Essays on Matching Theory

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I dedicate this dissertation to my parents, my wife, and newborn son.

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ABSTRACT

Matching theory is a rapidly growing field in economics that often deals with markets in which monetary transfers are forbidden. Hence, policy makers often use centralized procedures to organize markets and coordinate players' behavior. Three concerns play central roles in designing the procedures: efficiency, fairness, and incentive compatibility. These concerns are also what I focus on in my studies. Specifically, my dissertation consists of three original studies on the allocation of indivisible resources to agents. The first chapter studies school choice, which is a centralized market to assign students to public schools. I compare popular matching mechanisms used in school choice by accommodating the fact that students and their parents often have heterogeneous sophistication in understanding the mechanisms. In the second chapter I study abstract object allocation problem in which objects do not have priority rankings of agents. I want to show that the three objectives of efficiency, fairness, and incentive compatibility can be incompatible with each other: a mechanism that satisfies a minimal efficiency requirement and mild fairness requirements must be manipulable by some group of agents in a strong sense. Since the efficiency requirement is weak enough such that policy makers are likely to pursue, my results suggest that policy makers have to make a choice between fairness and group incentive compatibility. In the third chapter I study same object allocation problems except that some agents have private endowments. I propose a new mechanism that has desirable properties in efficiency, fairness, and incentive compatibility. In the following I provide more details of each chapter.

School choice is a trend in the K-12 public education of US and many other countries that allows children to choose schools across neighborhoods. In Chapter 1, "Level-k Reasoning in School Choice", I compare two matching algorithms that many cities use to assign children to public schools in school choice. The algorithms are called Boston Mechanism and Deferred Acceptance. BM is manipulable, while DA is strategy-proof. Recently several cities decide to switch from BM to DA to avoid manipulation. However, the effect of the switch has not been well understood. In this paper I use the level-k model to study the strategies used by parents in BM by taking account of the fact that parents often have different abilities to manipulate BM, which are due to their heterogeneous sophistication. Interestingly, I find that the level-k reasoning process in BM is analogous to the procedure of DA. This analogy provides a new way to understand how parents may behave in BM. Under some mild

assumption it implies that for any school choice problem and any sophistication distribution of parents, the assignment found by BM is never less efficient than the assignment found by DA. I also examine how parents' beliefs about others' sophistication affect their welfare. I find that, in general, a child is guaranteed to benefit from his parent's sophistication in BM only when his parent's level is high relative to others and his parent's belief about others' sophistication levels is accurate. The simulation results of my model exhibit patterns similar to empirical datasets.

Without monetary transfers, the concern of fairness motivates policy makers to use random assignments in objection allocation problems. In Chapter 2, "Efficient and Fair Assignment Mechanism is Strongly Group Manipulable", I study group incentive compatibility in random assignment mechanisms. I show that if a mechanism satisfies the minimal efficiency requirement (ex-post efficiency), then it cannot satisfy some mild fairness requirements and be minimally group incentive compatible simultaneously: by misreporting preferences, a group of agents can obtain lotteries that strictly first-order stochastically dominate the lotteries they obtain in the truth-telling case. Hence, fairness concerns may force policy maker to give up group incentive compatibility. My results hold as long as there are at least three agents and at least three objects, no matter outside option is available or not. Possibility results exist when there are only two objects and outside option is not available.

In some object allocation problems, some players have private endowments and are willing to bring them to the market in exchange for better ones. In Chapter 3, "A New Solution to the Random Assignment Problem with Private Endowment", I propose a new mechanism to solve the problems. Intuitively, in my mechanism the popularity of a private endowment plays the role of "price" in determining his owner's advantage in the market. Formally, the mechanism is a simultaneous eating algorithm, which generalizes Probabilistic Serial, by letting agents obtain additional eating speeds if their private endowments are consumed by others, and letting multiple agents trade their private endowments if they form cycles. This feature can be summarized by the idea of "you request my house - I get your speed". Indifferent preferences often cause difficulty in efficient random assignment mechanisms. Interestingly, I show that the same idea can also be used to deal with indifferent preferences in a straightforward way. It is in contrast to the mainstream method of iteratively solving maximum network flow problems in the literature.

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Chapter 1

LEVEL-K REASONING IN SCHOOL CHOICE

1.1 Introduction

Many countries of the world provide public education, which allows students to attend schools for free. In the traditional system of K-12 public education in many countries, students are simply assigned to schools according to their home locations. This has been changed in the recent school choice trend: students (actually their parents) can submit their preferences over schools to a school choice office, then the office runs a computer algorithm to find an assignment. It is believed that the freedom of express preferences gives parents more control over their children's education, and improves diversity in public schools. From the perspective of economics, a computer algorithm is a mechanism that maps the submitted preferences of students to an assignment. Therefore, school choice is a game for students.

There are two algorithms that are widely used in school choice: the *Boston Mechanism* and the *student-proposing deferred acceptance*. BM is a status quo algorithm which has existed for a long time in many cities. DA is a new algorithm, which was first proposed by Gale and Shapley (1962) and then adapted by Abdulkadiroğlu and Sönmez (2003) to school choice. DA is strategy-proof, which means that reporting true preferences is a weakly dominant strategy for students. But BM is manipulable, which means that students may obtain better assignments by reporting non-truthful preferences. This difference motivates some cities to switch from BM to DA. A lot of studies compare the two algorithms and want to find which one produces a better assignment for students. However, a major difficulty is that the strategies used by students in BM have not been understood well. Field and lab evidence shows that students are often boundedly rational and have different abilities to manipulate BM. Specifically, some students do not realize that school choice is a game, so they always report true preference. The remaining may realize it, but some of them play better strategies than the others. Thus, the purpose of this paper is to explore the strategies used by students in BM when they have heterogeneous sophistication, and to compare the two algorithms.

To model heterogeneous sophistication I use a nonequilibrium model called level-k. The model was first proposed by Stahl and Wilson (1994, 1995) and Nagel

(1995), and then developed by Ho, Camerer, and Weigelt (1998), Costa-Gomes, Crawford, and Broseta (2001), Costa-Gomes and Crawford (2006), Crawford and Iriberri (2007a, 2007b), and Arad and Rubinstein (2012), among many others. The model shows a good explanatory power in many experiments. In the school choice game, the setup of the model is as follows. Every student has a discrete sophistication level, which is his depth of strategic reasoning. If a student's level is zero, he is naive and does not make any strategic reasoning. If a student's level is a positive integer k, he makes strategic reasoning and chooses a best strategy based on his belief about others' levels. In the paper I consider two extreme settings of the beliefs of positive-level students. This consideration enables me to check the robustness of my results, and also examine the effect of beliefs on students' strategies and welfare. In the first setting, a level-k student believes that all others are level-k-1 irrespective of their true levels. It is the setting commonly used in the literature, so I call the model with this setting *original level-k*. In this setting a level-k student may overestimate some students' levels and underestimate some others'. By contrast, in the second setting, I let students have as accurate beliefs as possible. However, the spirit of the level-k model requires that a level-k student cannot believe that any other's level is k or higher.[1] So in the second setting, any level-k student has a correct belief about the levels of those whose levels are lower than k, and believes the remaining are level-k-1. I call the corresponding model the *informational level-k*.

Both DA and BM are preference revelation games. Since DA is strategy-proof, students report true preferences even though they have heterogeneous sophistication levels. So my main task in the paper is to use the two level-k models to analyze students's strategies in BM. Hence, in BM level-0 students report true preferences, while positive-level students behave strategically. To study the problem in the most transparent environment, I assume a complete information environment. That is, students commonly know each other's preferences and the priority rankings of students at all schools. Complete information is a widely used assumption in the matching theory literature. In this paper it removes the effect of probabilistic beliefs and risk attitudes on students' strategies. So I can focus on the effect of heterogeneous sophistication. In the following I briefly discuss my main results.

First, in both level-k models of BM, when students reason about the others' strategies, they essentially reason about the most preferred schools reported by the others. This is caused by the feature of BM that each school first admits those

[1] Otherwise, the level-k student should make more steps of reasoning and effectively has a level higher than k.

who report it as most preferred. Then I show that the reasoning process in both level-k models of BM is analogous to the procedure of DA. Specifically, when a positive-level student thinks about his best strategy, it is as if he runs some rounds of DA in his mind to decide which school he should report as most preferred; the higher is his level, the more rounds of DA he runs in his mind. I illustrate it in Section 1.3 through an example. Formal analysis is in Section 1.4 and Section 1.5.

Second, I compare the assignments found by DA and BM. I find that in both level-k models the assignment found by BM is not strictly Pareto dominated by the assignment found by DA for any level distribution of students. Specifically, in both level-k models of BM a student always reports a school weakly better than his assignment in DA as most preferred. So it is impossible that all students obtain worse assignments in BM than in DA. When all students have levels above some thresholds defined in the paper, all students report their assignments in DA as most preferred. Hence, BM finds the same assignment as DA does. If I further make a mild assumption about the strategies of positive-level students, then I prove that the assignment of BM is never Pareto dominated by that of DA.

Last, I examine the effect of a student's sophistication level and belief on his welfare in BM. This is to address the concern in practice that in BM, sophisticated students may take advantage of naive students and obtain better assignments. I find that in the original level-k model, a student of a higher level is not guaranteed to obtain a better assignment. It is because if a student has a higher level, he may overestimate more others' levels and choose an overcautious strategy. By contrast, in the informational level-k model a student never overestimates the others' levels. So when a student's level is sufficiently high, he has a correct belief about the true levels of most others. Therefore, he can choose a truly best strategy. I show that students of sufficiently high levels must have weakly better assignments in BM than in DA. So in this sense they have advantage in BM. Moreover, the contrast between the two models shows that correct beliefs play a crucial role in the existence of the advantage.

To quantify the difference between the assignments of DA and BM, I simulate it by randomly generating school choice problems and sophistication levels of students. The result shows that neither algorithm dominates the other. Specifically, there are always some percentage of students who prefer the assignments in BM and some percentage of students who prefer the assignments in DA. Both percentages are significantly above zero. However, the former percentage is often higher than the latter, so more students prefer BM. To examine the effect of sophistication in BM, I look at the assignments of students at each sophistication level and compare them with those in DA. I find that the average assignment of $L0$ students is worse than that of students of any positive level in BM, and is also worse than that of $L0$ students in DA. By contrast, the average assignment of students of any positive level is better in BM than in DA. So this implies that in BM naive students have a disadvantage while sophisticated students have an advantage. But there is a difference between the two level-k models of BM. In the original level-k model the average assignment of students is single-peaked, while in the informational level-k model the average assignment of students is monotonic in their levels. Hence, correct beliefs are beneficial to students. My simulation results are similar in some respects to recent empirical studies of He (2014) and Calsamiglia, Fu, and Güell (2015). Both papers estimate that replacing BM with DA will hurt more students than helping them, and an average student be worse off. But they do not estimate the possible heterogeneous sophistication distribution among strategic students.

In the rest of the paper, I present the school choice model in Section 1.2. Then I provide an example to illustrate BM and DA and the two level-k models in Section 1.3. The two level-k models of BM are in Section 1.4 and Section 1.5. Section 1.6 includes some discussions about models. Section 1.7 presents simulation results. Section 1.8 includes some extensions of the models. I discuss related literature in Section 1.9. Section 1.10 concludes. The appendix includes omitted proofs and additional results.

1.2 School Choice Model

A school choice problem consists of the following elements:

- a finite set of students I;
- a finite set of schools S;
- a capacity vector $Q_S = \{q_s\}_{s \in S}$ where q_s is the number of seats at school s;
- a priority profile of schools $\Pi_S = \{\pi_s\}_{s \in S}$ where π_s is the strict priority ranking of school s over students;
- a preference profile of students $P_I = \{P_i\}_{i \in I}$ where P_i is the strict preference ordering of student i over schools. Let R_i denote the associated weak preference ordering.

There are enough seats to admit all students such that $\sum_{s\in S} q_s = |I|$. This accommodates two cases. First, the law in many cities requires each student attend a public school, so it is natural to assume enough seats. Second, if students have outside options (private schools or studying at home), some school in S denotes the outside options. An *assignment* of students to schools is a function $\mu : I \rightarrow S$ such that $|\mu^{-1}(s)| \leq q_s$ for all $s \in S$. Here, $\mu(i)$ is the assignment of each i and $\mu^{-1}(s)$ is the set of students admitted by each s. I denote the set of all assignments by $\mathcal{M}$. A student i *justified envies* another student j in an assignment μ if $\mu(j)P_i\mu(i)$ and $i\pi_{\mu(j)}j$. That is, i has a higher priority than j at school $\mu(j)$ but i is assigned to a school worse than $\mu(j)$. An assignment μ is *wasteful* if $|\mu^{-1}(s)| < q_s$ and $sP_i\mu(i)$ for some s and some i. That is, s has empty seats and i prefers s to his assignment. An assignment is *stable* if it does not contain justified envies and is not wasteful.

An assignment μ *Pareto dominates* another assignment μ' if $\mu(i)R_i\mu'(i)$ for all $i \in I$, and $\mu(j)P_j\mu'(j)$ for some $j \in I$. If $\mu(i)P_i\mu'(i)$ for all $i \in I$, μ *strictly Pareto dominates* μ'. An assignment is *Pareto efficient* if it is not Pareto dominated by any other assignment. I use $\mathcal{P}$ to denote the set of all strict preference orderings of S, and use $\mathcal{O}$ to denote the set of all school choice problems. In practice Π_S is regulated and known by the school choice office. So throughout the paper I fix I, S, Q_S, Π_S, and denote a school choice problem simply by P_I. A school choice algorithm is a function $\psi : \mathcal{O} \rightarrow \mathcal{M}$ such that $\psi(P_I)$ is the assignment found for P_I. ψ is *Pareto efficient* or *stable* if $\psi(P_I)$ is Pareto efficient or stable for all P_I. ψ is *strategy-proof* if reporting true preferences is a weakly dominant strategy for all students. Formally, $\psi(P_I)(i)\ R_i\ \psi(\{P'_i, P_{-i}\})(i)$ for all $i \in I$, all $P_I \in \mathcal{P}^{|I|}$ and all $P'_i \in \mathcal{P}$.

1.3 Illustrating Example

Consider a school choice problem that contains three students Alex, Bob, and Charlie, and three schools X, Y and Z. Each school has only one seat. Table 1.1 lists the preferences of students and the priority rankings at schools.

Alex	Bob	Charlie	X	Y	Z
X	X	Y	Charlie	Bob	Alex
Y	Y	X	Alex	Alex	Bob
Z	Z	Z	Bob	Charlie	Charlie

Table 1.1: Illustrating Example

1.3.1 Procedures of BM and DA

In school choice each student is required to submit a list of schools, which is supposed to be his preference ordering, to an office. Then the office runs an algorithm to find an assignment. BM and DA are two most popular algorithms. They are run as follows. In the first round the applications of all students are simultaneously sent to the first schools in their reported lists, and schools tentatively admit applicants according to priority rankings. Then, the applications of rejected students are simultaneously sent to the next schools in their reported lists in the next round, and so on. BM and DA have a crucial difference since the second round: in BM once a school has no empty seats in some round, it cannot admit new applicants even though they have higher priorities than students admitted in earlier rounds; but in DA schools only consider the priority rankings of applicants without considering the timing of receiving their applications. It is well-known that DA always finds the *student-optimal stable assignment*, which Pareto dominates any other stable assignment. I denoted it by μ^{DA}.

The Procedures of BM and DA

Round 1: Each student applies to the first school in his reported list. Each school tentatively admits its applicants one by one according to its priority ranking until its all seats are occupied or all applicants are admitted. Remaining applicants, if any, are rejected. If all students are admitted, stop the procedure and finalize all assignments.

Round $r \geq 2$: Each rejected student applies to the next school in his reported list.

- In **BM**, each school with empty seats tentatively admits its applicants one by one according to its priority ranking until its all seats are occupied or all applicants are admitted. Remaining applicants, if any, are rejected. If all students are admitted, stop the procedure and finalize all assignments.
- In **DA**, each school that receives new applications considers its earlier admitted students and new applicants and admits them one by one according to its priority ranking until its all seats are occupied or all students are admitted. Remaining students, if any, are rejected. If all students are admitted, stop the procedure and finalize all assignments.

If all students in the example submit true preferences, Table 1.2 summarizes the procedures of BM and DA by listing the school that each student applies to in each round. In particular, although Bob has a higher priority than Charlie at Y, Bob loses his chance at Y in BM because he applies to Y in the second round while Charlie applies to Y in the first round.

Round	BM			DA		
	Alex	Bob	Charlie	Alex	Bob	Charlie
1	X	X	Y	X	X	Y
2		Y			Y	
3		Z				X
4				Y		
5				Z		

Table 1.2: Procedures of BM and DA

In BM if a student ranks a school higher in his reported list, his application will be sent to the school in an earlier round. So in general when students want to manipulate BM, they often misreport their top ranked schools, especially their most preferred schools. In this example if Bob reports Y as first choice, he will be admitted by Y in BM.

1.3.2 Level-k Model of BM

I use two level-k models to analyze the strategies of students in BM in the complete information environment. In both models a level-0 student is naive and reports true preferences. In the original level-k model a level-k student for any $k > 0$ believes others are level-k-1 and chooses a best strategy. In the informational level-k model a level-k student has a correct belief about the levels of those whose levels are lower than k and believes the remaining are level-k-1, then chooses a best strategy. The following presents the reasoning processes in the example.

1.3.2.1 Original Level-k Model of BM in the Example

- Level 0: If a student is level-0, he reports his true preferences in BM. In particular, he reports his most preferred school as first choice.
- Level 1: If a student is level-1, he believes the others are level-0 and chooses a best strategy. In the complete information environment he knows the true preferences reported by the other students. Then if it is Alex, in his best strategy Alex must report X as first choice since he believes that Bob also

reports X as first choice. If it is Bob, in his best strategy Bob must report Y as first choice. Otherwise, he will not be admitted by X but also lose the chance at Y. If it is Charlie, in his best strategy Charlie must report Y as first choice. Otherwise, Charlie will be admitted by another school that he reports as first choice. However, for each student it is uncertain that how he reports the whole preference orderings in his best strategy.

- Level 2: If a student is level-2, he believes the others are level-1 and chooses a best strategy. In the complete information environment by conducting the above level-1 reasoning process in his mind, he knows the first choices reported by the others at level 1. Although he is uncertain about their whole reported preference orderings, it is sufficient for him to choose his best strategy in BM. If it is Alex, Alex knows that X is the school he wants to obtain by using a best strategy, and he can obtain X for sure by reporting it as first choice. So I assume that Alex just reports X as first choice.[2] If it is Bob, in his best strategy Bob must report Y as first choice since he believes that Charlie also reports Y as first choice. If it is Charlie, in his best strategy he must report X as first choice. Otherwise, he will not be admitted by Y but also lose the chance at X. However, for each student it is still uncertain that how he reports the whole preference orderings in his best strategy.

- Level 3: If a student is level-3, he believes the others are level-2 and chooses a best strategy. In the complete information environment by conducting the above reasoning process in his mind, he knows the first choices reported by the others at level 2. If it is Alex, Alex knows that Z is the school he wants to obtain by using a best strategy, and he can obtain Z for sure by reporting it as first choice. So by my assumption Alex just reports Z as first choice. If it is Bob, Bob knows that Y is the school he wants to obtain by using a best strategy, and he can obtain Y for sure by reporting it as first choice. So by my assumption Bob just reports Y as first choice. If it is Charlie, in this best strategy Charlie must report X as first choice since he believes that Alex also reports X as first choice.

- Level $k \geq 4$: If a student is level-4, he believes the others are level-3 and chooses a best strategy. In the complete information environment by con-

[2]If Alex believes that Charlie reports the preference ordering of $Y \succ Z \succ X$ at level 1, he believes he can also obtain X by reporting the preference ordering of $Y \succ X \succ Z$. Hence, it is not 100% sure that Alex must report X as first choice in his best strategy, and so I make the assumption. Arguments to support the assumption are in Section 6.

ducting the above reasoning process in his mind, he knows the first choices reported by the others at level 3. By my assumption Alex still reports Z as first choice, Bob still reports Y as first choice, and Charlie still reports X as first choice. It is easy to see that same conclusions also apply to all levels higher than 4.

There are two observations from the above procedure. First, in the reasoning process a level-k student essentially reasons about the first choices reported by the others at level k-1. Second, by looking at the first choices reported by students, the level-k reasoning process is analogous to the procedure of DA. I illustrate it by Table 1.3. For each round of DA and each student, the school I list is the school admitting the student in the previous round, or the school the student applies to in the round.

	Alex	Bob	Charlie
Level 0:	X	X	Y
Level 1:	X	Y	Y
Level 2:	X	Y	X
Level 3:	Z	Y	X
Level $k \geq 4$:	Z	Y	X

(a) Original level-k model of BM

	Alex	Bob	Charlie
Round 1:	X	X	Y
Round 2:	X	Y	Y
Round 3:	X	Y	X
Round 4:	Y	Y	X
Round 5:	Z	Y	X

(b) The procedure of DA

Table 1.3: The original level-k reasoning process is analogous to the procedure of DA

1.3.2.2 Informational Level-k Model of BM in the Example

In the informational level-k model a positive-level student's strategy depends on the levels of those whose levels are lower than him. Hence, I assume a level distribution: Alex is level-1, Bob is level-0, and Charlie is level-2. As before, in the reasoning process students essentially reason about the first choices reported by the others. So Table 1.4 lists the first choices reported by students at each level.

	Alex	Bob	Charlie
Level 0:	X	X	Y
Level 1:	X		Y
Level 2:			Y

Table 1.4: Informational level-k reasoning process of BM

Since Charlie believes that Bob is level-0, he reports Y as first choice and obtains Y in BM. By contrast, in the previous model Charlie would believe that Bob is level-

1, so he would report X as first choice and obtain it. So Charlie obtains a better assignment by having a correct belief. In the paper I show that the level-k reasoning process in this model is also analogous to the procedure of DA. But correct beliefs of high-level students can bring them better assignments.

1.4 The Original Level-k Model of BM

From now on if a student is level-k, I simply say he is Lk. In the original level-k model $L0$ students report true preferences. An Lk student for any $k > 0$ believes the others are $Lk-1$. In the main content of this paper, I assume that the preference profile and priority rankings are common knowledge among students. Under this assumption each Lk student can infer others' strategies at lower levels, then chooses a best strategy at his level. I call the school an Lk student wants to obtain by reporting a best strategy *the best obtainable school at* Lk for him. In general, best strategies are not unique. In particular, any preference ordering that lists the best obtainable school at Lk as first choice is a best strategy. In this paper I make the assumption that an Lk student always reports his best obtainable school at Lk as first choice, and this is common knowledge among all positive-level students.

Best strategy selection assumption: An Lk student for any $k > 0$ reports his best obtainable school at Lk as first choice, and this is commonly known by students at positive levels.

In the illustrating example of Section 1.3, an Lkstudent often has to report his best obtainable school at Lk as first choice in his all best strategies. This happens when he believes that there are enough many other student who also report his best obtainable school at Lk as first choice, and when he believes that if he reports another school as first choice, he must obtain it. In Section 1.6.3 I discuss the validity of this assumption. For most results of the paper I do not assume how positive-level students report the whole preference orderings. This not only make my results robust to any further assumption, but also captures the uncertainty of students in the level-k reasoning process about the others' whole reported preference orderings.

By slightly adjusting the procedure of DA, I prove that the reasoning process in the original level-k model of BM can be understood through the adjusted procedure. Formally, I use s_i^k to denote the school each student i reports as first choice at any Lk. I call the adjusted procedure *Fast DA* since it usually runs faster than DA. In any round of Fast DA when an unassigned student applies to a new school, he skips any school s that has admitted q_s students each of whom has a higher priority than him at s. Fast DA always find the same assignment as DA.

Fast Deferred Acceptance

Round $r \geq 0$: Each unassigned student i applies to his most preferred school s that he has not applied to and has not admitted q_s students who all have higher priorities than i at s. Each school tentatively admits students according to its priority ranking. If all students are admitted after this round, stop the procedure.

Note that I index the first round of Fast DA by 0. I denote the last round of Fast DA by r^{FDA}. Then for each i and each $k \geq 0$ I define

$$a_i^k \equiv \begin{cases} \text{the school admitting } i \text{ in round } k-1, & \text{if } i \text{ is admitted in round } k-1, \\ \text{the school } i \text{ applies to in round } k, & \text{if } i \text{ is rejected in round } k-1, \\ a_i^{r^{FDA}}, & \text{if } k > r^{FDA}. \end{cases}$$

That is, a_i^k is the school that admits i in round $k-1$ of Fast DA, or the school i applies to in round k. If $k > r^{FDA}$, a_i^k is the school that finally admits i, which is just $\mu^{DA}(i)$. Now I prove that a_i^k is exactly the school each i reports as first choice at any Lk of the original level-k model of BM. So it is as if students run the procedure of Fast DA in their minds to do their level-k reasoning.

Proposition 1. *For any P_I, $s_i^k = a_i^k$ for all i and all $k \geq 0$.*

Proposition 1 implies that each i must report a weakly worse school as first choice at a higher level, but the school is no worse than $\mu^{DA}(i)$.

Corollary 1. *For any P_I and any i,*

(1) $s_i^k \ R_i \ s_i^{k+1} \ R_i \ \mu^{DA}(i)$ for all $k \geq 0$;

(2) there exists some finite $r_i \geq 0$ such that $s_i^k \ P_i \ \mu^{DA}(i)$ for all $k < r_i$, and $s_i^k = \mu^{DA}(i)$ for all $k \geq r_i$.

In general r_i depends on P_I, which I compress in notation for simplicity. There is an intuitive explanation of Corollary 1. When i has a higher level, in his belief others also have higher levels. So i believes that all students use more competitive strategies such that i's best strategy is to cautiously report a weakly worse school as first choice. However, since students compete with each other only through priority rankings in DA, the situation in DA is weakly more competitive than all possible situation in BM. Hence, i's reported first choice in BM is never worse than $\mu^{DA}(i)$.

