g)}$ , for  $j = 1, \dots, g-1$ ,
- $s_{i1} = \frac{(n-g)!}{(i+n-g-1)!} c^{(i-1)} \frac{b}{(n-g)}$ , for  $i = 1, \dots, g+1$ ,
- $s_{ij} = \frac{(n-g)!}{(i+n-g-1)!} c^{(i-1)} \left[ \frac{b}{(n-g)} + \frac{b}{c} \left( \sum_{l=g-j+2}^g \frac{(n-l)!}{(n-g)!} c^{(l-g)} \right) \right]$ ,  
for  $\begin{cases} i = 2, \dots, g+1, \\ 2 \leq j \leq i, \end{cases}$
- $s_{ij} = \frac{(n-g)!}{(i+n-g-1)!} c^{(i-1)} \left[ \frac{b}{(n-g)} + \frac{b}{c} \left( \sum_{l=g-i+2}^g \frac{(n-l)!}{(n-g)!} c^{(l-g)} \right) \right]$ ,  
for  $\begin{cases} i = 2, \dots, g, \\ j > i. \end{cases}$

The coefficients of the matrix  $S$  are of smaller complexity than the ones of the matrix  $T$  defined in Section 6.7. This justifies the remark that the computational complexities of Equation (6.27) and the following Equation (6.32) are nearly equivalent.

Having defined the matrix  $S$ , we can now readily express  $\vec{P}_L$  in terms of  $\vec{P}_0$ . This will give us a complete description of our finite queue system. Namely, we have:

$$\begin{cases} \vec{P}_{i_1} = T^{i_1} \vec{P}_0, & \text{for } 0 \leq i_1 < L; \\ \vec{P}_L = ST^{L-1} \vec{P}_0. \end{cases} \quad (6.32)$$

We must realize that limiting the queue length to  $L$  introduces blocking for customers of type II as well as for type I. Namely, a type II customer will be blocked if he finds a full queue upon his arrival. This new blocking probability is given by the following expression:

$$\begin{aligned} P_{B_{II}} &= \bar{U}^T \bar{P}_L \\ \Rightarrow P_{B_{II}} &= \bar{U}^T (ST^{L-1}) \bar{P}_0. \end{aligned} \quad (6.33)$$

We can verify that this expression goes to 0 when we let  $L \rightarrow \infty$ , since the ergodicity of the system implies that:

$$\lim_{L \rightarrow \infty} T^{L-1} = O.$$

Here  $O$  stands for the all zero matrix.

We can also obtain the probability of delay for customers of type II, which is still given by Equation (6.16):

$$P_{D_{II}} = 1 - \left( \sum_{l=0}^{n-g-1} \frac{a^l}{l!} \right) P(0,0). \quad (6.16)$$

However,  $P(0,0)$  is not given by (6.14) anymore, since we now assume a finite maximum queue length. The actual value of  $P(0,0)$  will have to be determined *a posteriori*, by normalizing the sum of the state probabilities to 1. Similarly, the expression giving the blocking probability for customers of type I will differ from Equation (6.15). However, we still have  $P_{B_I} = P_n$ , where:

$$P_n = \sum_{i_1=0}^L P(i_1, n), \quad \text{whereas} \quad P_n = \sum_{i_1=0}^{\infty} P(i_1, n) \quad \text{before.}$$

Let  $\bar{E}_{g+1}$  be the column vector of dimension  $(g+1)$  with all its components equal to 0 except the last one, which is taken equal to 1. We use  $\bar{E}_{g+1}$  to write:

$$\begin{aligned} P_n &= \sum_{i_1=0}^L \bar{E}_{g+1}^T \bar{P}_{i_1}, \\ \Rightarrow P_n &= \bar{E}_{g+1}^T \left[ \left( \sum_{i_1=0}^{L-1} T^{i_1} \right) + ST^{L-1} \right] \bar{P}_0 \\ \Rightarrow P_{B_I} &= \bar{E}_{g+1}^T \left[ (I - T)^{-1} (I - T^L) + ST^{L-1} \right] \bar{P}_0. \end{aligned} \quad (6.34)$$

This concludes the modifications that were needed in order to adapt the infinite-queue system of Section 6.7 to the finite queue case.

### **6.10 Summary of traffic policies.**

In Chapters 5 and 6, we have presented some traffic policies aimed at giving a higher level of protection to handoff calls. The motivation behind this higher level of protection was that handoff calls represent calls already in progress, and their blocking is extremely penalizing for the users. Furthermore, we wanted some policies that could be implemented at any development stage of a cellular system, without forcing any modifications on the mobile units. This led us to the concept of guard channels which can be implemented through simple modifications of a cellular equipment at the ECP; the DCS and the cell sites do not need to be modified at all (except if queueing of originating calls is allowed).

We first established in Chapter 5 that guard channels are efficient at sharply decreasing the blocking probability of handoff calls, while only slightly increasing the blocking probability of originating calls. We then proved that the possibility of queueing handoff calls was useful only in the case of cells with a large number of allocated frequency channels, more than in practice. The main reason for the limited applicability of queueing handoff calls is that the conditional average delay experienced by the calls that actually use the queueing facility is independent of the number of guard channels, and usually large enough to be noticed when occurring in the middle of a voice communication. We also noticed in Chapter 5 that the price paid for the introduction of guard channels and for the associated decrease in the blocking probability of handoff calls was a slight decrease in the total carried traffic.

The two previous remarks, on the limited efficiency of queueing handoff calls and the decrease in carried traffic due to the guard channels, motivated the introduction

of the queueing of originating calls as studied in Chapter 6. Queueing originating calls is predicted to have the advantage of increasing the total carried traffic, since the originating calls that were previously lost because of the presence of guard channels are now simply delayed and will ultimately be provided with service. On the other hand, the blocking of handoff calls is, as we saw in Chapter 6, only slightly increased, which has hardly any effect on the carried traffic. In any case, this can be taken care of by increasing the number of guard channels. Note that even without guard channels, queueing originating calls represents an improvement over the simple blocking system where calls that find all servers busy are blocked. This improvement is, from a traffic point of view, more important than when handoffs are queued, since the traffic due to originating calls will usually be larger than the traffic due to handoff calls. Furthermore, waits of a few seconds are much more acceptable for originating calls.

Note that our claim of increased carried traffic when using the above method implicitly implies that queued originating calls will always be served. That is, the average waiting of originating calls does not go to infinity. This could happen if the arrival rate of handoff calls were large enough so that the system would always have fewer than  $g$  servers free ( $g$  number of guard channels) with probability one. In this case, an originating call would, with probability one, never be served, and the queueing of originating calls would have no influence on the total carried traffic which here merely corresponds to the handoff calls that go through the system. Such a situation corresponds to a non-ergodic system. Equation (6.18) provides us with a way of estimating whether a given system is ergodic or not, as a function of the number of guard channels and the handoff and originating traffic intensities.

To summarize, we can say that guard channels with no queueing at all represent a very simple way of protecting handoff calls from high blocking. Such a policy is

probably best adapted to the first phases of a cellular system, where low-cost techniques are very important. The reason for this is that the policy can be implemented with simple software modifications in the fixed equipment (at the ECP or possibly also at the cell sites for some system configurations), and no modifications at all of the mobiles. In a more mature cellular system, it will be possible to add some buffer capacity in the system, in order to make the queueing of originating calls possible. This will further improve the perceived service quality and will also yield a higher carried traffic. This additional feature will probably be coupled with a dynamic channel assignment algorithm which will allow "busy" cells to temporarily borrow channels from other "idle" cells (this will however require more transmitters at the cell site). Such a joint policy will be able to maximize the carried traffic not only in a cell, but in the entire cellular system.

## CHAPTER 7

### SUMMARY AND CONCLUSION

This thesis has been mainly devoted to Cellular Radio, but some of the results that were derived have a wider range of possible applications. The traffic policies derived in Chapters 5 and 6 can be used in numerous non-cellular situations. For example, they are applicable to systems (hospital, fire department, etc.) that receive two types of calls, let us say regular and emergency calls, and want to avoid the blocking of emergency calls. In this case, the simple technique of guard channels without any queueing provides an adequate answer to the problem. One can also consider a business center that receives both regular business communications as well as intermittent updates from a stock exchange center. In such a situation the guard channel method together with the queueing of update calls can be efficiently used. Updates have a higher priority than regular calls, and furthermore, we do not want them blocked in any case. Also in many situations we find systems with two types of transmissions (or services). One of them is essential to the system (maintenance, for example) but does not generate any revenue. The other (typically more frequent) represents the revenue source of the system. In such circumstances, one would want to avoid the blocking of the first type of calls without losing any calls from the second type (more precisely without losing any revenue). The policies presented in Chapter 6, where guard channels are combined with the queueing of the second type of calls, are a possible solution to this problem. Furthermore, the condition we obtained on the ergodicity of such a queueing system provides some useful indications on the overall load of the system and the possible effect of the traffic policies on the system behavior.

To summarize, we have analyzed a cellular system with the use of both a simulation program and an analytic model. This provided us with an estimate of the



channel occupancy time distribution. Using this distribution, and modeling a cell as a multi-server facility, we derived some traffic policies in order to give a higher protection level to handoff calls. Some simple policies useful in a start-up configuration were first proposed. We then derived some more evolved policies that presented the additional advantage of increasing the total traffic carried, while still protecting handoff calls and only slightly delaying originating calls.

## APPENDIX A

### KOLMOGOROV-SMIRNOV GOODNESS-OF-FIT TEST

In this appendix we give a more detailed presentation of the Kolmogorov-Smirnov goodness-of-fit test mentioned in Chapter 3. It was used to compare the experimental distribution of the channel occupancy time obtained from the simulation to a theoretical exponential model. This test gives limits for the maximum difference allowable with a given "level of significance" between the simulated distribution and the theoretical model, under the assumption that the samples used in the simulation have been drawn from a population distributed according to the theoretical model.

We assume a population distributed according to a certain (continuous) distribution  $F(x)$ . We sample this population, say,  $m$  times (simulation results) and define the empirical distribution  $F_m(x)$  to be the step function:

$$F_m(x) = \begin{cases} 0, & \text{if } x < x_{(1)}; \\ \frac{k}{m}, & \text{if } x_{(k)} \leq x < x_{(k+1)}; \\ 1, & \text{if } x \geq x_{(k+1)}; \end{cases}$$

where the  $x_{(i)}$ 's are the sample values arranged in ascending order. We then define the two-tailed deviation  $D_m$  between the experimental distribution  $F_m(x)$  and the theoretical  $F(x)$ :

$$D_m = \max|F_m(x) - F(x)|.$$

Under the hypothesis that the samples have really been drawn from the theoretical distribution  $F(x)$ ,  $D_m$  will exceed a certain value, the significance limit, which is function of the sample size and the desired significance  $\alpha$ , only the fraction  $\alpha$  of the time. These significance limits can be shown to behave asymptotically like  $1/\sqrt{m}$ . We will call the significance limits for  $D_m$  as given by the Kolmogorov-Smirnov test the *K-S limits* (at significance  $\alpha$ ).

We now give a more detailed explanation of how the significance  $\alpha$  is related to the K-S limit. Recall that we initially assumed the following relation:

$$\alpha = P\{D_m \geq D_\alpha\}.$$

Here  $\alpha$  is the significance and  $D_\alpha$  is the significance limit. Suppose that we have a **substantial** value for the significance  $\alpha$  (around 0.5); this means that the probability that  $D_m$  exceeds  $D_\alpha$  is substantial. Therefore, if we obtain an empirical distribution  $F_m(x)$  for which  $D_m$  is smaller than  $D_{0.5}$ , we have a very strong indication that  $F(x)$  is the correct distribution for our population (but half the time we would have  $D_m > D_{0.5}$ , even if  $F(x)$  is correct). On the other hand, for **small** values of  $\alpha$ , obtaining a value of  $D_m$  smaller than  $D_\alpha$  does not provide a very good validation of the hypothesis that  $F(x)$  was the correct distribution; in fact, it tends to refute the hypothesis.

The most important point however is the existence of an asymptotic relation between  $\alpha$  and  $D_\alpha$ . Namely, Kolmogorov's theorem tells us that if the function  $F(x)$  is continuous, as it is in our application, then as  $m \rightarrow \infty$ , we have:

$$P\{\sqrt{m}D_m < z\} \rightarrow K(z) = \begin{cases} 0, & \text{for } z \leq 0; \\ \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2 z^2}, & \text{for } z > 0. \end{cases}$$

In our particular case, the number of samples (over 10,000) will be large enough so that the asymptotic relation can be taken as an equality.

## APPENDIX B

### HANDOFF PROBABILITIES FOR A SIMPLIFIED CELLULAR SYSTEM

In this appendix, we summarize the computational steps necessary to derive the different probabilities obtained in Sections 4.2 and 4.3. We start with the computation that led us to the probability that a call goes through at least one handoff. It is obtained from the service time density function and the distribution of the time to the first boundary crossing derived in Section 4.2. Namely, we have:

$$\begin{aligned}
 P_{\geq 1H} &= \int_0^k \mu e^{-\mu t} \left[ -\frac{t^2}{18k^2} + \left( \frac{2\sqrt{3}+3}{9k} \right) t \right] dt \\
 &+ \int_k^{k\sqrt{3}} \mu e^{-\mu t} \left[ -\frac{2}{9k^2} t^2 + \left( \frac{2\sqrt{3}+6}{9k} \right) t - \frac{1}{6} \right] dt \\
 &+ \int_{k\sqrt{3}}^{2k} \mu e^{-\mu t} \left[ -\frac{3}{18k^2} t^2 + \frac{2}{3k} t + \frac{1}{3} \right] dt + \int_{2k}^{\infty} \mu e^{-\mu t} dt.
 \end{aligned}$$

At this point, for convenience and further use, we will review the following definite integrals:

- $\int_a^b \mu e^{-\mu t} dt = [-e^{-\mu t}]_a^b$   
 $= e^{-\mu a} - e^{-\mu b},$
- $\int_a^b t \mu e^{-\mu t} dt = [-te^{-\mu t}]_a^b + \int_a^b \mu e^{-\mu t} dt$   
 $= ae^{-\mu t} - be^{-\mu t} + \frac{1}{\mu} (e^{-\mu a} - e^{-\mu b})$   
 $= \left( a + \frac{1}{\mu} \right) e^{-\mu a} - \left( b + \frac{1}{\mu} \right) e^{-\mu b},$
- $\int_a^b t^2 \mu e^{-\mu t} dt = [-t^2 e^{-\mu t}]_a^b + \frac{2}{\mu} \int_a^b t \mu e^{-\mu t} dt$   
 $= a^2 e^{-\mu a} - b^2 e^{-\mu b} + \frac{2}{\mu} \left[ \left( a + \frac{1}{\mu} \right) e^{-\mu a} - \left( b + \frac{1}{\mu} \right) e^{-\mu b} \right]$   
 $= \left( a^2 + \frac{2a}{\mu} + \frac{2}{\mu^2} \right) e^{-\mu a} - \left( b^2 + \frac{2b}{\mu} + \frac{2}{\mu^2} \right) e^{-\mu b},$
- $\int_a^b t^3 \mu e^{-\mu t} dt = [-t^3 e^{-\mu t}]_a^b + \frac{3}{\mu} \int_a^b t^2 \mu e^{-\mu t} dt$

$$\begin{aligned}
 &= (a^3 e^{-\mu a} - b^3 e^{-\mu b}) + \frac{3}{\mu} \left[ \left( a^2 + \frac{2a}{\mu} + \frac{2}{\mu^2} \right) e^{-\mu a} \right. \\
 &\quad \left. - \left( b^2 + \frac{2b}{\mu} + \frac{2}{\mu^2} \right) e^{-\mu b} \right] \\
 &= \left( a^3 + \frac{3a^2}{\mu} + \frac{6a}{\mu^2} + \frac{6}{\mu^3} \right) e^{-\mu a} - \left( b^3 + \frac{3b^2}{\mu} + \frac{6b}{\mu^2} + \frac{6}{\mu^3} \right) e^{-\mu b}.
 \end{aligned}$$

This then allows us to write:

$$\begin{aligned}
 P_{\geq 1H} &= -\frac{1}{18k^2} \left[ \frac{2}{\mu^2} - \left( k^2 + \frac{2k}{\mu} + \frac{2}{\mu^2} \right) e^{-\mu k} \right] + \left( \frac{2\sqrt{3}+3}{9k} \right) \left[ \frac{1}{\mu} - \left( k + \frac{1}{\mu} \right) e^{-\mu k} \right] \\
 &\quad - \frac{2}{9k^2} \left[ \left( k^2 + \frac{2k}{\mu} + \frac{2}{\mu^2} \right) e^{-\mu k} - \left( 3k^2 + \frac{2\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-\sqrt{3}\mu k} \right] \\
 &\quad + \left( \frac{2\sqrt{3}+6}{9k} \right) \left[ \left( k + \frac{1}{\mu} \right) e^{-\mu k} - \left( \sqrt{3}k + \frac{1}{\mu} \right) e^{-\sqrt{3}\mu k} \right] - \frac{1}{6} \left[ e^{-\mu k} - e^{-\sqrt{3}\mu k} \right] \\
 &\quad - \frac{3}{18k^2} \left[ \left( 3k^2 + \frac{2\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-\sqrt{3}\mu k} - 4 \left( 4k^2 + \frac{4k}{\mu} + \frac{2}{\mu^2} \right) e^{-2\mu k} \right] \\
 &\quad + \frac{2}{3k} \left[ \left( \sqrt{3}k + \frac{1}{\mu} \right) e^{-\sqrt{3}\mu k} - \left( 2k + \frac{1}{\mu} \right) e^{-2\mu k} \right] + \frac{1}{3} \left[ e^{-\sqrt{3}\mu k} - e^{-2\mu k} \right] + e^{-2\mu k}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow P_{\geq 1H} &= -\frac{1}{9k^2\mu^2} + \left( \frac{2\sqrt{3}+3}{9k\mu} \right) \\
 &\quad + e^{-2\mu k} \left[ \frac{2}{3} + \frac{2}{3k\mu} + \frac{1}{3k^2\mu^2} - \frac{4}{3} - \frac{2}{3k\mu} - \frac{1}{3} + 1 \right] \\
 &\quad + e^{-\mu k} \left[ \frac{1}{18} + \frac{1}{9k\mu} + \frac{1}{9k^2\mu^2} - \left( \frac{2\sqrt{3}+3}{9} \right) - \left( \frac{2\sqrt{3}+3}{9k\mu} \right) - \frac{2}{9} \right. \\
 &\quad \left. - \frac{4}{9k\mu} - \frac{4}{9k^2\mu^2} + \left( \frac{2\sqrt{3}+6}{9k\mu} \right) - \frac{1}{6} \right] \\
 &\quad + e^{-\sqrt{3}\mu k} \left[ \frac{2}{3} + \frac{4\sqrt{3}}{9k\mu} + \frac{4}{9k^2\mu^2} - \frac{2}{3} - \frac{2\sqrt{3}}{3} - \left( \frac{2\sqrt{3}+6}{9k\mu} \right) + \frac{1}{6} \right. \\
 &\quad \left. - \frac{1}{2} - \frac{\sqrt{3}}{3k\mu} - \frac{1}{3k^2\mu^2} + \frac{2\sqrt{3}}{3} + \frac{2}{3k\mu} + \frac{1}{3} \right] \\
 \Rightarrow P_{\geq 1H} &= -\frac{1}{9k^2\mu^2} + \left( \frac{2\sqrt{3}+3}{9k\mu} \right) + e^{-2\mu k} \left[ \frac{1}{3k^2\mu^2} \right] \\
 &\quad + e^{-\mu k} \left[ -\frac{1}{3k^2\mu^2} \right] + e^{-\sqrt{3}\mu k} \left[ \frac{1}{9k^2\mu^2} - \frac{\sqrt{3}}{9k\mu} \right].
 \end{aligned}$$

If we now make the substitution  $\alpha = k\mu$ , we get:

$$P_{\geq 1H}(\alpha) = \frac{1}{9\alpha^2} \left[ 3e^{-2\alpha} + e^{-\sqrt{3}\alpha} - 3e^{-\alpha} - 1 \right] + \frac{1}{9\alpha} \left[ 2\sqrt{3} + 3 - \sqrt{3}e^{-\sqrt{3}\alpha} \right].$$

If we want the probability that a call goes through at least two handoffs, we start with the following equation:

$$\begin{aligned} P_{\geq 2H} &= \int_0^k \mu e^{-\mu t} \frac{t^2}{18k^2} dt + \int_k^{\sqrt{3}k} \mu e^{-\mu t} \left[ \frac{2}{9k^2} t^2 - \frac{t}{3k} + \frac{1}{6} \right] dt \\ &+ \int_{\sqrt{3}k}^{2k} \mu e^{-\mu t} \left[ \frac{t^2}{6k^2} - \left( \frac{3-\sqrt{3}}{9k} \right) t \right] dt + \int_{2k}^{3k} \mu e^{-\mu t} \left[ \left( \frac{3+\sqrt{3}}{9k} \right) t - \frac{2}{3} \right] dt \\ &+ \int_{3k}^{2\sqrt{3}k} \mu e^{-\mu t} \left[ \frac{\sqrt{3}}{9k} t + \frac{1}{3} \right] dt + \int_{2\sqrt{3}k}^{\infty} \mu e^{-\mu t} dt. \end{aligned}$$