1.4.1 Efficiency Comparison between BM and DA

I use k_i to denote any i's level and use $k_I \equiv \{k_i\}_{i \in I}$ to denote any level distribution. If $k_i \geq r_i$ (the threshold defined in Corollary 1), i reports $\mu^{DA}(i)$ as first choice no matter how high k_i is. So I say i is *sufficiently sophisticated*. If $k_i < r_i$, I say i is insufficiently sophisticated. I use $\mu^{BM}_{k_I}$ to denote the assignment of BM for any P_I when the level distribution is k_I.

Although I do not characterize the whole preference orderings reported by positive-level students, characterizing their reported first choices is sufficient for me to make some comparison between $\mu^{BM}_{k_I}$ and μ^{DA}. Specifically, in the first round of BM there must be some students who are admitted by their reported first choices. By Corollary 1, for any i who is admitted by his reported first choice s, if i is insufficiently sophisticated, s must be strictly better than $\mu^{DA}(i)$; if i is sufficiently sophisticated, s must coincide with $\mu^{DA}(i)$. So I have the following proposition.

Proposition 2. *For any P_I:*

1. *$\mu^{BM}_{k_I}$ is not strictly Pareto dominated by μ^{DA} for any k_I;*
2. *If each student is sufficiently sophisticated, $\mu^{BM}_{k_I} = \mu^{DA}$;*
3. *If each student is insufficiently sophisticated, $\mu^{BM}_{k_I}$ is not Pareto dominated by μ^{DA}.*

So μ^{DA} can Pareto dominate $\mu^{BM}_{k_I}$ only when some students are insufficiently sophisticated while the others are sufficiently sophisticated. In the following I prove that if that happens, there must be some insufficiently sophisticated student i who prefers $\mu^{DA}(i)$ to $\mu^{BM}_{k_I}(i)$ but reports that $\mu^{BM}_{k_I}(i)$ is preferred to $\mu^{DA}(i)$.

Lemma 1. *For any P_I and any k_I, if μ^{DA} Pareto dominates $\mu^{BM}_{k_I}$, there exists some insufficiently sophisticated student i who reports some P'_i such that $\mu^{BM}_{k_I}(i)\ P'_i\ \mu^{DA}(i)$, but $\mu^{DA}(i)\ P_i\ \mu^{BM}_{k_I}(i)$.*

So if each insufficiently sophisticated student i reports the truthful preferences between $\mu^{DA}(i)$ and any school worse than $\mu^{DA}(i)$, then $\mu^{BM}_{k_I}$ is never Pareto dominated by μ^{DA}.

Proposition 3. *For any P_I and any k_I, if each insufficiently sophisticated student i reports a best preference ordering P'_i such that $\mu^{DA}(i)\ P_i\ s$ implies $\mu^{DA}(i)\ P'_i\ s$ for any $s \in S$, then $\mu^{BM}_{k_I}$ is not Pareto dominated by μ^{DA}.*

There is a simple best strategy P'_i for each insufficiently sophisticated i that satisfies the above condition: i only manipulates his first choice and reports the truthful preference ordering of the remaining schools. Formally, if i reports s as first choice, then sP'_is' for all $s' \neq s$, and $s'P'_is''$ if and only if $s'P_is''$ for all $s', s'' \neq s$. I call P'_i a *topping strategy*.

For any P_I, I call the ordering of the schools that any i applies to in the procedure of Fast DA the *expressed preferences of i in Fast DA*. If μ^{DA} is not Pareto efficient with respect to the expressed preferences of students in Fast DA, there exists a level distribution k_I such that all students obtain their reported first choice and $\mu^{BM}_{k_I}$ Pareto dominates μ^{DA}.

Proposition 4. *For any P_I, if μ^{DA} is not Pareto efficient with respect to the expressed preferences of students in Fast DA, then there exists some k_I such that $\mu^{BM}_{k_I}$ Pareto dominates μ^{DA}.*

1.4.2 Advantage of Sophisticated Students in BM

A popular concern in practice about BM is that a student may obtain a better assignment if he is more sophisticated. It is easy to show that this concern does not hold in general in the original level-k model of BM. It is because a student of a higher level may overestimate others' levels such that he chooses an overcautious strategy. For example, in the example of Section 1.3 if all students are level-0, Alex is admitted by X and Bob is admitted by Z. If Bob becomes $L1$, Bob is admitted by Y and becomes better off. But if Alex becomes level-3, Alex is admitted by Z and becomes worse off.

1.5 The Informational Level-k Model of BM

In the informational level-k model an Lk student for any $k > 0$ has a correct belief about the level of any Lk' student if $k' < k$, and believes the remaining are $Lk - 1$. So his strategy in BM depends on the level distribution k_I. Hence, in this section I use $\tilde{s}_i^k(k_I)$ to denote the first choice reported by any i at Lk for any $0 \leq k \leq k_i$. Interestingly, the level-k reasoning process in this model can still be understood through an adjusted procedure of DA. Formally, for any P_I and any k_I, I define:

Fast Deferred Acceptance*

Round $r \geq 0$: For each unassigned student i, if $k_i \geq r$, then i applies to her most preferred school s that he has not applied to and has not admitted q_s students who all have higher priorities than i at s. Each school tentatively admits students according to its priority ranking. If $k_i < r$ for all unassigned i, or all students are admitted after this round, stop the procedure.

Fast DA* is different from Fast DA in that an unassigned i cannot apply to a new school in any round $r > k_i$. Since its procedure depends on k_I, I denote its outcome by $\mu_{k_I}^{FDA^*}$. If some i is unassigned in $\mu_{k_I}^{FDA^*}$, I say i is admitted by $\emptyset$. Let $r_{k_I}^{FDA^*}$ denote the last round of Fast DA*. Then for each i and each $0 \leq k \leq k_i$ I define:

$$\tilde{a}_i^k(k_I) \equiv \begin{cases} \text{the school admitting } i \text{ in round } k-1, & \text{if } i \text{ is admitted in round } k-1, \\ \text{the school } i \text{ applies to in round } k, & \text{if } i \text{ is rejected in round } k-1, \\ \tilde{a}_i^{r_{k_I}^{FDA^*}}, & \text{if } k > r_{k_I}^{FDA^*}. \end{cases}$$

Proposition 5. *For any P_I and any k_I, $\tilde{s}_i^k(k_I) = \tilde{a}_i^k(k_I)$ for all i and all $0 \leq k \leq k_i$.*

By definition $\tilde{a}_i^{k_i}(k_I)$ is the last school that each i applies to in Fast DA*. It is also the first choice reported by each i in BM. So $\mu_{k_I}^{FDA^*}$ coincides with the assignment found by the first round of BM.

Corollary 2. *For any P_I and any k_I, $\mu_{k_I}^{FDA^*}$ is the assignment found by the first round of BM.*

If all students are sufficiently sophisticated, each i must finally apply to $\mu^{DA}(i)$ in Fast DA*. Then $\mu_{k_I}^{FDA^*}$ coincides with μ^{DA}. If some i is insufficiently sophisticated, since he applies to fewer schools than being sufficiently sophisticated, some other j of a level higher than i may therefore only apply to schools better than $\mu^{DA}(j)$ in

Fast DA*.[3] If all students are insufficiently sophisticated, each i must finally apply to a school better than $\mu^{DA}(i)$ in Fast DA*. So I have the following corollary.

Corollary 3. *For any P_I,*

(1) for any k_I, $\tilde{s}_i^k(k_I)\ R_i\ \tilde{s}_i^{k+1}(k_I)\ R_i\ \mu^{DA}(i)$ for all i and all $0 \leq k \leq k_i$;

(2) if each student is sufficiently sophisticated, $\mu_{k_I}^{FDA^*} = \mu^{DA}$;

(3) if each student is insufficiently sophisticated, $\tilde{s}_i^{k_i}(k_I)\ P_i\ \mu^{DA}(i)$ for all i.

1.5.1 Efficiency Comparison between BM and DA

For any P_I and any k_I, I denote the assignment of BM by $\tilde{\mu}_{k_I}^{BM}$. Using Corollary 3 I can prove the following result in the same way as in the previous section.

Proposition 6. *For any P_I:*

1. *$\tilde{\mu}_{k_I}^{BM}$ is not strictly Pareto dominated by μ^{DA} for any k_I;*
2. *If each student is sufficiently sophisticated, $\tilde{\mu}_{k_I}^{BM} = \mu^{DA}$;*
3. *If each student is insufficiently sophisticated, $\tilde{\mu}_{k_I}^{BM}$ is not Pareto dominated by μ^{DA};*
4. *If each positive-level i reports a best strategy P_i' such that $\mu^{DA}(i)\ P_i\ s$ implies $\mu^{DA}(i)\ P_i'\ s$ for any $s \in S$, then $\tilde{\mu}_{k_I}^{BM}$ is not Pareto dominated by μ^{DA} for any k_I.*

There is no counterpart of Proposition 4 in this section because if all students obtain their reported first choice, $\tilde{\mu}_{k_I}^{BM}$ coincides with μ^{DA}.

1.5.2 Advantage of Sophisticated Students in BM

To investigate the advantage of sophisticated students in BM, I compare the assignment of BM when j is Lk_j with the assignment of BM when j is Lk_j' for any $k_j' > k_j$. Since I only characterize the first choices reported by students, I investigate how the assignment found by the first round of BM changes when j's level is increased from Lk_j to Lk_j' for any $k_j' > k_j$. By Corollary 2 it is equivalent to investigating how the outcome of Fast DA* changes. My first result is as follows.

[3] In particular, even though j is sufficiently sophisticated, if his level is not high enough, i can be unassigned in Fast DA*. This is different from the previous model in which a sufficiently sophisticated j must be assigned to $\mu^{DA}(j)$ in the first round of BM.

Proposition 7. *For any P_I and any k_I, if any $j \in I$ becomes Lk'_j for any $k'_j > k_j$, then,*

- *if $\mu^{FDA^*}_{k_I}(j) \neq \emptyset$, $\tilde{\mu}^{BM}_{k_I} = \tilde{\mu}^{BM}_{(k'_j,k_{-j})}$;*
- *if $\mu^{FDA^*}_{k_I}(j) = \emptyset$, for any $i \in I$ such that $\mu^{FDA^*}_{k_I}(i) \neq \emptyset$ and $\mu^{FDA^*}_{(k'_j,k_{-j})}(i) \neq \emptyset$:*
  - *if $k_i \leq k_j + 1$, $\tilde{\mu}^{BM}_{k_I}(i) = \tilde{\mu}^{BM}_{(k'_j,k_{-j})}(i)$;*
  - *if $k_i > k_j + 1$, $\tilde{\mu}^{BM}_{k_I}(i) \; R_i \; \tilde{\mu}^{BM}_{(k'_j,k_{-j})}(i)$.*

The proof is as follows. If j is assigned in $\mu^{FDA^*}_{k_I}$, it means that j obtains his reported first choice. Then becoming Lk'_j does not change j's strategy as well as the others'. So the outcome of BM does not change. If j is unassigned in $\mu^{FDA^*}_{k_I}$, then by becoming Lk'_j, j will apply to more schools in Fast DA* than before. Then for any i such that $\mu^{FDA^*}_{k_I}(i) \neq \emptyset$ and $\mu^{FDA^*}_{(k'_j,k_{-j})}(i) \neq \emptyset$, if $k_i \leq k_j + 1$, the level change of j cannot affect the set of schools that i applies to in Fast DA*. So i's assignment does not change. If $k_i > k_j + 1$, since j applies to more schools than before in in Fast DA*, i will also apply to weakly more schools than before. So i's assignment must be weakly worse off.

For any P_I, define $\bar{r} \equiv \max_{k_I} r^{FDA^*}_{k_I}$. That is, $\bar{r}$ is the largest last round of Fast DA* for all possible k_I.[4] If any i's level is weakly higher than $\bar{r}$, i must be assigned in the outcome of Fast DA* irrespective of the others' levels. So I say i is *quasi-rational* if $k_i \geq \bar{r}$. A quasi-rational student is sophisticated enough in the sense that he always obtains his reported first choice. For any P_I and any k_I, I denote the set of quasi-rational students by M and the set of the remaining by N. Proposition 7 implies the following corollary.

Corollary 4. *For any P_I and any k_I, if $M \neq \emptyset$ and $N \neq \emptyset$, then if any $j \in N$ becomes Lk'_j for any $k'_j > k_j$,*

$$\tilde{\mu}^{BM}_{k_I}(i) \; R_i \; \tilde{\mu}^{BM}_{(k'_j,k_{-j})}(i) \textit{ for all } i \in M.$$

If all students in N become quasi-rational, the outcome of Fast DA* will coincide with μ^{DA}. Then Corollary 4 implies the following result.

Corollary 5. *For any P_I and any k_I, if $M \neq \emptyset$, all students in M obtain weakly better assignments in BM than in DA.*

[4]Given the set I of students and the set S of schools, $\bar{r} \leq |I| \cdot |S| - 1$.

For any k_I, define $\bar{k}_N \equiv \max_{i \in N} k_i$. If $M = \emptyset$ and there exists a unique $L\bar{k}_N$ student i, i must be assigned in $\mu^{FDA^*}_{k_I}$ and obtains an assignment weakly better than $\mu^{DA}(i)$. It is because no students other than i can apply to schools in round $\bar{k}_N$ of Fast DA*. Then if i is assigned in round $\bar{k}_N - 1$, i must still be assigned in round $\bar{k}_N$; if i is unassigned in round $\bar{k}_N - 1$, i must apply to a school in round $\bar{k}_N$ and is admitted. So Proposition 7 implies the following corollary.[5]

Corollary 6. *For any P_I and any k_I, if $M = \emptyset$ and there is a unique $L\bar{k}_N$ student i,*

(1) if any $j \in N \backslash \{i\}$ becomes Lk'_j for any $k_j < k'_j < k_i$, $\tilde{\mu}^{BM}_{k_I}(i)\ \ R_i\ \ \tilde{\mu}^{BM}_{(k'_j, k_{-j})}(i)$;

(2) i obtains a weakly better assignment in BM than in DA.

If $\mu^{FDA^*}_{k_I}(j) = \emptyset$, j may not be better off by becoming more sophisticated. It is shown by the following example. It is because the other students in N who have higher levels than j may respond to the level change of j by using more competitive strategies.

Example 1. *$I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ and $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. Each school has one seat. The preferences of students and the priority rankings of schools are shown in Table 1.5.*

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	π_{s_1}	π_{s_2}	π_{s_3}	π_{s_4}	π_{s_5}	π_{s_6}
s_1	s_2	s_3	s_4	s_1	s_1	i_6	i_3	i_4	i_5	i_2	$\vdots$
s_2	s_1	s_2	s_3	s_4	$\vdots$	$\vdots$	i_1	i_3	i_4	i_1	
s_5	s_3	$\vdots$	$\vdots$	$\vdots$			i_2	$\vdots$	$\vdots$	$\vdots$	
s_6	s_5						$\vdots$				
$\vdots$	$\vdots$										

Table 1.5: Example 1

Suppose i_1 is L0, i_2 is L2, and all others are quasi-rational. The first choices reported students are shown in Table 1.6a. If i_1 becomes quasi-rational, the first choices reported by students are shown in Table 1.6b.

If all positive-level students use topping strategies, then the outcomes of BM are shown in Table 1.7. It is easy to see that i_1 is worse off by becoming quasi-rational.

However, if j has the highest level among N, I prove that j must be weakly better off by becoming more sophisticated and using a strategy that satisfies a mild

[5] If there are multiple $L\bar{k}_N$ students, the corollary may not hold.

	i_1	i_2	i_3	i_4	i_5	i_6
$L0$:	s_1	s_2	s_3	s_4	s_1	s_1
$L1$:		s_2	s_3	s_4	s_4	s_1
$L2$:		s_2	s_3	s_3	s_4	s_1
$Lk \geq 3$:			s_2	s_3	s_4	s_1

(a) i_1 is $L0$

	i_1	i_2	i_3	i_4	i_5	i_6
$L0$:	s_1	s_2	s_3	s_4	s_1	s_1
$L1$:	s_2	s_2	s_3	s_4	s_4	s_1
$L2$:	s_2	s_5	s_3	s_3	s_4	s_1
$L3$:	s_2		s_2	s_3	s_4	s_1
$Lk \geq 4$:	s_6		s_2	s_3	s_4	s_1

(b) i_1 is quasi-rational

Table 1.6: Informational level-k model of BM in Example 1

i_1	i_2	i_3	i_4	i_5	i_6
s_5	s_6	s_2	s_3	s_4	s_1

(a) i_1 is $L0$

i_1	i_2	i_3	i_4	i_5	i_6
s_6	s_5	s_2	s_3	s_4	s_1

(b) i_1 is quasi-rational

Table 1.7: The assignments of BM

condition. Formally, I use $P_j^{k_j}$ and $P_j^{k'_j}$ to denote the preference orderings reported by j at Lk_j and Lk'_j respectively. If s is the first choice in $P_j^{k'_j}$, I say $P_j^{k'_j}$ satisfies *worse-rank invariance* if for any s' such that sP_js', s' has the same rank in $P_j^{k_j}$ and $P_j^{k'_j}$.

Proposition 8. *For any P_I and any k_I, if any $j \in N$ of $k_j = \bar{k}_N$ becomes Lk'_j for any $k'_j > k_j$ and his strategy satisfies worse-rank invariance, then*

$$\tilde{\mu}^{BM}_{(k'_j, k_{-j})}(j)\ R_j\ \tilde{\mu}^{BM}_{k_I}(j).$$

If j uses topping strategies at both Lk_j and Lk'_j, the worse-rank invariance condition is satisfied. If j becomes quasi-rational, since j must be admitted by his reported first choice, j is weakly better off even if his strategy does not satisfy worse-rank invariance.

1.6 Discussion

1.6.1 Insights from the Two Level-k Models

In both level-k models of BM the iterated reasoning process is analogous to the procedure of DA. Based on it I prove that BM is not (strictly) Pareto dominated by DA. Since the two models use extreme belief settings, I believe similar results will still hold in a level-k model with any other intermediate belief setting. The two models are different in the advantage of sophisticated students. In the original level-k model of BM sophisticated students do not have definite advantage because they may overestimate others' levels, while in the informational level-k model a student has a definite advantage in BM only when his level is high enough relative to the others. Hence, the two model together imply that both high sophistication and accurate belief are crucial for a student to have any advantage in BM.

1.6.2 Comparison with Nash Equilibrium Models

It is interesting to compare my results with those of Nash equilibrium models in the complete information environment. By assuming all students are rational, Ergin and Sönmez (2006) prove that every NE outcome of BM is a stable assignment with respect to true preferences of students. Since DA always finds the student-optimal stable assignment, it implies that BM is weakly Pareto dominated by DA. However, in the two level-k models if students are sufficiently sophisticated, BM finds the same assignment as DA. Hence, when students are very sophisticated and are more likely to use the level-k reasoning than using the circular equilibrium reasoning, students are more likely to coordinate on the stable-optimal stable assignment.

Pathak and Sönmez (2008) use a NE model to show that rational students can take advantage of naive students in BM. In their model students are either rational and naive, and rational students commonly know the identifies of naive students. By assuming that the best NE outcome of BM is always realized, they prove that there exists a conflict of interest between naive students and rational students. In particular, rational students obtain weakly better assignments in BM than in DA. This dichotomous sophistication distribution can be seen as a special case of my models. Indeed, I prove that if students are either $L0$ or quasi-rational, then the outcome of BM in the informational level-k model is exactly the best NE outcome in the BM game where $L0$ students are naive and quasi-rational students are rational. Then the results of Pathak and Sönmez are corollaries of mine.

Proposition 9. *For any P_I and any k_I, if $N, M \neq \emptyset$ and $k_N = \mathbf{0}$, then $\tilde{\mu}^{BM}_{k_I}$ is the best NE outcome in the BM game where N are naive and M are rational.*

In Appendix A.2 I use a new method to characterize the set of NE outcomes of BM when N are naive and M are rational. Proposition 9 is a corollary of the characterization (Proposition 15). Then by Proposition 7 if any $j \in N$ becomes quasi-rational, all students in M are weakly worse off in BM. Since $k_N = \mathbf{0}$, each $j \in N$ has the highest level in N. So by Proposition 8 if any $j \in N$ becomes quasi-rational, j must be weakly better off in BM. Hence, I obtain the main results of Pathak and Sönmez.[6]

Abdulkadiroğlu, Che, and Yasuda (2011) analyze a special incomplete information environment in which schools do not prioritize students and students share a common ordinal preference ordering over schools (but may have different cardinal utilities). They prove that in any symmetric Bayesian NE of BM students have weakly higher utilities in BM than in DA, and if there exist naive students, they can benefit from the equilibrium strategies of rational students.[7] I want to argue that the driving force behind their result is the special priority and preference assumption, not the incomplete information environment. Specifically, in the no priorities environment any two students of same cardinal utilities are assumed to play same strategies and also treated equally by schools. So when a rational student calculates his best strategy, he does not need to know the identities of others if he knows the distribution of cardinal utilities in the student population. Hence, assuming common knowledge of the cardinal utility distribution in the incomplete information environment is similar to assuming complete information. In Section 1.8 I provide a preliminary analysis of a level-k model of BM in the incomplete information environment.

1.6.3 Validity of My Assumptions about Students' Strategies

$L0$ strategy Intuitively, $L0$ strategy captures the instinct response of a player to a game. Since school choice is a preference revelation game, I believe my assumption that $L0$ students report true preferences is valid. In other games such as "p-beauty contest",[8] there is no natural focal point for $L0$ players, so it is often assumed $L0$ players use a random strategy.[9]

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[6] In Appendix A.2 I show that the other results of Pathak and Sönmez are also proved easily by my method.

[7] Abdulkadiroğlu, Che, and Yasuda (2011) also consider the complete information environment with strict priorities. They prove that if any naive student becomes rational, the other naive students must be weakly worse off in the unique NE outcome of BM. So naive students suffer from the existence of rational students. In Appendix A.5 I show that this result is actually *incorrect*.

[8] In the game each player is asked to propose an integer between 0 and 100. The winner is the one whose proposal is closest to a multiple p of the group average.

[9] In some games it is believed that some strategies are more likely to become instinct responses than the others. In the literature they are called *salient strategies*. For example, Crawford and Iriberri (2007a) point out the framing effects in the experiments of "hide-and-seek" games. By suitably adapting $L0$ behavior to salient strategies, they show that the level-k model can well explain the experimental dataset. Arad and Rubinstein (2012) conduct experiments of the "11-20" game to estimate the levels of players. In the game each of two players reports an integer between 11 and 20 and obtains an amount of dollars equaling his report; a player can win additional 20 dollars if his report is one less than the other's. Since the game rule is straightforward, Arad and Rubinstein argue that it is very natural for a naive player to report 20.

Lk **strategy** I assume that an Lk student for any $k > 0$ report their best obtainable schools as first choice. In many situations students have to do that in their best strategies. In other situations I believe my assumption is still reasonable. First, because first choice plays the most important role in determining a student's assignment in BM, it is natural that students are attracted to focus on first choice. This is supported by lab evidence. In the experiment of Chen and Sönmez (2006), 70.8% of students receive their reported first choices in BM, but only 28.5% receive their true first choices. So over 40% of students manipulate and obtain their first choices. Second, in practice students may be advertised/convinced to focus on first choice. For example, Boston provided a reference material to students in 2004 that suggested students to strategically choose their first choices. In Seattle and Tampa-St. Petersburg similar suggestions appear in local press (Abdulkadiroğlu et al. 2005). Last, as illustrated in the example of Section 1.3, in the level-k reasoning process students are often uncertain about the whole preferences reported by others at lower levels. If students is risk-averse and considers the worst case, they should assume that others optimally manipulate their first choices. Then my assumption captures such worst-case consideration.

1.7 Simulation

In previous sections I allow the level distribution to be arbitrary. But many experiments have found that subjects' levels are often not high. So in this section I do simulations by randomly generating students' levels from a reasonable distribution.

1.7.1 Setup

There are 1000 students and 20 schools. Each school has 50 seats. Although my models do not involve in utilities, I randomly generate the utilities of students and schools to generate preferences and priority rankings. Formally, the utility function of each student i is denoted by U_i and the utility function of each school s is denoted by U_s. Each utility consists of a private-value component and a common-value component:

$$U_i(s) \equiv \alpha U(s) + (1-\alpha)\epsilon_i(s),$$
$$U_s(i) \equiv \beta U(i) + (1-\beta)\epsilon_s(i),$$

where $U(s)$ and $U(i)$ are the common values of each s and each i, while $\epsilon_i(s)$ is the private value of each s in the utility of each i and $\epsilon_s(i)$ is the private value of each i in the utility of each s. All U and ϵ are independently and identically drawn from the uniform distribution on $[0, 1]$. $\alpha, \beta \in [0, 1]$ are correlation coefficients. In my simulation I vary the values of α, β from 0 to 1 in steps of .2. So $\alpha, \beta \in \{0, .2, .4, .6, .8, 1\}$. Students' preferences and schools' priority rankings are generated as:

$$P_i : s_a \, P_i \, s_b \iff U_i(s_a) > U_i(s_b),$$
$$\pi_s : i_a \, \pi_s \, i_b \iff U_s(i_a) > U_s(i_b).$$

For every value pair of (α, β) I randomly generate 1000 markets. In each market I draw the levels of students independently and identically from the Poisson distribution with a mean of 2. This distribution is consistent with the estimation of level distribution in multiple experiments.[10] In particular, in this distribution the probabilities for $L0$ to $L4$ are respectively .135, .271, .271, .180, .090.

In previous sections I do not assume how positive-level students report whole preferences. In the simulation I consider two strategy settings. In the first setting positive-level students use topping strategies. That is, they report true preferences over the schools other than reported first choices. In the second setting they report random preference orderings over the schools other than reported first choices which are independently and identically drawn from the uniform distribution. I call them *random strategies*. The two settings enable me to check the robustness of simulation results.

1.7.2 Result

There are 36 pairs of (α, β). For convenience I first report the simulation results corresponding to $(\alpha, \beta) = (.4, .4)$. The results for other pairs are similar. To measure the welfare of students in BM and DA, I calculate the ranks of their assignments in their true preferences. Table 1.8 reports the rank distribution and the average rank in the two level-k models of BM with the counterparts in DA, as well as the percentage of students that obtain better assignments in BM and the percentage of students that obtain better assignments in DA.