This then gives:

$$\begin{aligned} P_{\geq 2H} &= \frac{1}{18k^2} \left[ \frac{2}{\mu^2} - \left( k^2 + \frac{2k}{\mu} + \frac{2}{\mu^2} \right) e^{-\mu k} \right] \\ &+ \frac{2}{9k^2} \left[ \left( k^2 + \frac{2k}{\mu} + \frac{2}{\mu^2} \right) e^{-\mu k} - \left( 3k^2 + \frac{2\sqrt{3}k}{\mu} + \frac{2}{\mu^2} e^{-\sqrt{3}\mu k} \right) \right] \\ &- \frac{1}{3k} \left[ \left( k + \frac{1}{k} \right) e^{-\mu k} - \left( \sqrt{3}k + \frac{1}{\mu} \right) e^{-\sqrt{3}\mu k} \right] + \frac{1}{6} \left[ e^{-\mu k} - e^{-\sqrt{3}\mu k} \right] \\ &+ \frac{1}{6k^2} \left[ \left( 3k^2 + \frac{2\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-\sqrt{3}\mu k} - \left( 4k^2 + \frac{4k}{\mu} + \frac{2}{\mu^2} \right) e^{-2\mu k} \right] \\ &- \left( \frac{3-\sqrt{3}}{9k} \right) \left[ \left( \sqrt{3}k + \frac{1}{\mu} \right) e^{-\sqrt{3}\mu k} - \left( 2k + \frac{1}{\mu} \right) e^{-2\mu k} \right] \\ &+ \left( \frac{3+\sqrt{3}}{9k} \right) \left[ \left( 2k + \frac{1}{\mu} \right) e^{-2\mu k} - \left( 3k + \frac{1}{\mu} \right) e^{-3\mu k} \right] - \frac{2}{3} \left[ e^{-2\mu k} - e^{-3\mu k} \right] \\ &+ \frac{\sqrt{3}}{9k} \left[ \left( 3k + \frac{1}{\mu} \right) e^{-3\mu k} - \left( 2\sqrt{3}k + \frac{1}{\mu} \right) e^{-2\sqrt{3}\mu k} \right] \\ &+ \frac{1}{3} \left[ e^{-3\mu k} - e^{-2\sqrt{3}\mu k} \right] + e^{-2\sqrt{3}\mu k} \end{aligned}$$

$$\Rightarrow P_{\geq 2H} = \frac{1}{9k^2\mu^2} + e^{-\mu k} \left[ -\frac{1}{18} - \frac{1}{9k\mu} - \frac{1}{9k^2\mu^2} + \frac{2}{9} \right]$$

$$\begin{aligned}
 & + \frac{4}{9k\mu} + \frac{4}{9k^2\mu^2} - \frac{1}{3} - \frac{1}{3k\mu} + \frac{1}{6} \Big] \\
 & + e^{-\sqrt{3}\mu k} \left[ \frac{2}{3} - \frac{4\sqrt{3}}{9k\mu} - \frac{4}{3k^2\mu^2} + \frac{\sqrt{3}}{3} + \frac{1}{3k\mu} - \frac{1}{6} \right. \\
 & \quad \left. + \frac{1}{2} + \frac{\sqrt{3}}{3k\mu} + \frac{1}{3k^2\mu^2} - \frac{\sqrt{3}}{3} - \frac{1}{3k\mu} + \frac{1}{3} + \frac{\sqrt{3}}{9k\mu} \right] \\
 & + e^{-2\mu k} \left[ -\frac{2}{3} - \frac{2}{3k\mu} - \frac{1}{3k^2\mu^2} + \frac{2}{3} + \frac{1}{3k\mu} - \frac{2\sqrt{3}}{9} \right. \\
 & \quad \left. - \frac{\sqrt{3}}{9k\mu} + \frac{2}{3} + \frac{1}{3k\mu} + \frac{2\sqrt{3}}{9} + \frac{\sqrt{3}}{9k\mu} - \frac{2}{3} \right] \\
 & + e^{-3\mu k} \left[ -1 - \frac{1}{3k\mu} - \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{9k\mu} + \frac{2}{3} + \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{9k\mu} + \frac{1}{3} \right] \\
 & + e^{-2\sqrt{3}\mu k} \left[ -\frac{2}{3} - \frac{\sqrt{3}}{9k\mu} - \frac{1}{3} + 1 \right] \\
 \Rightarrow P_{\geq 2H} & = \frac{1}{9k^2\mu^2} + e^{-\mu k} \left[ \frac{1}{3k^2\mu^2} \right] + e^{-\sqrt{3}\mu k} \left[ -\frac{1}{9k^2\mu^2} \right] + e^{-2\mu k} \left[ -\frac{1}{3k^2\mu^2} \right] \\
 & + e^{-3\mu k} \left[ -\frac{1}{3k\mu} \right] + e^{-2\sqrt{3}\mu k} \left[ -\frac{\sqrt{3}}{9k\mu} \right].
 \end{aligned}$$

Making the usual substitution  $\alpha = k\mu$ , we get:

$$P_{\geq 2H}(\alpha) = \frac{1}{9\alpha^2} \left[ 1 + 3e^{-\alpha} - e^{\sqrt{3}\alpha} - 3e^{-2\alpha} \right] - \frac{1}{9\alpha} \left[ 3e^{-3\alpha} + \sqrt{3}e^{-2\sqrt{3}\alpha} \right].$$

The probability of at least three handoffs is obtained in a similar matter, starting from the following equation:

$$\begin{aligned}
 P_{\geq 3H} & = \int_{\sqrt{3}k}^{3k} \mu e^{-\mu t} \left[ -\frac{t^2}{18k^2} + \frac{2\sqrt{3}}{9k}t - \frac{1}{2} \right] dt \\
 & + \int_{3k}^{2\sqrt{3}k} \mu e^{-\mu t} \left[ -\frac{t^2}{18k^2} + \left( \frac{3+2\sqrt{3}}{9k} \right) t - \frac{3}{2} \right] dt \\
 & + \int_{2\sqrt{3}k}^{4k} \mu e^{-\mu t} \left[ \left( \frac{3+\sqrt{3}}{9k} \right) t - \frac{3}{2} \right] dt \\
 & + \int_{4k}^{5k} \mu e^{-\mu t} \left[ -\frac{t^2}{6k^2} + \left( \frac{15+\sqrt{3}}{9k} \right) t - \frac{25}{6} \right] dt \\
 & + \int_{5k}^{3\sqrt{3}k} \mu e^{-\mu t} \left[ \frac{\sqrt{3}}{9k}t \right] dt + \int_{3\sqrt{3}k}^{\infty} \mu e^{-\mu t} dt.
 \end{aligned}$$

This then gives:

$$\begin{aligned}
 P_{\geq 3H} = & -\frac{1}{18k^2} \left[ \left( 3k^2 + \frac{2\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-\sqrt{3}\mu k} - \left( 12k^2 + \frac{4\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-2\sqrt{3}\mu k} \right] \\
 & + \frac{2\sqrt{3}}{9k} \left[ \left( \sqrt{3}k + \frac{1}{\mu} \right) e^{-\sqrt{3}\mu k} - \left( 2\sqrt{3}k + \frac{1}{\mu} \right) e^{-2\sqrt{3}\mu k} \right] \\
 & - \frac{1}{2} \left[ e^{-\sqrt{3}\mu k} - e^{-2\sqrt{3}\mu k} \right] + \frac{1}{3k} \left[ \left( 3k + \frac{1}{\mu} \right) e^{-3\mu k} - \left( 2\sqrt{3}k + \frac{1}{\mu} \right) e^{-2\sqrt{3}\mu k} \right] \\
 & - \left[ e^{-3\mu k} - e^{-2\sqrt{3}\mu k} \right] + \left( \frac{3 + \sqrt{3}}{9k} \right) \left[ \left( 2\sqrt{3}k + \frac{1}{\mu} \right) e^{-2\sqrt{3}\mu k} - \left( 4k + \frac{1}{\mu} \right) e^{-4\mu k} \right] \\
 & - \frac{3}{2} \left[ e^{-2\sqrt{3}\mu k} - e^{-4\mu k} \right] \\
 & - \frac{1}{6k^2} \left[ \left( 16k^2 + \frac{8k}{\mu} + \frac{2}{\mu^2} \right) e^{-4\mu k} - \left( 25k^2 + \frac{10k}{\mu} + \frac{2}{\mu^2} \right) e^{-5\mu k} \right] \\
 & + \left( \frac{15 + \sqrt{3}}{9k} \right) \left[ \left( 4k + \frac{1}{\mu} \right) e^{-4\mu k} - \left( 5k + \frac{1}{\mu} \right) e^{-5\mu k} \right] - \frac{25}{6} \left[ e^{-4\mu k} - e^{-5\mu k} \right] \\
 & + \frac{\sqrt{3}}{9k} \left[ \left( 5k + \frac{1}{\mu} \right) e^{-5\mu k} - \left( 3\sqrt{3}k + \frac{1}{\mu} \right) e^{-3\sqrt{3}\mu k} \right] + e^{-3\sqrt{3}\mu k}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow P_{\geq 3H} = & e^{-\sqrt{3}\mu k} \left[ -\frac{1}{6} - \frac{\sqrt{3}}{9k\mu} - \frac{1}{9k^2\mu^2} + \frac{2}{3} + \frac{2\sqrt{3}}{9k\mu} - \frac{1}{2} \right] \\
 & + e^{-2\sqrt{3}\mu k} \left[ \frac{2}{3} + \frac{2\sqrt{3}}{9k\mu} + \frac{1}{9k^2\mu^2} - \frac{4}{3} - \frac{2\sqrt{3}}{9k\mu} \right] \\
 & + e^{-2\sqrt{3}\mu k} \left[ +\frac{1}{2} - \frac{2\sqrt{3}}{3} + \frac{2}{3} + \frac{1}{3k\mu} + \frac{\sqrt{3}}{9k\mu} - \frac{3}{2} \right] \\
 & + e^{-4\mu k} \left[ -\frac{4}{3} - \frac{4\sqrt{3}}{9} - \frac{(3 + \sqrt{3})}{9k\mu} + \frac{3}{2} - \frac{8}{3} - \frac{4}{3k\mu} \right. \\
 & \quad \left. - \frac{1}{3k^2\mu^2} + \frac{20}{3} + \frac{4\sqrt{3}}{9} + \frac{5}{3k\mu} + \frac{\sqrt{3}}{9k\mu} - \frac{25}{6} \right] \\
 & + e^{-5\mu k} \left[ \frac{25}{6} + \frac{5}{3k\mu} + \frac{1}{3k^2\mu^2} - \frac{25}{3} - \frac{5\sqrt{3}}{9} \right. \\
 & \quad \left. + e^{-5\mu k} \left[ -\frac{5}{3k\mu} - \frac{\sqrt{3}}{9k\mu} + \frac{25}{6} + \frac{5\sqrt{3}}{9} + \frac{\sqrt{3}}{9k\mu} \right] \right] \\
 & + e^{-3\sqrt{3}\mu k} \left[ -1 - \frac{\sqrt{3}}{9k\mu} + 1 \right] + e^{-3\mu k} \left[ 1 + \frac{1}{3k\mu} - 1 \right]
 \end{aligned}$$



$$\Rightarrow P_{\geq 3H} = e^{-\sqrt{3}\mu k} \left[ -\frac{1}{9k^2\mu^2} + \frac{\sqrt{3}}{9k\mu} \right] + e^{-2\sqrt{3}\mu k} \left[ \frac{1}{9k^2\mu^2} + \frac{\sqrt{3}}{9k\mu} \right] \\ + e^{-4\mu k} \left[ -\frac{1}{3k^2\mu^2} \right] + e^{-5\mu k} \left[ \frac{1}{3k^2\mu^2} \right] + e^{-3\sqrt{3}\mu k} \left[ -\frac{\sqrt{3}}{9k\mu} \right] + e^{-3\mu k} \left[ \frac{1}{3k\mu} \right].$$

This gives after letting  $\alpha = k\mu$ :

$$P_{\geq 3H}(\alpha) = \frac{1}{9\alpha^2} \left[ -e^{-\sqrt{3}\alpha} + e^{-2\sqrt{3}\alpha} - 3e^{-4\alpha} + 3e^{-5\alpha} \right] \\ + \frac{1}{9\alpha} \left[ \sqrt{3}e^{-\sqrt{3}\alpha} + 3e^{-3\alpha} + \sqrt{3}e^{-2\sqrt{3}\alpha} - \sqrt{3}e^{-3\sqrt{3}\alpha} \right].$$

This concludes the computation of the probability that a call goes through at least three handoffs. We now start similar computations for the general case by first computing the conditional probabilities that a call goes through at least  $n$  handoffs given the direction of motion of the associated mobile, and the fact that it is located in the first quadrant of the cell. In the case of a mobile moving in direction (1), we get:

$n$  odd:

$$\bar{P}_{\geq nH/(1)} = \int_{(n-1)\frac{\sqrt{3}}{2}k}^{n\frac{\sqrt{3}}{2}k} \mu e^{-\mu t} \left[ -\frac{4}{9k^2}t^2 + \frac{4\sqrt{3}n}{9k}t - \left( \frac{n^2-1}{3} \right) \right] dt \\ + \frac{1}{3} \int_{n\frac{\sqrt{3}}{2}k}^{(n-1)\sqrt{3}k} \mu e^{-\mu t} dt \\ + \int_{(n-1)\sqrt{3}k}^{(2n-1)\frac{\sqrt{3}}{2}k} \mu e^{-\mu t} \left[ \frac{4\sqrt{3}}{9k}t - \frac{(4n-5)}{3} \right] dt + \int_{(2n-1)\frac{\sqrt{3}}{2}k}^{\infty} \mu e^{-\mu t} dt.$$

This then gives:

$$\bar{P}_{\geq nH/(1)} = -\frac{4}{9k^2} \left[ \left( \frac{3(n-1)^2k^2}{4} + \frac{(n-1)\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} \right. \\ \left. - \left( \frac{3n^2k^2}{4} + \frac{n\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-n\frac{\sqrt{3}}{2}\mu k} \right] \\ + \frac{4\sqrt{3}n}{9k} \left[ \left( \frac{(n-1)\sqrt{3}k}{2} + \frac{1}{\mu} \right) e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} - \left( \frac{n\sqrt{3}k}{2} + \frac{1}{\mu} \right) e^{-n\frac{\sqrt{3}}{2}\mu k} \right] \\ + \frac{4\sqrt{3}n}{9k} \left[ -\left( \frac{n^2-1}{3} \right) \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} - e^{-n\frac{\sqrt{3}}{2}\mu k} \right] \right]$$

$$\begin{aligned}
 & + \frac{1}{3} \left[ e^{-n\frac{\sqrt{3}}{2}\mu k} - e^{-(n-1)\sqrt{3}\mu k} \right] \\
 & + \frac{4\sqrt{3}}{9k} \left[ \left( (n-1)\sqrt{3}k + \frac{1}{\mu} \right) e^{-(n-1)\sqrt{3}\mu k} \right. \\
 & \quad \left. - \left( \frac{(2n-1)\sqrt{3}k}{2} + \frac{1}{\mu} \right) e^{-(2n-1)\frac{\sqrt{3}}{2}\mu k} \right] \\
 & - \frac{(4n-5)}{3} \left[ e^{-(n-1)\sqrt{3}\mu k} - e^{-(2n-1)\frac{\sqrt{3}}{2}\mu k} \right] + e^{-(2n-1)\frac{\sqrt{3}}{2}\mu k}.
 \end{aligned}$$

Letting  $\alpha = k\mu$ , we get:

$$\begin{aligned}
 \tilde{P}_{\geq nH/(1)}(\alpha) & = -e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} \left[ \frac{(n-1)^2}{3} + \frac{4\sqrt{3}(n-1)}{9\alpha} + \frac{8}{9\alpha^2} \right. \\
 & \quad \left. - \frac{2n(n-1)}{3} - \frac{4\sqrt{3}n}{9\alpha} + \frac{n^2}{3} - \frac{1}{3} \right] \\
 & + e^{-n\frac{\sqrt{3}}{2}\alpha} \left[ \frac{n^2}{3} + \frac{4\sqrt{3}n}{9\alpha} + \frac{8}{9\alpha^2} - \frac{2n^2}{3} - \frac{4\sqrt{3}n}{9\alpha} + \frac{n^2}{3} - \frac{1}{3} + \frac{1}{3} \right] \\
 & + e^{-(n-1)\sqrt{3}\alpha} \left[ -\frac{1}{3} + \frac{4(n-1)}{3} + \frac{4\sqrt{3}}{9\alpha} - \frac{(4n-5)}{3} \right] \\
 & + e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \left[ -\frac{2(2n-1)}{3} - \frac{4\sqrt{3}}{9\alpha} + \frac{(4n-5)}{3} + 1 \right] \\
 \\
 \Rightarrow \tilde{P}_{\geq nH/(1)}(\alpha) & = \frac{8}{9\alpha^2} \left[ e^{-n\frac{\sqrt{3}}{2}\alpha} - e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} \right] \\
 & + \frac{4\sqrt{3}}{9\alpha} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-(n-1)\sqrt{3}\alpha} - e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \right]. \quad (n, \text{ odd})
 \end{aligned}$$

$n$  even:

$$\begin{aligned}
 \tilde{P}_{\geq nH/(1)} & = \int_{(n-2)\frac{\sqrt{3}}{2}k}^{(n-1)\frac{\sqrt{3}}{2}k} \mu e^{-\mu t} \left[ \frac{2}{9k^2}t^2 - \frac{2\sqrt{3}(n-2)}{9k}t + \frac{(n-2)^2}{6} \right] dt \\
 & + \int_{(n-1)\frac{\sqrt{3}}{2}k}^{n\frac{\sqrt{3}}{2}k} \mu e^{-\mu t} \left[ -\frac{2}{9k^2}t^2 - \frac{2\sqrt{3}n}{9k}t + \frac{(n-2)^2}{6} \right] dt + \int_{n\frac{\sqrt{3}}{2}k}^{(n-1)\sqrt{3}k} \mu e^{-\mu t} dt \\
 & + \int_{(n-1)\sqrt{3}k}^{(2n-1)\frac{\sqrt{3}}{2}k} \mu e^{-\mu t} \left[ \frac{4\sqrt{3}}{9k}t - \frac{(4n-5)}{3} \right] dt + \int_{(2n-1)\frac{\sqrt{3}}{2}k}^{\infty} \mu e^{-\mu t} dt.
 \end{aligned}$$

This gives:

$$\begin{aligned}
 \Rightarrow \tilde{P}_{\geq nH/(1)} &= \frac{2}{9k^2} \left[ \left( \frac{3(n-2)^2k^2}{4} + \frac{(n-2)\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-(n-2)\frac{\sqrt{3}}{2}\mu k} \right. \\
 &\quad - \left. \left( \frac{3(n-1)^2k^2}{4} + \frac{(n-1)\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} \right] \\
 &\quad - \frac{2\sqrt{3}(n-2)}{9k} \left[ \left( \frac{(n-2)\sqrt{3}k}{2} + \frac{1}{\mu} \right) e^{-(n-2)\frac{\sqrt{3}}{2}\mu k} \right. \\
 &\quad \quad \left. - \left( \frac{(n-1)\sqrt{3}k}{2} + \frac{1}{\mu} \right) e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} \right] \\
 &\quad + \frac{(n-2)^2}{6} \left[ e^{-(n-2)\frac{\sqrt{3}}{2}\mu k} - e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} \right] \\
 &\quad - \frac{2}{9k^2} \left[ \left( \frac{3(n-1)^2k^2}{4} + \frac{(n-1)\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} \right. \\
 &\quad \quad \left. - \left( \frac{3n^2k^2}{4} + \frac{n\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-n\frac{\sqrt{3}}{2}\mu k} \right] \\
 &\quad + \frac{2\sqrt{3}n}{9k} \left[ \left( \frac{\sqrt{3}(n-1)k}{2} + \frac{1}{\mu} \right) e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} - \left( \frac{n\sqrt{3}k}{2} + \frac{1}{\mu} \right) e^{-n\frac{\sqrt{3}}{2}\mu k} \right] \\
 &\quad + \left( \frac{n^2-2}{6} \right) \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} - e^{-n\frac{\sqrt{3}}{2}\mu k} \right] + \frac{1}{3} \left[ e^{-n\frac{\sqrt{3}}{2}\mu k} - e^{-(n-1)\sqrt{3}\mu k} \right] \\
 &\quad + \frac{4\sqrt{3}}{9k} \left[ \left( (n-1)\sqrt{3}k + \frac{1}{\mu} \right) e^{-(n-1)\sqrt{3}\mu k} \right. \\
 &\quad \quad \left. - \left( \frac{(2n-1)\sqrt{3}k}{2} + \frac{1}{\mu} \right) e^{-(2n-1)\frac{\sqrt{3}}{2}\mu k} \right] \\
 &\quad - \frac{(4n-5)}{3} \left[ e^{-(n-1)\sqrt{3}\mu k} - e^{-(2n-1)\frac{\sqrt{3}}{2}\mu k} \right] + e^{-(2n-1)\frac{\sqrt{3}}{2}\mu k}.
 \end{aligned}$$

With  $\alpha = k\mu$ , this gives:

$$\begin{aligned}
 P_{\geq nH/(1)}(\alpha) &= e^{-(n-2)\frac{\sqrt{3}}{2}\alpha} \left[ \frac{(n-2)^2}{6} + \frac{2\sqrt{3}(n-2)}{9\alpha} + \frac{4}{9\alpha^2} \right. \\
 &\quad \left. - \frac{(n-2)^2}{3} - \frac{2\sqrt{3}(n-2)}{9\alpha} + \frac{(n-2)^2}{6} \right] \\
 &\quad + e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} \left[ -\frac{(n-1)^2}{6} - \frac{2\sqrt{3}(n-1)}{9\alpha} - \frac{4}{9\alpha^2} + \frac{(n-1)(n-2)}{3} \right. \\
 &\quad \left. + \frac{2\sqrt{3}(n-2)}{9\alpha} - \frac{(n-2)^2}{6} - \frac{(n-1)^2}{6} - \frac{2\sqrt{3}(n-1)}{9\alpha} \right]
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{4}{9\alpha^2} + \frac{n(n-1)}{3} + \frac{2\sqrt{3}n}{9\alpha} - \frac{(n-2)^2}{6} \Big] \\
 & + e^{-n\frac{\sqrt{3}}{2}\alpha} \left[ \frac{n^2}{6} + \frac{2\sqrt{3}n}{9\alpha} + \frac{4}{9\alpha^2} - \frac{n^2}{3} - \frac{2\sqrt{3}n}{9\alpha} + \frac{n^2-2}{6} + \frac{1}{3} \right] \\
 & + e^{-(n-1)\sqrt{3}\alpha} \left[ -\frac{1}{3} + \frac{4(n-1)}{3} + \frac{4\sqrt{3}}{9\alpha} - \frac{(4n-5)}{3} \right] \\
 & + e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \left[ -\frac{2(2n-1)}{3} - \frac{4\sqrt{3}}{9\alpha} + \frac{(4n-5)}{3} + 1 \right].
 \end{aligned}$$

The above finally gives:

$$\begin{aligned}
 \tilde{P}_{\geq nH/(1)}(\alpha) &= \frac{4}{9\alpha^2} \left[ e^{-n\frac{\sqrt{3}}{2}\alpha} - 2e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-(n-2)\frac{\sqrt{3}}{2}\alpha} \right] \\
 &+ \frac{4\sqrt{3}}{9\alpha} \left[ e^{-(n-1)\sqrt{3}\alpha} - e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \right]. \quad (n, \text{ even})
 \end{aligned}$$

The same procedure is repeated for the case of direction (2):

n odd:

$$\begin{aligned}
 \tilde{P}_{\geq nH/(2)} &= \int_{(n-1)\frac{\sqrt{3}}{2}k}^{n\frac{\sqrt{3}}{2}k} \mu e^{-\mu t} \left[ \frac{2}{9k^2}t^2 - \frac{2\sqrt{3}(n-1)}{9k}t + \frac{(n-1)^2}{6} \right] dt \\
 &+ \int_{n\frac{\sqrt{3}}{2}k}^{(n+1)\frac{\sqrt{3}}{2}k} \mu e^{-\mu t} \left[ -\frac{2}{9k^2}t^2 - \frac{2\sqrt{3}(n+1)}{9k}t - \left( \frac{(n+1)^2-2}{6} \right) \right] dt \\
 &+ \frac{1}{3} \int_{(n+1)\frac{\sqrt{3}}{2}k}^{(2n-1)\frac{\sqrt{3}}{2}k} \mu e^{-\mu t} dt \\
 &+ \int_{(2n-1)\frac{\sqrt{3}}{2}k}^{n\sqrt{3}k} \mu e^{-\mu t} \left[ \frac{4\sqrt{3}}{9k}t - \frac{(4n-3)}{3} \right] dt + \int_{n\sqrt{3}k}^{\infty} \mu e^{-\mu t} dt \\
 \Rightarrow \tilde{P}_{\geq nH/(2)} &= \frac{2}{9k^2} \left[ \left( \frac{3(n-1)^2k^2}{4} + \frac{(n-1)\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} \right. \\
 &\quad \left. - \left( \frac{3n^2k^2}{4} + \frac{n\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-n\frac{\sqrt{3}}{2}\mu k} \right] \\
 &- \frac{2\sqrt{3}(n-1)}{9k} \left[ \left( \frac{(n-1)\sqrt{3}k}{2} + \frac{1}{\mu} \right) e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} \right. \\
 &\quad \left. - \left( \frac{n\sqrt{3}k}{2} + \frac{1}{\mu} \right) e^{-n\frac{\sqrt{3}}{2}\mu k} \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{(n-1)^2}{6} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} - e^{-n\frac{\sqrt{3}}{2}\mu k} \right] \\
 & - \frac{2}{9k^2} \left[ \left( \frac{3n^2k^2}{4} + \frac{n\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-n\frac{\sqrt{3}}{2}\mu k} \right. \\
 & \quad \left. - \left( \frac{3(n+1)^2k^2}{4} + \frac{(n+1)\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-(n+1)\frac{\sqrt{3}}{2}\mu k} \right] \\
 & + \frac{2\sqrt{3}(n+1)}{9k} \left[ \left( \frac{\sqrt{3}nk}{2} + \frac{1}{\mu} \right) e^{-n\frac{\sqrt{3}}{2}\mu k} \right. \\
 & \quad \left. - \left( \frac{(n+1)\sqrt{3}k}{2} + \frac{1}{\mu} \right) e^{-(n+1)\frac{\sqrt{3}}{2}\mu k} \right] \\
 & - \left( \frac{(n+1)^2 - 2}{6} \right) \left[ e^{-n\frac{\sqrt{3}}{2}\mu k} - e^{-(n+1)\frac{\sqrt{3}}{2}\mu k} \right] \\
 & + \frac{1}{3} \left[ e^{-(n+1)\frac{\sqrt{3}}{2}\mu k} - e^{-(2n-1)\frac{\sqrt{3}}{2}\mu k} \right] \\
 & + \frac{4\sqrt{3}}{9k} \left[ \left( \frac{(2n-1)\sqrt{3}k}{2} + \frac{1}{\mu} \right) e^{-(2n-1)\frac{\sqrt{3}}{2}\mu k} - \left( n\sqrt{3}k + \frac{1}{\mu} \right) e^{-n\sqrt{3}\mu k} \right] \\
 & - \frac{(4n-3)}{3} \left[ e^{-(2n-1)\frac{\sqrt{3}}{2}\mu k} - e^{-n\sqrt{3}\mu k} \right] + e^{-n\sqrt{3}\mu k}.
 \end{aligned}$$

With  $\alpha = k\mu$ , this gives:

$$\begin{aligned}
 P_{\geq nH/(2)}(\alpha) & = e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} \left[ \frac{(n-1)^2}{6} + \frac{2\sqrt{3}(n-1)}{9\alpha} + \frac{4}{9\alpha^2} \right. \\
 & \quad \left. - \frac{(n-1)^2}{3} - \frac{2\sqrt{3}(n-1)}{9\alpha} + \frac{(n-1)^2}{6} \right] \\
 & + e^{-n\frac{\sqrt{3}}{2}\alpha} \left[ -\frac{n^2}{6} - \frac{2\sqrt{3}n}{9\alpha} - \frac{4}{9\alpha^2} + \frac{n(n+1)}{3} + \frac{2\sqrt{3}(n-1)}{9\alpha} \right. \\
 & \quad \left. - \frac{(n-1)^2}{6} - \frac{n^2}{6} - \frac{2\sqrt{3}n}{9\alpha} - \frac{4}{9\alpha^2} \right. \\
 & \quad \left. + \frac{n(n+1)}{3} + \frac{2\sqrt{3}(n+1)}{9\alpha} - \frac{(n+1)^2 - 2}{6} \right] \\
 & + e^{-(n+1)\frac{\sqrt{3}}{2}\alpha} \left[ \frac{(n+1)^2}{6} + \frac{2\sqrt{3}(n+1)}{9\alpha} + \frac{4}{9\alpha^2} - \frac{(n+1)^2}{3} \right. \\
 & \quad \left. - \frac{2\sqrt{3}(n+1)}{9\alpha} + \frac{(n+1)^2 - 2}{6} + \frac{1}{3} \right] \\
 & + e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \left[ -\frac{1}{3} + \frac{2(2n-1)}{3} + \frac{4\sqrt{3}}{9\alpha} - \frac{(4n-3)}{3} \right]
 \end{aligned}$$

$$+ e^{-2\sqrt{3}\alpha} \left[ -\frac{4n}{3} - \frac{4\sqrt{3}}{9\alpha} + \frac{(4n-3)}{3} + 1 \right].$$

The above finally gives:

$$\begin{aligned} \bar{P}_{\geq nH/(2)}(\alpha) &= \frac{4}{9\alpha^2} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} - 2e^{-n\frac{\sqrt{3}}{2}\alpha} + e^{-(n+1)\frac{\sqrt{3}}{2}\alpha} \right] \\ &+ \frac{4\sqrt{3}}{9\alpha} \left[ e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} - e^{-n\sqrt{3}\alpha} \right]. \end{aligned} \quad (n, \text{ odd})$$

$n$  even:

$$\begin{aligned} \bar{P}_{\geq nH/(2)} &= \int_{(n-1)\frac{1}{2}\sqrt{3}k}^{n\frac{1}{2}\sqrt{3}k} \mu e^{-\mu t} \left[ \frac{4}{9k^2} t^2 - \frac{4\sqrt{3}n}{9k} t + \left( \frac{(n-1)^2}{3} \right) \right] dt \\ &+ \frac{1}{3} \int_{n\frac{1}{2}\sqrt{3}k}^{(2n-1)\frac{\sqrt{3}}{2}k} \mu e^{-\mu t} dt \\ &+ \int_{(2n-1)\frac{\sqrt{3}}{2}k}^{n\sqrt{3}k} \mu e^{-\mu t} \left[ \frac{4\sqrt{3}}{9k} t - \frac{(4n-3)}{3} \right] dt + \int_{n\sqrt{3}k}^{\infty} \mu e^{-\mu t} dt. \end{aligned}$$

This then gives:

$$\begin{aligned} \bar{P}_{\geq nH/(2)} &= \frac{4}{9k^2} \left[ \left( \frac{3(n-1)^2 k^2}{4} + \frac{(n-1)\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} \right. \\ &\quad \left. - \left( \frac{3n^2 k^2}{4} + \frac{n\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-n\frac{\sqrt{3}}{2}\mu k} \right] \\ &- \frac{4\sqrt{3}(n-1)}{9k} \left[ \left( \frac{(n-1)\sqrt{3}k}{2} + \frac{1}{\mu} \right) e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} \right. \\ &\quad \left. - \left( \frac{n\sqrt{3}k}{2} + \frac{1}{\mu} \right) e^{-n\frac{\sqrt{3}}{2}\mu k} \right] \\ &+ \left( \frac{(n-1)^2}{3} \right) \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\mu k} - e^{-n\frac{\sqrt{3}}{2}\mu k} \right] + \frac{1}{3} \left[ e^{-n\frac{\sqrt{3}}{2}\mu k} - e^{-(2n-1)\frac{\sqrt{3}}{2}\mu k} \right] \\ &+ \frac{4\sqrt{3}}{9k} \left[ \left( \frac{(2n-1)\sqrt{3}}{2} k + \frac{1}{\mu} \right) e^{-(2n-1)\frac{\sqrt{3}}{2}\mu k} - \left( n\sqrt{3}k + \frac{1}{\mu} \right) e^{-n\sqrt{3}\mu k} \right] \\ &- \frac{(4n-3)}{3} \left[ e^{-(2n-1)\frac{\sqrt{3}}{2}\mu k} - e^{-n\sqrt{3}\mu k} \right] + e^{-n\sqrt{3}\mu k}. \end{aligned}$$

After letting  $\alpha = k\mu$ , we obtain:

$$\bar{P}_{\geq nH/(2)}(\alpha) = e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} \left[ \frac{(n-1)^2}{3} + \frac{4\sqrt{3}(n-1)}{9\alpha} + \frac{8}{9\alpha^2} \right]$$

$$\begin{aligned}
 & -\frac{2(n-1)^2}{3} - \frac{4\sqrt{3}(n-1)}{9\alpha} + \frac{(n-1)^2}{3} \\
 & + e^{-n\frac{\sqrt{3}}{2}\alpha} \left[ -\frac{n^2}{3} - \frac{4\sqrt{3}n}{9\alpha} - \frac{8}{9\alpha^2} + \frac{2n(n-1)}{3} \right. \\
 & \quad \left. + \frac{4\sqrt{3}(n-1)}{9\alpha} - \frac{(n-1)^2}{3} + \frac{1}{3} \right] \\
 & + e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \left[ -\frac{1}{3} + \frac{2(2n-1)}{3} + \frac{4\sqrt{3}}{9\alpha} - \frac{(4n-3)}{3} \right] \\
 & + e^{-n\sqrt{3}\alpha} \left[ -\frac{4n}{3} - \frac{4\sqrt{3}}{9\alpha} + \frac{(4n-3)}{3} + 1 \right] \\
 \Rightarrow \bar{P}_{\geq nH/(2)}(\alpha) &= \frac{8}{9\alpha^2} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} - e^{-n\frac{\sqrt{3}}{2}\alpha} \right] \\
 & - \frac{4\sqrt{3}}{9\alpha} \left[ e^{-n\frac{\sqrt{3}}{2}\alpha} - e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-n\sqrt{3}\alpha} \right]. \quad (n, \text{ even})
 \end{aligned}$$

We now treat the case of direction (3) starting again from the distribution of the time to the  $n^{\text{th}}$  boundary crossing:

$n$  odd:

$$\begin{aligned}
 \bar{P}_{\geq nH/(3)} &= \int_{(3n-3)\frac{1}{2}k}^{(3n-2)\frac{1}{2}k} \mu e^{-\mu t} \left[ \frac{4}{3k}t - 2(n-1) \right] dt \\
 & + \int_{(3n-2)\frac{1}{2}k}^{(3n-1)\frac{1}{2}k} \mu e^{-\mu t} \left[ -\frac{4}{3k^2}t^2 + \frac{4(3n-1)}{3k}t - \left( \frac{(3n-1)^2 - 3}{3} \right) \right] dt \\
 & + \int_{(3n-1)\frac{1}{2}k}^{\infty} \mu e^{-\mu t} dt, \\
 \bar{P}_{\geq nH/(3)} &= \frac{4}{3k} \left[ \left( \frac{(3n-3)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-3)}{2}\mu k} - \left( \frac{(3n-2)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-2)}{2}\mu k} \right] \\
 & - 2(n-1) \left[ e^{-(3n-3)\frac{1}{2}\mu k} - e^{-\frac{(3n-2)}{2}\mu k} \right] \\
 & - \frac{4}{3k^2} \left[ \left( \frac{(3n-2)^2 k^2}{2} + \frac{(3n-2)k}{\mu} + \frac{2}{\mu^2} \right) e^{-\frac{(3n-2)}{2}\mu k} \right. \\
 & \quad \left. - \left( \frac{(3n-1)^2 k^2}{4} + \frac{(3n-1)k}{\mu} + \frac{2}{\mu^2} \right) e^{-\frac{(3n-1)}{2}\mu k} \right]
 \end{aligned}$$

$$+ \frac{4(3n-1)}{3k} \left[ \left( \frac{(3n-2)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-2)}{2}\mu k} - \left( \frac{(3n-1)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-1)}{2}\mu k} \right] \\ - \left( \frac{(3n-1)^2 - 3}{3} \right) \left[ e^{-\frac{(3n-2)}{2}\mu k} - e^{-\frac{(3n-1)}{2}\mu k} \right] + e^{-\frac{(3n-1)}{2}\mu k}.$$

After letting  $\alpha = k\mu$ , we have:

$$\tilde{P}_{\geq nH/(3)}(\alpha) = e^{-\frac{(3n-3)}{2}\alpha} \left[ 2(n-1) + \frac{4}{3\alpha} - 2(n-1) \right] \\ + e^{-\frac{(3n-2)}{2}\alpha} \left[ -\frac{2(3n-2)}{3} - \frac{4}{3\alpha} + 2(n-1) - \frac{(3n-2)^2}{3} - \frac{4(3n-2)}{3\alpha} \right. \\ \left. - \frac{8}{3\alpha^2} - \frac{2(3n-1)(3n-2)}{2} + \frac{4(3n-1)}{3\alpha} - \frac{(3n-1)^2}{3} + 1 \right] \\ + e^{-\frac{(3n-1)}{2}\alpha} \left[ \frac{(3n-1)^2}{3} + \frac{4(3n-1)}{3\alpha} + \frac{8}{3\alpha^2} - \frac{2(3n-1)^2}{3} \right. \\ \left. - \frac{4(3n-1)}{3\alpha} + \frac{(3n-1)^2}{3} - 1 + 1 \right]$$

$$\Rightarrow \tilde{P}_{\geq nH/(3)}(\alpha) = \frac{8}{3\alpha^2} \left[ e^{-(3n-1)\frac{1}{2}\alpha} - e^{-(3n-2)\frac{1}{2}\alpha} \right] + \frac{4}{3\alpha} \left[ e^{-\frac{(3n-3)}{2}\alpha} \right]. \quad (n, \text{ odd})$$

$n$  even:

$$\tilde{P}_{\geq nH/(3)} = \int_{(3n-4)\frac{1}{2}k}^{(3n-2)\frac{1}{2}k} \mu e^{-\mu t} \left[ \frac{2}{3k^2}t^2 - \frac{2(3n-4)}{3k}t + \frac{(3n-4)^2}{6} \right] dt \\ + \int_{(3n-2)\frac{1}{2}k}^{(3n-1)\frac{1}{2}k} \mu e^{-\mu t} \left[ -\frac{4}{3k^2}t^2 + \frac{4(3n-1)}{3k}t - \frac{(3n-1)^2}{3} + 1 \right] dt \\ + \int_{(3n-1)\frac{1}{2}k}^{\infty} \mu e^{-\mu t} dt.$$

This yields:

$$\tilde{P}_{\geq nH/(3)} = \frac{2}{3k^2} \left[ \left( \frac{(3n-4)^2k^2}{4} + \frac{(3n-4)k}{\mu} + \frac{2}{\mu^2} \right) e^{-\frac{(3n-4)}{2}\mu k} \right. \\ \left. - \left( \frac{(3n-2)^2k^2}{4} + \frac{(3n-2)k}{\mu} + \frac{2}{\mu^2} \right) e^{-\frac{(3n-2)}{2}\mu k} \right] \\ - \frac{2(3n-4)}{3k} \left[ \left( \frac{(3n-4)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-4)}{2}\mu k} - \left( \frac{(3n-2)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-2)}{2}\mu k} \right]$$



$$\begin{aligned}
 & + \frac{(3n-4)^2}{6} \left[ e^{-\frac{(3n-4)}{2}\mu k} - e^{-\frac{(3n-2)}{2}\mu k} \right] \\
 & - \frac{4}{3k^2} \left[ \left( \frac{(3n-2)^2 k^2}{4} + \frac{(3n-2)k}{\mu} + \frac{2}{\mu^2} \right) e^{-\frac{(3n-2)}{2}\mu k} \right. \\
 & \quad \left. - \left( \frac{(3n-1)^2 k^2}{4} + \frac{(3n-1)k}{\mu} + \frac{2}{\mu^2} \right) e^{-\frac{(3n-1)}{2}\mu k} \right] \\
 & + \frac{4(3n-1)}{3k} \left[ \left( \frac{(3n-2)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-2)}{2}\mu k} - \left( \frac{(3n-1)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-1)}{2}\mu k} \right] \\
 & - \left( \frac{(3n-1)^2}{3} - 1 \right) \left[ e^{-\frac{(3n-2)}{2}\mu k} - e^{-\frac{(3n-1)}{2}\mu k} \right] + e^{-\frac{(3n-1)}{2}\mu k}.
 \end{aligned}$$

With  $\alpha = k\mu$ , this gives:

$$\begin{aligned}
 \tilde{P}_{\geq nH/(3)}(\alpha) &= e^{-\frac{(3n-4)}{2}\alpha} \left[ \frac{(3n-4)^2}{6} + \frac{2(3n-4)}{3\alpha} + \frac{4}{3\alpha^2} \right. \\
 & \quad \left. - \frac{(3n-4)^2}{3} - \frac{2(3n-4)}{3\alpha} + \frac{(3n-4)^2}{6} \right] \\
 & + e^{-\frac{(3n-2)}{2}\alpha} \left[ -\frac{(3n-2)^2}{6} - \frac{2(3n-2)}{3\alpha} - \frac{4}{3\alpha^2} + \frac{(3n-4)(3n-2)}{3} \right. \\
 & \quad + \frac{2(3n-4)}{3\alpha} - \frac{(3n-4)^2}{6} - \frac{(3n-2)^2}{3} - \frac{4(3n-2)}{3\alpha} \\
 & \quad \left. - \frac{8}{3\alpha^2} + \frac{2(3n-1)(3n-2)}{3} + \frac{4(3n-1)}{3\alpha} - \frac{(3n-1)^2}{3} + 1 \right] \\
 & + e^{-\frac{(3n-1)}{2}\alpha} \left[ \frac{(3n-1)^2}{3} + \frac{4(3n-1)}{3\alpha} + \frac{8}{3\alpha^2} - \frac{2(3n-1)^2}{3} \right. \\
 & \quad \left. - \frac{4(3n-1)}{3\alpha} + \frac{(3n-1)^2}{3} - 1 + 1 \right] \\
 \Rightarrow \tilde{P}_{\geq nH/(3)}(\alpha) &= \frac{4}{3\alpha^2} \left[ e^{-\frac{(3n-4)}{2}\alpha} - 3e^{-\frac{(3n-2)}{2}\alpha} + 2e^{-\frac{(3n-1)}{2}\alpha} \right]. \quad (n, \text{ even})
 \end{aligned}$$