There are three observations from Table 1.8. First, the rank distributions in the two level-k models of BM are very close, and the difference between the topping strategy setting and the random strategy setting is small. Second, BM produces

[10] Camerer, Ho, and Chong (2004) use the Poisson Cognitive Hierarchy model to estimate multiple games and find the median estimation of the Poisson mean is 1.61. Arad and Rubinstein (2012) find the best estimate of the Poisson mean for the "11-20" game is 2.36.

Rank	BM (%) (topping)	BM (%) (random)		DA (%)
1	25.72	25.62	>	18.44
2	18.02	17.55	>	15.77
3	13.93	12.30	<	13.24
4	10.65	9.35	<	11.11
5	8.04	6.71	<	9.35
6	6.08	4.82	<	7.81
7	4.46	3.53	<	6.27
8	3.27	2.64	<	4.91
9	2.40	2.01	<	3.71
10	1.71	1.62	<	2.72
11	1.27	1.36	<	1.95
12	.95	1.21	<	1.41
13	.72	1.10	<	1.03
14	.59	1.05	<	.75
15	.46	1.03	<	.53
16	.40	1.08	>	.38
17	.34	1.18	>	.27
18	.30	1.36	>	.18
19	.29	1.66	>	.11
20	.40	2.13	>	.05
Avg rank	4.0	4.9		4.6

Topping: 32.8% prefer BM > 12.6% prefer DA

Random: 30.7% prefer BM > 18.2% prefer DA

(a) Original level-k

Rank	BM (%) (topping)	BM (%) (random)		DA (%)
1	26.44	26.07	>	18.44
2	18.36	17.85	>	15.77
3	14.00	12.92	<	13.24
4	10.54	9.00	<	11.11
5	7.77	6.13	<	9.35
6	5.73	4.21	<	7.81
7	4.18	2.95	<	6.27
8	3.11	2.26	<	4.91
9	2.31	1.84	<	3.71
10	1.70	1.57	<	2.72
11	1.28	1.40	<	1.95
12	.98	1.29	<	1.41
13	.75	1.24	<	1.03
14	.61	1.20	<	.75
15	.50	1.23	<	.53
16	.42	1.29	>	.38
17	.35	1.42	>	.27
18	.30	1.63	>	.18
19	.29	1.99	>	.11
20	.38	2.51	>	.05
Avg rank	4.0	5.1		4.6

Topping: 35.1% prefer BM > 14.1% prefer DA

Random: 32.5% prefer BM > 21.1% prefer DA

(b) Informational level-k

Table 1.8: The rank distribution in BM and DA when $(\alpha, \beta) = (.4, .4)$

more extreme assignments than DA: in BM more students obtain high or low-ranked assignments, while in DA more students obtain medium-ranked assignments. Third, although the comparison between the average ranks in BM and DA depends on the two strategy settings, there are always more students who prefer BM than those who prefer DA. So neither BM nor DA dominates the other, but more students prefer BM.

Result 1. *For* $(\alpha, \beta) = (.4, .4)$*:*

1. The rank distribution in BM and DA does not depend on the two level-k models and the two strategy settings;

2. BM produces more extreme assignments than DA;

3. There are more students who prefer BM than those who prefer DA.

Level	Average rank of assignments				
	Original level-k of BM		Informational level-k of BM		DA
	topping	random	topping	random	
0	6.0	6.6	6.0	6.5	4.59
1	4.5	6.9	4.5	7.2	4.58
2	3.2	3.8	3.1	3.3	4.59
3	3.2	3.2	2.94	3.0	4.59
4	3.6	3.6	2.94	3.0	4.59
5	3.9	3.9	2.93	2.97	4.60
⋮	⋮	⋮	⋮	⋮	⋮

Table 1.9: Average assignment rank at each level when $(\alpha, \beta) = (.4, .4)$

Level	Prefer BM|DA (%)			
	Original level-k of BM		Informational level-k of BM	
	topping	random	topping	random
0	19.97|41.19	17.11|45.41	19.53|41.39	16.75|45.39
1	31.19|21.15	26.52|36.00	29.18|22.23	23.78|38.68
2	42.20|4.55	40.84|8.45	38.66|7.82	36.44|14.49
3	37.16|.06	37.16|.09	43.03|1.61	42.66|2.87
4	27.62|≈ 0	27.50|≈ 0	44.06|.17	43.78|.26
5	21.00|≈ 0	20.33|≈ 0	44.47|≈ 0	44.11|≈ 0
⋮	⋮	⋮	⋮	⋮

Table 1.10: Percentage who prefer BM/DA at each level when $(\alpha, \beta) = (.4, .4)$

To examine how a student's welfare depends on his sophistication level, for the students at each level from $L0$ to $L5$, in Table 1.9 I report the average rank of their assignments in BM and DA, while in Table 1.10 I report the percentage of students who obtain better assignments in BM and the percentage of students who obtain better assignments in DA. In DA, the average rank for all levels is almost always 4.59, but in BM the average rank depends on level. In particular, in BM the average rank for level-0 students in BM is lower than the average rank for any positive-level students, and is also lower than the average rank in DA. By contrast, positive-level students have higher average ranks in BM than in DA. Similar can also be observed from Table 1.10: among level-0 students more prefer DA, while among positive-level students more prefer BM.

However, Table 1.9 shows an important difference between the two level-k models. In the original level-k model the average rank in BM has a single peak at $L3$. This is because in my chosen distribution level-3 students correctly believe the largest number of students' levels. By contrast, in the informational level-k model the average rank increases in level. So higher-level students on average obtain better assignments. This is similarly observed in Table 1.10. So I have the following result.

Result 2. *For* $(\alpha, \beta) = (.4, .4)$*:*

1. Level-0 students are on average better off in DA, while positive-level students are on average better off in BM;

2. The advantage of positive-level students is single-peaked in the original level-k model of BM, but increases in level in the informational level-k model of BM.

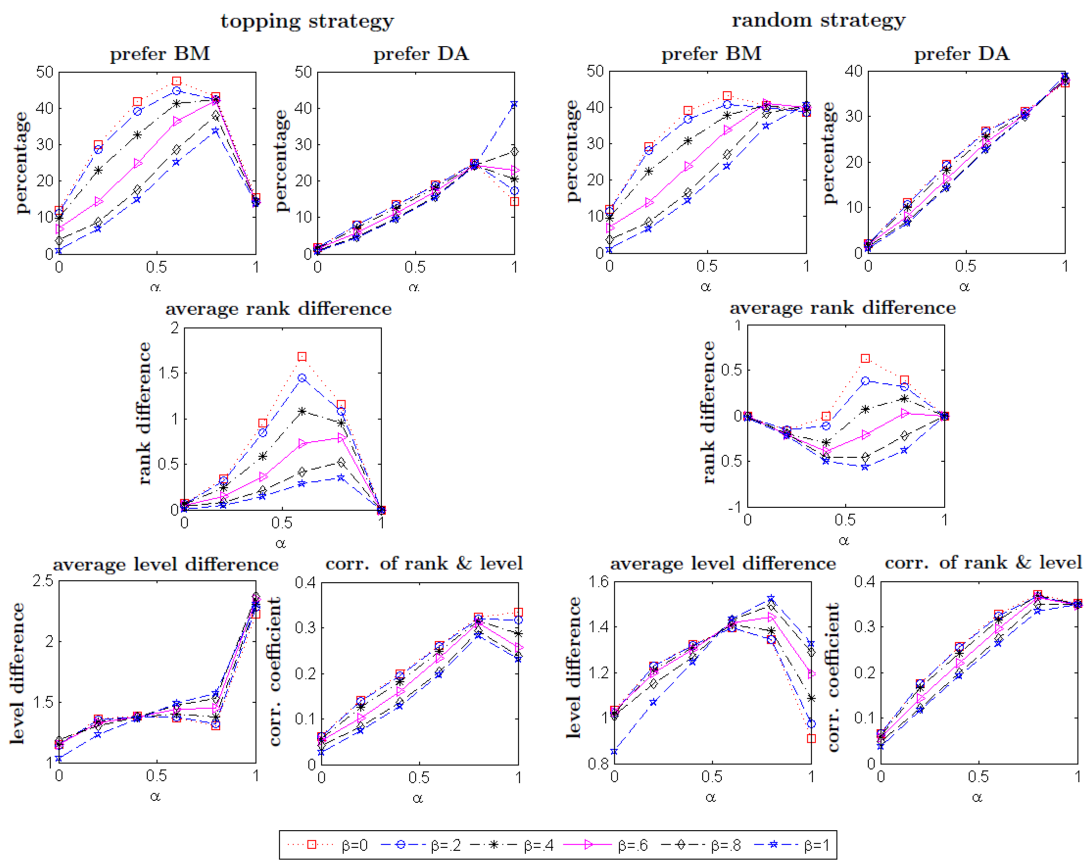


Figure 1.1: Simulation results in original level-k

Figure 1.1 summarizes the simulation results for all pairs of (α, β) in the original level-k model. The simulation results for the informational level-k model are almost same and reported in Appendix A.3.[11] There are five subfigures for each strategy

[11] Ashlagi, Kanoria, and Leshno (2015) show that unbalanced markets perform very differently from balanced markets. In Appendix A.3 I also report the simulation results for unbalanced markets by setting $U(s) = \epsilon_i(s) = 0$ for some s and all i. In this way s becomes the worst school in all

setting. In each subfigure the horizontal axis is the value of α, and the six lines correspond to the six values of β. For most pairs of (α, β), there are more students who prefer BM than those who prefer DA.[12] Average rank difference is equal to the average rank of all students' assignments in DA minus the average rank of all students' assignments in BM. I use it to compare the average welfare of students in BM and DA. If it is positive, students are on average better off in BM; vice versa. When topping strategies are used, students are on average better off in BM for all pairs of (α, β),[13] while when random strategies are used, the answer depends on the pairs of (α, β). So in general it is uncertain that which algorithm gives students a higher average welfare. The last two subfigures examine the advantage of sophisticated students in BM. Average level difference is equal to the average level of those who prefer BM minus the average level of those who prefer DA. In the figure it is always positive. Corr. of rank & level reports the correlation coefficient between the preference ranks of students' assignments in BM and their levels. In the figure it is always positive and significantly above zero for most pairs of (α, β). Hence, both subfigures suggest that positive-level students have advantage in BM.

Result 3. *For all pairs of* (α, β)*:*

1. *There are more students who prefer BM than those who prefer DA;*
2. *Positive-level students have advantage in BM.*

1.7.3 Comparison with Empirical Estimation

It is interesting to compare my simulation results with recent empirical estimations conducted by He (2014) and Calsamiglia, Fu, and Güell (2015). He uses the dataset from Beijing of China, and Calsamiglia et al. use the dataset from Barcelona of Spain. Both cities implemented some kind of BM in their school choice programs. The two studies accommodate the fact that students have heterogeneous sophistication types. After estimating the preferences of students they

students' preferences and plays the role of "unassigned". Unbalanced markets make some of the simulation results sharper, but my qualitative conclusions do not change.

[12] $\alpha = 1$ in the topping strategy setting is an exception. It is because when $\alpha = 1$ students have identical preferences, and by using topping strategies they report highly correlated preferences in BM. So the assignment in BM is mainly determined by priority rankings of schools, which is further determined by β. By contrast, in the random strategy setting students report weakly correlated preferences.

[13] When $\alpha = 1$, students have identical preferences, and any assignment has the same average rank. So the average rank difference is zero. While when $\alpha = 0$, students have uncorrelated preferences. Then almost all of them obtain their most preferred schools in both BM and DA. So the average rank difference is almost zero for any β.

conduct counter-factual analyses to predict the effect of replacing BM with DA in the two cities. Specifically, He develops an approach to estimate the preferences of students without having to estimate their sophistication distribution. In his counter-factual analysis he only considers the welfare of naive students and rational students. Calsamiglia et al. estimate both the preferences of students and their sophistication types. But they assume that there are only two sophistication types: being naive or strategic. The results of the two studies are summarized in Table 1.11. Both studies predict that replacing BM with DA will hurt more students of any sophistication type than benefiting them, and an average student of any sophistication type will have a welfare loss equivalent to either some increase in school distance or some increase in school fee.

	He (2014)		Calsamiglia, Fu, and Güell (2015)		
Effect	naive	rational	naive	strategic	all
Benefit	27%	15%	8.6%	9.6%	9.5%
Hurt	55%	66%	35.4%	28.2%	28.5%
Average utility loss	8% ↑	40% ↑	€117 ↑	€57 ↑	€60 ↑
	in school distance		in school fee		

Table 1.11: Empirical estimation of the effect of replacing BM with DA

My simulation results are consistent with the two studies in that we all predict that there are more sophisticated students who will be hurt by the replacement than those who will be helped, and the average welfare of sophisticated students will be reduced by the replacement. However, my simulations predict that the replacement will help naive students in general (the only exception happens when students have uncorrelated preferences). A possible reason for this difference is that the two studies do not consider the possible heterogeneous sophistication levels among strategic students.[14] Hence, I hope my models and simulation results can motivate future empirical research to address this problem.

[14] Although He somehow considers it, he does not model the sophistication distribution of strategic students. In his second counter-factual analysis of replacing BM with DA, He assumes that students are either naive or rational. He finds that the result depends on the proportion of naive students and is different from his first counter-factual analysis. So he concludes that it is important to allow additional sophistication types beyond naivety or rationality.

1.8 Extension

1.8.1 Constrained School Choice

There are many schools in a city, but in some cities students are constrained to report only a few schools in their submitted preferences (Haeringer and Klijn 2009; Calsamiglia, Haeringer, and Klijn 2010). Under this constraint it is impossible for students to report true preferences in any algorithm. In particular, students cannot report true preferences in DA. So there should be a level-k model of DA to analyze the strategies of students. Any such model needs to specify the default strategies of $L0$ students and the best strategies of Lk students. However, it is not clear what specifications are reasonable. In Appendix A.4, I analyze an original level-k model of DA by assuming that $L0$ students report true preferences truncated by the constraint and Lk students use topping strategies. I show that the reasoning process is similar to that in BM. By contrast, since students manipulate BM through misreporting first choice, the previous level-k models of BM still hold.

1.8.2 Incomplete Information

In this section I present an original level-k model of BM in the incomplete information environment. To simplify the analysis I assume that schools do not exogenously prioritize students and draw priority rankings randomly from uniform distributions. This is to capture the fact that schools often very coarsely prioritize students and break the ties by lotteries.

Denote the cardinal utility vector of each student i by $\mathbf{v}^i \equiv (v_s^i)_{s\in S}$, where v_s^i is the utility of obtaining s. $\mathbf{v}^i$ is also called the type of i. $\mathbf{v}^i$ is drawn from the type space $\mathcal{V} \equiv \{(v_s)_{s\in S} \in [0,1]^{|S|} : v_s \neq v_{s'}, \forall s, s' \in S\}$ according to a probability distribution f. I assume f is public information and has full support. That is, $f(\mathbf{v}^i) > 0$ for all $\mathbf{v}^i \in \mathcal{V}$. Let $P_\mathbf{v}$ be the preference ordering induced by any $\mathbf{v} \in \mathcal{V}$.

I use $P_\mathbf{v}^k$ to denote the preferences reported by any type-$\mathbf{v}$ students at any Lk. As before $L0$ students report true preferences. So $P_\mathbf{v}^0 = P_\mathbf{v}$. I assume positive-level students are risk-neutral, so they choose strategies to maximize their expected utilities. Then for any $\mathbf{v} \in \mathcal{V}$ and any $k > 0$,

$$P_\mathbf{v}^k \equiv \arg\max_{P^\star \in \mathcal{P}} \mathrm{EU}_\mathbf{v}^k(P^\star),$$

where $\mathrm{EU}_\mathbf{v}^k(P^\star)$ is the expected utility of type-$\mathbf{v}$ students by reporting $P^\star$. Specifically, let $\mu^{BM}(P_{\mathbf{v}^{-i}}^{k-1}, P^\star)$ be the random outcome of BM if a type-$\mathbf{v}$ students i reports $P^\star$ and the others report $P_{\mathbf{v}^{-i}}^{k-1}$. Let $\mu^{BM}(P_{\mathbf{v}^{-i}}^{k-1}, P^\star)(i)(s)$ be the probability that i

obtains any $s \in S$. Then,

$$\begin{aligned}\mathrm{EU}^k_{\mathbf{v}}(P^\star) &= \int_{\mathbf{v}^{-i}\in\mathcal{V}^{|I|-1}} \sum_{s\in S} \left[\mu^{BM}(P^{k-1}_{\mathbf{v}^{-i}}, P^\star)(i)(s)\cdot v_s\right] f(\mathbf{v}^{-i})d\mathbf{v}^{-i} \\ &= \sum_{s\in S}\left[\int_{\mathbf{v}^{-i}\in\mathcal{V}^{|I|-1}} \mu^{BM}(P^{k-1}_{\mathbf{v}^{-i}}, P^\star)(i)(s) f(\mathbf{v}^{-i})d\mathbf{v}^{-i}\right] v_s.\end{aligned}$$

When $k = 1$, $P^0_{\mathbf{v}^{-i}}$ are the true preferences of the students other than i. So $\mathrm{EU}^1_{\mathbf{v}}(P^\star)$ is well-defined. Since $f(\mathbf{v}^{-i}) = \prod_{j\neq i} f(\mathbf{v}^j) > 0$ for all $\mathbf{v}^{-i} \in \mathcal{V}^{|I|-1}$, with probability one there is a unique $P^1_{\mathbf{v}}$ that maximizes $\mathrm{EU}^1_{\mathbf{v}}(P^\star)$. If $P^1_{\mathbf{v}}$ is not unique, choose an arbitrary best strategy. When $k \geq 2$, since $f(\mathbf{v}^{-i}) > 0$ for all $\mathbf{v}^{-i} \in \mathcal{V}^{|I|-1}$ and $P^{k-1}_{\mathbf{v}}$ is generally unique, $\mathrm{EU}^k_{\mathbf{v}}(P^\star)$ is still well-defined. Then $P^k_{\mathbf{v}}$ is still generically unique. If it is not unique, choose an arbitrary best strategy.

As shown above, students' strategies depend on their beliefs and cardinal utilities. Without any additional assumptions it is hard to characterize their strategies and compare the outcome of BM with that of DA. I leave it for future research.

1.9 Related Literature

There are a lot of related papers in the matching theory and especially school choice literature. In Section 1.6 I have discussed related NE models. Troyan (2012) generalizes the idea of Abdulkadiroğlu, Che, and Yasuda (2011) by relaxing the no priorities assumption to coarse priorities. Featherstone and Niederle (2014) use experiments to test the idea of Abdulkadiroğlu et al.. They design a simple environment in which there is a unique non-truth-telling Bayesian NE in BM, and find that subjects fail to coordinate on the equilibrium even with feedback and repetition. Haeringer and Klijn (2009) analyze the NE outcomes of popular matching algorithms in constrained school choice, and find it is hard to compare BM and DA based on NE outcomes. Specifically, they prove that the set of NE outcomes of BM is equal to the set of stable assignments, but the set of NE outcomes of DA is a superset of stable assignments. Calsamiglia, Haeringer, and Klijn (2010) use experiments to study constrained school choice, and find that constraints significantly reduce the efficiency of BM and DA as well as the proportion of truth telling in DA.

Basteck and Mantovani (2016) and Dur, Hammond, and Morrill (2015) use experimental and empirical datasets respectively to examine the advantage of sophisticated students in BM. Basteck and Mantovani first measure the cognitive abilities of subjects by standard tests in labs, then let them play the BM and DA games. Although test scores have a wide range, they classify subjects into two groups:

high-ability group of the top half scores and low-ability group of the remaining. By matching subjects' performance in the two games with their groups, Basteck and Mantovani find that the low-ability group have significantly lower payoffs than the high-ability group in BM, but the difference is small in DA. Meanwhile, the average payoff of all subjects is higher in BM than in DA. Dur et al. obtain an interesting dataset from Wake County of North Carolina. In the city students have two weeks to submit or revise their preferences as many times as they want through an online system. Once a student logs into the system, he can see the number of students who have reported each school as first choice. In the dataset an average student visits the system 4.61 times with a standard deviation of 8.65; 60.7% of students visit the system more than once. Dur et al. interpret those who visit once as naive and interpret the remaining as sophisticated. They find that sophisticated students have better assignments than naive students.

Agarwal and Somaini (2014) find that BM is significantly manipulated in the dataset from Cambridge of Massachusetts. So they estimate the preferences of students by assuming all of them are strategic. They predict that replacing BM with DA will make students on average worse off. Top Trading Cycle (Shapley and Scarf 1974) is another popular matching algorithm, but is not widely used in school choice. One reason is that it is hard to explain the role of priorities in TTC to schools and students (Pathak 2016). So I do not analyze it in the paper. Since TTC is strategy-proof, students report true preferences even though they have heterogeneous sophistication levels. Bade (2016b) is the only paper I am aware of that discusses the properties of matching algorithms when players are boundedly rational. She studies the problem of assigning indivisible objects to players in the no priorities environment, and examines whether the large set of hierarchical exchange algorithms are still Pareto efficient.

1.10 Conclusion

In this paper I study how students behave in a manipulable school choice algorithm known as Boston mechanism when students have heterogeneous sophistication, and then compare BM with Deferred Acceptance. In the two level-k models with extreme belief settings, it is robust that BM is not Pareto dominated by DA. However, whether the advantage of sophisticated students exists depends on how accurate their beliefs are. My results provide a new perspective on the comparison between BM and DA in school choice.

Pathak and Sönmez (2013) argue that the main motivation behind multiple school choice reforms in practice is to reduce the manipulability of algorithms in use. This seems to imply that the strategy-proofness of DA is sufficient to support its replacement of BM. However, lab and field evidence have revealed that some players do not understand the strategy-proofness of DA and attempt to manipulate DA: in the experiment of Chen and Sönmez (2006) 36% of subjects attempt to manipulate DA; in the survey conducted by Rees-Jones (2015) from the participants in the 2012 National Resident Matching Program, around 5% of respondents report that they attempt to manipulate DA. Ashlagi and Gonczarowski (2015) prove that DA is not obviously strategy-proof, a property that is supposed to be easier to understand than strategy-proofness (Li 2015). This implies that understanding the strategy-proofness of DA may require some level of sophistication. Hence, a natural question for future research is to understand the behavior of players in DA when they have different abilities to understand it.

Chapter 2

EFFICIENT AND FAIR ASSIGNMENT MECHANISM IS STRONGLY GROUP MANIPULABLE

2.1 Introduction

Assigning indivisible objects to agents without using monetary transfers is a common theme in many matching markets. This paper studies the simplest form of the problem: agents have strict preferences on objects, while objects do not discriminate agents by priority rankings. Efficiency, fairness, and (group) incentive compatibility are three most important objectives when policy makers design mechanisms to solve the problem. In this paper I show that in the presence of a minimal efficiency requirement, fairness and group incentive compatibility are not compatible. Specifically, since objects are indivisible, fairness motivates policy makers to use random assignment mechanisms. I prove that if a mechanism satisfies ex-post efficiency and some mild fairness criteria, no matter outside option is available or not, it must be *strongly group manipulable*: by misreporting preferences all agents of a manipulating group obtain lotteries that strictly first-order stochastically dominate the lotteries they obtain in the truth-telling case. When outside option is available, the manipulating group can be as small as containing only two agents. Since ex-post efficiency is a very weak requirement that policy makers are very likely to pursue, my results tell policy makers that they have to make a choice between fairness and group incentive compatibility.

To illustrate my results, let us look at Random Serial Dictatorship (RSD) and Probabilistic Serial (PS), which are two popular mechanisms in the literature. For the two preference profiles in Table 2.1, both RSD and PS find the assignments shown in Table 2.2. The lotteries i_1, i_2 obtain in $\succ_I^\circ$ strictly first-order stochastically dominate the lotteries they obtain in $\succ_I^*$. Therefore, I say i_1, i_2 can strongly group manipulate RSD and PS in $\succ_I^*$ by misreporting their preferences as in $\succ_I^\circ$.

It is well-known that RSD and PS are ex-post efficient and satisfy some existing fairness criteria. In particular, RSD satisfies *equal treatment of equals* and *weak envy-freeness*, while PS satisfies a stronger criterion of *envy-freeness*. Nevertheless, I show that they also satisfy three new fairness criteria proposed by this paper, and it is actually the combination of ex-post efficiency and some existing and new

$\succ^*_{i_1}$	$\succ^*_{i_2}$	$\succ^*_j$
o_1	o_1	o_2
o_2	o_2	o_3
o_3	o_3	o_1
$\vdots$	$\vdots$	$\vdots$

(a) $\succ^*_I$

$\succ^\circ_{i_1}$	$\succ^\circ_{i_2}$	$\succ^*_j$
o_2	o_2	o_2
o_1	o_1	o_3
o_3	o_3	o_1
$\vdots$	$\vdots$	$\vdots$

(b) $\succ^\circ_I$

Table 2.1: Two preference profiles

	i_1	i_2	j
o_1	$\frac{1}{2}$	$\frac{1}{2}$	
o_2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
o_3	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

(a) RSD($\succ^*_I$)=PS($\succ^*_I$)

	i_1	i_2	j
o_1	$\frac{1}{2}$	$\frac{1}{2}$	
o_2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
o_3	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$

(b) RSD($\succ^\circ_I$)=PS($\succ^\circ_I$)

Table 2.2: The assignments for the two preference profiles

fairness criteria that makes a mechanism inevitably be strongly group manipulable. Therefore, my results provide a better understanding of random assignment mechanisms.