We finally treat the case of direction(4), starting with the following equation:

$n$  odd:

$$\begin{aligned}
 \tilde{P}_{\geq nH/(4)} &= \int_{(3n-2)\frac{1}{2}k}^{(3n-1)\frac{1}{2}k} \mu e^{-\mu t} \left[ \frac{4}{3k^2} t^2 - \frac{4(3n-2)}{3k} t + \frac{(3n-2)^2}{3} \right] dt \\
 & + \int_{(3n-1)\frac{1}{2}k}^{(3n+1)\frac{1}{2}k} \mu e^{-\mu t} \left[ -\frac{2}{3k^2} t^2 + \frac{2(3n+1)}{3k} t - \frac{(3n+1)^2}{6} + 1 \right] dt \\
 & + \int_{(3n+1)\frac{1}{2}k}^{\infty} \mu e^{-\mu t} dt.
 \end{aligned}$$

This gives:

$$\begin{aligned}
 \bar{P}_{\geq nH/(4)} &= \frac{4}{3k^2} \left[ \left( \frac{(3n-2)^2 k^2}{4} + \frac{(3n-2)k}{\mu} + \frac{2}{\mu^2} \right) e^{-\frac{(3n-2)}{2} \mu k} \right. \\
 &\quad \left. - \left( \frac{(3n-1)^2 k^2}{4} + \frac{(3n-1)k}{\mu} + \frac{2}{\mu^2} \right) e^{-\frac{(3n-1)}{2} \mu k} \right] \\
 &\quad - \frac{4(3n-2)}{3k} \left[ \left( \frac{(3n-2)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-2)}{2} \mu k} - \left( \frac{(3n-1)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-1)}{2} \mu k} \right] \\
 &\quad + \frac{(3n-2)^2}{3} \left[ e^{-\frac{(3n-2)}{2} \mu k} - e^{-\frac{(3n-1)}{2} \mu k} \right] \\
 &\quad - \frac{2}{3k^2} \left[ \left( \frac{(3n-1)^2 k^2}{4} + \frac{(3n-1)k}{\mu} + \frac{2}{\mu^2} \right) e^{-\frac{(3n-1)}{2} \mu k} \right. \\
 &\quad \left. - \left( \frac{(3n+1)^2 k^2}{4} + \frac{(3n+1)k}{\mu} + \frac{2}{\mu^2} \right) e^{-\frac{(3n+1)}{2} \mu k} \right] \\
 &\quad + \frac{2(3n+1)}{3k} \left[ \left( \frac{(3n-1)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-1)}{2} \mu k} - \left( \frac{(3n+1)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n+1)}{2} \mu k} \right] \\
 &\quad - \left( \frac{(3n+1)^2}{3} - 1 \right) \left[ e^{-\frac{(3n-1)}{2} \mu k} - e^{-\frac{(3n+1)}{2} \mu k} \right] + e^{-\frac{(3n+1)}{2} \mu k}.
 \end{aligned}$$

After setting  $\alpha = k\mu$ , we get:

$$\begin{aligned}
 \bar{P}_{\geq nH/(4)}(\alpha) &= e^{-\frac{(3n-2)}{2} \alpha} \left[ \frac{(3n-2)^2}{3} + \frac{4(3n-2)}{3\alpha} + \frac{8}{3\alpha^2} \right. \\
 &\quad \left. - \frac{2(3n-2)^2}{3} - \frac{4(3n-2)}{3\alpha} + \frac{(3n-2)^2}{3} \right] \\
 &\quad + e^{-\frac{(3n-1)}{2} \alpha} \left[ -\frac{(3n-1)^2}{3} - \frac{4(3n-1)}{3\alpha} - \frac{8}{3\alpha^2} + \frac{2(3n-2)(3n-1)}{3} \right. \\
 &\quad \left. + \frac{4(3n-2)}{3\alpha} - \frac{(3n-2)^2}{3} - \frac{(3n-1)^2}{6} - \frac{2(3n-1)}{3\alpha} - \frac{4}{3\alpha^2} \right. \\
 &\quad \left. + \frac{(3n+1)(3n-1)}{3} + \frac{2(3n+1)}{3\alpha} - \frac{(3n+1)^2}{6} + 1 \right] \\
 &\quad + e^{-\frac{(3n+1)}{2} \alpha} \left[ \frac{(3n+1)^2}{6} + \frac{2(3n+1)}{3\alpha} + \frac{4}{3\alpha^2} - \frac{(3n+1)^2}{3} \right. \\
 &\quad \left. - \frac{2(3n+1)}{3\alpha} + \frac{(3n+1)^2}{6} - 1 + 1 \right] \\
 \Rightarrow \bar{P}_{\geq nH/(4)}(\alpha) &= \frac{4}{3\alpha^2} \left[ 2e^{-\frac{(3n-2)}{2} \alpha} - 3e^{-\frac{(3n-1)}{2} \alpha} + e^{-\frac{(3n+1)}{2} \alpha} \right]. \quad (n, \text{ odd})
 \end{aligned}$$

$n$  even:

$$\begin{aligned} \bar{P}_{\geq nH/(4)} &= \int_{\frac{(3n-2)k}{2}}^{\frac{(3n-1)k}{2}} \mu e^{-\mu t} \left[ \frac{4}{3k^2} t^2 - \frac{4(3n-2)}{3k} t + \frac{(3n-2)^2}{3} \right] dt \\ &+ \int_{\frac{3n k}{2}}^{\frac{(3n-1)k}{2}} \mu e^{-\mu t} \left[ \frac{4}{3k} t - (2n-1) \right] dt + \int_{\frac{3n k}{2}}^{\infty} \mu e^{-\mu t} dt. \end{aligned}$$

This gives:

$$\begin{aligned} \bar{P}_{\geq nH/(4)} &= \frac{4}{3k^2} \left[ \left( \frac{(3n-2)^2 k^2}{4} + \frac{(3n-2)k}{\mu} + \frac{2}{\mu^2} \right) e^{-\frac{(3n-2)}{2} \mu k} \right. \\ &\quad \left. - \left( \frac{(3n-1)^2 k^2}{4} + \frac{(3n-1)k}{\mu} + \frac{2}{\mu^2} \right) e^{-\frac{(3n-1)}{2} \mu k} \right] \\ &- \frac{4(3n-2)}{3k} \left[ \left( \frac{(3n-2)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-2)}{2} \mu k} - \left( \frac{(3n-1)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-1)}{2} \mu k} \right] \\ &+ \frac{(3n-2)^2}{3} \left[ e^{-\frac{(3n-2)}{2} \mu k} - e^{-\frac{(3n-1)}{2} \mu k} \right] \\ &+ \frac{4}{3k} \left[ \left( \frac{(3n-1)k}{2} + \frac{1}{\mu} \right) e^{-\frac{(3n-1)}{2} \mu k} - \left( \frac{3nk}{2} + \frac{1}{\mu} \right) e^{-\frac{3n}{2} \mu k} \right] \\ &- (2n-1) \left[ e^{-\frac{(3n-1)}{2} \mu k} - e^{-\frac{3n}{2} \mu k} \right] + e^{-\frac{3n}{2} \mu k}. \end{aligned}$$

Again, after setting  $\alpha = k\mu$ , we obtain:

$$\begin{aligned} \bar{P}_{\geq nH/(4)}(\alpha) &= e^{-\frac{(3n-2)}{2} \alpha} \left[ \frac{(3n-2)^2}{3} + \frac{4(3n-2)}{3\alpha} + \frac{8}{3\alpha^2} \right. \\ &\quad \left. - \frac{2(3n-2)^2}{3} - \frac{4(3n-2)}{3\alpha} + \frac{(3n-2)^2}{3} \right] \\ &+ e^{-\frac{(3n-1)}{2} \alpha} \left[ -\frac{(3n-1)^2}{3} - \frac{4(3n-1)}{3\alpha} - \frac{8}{3\alpha^2} \right. \\ &\quad \left. + \frac{2(3n-2)(3n-1)}{3} + \frac{4(3n-2)}{3\alpha} - \frac{(3n-2)^2}{3} \right. \\ &\quad \left. - \frac{(3n-2)^2}{3} + \frac{2(3n-1)}{3} + \frac{4}{3\alpha} - (2n-1) \right] \\ &+ e^{-\frac{3n}{2} \alpha} \left[ -2n - \frac{4}{3\alpha} + (2n-1) + 1 \right] \\ \Rightarrow \bar{P}_{\geq nH/(4)}(\alpha) &= \frac{8}{\alpha^2} \left[ e^{-\frac{(3n-2)}{2} \alpha} - e^{-\frac{(3n-1)}{2} \alpha} \right] - \frac{4}{3\alpha} \left[ e^{-\frac{3n}{2} \alpha} \right]. \quad (n, \text{ even}) \end{aligned}$$

This finishes the computation of the conditional probabilities of at least  $n$  handoffs given the direction of the mobile and the fact that it is located in the first quadrant. We will now derive the probability that a call goes through at least  $n$  handoffs simply given the direction of motion of the associated vehicle, without any restriction on its location.

Due to the symmetry of the hexagonal cell with respect to the  $x$  and  $y$  axis, we have:

$$\begin{aligned} P_{\geq nH/(1)}(\alpha) &= P_{\geq nH/(2)}(\alpha) \\ &= \frac{1}{2}\bar{P}_{\geq nH/(1)}(\alpha) + \frac{1}{2}\bar{P}_{\geq nH/(2)}(\alpha); \end{aligned}$$

similarly,

$$\begin{aligned} P_{\geq nH/(3)}(\alpha) &= P_{\geq nH/(4)}(\alpha) \\ &= \frac{1}{2}\bar{P}_{\geq nH/(3)}(\alpha) + \frac{1}{2}\bar{P}_{\geq nH/(4)}(\alpha). \end{aligned}$$

Let us again distinguish between the cases  $n$  odd and  $n$  even, starting with  $n$  odd:

$$\begin{aligned} P_{\geq nH/(1)}(\alpha) &= \frac{4}{9\alpha^2} \left[ e^{-n\frac{\sqrt{3}}{2}\alpha} - e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} \right] \\ &+ \frac{2\sqrt{3}}{9\alpha} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-(n-1)\sqrt{3}\alpha} - e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \right] \\ &+ \frac{2}{9\alpha^2} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} - 2e^{-n\frac{\sqrt{3}}{2}\alpha} + e^{-(n+1)\frac{\sqrt{3}}{2}\alpha} \right] \\ &+ \frac{2\sqrt{3}}{9\alpha} \left[ -e^{-n\sqrt{3}\alpha} + e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \right] \\ \Rightarrow P_{\geq nH/(1)}(\alpha) &= P_{\geq nH/(2)}(\alpha) \\ &= \frac{2}{9\alpha^2} \left[ -e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-(n+1)\frac{\sqrt{3}}{2}\alpha} \right] \\ &+ \frac{2\sqrt{3}}{9\alpha} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-(n-1)\sqrt{3}\alpha} - e^{-n\sqrt{3}\alpha} \right]. \quad (n, \text{ odd}) \end{aligned}$$

The analogous expression is obtained for directions (3) and (4):

$$\begin{aligned} P_{\geq nH/(3)}(\alpha) &= \frac{4}{3\alpha^2} \left[ e^{-\frac{(3n-1)}{2}\alpha} - e^{-\frac{(3n-2)}{2}\alpha} \right] + \frac{2}{3\alpha} \left[ e^{-\frac{(3n-3)}{2}\alpha} \right] \\ &+ \frac{2}{3\alpha^2} \left[ 2e^{-\frac{(3n-2)}{2}\alpha} - 3e^{-\frac{(3n-1)}{2}\alpha} + e^{-\frac{(3n+1)}{2}\alpha} \right] \end{aligned}$$

$$\begin{aligned}
 \Rightarrow P_{\geq nH/(3)}(\alpha) &= P_{\geq nH/(4)}(\alpha) \\
 &= \frac{2}{3\alpha^2} \left[ -e^{-\frac{(3n-1)}{2}\alpha} + e^{-\frac{(3n+1)}{2}\alpha} \right] \\
 &\quad + \frac{2}{9\alpha} \left[ e^{-\frac{(3n-3)}{2}\alpha} \right]. \quad (n, \text{ odd})
 \end{aligned}$$

The case  $n$  even is treated in complete parallelism with what we have just done:

$$\begin{aligned}
 P_{\geq nH/(1)}(\alpha) &= \frac{2}{9\alpha^2} \left[ e^{-(n-2)\frac{\sqrt{3}}{2}\alpha} - 2e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} + e^{-n\frac{\sqrt{3}}{2}\alpha} \right] \\
 &\quad + \frac{2\sqrt{3}}{9\alpha} \left[ e^{-(n-1)\sqrt{3}\alpha} - e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \right] \\
 &\quad + \frac{4}{9\alpha^2} \left[ e^{-(n-1)\frac{\sqrt{3}}{2}\alpha} - e^{-n\frac{\sqrt{3}}{2}\alpha} \right] \\
 &\quad - \frac{2\sqrt{3}}{9\alpha} \left[ e^{-n\frac{\sqrt{3}}{2}\alpha} + e^{-n\sqrt{3}\alpha} - e^{-(2n-1)\frac{\sqrt{3}}{2}\alpha} \right]
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow P_{\geq nH/(1)}(\alpha) &= P_{\geq nH/(2)}(\alpha) \\
 &= \frac{2}{9\alpha^2} \left[ e^{-(n-2)\frac{\sqrt{3}}{2}\alpha} - e^{-n\frac{\sqrt{3}}{2}\alpha} \right] \\
 &\quad + \frac{2\sqrt{3}}{9\alpha} \left[ -e^{-n\frac{\sqrt{3}}{2}\alpha} + e^{-(n-1)\sqrt{3}\alpha} - e^{-n\sqrt{3}\alpha} \right]. \quad (n, \text{ even})
 \end{aligned}$$

The same method is used again for directions (3) and (4):

$$\begin{aligned}
 P_{\geq nH/(3)}(\alpha) &= \frac{2}{3\alpha^2} \left[ e^{-\frac{(3n-4)}{2}\alpha} - 3e^{-\frac{(3n-2)}{2}\alpha} + 2e^{-\frac{(3n-1)}{2}\alpha} \right] \\
 &\quad + \frac{4}{3\alpha^2} \left[ e^{-\frac{(3n-2)}{2}\alpha} - e^{-\frac{(3n-1)}{2}\alpha} \right] + \frac{2}{3\alpha} \left[ e^{-\frac{3n}{2}\alpha} \right]
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow P_{\geq nH/(3)}(\alpha) &= P_{\geq nH/(4)}(\alpha) \\
 &= \frac{2}{3\alpha^2} \left[ e^{-\frac{(3n-4)}{2}\alpha} - e^{-\frac{(3n-2)}{2}\alpha} \right] - \frac{2}{3\alpha} \left[ e^{-\frac{3n}{2}\alpha} \right]. \quad (n, \text{ even})
 \end{aligned}$$

We have provided at last all the computational steps needed in the derivations of the different handoff probabilities. This terminates Appendix B.

## APPENDIX C

### AVERAGE NUMBER OF HANDOFFS

In this appendix, we summarize the computational steps necessary to analytically derive the average number of handoffs experienced by a random call under the assumptions of Chapter 4. We start with the usual expression for the mean:

$$\begin{aligned}\bar{h} &= 2 \sum_{m=0}^{\infty} m P_{2mH}(\alpha) + \sum_{m=0}^{\infty} m P_{(2m+1)H}(\alpha) + \sum_{m=0}^{\infty} P_{(2m+1)H}(\alpha) \\ &= 2 \sum_{m=0}^{\infty} m \left( P_{2mH}(\alpha) + P_{(2m+1)H}(\alpha) \right) + \sum_{m=0}^{\infty} P_{(2m+1)H}(\alpha).\end{aligned}$$

We recall from Section 4.3, Equations (4.16) and (4.17), that we have:

$$\begin{aligned}P_{(2m+1)H}(\alpha) &= e^{-\sqrt{3}m\alpha} \left[ -\frac{2}{9\alpha^2} \left( 1 - e^{-\sqrt{3}\alpha} \right) + \frac{\sqrt{3}}{9\alpha} \left( 1 + e^{-\sqrt{3}\alpha} \right) \right] \\ &\quad + e^{-2\sqrt{3}m\alpha} \left[ \frac{\sqrt{3}}{9\alpha} \left( 1 - e^{-\sqrt{3}\alpha} \right)^2 \right] + e^{-3m\alpha} \left[ -\frac{2}{3\alpha^2} \left( e^{-\alpha} - e^{-2\alpha} \right) \right],\end{aligned}$$

and

$$\begin{aligned}P_{2mH}(\alpha) &= e^{-\sqrt{3}m\alpha} \left[ \frac{1}{9\alpha^2} \left( e^{\sqrt{3}\alpha} - e^{-\sqrt{3}\alpha} \right) - \frac{2\sqrt{3}}{9\alpha} \right] \\ &\quad + e^{-2\sqrt{3}m\alpha} \left[ \frac{\sqrt{3}}{9\alpha} \left( e^{\sqrt{3}\alpha} + e^{-\sqrt{3}\alpha} - 2 \right) \right] \\ &\quad + e^{-3m\alpha} \left[ \frac{1}{3\alpha^2} \left( e^{2\alpha} - e^{-2\alpha} - e^{\alpha} + e^{-\alpha} \right) - \frac{2}{3\alpha} \right].\end{aligned}$$

Let us first compute the quantity in parenthesis in the above expression of  $\bar{h}$ :

$$\begin{aligned}P_{2mH}(\alpha) + P_{(2m+1)H}(\alpha) &= \\ &= e^{-\sqrt{3}m\alpha} \left[ \frac{1}{9\alpha^2} \left( e^{\sqrt{3}\alpha} + e^{-\sqrt{3}\alpha} - 2 \right) - \frac{\sqrt{3}}{9\alpha} \left( 1 - e^{-\sqrt{3}\alpha} \right) \right] \\ &\quad + e^{-2\sqrt{3}m\alpha} \left[ \frac{\sqrt{3}}{9\alpha} \left( e^{\sqrt{3}\alpha} + e^{-2\sqrt{3}\alpha} - e^{-\sqrt{3}\alpha} - 1 \right) \right] \\ &\quad + e^{-3m\alpha} \left[ \frac{1}{3\alpha^2} \left( e^{2\alpha} + e^{-2\alpha} - e^{\alpha} - e^{-\alpha} \right) - \frac{1}{3\alpha} \left( 1 - e^{-3\alpha} \right) \right].\end{aligned}$$