Two of the three new fairness criteria are weaker than envy-freeness, and the other one is independent of envy-freeness. Specifically, *equal top-assignment of equal tops* is a fairness notion defined only on agents' top choices. It requires that any two agents who mostly prefers the same object must receive equal probability of the object. In other words, the differences in their lotteries exist only in the assignments of objects other than their common top choice. As discussed in the paper, this criterion and equal treatment of equals can be seen as two extreme special cases of a general fairness criterion, which requires that if any two agents have equal upper contour sets of some object and have equal preferences on the upper contour set, then they receive equal probability of each object in the upper contour set. *Uniform tail-assignment of uniform tails* requires that if all agents have equal upper contour sets of some object and have equal preferences on the remaining objects, then they receive equal probability of each remaining object. In other words, the differences in agents' lotteries exist only in the assignments of the upper contour set. It is different from previous criteria in that its restriction is imposed on all agents' preferences. Both new criteria are weaker than envy-freeness, and are independent of equal treatment of equals or weak envy-freeness. *Top advantage* is also a fairness notion defined on agents' top choices, and can be seen as a complement of equal top-assignment of equal tops. It requires that any agent receives a higher probability of his most preferred object than any other agent who has a different most preferred object if the latter agent receives a positive probability of the former agent's most preferred object. Top advantage is independent of envy-freeness (therefore, as well as others).

The paper explicitly distinguishes the cases that outside option is available or not. When outside option is available, agents can prefer outsdie option to an object. My first result (Section 2.3) proves that in the simple environment of three agents and at least three objects, no matter outside option is available or not, any ex-post efficient mechanism that satisfies equal top-assignment of equal tops, top advantage, and either equal treatment of equals or weak envy-freeness must be strongly group manipulable. In this environment Bogomolnaia and Moulin (2001) and Nesterov (2016) respectively prove that RSD is the only ex-post efficient and strategy-proof mechanism that satisfies equal treatment of equals or weak envy-freeness. Therefore, there is a trade-off between individual incentive compatibility and minimal group incentive compatibility.

My second result (Section 2.4) considers the environment of at least four agents and at least three objects. If outside option is available, I prove that the first result still holds. It is because the previous three-agent environment can be embedded into this general environment. However, if outside option is not available, I prove that a mechanism is still strongly group manipulable if it further satisfies uniform tail-assignment of uniform tails. As a corollary, no matter outside option is available or not, when there are at three agents and at least three objects, any ex-post efficient mechanism that satisfies envy-freeness and top advantage is strongly group manipulable.

The above negative results hold when there are at least three objects. Thus, they motivate me to consider the two-object environment and investigate whether any positive result can hold. My third result (Section 2.5) proves that when there are at least three agents and outside option is not available, any ex-post efficient mehcanism that satisfies equal treatment of equals and top advantage must not be strongly group manipulable. If a mechanism further satisfies uniform tail-assignment of uniform tails, it must be group strategy-proof. However, if outside option is available, previous negative results still hold.

Related Literature

Group manipulation has been studied a lot for deterministic mechanisms (Svensson 1999; Pápai 2000; Pycia and Ünver 2017). However, the literature mainly focus on individual manipulation for random mechanisms. To the best of my knowledge, Bade (2016a) provides the only study of group manipulation in random mechanisms. She proves that when there are at least three objects, any ex-post efficient mechanism that satisfies equal treatment of equals is not group strategy-proof: by misreporting preferences some members of a manipulating group obtain lotteries that are not strictly first-order stochastically dominated by lotteries in the truth-telling case, while the remaining members obtain lotteries same as the truth-telling case. Therefore, group strategy-proofness is a very strong requirement: no matter what cardinal utilities a group of agents may have behind their preferences, none of them can obtain higher expected utilities without hurting others by misreporting preferences. Accordingly, it is a weak statement that a mechanism is not group strategy-proof. For example, it is known that PS is not strategy-proof. Thus, Bade's result does not have any new implication for PS. By contrast, by imposing more (mild) fairness criteria than Bade, I prove stronger theorems, which imply that no matter what cardinal utilities a manipulating group may have behind their preferences, all of them must obtain higher expected utilities by misreporting preferences. In particular, although PS is not strongly manipulable by individuals (because PS is weakly strategy-proof), my theorems imply that PS is strongly manipulable by groups. Moreover, in the two-object environment I prove that by adding some new fairness criteria I introduce to Bade's theorem, the theorem becomes positive, that is, any mechanism must be group strategy-proof. This result is an interesting complement of Bade's result.

RSD attracts a lot of attention in the literature because of its strategy-proofness and good fairness properties. Bade (2016c) proves that any random mechanism defined by uniformly randomizing the roles of agents in a Pareto efficient, strategy-proof and non-bossy deterministic mechanism is equivalent to RSD. Li (2015) proves that RSD is obviously strategy-proof if agents take turns to choose objects. Pycia and Troyan (2016) further prove that RSD is the only mechanism that is obviously strategy-proof, ex-post efficient and symmetric.[1] These results make RSD stand out from random mechanisms. However, in Section 2.6, I prove that both PS and RSD satisfy the three new fairness criteria. So both mechanisms are strongly group manipulable as long as there are at least three objects. It strengthens the implication of this paper that group incentive compatibility is hard to achieve if policy makers

[1] Simply speaking, a mechanism is symmetric if its underlying game is anonymous to the roles of agents. It implies equal treatment of equals.

want to pursue strategy-proofness and fairness in the way suggested by Bade or Pycia and Troyan.

The literature (Zhou 1990; Bogomolnaia and Moulin 2001; Martini 2016; Nesterov 2016) have proved multiple impossibility theorems regarding the tension between efficiency and fairness for strategy-proof mechanisms. Nesterov proves that when there are at least three agents, any ex-post efficient and upper-envy-free mechanism must not be upper-shuffle-proof. Upper-envy-freeness are stronger than equal-top assignment of equal tops, equal treatment of equals, and uniform tail-assignment of uniform tails, while upper-shuffle-proofness is weaker than strategy-proofness. As a corollary, any ex-post efficient mechanism that satisfies envy-freeness is not strategy-proof. In comparison, my Corollary 8 proves that if the mechanism further satisfies top advantage, it must be strongly group manipulable, which implies that the mechanism is even not weakly strategy-proof.

Zhou (1990) proves that when there are at least three agents, any ex-ante efficient and symmetric (agents of identical cardinal utilities obtain equal expected utility) mechanism is not strategy-proof. Bogomolnaia and Moulin (2001) prove that when there are at least four agents, any ordinally efficient mechanism that satisfies equal treatment of equals is not strategy-proof. Martini (2016) strengthens their result by weakening ordinal efficiency to non-wastefulness. Zhou, BM and Nesterov all assume that objects are as many as agents. This assumption allows the authors to easily embed a small-size problem into a large-size problem.[2] Martin does not make this assumption, but he assumes that outside option is available, which is crucial for his result.

2.2 Definition

2.2.1 Object allocation problem

A finite set of heterogeneous indivisible objects O is assigned to a finite set of agents I. I assume that $|O| \geq 2$ and $|I| \geq 3$. A generic object is denoted by o or o' and a generic agent is denoted by i or j. Each $i \in I$ demands only one object and each $o \in O$ has only one copy. When an agent does not obtain an object, he is said to obtain the virtual object $\emptyset$. So $\emptyset$ plays the role of outside option. Let $\hat{O} \equiv O \cup \{\emptyset\}$. Each $i \in I$ has a strict preference ordering $\succ_i$ of $\hat{O}$, with the associated

[2] For example, after an impossibility result is proved for a three-agent (or four-agent) case, to prove it also holds for more agents, both Bogomolnaia and Moulin (2001) and Nesterov (2016) construct a preference profile in which three (or four) agents essentially constitute a subproblem by requiring the other agents mostly prefer objects of same indexes with them.

weak ordering denoted by $\succsim_i$. Let $SU(\succ_i, o) \equiv \{o' \in \hat{O} : o' \succ_i o\}$ be the **strict upper contour set of** o **at** $\succ_i$. Let $\succ_i |_{O'}$ be the **restriction of any** $\succ_i$ **to any** $O' \subseteq \hat{O}$. That is, for any $o, o' \in O'$, $o \succ_i |_{O'} o'$ if and only if $o \succ_i o'$. Any o is **acceptable** to i if any $o \succsim_i \emptyset$. For any $J \subseteq I$, let $\succ_J \equiv \{\succ_i\}_{i \in J}$ be the preference profile of the agents in J.

A **deterministic assignment** π is a function from I to $\hat{O}$ such that $\pi(i) = \pi(j)$ for any distinct i, j if and only if $\pi(i) = \pi(j) = \emptyset$. Let Π be the set of all deterministic assignments. A **random assignment** p is a probability distribution on Π such that $p(\pi)$ is the probability that π is realized, $p_i \in \Delta\hat{O}$ is the lottery assigned to i, and $p_{i,o}$ is the probability that i obtains o. Hence, $\Delta\Pi$ is the set of all random assignments. From now on whenever I refer to an assignment, I mean it is a random assignment. For any two lotteries $p_i, q_i \in \Delta\hat{O}$, p_i **strictly first-order stochastically dominates** q_i for any i, denoted by $p_i \succ_i q_i$, if $\sum_{o' \succ_i o} p_{i,o'} \geq \sum_{o' \succ_i o} q_{i,o'}$ for all o and $\sum_{o' \succ_i o} p_{i,o'} > \sum_{o' \succ_i o} q_{i,o'}$ for some o. I use $p_i \succeq_i q_i$ to denote that either $p_i \succ_i q_i$ or $p_i = q_i$, and use $p_i \not\succ_i q_i$ and $p_i \not\succeq_i q_i$ to denote that $p_i \succ_i q_i$ and $p_i \succeq_i q_i$ do not hold respectively.

Let $\mathcal{P}$ be any domain of agents' preferences. In this paper I focus on two preference domains: the universal domain $\mathcal{U}$, which contains all strict preference orderings of $\hat{O}$, and the no outside-option domain $\mathcal{Q}$, which contains all strict preference orderings of $\hat{O}$ that rank $\emptyset$ as worst.

Given any $\mathcal{P}$, a **random assignment mechanism** ρ is a function from $\mathcal{P}^{|I|}$ to $\Delta\Pi$ such that $\rho(\succ_I)$ is the random assignment found by ρ for any $\succ_I$. Let $\rho(\succ_I)(\pi)$ be the probability of any π in $\rho(\succ_I)$, $\rho_i(\succ_I) \in \Delta\hat{O}$ be the lottery assigned to any i, and $\rho_{io}(\succ_I)$ be the probability of o that any i obtains.

2.2.2 Group manipulation

I define two group manipulation concepts. In the weak one, a group manipulates a mechanism if by misreporting preferences some member of the group obtains new lottery that is not strictly first-order stochastically dominated by the lottery in the truth-telling case, while the other members obtain lotteries same with the truth-telling case. In the strong one, a group manipulates a mechanism if by misreporting preferences every member of the group obtains a new lottery that strictly first-order stochastically dominates the lottery in the truth-telling case. So they respectively induce a strong and a weak group strategy-proofness concept.

Formally, a group $J \subseteq I$ **weakly group manipulate** a mechanism ρ at $\succ_I$ if by reporting some $\succ'_J$, $\{i \in J : \rho_i(\succ_I) \neq \rho_i(\succ_{I\setminus J}, \succ'_J)\}$ is nonempty, and for every i in the set, $\rho_i(\succ_I) \not\succ_i \rho_i(\succ_{I\setminus J}, \succ'_J)$. Then ρ is **group strategy-proof** if it is not weakly group manipulable. When J is restricted to be singleton, group strategy-proofness reduces to **strategy-proofness**. A group $J \subseteq I$ **strongly group manipulate** ρ at $\succ_I$ if by reporting some $\succ'_J$, $\rho_i(\succ_{I\setminus J}, \succ'_J) \succ_i \rho_i(\succ_I)$ for all $i \in J$. Then ρ is **minimally group strategy-proof** if it is not strongly group manipulable. When J is restricted to be singleton, minimal group strategy-proofness reduces to **weak strategy-proofness** defined by Bogomolnaia and Moulin (2001).

In the paper I will prove that an efficient and fair mechanism is strongly group manipulable. By the definition, claiming strong group manipulability is a robust statement: no matter what cardinal utilities agents may have behind their preferences, a manipulating group always want to manipulate the mechanism to obtain higher expected utilities. By contrast, when claiming a mechanism is weakly group manipulable (Bade 2016a), it only guarantees that a manipulating group want to manipulate the mechanism when they have some proper cardinal utilities.

To further illustrate that strong group manipulation is indeed the strongest manipulation concept in the environment, I define two intermediate concepts. Formally, at any $\succ_I$, a group J **I-intermediately group manipulate** a mechanism ρ if by reporting some $\succ'_J$, $\rho_i(\succ_I) \not\succeq_i \rho_i(\succ_{I\setminus J}, \succ'_J)$ for all $i \in J$. That is, every member of J obtains a new lottery that is not strictly first-order stochastically dominated by the truth-telling case. Therefore, this concept is a stronger version of weak group manipulation. At any $\succ_I$, a group J **II-intermediately group manipulate** ρ if by reporting some $\succ'_J$, $\rho_i(\succ_{I\setminus J}, \succ'_J) \succeq_i \rho_i(\succ_I)$ for all $i \in J$, and $\rho_j(\succ_{I\setminus J}, \succ'_J) \succ_j \rho_j(\succ_I)$ for at least one $j \in J$. That is, at least one member of J obtains new lottery that strictly first-order stochastically dominates the lottery in the truth-telling case, and the other members obtain lotteries same with the truth-telling case. Therefore, this concept is a weaker version of strong group manipulation. It is obvious that the relations between the multiple concepts are as follows.

$$\begin{tikzcd}
& \text{II-intermediate GM} \arrow[dr] & \\
\text{Strong GM} \arrow[ur] \arrow[dr] & & \text{Weak GM} \\
& \text{I-intermediate GM} \arrow[ur] &
\end{tikzcd}$$

In the following I will define ex-post efficiency and several fairness criteria. A mechanism ρ is said to satisfy an efficiency or fairness criterion if $\rho(\succ_I)$ satisfies the criterion for all $\succ_I$.

2.2.3 Ex-post efficiency

For any $\succ_I$, a deterministic assignment π is **Pareto efficient** if there does not exist another π' such that $\pi'(i) \succsim_i \pi(i)$ for all i and $\pi'(j) \succ_j \pi(j)$ for some j. An assignment p is **ex-post efficient** if for any π such that $\rho(\pi) > 0$, π is Pareto efficient. In an ex-post efficient mechanism every agent must be assigned an acceptable object. In the literature there are stronger efficiency criteria such as ordinal efficiency and ex-ante efficiency. However, since I want to show the tension between fairness and group incentive compatibility, I use ex-post efficiency as the weakest efficiency requirement.

2.2.4 Fairness

For any assignment p and any preference profile $\succ_I$, I first define three fairness criteria that have been studied a lot in the literature.

1. p is **envy-free** if $p_i \geq_i p_j$ for any distinct i, j. That is, every i thinks that his lottery is weakly better than any other j's.
2. p is **weakly envy-free** if $p_j \not\succ_i p_i$ for any distinct i, j. That is, every i does not think that any other j's lottery is strictly better than his.
3. p satisfies **equal treatment of equals** if $p_i = p_j$ for any distinct i, j such that $\succ_i = \succ_j$. That is, any two agents of equal preference orderings obtain equal lotteries.

In this paper I introduce three new fairness criteria. They capture some crucial properties of RSD and PS that make them be strongly group manipulable, and play important roles in my theorems.

4 p satisfies **equal top-assignment of equal tops** if $p_{i,o} = p_{j,o}$ for any two distinct i, j that mostly prefer the same object o.

5 p satisfies **uniform tail-assignment of uniform tails** if there exists some $o \in \hat{O}$ such that $SU(\succ_i, o) = SU(\succ_j, o)$ and $\succ_i |_{\hat{O}\backslash SU(\succ_i, o)} = \succ_j |_{\hat{O}\backslash SU(\succ_j, o)}$ for all distinct i, j, then $p_{io'} = p_{jo'}$ for all $o' \in \hat{O}\backslash SU(\succ_i, o)$. That is, if all agents' preferences have equal strict upper contour sets of some object o, and their preferences on the remaining objects are identical, then all agents obtain equal probability of each remaining object.

6 p satisfies **top advantage** if for any distinct i, j such that i mostly prefers some o while j does not, then $p_{j,o} > 0$ implies that $p_{i,o} > p_{j,o}$. That is, if any j obtains positive probability of some o that he does not mostly prefer, then any i who mostly prefers o must obtain a higher probability of o than j.

The common idea behind 3, 4, and 5 is that if agents have somehow "equal" preferences, they should receive somehow "equal" lotteries. Their differences are that, equal treatment of equals imposes restriction on agents whose whole preferences are equal, and accordingly requires that those agents obtain equal probabilities of all objects; equal top-assignment of equal tops imposes restriction on agents who top choices are equal, and accordingly requires that those agents obtain equal probabilities of the common top choice; while uniform tail-assignment of uniform tails has a bite only when all agents' preferences have equal tails, and accordingly requires that all agents receive equal probabilities of the objects in the common tail.

Criteria 3-5 are related to two other introduced by Nesterov (2016). Specifically, p satisfies **strong equal treatment of equals** if any two agents with equal preferences from their most preferred objects down to some particular object obtain equal probabilities of the objects from their most preferred to the particular one. p is **upper envy-free** if any two agents whose preferences have equal strict upper contour sets of some object o obtain equal probability of o.[3] It is easy to see that equal 3 and 4 are two extreme special cases of strong equal treatment of equals by restricting attention to agents' top choices or whole preferences, while 5 is implied by upper envy-freeness by restricting attention to all agents. Nesterov proves that envy-freeness implies upper envy-freeness, which further implies strong equal treatment of equals. Therefore, envy-freeness implies 4 and 5. 4 is also implied by **ordinal fairness** introduced by Hashimoto et al. (2014), which requires that any agent's surplus at any o that he obtains positive probability (i.e., total probability of received objects weakly better than o) should be no greater than any other agent's surplus at o. Thus, I can summarize these observations in Figure 2.1.

The idea behind top advantage is different from 3-5. As the name suggest, it requires agents have advantage at their top choices.[4] It is independent of 1-5, and can

[3]Formally, p is **upper envy-free** if $SU(\succ_i, o) = SU(\succ_j, o)$ implies that $p_{i,o} = p_{j,o}$, while p satisfies **strong equal treatment of equals** if $SU(\succ_i, o) = SU(\succ_j, o)$ and $\succ_i |_{SU(\succ_i,o)} = \succ_j |_{SU(\succ_j,o)}$ imply that $p_{i,o'} = p_{j,o'}$ for all $o' \in SU(\succ_i, o) \cup \{o\}$.

[4]It has a flavor somehow similar to **favoring higher ranks** introduced by Kojima and Ünver (2014) for deterministic assignments, which requires that if an agent does not obtain an object preferred to his assignment, then the object must be assigned to another agent who ranks the object

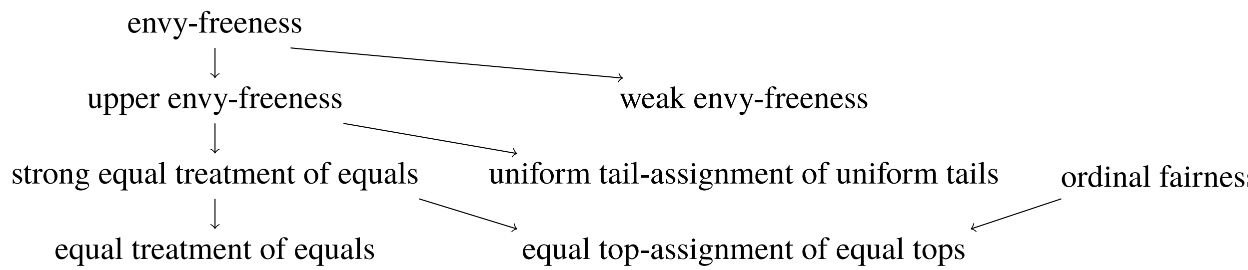


Figure 2.1: Relations between multiple fairness criteria

be seen as a complement of equal top-assignment of equal tops. When each agent obtains positive probability of his most preferred object, ordinal fairness implies top advantage. If an ex-post efficient assignment satisfies both top advantage and equal top-assignment of equal tops, then ex-post efficiency guarantees that every agent's most preferred object must be assigned, and equal top-assignment of equals and top advantage guarantee that every agent must obtain positive probability of his most preferred object, which is greater than that of any other agent who mostly prefers a different object. This fact plays an important role in the proofs of my theorems.

2.3 Impossibility theorems for $|I| = 3$ and $|O| \geq 3$

In this section I consider the simple environment of three agents and at least three objects. My first theorem is as follows.

Theorem 1. *When $|I| = 3$ and $|O| \geq 3$, in $\mathcal{Q}$ (therefore, also in $\mathcal{U}$), an ex-post efficient mechanism that satisfies equal treatment of equals (or weak envy-freeness), equal top-assignment of equal tops, and top advantage is strongly group manipulable.*

In this simple environment an ex-post efficient mechanism that satisfies equal top-assignment of equal tops and weak envy-freeness must also satisfy equal treatment of equals. So in the above theorem equal treatment of equals can be replaced by weak envy-freeness.

In the proof I construct two preference profiles such that if a mechanism satisfies ex-post efficiency and equal treatment of equals, then the mechanism is minimally group strategy-proof only if it finds same assignments for the two preference profiles. However, if the mechanism further satisfies equal top-assignment of equal tops and top advantage, it must find different assignments for them. Therefore, the mechanism

at least as high as the initial agent.

is strongly group manipulable. The proof is also a good illustration of the roles of new fairness criteria I introduce.

In this simple environment, Bogomolnaia and Moulin (2001) and Nesterov (2016) respectively prove that RSD is the only ex-post efficient and strategy-proof mechanism that satisfies equal treatment of equals or weak envy-freeness. Therefore, I have the following corollary.

Corollary 7. *When $|I| = 3$ and $|O| \geq 3$, in $\mathcal{Q}$ (therefore, also in $\mathcal{U}$), an ex-post efficient mechanism that satisfies equal treatment of equals (or weak envy-freeness) cannot be strategy-proof and minimally group strategy-proof simultaneously.*

Thus, there is a trade-off between individual incentive compatibility and minimal group incentive compatibility.

2.4 Impossibility theorems for $|I| \geq 4$ and $|O| \geq 3$

2.4.1 Universal preference domain $\mathcal{U}$

When the preference domain is $\mathcal{U}$, I can construct preference profiles subsume those in the proof of Theorem 1 by requiring that three particular agents prefer three particular objects to the remaining ones, while the other agents do not accept the three objects. Then the previous theorems still hold in this section. Similar constructions are also used by other papers which assume that outside option is available or objects are as many as agents (Erdil 2014; Martini 2016; Nesterov 2016).

Theorem 2. *When $|I| \geq 4$ and $|O| \geq 3$, in $\mathcal{U}$, an ex-post efficient mechanism that satisfies equal treatment of equals (or weak envy-freeness), equal top-assignment of equal tops, and top advantage is strongly group manipulable.*

2.4.2 No outside-option domain $\mathcal{Q}$

If the preference domain is $\mathcal{Q}$, agents must report all objects as acceptable. So the preference profiles constructed in Section 2.4.1 are not allowed in this domain. Moreover, when a group misreport their preferences, the fairness criteria in Theorem 2 do not have useful restrictions on the change of the assignments. In the following I prove that if a mechanism further satisfies uniform tail-assignment of uniform tails, I can recover the previous impossibility result.

Theorem 3. *When $|I| \geq 4$ and $|O| \geq 3$, in $\mathcal{Q}$, an ex-post efficient mechanism that satisfies equal treatment of equals (or weak envy-freeness), equal top-assignment of* *equal tops, top advantage, and uniform tail-assignment of uniform tails is strongly group manipulable.*

Since envy-freeness implies all fairness criteria in Theorem 1-3 except for top advantage, I have the following corollary.

Corollary 8. *When $|I| \geq 3$ and $|O| \geq 3$, in $\mathcal{Q}$ (therefore, also in $\mathcal{U}$), an ex-post efficient mechanism that satisfies envy-freeness and top advantage is strongly group manipulable.*

Nesterov (2016) proves that when there are at least three agents and objects are as many as agents, any ex-post efficient and envy-free mechanism is not strategy-proof. Therefore, by adding top advantage I obtain a strong impossibility theorem regarding group incentive compatibility. Bade (2016a) proves that any ex-post efficient mechanism that satisfies equal treatment of equals is weakly group manipulable. So by using more mild fairness criteria than her, I obtain a much stronger negative result (see discussion in Section 2.2).

2.5 (Im)possibility theorems for $|I| \geq 3$ and $|O| = 2$

The previous negative results are proved in the environment of at least three objects. This motivates me to examine the environment of only two objects. Interestingly, I show that ex-post efficiency and some fairness criteria can guarantee that a mechanism is (minimally) group strategy-proof, if outside option is not available. In the absence of outside option, every agent has only one way to misreport preferences. Therefore, a proper combination of fairness criteria can have strict restriction on the possible lotteries agents obtain by misreporting preferences.

Theorem 4. *When $|I| \geq 3$ and $|O| = 2$, in $\mathcal{Q}$, any ex-post efficient mechanism that satisfies equal treatment of equals and top advantage must be minimally group strategy-proof.*

As before, equal treatment of equals can be replaced by weak envy-freeness and equal top-assignment of equal tops.

If a mechanism further satisfies uniform tail-assignment of uniform tails, then the mechanism must be group strategy-proof. This is an interesting complement of Bade's negative result, which is proved in the environment of at least three objects.

Theorem 5. *When $|I| \geq 3$ and $|O| = 2$, in $\mathcal{Q}$, any ex-post efficient mechanism that satisfies equal treatment of equals, top advantage, and uniform tail-assignment of uniform tails must be group strategy-proof.*

However, if outside option is available, the previous negative result still holds. The proof is also similar as before.

Theorem 6. *When $|I| \geq 3$ and $|O| = 2$, in $\mathcal{U}$, an ex-post efficient mechanism that satisfies equal treatment of equals (or weak envy-freeness), equal top-assignment of equal tops, and top advantage is strongly group manipulable.*

2.6 Discussion

2.6.1 RSD and PS

RSD and PS are two popular mechanisms in the object allocation problem.[5] RSD is ex-post efficient and strategy-proof, and satisfies weak envy-freeness and equal treatment of equals. PS is ordinally efficient and weakly strategy-proof, and satisfies envy-freeness. In the following I prove that they satisfy the three new fairness criteria I introduce. So they are strongly group manipulable in general environments.

Proposition 10. *RSD and PS satisfy equal top-assignment of equal tops, top advantage, and uniform tail-assignment of uniform tails.*

RSD attracts a lot of attention in literature because it is strategy-proof and treats agents in a symmetric way, which has a strong implication on fairness. However, its strong group manipulability is often ignored. PS satisfies a stronger efficiency criterion (ordinal efficiency) than RSD. Although it is not strategy-proof, it is proved to be weakly strategy-proof (Bogomolnaia and Moulin 2001), and is asymptotically equivalent to RSD (Che and Kojima 2010). However, I show that PS is strongly group manipulable.

2.6.2 Independence of axioms

A complete analysis requires me to show that the efficiency/fairness criteria in my theorems are also necessary. I do this below for the two possibility theorems in Section 2.5. Ex-post efficiency is obviously necessary for impossibility theorems:

[5] Competitive Equilibrium from Equal Income (CEEI, Hylland and Zeckhauser 1979) is also a desirable mechanism. But CEEI requires agents to report cardinal utilities, which is beyond the framework of this paper and also hard to implement in practice.

the mechanism that always assigns all agents the virtual object $\emptyset$ trivially satisfies all fairness criteria and is group strategy-proof. However, it is hard to show that each fairness criterion is also necessary for these theorems. That is, it is hard to propose an ex-post efficient mechanism that satisfies other fairness criteria and prove it is minimally group strategy-proof. The reason is that, for deterministic mechanisms there is a equivalence result between group strategy-proofness and the combination of strategy-proofness and non-bossiness, but for random mechanisms there is no similar equivalence result between minimal group strategy-proofness and the combination of (weak) strategy-proofness and some easy-to-check condition (e.g., non-bossiness). For example, RSD is strategy-proof and non-bossy, but not minimally group strategy-proof. So to prove minimal group strategy-proofness, I need to verify all possible deviations of all possible groups, which is difficult in the general environment of at least three agents and at least three objects (it is possible to do so for the possibility theorems in the two-object environment because each agent has only one way to misreport preferences).

Below I show that each efficiency/fairness criterion in the two possibility theorems is necessary.

(1) Ex-post efficiency. The mechanism that assigns all agents $\emptyset$ for some preference profile and assigns the outcome of RSD for other preference profiles obviously satisfies equal treatment of equals, top advantage and uniform tail-assignment of uniform tails. However, it is strongly group manipulable.

(2) Equal treatment of equals. Consider the mechanism ρ that finds the outcome of RSD for all other preference profiles but finds the assignments for the following two particular preference profiles shown below. In the table o^q means that the corresponding agent obtains probability q of o.

$\succ^*_{i_1}$	$\succ^*_{i_2}$	$\succ^*_j$
$o_1^{1/3}$	$o_1^{2/3}$	$o_2^{2/3}$
$o_2^{1/3}$	o_2^{0}	o_1^{0}

(a) $\rho(\succ^*_I)$

$\succ^\circ_{i_1}$	$\succ^*_{i_2}$	$\succ^*_j$
$o_2^{2/9}$	$o_1^{5/9}$	$o_2^{2/3}$
$o_1^{4/9}$	$o_2^{1/9}$	o_1^{0}

(b) $\rho(\succ^\circ_I)$

It is obvious that ρ satisfies top advantage and uniform tail-assignment of uniform tails but not equal treatment of equals. However, i_1 can strongly group manipulate ρ at $\succ^*_I$ by reporting $\succ^\circ_{i_1}$.

(3) Top advantage. Consider the mechanism ρ that finds the outcome of RSD for all other preference profiles but finds the following assignments for two particular preference profiles shown below.

$\succ^*_{i_1}$	$\succ^*_{i_2}$	$\succ^*_j$
$o_1^{2/9}$	$o_1^{2/9}$	$o_2^{1/9}$
$o_2^{4/9}$	$o_2^{4/9}$	$o_1^{5/9}$

(a) $\rho(\succ^*_I)$

$\succ^\circ_{i_1}$	$\succ^\circ_{i_2}$	$\succ^*_j$
$o_2^{1/3}$	$o_2^{1/3}$	$o_2^{1/3}$
$o_1^{1/3}$	$o_1^{1/3}$	$o_1^{1/3}$

(b) $\rho(\succ^\circ_I)$

It is obvious that ρ satisfies equal treatment of equals and uniform tail-assignment of uniform tails but not top advantage. i_1, i_2 can strongly group manipulate ρ at $\succ^*_I$ by reporting $(\succ^\circ_{i_1}, \succ^\circ_{i_2})$.

(4) Uniform tail-assignment of uniform tails. Consider the mechanism ρ that finds the outcome of RSD for all other preference profiles but finds the following assignments for two particular preference profiles shown below.

$\succ^*_{i_1}$	$\succ^*_{i_2}$	$\succ^*_j$
$o_1^{4/9}$	$o_1^{4/9}$	$o_2^{8/9}$
$o_2^{1/18}$	$o_2^{1/18}$	$o_1^{1/9}$

(a) $\rho(\succ^*_I)$

$\succ^\circ_{i_1}$	$\succ^\circ_{i_2}$	$\succ^*_j$
$o_2^{1/3}$	$o_2^{1/3}$	$o_2^{1/3}$
$o_1^{1/3}$	$o_1^{1/3}$	$o_1^{1/3}$

(b) $\rho(\succ^\circ_I)$

It is obvious that ρ satisfies equal treatment of equals and top advantage but not uniform tail-assignment of uniform tails. i_1, i_2 can weakly group manipulate ρ at $\succ^*_I$ by reporting $(\succ^\circ_{i_1}, \succ^\circ_{i_2})$.

2.6.3 Other efficiency criteria

Since fairness is the motivation for using random mechanisms, this paper focuses on the tension between fairness and group incentive compatibility and chooses the weakest efficiency requirement, i.e., ex-post efficiency. Of course, it is an interesting exercise to examine whether the combination of a stronger efficiency criterion and weaker fairness criteria than I use still produces impossibility theorems.[6] This is left for future research.

[6] I pursue this direction but fail to find interesting results. Nesterov (2016) proves that when there are at least four agents, the combination of ordinal efficiency and weak envy-freeness (or equal division lower bound) is incompatible with strategy-proofness. Strategy-proofness is a strong requirement for individual incentive compatibility. Hence, by assuming a mechanism satisfies the three properties Nesterov can pin down the assignments for some preference profiles and find contradictions. By contrast, minimal group strategy-proofness is a rather weak requirement. It is not handy to use along with ordinal efficiency and some fairness criteria to find contradictions.

2.6.4 Restricted preference domain

In the paper I only consider the universal preference domain and the no outside-option domain. In some applications the preferences of agents may belong to a further restricted domain. Then we may hope that the impossibility theorems disappear in these domains. However, a careful examination of my proofs should convince the reader that only the following three preference orderings are essential for Theorem 1 and 3 (Theorem 3 also holds in the universal domain $\mathcal{U}$). The three orderings differ only in the rank of three objects. So as long as a domain allows such a small variation, my impossibility theorems still hold. For example, the single-peaked preference domain, which is often used in spatial competition and voting theory, and the single-dipped preference domain both allow such a variation.

$$\begin{array}{ccc} o_1 & o_2 & o_2 \\ o_2 & o_1 & o_3 \\ o_3 & o_3 & o_1 \\ \vdots & \vdots & \vdots \end{array}$$

Chapter 3

A NEW SOLUTION TO THE RANDOM ASSIGNMENT PROBLEM WITH PRIVATE ENDOWMENT

3.1 Introduction

In many matching markets the central question is how to assign indivisible objects to agents without using monetary transfers. Examples include the assignment of public school seats to children, the assignment of on-campus apartments to students, and the assignment of donated kidneys to patients. Depending on the ownership structure in the problems, they are often classified as *house allocation problem*, *house allocation problem with existing tenants* and *house exchange*.

In this paper I make two contributions to the literature. First, I propose a generalization of the Probabilistic Serial (PS) mechanism to solve house allocation problems with existing tenants in the strict preferences environment. I denote my generalization by PS^E. Second, I propose a new method to adapt PS^E (and also PS) to deal with indifferent preferences. My method is more straightforward to understand and easier to implement than the current method in the literature. Interestingly, both of my contributions are driven the same idea I call "you request my house - I get your speed". In the following I elaborate on each contribution.

Bogomolnaia and Moulin (2001) propose PS to solve house allocation problems. PS is implemented in a simple way: imagine each object as a "divisible cake" and let agents "eat" objects according to their preference orderings with equal speeds; each agent's consumption is the random assignment he obtains in PS. Compared to previous mechanisms such as Random Priority (RP, Abdulkadiroğlu and Sönmez 1998), the most important property of PS is *ordinal efficiency*. It is a desirable efficiency notion between ex ante efficiency[1] and ex post efficiency (RP is only ex post efficient). However, if an agent has a private endowment, he may obtain a positive fraction of an object worse than his private endowment in PS. So PS is not *individually rational* for house allocation problems with existing tenants.

My proposed PS^E deviates from PS in two aspects. First, at any time of the procedure if the private endowment of an agent is eaten by other agents, the agent

[1]When the cardinal utilities of agents are known, Hylland and Zeckhauser (1979) use the peseudo competitive market to obtain ex ante efficient random assignments.

can instantly get an additional eating speed, which is equal to the total speed at which his private endowment is being eaten. This is what I mean "you request my house - I get your speed". Second, at any time of the procedure if several existing tenants (those with private endowments) want to consume each other's private endowment such that they form a cycle, I let them trade the fractions of their private endowments instantly. This is similar to the Top Trading Cycle (TTC) mechanism proposed by Shapley and Scarf (1974). But in the paper I show that this aspect is also the outcome of "you request my house - I get your speed". Both aspects guarantee that the demand of an existing tenant must be satisfied weakly before his private endowment is exhausted. So PS^E must be *individually rational*. PS^E is *ordinally efficient* since it is a still simultaneous eating algorithm.[2] PS^E is also *envy-free among new agents* (those without private endowments) since they always have equal eating speeds.

The motivation behind PS^E is not just to satisfy individual rationality; indeed there are infinitely many generalizations of PS to satisfy individual rationality. I argue that PS^E is a natural generalization of PS by fully allowing existing tenants to trade their private endowments with the others. To illustrate it, if the objects without owners (I call *social endowments*) are regarded as owned by all agents collectively, then an algorithm satisfies "you request my house - I get your speed" (each agent's eating speed comes from the transfer of his endowments) if and only if it is PS^E. Hence, PS^E is essentially a dynamic process of trading eating speeds.

To support the above argument in another way, I prove that PS^E generalizes two equivalence theorems of PS in the literature, which are summarized in Figure 3.1. Here, "$a \Leftrightarrow b$" means that a is equivalent to b, "$a \underset{\text{asym.}}{\Leftrightarrow} b$" means that a is asymptotically equivalent to b, and "$a \longrightarrow b$" means that a is generalized to b. Specifically, Kesten (2009) proposes $TTCfED$ for house allocation problems. It proceeds by first assigning the fractions of all houses equally to all agents, then letting agents trade their fractional endowments as in TTC. Kesten proves that PS is equivalent to $TTCfED$. I show that this result still holds between PS^E and a direct generalization of $TTCfED$ I call TTC^E.[3] Che and Kojima (2010) prove that PS is asymptotically equivalent to RP. When there are private endowments, Abdulkadiroğlu and Sönmez (1999) generalize RP to the *random "you request my house-I get your turn" (YRMH-IGYT)* mechanism. By using the technique of

[2] When existing-tenant cycles are traded instantly, it is as if that existing tenants in the cycles have infinitely large eating speeds.

[3] In TTC^E only the fractions of social endowments are equally assigned to all agents.

Che and Kojima I similarly prove that PS^E is asymptotically equivalent to random YRMH-IGYT. The intuition behind it is that the additional eating speeds of existing tenants in PS^E correspond to their additional chances to request objects in random YRMH-IGYT in large markets.

$$\begin{tikzcd}
PS \arrow[r, Leftrightarrow, "\text{Kesten (2009)}"] \arrow[d, "\text{this paper}"] & TTCfED \arrow[d, "\text{this paper}"] & PS \arrow[r, Leftrightarrow, "\text{Che and Kojima (2010)}", "\text{asym.}"'] \arrow[d, "\text{this paper}"] & RP \arrow[d, "\text{Abdulkadiroğlu and Sönmez (1999)}"] \\
PS^E \arrow[r, Leftrightarrow, "\text{this paper}"] & TTC^E & PS^E \arrow[r, Leftrightarrow, "\text{this paper}", "\text{asym.}"'] & \text{random YRMH-IGYT}
\end{tikzcd}$$

Figure 3.1: Equivalence theorems

Since PS^E subsumes PS as a special case, it cannot be strategy-proof. But I prove that it satisfies *bounded invariance*, which is proposed by Bogomolnaia and Heo (2012), along with ordinal efficiency and envy-freeness, to characterize PS. This property implies that PS^E cannot be weakly manipulated by a class of strategies I call *dropping strategies*. In particular, PS^E cannot be manipulated by *truncation strategies*, which have been well studied in the literature (Roth and Rothblum 1999; Ehlers 2008; Coles and Shorrer 2014; Castillo and Dianat 2016).

I compare PS^E with other mechanisms in the literature. Yılmaz (2010) proposes another generalization of PS called *Individually Rational Probabilistic Serial* (PS^{IR}). PS^{IR} minimally deviates from PS to accommodate the individual rationality constraint. So PS^{IR} satisfies a stronger fairness notion (no justified-envy) than PS^E. However, PS^{IR} has a worse incentive property than PS^E. In particular, it can be manipulated by truncation strategies. I have more discussion in Section 3.6. In a not circulated paper Sethuraman (2001) proposes a generalization of PS called *uniform eating rate* (UER) to satisfy individual rationality. According to the description of Yılmaz (2010), UER is close to PS^E: UER lets an existing tenant's eating speed be one as long as his individual rationality is not to be violated; otherwise, UER lets his eating speed equal the total speed at which his private endowment is being eaten. But there is no any characterization of UER which supports its selection from infinitely many algorithms of satisfying individual rationality. Random YRMH-IGYT is desirable because of strategy-proofness. But it has weaker efficiency and fairness properties than PS^E.

In my second contribution, I use the idea of "you request my house - I get your speed" to solve the difficulty caused by indifferent preferences. Katta and Sethuraman (2006) deal with indifferences in PS through iteratively solving maximum network flow problems. This method is further used by Yılmaz (2009) to deal with indifferent preferences in PS^{IR}, and used by Athanassoglou and Sethuraman (2011) to deal with indifferent preferences in their generalization of PS^{IR}. Compared with the previous method, mine is straightforward to understand and easy to implement.

Specifically, suppose two agents i, j have preferences over two objects h, h' such that $h \sim_i h'$ and $h \succ_j h'$. If i uses an arbitrary tie-breaking rule such that he first eats h, then eats h', then the resulting random assignment in PS^E (and PS) is not ordinally efficient since i, j can exchange an equal fraction of h and h' such that j is strictly better off and i is not worse off. In my method, when h is exhausted, since h' remains and $h \sim_i h'$, I let i label his consumption of h as "available" for others to consume. Since j strictly prefers h to h', j will eat i's consumption of h. Then I compensate i by letting he eat h' with an additional speed that equals the speed at which i's consumption of h is being eaten. If h' is j's private endowment, then i and j form a cycle when j consumes i's consumption of h. Then the cycle is traded immediately. In either case i's welfare is exactly compensated when his consumption of h is consumed by others. My method incorporates an adaption of PS as a special case.

Related Literature There are a lot of papers studying random assignment problems and especially PS. I only discuss some of them here. Since PS is not strategy-proof, Ekici and Kesten (2015) use multiple equilibrium solutions to study the possible outcome of PS. Hugh-Jones, Kurino, and Vanberg (2014) use laboratory experiments to study the incentive property of PS. They find that manipulation is a significant problem in PS. Dogan, Dogan, and Yildiz (2016) propose a new efficiency criterion for random assignments. By the criterion PS can be improved in efficiency without sacrificing fairness. Hashimoto et al. (2014) and Bogomolnaia (2015) independently characterize PS.

Several papers have extended PS to general environments. Kojima (2009) and Heo (2014) extend PS to the environment in which agents demand more than one objects. But in the former paper agents have equal demands, but in the latter agents may have heterogeneous demands. Budish et al. (2013) extend PS to the environment in which agents have multi-item demands and random assignments are subject to exogenous constraints. They show that only a special structure of contraints can be solved. Balbuzanov (2014) extends PS to kidney exchange problems in which the length of trading cycles is constrained.

Kesten and Ünver (2015) is different from the above papers in that they extend the Deferred Acceptance algorithm (Gale and Shapley 1962) to solve a random assignment problem in which objects have coarse priority rankings of agents. A house allocation problem with existing tenants can be seen as a special case in their environment.

The rest of the paper is organized as follows. Section 3.2 describes the assignment problem and defines some concepts. Section 3.3 proposes PS^E under the strict preferences environment. Section 3.4 discusses the properties of PS^E. Section 3.5 proves the two equivalence theorems. Section 3.6 compares PS^E with the other mechanisms. Section 3.7 presents my method of dealing with indifferent preferences. Section 3.8 concludes. All proofs are in the appendix.

3.2 House Allocation Problem with Existing Tenants

3.2.1 The Model

A *house allocation problem with existing tenants* is a four-tuple $m = \{I, H, \pi, \succsim_I\}$ where I is a finite set of agents, H is a finite set of houses, $\pi : I \to H$ is an endowment function, and $\succsim_I = (\succsim_i)_{i \in I}$ is the preference profile of all agents. There is a null house h_0 in H such that if $\pi(i) = h_0$, then i has no private endowment. Otherwise, $\pi(i)$ is the private endowment of i. Each non-null house has only one copy and can be owned by at most one agent. So $\pi(i) \neq \pi(j)$ for all distinct $i, j \in I$ unless $\pi(i) = \pi(j) = h_0$. The agents who own non-null houses are called *existing tenants*. Their private endowments are called *occupied houses*. The set of existing tenants is denoted by I_E, and the set of their private endowments is denoted by H_O. The remaining agents and houses are called *new agents* and *vacant houses*, and their sets are denoted by I_N and H_V respectively. For convenience I also call H_V *social endowments*. Each agent i demands one house and has a preference relation $\succsim_i$ over H with the strict part denoted by $\succ_i$. I do not assume that $\succsim_i$ is strict, but I assume that no agent is indifferent between a real house $h \in H \backslash \{h_0\}$ and the null house h_0. Every house h satisfying $h \succsim_i \pi(i)$ is called *acceptable* to i. I sometimes denote a problem by $\{\succsim_I\}$ if there is no confusion to omit other elements. Let $\mathcal{M}$ denote the set of all house allocation problems with existing tenants, and $\mathcal{R}$ denote the set of all preference relations over H.

3.2.2 Random Assignment and Other Concepts

A *random assignment* is a matrix $q = (q_{ih})_{i \in I, h \in H}$ such that $\sum_{i \in I} q_{ih} \leq 1$ for $\forall h \in H \backslash \{h_0\}$ and $\sum_{h \in H} q_{ih} = 1$ for $\forall i \in I$. Here q_{ih} is the probability that i is assigned the house h, and q_{ih_0} is the probability that i is not assigned any house. So $q_i = (q_{ih})_{h \in H}$ is the lottery assigned to i. If $q_{ih} \in \{0, 1\}$ for all $i \in I$ and all $h \in H$, then q is a *deterministic assignment*.

A lottery q_i is *individually rational* for agent i if $h \succsim_i \pi(i)$ for all $h \in H$ such that $q_{ih} > 0$. That is, i is never assigned an unacceptable house with a positive probability. A random assignment q is *individually rational* if q_i is individually rational for every agent i. Given $\succsim_i$, I compare any two lotteries assigned to i in the sense of first-order stochastic dominance. Formally, a lottery q_i *first-order stochastically dominates* another lottery q'_i for i, denoted by $q_i \trianglerighteq_i q'_i$, if

$$\sum_{h' \succsim_i h} q_{ih'} \geq \sum_{h' \succsim_i h} q'_{ih'} \text{ for } \forall h \in H.$$

If the above inequality holds strictly for an acceptable h, I say q_i strictly stochastically dominates q'_i and denote it by $q_i \triangleright_i q'_i$. For any two random assignments q and q', I say q strictly stochastically dominates q' if $q_i \trianglerighteq_i q'_i$ for all i and $q_j \triangleright_j q'_j$ for some j. I denote it by $q \triangleright q'$. A random assignment q is *ordinally efficient* if there does not exist q' such that $q' \triangleright q$.

In a random assignment q an agent i *envies* another agent j if $q_i \trianglerighteq_i q_j$ does not hold, and i *weakly envies* j if $q_j \triangleright_i q_i$ holds. Then q is *envy-free* if any i does not envy any other j, and is *weakly envy-free* if any i does not weakly envy any other j. Similarly, q is *new-agent envy-free* if any new agent i does not envy any other new agent j, and q is *weakly new-agent envy-free* if any new agent i does not weakly envy any other new agent j.

For every problem $m \in \mathcal{M}$, let $Q(m)$ be the set of all random assignments for m. Let $Q := \bigcup_{m \in \mathcal{M}} Q(m)$ be the set of all possible random assignments. A *random assignment mechanism* is a function $\varphi : \mathcal{M} \to Q$ such that $\varphi(m) \in Q(m)$ for $\forall m \in \mathcal{M}$. The above paragraphs have defined multiple properties of random assignments. A mechanism φ is said to have one property if for all $m \in \mathcal{M}$, $\varphi(m)$ has the property. A mechanism φ is *boundedly invariant* if for any h, if any agent i in any problem m reports a preference relation $\succsim'_i$ that coincides with his true preference relation $\succsim_i$ at all houses weakly better than h, the assignments of houses weakly better than h in $\succsim_i$ do not change in $\varphi(m)$. Formally, if letting $U(\succsim_i, h) := \{h' \in H \mid h' \succsim_i h\}$ be the upper contour set of $\succsim_i$ at h and let $\succsim_i |_{U(\succsim_i, h)}$ be the restriction of $\succsim_i$ to $U(\succsim_i, h)$, then for $\forall h$ and $\forall \succsim'_i \in \mathcal{R}$ such that $U(\succsim'_i, h) = U(\succsim_i, h)$ and $\succsim'_i |_{U(\succsim'_i, h)} = \succsim_i |_{U(\succsim_i, h)}$, $\varphi_{jh'}(\{\succsim_I\}) = \varphi_{jh'}(\{\succsim'_i, \succsim_{-i}\})$ for $\forall j \in I$ and $\forall h' \in U(\succsim_i, h)$. Finally, a mechanism φ is *strategy-proof* if $\varphi_i(\{\succsim_I\}) \trianglerighteq_i \varphi_i(\{\succsim'_i, \succsim_{-i}\})$ for $\forall \succsim'_i \in \mathcal{R}$,

$\forall i \in I$ and $\forall m = \{\succsim_I\}$, and φ is *weakly strategy-proof* if for $\forall i \in I$ there does not exist $\succsim_i' \in \mathcal{R}$ such that $\varphi_i(\succsim_i', \succsim_{-i}) \rhd_i \varphi_i(\{\succsim_I\})$.

3.3 The PS^E Mechanism

From this section to Section 3.6 I assume all agents' preferences are strict. I deal with weak preferences in Section 3.7. As mentioned before, PS^E generalizes PS with two new features. First, at any time $t \in [0,1]$, if an existing tenant's private endowment is being eaten by other agents, the existing tenant can immediately get an additional eating speed which equals the total speed at which his private endowment is being eaten. I call this feature "you request my house - I get your speed". Second, at any time $t \in [0,1]$, if several existing tenants want to consume each other's private endowment such that they form a cycle, they trade the fractions of their private endowments instantly. How much can be traded depends on the remainder of each private endowment and the residual demand of each existing tenant in the cycle. The second feature is similar to TTC, but PS^E can still be seen as a simultaneous eating algorithm defined by Bogomolnaia and Moulin (BM hereafter):[4] when there are cycles among some existing tenants, let their eating speeds be infinitely large. Before giving the formal definition I first illustrate PS^E through a simple example.

Example 2. *A problem consists of $H = \{h_0, h_1, \ldots, h_6\}$ and $I = \{i_1, i_2, \ldots, i_6\}$. i_1, i_2, i_3, i_4, i_5 are existing tenants and own h_1, h_2, h_3, h_4, h_5 respectively. i_6 is a new agent and h_6 is a vacant house. The following table is the preference profile of all agents where $\succsim_o$ is i_o's preference list ($o = 1, \ldots, 6$). Boxed houses are private endowments of the corresponding agents. Unacceptable houses for existing tenants are omitted from their preference lists.*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$	$\succsim_5$	$\succsim_6$
h_2	h_3	h_1	h_2	h_1	h_3
h_3	$\boxed{h_2}$	h_5	h_6	h_6	h_4
$\boxed{h_1}$		$\boxed{h_3}$	h_5	h_4	h_5
			$\boxed{h_4}$	$\boxed{h_5}$	h_0

PS^E will solve this problem in the following steps.

Step 1: *At $t = 0$, i_1, i_4 want to consume h_2, i_2, i_6 want to consume h_3, and i_3, i_5 want to consume h_1.*

[4] Formally, a *simultaneous eating algorithm* is identified by a profile of eating speed functions $\{s_i(t)\}_{i \in I}$ where $s_i(t) : [0,1] \to \mathbb{R}_+$ is a measurable function such that $\int_0^1 s_i(t)dt = 1$.

There is a cycle consisting of existing tenants i_1, i_2, i_3 and their private endowments. Trade this cycle instantly. Since every private endowment in the cycle is never consumed and every existing tenant in the cycle demands one house, I trade one unit of house in the cycle. After the trade i_1, i_2, i_3 own h_2, h_3, h_1 respectively and they stop consuming other houses. Note that after the trading the time is still at $t = 0$ since the trading happens instantly.