We will require two useful and well-known summations:

$$\sum_{m=0}^{\infty} m (e^{-rz})^m = \frac{e^{-rz}}{(1 - e^{-rz})^2},$$

$$\sum_{m=0}^{\infty} (e^{-rz})^m = \frac{1}{1 - e^{-rz}}.$$

Using these results we get the following expression for  $\bar{h}$ :

$$\begin{aligned} \bar{h} &= 2 \left[ \frac{e^{-\sqrt{3}\alpha}}{(1 - e^{-\sqrt{3}\alpha})^2} \right] \cdot \left[ \frac{1}{9\alpha^2} (e^{\sqrt{3}\alpha} + e^{-\sqrt{3}\alpha} - 2) - \frac{\sqrt{3}}{9\alpha} (1 - e^{-\sqrt{3}\alpha}) \right] \\ &+ 2 \left[ \frac{e^{-2\sqrt{3}\alpha}}{(1 - e^{-2\sqrt{3}\alpha})^2} \right] \cdot \left[ \frac{\sqrt{3}}{9\alpha} (e^{\sqrt{3}\alpha} + e^{-2\sqrt{3}\alpha} - e^{-\sqrt{3}\alpha} - 1) \right] \\ &+ 2 \left[ \frac{e^{-3\alpha}}{(1 - e^{-3\alpha})^2} \right] \cdot \left[ \frac{1}{3\alpha^2} (e^{2\alpha} + e^{-2\alpha} - e^{\alpha} - e^{-\alpha}) - \frac{1}{3\alpha} (1 - e^{-3\alpha}) \right] \\ &+ \left[ \frac{1}{1 - e^{-\sqrt{3}\alpha}} \right] \cdot \left[ -\frac{2}{9\alpha^2} (1 - e^{-\sqrt{3}\alpha}) + \frac{\sqrt{3}}{9\alpha} (1 + e^{-\sqrt{3}\alpha}) \right] \\ &+ \left[ \frac{1}{1 - e^{-2\sqrt{3}\alpha}} \right] \cdot \left[ \frac{\sqrt{3}}{9\alpha} (1 - e^{-\sqrt{3}\alpha})^2 \right] \\ &+ \left[ \frac{1}{1 - e^{-3\alpha}} \right] \cdot \left[ -\frac{2}{3\alpha^2} (e^{-\alpha} - e^{-2\alpha}) + \frac{1}{3\alpha} (1 + e^{-3\alpha}) \right]. \\ \Rightarrow \bar{h} &= \frac{2}{9\alpha^2} - \frac{2\sqrt{3}}{9\alpha} \cdot \frac{e^{-\sqrt{3}\alpha}}{(1 - e^{-\sqrt{3}\alpha})} - \frac{2\sqrt{3}}{9\alpha} \cdot \frac{e^{-2\sqrt{3}\alpha}}{(1 - e^{-2\sqrt{3}\alpha})} + \frac{2\sqrt{3}}{9\alpha} \cdot \frac{e^{-\sqrt{3}\alpha}}{(1 - e^{-2\sqrt{3}\alpha})} \\ &+ \frac{2}{3\alpha^2} \cdot \frac{e^{-\alpha}}{(1 - e^{-3\alpha})} - \frac{2}{3\alpha^2} \cdot \frac{e^{-2\alpha}}{(1 - e^{-3\alpha})} - \frac{2}{3\alpha} \cdot \frac{e^{-3\alpha}}{(1 - e^{-3\alpha})} - \frac{2}{9\alpha^2} \\ &+ \frac{\sqrt{3}}{9\alpha} \cdot \frac{(1 + e^{-\sqrt{3}\alpha})}{(1 - e^{-\sqrt{3}\alpha})} + \frac{\sqrt{3}}{9\alpha} \cdot \frac{(1 - e^{-\sqrt{3}\alpha})^2}{(1 - e^{-2\sqrt{3}\alpha})} \\ &- \frac{2}{3\alpha^2} \cdot \frac{(e^{-\alpha} - e^{-2\alpha})}{(1 - e^{-3\alpha})} + \frac{1}{3\alpha} \cdot \frac{(1 + e^{-3\alpha})}{(1 - e^{-3\alpha})} \\ \Rightarrow \bar{h} &= \frac{1}{(1 - e^{-\sqrt{3}\alpha})} \cdot \left[ -\frac{2\sqrt{3}}{9\alpha} e^{-\sqrt{3}\alpha} + \frac{\sqrt{3}}{9\alpha} (1 + e^{-\sqrt{3}\alpha}) \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{(1 - e^{-2\sqrt{3}\alpha})} \cdot \left[ -\frac{2\sqrt{3}}{9\alpha} e^{-2\sqrt{3}\alpha} + \frac{2\sqrt{3}}{9\alpha} e^{-\sqrt{3}\alpha} \right. \\
 & \quad \left. + \frac{\sqrt{3}}{9\alpha} - \frac{2\sqrt{3}}{9\alpha} e^{-\sqrt{3}\alpha} + \frac{\sqrt{3}}{9\alpha} e^{-2\sqrt{3}\alpha} \right] \\
 & + \frac{1}{(1 - e^{-3\alpha})} \cdot \left[ \frac{2}{3\alpha^2} e^{-\alpha} - \frac{2}{3\alpha^2} e^{-2\alpha} - \frac{2}{3\alpha} e^{-3\alpha} - \frac{2}{3\alpha^2} e^{-\alpha} \right. \\
 & \quad \left. + \frac{2}{3\alpha^2} e^{-2\alpha} + \frac{1}{3\alpha} + \frac{1}{3\alpha} e^{-3\alpha} \right] \\
 \Rightarrow \bar{h} & = \frac{\sqrt{3}}{9\alpha} \cdot \frac{(1 - e^{-\sqrt{3}\alpha})}{(1 - e^{-3\alpha})} + \frac{\sqrt{3}}{9\alpha} \cdot \frac{(1 - e^{-2\sqrt{3}\alpha})}{(1 - e^{-2\sqrt{3}\alpha})} + \frac{1}{3\alpha} \cdot \frac{(1 - e^{-3\alpha})}{(1 - e^{-3\alpha})} \\
 & \Rightarrow \bar{h} = \frac{3 + 2\sqrt{3}}{9\alpha}.
 \end{aligned}$$

We have provided at last all the computational steps needed in the derivation of the average number of handoffs. This terminates Appendix C.



## APPENDIX D.

### VERIFICATION OF TRAFFIC CONSERVATION WITHIN A CELL

This appendix is devoted to the analytic check of the fact that the traffic handled by a cell remains equal to  $\lambda/\mu$  under the assumptions of Chapter 4. We will use results established in Section 4.5, and compute the following traffic components:

$T_{O_1}$  : Traffic due to calls originated and terminated inside the cell.

$T_{O_2}$  : Traffic due to calls originated inside the cell and handed off to another cell.

$T_{H_1}$  : Traffic due to calls handed off to the cell and terminated inside the cell.

$T_{H_2}$  : Traffic due to calls handed off to the cell and handed off to another cell.

Let us start with calls originated inside the cell that also terminate inside the cell, and among these let us first consider those associated with mobiles moving in directions (1) or (2). We found in Section 4.5 the probability that a call originated inside the cell moving in directions (1) or (2) would remain less than  $t$  in the cell. If we now call  $t$  the service time of such a call, the probability that it terminates inside the cell is simply given by the complement of the (see Section 4.5) probability of leaving the cell before  $t$ . Namely, we get:

$$P_{O_1/(1)\&(2)}(t) = \begin{cases} 0.0, & k\sqrt{3} \leq t; \\ 1 - \frac{4\sqrt{3}}{9} \left(\frac{t}{k}\right) + \left(\frac{t}{3k}\right)^2, & 0 \leq t \leq k\sqrt{3}. \end{cases}$$

Since the above type of call contributes to the traffic handled by the cell during their entire service duration, and since we have an average of  $\lambda/2$  arrivals inside the cell moving in directions (1) or (2), we get the following traffic contribution from these calls:

$$T_{O_1}^{(1)\&(2)} = \frac{\lambda}{2} \int_0^{k\sqrt{3}} t \mu e^{-\mu t} \left[ 1 - \frac{4\sqrt{3}}{9} \left(\frac{t}{k}\right) + \left(\frac{t}{3k}\right)^2 \right] dt.$$

This gives:

$$\begin{aligned}
 T_{O_1}^{(1)\&(2)} &= \frac{\lambda}{2} \left( \left[ \frac{1}{\mu} - \left( k\sqrt{3} + \frac{1}{\mu} \right) e^{-k\sqrt{3}\mu} \right] \right. \\
 &\quad - \frac{4\sqrt{3}}{9k} \left[ \frac{2}{\mu^2} - \left( 3k^2 + \frac{2\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-k\sqrt{3}\mu} \right] \\
 &\quad \left. + \frac{1}{9k^2} \left[ \frac{6}{\mu^3} - \left( 3\sqrt{3}k^3 + \frac{9k^2}{\mu} + \frac{6\sqrt{3}k}{\mu^2} + \frac{6}{\mu^3} \right) e^{-k\sqrt{3}\mu} \right] \right) \\
 &= \frac{\lambda}{\mu} \left( \left[ \frac{1}{\mu} - \frac{8\sqrt{3}}{9k\mu^2} + \frac{2}{3k^2\mu^3} \right] + e^{-k\sqrt{3}\mu} \left[ -k\sqrt{3} - \frac{1}{\mu} + \frac{4\sqrt{3}k}{3} + \frac{8}{3\mu} + \frac{8\sqrt{3}}{9k\mu^2} \right. \right. \\
 &\quad \left. \left. - \frac{\sqrt{3}k}{3} - \frac{1}{\mu} - \frac{2\sqrt{3}}{3k\mu^2} - \frac{2}{3k^2\mu^3} \right] \right) \\
 \Rightarrow T_{O_1}^{(1)\&(2)} &= \frac{\lambda}{2} \left( \left[ \frac{1}{\mu} - \frac{8\sqrt{3}}{9k\mu^2} + \frac{2}{3k^2\mu^3} \right] + e^{-k\sqrt{3}\mu} \left[ \frac{2}{3\mu} + \frac{2\sqrt{3}}{9k\mu^2} - \frac{2}{3k^2\mu^3} \right] \right).
 \end{aligned}$$

We now consider calls moving in directions (3) or (4), originated inside the cell and that also terminate inside the cell. We recall that in Section 4.5 we obtained the probability that a mobile originated inside the cell would remain less than  $t$  in the cell. Similarly as for directions (1) and (2), taking the complement of this probability gives us the probability that a call originating inside the cell with service time equal to  $t$  terminates inside the cell. This gives:

$$P_{O_1/(3)\&(4)}(t) = \begin{cases} 0.0, & 2k \leq t; \\ \frac{4}{3} - \frac{4}{3} \left( \frac{t}{k} \right) + 3 \left( \frac{t}{3k} \right)^2, & k \leq t \leq 2k; \\ 1 - \frac{2}{3} \left( \frac{t}{k} \right), & 0 \leq t \leq k. \end{cases}$$

All these calls also contribute to the traffic handled by the cell with their whole service time, so we have:

$$\begin{aligned}
 T_{O_1}^{(3)\&(4)} &= \frac{\lambda}{2} \left( \int_0^k t\mu e^{-\mu t} \left[ 1 - \frac{2}{3} \left( \frac{t}{k} \right) \right] dt \right. \\
 &\quad \left. + \int_k^{2k} t\mu e^{-\mu t} \left[ \frac{4}{3} - \frac{4}{3} \left( \frac{t}{k} \right) + 3 \left( \frac{t}{3k} \right)^2 \right] dt \right).
 \end{aligned}$$

This gives:

$$\begin{aligned}
 T_{O_1}^{(3)\&(4)} &= \frac{\lambda}{2} \left( \left[ \frac{1}{\mu} - \left( k + \frac{1}{\mu} \right) e^{-k\mu} \right] - \frac{2}{3k} \left[ \frac{2}{\mu^2} - \left( k^2 + \frac{2k}{\mu} + \frac{2}{\mu^2} \right) e^{-k\mu} \right] \right. \\
 &\quad + \frac{4}{3} \left[ \left( k + \frac{1}{\mu} \right) e^{-k\mu} - \left( 2k + \frac{1}{\mu} \right) e^{-2k\mu} \right] \\
 &\quad - \frac{4}{3k} \left[ \left( k^2 + \frac{2k}{\mu} + \frac{2}{\mu^2} \right) e^{-k\mu} - \left( 4k^2 + \frac{4k}{\mu} + \frac{2}{\mu^2} \right) e^{-2k\mu} \right] \\
 &\quad + \frac{1}{3k^2} \left[ \left( k^3 + \frac{3k^2}{\mu} + \frac{6k}{\mu^2} + \frac{6}{\mu^3} \right) e^{-k\mu} \right. \\
 &\quad \quad \left. - \left( 8k^3 + \frac{12k^2}{\mu} + \frac{12k}{\mu^2} + \frac{6}{\mu^3} \right) e^{-2k\mu} \right] \Bigg) \\
 &= \frac{\lambda}{2} \left( \left[ \frac{1}{\mu} - \frac{4}{3k\mu^2} \right] + e^{-k\mu} \left[ -k - \frac{1}{\mu} + \frac{2k}{3} + \frac{4}{3\mu} + \frac{4}{3k\mu^2} + \frac{4k}{3} + \frac{4}{3\mu} \right. \right. \\
 &\quad \quad \left. \left. - \frac{4k}{3} - \frac{8}{3\mu} - \frac{8}{3k\mu^2} + \frac{k}{3} + \frac{1}{\mu} + \frac{2}{k\mu^2} + \frac{2}{k^2\mu^3} \right] \right. \\
 &\quad \left. + e^{-2k\mu} \left[ -\frac{8k}{3} - \frac{4}{3\mu} + \frac{16k}{3} + \frac{16}{3\mu} + \frac{8}{3k\mu^2} \right. \right. \\
 &\quad \quad \left. \left. - \frac{8k}{3} - \frac{4}{\mu} - \frac{4}{k\mu^2} - \frac{2}{k^2\mu^3} \right] \right) \\
 \Rightarrow T_{O_1}^{(3)\&(4)} &= \frac{\lambda}{2} \left( \left[ \frac{1}{\mu} - \frac{4}{3k\mu^2} \right] + e^{-k\mu} \left[ \frac{2}{3k\mu^2} + \frac{2}{k^2\mu^3} \right] + e^{-2k\mu} \left[ -\frac{4}{3k\mu^2} - \frac{2}{k^2\mu^3} \right] \right).
 \end{aligned}$$

So we finally have:

$$\begin{aligned}
 T_{O_1} &= T_{O_1}^{(1)\&(2)} + T_{O_1}^{(3)\&(4)} \\
 \Rightarrow T_{O_1} &= \frac{\lambda}{2} \left( \left[ \frac{2}{\mu} - \frac{(12 + 8\sqrt{3})}{9k\mu^2} + \frac{2}{3k^2\mu^3} \right] + e^{-k\mu} \left[ \frac{2}{3k\mu^2} + \frac{2}{k^2\mu^3} \right] \right. \\
 &\quad \left. + e^{-k\sqrt{3}\mu} \left[ \frac{2}{3\mu} + \frac{2\sqrt{3}}{9k\mu^2} - \frac{2}{3k^2\mu^3} \right] - e^{-2k\mu} \left[ \frac{4}{3k\mu^2} + \frac{2}{k^2\mu^3} \right] \right).
 \end{aligned}$$

We now proceed with the computation of  $T_{O_2}$ , which is fortunately easily deduced from the computation of  $T_{O_1}$ . Namely, the probability that a call originating inside the cell with a given service time  $t$  terminates outside the cell is simply given

by the complement of the corresponding probabilities for the case  $T_{O_1}$ . That is, we have:

$$P_{O_2/(1)\&(2)}(t) = 1 - P_{O_1/(1)\&(2)}(t),$$

$$P_{O_2/(3)\&(4)}(t) = 1 - P_{O_1/(3)\&(4)}(t).$$

The basic difference between the computations of  $T_{O_1}$  and  $T_{O_2}$  is that for  $T_{O_2}$  the calls that need to be taken into account do not contribute to the traffic handled by the cell for their whole service time. Their contribution is limited in any case to their time spent inside the cell, which in turn will depend on their location.

We recall from Figure 4.13 of Section 4.5 that the above probabilities were obtained by considering possible areas of locations for the mobiles. We now want the traffic contribution of such calls, or, in other words, we want the product of the average arrival rates of such calls and the average holding time or time they spend in the cell. The physical location of the mobiles and the service of the associated call being two independent processes, this product can be rewritten as:

$$T_{O_2} = \int_{\text{all } t} [\text{arrival rate}(t)] \cdot [\text{average time spent in the cell}(t)] \cdot s(t) dt,$$

where  $s(t)$  is the service time density function. The arrival rate is indicated as a function of the service time  $t$ , since for different values of  $t$  we will have different values for the probability that the call terminates outside the cell. This in turn means a different arrival rate of the calls contributing to  $T_{O_2}$ . Similarly, for different service times, the possible areas of origination for the call will be different, implying therefore different average holding times for these calls.

The arrival rate is easily obtained by taking the product of the arrival rate in the cell of calls moving in directions (1) or (2) ( (3) or (4) ) and the above probability  $P_{O_2/(1)\&(2)}(t)$  ( $P_{O_2/(3)\&(4)}(t)$ ) respectively. In order to obtain the expression of the corresponding average time spent in the cell we will have to distinguish between the

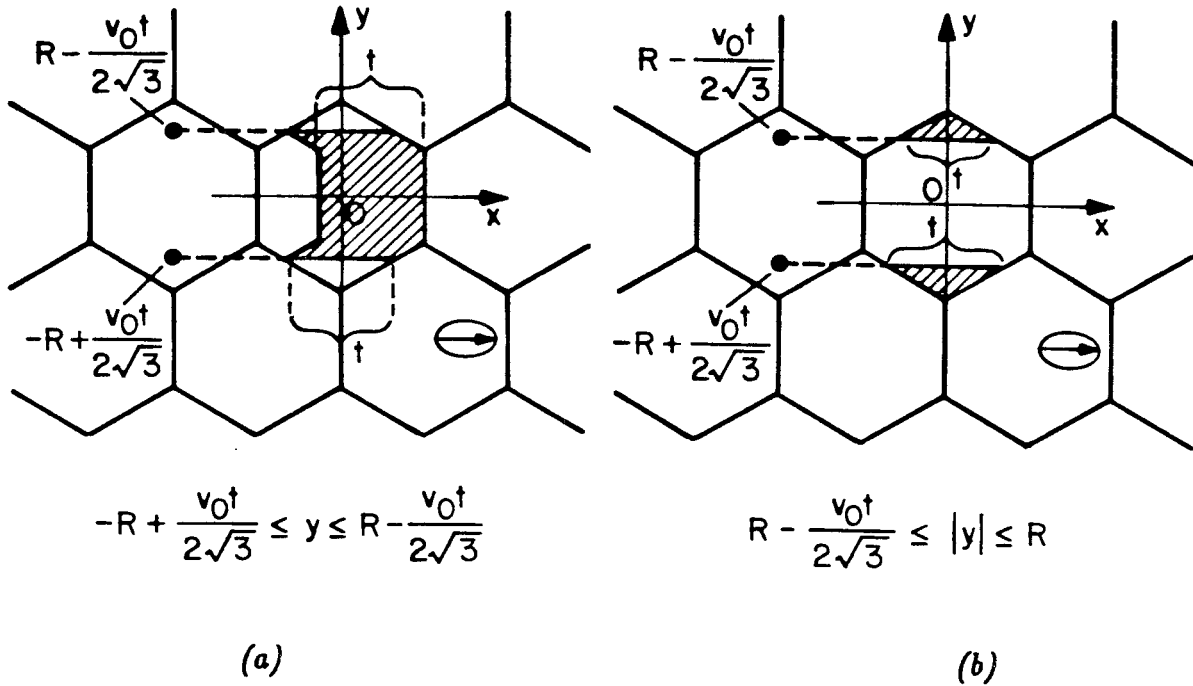


Fig. D1. Calls originated inside the cell and handed off to another cell.

Directions (1) and (2), service time  $t$ :  $0 \leq t \leq k\sqrt{3}$ .

cases of directions (1) or (2), and directions (3) or (4). Let us first proceed with directions (1) or (2).

For the values of  $t$  in the range  $k\sqrt{3} \leq t$ , we are in the case where all the calls moving in directions (1) or (2) need to be taken into account, since all calls with a service time greater than  $k\sqrt{3}$  will terminate outside the cell, and therefore the average time spent inside the cell is the average time needed to cross the cell in the  $x$  direction. This is easily found to be equal to  $4\sqrt{3}k/9$  while the corresponding arrival rate is simply  $\lambda/2$ . If we now consider  $t$  in the range  $0 \leq t \leq k\sqrt{3}$ , we need to consider two distinct regions inside the shaded area of Figure 4.13(a). This yields two different average times spent in the cell. The first region, illustrated in Figure D1(a), corresponds to calls with an ordinates in the range  $-R + \frac{v_0 t}{2\sqrt{3}} \leq y \leq R - \frac{v_0 t}{2\sqrt{3}}$ . These calls will remain on the average  $t/2$  time units in the cell. The

corresponding arrival rate is deduced from  $P_{O_2/(1)\&(2)}(t)$  and found equal to:

$$\lambda_{O_2/(1)\&(2)}^{(1)} = \frac{\lambda}{2} \left[ \frac{4\sqrt{3}}{9} \left( \frac{t}{k} \right) - 2 \left( \frac{t}{3k} \right)^2 \right].$$

Let us now consider the calls with ordinate in the range  $R - \frac{v_0 t}{2\sqrt{3}} \leq |y| \leq R$ . They correspond to calls located in the shaded area of Figure D1(b) and we can again compute the corresponding arrival rate using  $P_{O_2/(1)\&(2)}(t)$ . Namely we have:

$$\lambda_{O_2/(1)\&(2)}^{(2)} = \frac{\lambda}{2} \left[ \left( \frac{t}{3k} \right)^2 \right].$$

Furthermore, we easily see that this type of call remains on the average  $t/3$  time units in the cell.