Step 2: *Still at $t = 0$, i_4, i_5 want to consume h_6 and i_6 wants to consume h_4.*

There is no cycle. i_6 eats h_4 with speed one. i_5 eats h_6 also with speed one. But i_4 eats h_6 with speed two since his private endowment h_4 is being eaten with a total speed of one.

At $t = 1/3$, h_6 is exhausted. Then i_4 consumes $2/3$ of h_6, i_5 consumes $1/3$ of h_6, and i_6 consumes $1/3$ of h_4

Step 3: *At $t = 1/3$, i_4 wants to consume h_5 and i_5, i_6 want to consume h_4.*

There is a cycle consisting of existing tenants i_4, i_5 and their private endowments. h_4 has a remainder of $2/3$, h_5 has remainder of 1, while i_4's remaining demand is $1/3$ and i_5's remaining demand is $2/3$. So I trade $1/3$ of each house in the cycle. After the trade i_4 gets $1/3$ of h_5 and stops consuming other houses, and i_5 gets $1/3$ of h_4. Note that the time is at $t = 1/3$.

Step 4: *Still at $t = 1/3$, both i_5 and i_6 want to consume h_4. There is no cycle, so each of them eats h_4 with speed one. At $t = 1/2$, h_4 is exhausted. Then each of i_5 and i_6 gets $1/6$ of h_4.*

Step 5: *At $t = 1/2$, both i_5 and i_6 want to consume h_5. Since h_5 is the private endowment of i_5, I say there is a i_5's self-cycle. Since h_5 has a remainder of $2/3$ and i_5's remaining demand is $1/2$, I trade $1/6$ of h_5 in the cycle. After the trade i_5 gets $1/6$ of h_5 and stops consuming other houses.*

Step 6: *Still at $t = 1/2$, i_6 is the only agent, so he eats h_5 with speed one. At $t = 1$, h_5 is exhausted and i_6's remaining demand is fulfilled.*

The above steps are summarized in Table 3.1.

3.3.1 Formal Definition

I present the formal definition of PS^E in this section. As in the above example, I track the procedure of PS^E by discrete steps at which some houses are exhausted or some agents' demands are fulfilled.

Step d: what happened	i_1	i_2	i_3	i_4	i_5	i_6
1: $i_1 - i_2 - i_3 - i_1$ cycle	h_2	h_3	h_1			
2: eating				$2/3h_6$	$1/3h_6$	$1/3h_4$
3: $i_4 - i_5 - i_4$ cycle				$1/3h_5$	$1/3h_4$	
4: eating					$1/6h_4$	$1/6h_4$
5: i_5's self-cycle					$1/6h_5$	
6: eating						$1/2h_5$

Table 3.1: The procedure of PS^E in Example 2.

d: steps; t_d: the time at which step d ends;
$r_i(d)$: i's residual demand when step d ends; $r_h(d)$: remainder of h when step d ends;
$s_i(t)$: i's eating speed at t; $s_h(t)$: total speed at which h is being eaten at t;
$A_h(t)$: set of agents who point to h at t;
$H(d)/I(d)/I_N(d)/I_E(d)$: remaining houses/agents/new agents/existing tenants when step d ends.

Initialization: $I(0) = I$, $H(0) = H$, $r_i(0) = 1$ for $\forall i \in I$, $r_h(0) = 1$ for $\forall h \in H\backslash\{h_0\}$, $t_0 = 0$.

Step $d \geq 1$: If $I(d-1) = \emptyset$ or $H(d-1) = \emptyset$, stop. Otherwise, proceed to the Pointing stage.

- *Pointing*: Every $i \in I(d-1)$ points to his most preferred house in $H(d-1)$. If it is h_0, let i point to his own copy of h_0. Every occupied house in $H(d-1)$ point to its owner if its owner is in $I(d-1)$. Go to the Consuming stage.
- *Consuming*: If there exist cycles consisting of existing tenants and their private endowments such as $h_1 \to i_1 \to h_2 \to i_2 \to \cdots \to i_k \to h_1$, go to the Trading Cycle stage. Otherwise, go to the Eating stage.
  - *Trading Cycle*: Trade the above cycles instantly. For every cycle c, the trading quota of c is defined as

    $$TQ(c) := \min\{\min_{i\in I(c)} r_i(d-1), \min_{h\in H(c)} r_h(d-1)\},$$

    where $I(c)$ and $H(c)$ are the set of existing tenants and the set of occupied houses involved in c respectively. So every agent in $I(c)$ obtains $TQ(c)$ of the house he points to. Then $r_h(d) := r_h(d-1) - TQ(c)$ for $\forall h \in H(c)$ and $r_i(d) := r_i(d-1) - TQ(c)$ for $\forall i \in I(c)$.

    For every house h and every agent i not involved in any cycle, $r_h(d) := r_h(d-1)$ and $r_i(d) := r_i(d-1)$. Agents in $G(d) := \{i \in I(d-1) : r_i(d) =$

$0\}$ are full. Then $I(d) := I(d-1)\backslash G(d)$ and $H(d) := H(d-1)\backslash\{h \in H(d-1) : r_h(d) = 0\}$. Step d ends at time $t_d := t_{d-1}$. Go to step $d+1$.

– Eating: All agents simultaneously eat the houses they point to with eating speeds specified as follows.

For $t \geq t_{d-1}$, $s_i(t) := 1$ for $\forall i \in I_N(d-1)$, and $s_j(t) := s_{\pi(j)}(t) + 1$ for $\forall j \in I_E(d-1)$ where $s_{\pi(j)}(t) := \sum_{i \in A_{\pi(j)}(t_{d-1})} s_i(t)$ is the total speed at which $\pi(j)$ is being eaten.

Define $t_d := \min\{t_a : r_a(d-1) - s_a(t)(t_a - t_{d-1}) = 0, a \in H(d-1) \cup I(d-1)\}$. That is, step d ends when a house in $H(d-1)$ is exhausted or an agent in $I(d-1)$ is full, depending on which happens earlier.

Then $r_h(d) := r_h(d-1) - s_h(t_{d-1})(t_d - t_{d-1})$ for $\forall h \in H(d-1)$ and $r_i(d) := r_i(d-1) - s_i(t_{d-1})(t_d - t_{d-1})$ for $\forall i \in I(d-1)$. Agents in $G(d) := \{i \in I(d-1) : r_i(d) = 0\}$ are full. Then $I(d) := I(d-1)\backslash G(d)$ and $H(d) := H(d-1)\backslash\{h \in H(d-1) : r_h(d) = 0\}$. Go to step $d+1$.

3.3.2 Characterization of PS^E

Now I show that PS^E is characterized by the idea of "you request my house - I get your speed" among all simultaneous eating algorithms.[5] Specifically, by treating social endowments as equally owned by all agents, I say a simultaneous eating algorithm satisfies "you request my house - I get your speed" if each agent's eating speed at any time comes from the transfer of their endowments (either social or private). In particular, each social endowment uniformly transfers the total speed at which it is being eaten to its each owner, and each private endowment transfers the total speed at which it is being eaten only to its unique owner. If the owner of a private endowment is satisfied at any time, his private endowment is treated as a social endowment owned by the remaining agents.

Formally, at any $t \in [0,1]$ let $E_i(t)$ be the set of endowments of each agent i, which includes social endowments and his private endowment. Let $O_h(t)$ be the set of owners of each house h. $A_h(t)$, $s_i(t)$ and $s_h(t)$ are defined as before. Then, a simultaneous eating algorithm satisfies "you request my house - I get your speed" if for each remaining h and each remaining i at t,

$$(1) \quad s_h(t) = \textstyle\sum_{i \in A_h(t)} s_i(t); \qquad (2) \quad s_i(t) = \textstyle\sum_{h \in E_i(t)} s_h(t)/|O_h(t)|.$$

[5]Bogomolnaia and Heo (2012) have shown that every random assignment can be seen as the outcome of a simultaneous eating process. So I essentially characterize PS^E among all mechanisms.

It is interesting to observe that (1) and (2) are similar to market equilibrium conditions: if I imagine each agent's eating speed as his budget, then at any time t each agent spends his budget on his most preferred house; on the other hand, each agent's budget comes from the money other agents spend on his endowments. Hence, (1) and (2) characterize an equilibrium of trading eating speeds.

Proposition 11. *A simultaneous eating algorithm satisfies "you request my house - I get your speed" if and only if it is equivalent to* PS^E.

By saying two mechanisms are equivalent I mean they always find the same assignment for the same problem. The proof is rather simple. At any time t, if there is no cycle among existing tenants, I can normalize the eating speed of every new agent i, which is equal to $\sum_{h\in E_i(t)} s_h(t)/|O_h(t)|$, to one. Then the eating speed of every existing tenant j is $s_j(t) = s_{\pi(j)}(t) + \sum_{h\in E_i(t)} s_h(t)/|O_h(t)| = s_{\pi(j)}(t) + 1$. If there are cycles, let a typical cycle be $\pi(j_1) \longrightarrow j_1 \longrightarrow \cdots \longrightarrow \pi(j_n) \longrightarrow j_n \longrightarrow \pi(j_1)$. Then conditions (1) and (2) implies that $s_{j_1}(t) \le \cdots \le s_{j_n}(t) \le s_{j_1}(t)$. So $s_{j_1}(t) \cdots = s_{j_n}(t)$. However, since $s_{j_2}(t) = s_{\pi(j_2)}(t) + \sum_{h\in E_i(t)}(s_h(t)/|O_h(t)|) \ge s_{j_1}(t) + s_i(t)$, it must be that $s_i(t) = 0$ for every new agent i. Hence, for any existing tenant j who is not involved in any cycle, $s_{\pi(j)} = 0$, which implies that $s_j(t) = s_{\pi(j)}(t) + s_i(t) = 0$. So it is equivalent to trading the cycle instantly.

It is easy to see that when there are no private endowments, conditions (1) and (2) also characterize PS.

3.4 The Properties of PS^E

In this section I discuss the properties of PS^E in efficiency, fairness, and manipulability. First, since PS^E is a simultaneous eating algorithm, it must be *ordinally efficient*. Second, PS^E is individually rational since existing tenants are satisfied no later than their private endowments are exhausted. Third, although PS^E does not satisfy envy-freeness, it satisfies *new-agent envy-freeness* since new agents always have equal eating speeds.

Proposition 12. *The* PS^E *mechanism is ordinally efficient, individually rational, and new-agent envy-free.*

BM prove that PS is weakly strategy-proof. However, the following example shows that PS^E is not.

Example 3. *A problem consists of* $H = \{h_0, h_1, \ldots, h_9\}$ *and* $I = \{i_1, i_2, \ldots, i_8\}$. i_1, i_2 *own* h_1, h_2 *respectively. The preference profile is as follows.* $\succsim_o$ *is the true preference relation of agent* i_o *(*$o = 1, \ldots, 9$*). Agents* i_3, i_4 *have identical preferences, and* i_5, i_6, i_7 *have identical preferences.*

$\succsim_1$	$\succsim_2$	$\succsim_3 / \succsim_4$	$\succsim_5 / \succsim_6 / \succsim_7$	$\succsim_8$	$\succsim'_8$
h_7	h_7	h_1	h_2	h_1	h_1
h_8	h_9	h_3	h_3	h_8	h_3
$\boxed{h_1}$	$\boxed{h_2}$	h_4	h_4	h_3	h_4
		h_5	h_5	h_4	h_8
		h_6	h_6	h_5	h_5
		h_0	h_0	h_6	h_6
				h_0	h_0

When all agents report their true preferences, the procedure of PS^E *and the assignment it finds are shown by Table 3.2.*

time	i_1	i_2	i_3/i_4	$i_5/i_6/i_7$	i_8
1/8	$1/2h_7$	$1/2h_7$	$1/8h_1$	$1/8h_2$	$1/8h_1$
+1/8	$1/2h_8$	$1/2h_9$	$1/8h_1$	$1/8h_2$	$1/8h_1$
+1/7			$1/7h_3$	$1/7h_3$	$1/7h_8$
+1/7			$1/7h_4$	$1/7h_4$	$1/7h_8$
+1/7			$1/7h_5$	$1/7h_5$	$1/7h_8$
+1/14			$1/14h_6$	$1/14h_6$	$1/14h_8$
+1/16			$1/16h_6$	$1/16h_6$	$1/16h_6$

Table 3.2: The procedure of PS^E in Example 3

At $t = 1/4$, h_1 *is exhausted and* i_1, i_2 *are full and stop consuming. Among the remaining agents only* i_8 *wants to consume* h_8. *But the remaining* $1/2h_8$ *can not satisfy the residual demand of* i_8 *which is* $3/4$. *At this point if* i_8 *strategically chooses to eat* h_3, *then eats* h_4 *at* $t = 3/8$ *and returns to* h_8 *at* $t = 1/2$, *his residual demand can be exactly satisfied by* $1/2h_8$. *So by using this strategy* i_8 *can eat more fractions of* h_3 *and* h_4 *without losing any fraction of* h_8. *So* i_8 *can manipulate* PS^E *by reporting* $\succsim'_8$, *and* PS^E *is not weakly strategy-proof.*

In the above example i_8 manipulates PS^E by reshuffling his preferences. This kind of manipulation happens because some existing tenants may leave the algorithm earlier than others. However, I prove that PS^E is *boundedly invariant*. This property means that the temporary assignment at any step of PS^E is determined only by the preferences revealed by the step. This implies that PS^E cannot be manipulated by a class of strategies I call *dropping strategies*. Specifically, $\succsim_i'$ is a *dropping strategy* of $\succsim_i$ if it is obtained by dropping some houses from the set of acceptable houses in $\succsim_i$: $U(\succsim_i', \pi(i)) \subsetneqq U(\succsim_i, \pi(i))$ and $\succsim_i' |_{U(\succsim_i', \pi(i))} = \succsim_i |_{U(\succsim_i', \pi(i))}$. Then a mechanism φ is *weakly dropping-strategy-proof* if $\varphi(\{\succsim_{-i}, \succsim_i'\}) \rhd_i \varphi(\{\succsim_I\})$ does not hold for any i and any dropping strategy $\succsim_i'$ in any problem $\{\succsim_I\}$. I prove that PS^E is weakly dropping-strategy-proof.

Truncation strategies are special cases of dropping strategies: $\succsim_i'$ is a truncation strategy of $\succsim_i$ if there exists some $h \succsim_i \pi(i)$ such that $\succsim_i' |_{U(\succsim_i', \pi(i))} = \succsim_i |_{U(\succsim_i, h)}$. A mechanism φ is *truncation-strategy-proof* if $\varphi_i(\{\succsim_I\}) \unrhd_i \varphi_i(\{\succsim_i', \succsim_{-i}\})$ for any i and any truncation strategy $\succsim_i'$ in any problem $\{\succsim_I\}$. Hence, PS^E is truncation-strategy-proof.

Proposition 13. *The PS^E mechanism is boundedly invariant, weakly dropping-strategy-proof and truncation-strategy-proof.*

3.5 Equivalence Theorems

In this section I prove two equivalence theorems, both generalizing the counterparts of PS. In the first theorem I prove that PS^E is equivalent to a probabilistic version of TTC, which generalizes a result of Kesten (2009). In the second theorem I prove that PS^E is asymptotically equivalent to random YRMH-IGYT, which generalizes the result of Che and Kojima (2010). I discuss the two theorems one by one.

When there are no private endowments, Kesten proves that PS is equivalent to *TTC from Equal Division* ($TTCfED$). $TTCfED$ proceeds in two steps: at the first step the fractions of all vacant house are uniformly assigned to all agents such that each agent has the same endowment profile; at the second step agents trade their fractional endowments as in TTC. In this paper I repeat the two steps except that existing tenants have private endowments. I call the corresponding mechanism *TTC from Equal Division of Social Endowments* and denote it by TTC^E.

TTC from Equal Division of Social Endowments

$d/I(d)/H(d)/r_i(d)$: have the same interpretations as in PS^E;

$e_i(d)$: the endowment profile of agent i when step d ends;

$\{i_h \cdot \Delta h : \forall \Delta h \in e_i(d)\}$: set of pseudo-agents representing i when step d ends. Here i_h is the pseudo-agent holding Δh.[6] i is called i_h's host. Pseudo-agents that hold vacant houses are also called new pseudo-agents, and pseudo-agents that hold occupied houses are also called pseudo-tenants.

Initialization: $I(0) = I$, $H(0) = H$, $e_i(0) = \{\frac{1}{|I|}h\}_{h\in H_V} \cup \{\pi(i)\}$, and $r_i(0) = 1$ for $\forall i \in I(0)$.

Step $d \geq 1$: If $I(d-1) = \emptyset$ or $H(d-1) = \emptyset$, stop. Otherwise, proceed to the following steps.

- *Pointing*: For every $i \in I(d-1)$ and i's every pseudo-agent i_h, if h is i's most preferred house, let i_h point to himself. Otherwise, let i_h point to all pseudo-agents $j_{h'}$ such that $j \neq i$ and h' is i's most preferred house in $H(d-1)$. There will be multiple cycles.

- *Selecting Cycles*: I select the following three types of cycles to trade:

  (i) existing-tenant cycles: the cycles consisting only of pseudo-tenants;

  (ii) new-agent self-cycles: the cycles formed by new pseudo-agents pointing to themselves;

  (iii) feasible new-agent cycles: the cycles involving at most two new pseudo-agents and not contained in (i) and (ii).

- *Trading*: For every selected cycle c, the trading quota of c is

  $$TQ(c) := \min\{\min_{i\in I(c)} r_i(d-1), \min_{h\in H(c)} \Delta h\},$$

  where $I(c)$ is the set of hosts of the pseudo-agents involved in c, $H(c)$ is the set of houses involved in c, and Δh is the amount of h held by the relevant pseudo-agent involved in c. Then,

  (1) Cycles of types (i) and (ii) are traded immediately with their trading quotas.

  (2) Cycles of type (iii) are traded with a common quota, which equals the smallest trading quota of all type (iii) cycles.

  Any house a pseudo-agent obtains by trading cycles belongs to his host.

[6] For example, if $e_i(d) = \{h_1, 1/2h_2, 1/3h_3\}$, then i is represented by three pseudo-agents: $i_{h_1} \cdot h_1$, $i_{h_2} \cdot 1/2h_2$ and $i_{h_3} \cdot 1/3h_3$.

- *Leaving*: If a pseudo-agent uses up the house he holds, the pseudo-agent is removed. When an agent i's demand is satisfied, i.e. $r_i(d) = 0$, i leaves the algorithm along with his all pseudo-agents. The remaining endowments of i, if any, are regarded as social endowments and uniformly assigned to the remaining agents in $I(d)$. Go to step $d + 1$.

Appendix C.3.2 presents the procedure of using TTC^E to solve Example 2. Now I prove that PS^E is equivalent to TTC^E.[7]

Theorem 7. *The PS^E mechanism is equivalent to the TTC^E mechanism.*

This theorem implies that PS^E can be equivalently seen as a dynamic process of trading ownerships. In Section 3.3.2 I have shown that PS^E can be seen as a dynamic process of trading eating speeds. In an eating algorithm agents spend their time on the houses they want to consume. If an agent consumes a house, he gives up the time to consume other houses. So trading eating speeds is equivalent to trading ownerships. Hence, Proposition 11 and Theorem 7 illustrate the same feature of PS^E.

Although PS is ex ante more efficient than RP, Che and Kojima prove that PS and RP are asymptotically equivalent if the market size properly grows. Using their proof method I similarly prove that PS^E is asymptotically equivalent to random YRMH-IGYT,[8] which generalizes RP to house allocation problems with existing tenants. The intuition behind the result is straightforwardly illustrated by the analogy between "you request my house - I get your speed" and "you request my house - I get your turn". Specifically, the advantage of an existing tenant in PS^E exactly corresponds to the advantage of the same existing tenant in random YRMH-IGYT when the market size is infinitely large.

Theorem 8. *The PS^E mechanism is asymptotically equivalent to the random "you request my house - I get your turn" mechanism.*

[7] In my proof I make a slight adjustment of TTC^E such that PS^E is equivalent to TTC^E step by step.

[8] Random YRMH-IGYT proceeds as follows. First randomly draw an ordering of all agents from the uniform distribution. Then let agents sequentially obtain their most preferred objects among remaining ones according to the ordering. But if an agent wants to obtain the private endowment of an existing tenant who has not obtained an object, let the existing tenant points to his most preferred object. If the object is the private endowment of another existing tenant who has not obtained an object, repeat the process until I have a chain or cycle such that all agents in the chain or cycle obtain the objects they want.

Since random YRMH-IGYT is strategy-proof, Theorem 8 implies that PS^E is asymptotically strategy-proof. Liu and Pycia (2013) prove that in house allocation problems all mechanisms that are asymptotically ordinally efficient, asymptotically strategy-proof and treat equals equally, must be asymptotically equivalent under some regularity condition. Because of the existence of existing tenants, Theorem 8 is not implied by their result.

3.6 Comparison with Other Mechanisms

In this section I compare PS^E with other mechanisms in the literature. Yılmaz (2010) proposes the PS^{IR} mechanism, which is the minimal deviation from PS by satisfying the *individual rationality* (IR) constraint. Specifically, PS^{IR} proceeds by letting agents eat their most preferred houses with equal speeds, but if at any time the IR constraint of some group of existing tenants binds, PS^{IR} isolates the group and their remaining acceptable houses as a sub-problem by blocking other agents from consuming those houses. To illustrate it I present the procedure of PS^{IR} in solving Example 2 in Appendix C.3.1. Yilmaz proves that PS^{IR} is ordinally efficient and satisfies a fairness notion called *no justified-envy* (NJE). Formally, an assignment q satisfies NJE if for any two agents i, j, if q_i is individually rational for j, then i does not envy j. Since the IR constraint of new agents never binds, PS^{IR} must be new-agent envy-free. So PS^{IR} has a stronger fairness property than PS^E. On the other hand, PS^{IR} has worse incentive property than PS^E. In particular, PS^{IR} is not *boundedly invariant* (see Appendix C.3.1), and can be manipulated by truncation strategies.

PS^E and PS^{IR} are different in their treatments of private ownerships. In PS^E existing tenants can trade their ownerships of private endowments with others, but cannot in PS^{IR}. In PS^{IR} existing tenants have advantages over new agents only to the extent that their IR constraint is respected. This difference implies that PS^E and PS^{IR} should be used in different applications. To illustrate it I construct the following example.

Example 4. *There are three agents and two houses. Their preferences and the assignments found by PS^E and PS^{IR} are shown in Table 3.3.*

At $t = 0$, i_1, i_3 want to consume h_2 and i_2 wants to consume h_1:

• *In PS^{IR} all agents have the same eating speed of one. So at $t = 1/2$, h_2 is exhausted. Then to satisfy the IR of i_1, the remaining $1/2 h_1$ is exclusively given to i_1.*

Problem			PS^{IR}			PS^{E}		
$\succsim_1$	$\succsim_2$	$\succsim_3$	i_1	i_2	i_3	i_1	i_2	i_3
h_2	h_1	h_2	$1/2h_2$	$1/2h_1$	$1/2h_2$	$2/3h_2$	$2/3h_1$	$1/3h_2$
$\boxed{h_1}$	h_0	h_0	$1/2h_1$	$1/2h_0$	$1/2h_0$	$1/3h_1$	$1/3h_0$	$2/3h_0$

Table 3.3: Example 4

• In PS^E agent i_1 has an eating speed of two and others have an eating speed of one. At $t = 1/3$, h_2 is exhausted. Then i_1 consumes $1/3h_1$ by trading a self-cycle. Lastly i_2 consumes $1/3h_1$.

The two ranodm assignments are implemented by putting different probabilities on two deterministic assignments:

$$PS^{IR} = 1/2\begin{pmatrix} i_1 & i_2 & i_3 \\ h_2 & h_1 & h_0 \end{pmatrix} + 1/2\begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_0 & h_2 \end{pmatrix},$$

$$PS^{E} = 2/3\begin{pmatrix} i_1 & i_2 & i_3 \\ h_2 & h_1 & h_0 \end{pmatrix} + 1/3\begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_0 & h_2 \end{pmatrix}.$$

If this example is the assignment of on-campus apartments, then the school controls all apartments and can give existing tenants only squatting rights over their current apartments. In this case PS^{IR} is more appropriate since it gives more fairness among agents. In the above example i_1, i_3 have the same chance of obtaining h_2.

If this example is a kidney exchange problem in which h_2 is a non-directional altruistic kidney that is compatible with both i_1 and i_3, while h_1 is the kidney donated to i_1 by his families, but it is incompatible with i_1 yet compatible with i_2. In this case i_1 controls h_1 and can bring h_1 away if he wants to. Since maximizing the number of successful transplants is the main objective in this situation, PS^E is more appropriate since with $2/3$ probability there are two transplants and with $1/3$ probability there is only one transplant. In general by giving existing tenants advantages PS^E can incentivize them to bring their donated kidneys to the exchange program.

According to the description of Yılmaz (2010), UER of Sethuraman (2001) also trades cycles between existing tenants. But when there are no cycles at any step d, the eating speeds of agents are specified as follows: for $t \geq t_d$, $s_i(t) = 1$

Mechanism	Efficiency	Manipulability	Fairness
PS^E	Ordinally efficient	Weakly dropping-strategy-proof, truncation-strategy-proof, asymptotically strategy-proof	New-agent envy-free
PS^{IR}	Ordinally efficient	None	No justified envy
UER	Ordinally efficient	Weakly dropping-strategy-proof, truncation-strategy-proof	New-agent envy-free
Random YRMH-IGYT	Ex post efficient	Strategy-proof	Weakly new-agent envy-free

Table 3.4: Comparison between PS^E and other mechanisms

for any $i \in I_N(d-1)$, $s_j(t) = 1$ if $r_j(d-1) < r_{\pi(j)}(d-1)$ and $s_j(t) = s_{\pi(j)}(t)$ if $r_j(d-1) = r_{\pi(j)}(d-1)$ for any $j \in I_E(d-1)$. That is, the eating speed of any existing tenant j becomes equal to the total speed at which his private endowment is being eaten only when his IR constraint is to be violated. There is no any known characterization of UER. In Example 4 UER finds the same assignment as PS^{IR} does. Random YRMH-IGYT is a desirable mechanism because it is strategy-proof. But it is only ex post efficient and weakly new-agent envy-free. It approximates PS^E in large markets under some conditions. I summarize the properties of these mechanisms in Table 3.4.

3.7 PS^E under Weak Preferences

In this section I use the idea of "you request my house - I get your speed" to adapt PS^E to weak preferences. It incorporates an adaption of PS to weak preferences.[9] I briefly describe my method as follows.

3.7.1 The Idea

I first arbitrarily choose an exogenous ordering $>_H$ of all houses and an exogenous ordering $>_I$ of all agents. They are used to break ties when necessary. At any step of PS^E, if an agent i is indifferent between two remaining houses, say h and h', then if $h >_H h'$, I let i point to h and consume it. Then at some latter step if h is exhausted but h' remains, I let i point to h' and label his consumption of h, denoted by h^i, as "available" for other agents to consume. If h^i is indeed consumed by other agents, I compensate i by letting he consume h' with an additional speed, which equals the speed at which h^i is being consumed. In this way i is exactly compensated.

[9]When there are no social endowments and all agents are existing tenants, PS^E coincides with TTC. So it also incorporates an adaption of TTC to weak preferences, which turns out to be very close to Jaramillo and Manjunath (2012).

The above describes a simple case. In general more complicated cases can appear. For example, when agent i labels his consumption h^i as available, suppose another agent k is indifferent between h and some house h'' in his consumption. Then k must most prefer h among all remaining houses and all consumptions that are labeled as available.[10] Then I let k point to i's consumption h^i and label his consumption of h'' as available. If there exists another agent j who also labels his consumption h^j as available, I let j point to h^i if $i >_I j$. After h''^k is labeled as available, other agents may be further induced to label their consumptions as available. So in general a chain can exist that looks like

$$i_1 \to h_2^{i_2} \to i_2 \to h_3^{i_3} \to i_3 \to \cdots \to h_m^{i_m} \to i_m \to h_{m+1},$$

where every $i_o (o = 2, \ldots, m)$ labels his consumption $h_o^{i_0}$ as available because $h_{o+1} \sim_{i_o} h_o$, h_{m+1} is a remaining house, and i_1 strictly prefers h_2 to all remaining houses and points to $h_2^{i_2}$ after breaking any possible ties. Then there are two cases to consider:

- If i_1 eats $h_2^{i_2}$, then every i_o $(o = 2, \ldots, m-1)$ eats $h_{o+1}^{i_{o+1}}$ and i_m eats h_{m+1} with additional speeds as specified in PS^E. But if some agent $i_o (o = 2, \ldots, m)$ in the chain has been full, his eating speed will be equal to the speed at which his consumption $h_o^{i_o}$ is consumed. So i_o will never over-consume.

- If i_1 is an existing tenant and h_{m+1} is i_1's private endowment, the above chain becomes a cycle. Then I trade the cycle immediately. But when calculating how much house can be traded I ignore the residual demands of all $i_o (o = 2, \ldots, m)$ since the trade does not increase their total consumptions.

In the above chain every $i_o (o = 1, \ldots, m)$ is path-linked to a remaining house h_{m+1} through the chain. In general only this kind of chains are *indispensable* to keep ordinal efficiency of PS^E. Any exchange of available consumptions among agents is unnecessary and may mess the algorithm up. So in my formal definition below, at every step I introduce a pointing stage in which I carefully construct the chains to exclude the possibility that agents point to each other's available consumptions such that they form an unnecessary cycle.

[10] It is because in the procedure of PS^E agents always consume their most preferred houses. Since k consumed h'' earlier and h'' is as good as h, h must be a best house for k at that step.

3.7.2 Formal Definition

There are three stages at every step: *Pointing* stage, *Consuming* stage, and *Labeling* stage.

d: step;

$h^0(d)$: The remaining (fractional) house h when step d ends;

$H(d) := \{h^0(d) : \ h \in H, h^0(d) > 0\}$: set of remaining houses when step d ends;

$I(d)$: set of remaining agents when step d ends;

$h^i(d)$: agent i's consumption of house h when step d ends;

$p_i(d)$: the (fractional) house agent i points to at step d;

$C_i(d) := \{h^i(d) : h \in H, h^i(d) > 0\}$: agent i's consumption profile when step d ends;

$H_i(d)$: set of consumptions that agent i labels as available when step d ends;

Initialization: $H(0) = H$, $I(0) = I$, and $c_i(0) = a_i(0) = \emptyset$ for $\forall i \in I$. Arbitrarily choose an ordering $>_H$ of all houses and an ordering $>_I$ of all agents.

Step $d \geq 1$:

- *Pointing*:

  Define the menu of every $i \in I(d-1)$ as $M_i(d) := H(d-1) \cup H_{-i}(d-1)$, where $H_{-i}(d-1) := \bigcup_{j \in I(d-1): j \neq i} H_j(d-1)$. Then the set of i's most preferred (fractional) houses in $M_i(d)$ is $Ch_i(M_i(d)) := \arg \underset{h \in M_i(d)}{\succsim_i} - \max h$.[11]

  Round 0: For every $i \in I(d-1)$, if the (fractional) house he points to at step $d-1$ is still in $M_i(d)$, let i still point to the house. Formally, if $p_i(d-1) = h^j(d-2)$ and $h^j(d-1) \in M_i(d)$, then $p_i(d) = h^j(d-1)$. Denote the set of all such agents by $P_0(d)$. It is obvious that $P_0(1) = \emptyset$.

  Round 1: For every $i \in I(d-1) \backslash P_0(d)$, if $Ch_i(M_i(d)) \cap H(d-1) \neq \emptyset$, let i point to the house in $Ch_i(M_i(d)) \cap H(d-1)$ that is ranked highest in $>_H$. Formally,

$$p_i(d) := \underset{h^0(d-1) \in Ch_i(M_i(d)) \cap H(d-1)}{\arg >_H - \max} h^0(d-1).$$

[11] Note that $M_i(d)$ may contain more than one fractions of the same house, for example, $h^0(d-1)$ and $h^j(d-1)$. Then i is indifferent between them.

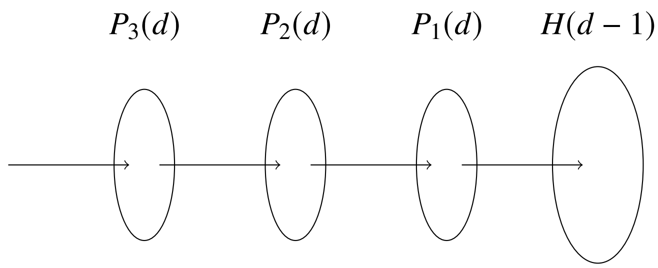


Figure 3.2: Illustration of the chains I construct

Note that some agents in $P_0(d)$ may also point to some houses in $H(d-1)$. Then I denote the set of all agents that point to some houses in $H(d-1)$ by $P_1(d)$.

Round 2: For every $i \in I(d-1) \backslash \{P_0(d) \cup P_1(d)\}$, $Ch_i(M_i(d)) \subset H_{-i}(d-1)$. Then for every i who most prefers some available consumption held by $P_1(d)$, that is, $Ch_i(M_i(d)) \cap (\cup_{j \in P_1(d)} H_j(d-1)) \neq \emptyset$, I let

$$p_i(d) := \underset{h^j(d-1) \in Ch_i(M_i(d)) \cap a_j(d-1)}{\arg >_H - \max} h^j(d-1), \tag{3.1}$$

$$\text{where } j := \underset{j' \in J_i(d)}{\arg >_I - \max} j', \tag{3.2}$$

$$\text{where } J_i(d) := \underset{j'' \in P_1(d): Ch_i(M_i(d)) \cap H_{j''}(d-1) \neq \emptyset}{\arg >_H - \max} p_{j''}(d). \tag{3.3}$$

That is, among the set of agents in $P_1(d)$ who hold i's most preferred available consumption, $J_i(d)$ are those who point to the house in $H(d-1)$ that is ranked highest in $>_H$. Then among $J_i(d)$ I choose the agent j who is ranked highest in $>_I$. Finally, among i's most preferred available consumptions held by j, i points to the consumption that is ranked highest in $>_H$.

I denote the set of the agents discussed in this around, and those in $P_0(d)$ who also point to some available consumption held by $P_1(d)$, by $P_2(d)$.

Round 3: For every $i \in I(d-1) \backslash \{P_0(d) \cup P_1(d) \cup P_2(d)\}$ who most prefers some available consumption held by $P_2(d)$, that is, $Ch_i(M_i(d)) \cap (\cup_{j \in P_2(d)} H_j(d-1)) \neq \emptyset$, I let

$$p_i(d) := \underset{h^j(d-1) \in Ch_i(M_i(d)) \cap H_j(d-1)}{\arg >_H - \max} h^j(d-1), \tag{3.4}$$

$$\text{where } j := \underset{j' \in J_i(d)}{\arg >_I - \max} j', \tag{3.5}$$

$$\text{where } J_i(d) := \underset{j'' \in P_2(d): Ch_i(M_i(d)) \cap H_{j''}(d-1) \neq \emptyset}{\arg >_H - \max} p^2_{j''}(d). \tag{3.6}$$

Here $p^2_{j''}(d)$ is the house that the owner of $p_{j''}(d)$ points to at step d. In other words, it is the house in $H(d-1)$ to which $j'' \in P_2(d)$ is linked through the chains I construct. Then the explanation of (4)-(6) is that, among the set of agents in $P_2(d)$ who hold i's most preferred available consumption, $J_i(d)$ are those who are linked to a house in $H(d-1)$ that is ranked highest in $>_H$. Then among $J_i(d)$ I choose the agent j that is ranked highest in $>_I$. Finally, among i's most preferred available consumptions held by j, i points to the consumption that is ranked highest in $>_H$.

I denote the set of the agents discussed in this around, and those in $P_0(d)$ who also point to some available consumption held by $P_2(d)$, by $P_3(d)$.

. . .

Round n: For every $i \in I(d-1) \setminus \cup_{x=1}^{n-1} P_x(d)$ who most prefers some available consumption held by $P_{n-1}(d)$, that is, $Ch_i(M_i(d)) \cap (\cup_{j \in P_{n-1}(d)} H_j(d-1)) \neq \emptyset$, I let

$$p_i(d) := \underset{h^j(d-1) \in Ch_i(M_i(d)) \cap H_j(d-1)}{\arg >_H - \max} h^j(d-1), \tag{3.7}$$

$$\text{where } j := \underset{j' \in J_i(d)}{\arg >_I - \max} j', \tag{3.8}$$

$$\text{where } J_i(d) := \underset{j'' \in P_{n-1}(d) : Ch_i(M_i(d)) \cap H_{j''}(d-1) \neq \emptyset}{\arg >_H - \max} p^{n-1}_{j''}(d). \tag{3.9}$$

Here $p^{n-1}_{j''}(d)$ is the house in $H(d-1)$ to which $j'' \in P_{n-1}(d)$ is linked through the chains I construct. Then the explanation of (7)-(9) is that, among the set of agents in $P_{n-1}(d)$ who hold i's most preferred available consumption, $J_i(d)$ are those who are linked to a house in $H(d-1)$ that is ranked highest in $>_H$. Then among $J_i(d)$ I choose the agent j that is ranked highest in $>_I$. Finally, among i's most preferred available consumptions held by j, i points to the consumption that is ranked highest in $>_H$.

I denote the set of the agents discussed in this around, and those in $P_0(d)$ who also point to some available consumption held by $P_{n-1}(d)$, by $P_n(d)$.

Since there are finite agents, the above procedure must stop in some finite rounds. Then every agent in $I(d-1)$ points to some house. Then I let every private endowment and available consumption point to its owner.

- *Consuming*:

  Run the consuming stage of PS^E with the following two remarks:

- An agent whose demand has been satisfied can consume again only if his available consumptions are consumed by other agents. His eating speed equals the total speed at which his consumptions are consumed.
- If an agent is involved in a cycle along with his available consumption, his residual demand is omitted in calculating the trading quota of the cycle.

This stage ends if an available consumption or a remaining house is exhausted, or a cycle is traded, or an agent's demand is satisfied. Let $C_i(d)$ be the new consumption profile of every $i \in I(d-1)$ and $H(d)$ be the new set of remaining houses. If $H(d) = \emptyset$, stop the algorithm.

- *Labeling*:

I update the available consumption sets of agents in $I(d-1)$ sequentially in the following rounds.

Round 1: For every $i \in I(d-1)$, the available consumption set of i is $H_i(d) := \{h^i(d) : h^i(d) \in C_i(d)$ and $h \sim_i \hat{h}$ for some $\hat{h}^0(d) \in H(d)\}$. Denote the set of such agents by $L_1(d)$.

Round 2: For every $i \in I(d-1)\backslash L_1(d)$, the available consumption set of i is $H_i(d) := \{h^i(d) : h^i(d) \in C_i(d)$ and $h \sim_i \hat{h}$ for some $\hat{h}^k(d) \in \bigcup_{j \in L_1(d)} H_j(d)\}$. Denote the set of such agents by $L_2(d)$.

...

Round n: For every agent $i \in I(d-1)\backslash\{L_1(d) \cup L_2(d) \cup \cdots \cup L_{n-1}(d)\}$, the available consumption set of i is $H_i(d) := \{h^i(d) : h^i(d) \in C_i(d)$ and $h \sim_i \hat{h}$ for some $\hat{h}^k(d) \in \bigcup_{j \in L_{n-1}(d)} H_j(d)\}$. Denote the set of such agents by $L_n(d)$.

Since there are finite agents, the above process must finish in finite rounds. Then any agent who is full and has an empty set of available consumptions leaves the algorithm with his consumption. Then the set of remaining agents is denoted by $I(d)$. If $I(d) = \emptyset$, stop the algorithm. Otherwise, go to step $d+1$.

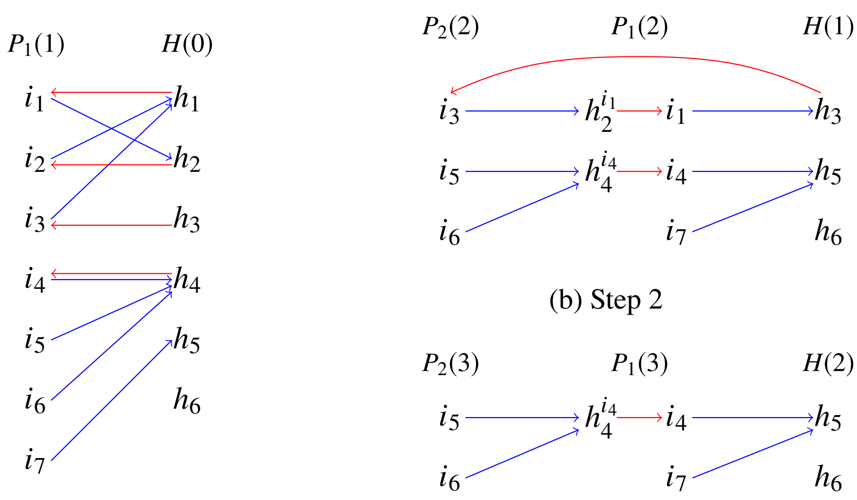
The following example illustrates the algorithm.

Example 5. *A problem consists of $H = \{h_0, h_1, \ldots, h_6\}$ and $I = \{i_1, i_2, \ldots, i_7\}$. i_1, i_2, i_3, i_4 are existing tenants and own h_1, h_2, h_3, h_4 respectively. i_5, i_6, i_7 are new agents and h_5, h_6 are vacant houses. The following table is the preference profile of* *all agents where* $\succsim_o$ *is* i_o*'s preference list. Unacceptable houses for existing tenants are omitted from their preference lists.*

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$	$\succsim_5$	$\succsim_6$	$\succsim_7$
h_2, h_3	h_1	h_1	$\boxed{h_4}, h_5$	h_4, h_5	h_4, h_6	h_5
$\boxed{h_1}$	h_3	h_2		h_6	h_5	h_6
	$\boxed{h_2}$	$\boxed{h_3}$		h_0	h_0	h_0

The two exogenous orderings are: $i_1 >_I i_2 >_I i_3 >_I i_4 >_I i_5 >_I i_6$ *and* $h_0 >_H h_1 >_H h_2 >_H h_3 >_H h_4 >_H h_5 >_H h_6$.

Step 1: *The pointing stage is shown as the following graph. There are two cycles:* $i_1 \to h_2 \to i_2 \to h_1 \to i_1$ *and* $i_4 \to h_4 \to i_4$*. After trading these cycles,* i_1 *gets* h_2 *and labels it as available since* $h_2 \sim_1 h_3$*,* i_2 *gets* h_1 *and leaves the algorithm, and* i_4 *gets* h_4 *and labels it as available since* $h_4 \sim_4 h_5$.



(a) Step 1

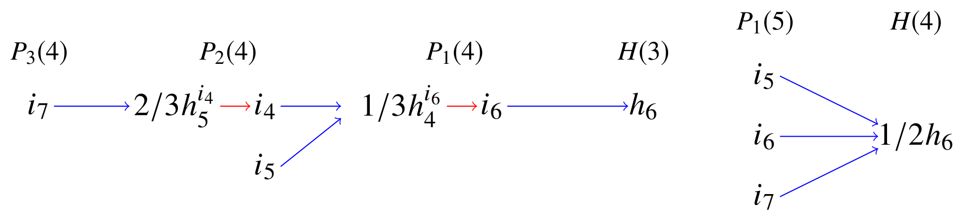
(b) Step 2

(c) Step 3

Figure 3.3: Steps 1, 2, and 3

Step 2: *The pointing stage is shown as the following graph. In particular,* i_5, i_6, i_7 *point to the same houses as they did at step 1. There is one cycle:* $i_1 \to h_3 \to i_3 \to h_2^{i_1} \to i_1$*. After trading the cycle* i_3 *gets* h_2 *and leaves the algorithm. Then* i_1 *gets* h_3 *and also leaves the algorithm.* i_4 *still labels his consumption of* h_4 *as available since* $h_4 \sim_4 h_5$.

Step 3: i_4, i_5, i_6, i_7 *point to the same houses as they did in Step 2. Since* i_4 *is full,* i_4*'s eating speed is two. Every other agent' eating speed is one. So* h_5 *is* *exhausted at $t = 1/3$, and i_4 gets $2/3h_5$, i_5, i_6 each get $1/3h_4$, and i_7 gets $1/3h_5$. In round 1 of the labeling stage i_6 labels his $1/3h_4$ as available since $h_4 \sim_6 h_6$. Then in round 2 both i_4 and i_5 label all of their consumption as available.*



(a) Step 4

(b) Step 5

Figure 3.4: Steps 4 and 5

Step 4: *The pointing stage is shown as the following graph. Since i_4 is full, his eating speed is one. Thus i_6's eating speed is three. Every other agent's eating speed is one. At $t = 1/2$, i_6's $1/3h_4$ is exhausted. Then in the labeling stage only i_6 labels his $1/2h_6$ as available. Since i_4 is full, he leaves the algorithm.*

Step 5: *All of i_5, i_6, i_7 point to h_6. At $t = 2/3$, h_6 is exhausted. Stop the algorithm.*

The final assignment is shown as follows:

i_1	i_2	i_3	i_4	i_5	i_6	i_7
h_3	h_1	h_2	$1/2h_4$	$1/2h_4$	$2/3h_6$	$1/2h_5$
			$1/2h_5$	$1/6h_6$		$1/6h_6$

3.7.3 Properties

I prove that PS^E under weak preferences has the same properties as before.

Proposition 14. *The PS^E mechanism under weak preferences is ordinally efficient, individually rational, new-agent envy-free, boundedly invariant, weakly dropping-strategy-proof, and truncation-strategy-proof.*

The equivalence theorems proved before may not hold, depending on how weak preferences are dealt with in TTC^E and Random YRMH-IGYT. That is, an agent may obtain different assignments in PS^E and TTC^E (or random YRMH-IGYT). But I conjecture that every agent can have (asymptotically) equal welfare in PS^E and TTC^E (or random YRMH-IGYT).

3.8 Conclusion

This paper proposes a new mechanism to solve random assignment problems in which some agents have private endowments. The new mechanism PS^E generalizes PS by letting agents benefit from the popularity of their private endowments, which is illustrated by the idea of “you request my house - I get your speed”. Interestingly, the same idea can also be used to deal with weak preferences in a simple way.

There are two directions for future research. Athanassoglou and Sethuraman (2011) generalize PS^{IR} to a fractional endowment setting and prove that the mechanism satisfies NJE. However, since two agents of equal endowments bring the same resources to the problem, they also believe that *equal-endowment no envy (EENE)*, which is not satisfied by their mechanism, is a more reasonable fairness criterion than NJE. It is interesting to extend PS^E to their setting such that EENE is satisfied. In another direction, it is interesting to examine whether PS can be extended to solve the random assignment problems with coarse priorities proposed by Kesten and Ünver (2015).

Appendix A

APPENDIX TO CHAPTER 1

A.1 Omitted Proofs

Proof of Proposition 1

For any $k \geq 0$, if all students are Lk, I denote the assignment found by the first round of BM by μ^k. In the following by induction I prove that μ^k coincides with the assignment found by round k of Fast DA. Then the proposition follows.

- At $L0$, each i reports his most preferred school as first choice. On the other hand, in round 0 of Fast DA each i applies to his most preferred school. So it is obvious that $s_i^0 = a_i^0$ and μ^0 coincides with the assignment found by round 0 of Fast DA.

- Suppose for all $r \leq k$ for some $k \geq 0$, μ^r coincides with the assignment found by round r of Fast DA. Now I consider $k+1$.

  If $\mu^k(i) = s_i^k$, which means that i is admitted by his reported first choice at Lk in μ^k, then s_i^k must be i's best obtainable school at $Lk+1$. On the other hand, $\mu^k(i) = s_i^k$ implies that i is admitted in round k of Fast DA. So $a_i^{k+1} = a_i^k = s_i^k = s_i^{k+1}$. If $\mu^k(i) = \emptyset$, by the induction assumption the school i will report at $Lk+1$ is also the school that i will apply to in round $k+1$ of Fast DA. So $s_i^{k+1} = a_i^{k+1}$.

  Hence, μ^{k+1} coincides with the assignment found by round $k+1$ of Fast DA. Then by induction, $s_i^k = a_i^k$ for all i and all $k \leq r_{FDA}$. For all $k > r_{FDA}$, i always reports $s_i^{r_{FDA}}$ as first choice at Lk. So $s_i^k = a_i^k$ for all i and all $k > r_{FDA}$.

Proof of Lemma 1

I prove it by contradiction. Suppose every insufficiently sophisticated i whose preferences satisfy $\mu^{DA}(i)\ P_i\ \mu^{BM}_{k_I}(i)$ reports a preference ordering P'_i such that $\mu^{DA}(i)\ P'_i\ \mu^{BM}_{k_I}(i)$. Let i_1 be any such student. Since $\mu^{DA}(i_1)\ P'_i\ \mu^{BM}_{k_I}(i_1)$, i_1 must be rejected by $\mu^{DA}(i_1)$ in some round of BM. Denote the round by r^1. $\mu^{DA}(i_1)$ must admit $q_{\mu^{DA}(i_1)}$ students in $\mu^{BM}_{k_I}$, and there is some i_2 who is admitted by $\mu^{DA}(i_1)$ in $\mu^{BM}_{k_I}$ but $\mu^{DA}(i_2) \neq \mu^{DA}(i_1)$. Since $\mu^{BM}_{k_I}$ Pareto dominates μ^{DA}, i_2 must prefer

$\mu^{DA}(i_2)$ to $\mu^{DA}(i_1)$. By assumption, $\mu^{DA}(i_2)P'_{i_2}\mu^{DA}(i_1)$. Since i_2 must apply to $\mu^{DA}(i_1)$ in a round no latter than r^1, i_2 must apply to $\mu^{DA}(i_2)$ in some earlier round r^2 such that $r^2 < r^1$. By the same argument as before, there is some i_3 who is admitted by $\mu^{DA}(i_2)$ in $\mu^{BM}_{k_I}$ but $\mu^{DA}(i_3) \neq \mu^{DA}(i_2)$. Then i_3 must prefer $\mu^{DA}(i_3)$ to $\mu^{DA}(i_2)$. Hence, i_3 must be rejected by $\mu^{DA}(i_3)$ in some round r^3 of BM, and $r^3 < r^2$. Denote by r^x the earliest round in which some student i_x is rejected by $\mu^{DA}(i_x)$ in BM. As before, there is some student i_{x+1} who is admitted by $\mu^{DA}(i_x)$ in $\mu^{BM}_{k_I}$ but $\mu^{DA}(i_{x+1}) \neq \mu^{DA}(i_x)$. Hence, i_{x+1} must apply to $\mu^{DA}(i_{x+1})$ and be rejected in a round earlier than r^x, which contradicts the definition of r^x.

Proof of Proposition 4

Let $\tilde{P}_i$ be the expressed preference ordering of i in Fast DA. Let $\tilde{\mu}$ be a Pareto efficient assignment that Pareto dominates μ^{DA} with respect to $\{\tilde{P}_i\}_{i\in I}$. $\tilde{I} \equiv \{i \in I : \tilde{\mu}(i)\ \tilde{P}_i\ \mu^{DA}(i)\}$ is the set of students who are better off in $\tilde{\mu}$ than in μ^{DA} with respect to $\{\tilde{P}_i\}_{i\in I}$. Since every $i \in \tilde{I}$ must apply to $\tilde{\mu}(i)$ in some round of Fast DA, i must report $\tilde{\mu}(i)$ as first choice at some level in BM. Then for any level distribution in which every $i \in \tilde{I}$ reports $\tilde{\mu}(i)$ as first choice and every $j \in I\backslash\tilde{I}$ reports $\mu^{DA}(j)$ as first choice, the outcome of BM coincides with $\tilde{\mu}$. Since $\tilde{\mu}$ also Pareto dominates μ^{DA} with respect to P_I, I finish the proof.