We are now able to write down the expression of  $T_{O_2}$  for calls moving in directions (1) or (2), and we get:

$$T_{O_2}^{(1)\&(2)} = \frac{\lambda}{2} \left( \int_0^{k\sqrt{3}t} \frac{t}{2} \mu e^{-\mu t} \left[ \frac{4\sqrt{3}}{9} \left( \frac{t}{k} \right) - 2 \left( \frac{t}{3k} \right)^2 \right] dt + \int_0^{k\sqrt{3}t} \frac{t}{3} \mu e^{-\mu t} \left( \frac{t}{3k} \right)^2 dt + \int_{k\sqrt{3}t}^{\infty} \frac{4\sqrt{3}k}{9} \mu e^{-\mu t} dt \right).$$

This gives:

$$\begin{aligned} T_{O_2}^{(1)\&(2)} &= \frac{\lambda}{2} \left( \frac{2\sqrt{3}}{9k} \left[ \frac{2}{\mu^2} - \left( 3k^2 + \frac{2\sqrt{3}k}{\mu} + \frac{2}{\mu^2} \right) e^{-k\sqrt{3}\mu} \right] \right. \\ &\quad - \frac{1}{9k^2} \left[ \frac{6}{\mu^3} - \left( 3\sqrt{3}k^3 + \frac{9k^2}{\mu} + \frac{6\sqrt{3}k}{\mu^2} + \frac{6}{\mu^3} \right) e^{-k\sqrt{3}\mu} \right] \\ &\quad - \frac{1}{27k^2} \left[ \frac{6}{\mu^3} - \left( 3\sqrt{3}k^3 + \frac{9k^2}{\mu} + \frac{6\sqrt{3}k}{\mu^2} + \frac{6}{\mu^3} \right) e^{-k\sqrt{3}\mu} \right] \\ &\quad \left. + \frac{4\sqrt{3}k}{9} \left[ e^{-k\sqrt{3}\mu} \right] \right) \\ &= \frac{\lambda}{2} \left( \left[ \frac{4\sqrt{3}}{9k\mu^2} - \frac{4}{9k^2\mu^3} \right] \right. \\ &\quad + e^{-k\sqrt{3}\mu} \left[ -\frac{2\sqrt{3}k}{3} - \frac{4}{3\mu} - \frac{4\sqrt{3}}{9k\mu^2} + \frac{\sqrt{3}k}{3} + \frac{1}{\mu} + \frac{2\sqrt{3}}{3k\mu^2} \right. \\ &\quad \left. \left. + \frac{2}{3k^2\mu^3} - \frac{\sqrt{3}k}{9} - \frac{1}{3\mu} - \frac{2\sqrt{3}}{9k\mu^2} - \frac{2}{9k^2\mu^3} + \frac{4\sqrt{3}k}{9} \right] \right) \end{aligned}$$

$$\Rightarrow T_{O_2}^{(1)\&(2)} = \frac{\lambda}{2} \left( \left[ \frac{4\sqrt{3}}{9k\mu^2} - \frac{4}{9k^2\mu^3} \right] + e^{-k\sqrt{3}\mu} \left[ -\frac{2}{3\mu} + \frac{4}{9k^2\mu^3} \right] \right).$$

We now proceed in a similar fashion with the case of directions (3) or (4). Calls with service time equal to  $t$  and originated inside the cell will terminate outside the cell with probability  $P_{O_2/(3)\&(4)}(t)$ , which corresponds to the shaded areas of Figures 4.14(a1) and 4.14(a2). If  $t$  is in the range  $2k \leq t$ , then calls with such a service time always terminate outside the cell and we have a corresponding arrival rate equal to  $\lambda/2$ . The average time spent in the cell by such calls is simply the average time needed to cross the cell when moving in directions (3) or (4) and is found equal to  $7k/9$ .

In the case where  $t$  is in the range  $k \leq t \leq 2k$ , we need to distinguish two regions for the possible location of the mobile.

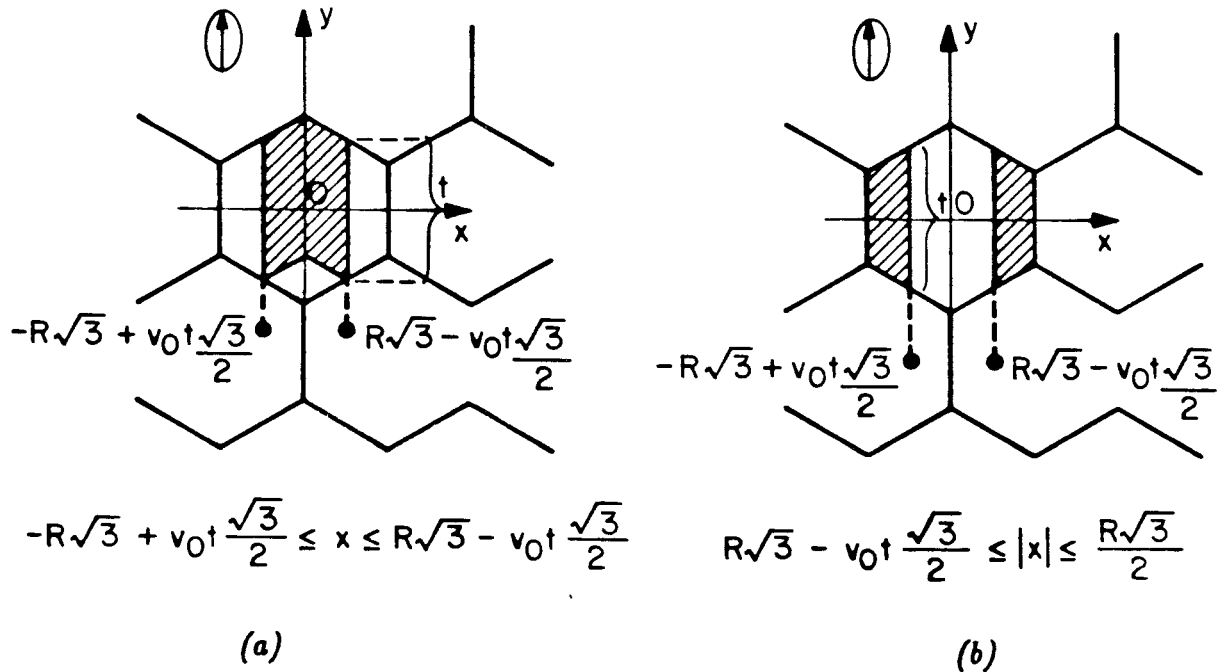


Fig. D2. Calls originated inside the cell and handed off to another cell.

Directions (3) and (4), service time  $t$ :  $k \leq t \leq 2k$ .

When the mobile has an abscissa  $x$  in the range  $0 \leq |x| \leq \sqrt{3} \left( R - \frac{v_0 t}{2} \right)$  (see Figure D2(a)), a mobile will remain in the cell on the average  $t/2$  time units. Using  $P_{O_2/(3)\&(4)}(t)$ , we find a corresponding arrival rate equal to:

$$\lambda_{O_2/(3)\&(4)}^{(1)} = \frac{\lambda}{2} \left[ \frac{4}{3} \left( \frac{t}{k} \right) - 6 \left( \frac{t}{3k} \right)^2 \right].$$

If on the other hand the abscissa  $x$  of the mobile is in the range  $\sqrt{3} \left( R - \frac{v_0 t}{2} \right) \leq |x| \leq \frac{R\sqrt{3}}{2}$  (see Figure D2(b)), we can directly compute the product of the arrival rate and the average time spent in the cell. Namely, the product of the average distance to a cell boundary for the calls located in the shaded area of Figure D2(b) and the probability that a call is actually located in this area is given by:

$$P_{A_s} \cdot d_{avg} = \int_{\sqrt{3}(R-\frac{v_0 t}{2})}^{\frac{R\sqrt{3}}{2}} \frac{8\sqrt{3}}{9R^2} \left( R - \frac{\sqrt{3}}{3}x \right) \left[ \frac{1}{2 \left( R - \frac{\sqrt{3}}{3}x \right)} \int_0^{2 \left( R - \frac{\sqrt{3}}{3}x \right)} y dy \right] dx,$$

where the factor

$$\frac{8\sqrt{3}}{9R^2} \left( R - \frac{\sqrt{3}}{3}x \right) dx$$

expresses the probability that a mobile has abscissa in the range  $x$  and  $x + dx$  inside the shaded area of Figure D2(b), while the factor

$$\frac{dy}{2 \left( R - \frac{\sqrt{3}}{3}x \right)}$$

expresses the fact that for a given abscissa  $x$  the mobiles are uniformly distributed with respect to  $y$ .  $P_{A_s}$  simply stands for the probability that the mobile is located in the shaded area of Figure D2(b). Note that in the above expression, due to the symmetry of the hexagonal cell with respect to the  $y$ -axis, we have limited ourself to the case  $x \geq 0$ . After some simple algebraic manipulations we obtain the average distance to a cell boundary, and therefore the product of the average time



spent in the cell by the mobiles located in the shaded area of Figure D2(b) and the corresponding arrival rate. Namely, we get:

$$\lambda_{O_2/(3)\&(4)}^{(2)} \cdot t_{avg} = \frac{k}{9} \left[ \left( \frac{t}{k} \right)^3 - 1 \right] \cdot \frac{\lambda}{2}.$$

Finally, if we consider the case where  $t$  is in the range  $0 \leq t \leq k$ , the average arrival rate is found, again using  $P_{O_2/(3)\&(4)}(t)$ , to be equal to

$$\lambda_{O_2/(3)\&(4)}^{(3)} = \frac{\lambda}{2} \left[ \frac{2}{3} \left( \frac{t}{k} \right) \right],$$

and the corresponding average time spent in the cell is simply  $t/2$  time units.

We are now in a position to compute  $T_{O_2}^{(3)\&(4)}$ , which can be written as:

$$\begin{aligned} T_{O_2}^{(3)\&(4)} = & \frac{\lambda}{2} \left( \int_0^k \frac{t}{2} \mu e^{-\mu t} \left[ \frac{2}{3} \left( \frac{t}{k} \right) \right] dt + \int_k^{2k} \frac{t}{2} \mu e^{-\mu t} \left[ \frac{4}{3} \left( \frac{t}{k} \right) - 6 \left( \frac{t}{3k} \right)^2 \right] dt \right. \\ & \left. + \int_k^{2k} \frac{k}{9} \left[ \left( \frac{t}{k} \right)^3 - 1 \right] \mu e^{-\mu t} dt + \int_{2k}^{\infty} \frac{7k}{9} \mu e^{-\mu t} dt \right). \end{aligned}$$

This gives the following expression:

$$\begin{aligned} T_{O_2}^{(3)\&(4)} = & \frac{\lambda}{2} \left( \frac{1}{3k} \left[ \frac{2}{\mu^2} - \left( k^2 + \frac{2k}{\mu} + \frac{2}{\mu^2} \right) e^{-k\mu} \right] \right. \\ & + \frac{2}{3k} \left[ \left( k^2 + \frac{2k}{\mu} + \frac{2}{\mu^2} \right) e^{-k\mu} - \left( 4k^2 + \frac{4k}{\mu} + \frac{2}{\mu^2} \right) e^{-2k\mu} \right] \\ & - \frac{1}{3k^2} \left[ \left( k^3 + \frac{3k^2}{\mu} + \frac{6k}{\mu^2} + \frac{6}{\mu^3} \right) e^{-k\mu} \right. \\ & \quad \left. - \left( 8k^3 + \frac{12k^2}{\mu} + \frac{12k}{\mu^2} + \frac{6}{\mu^3} \right) e^{-2k\mu} \right] - \frac{k}{9} [e^{-k\mu} - e^{-2k\mu}] \\ & + \frac{1}{9k^2} \left[ \left( k^3 + \frac{3k^2}{\mu} + \frac{6k}{\mu^2} + \frac{6}{\mu^3} \right) e^{-k\mu} \right. \\ & \quad \left. - \left( 8k^3 + \frac{12k^2}{\mu} + \frac{12k}{\mu^2} + \frac{6}{\mu^3} \right) e^{-2k\mu} \right] + \frac{7k}{9} [e^{-2k\mu}] \Bigg) \\ = & \frac{\lambda}{2} \left( \left[ \frac{2}{3k\mu^2} \right] + e^{-k\mu} \left[ -\frac{k}{3} - \frac{2}{3\mu} - \frac{2}{3k\mu^2} - \frac{k}{3} - \frac{1}{\mu} - \frac{2}{k\mu^2} - \frac{2}{k^2\mu^3} - \frac{k}{9} \right] \right. \end{aligned}$$

$$\begin{aligned}
 & + \frac{k}{9} + \frac{1}{3\mu} + \frac{2}{3k\mu^2} + \frac{2}{3k^2\mu^3} + \frac{2k}{3} + \frac{4}{3\mu} + \frac{4}{3k\mu^2} \Big] \\
 & + e^{-2k\mu} \left[ -\frac{8k}{3} - \frac{8}{3\mu} - \frac{4}{3k\mu^2} + \frac{8k}{3} + \frac{4}{\mu} + \frac{4}{k\mu^2} + \frac{2}{k^2\mu^3} \right. \\
 & \quad \left. + \frac{k}{9} - \frac{8k}{9} - \frac{4}{3\mu} - \frac{4}{3k\mu^2} - \frac{2}{3k^2\mu^3} + \frac{7k}{9} \right] \Big) \\
 \Rightarrow T_{O_2}^{(3)\&(4)} &= \frac{\lambda}{2} \left( \left[ \frac{2}{3k\mu^2} \right] - e^{-k\mu} \left[ \frac{2}{3k\mu^2} + \frac{4}{3k^2\mu^3} \right] + e^{-2k\mu} \left[ \frac{4}{3k\mu^2} + \frac{4}{3k^2\mu^3} \right] \right).
 \end{aligned}$$

This at last provides the total traffic due to calls that originated inside the cell and terminated outside the cell:

$$\begin{aligned}
 T_{O_2} &= T_{O_2}^{(1)\&(2)} + T_{O_2}^{(3)\&(4)} \\
 \Rightarrow T_{O_2} &= \frac{\lambda}{2} \left( \left[ \frac{(4\sqrt{3}+6)}{9k\mu^2} - \frac{4}{9k^2\mu^3} \right] - e^{-k\mu} \left[ \frac{2}{3k\mu^2} + \frac{4}{3k^2\mu^3} \right] \right. \\
 & \quad \left. + e^{-k\sqrt{3}\mu} \left[ -\frac{2}{3\mu} + \frac{4}{9k^2\mu^3} \right] + e^{-2k\mu} \left[ \frac{4}{3k\mu^2} + \frac{4}{3k^2\mu^3} \right] \right).
 \end{aligned}$$

From this we can obtain the overall traffic contribution  $T_O$  of calls that originated inside the cell. Namely,

$$\begin{aligned}
 T_O &= T_{O_1} + T_{O_2} \\
 \Rightarrow T_O &= \frac{\lambda}{2} \left( \left[ \frac{2}{\mu} - \frac{(4\sqrt{3}+6)}{9k\mu^2} + \frac{2}{9k^2\mu^3} \right] + e^{-k\mu} \left[ \frac{2}{3k^2\mu^3} \right] \right. \\
 & \quad \left. + e^{-k\sqrt{3}\mu} \left[ \frac{2\sqrt{3}}{9k\mu^2} - \frac{2}{9k^2\mu^3} \right] - e^{-2k\mu} \left[ \frac{2}{3k^2\mu^3} \right] \right).
 \end{aligned}$$

At this point, we concentrate on the traffic contribution of calls that are handed off to the cell. Similarly as for the case of calls that originated inside the cell, we will distinguish between calls that terminate inside the cell and calls that are further handed off to another cell. We recall that in Section 4.5 we computed the arrival rates due to handoffs for the cases of directions (1) or (2) and (3) or (4).

Furthermore, these arrival rates are uniform with respect to the differential intervals  $dx$  and  $dy$ . Namely, we know that the average number of calls handed off to the cell in an interval along the  $y$ -axis  $dy$  in the range  $-R$  to  $R$  is given by:

$$h_y = \eta_{(1)\&(2)} \frac{dy}{2R} = \frac{2\sqrt{3}}{9} \frac{\lambda}{k\mu} \frac{dy}{2R},$$

similarly, the average number of calls handed off to the cell in an interval along the  $x$ -axis  $dx$  in the range  $-R\sqrt{3}/2$  to  $R\sqrt{3}/2$  is given by:

$$h_x = \eta_{(3)\&(4)} \frac{dx}{R\sqrt{3}} = \frac{1}{3} \frac{\lambda}{k\mu} \frac{dx}{R\sqrt{3}}.$$

We will start with the traffic contribution of calls handed off to the cell that will terminate inside the cell. First consider calls handed off to the cell which are associated to a mobile moving in directions (1) or (2). Such calls will enter the cell at a given starting ordinate  $y$  on the cell boundary. Due to the symmetry of the hexagonal cell, we can restrict our study to the case  $0 \leq y \leq R$ . Within this range we will distinguish two possibilities for  $y$ ,  $0 \leq y \leq R/2$  and  $R/2 \leq y \leq R$ .

If  $y$  is in the range  $0 \leq y \leq R/2$ , a handoff call located on the cell boundary and moving in directions (1) or (2) will terminate inside the cell if and only if its service time  $t$  satisfies the following inequality:

$$0 \leq t \leq k\sqrt{3}.$$

Similarly, if  $y$  is in the range  $R/2 \leq y \leq R$ , a corresponding handoff call moving in directions (1) or (2) will terminate inside the cell if its service time  $t$  satisfies the inequality:

$$0 \leq t \leq 2\sqrt{3} \left( k - \frac{y}{v_0} \right).$$

We can now compute the traffic contribution of handoff calls that terminate inside the cell. We get:

$$T_{H_1}^{(1)\&(2)} = \frac{2\sqrt{3}\lambda}{9k\mu R} \left( \int_0^{R/2} \left[ \int_0^{k\sqrt{3}} t\mu e^{-\mu t} dt \right] dy + \int_{R/2}^R \left[ \int_0^{2\sqrt{3}\left(k-\frac{y}{v_0}\right)} t\mu e^{-\mu t} dt \right] dy \right).$$

This then gives:

$$\begin{aligned} T_{H_1}^{(1)\&(2)} &= \frac{2\sqrt{3}\lambda}{9k\mu R} \left( \frac{R}{2} \left[ \frac{1}{\mu} - \left( k\sqrt{3} + \frac{1}{\mu} \right) e^{-k\sqrt{3}\mu} \right] \right. \\ &\quad \left. + \int_{R/2}^R \left[ \frac{1}{\mu} - \left( 2\sqrt{3}k - 2\sqrt{3}\frac{y}{v_0} + \frac{1}{\mu} \right) e^{-2\sqrt{3}k\mu} e^{2\sqrt{3}\mu\frac{y}{v_0}} \right] dy \right) \\ &= \frac{\sqrt{3}\lambda}{9k\mu} \left[ \frac{1}{\mu} - \left( k\sqrt{3} + \frac{1}{\mu} \right) e^{-k\sqrt{3}\mu} \right] \\ &\quad + \frac{2\sqrt{3}\lambda}{9k\mu R} \left( \frac{R}{2\mu} - e^{-2\sqrt{3}k\mu} \left[ \frac{R}{\mu} \int_{R/2}^R \frac{2\sqrt{3}\mu}{v_0} e^{2\sqrt{3}\mu\frac{y}{v_0}} dy \right. \right. \\ &\quad \left. \left. - \frac{1}{\mu} \int_{R/2}^R \frac{2\sqrt{3}\mu y}{v_0} e^{2\sqrt{3}\mu\frac{y}{v_0}} dy + \frac{1}{\mu} \int_{R/2}^R e^{2\sqrt{3}\mu\frac{y}{v_0}} dy \right] \right) \end{aligned}$$

$$\begin{aligned} \Rightarrow T_{H_1}^{(1)\&(2)} &= \frac{\sqrt{3}\lambda}{9k\mu} \left[ \frac{2}{\mu} - \left( k\sqrt{3} + \frac{1}{\mu} \right) e^{-k\sqrt{3}\mu} \right] \\ &\quad - \frac{2\sqrt{3}\lambda}{9k\mu R} e^{-2\sqrt{3}k\mu} \left( \frac{R}{\mu} \left[ e^{2\sqrt{3}k\mu} - e^{k\sqrt{3}\mu} \right] \right. \\ &\quad \left. - \frac{1}{\mu} \left[ \left( R - \frac{\sqrt{3}v_0}{6\mu} \right) e^{2\sqrt{3}k\mu} - \left( \frac{R}{2} - \frac{\sqrt{3}v_0}{6\mu} \right) e^{k\sqrt{3}\mu} \right] \right. \\ &\quad \left. \frac{v_0\sqrt{3}}{6\mu^2} \left[ e^{2\sqrt{3}k\mu} - e^{k\sqrt{3}\mu} \right] \right) \\ &= \frac{\sqrt{3}\lambda}{9k\mu} \left( \frac{2}{\mu} - \left( k\sqrt{3} + \frac{1}{\mu} \right) e^{-k\sqrt{3}\mu} - \frac{2}{\mu} + \frac{2}{\mu} e^{-k\sqrt{3}\mu} + \frac{2}{\mu} - \frac{\sqrt{3}}{3k\mu^2} \right. \\ &\quad \left. - \frac{1}{\mu} e^{-k\sqrt{3}\mu} + \frac{\sqrt{3}}{3k\mu^2} e^{-k\sqrt{3}\mu} - \frac{\sqrt{3}}{3k\mu^2} + \frac{\sqrt{3}}{3k\mu^2} e^{-k\sqrt{3}\mu} \right) \\ &= \lambda \left( \left[ \frac{2\sqrt{3}}{9k\mu^2} - \frac{2}{9k^2\mu^3} \right] \right. \\ &\quad \left. + e^{-k\sqrt{3}\mu} \left[ -\frac{1}{3\mu} - \frac{\sqrt{3}}{9k\mu^2} + \frac{2\sqrt{3}}{9k\mu^2} - \frac{\sqrt{3}}{9k\mu^2} + \frac{1}{9k^2\mu^3} + \frac{1}{9k^2\mu^3} \right] \right) \end{aligned}$$

$$\Rightarrow T_{H_1}^{(1)\&(2)} = \frac{\lambda}{2} \left( \left[ \frac{4\sqrt{3}}{9k\mu^2} - \frac{4}{9k^2\mu^3} \right] + e^{-k\sqrt{3}\mu} \left[ -\frac{2}{3\mu} + \frac{4}{9k^2\mu^3} \right] \right).$$

Now consider calls handed off to the cell which terminate inside the cell and are associated with a mobile moving in directions (3) or (4). Such calls will enter the cell at a given starting abscissa  $x$  on the cell boundary. Due to the symmetry of the hexagonal cell we can restrict our study to the case  $0 \leq x \leq R\sqrt{3}/2$ . We easily see that a call with service time equal to  $t$  and located at abscissa  $x$  ( $0 \leq x \leq R\sqrt{3}/2$ ) on the cell boundary will terminate inside the cell if and only if its service time satisfies the inequality:

$$0 \leq t \leq 2 \left( k - \frac{\sqrt{3} x}{3 v_0} \right).$$

The traffic contribution of this type of calls can therefore be expressed as:

$$T_{H_1}^{(3)\&(4)} = \frac{\lambda}{3k\mu} \left( \frac{2}{R\sqrt{3}} \int_0^{R\sqrt{3}/2} \left[ \int_0^{2\left(k - \frac{\sqrt{3}x}{3v_0}\right)} t\mu e^{-\mu t} dt \right] dx \right).$$

This gives:

$$\begin{aligned} T_{H_1}^{(3)\&(4)} &= \frac{2\lambda\sqrt{3}}{9k\mu R} \left( \int_0^{R\sqrt{3}/2} \left[ \frac{1}{\mu} - \left( 2 \left[ k - \frac{\sqrt{3}x}{3v_0} \right] + \frac{1}{\mu} \right) e^{-2k\mu} e^{\frac{2\sqrt{3}\mu x}{3v_0}} \right] dx \right) \\ &= \frac{\lambda}{3k\mu^2} - \frac{2\lambda\sqrt{3}}{9k\mu R} \left( \int_0^{R\sqrt{3}/2} 2 \left[ k - \frac{\sqrt{3}x}{3v_0} \right] e^{-2\mu\left(k - \frac{\sqrt{3}x}{3v_0}\right)} \right. \\ &\quad \left. + \frac{\sqrt{3}v_0}{2\mu^2} e^{-2k\mu} \int_0^{R\sqrt{3}/2} \frac{2\sqrt{3}\mu}{3v_0} e^{\frac{2\sqrt{3}\mu x}{3v_0}} dx \right) \\ &= \frac{\lambda}{3k\mu^2} - \frac{2\lambda\sqrt{3}}{9k\mu R} \left( \frac{\sqrt{3}v_0}{2\mu} \int_k^{2k} u\mu e^{-\mu u} du + \frac{\sqrt{3}v_0}{2\mu^2} e^{-2k\mu} \int_0^k \mu e^{\mu v} dv \right) \\ &= \frac{\lambda}{3k\mu^2} - \frac{\lambda}{3k^2\mu^2} \left( \left[ \left( k + \frac{1}{\mu} \right) e^{-k\mu} - \left( 2k + \frac{1}{\mu} \right) e^{-2k\mu} \right] + \frac{1}{\mu} e^{-2k\mu} [e^{k\mu} - 1] \right) \\ \Rightarrow T_{H_1}^{(3)\&(4)} &= \frac{\lambda}{2} \left( \left[ \frac{2}{3k\mu^2} \right] - e^{-k\mu} \left[ \frac{2}{3k\mu^2} + \frac{4}{3k^2\mu^3} \right] + e^{-2k\mu} \left[ \frac{4}{3k\mu^2} + \frac{4}{3k^2\mu^3} \right] \right). \end{aligned}$$

We can use this to express the total traffic contribution of calls handed off to the cell that also terminate inside the cell in terms of the two traffic components we have just computed. Namely, we have:

$$\begin{aligned}
 T_{H_1} &= T_{H_1}^{(1)\&(2)} + T_{H_1}^{(3)\&(4)} \\
 \Rightarrow T_{H_1} &= \frac{\lambda}{2} \left( \left[ \frac{(4\sqrt{3} + 6)}{9k\mu^2} - \frac{4}{9k^2\mu^3} \right] - e^{-k\mu} \left[ \frac{2}{3k\mu^2} + \frac{4}{3k^2\mu^3} \right] \right. \\
 &\quad \left. + e^{-k\sqrt{3}\mu} \left[ -\frac{2}{3\mu} + \frac{4}{9k^2\mu^3} \right] + e^{-2k\mu} \left[ \frac{4}{3k\mu^2} + \frac{4}{3k^2\mu^3} \right] \right).
 \end{aligned}$$

This finishes our study of the traffic contribution of calls handed off to the cell that terminate inside the cell.

We now consider the last category of calls, namely, the calls that are handed off to the cell and then further handed off to another cell. The traffic contribution from these calls can be obtained using the results derived for  $T_{H_1}$ . For example, consider the case of a call with service time  $t$ , a call associated with a mobile moving in directions (1) or (2). Let the call enter the cell at ordinate  $y$  on the cell boundary. We know that if  $y$  is in the range  $0 \leq y \leq R/2$ , then the call will be further handed off if its service time satisfies:

$$t \geq k\sqrt{3}.$$

In this case its total time spent in the cell will always be exactly equal to  $k\sqrt{3}$ .

Similarly, if  $y$  is in the range  $R/2 \leq y \leq R$ , then a handoff call will be further handed off to another cell if its service time satisfies:

$$t \geq 2\sqrt{3} \left( k - \frac{y}{v_0} \right).$$

The service time spent in the cell is in this case equal to  $2\sqrt{3}(k - y/v_0)$ . Note that due to the symmetry of the hexagonal cell we again limit ourselves to the case  $0 \leq y \leq R$ .

We can now obtain the traffic contribution of calls handed off to the cell, moving in directions (1) or (2), and that are further handed off to another cell. Namely, we have:

$$T_{H_2}^{(1)\&(2)} = \frac{2\sqrt{3}\lambda}{9k\mu R} \left( \int_0^{R/2} \left[ \int_{k\sqrt{3}}^{\infty} k\sqrt{3}\mu e^{-\mu t} dt \right] dy + \int_{R/2}^R \left[ \int_{2\sqrt{3}\left(k-\frac{y}{v_0}\right)}^{\infty} 2\sqrt{3}\left(k-\frac{y}{v_0}\right)\mu e^{-\mu t} dt \right] dy \right).$$

This gives:

$$\begin{aligned} T_{H_2}^{(1)\&(2)} &= \frac{2\sqrt{3}\lambda}{9k\mu R} \left( \frac{kR\sqrt{3}}{2} e^{-k\sqrt{3}\mu} + 2\sqrt{3}k e^{-2\sqrt{3}k\mu} \int_{R/2}^R e^{2\sqrt{3}\mu\frac{y}{v_0}} dy \right. \\ &\quad \left. - \frac{2\sqrt{3}}{v_0} e^{-2\sqrt{3}k\mu} \int_{R/2}^R y e^{2\sqrt{3}\mu\frac{y}{v_0}} dy \right) \\ &= \lambda \left( \frac{1}{3\mu} e^{-k\sqrt{3}\mu} + \frac{4}{3R\mu} e^{-2\sqrt{3}k\mu} \frac{v_0}{2\sqrt{3}\mu} \left[ e^{2\sqrt{3}k\mu} - e^{k\sqrt{3}\mu} \right] \right. \\ &\quad \left. - \frac{4}{3k^2\mu} e^{-2\sqrt{3}k\mu} \frac{v_0}{2\sqrt{3}\mu} \left[ e^{2\sqrt{3}k\mu} \left( R - \frac{v_0}{2\sqrt{3}\mu} \right) - e^{k\sqrt{3}\mu} \left( \frac{R}{2} - \frac{v_0}{2\sqrt{3}\mu} \right) \right] \right) \\ &= \lambda \left( \left[ \frac{2\sqrt{3}}{9k\mu^2} - \frac{2\sqrt{3}}{9k\mu^2} + \frac{1}{9k^2\mu^3} \right] + e^{-k\sqrt{3}\mu} \left[ \frac{1}{3\mu} - \frac{2\sqrt{3}}{9k\mu^2} + \frac{\sqrt{3}}{9k\mu^2} - \frac{1}{9k^2\mu^3} \right] \right) \\ &\Rightarrow T_{H_2}^{(1)\&(2)} = \frac{\lambda}{2} \left( \left[ \frac{2}{9k^2\mu^3} \right] + e^{-k\sqrt{3}\mu} \left[ \frac{2}{3\mu} - \frac{2\sqrt{3}}{9k\mu^2} - \frac{2}{9k^2\mu^3} \right] \right). \end{aligned}$$

The case of calls associated with mobiles moving in directions (3) or (4) is very similar. Using the results established for  $T_{H_1}$  we know that a call with service time equal to  $t$  and moving in directions (3) or (4), which reaches the cell boundary at abscissa  $x$  (we can assume  $0 \leq x \leq R\sqrt{3}/2$  by symmetry) will be handed off to another cell if and only if its service time satisfies:

$$t \geq 2 \left( k - \frac{\sqrt{3}}{3} \frac{x}{v_0} \right).$$

In this case, the corresponding time spent in the cell is always equal to the right-hand side of the above inequality. The traffic contribution of this type of call is then given by the following expression:

$$T_{H_2}^{(3)\&(4)} = \frac{\lambda}{3k\mu} \left( \frac{2}{R\sqrt{3}} \int_0^{R\frac{\sqrt{3}}{2}} \left[ \int_{2(k-\frac{\sqrt{3}x}{3v_0})}^{\infty} 2 \left( k - \frac{\sqrt{3}x}{3v_0} \right) \mu e^{-\mu t} dt \right] dx \right).$$

This gives:

$$\begin{aligned} T_{H_2}^{(3)\&(4)} &= \frac{2\lambda\sqrt{3}}{9k\mu R} \left( \int_0^{R\frac{\sqrt{3}}{2}} 2 \left[ k - \frac{\sqrt{3}x}{3v_0} \right] e^{-2\mu \left( k - \frac{\sqrt{3}x}{3v_0} \right)} dx \right) \\ &= \frac{\lambda}{3k^2\mu^2} \left( \int_k^{2k} u\mu e^{-\mu u} du \right) \\ &= \frac{\lambda}{3k^2\mu^2} \left( \left[ k + \frac{1}{\mu} \right] e^{-k\mu} - \left[ 2k + \frac{1}{\mu} \right] e^{-2k\mu} \right) \\ \Rightarrow T_{H_2}^{(3)\&(4)} &= \frac{\lambda}{2} \left( e^{-k\mu} \left[ \frac{2}{3k\mu^2} + \frac{2}{3k^2\mu^3} \right] - e^{-2k\mu} \left[ \frac{4}{3k\mu^2} + \frac{2}{3k^2\mu^3} \right] \right). \end{aligned}$$

So, after summing the two expressions obtained for the cases of calls moving in directions (1) or (2) and (3) or (4), we obtain the traffic contribution of calls that are first handed off to the cell and then further handed off to another cell:

$$\begin{aligned} T_{H_2} &= T_{H_2}^{(1)\&(2)} + T_{H_2}^{(3)\&(4)} \\ \Rightarrow T_{H_2} &= \frac{\lambda}{2} \left( \left[ \frac{2}{9k^2\mu^3} \right] + e^{-k\mu} \left[ \frac{2}{3k\mu^2} + \frac{2}{3k^2\mu^3} \right] \right. \\ &\quad \left. + e^{-k\sqrt{3}\mu} \left[ \frac{2}{3\mu} - \frac{2\sqrt{3}}{9k\mu^2} - \frac{2}{9k^2\mu^3} \right] - e^{-2k\mu} \left[ \frac{4}{3k\mu^2} + \frac{2}{3k^2\mu^3} \right] \right). \end{aligned}$$

This allows us to obtain the total traffic contribution  $T_H$  of calls handed off to the cell. Namely,

$$\begin{aligned} T_H &= T_{H_1} + T_{H_2} \\ \Rightarrow T_H &= \frac{\lambda}{2} \left( \left[ \frac{(4\sqrt{3}+6)}{9k\mu^2} - \frac{2}{9k^2\mu^3} \right] - e^{-k\mu} \left[ \frac{2}{3k^2\mu^3} \right] \right. \\ &\quad \left. - e^{-k\sqrt{3}\mu} \left[ \frac{2\sqrt{3}}{9k\mu^2} - \frac{2}{9k^2\mu^3} \right] + e^{-2k\mu} \left[ \frac{2}{3k^2\mu^3} \right] \right). \end{aligned}$$



We are at last in a position to verify that the total traffic handled by the cell is still equal to  $\lambda/\mu$ . If we call  $T$  this overall traffic, we have:

$$\begin{aligned} T &= T_O + T_H \\ \Rightarrow T &= \frac{\lambda}{2} \left( \left[ \frac{2}{\mu} - \frac{(4\sqrt{3}+6)}{9k\mu^2} + \frac{2}{9k^2\mu^3} + \frac{(4\sqrt{3}+6)}{9k\mu^2} - \frac{2}{9k^2\mu^3} \right] \right. \\ &\quad + e^{-k\mu} \left[ \frac{2}{3k^2\mu^3} - \frac{2}{3k^2\mu^3} \right] + e^{-k\sqrt{3}\mu} \left[ \frac{2\sqrt{3}}{9k\mu^2} - \frac{2}{9k^2\mu^3} - \frac{2\sqrt{3}}{9k\mu^2} + \frac{2}{9k^2\mu^3} \right] \\ &\quad \left. - e^{-2k\mu} \left[ \frac{2}{3k^2\mu^3} - \frac{2}{3k^2\mu^3} \right] \right) \\ &= \frac{\lambda}{2} \cdot \frac{2}{\mu} \\ &\Rightarrow T = \frac{\lambda}{\mu}. \end{aligned}$$

This finishes our check that the total traffic carried by a cell remained equal to  $\lambda/\mu$ .

**APPENDIX E**  
**TWO-DIMENSIONAL Z-TRANSFORM**  
**OF A QUEUEING-BLOCKING SYSTEM**  
**WITH TWO ARRIVAL STREAMS AND GUARD CHANNELS**

In this appendix we want to derive the two-dimensional Z-Transform corresponding to the queueing system described in Figure 6.2 of Chapter 6. We start with the general balance Equation (*Bg*) given in the Introduction and describing the modified queueing system:

$$\begin{aligned}
 [b + c\bar{\delta}_{i_2, n} + (1 - \delta_{i_1, n-g}\delta_{i_2, n-g})i_2] P(i_1, i_2) = \\
 (i_2 + 1)\bar{\delta}_{i_2, n}P(i_1, i_2 + 1) \\
 + c\bar{\delta}_{i_2, n-g}P(i_1, i_2 - 1) \\
 + b\bar{\delta}_{i_1, n-g}P(i_1 - 1, i_2) \\
 + (n - g)\delta_{i_2, n-g}P(i_1 + 1, i_2).
 \end{aligned}$$

At this point we make the convenient change of index:  $i_2 = i_2 - (n - g)$ . This gives a simplified general balance equation:

$$[b + c\bar{\delta}_{i_2, g} + (1 - \delta_{i_1, 0}\delta_{i_2, 0})(n - g + i_2)] P(i_1, i_2) = \tag{e1}$$

$$(n - g + 1 + i_2)\bar{\delta}_{i_2, g}P(i_1, i_2 + 1) \tag{e2}$$

$$+ c\bar{\delta}_{i_2, 0}P(i_1, i_2 - 1) \tag{e3}$$

$$+ b\bar{\delta}_{i_1, 0}P(i_1 - 1, i_2) \tag{e4}$$

$$+ (n - g)\delta_{i_2, 0}P(i_1 + 1, i_2) \tag{e5}$$

We now sum the above expression for all values of  $i_1$  ranging from 0 to  $\infty$  and all values of  $i_2$  ranging from 0 to  $g$ . If we multiply each term of the sum by the corresponding factor  $Z_1^{i_1} Z_2^{i_2}$  we then get the Z-Transform equation. Furthermore,

if  $F(Z_1, Z_2)$  is the overall Z-Transform and the  $F_i(Z_1, Z_2)$ 's are the restricted Z-Transforms corresponding to each  $(e_i)$  ( $i = 1, \dots, 5$ ), we get:

$$(e1) \implies F_1(Z_1, Z_2) = [b + c + n - g]F(Z_1, Z_2) + Z_2 \frac{\partial F(Z_1, Z_2)}{\partial Z_2} - cZ_2^g \left( \sum_{i_1=0}^{\infty} P(i_1, g) Z_1^{i_1} \right) - (n - g)P(0, 0)$$

$$(e2) \implies F_2(Z_1, Z_2) = (n - g)Z_2^{-1}F(Z_1, Z_2) + \frac{\partial F(Z_1, Z_2)}{\partial Z_2} - (n - g)Z_2^{-1} \sum_{i_1=0}^{\infty} P(i_1, 0) Z_1^{i_1}$$

$$(e3) \implies F_3(Z_1, Z_2) = cZ_2F(Z_1, Z_2) - cZ_2^{(g+1)} \sum_{i_1=0}^{\infty} P(i_1, g) Z_1^{i_1}$$

$$(e4) \implies F_4(Z_1, Z_2) = bZ_1F(Z_1, Z_2)$$

$$(e5) \implies F_5(Z_1, Z_2) = (n - g)Z_1^{-1} \sum_{i_1=0}^{\infty} P(i_1, 0) Z_1^{i_1} - (n - g)Z_1^{-1}P(0, 0)$$

We can now use the fact that we have:

$$F_1(Z_1, Z_2) = F_2(Z_1, Z_2) + F_3(Z_1, Z_2) + F_4(Z_1, Z_2) + F_5(Z_1, Z_2)$$

as well as the following relations:

- $\sum_{i_1=0}^{\infty} P(i_1, g) Z_1^{i_1} = \frac{1}{g!} \frac{\partial^g F(Z_1, Z_2)}{\partial Z_2^g}$
- $P(0, 0) = F(0, 0)$
- $\sum_{i_1=0}^{\infty} P(i_1, 0) Z_1^{i_1} = F(Z_1, 0)$

We obtain:

$$\begin{aligned} & [(b + c + n - g) - (n - g)Z_2^{-1} - cZ_2 - bZ_1] F(Z_1, Z_2) + (Z_2 - 1) \frac{\partial F(Z_1, Z_2)}{\partial Z_2} \\ & - \frac{cZ_2^g}{g!} (1 - Z_2) \frac{\partial^g F(Z_1, Z_2)}{\partial Z_2^g} = (n - g)F(0, 0) - (n - g)Z_1^{-1}F(0, 0) \\ & + (n - g)Z_1^{-1}F(Z_1, 0) - (n - g)Z_2^{-1}F(Z_1, 0) \end{aligned}$$

After multiplication by a factor  $Z_1 Z_2$  we get the following equation:

$$\begin{aligned}
 & [aZ_1 Z_2(1 - Z_1 - Z_2) + (n - g)Z_1(Z_2 - 1)] F(Z_1, Z_2) \\
 & + Z_1 Z_2(Z_2 - 1) \frac{\partial F(Z_1, Z_2)}{\partial Z_2} - \frac{cZ_1 Z_2^{(g+1)}}{g!} (1 - Z_2) \frac{\partial^g F(Z_1, Z_2)}{\partial Z_2^g} = \\
 & (n - g)(Z_2 - Z_1)F(Z_1, 0) + (n - g)Z_2(Z_1 - 1)F(0, 0) \qquad (e6)
 \end{aligned}$$

where:

$$\bullet \quad a = b + c, \quad \bullet \quad b = \frac{\lambda}{\mu}, \quad \bullet \quad c = \frac{\gamma}{\mu}.$$

Equation (e6) is the  $g^{th}$  order differential satisfied by the Z-Transform of the modified system described in Figure 6.2. For simplicity, the index  $i_2$  now ranges from 0 to  $g$  instead of  $n - g$  to  $n$ .

## APPENDIX F.

### JUSTIFICATION OF THE Z-TRANSFORM METHOD

In this appendix we justify more rigorously the fact that the coefficients given by Equation (6.7) of Section 6.3 actually satisfy Equation (6.1) of Section 6.2.

Recall that we started by summing Equation (6.1) for all values of the index  $k$  ranging from 0 to  $\infty$ . This led us to the differential equation given by Equation (6.2). In between, we took a critical step by assuming the existence of the Z-Transform associated to the sequence of the  $q_k$ 's. This allowed us to identify the  $q_k$ 's with the coefficients of the asymptotic power expansion of the solution  $Q(Z)$  we obtained. However, the Z-Transform associated with the  $q_k$ 's will typically have an empty region of convergence, as we can see through a simple example.

Let us choose the particular case  $b = c = \frac{1}{2}$  for our traffic coefficients. Note that since the total number of channels  $n$  can be chosen as large as we wish, this is not a very constraining situation from a traffic point of view. With these values for  $b$  and  $c$ , Equation (6.1) can be rewritten in the form:

$$q_k = 2(n + 2 - k)(q_{k-1} - q_{k-2}), \quad k \geq 2.$$

Let us work out the value of a few terms, starting from  $k = n + 2$ .

$$q_{n+2} = 0$$

$$q_{n+3} = 2q_{n+1}$$

$$q_{n+4} = -2 \times 2^2 q_{n+1}$$

$$q_{n+5} = 3 \times 2[2 \times 2^2 + 2]q_{n+1}$$

$$= [3! \times 2^3 + \dots]q_{n+1}$$

$$q_{n+6} = -4 \times 2[3! \times 2^3 + \dots]q_{n+1}$$

$$= -[4! \times 2^4 + \dots]q_{n+1}$$

$$q_{n+7} = -5 \times 2[-4! \times 2^4 - \dots]q_{n+1}$$

$$= [5! \times 2^5 + \dots]q_{n+1}$$

In general we easily get:

$$|q_{n+k+2}| \geq k!2^k |q_{n+1}|.$$

The above inequality clearly forces the power series associated to the  $q_k$ 's to have an empty region of convergence, unless  $q_n$  is 0 from some point on. This is due to the fact that if it were to converge in a given region, it would then be absolutely convergent inside this region. This would then imply that the series  $k!2^k Z^k$  converges for  $Z$  small enough, which is clearly false.

Despite this empty region of convergence of the Z-Transform for most cases, we still claim that the coefficients given by Equation (6.7) do satisfy Equation (6.1) with its initial condition. This can be checked by means of examples, but it also has a rigorous explanation.

Let us consider Equation (6.2) independently of how it was obtained. We have a first order differential equation with initial condition  $Q(0) = q_0$ . We will try to obtain a solution that is continuous at the origin. If we find one, Equation (6.3) then immediately tells us that this solution would be continuous in the whole region  $|Z| < 1$ .

Suppose (to be shown later) that our solution has an asymptotic power expansion at the origin up to rank  $n$  of the form:

$$\begin{aligned} Q(Z) &= \sum_{k=0}^n t_k Z^k + A(Z) \\ &\asymp \sum_{k=0}^n t_k Z^k + O(Z^{n+1}), \end{aligned} \tag{f1}$$

where

$$A(Z) \asymp O(Z^{n+1}).$$

Here we assume that the term  $A(Z)$  is differentiable at the origin and such that:

$$\frac{d[A(Z)]}{dZ} \asymp O(Z^n). \tag{f2}$$

We can then write the asymptotic expansion:

$$\begin{aligned}\dot{Q}(Z) &= \sum_{k=1}^n kt_k Z^{k-1} + \frac{d[A(Z)]}{dZ} \\ &\asymp \sum_{k=0}^{n-1} (k+1)t_{k+1} Z^k + O(Z^n).\end{aligned}$$

We can now use the expressions of  $Q(Z)$  and  $\dot{Q}(Z)$  in Equation (6.2), which we know must be satisfied since  $Q(Z)$  was obtained as its solution. This gives:

$$\begin{aligned}&\left[ \sum_{k=0}^n t_k Z^k + O(Z^{n+1}) \right] [1 - \alpha Z + \beta Z^2] \\ &+ \left[ \sum_{k=0}^{n-1} (k+1)t_{k+1} Z^k + O(Z^n) \right] \delta Z^2 [1 - Z] \asymp q_0 [1 - z]\end{aligned}$$

Now by continuity we can identify the coefficients of  $Z^k$  for  $k = 0, \dots, n-1$ . This gives:

$$\bullet t_1 - \alpha t_0 = -t_0$$

$$\Rightarrow t_1 = (\alpha - 1)t_0$$

$$\bullet t_0 = q_0$$

and for  $2 \leq k \leq n-2$ ,

$$\bullet t_k - \alpha t_{k-1} + \beta t_{k-2} + \delta(k-1)t_{k-1} - \delta(k-2)t_{k-2} = 0$$

$$\Rightarrow t_k = (\alpha + \delta - \delta k)t_{k-1} - (\beta + 2\delta - \delta k)t_{k-2}$$

Here we have:

$$\begin{aligned}\bullet \alpha + \delta - \delta k &= nc^{-1} + (r+1) + c^{-1} - c^{-1}k \\ &= (n+1)c^{-1} + (r+1) - c^{-1}k \\ &= \bar{\alpha} - k\bar{\delta}\end{aligned}$$

$$\begin{aligned}\bullet \beta + 2\delta - \delta k &= nc^{-1} + 2c^{-1} - c^{-1}k \\ &= (n+2)c^{-1} - c^{-1}k \\ &= \bar{\beta} - k\bar{\delta}\end{aligned}$$

$$\bullet \alpha - 1 = nc^{-1} + r, \quad \text{where } r = \frac{b}{c}$$

This allows us to conclude that the  $t_k$ 's satisfy Equation (6.1) and its initial condition. We now have to prove that the solution  $Q(Z)$  given by Equation (6.5) satisfies the conditions (f1) and (f2).

Recall that we proved in Section 6.2 that the function  $Q(Z)$  given by Equation (6.5) was a continuous solution of the differential Equation (6.2) for  $|Z| < 1$ . There, we had:

$$Q(Z) = q_0(a) + q_0(b),$$

$$(a) = c^{-b} \frac{Z^{(n+b)}}{(1-Z)^b} e^{-c[Z^{-1}-1]} \Gamma(b, c[Z^{-1}-1]) \cdot \Sigma,$$

$$(b) = Z^n \left[ \sum_{i=0}^n \binom{n}{i} c^{-i} \left( \sum_{j=0}^i (b+j+1)_{i-j} (c[Z^{-1}-1])^j \right) \right],$$

and  $\Sigma$  is a finite summation of constant terms.

Part (b) provides us with the desired asymptotic expansion of  $Q(Z)$  of rank  $n$  at the origin. But, for the expansion to be of the form given by (f1), we need to prove that (a) is  $O(Z^{n+1})$  and that it satisfies (f2). As we saw in Section 6.2, we have the following asymptotic expansion for  $\Gamma(\alpha, x)$  as  $x \rightarrow \infty$  ([49], p. 135 Eq. 6):

$$\Gamma(\alpha, x) \asymp x^{(\alpha-1)} e^{-x} (1 + O(x^{-1})).$$

In our case we have:

$$Z \rightarrow 0 \iff c[Z^{-1}-1] \rightarrow \infty.$$

We can then write:

$$\begin{aligned} \Gamma(b, c[Z^{-1}-1]) &\asymp c^{(b-1)} (Z^{-1}-1)^{(b-1)} e^{-c[Z^{-1}-1]} (1 + O(Z)), \\ &\asymp c^{(b-1)} Z^{(-b+1)} (1-Z)^{(b-1)} (1 + O(Z)). \end{aligned} \tag{f3}$$

If we use Equation (f3) in the expression of (a), we get:

$$\begin{aligned} (a) &\asymp \frac{q_0 c^{-1}}{(1-Z)} \cdot \Sigma \cdot Z^{n+1} (1 + O(Z)), \\ &\asymp O(Z^{n+1}). \end{aligned}$$



We now have to verify that (a) is differentiable at the origin and satisfies (f2).

We can note that (a) is formed by the product of four functions of  $Z$ .

$$\begin{aligned} f_1(Z) &= \Sigma \cdot Z^{n+b}, \\ f_2(Z) &= (1-Z)^{-b}, \\ f_3(Z) &= e^{c[Z^{-1}-1]}, \\ f_4(Z) &= c^{-b} \Gamma(b, c[Z^{-1}-1]). \end{aligned}$$

So if we differentiate (a) we get:

$$(\dot{a}) = D_1 + D_2 + D_3 + D_4.$$

where

$$\begin{aligned} D_1 &= \dot{f}_1(Z) f_2(Z) f_3(Z) f_4(Z), \\ &= \frac{(n+b)Z^{(n+b-1)}}{(1-Z)^b} \cdot \Sigma \cdot e^{c[Z^{-1}-1]} c^{-b} \Gamma(b, c[Z^{-1}-1]), \\ &\asymp \frac{(n+b)Z^n}{(1-Z)} c^{-1} \cdot \Sigma \cdot (1 + O(Z)), \\ &\asymp O(Z^n). \end{aligned}$$

Here the last two equalities are obtained by using Equation (f3). Continuing, we have:

$$\begin{aligned} D_2 &= f_1(Z) \dot{f}_2(Z) f_3(Z) f_4(Z), \\ &= b \frac{Z^{(n+b)}}{(1-Z)^{(b+1)}} \cdot \Sigma \cdot e^{c[Z^{-1}-1]} c^{-b} \Gamma(b, c[Z^{-1}-1]), \\ &\asymp bc^{-1} \cdot \Sigma \cdot \frac{Z^{n+1}}{(1-Z)^2} (1 + O(Z)), \\ &\asymp O(Z^{n+1}). \end{aligned}$$

The last two equalities have also been obtained using (f3). Again,

$$\begin{aligned} D_3 &= f_1(Z) f_2(Z) \dot{f}_3(Z) f_4(Z), \\ &= -\Sigma \cdot \frac{Z^{(n+b)}}{(1-Z)^b} c Z^{-2} e^{c[Z^{-1}-1]} c^{-b} \Gamma(b, c[Z^{-1}-1]), \\ &\asymp -\Sigma \cdot \frac{Z^{n-1}}{(1-Z)} (1 + O(Z)), \\ &\asymp -\Sigma \cdot \frac{Z^{n-1}}{(1-Z)} + O(Z^n). \end{aligned}$$

Here again, the last two equalities were obtained by using (f3).

In order to compute  $D_4$ , we will make use of the following relation ([49], p. 135 Equation 8):

$$\frac{d\Gamma(\alpha, x)}{dx} = -x^{(\alpha-1)}e^{-x}.$$

This gives:

$$\begin{aligned} D_4 &= f_1(Z)f_2(Z)f_3(Z)f_4(Z), \\ &= \Sigma \cdot \frac{Z^{(n+b)}}{(1-Z)^b} e^{c[Z^{-1}-1]} c^{-b} c^b (1-Z)^{(b-1)} Z^{-(b+1)} e^{-c[Z^{-1}-1]}, \\ &= \Sigma \cdot \frac{Z^{n-1}}{(1-Z)}. \end{aligned}$$

So we finally get:

$$\begin{aligned} (\dot{a}) &= D_1 + D_2 + D_3 + D_4, \\ &\asymp O(Z^n) + O(Z^{n+1}) + O(Z^n) - \Sigma \cdot \frac{Z^{n-1}}{(1-Z)} + \Sigma \cdot \frac{Z^{n-1}}{(1-Z)}, \\ &\asymp O(Z^n). \end{aligned}$$

This is the desired result, and this completes the proof that the coefficients given by Equation (6.7) of Section 6.2 satisfy Equation (6.1).

Note that the fact that we were not able to extend the result for  $k = n$  simply says that this method cannot be used when all channels are guard channels. Such a system is however a simple blocking system for customers of type I and can therefore easily be solved by classical methods. Furthermore, if the system is considered as a service facility for the two types of customers I and II, it is non-ergodic. This is due to the fact that customers of type II will never be served (no channels available to them) and thus the queue length will diverge if the type II-arrival rate is not 0.

We can also note that the above proof will also be valid for the case of the general state probabilities of Section 6.7. The reason for this is that the solution of the corresponding differential Equation (6.22) has the exact same form as in Section 6.2, except for a multiplicative constant.

## REFERENCES

- [1] W. R. Young; "Advanced mobile phone service: Introduction, background, and objectives." *Bell Syst. Tech. J.*, vol. 58, pp. 1-14, January 1979.
- [2] V. H. Mac Donald; "The cellular concept." *Bell Syst. Tech. J.*, vol. 58, pp. 15-42, January 1979.
- [3] K. Daikoku and H. Ohdate; "Optimal design for cellular mobile systems." *IEEE Trans. Veh. Technol.*, vol. VT-34, pp. 3-12, February 1985.
- [4] *Autoplex Cellular Telecommunications Systems*. System 100 fundamentals, Western Electric, 1983.
- [5] M. Sengoku; "Telephone traffic in a mobile radio communication system using dynamic frequency assignments." *IEEE Trans. Veh. Technol.*, vol. VT-29, pp. 270-278, May 1980.
- [6] M. Sengoku, M. Kurata and Y. Kajitani; "Rearrangement of switching networks and its application to mobile radio communication system." *Proc. Int. Conf. Circuits Comput., ICCO 80*, October 1980, vol. 1, pp. 402-405.
- [7] K. Suzuki, E. Niikura and N. Morita; "A new method which optimizes frequency reuse in cellular radio systems." *Proc 34th Veh. Technol. Conf.*, Pittsburgh, May 1984, pp. 322-327.
- [8] H. Pfannschmidt and G. Rolle; "Architecture and Radio Technology of EC 900 - A new high capacity automatic mobile telephone system." *Proc. Int. Switch. Sympos. (ISS 84)*, Florence, Italy, May 1984, Session 32 B, paper 2.
- [9] J. F. Whitehead; "Cellular spectrum efficiency via reuse planning." *Proc. 35th Veh. Technol. Conf.*, Boulder, May 1985, pp. 16-20.

- [10] R. W. Lorenz; "Frequency planning of cellular radio by the use of topographical data base." *Proc. 35th Veh. Technol. Conf.*, Boulder, May 1985, pp. 1-5.
- [11] H. Sekiguchi, H. Ishikawa, M. Koyama and H. Sawada; "Techniques for increasing frequency spectrum utilization in mobile radio communication system." *Proc. 35th Veh. Technol. Conf.*, Boulder, May 1985, pp. 26-31.
- [12] C. E. Sundberg; "On continuous phase modulation in cellular digital mobile radio system." *Bell Syst. Tech. J.*, vol. 62, pp. 2067-2089, September 1983.
- [13] C. Stevenson; "An improved modulation format and signal processing scheme for ACSB systems." *Proc. 34th Veh. Technol. Conf.*, Pittsburgh, May 1984, pp. 109-113.
- [14] S. Carney; "Co-channel interference comparison for conventional land mobile frequency modulation and 2400 BPS LPC-FSK system." *Proc. 34th Veh. Technol. Conf.*, Pittsburgh, May 1984, pp. 131-135.
- [15] S. Goode; "A comparison of Gaussian minimum shift keying to frequency shift keying for land mobile radio." *Proc. 34th Veh. Technol. Conf.*, Pittsburgh, May 1984, pp. 136-141.
- [16] K. -S. Chung; "Generalized tamed frequency modulation and its application for mobile radio communications." *IEEE J. Select. Areas Commun.*, Special Issue on Mobile Radio Communications, vol. SAC-2, pp. 487-497, July 1984.
- [17] D. D. Crombie; "Review of protection measurements and spectrum efficiencies on ACSB and FM mobile radio systems." *Proc. 35th Veh. Technol. Conf.*, Pittsburgh, May 1984, pp. 211-217.
- [18] K. Murota; "Spectrum efficiency of GMSK land mobile radio." *IEEE Trans. Veh. Technol.*, vol. VT-34, pp. 69-75, May 1985.

- [19] C. E. Sundberg; "Alternative cell configuration for digital mobile radio systems." *Bell Syst. Tech. J.*, vol. 62, pp. 2037-2065, September 1983.
- [20] Y. -S. Yeh and D. O. Reudink; "Efficient spectrum utilization for mobile radio systems using space diversity." *IEEE Trans. Commun.*, vol. COM-30, pp. 447-455, March 1982.
- [21] *IEEE Commun. Mag.*, Special Issue on Mobile Radio Communications, vol. 24, February 1986.
- [22] P. S. Henry and B. S. Glance; "A new approach to high-capacity digital mobile radio." *Bell Syst. Tech. J.*, vol. 60, pp. 1891-1904, October 1981.
- [23] B. S. Glance; "BER degradations caused by switching in digital mobile radio systems using base station diversity." *AT&T Bell Lab. Tech. J.*, vol. 63, pp. 545-564, April 1984.
- [24] T. Miki and H. Mata; "Performance of 16 kbits/sec GMSK transmission with postdetection selection diversity in land mobile radio." *IEEE J. Select. Areas Commun.*, Special Issue on Mobile Radio Communications, vol. SAC-2, pp. 512-517, July 1984.
- [25] J. H. Winters; "Optimum combining in digital mobile radio with cochannel interference." *IEEE J. Select. Areas Commun.*, Special Issue on Mobile Radio Communications, vol. SAC-2, pp. 528-539, July 1984.
- [26] G. L. Turin; "Commutation signaling-An antimultipath technique." *IEEE J. Select. Areas Commun.*, Special Issue on Mobile Radio Communications, vol. SAC-2, pp. 548-562, July 1984.
- [27] D. Verhulst, M. Mouly and J. Szpirglas; "Slow frequency hopping multiple access for digital cellular radiotelephone." *IEEE J. Select. Areas Commun.*, Special Issue on Mobile Radio Communications, vol. SAC-2, pp. 563-574, July 1984.

- [28] *Proc. Nordic Seminar on Digital Land Mobile Radiotelecommunications*. Espoo, Finland, February 1985.
- [29] K. Chaddock; "A proposed architecture for a mobile telephone system using multi-beam satellite coverage." *Proc. 34th Veh. Technol. Conf.*, Pittsburgh, May 1984, pp. 104-108.
- [30] V. O. K. Li and T. -Y. Yan; "Adaptive mobile access protocol (AMAP) for the message service of a land mobile satellite experiment (MSAT-X)." *IEEE J. Select. Areas Commun.*, Special Issue on Mobile Radio Communications, vol. SAC-2, pp. 621-627, July 1984.
- [31] W. J. Vogel and E. K. Smith; "Theory and measurements of propagation for satellite to land mobile communication at UHF." *Proc. 34th Veh. Technol. Conf.*, Boulder, May 1985, pp. 218-226.
- [32] A. Salmasi; "Omninet's STARSAT-An advanced mobile satellite communications system design." *Proc. 34th Veh. Technol. Conf.*, Boulder, May 1985, pp. 232-237.
- [33] G. R. Cooper and R. W. Nettleton; "A spread-spectrum technique for high-capacity mobile communications." *IEEE Trans. Veh. Technol.*, vol. VT-27, pp. 264-275, November 1978.
- [34] D. J. Goodman, P. S. Henry and V. K. Prabhu; "Frequency-hopped multilevel FSK for mobile radio." *Bell Syst. Tech. J.*, vol. 59, pp. 1257-1275, September 1980.
- [35] W. W. Chapman; "Spread spectrum applications to mobile radio: Past, present, future." *IEEE Veh. Technol. Newsletter*, February 1985.

- [36] E. A. Gerianotis and M. B. Pursley; "Performance of coherent direct-sequence spread-spectrum communications over specular multipath fading channel." *IEEE Trans. Commun.*, vol. COM-33, pp. 502-508, June 1985.
- [37] E. A. Gerianotis and M. B. Pursley; "Performance of non-coherent direct-sequence spread-spectrum communications over specular multipath fading channel." *IEEE Trans. Commun.*, vol. COM-34, pp. 219-226, March 1986.
- [38] *Federal Communications Commission (FCC)*; Washington, D. C. 20554, Report No. 81-161 29244, §11.
- [39] *Federal Communications Commission, Technical Standards Branch, U.S. Office of Science and Technology (OST)*; Washington, D. C. 20554, OST Bulletin No. 53.
- [40] *Advanced mobile phone service—A brief system description*. Technical documentation, Bell Laboratories.
- [41] H. Inose; *An Introduction to Digital Integrated Communications Systems*. University of Tokyo Press, 1979.
- [42] L. Kleinrock; *Queueing Systems, Volume 1: Theory*. Wiley, New York, 1975.
- [43] O. Avellaneda, R. Pandya and G. Brody; "Traffic modeling of a cellular mobile radio system." *Proc. 11th Int. Teletraffic Congress (ITC 11)*, Kyoto, Japan, September 1985, vol. 1, pp. 2.4B-4-1-2.4B-4-7.
- [44] H. Blecher; "Advanced mobile phone service." *IEEE Trans. Veh. Technol.*, vol. VT-29, pp. 238-244, May 1980.
- [45] W. H. Beyer; *Handbook of Tables for Probability and Statistics*. The Chemical Rubber Co, Cleveland, Ohio, 1966.
- [46] B. V. Gnedenko; *The Theory of Probability*. Translated by B. D. Seckler, Chelsea Publishing Company, New York 1962.

- [47] J. N. Franklin; *Matrix Theory*. Prentice-Hall, Englewood Cliffs, New Jersey, 1968.
- [48] I. S. Gradshteyn and I. M. Ryzhik; *Table of Integrals, Series and Products*. Translated from Russian by Academic Press, New York and London, 1965.
- [49] Bateman Manuscript Project; *Higher Transcendental Functions, Volume II*. McGraw-Hill, New York, 1953.
- [50] W. Feller; *An Introduction to Probability Theory and its Applications, Volume 2*, Wiley, New York, 1966.
- [51] A. T. Bharucha-Reid; *Elements of the Theory of Markov Processes and their Applications*. McGraw-Hill, New York, 1960.
- [52] B. C. Kuo; *Automatic Control Systems*. Prentice-Hall, Englewood Cliffs, New Jersey, 1975.
- [53] H. D. Ide and R. G. Scherer; "On application and performance of cut-off priority queues in switching system with overload protection." *Proc. 11th Int. Teletraffic Congress (ITC 11)*, Kyoto, Japan, September 1985, vol. 1, pp. 2.1B-2-1-2.1B-2-7.