Proof of Proposition 5

The proof is similar to that of Proposition 1. The only difference is that every Lk student has a correct belief about the level of every Lk' student if $k' < k$. Therefore, in Fast DA* every Lk' student cannot apply to new schools after round k', and this fact is known to every Lk student.

Proof of Proposition 6

1. By Corollary 3, any i at any level must report a school no worse than $\mu^{DA}(i)$ as first choice. So $\tilde{\mu}^{BM}_{k_I}$ is never strictly Pareto dominated by μ^{DA}.

2. If each student is sufficiently sophisticated, the outcome of Fast DA* coincides with μ^{DA}. So $\tilde{\mu}^{BM}_{k_I} = \mu^{DA}$.

3. If each student is insufficiently sophisticated, in the original level-k model each i, with the belief that the others are $Lk_i - 1$, must report a school strictly

better than $\mu^{DA}(i)$ as first choice. In the informational level-k model each i never overestimates the others' levels. So i must still report a school strictly better than $\mu^{DA}(i)$ as first choice. Therefore, $\tilde{\mu}^{BM}_{k_I}$ is not Pareto dominated by μ^{DA}.

4. The proof of Lemma 1 implies that if $\tilde{\mu}^{BM}_{k_I}$ is Pareto dominated by μ^{DA}, there must exist some positive-level i who reports some P'_i such that $\mu^{BM}_{k_I}(i)\ P'_i\ \mu^{DA}(i)$ but $\mu^{DA}(i)\ P_i\ \mu^{BM}_{k_I}(i)$. So if each positive-level i reports some P'_i such that for any $s \in S$, $\mu^{DA}(i)\ P'_i\ s$ if $\mu^{DA}(i)\ P_i\ s$, then $\tilde{\mu}^{BM}_{k_I}$ must not be Pareto dominated by μ^{DA}.

Proof of Proposition 8

As proved before, if j is admitted by his reported first choice when being Lk_j, becoming a higher level does not change the outcome of BM. If j is rejected by his reported first choice when being Lk_j, let s be the school that finally admits j in BM. Then s must have empty seats after the first round of BM. In other words, j is unassigned in $\mu^{FDA^*}_{k_I}$ and s has empty seats in $\mu^{FDA^*}_{k_I}$. Now suppose j becomes Lk'_j for any $k'_j > k_j$. To study how the outcome of the first round of BM changes, I study how the outcome of Fast DA* changes. In Fast DA*, j will apply to some new schools after round k_j. Let the sequence of these schools be $\{s_1, \cdots, s_v\}$. Since $k_j = \bar{k}_N$, no student in $N\backslash\{j\}$ will apply to new schools after round k_j of Fast DA*. So to obtain the new outcome of Fast DA*, I can start with the old outcome $\mu^{FDA^*}_{k_I}$ and let j apply to the schools in the sequence one by one. When j applies to a school in the sequence, j is either rejected immediately, or is tentatively accepted and induces a rejection-application chain.

Let s_a be the first school in the sequence that accepts j tentatively. Since s has empty seats, s will accept j immediately if j applies to s. So s_a must be better than s. If s_a finally accepts j, which means that j is better off, I finish the proof. If s_a finally rejects j, then in the induced rejection-application chain there must be a student who replaces j since he has a higher priority than j at s_a. Since any student in $N\backslash\{j\}$ cannot apply to a new school if being rejected, the chain cannot involve any student in $N\backslash\{j\}$. The chain must neither involve any school with empty seats since any student applying to such a school must be admitted immediately. So after j is rejected by s_a, the only change in the outcome of Fast DA* is that some quasi-rational students exchange their seats. The set of unassigned students and the set of empty seats are the same as before. Then I can repeat the above analysis for s_{a+1} and all following schools.

If j is rejected by all schools in the sequence, those schools must be strictly better than s, and s_v is just the first choice reported by j when being Lk'_j. So all schools weakly better than s_v must be exhausted in the new outcome of Fast DA*, and the set of unassigned students and the set of empty seats in the new outcome of Fast DA* are same as before. Note that all unassigned students will report the same preferences as before since their beliefs do not change. Hence, if j uses a strategy satisfying the worse-rank invariance condition, j will apply to every school worse than s_v in the same round of BM as before, and every other unassigned student will apply to the same school in the same round of BM as before. So j must still be admitted by s. Hence, I finish the proof.

A.2 Results of Pathak and Sönmez (2008) are Corollaries

I prove that all results of Pathak and Sönmez (2008) are either implied by mine or can be proved easily using my method.

For any P_I, if a nonempty $N \subsetneq I$ is the set of naive students and $M = I \backslash N$ is the set of rational students, then I prove that the set of NE outcomes of BM can be found by the following procedure.

- Construct an artificial economy $(\{P^1_j\}_{j \in N}, \{P_\ell\}_{\ell \in M})$ where P^1_j only lists the most preferred school of j. Let $\mathcal{M}_1$ be the set of stable assignments in this economy.
- For each $\mu \in \mathcal{M}_1$, finalize the assignments of all assigned students and reduce the number of seats at each school accordingly. Then run BM in the economy consisting of remaining unassigned students and unfilled seats. Denote the final assignment by $f(\mu)$. Define $\mathcal{M}_2 \equiv \{f(\mu) : \mu \in \mathcal{M}_1\}$.

Proposition 15. *$\mathcal{M}_2$ is the set of NE outcomes of BM in the economy P_I when N is naive and M is rational.*

Proof. By Ergin and Sönmez (2006), $\mathcal{M}_1$ is the set of NE outcomes of BM in the artificial economy if all students are rational. Since in the artificial economy each $j \in N$ only lists his most preferred school in his preference ordering, in each NE j must still only lists his most preferred school in his reported preference ordering. For each $\mu \in \mathcal{M}_1$, the students in M must be assigned in μ. So they obtain the same assignments in μ and $f(\mu)$. Hence, in P_I it must be a NE that each $j \in N$ reports P_j and each $\ell \in M$ reports $\mu(\ell)$ as first choice. So $f(\mu)$ is a NE outcome of BM in P_I.

Conversely, for each NE outcome μ in P_I, it is a NE that each $j \in N$ reports P_j and each $\ell \in M$ reports $\mu(\ell)$ as first choice. So in the artificial economy it is still a NE that each $j \in N$ reports P_j^1 and each $\ell \in M$ reports $\mu(\ell)$ as first choice. The corresponding NE outcome in the artificial economy is $f^{-1}(\mu)$. So I finish the proof. □

I denote the student-optimal stable assignment in the artificial economy by μ^*. Then $f(\mu^*)$ must be the student-optimal stable assignment in P_I. So $f(\mu^*) = \mu^{DA}$. On the other hand, in the informational level-k model of BM, if $k_N = \mathbf{0}$, then all students in N can only apply to their most preferred schools in Fast DA*. So the outcome of Fast DA* is μ^*. Hence, μ^* is also the assignment found by the first round of BM. Then the outcome of BM must be $f(\mu^*)$, which is same with μ^{DA}. This proves Proposition 9. So the main results of Pathak and Sönmez are implied by Proposition 5 and Proposition 6.

Corollary 9. *(Propositions 3 and 4 of Pathak and Sönmez (2008)) Rational students are weakly better off in the student-optimal NE outcome of BM than in the outcome of DA. A naive student weakly benefits from becoming rational and all rational students weakly suffer.*

Pathak and Sönmez also prove that each $j \in N$ obtains the same assignment in all NE outcomes of BM in P_I. I show that it is straightforwardly implied by the rural hospital theorem (Roth 1986). Specifically, for each $\mu \in \mathcal{M}_1$, all students in M must be assigned in μ. Each $j \in N$ is either assigned to his most preferred school or unassigned in μ. By the rural hospital theorem, if j is assigned in one stable assignment, j must be assigned in all stable assignments. Hence, if j is assigned in one μ, j must be assigned to his most preferred school in all $\mu \in \mathcal{M}_1$. On the other hand, by the rural hospital theorem each school must admit the same number of students in all $\mu \in \mathcal{M}_1$. Hence, the number of unassigned students and the number of empty seats at each school are same in all $\mu \in \mathcal{M}_1$. Then at the second step of the above procedure each unassigned $j \in N$ must obtain the same assignment in all $\mu \in \mathcal{M}_2$.

Corollary 10. *(Proposition 2 of Pathak and Sönmez (2008)) Each $j \in N$ is admitted by the same school in all NE outcomes of BM in P_I*

A.3 Additional Simulation Results

Figure A.1 reports the simulation results for the informational level-k model. The figure is very close to the original level-k model in Figure 1.1. To check the robustness of my results I also consider unbalanced markets in which students are more than schools. In particular, I let some school play the role of being unassigned by letting all students least prefer it. Figure A.2 reports the simulation results for the original level-k model in unbalanced markets. The results for the informational level-k model in unbalanced markets are very similar and omitted.

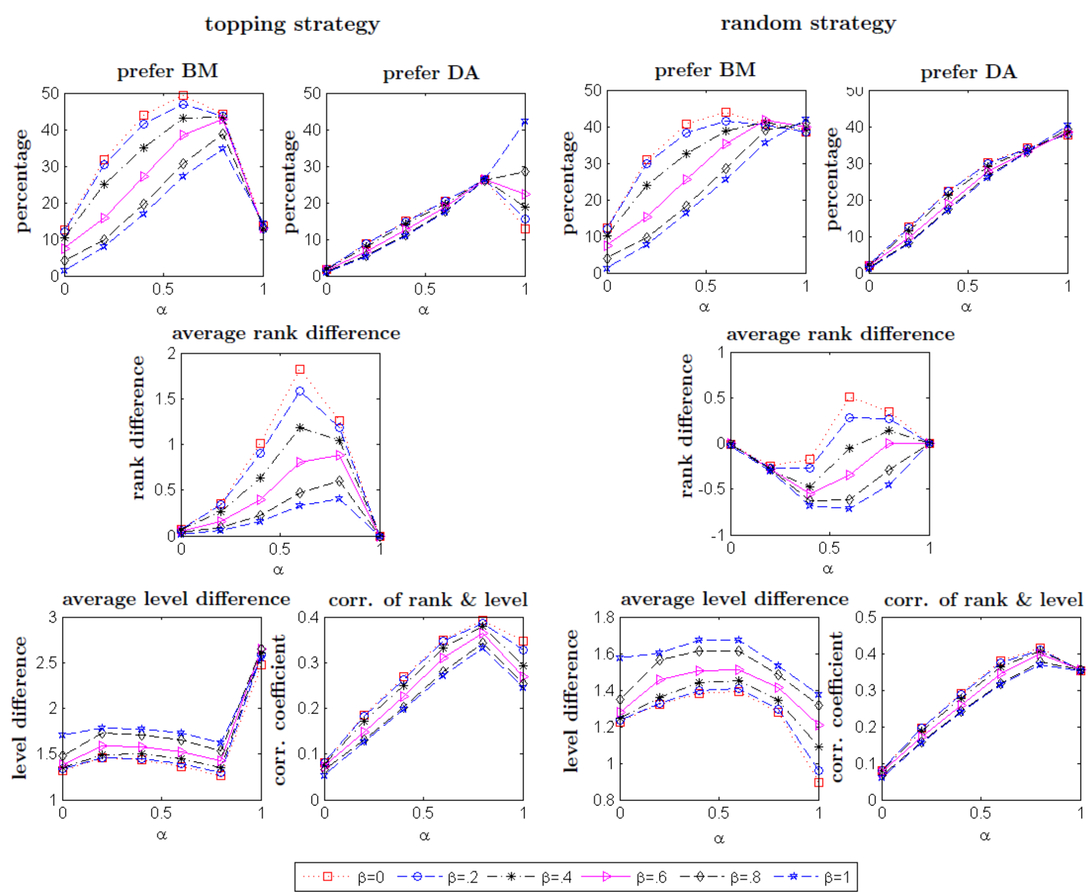


Figure A.1: Informational level-k model of BM

A.4 A Level-k Model of Constrained DA

Let $c < |S|$ be the constraint on the length of preference orderings that students can report. I analyze an original level-k model of constrained DA by assuming that $L0$ students report their top c choices and positive-level students use topping

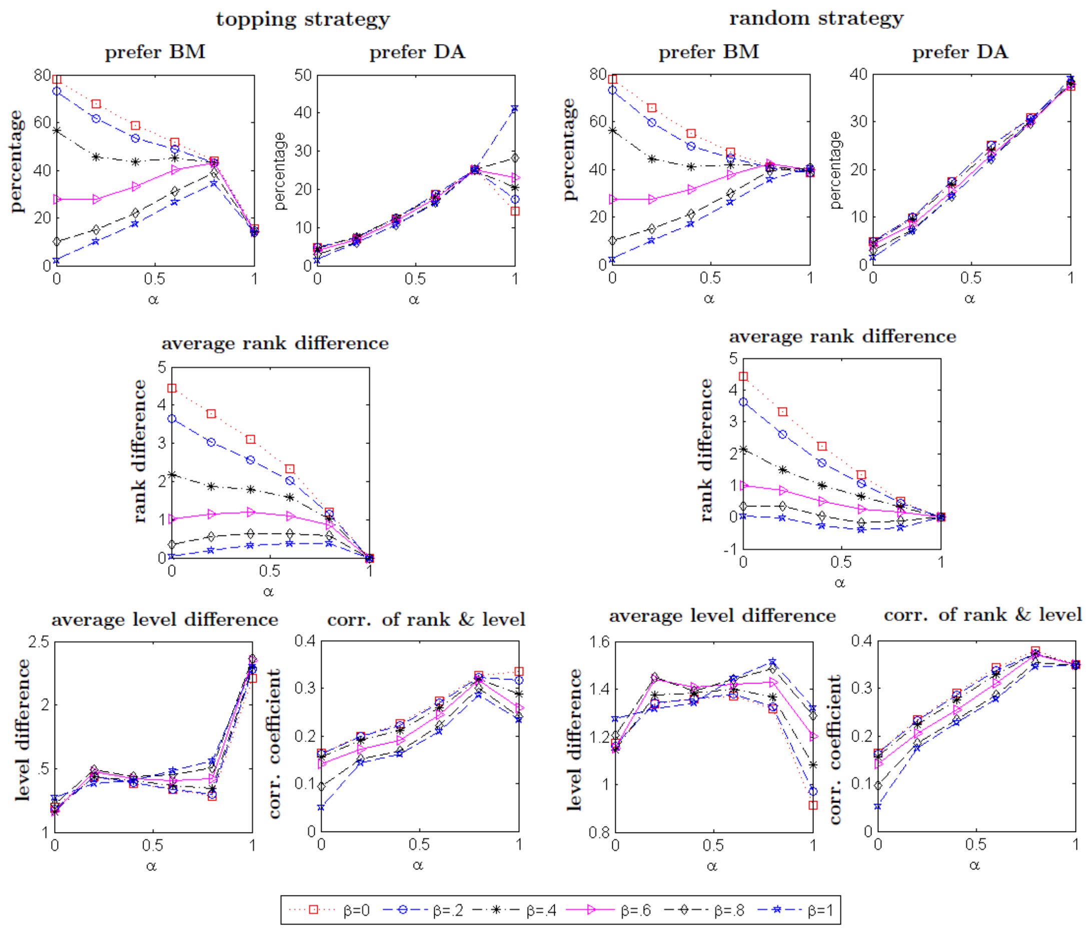


Figure A.2: Original Level-k model of BM in unbalanced markets

strategies.[1] Formally, for any $P \in \mathcal{P}$, let P^c only rank the top c choices. Denote the outcome of DA by μ^c if each i reports P_i^c. If $\mu^c(i) \neq \emptyset$, then $\mu^c(i)$ must be weakly better than $\mu^{DA}(i)$. If all students are assigned in μ^c, then $\mu^c = \mu^{DA}$. Let s_i^k be the first choice reported by each i at Lk, then I have the following result.

Proposition 16. *For any P_I and any i,*

(1) $s_i^k \; R_i \; s_i^{k+1} \; R_i \; \mu^{DA}(i)$ for all $i \in I$ and all $k \geq 0$;

(2) There exists some finite $r_i^{DA} \geq 0$ for each i such that $s_i^k = \mu^{DA}(i)$ for all $k \geq r_i^{DA}$.

Proof. At $L0$, each i reports P_i^c. So s_i^0 is the most preferred school of i. Denote the outcome of DA by μ^0 if all students are $L0$. It is obvious that $\mu^0 = \mu^c$. If any school

[1] That is, they report their best obtainable schools as first choices and report the true preference ordering of the remaining $c-1$ top choices.

s admits q_s students in μ^0, denote the priority rank of the lowest-priority admitted student by z_s^0. Otherwise, define $z_s^0 \equiv |I|$. So z_s^0 is the threshold priority rank to enter s in μ^0. Define z_s^{DA} similarly for μ^{DA}. It is obvious that $z_s^0 \geq z_s^{DA}$ for all s.

At $L1$, for each i, if $\mu^0(i) \neq \emptyset$, then $\mu^0(i)$ is the best obtainable school for i. Hence, $s_i^1 = \mu^0(i)$. If $\mu^0(i) = \emptyset$, then i will report a new best obtainable school s_i^1 as first choice. Since $z_s^0 \geq z_s^{DA}$ for all s, $\mu^{DA}(i)$ must be obtainable for i. So s_i^1 must be weakly better than $\mu^{DA}(i)$. Denote the outcome of DA by μ^1 if all students are $L1$. Denote the threshold priority rank to enter s in μ^1 by z_s^1. Then $z_s^1 \leq z_s^0$ for all s, that is, all thresholds are weakly higher. By using topping strategies each i is either unassigned in μ^1 or admitted by a school weakly better than $\mu^{DA}(i)$. So $z_s^{DA} \leq z_s^1$ for all s.

At Lk for any $k \geq 2$, suppose it is true that $s_i^{k'-1} \ R_i \ s_i^{k'} \ R_i \ \mu^{DA}(i)$ and $z_s^{DA} \leq z_s^{k'} \leq z_s^{k'-1}$ for all i, all s and all $k' < k$. Then for each i, any s better than s_i^{k-1} must be not obtainable for i. If $\mu^{k-1}(i) \neq \emptyset$, then $\mu^{k-1}(i)$ is the best obtainable school for i. If $\mu^{k-1}(i) = \emptyset$, i will report a new first choice at Lk. But the school must be weakly better than $\mu^{DA}(i)$ since $z_s^{DA} \leq z_s^{k-1}$ for all s. Hence, it is still true that $s_i^{k-1} \ R_i \ s_i^k \ R_i \ \mu^{DA}(i)$ and $z_s^{DA} \leq z_s^k \leq z_s^{k-1}$ for all i and all s. Then by induction, $s_i^k \ R_i \ s_i^{k+1} \ R_i \ \mu^{DA}(i)$ for all $i \in I$ and all $k \geq 0$.

When k is high enough, all students at Lk must be assigned in μ^k. Then μ^k must weakly Pareto dominate μ^{DA}. Since $\mu^k(i)$ is the best obtainable school for each i at $Lk+1$, μ^k must be stable. So $\mu^k = \mu^{DA}$. Hence, there exists some finite $r_i^{DA} \geq 0$ for each i such that $s_i^k = \mu^{DA}(i)$ for all $k \geq r_i^{DA}$. □

The above original level-k model of constrained DA looks like the corresponding model of BM. If all students have high enough levels, BM and DA have same outcomes, which are μ^{DA}. If some students have low levels, the comparison between BM and DA is ambiguous. Therefore, I do simulations to compare them by choosing $c = 5$ and letting positive-level students use topping strategies. Figure A.3 shows that DA is more efficient than BM. Specifically, there are more students who prefer DA than those who prefer BM. Sophistication level is also positively correlated with welfare in both BM and DA. However, as discussed before, the level-k model of constrained DA depends on the specification of students' best strategies at positive levels. By using topping strategies positive-level students may report non-truthful rankings of the schools they report, which are proved to be weakly dominated by truthful rankings of the schools they report (Calsamiglia, Haeringer, and Klijn 2010).

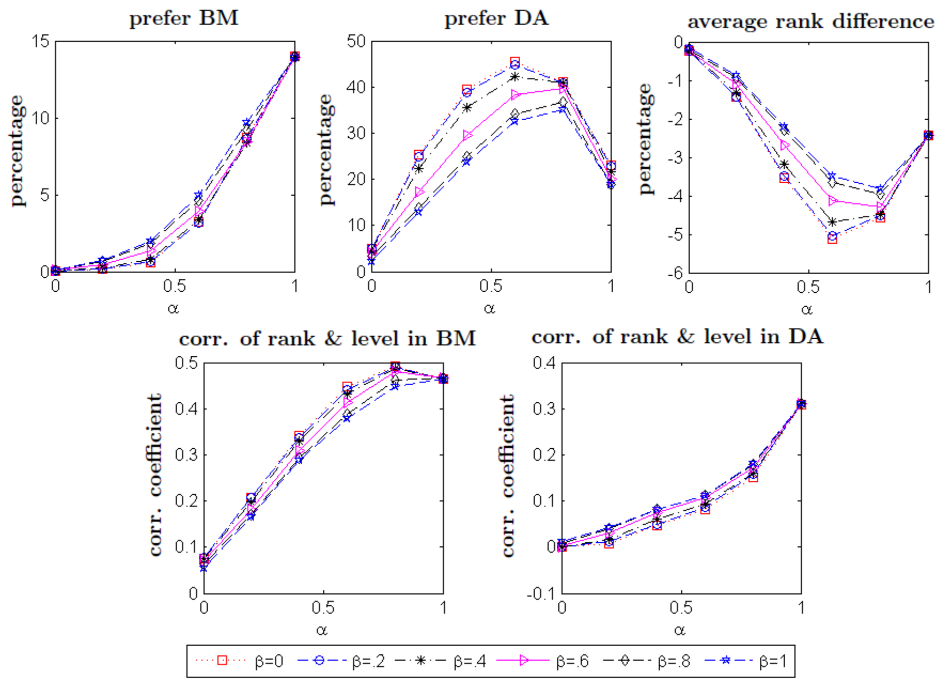


Figure A.3: Compare BM and DA in constrained school choice.

A.5 Correction of Proposition 1 of Abdulkadiroğlu, Che, and Yasuda (2011)

In the strict priorities and complete information environment, Abdulkadiroğlu, Che, and Yasuda (2011) prove their Proposition 1 in their on-line appendix, which states that if students have common preference orderings and are either naive or rational, then if any naive student becoming rational, in the unique NE outcome of BM all other naive students are weakly worse off. I show that this statement is incorrect. This is illustrated by the following example.

Example 6. *There are four schools* $\{s_1, s_2, s_3, s_4\}$ *and four students* $\{i_1, i_2, i_3, i_4\}$*. Each school has only one seat. Students have the common preference ordering* $s_1 \succ s_2 \succ s_3 \succ s_4$*. The priority rankings are shown in Table A.1. Suppose all students are naive, then they report true preferences. The assignment of BM is shown in the table. Now if* i_2 *becomes rational,* i_2 *will report* s_2 *as first choice since* s_1 *must be obtained by* i_1*. Then the new outcome of BM is also shown in the table. It is easy to see that* i_2, i_4 *are better off,* i_1 *remains same, and* i_3 *is worse off.*

Now I provide a detailed analysis of the statement using my method in Appendix A.2. When some naive student j becomes rational, by Proposition 9, it is equivalent to the situation that $k_N = \mathbf{0}$ in the informational level-k model, but then some $j \in N$ becomes quasi-rational. Suppose there are m schools in total and the common

π_{s_1}	π_{s_2}	π_{s_3}	π_{s_4}
i_1	i_3	i_2	i_1
i_2	i_2	i_4	i_2
i_3	i_4	i_3	i_3
i_4	i_1	i_1	i_4

(a) Priority rankings

i_1	i_2	i_3	i_4
s_1	s_3	s_2	s_4

(b) All students are naive

i_1	i_2	i_3	i_4
s_1	s_2	s_4	s_3

(c) i_2 is rational, the others are naive

Table A.1: Counterexample

preference ordering is $s_1 \succ s_2 \succ \cdots \succ s_m$. As proved before, if j is assigned in Fast DA* when being $L0$, becoming quasi-rational does not change the outcome of BM. If j is unassigned in Fast DA* when being $L0$, let $\{s_v, s_{v+1}, \cdots, s_m\}$ be the set of schools that have empty seats in the outcome of Fast DA*. Let the number of empty seats of s_v be e_{s_v}. Since students have common preferences, the number of empty seats of each s_a with $a > v$ must be q_{s_a}. Let s_u be the school that admits j in the outcome of BM. So $u \in [v, m]$.

By becoming quasi-rational j must be assigned in Fast DA*. So from the second round of BM on there will be one fewer empty seat and one fewer unassigned student than before. Since students have common preferences, the missing empty seat must belong to s_v. Hence, s_v has $e_{s_v} - 1$ empty seats now. So some student j_1 who was admitted by s_v in the second round of BM before will be rejected by s_v now. Then j_1 will apply to s_{v+1}, and will either be rejected or replace another student who was admitted by s_{v+1} before. If I repeat this argument, there must be exactly one student j_a who was admitted by some school in $\{s_v, \cdots, s_{u-1}\}$ but now is rejected. Moreover, all the students who were admitted by the schools in $\{s_v, \cdots, s_{u-1}\}$ before are weakly worse off and some is strictly worse off (e.g., j_1 and j_a). Then j_a will apply to s_u. Since j is assigned in the first round, compared to the before situation j_a essentially replaces j among those who applied to s_u before. Let j_b be the highest-priority student at s_u who was rejected by s_u before. If j_b has a higher priority than j_a, then j_b will be admitted by s_u and be strictly better off (this is what happens in the above example). Then j_a will apply to s_{u+1} and essentially replace j_b among those who applied to s_{u+1} before. On the other hand, if j_b has a lower priority than j_a at s_u, then j_a is admitted by s_u. Then the assignments of all naive students who were admitted by the schools in $\{s_{u+1}, \cdots, s_m\}$ do not change.

APPENDIX TO CHAPTER 2

Proof of Theorem 1

Let $I = \{i_1, i_2, j\}$ and $O = \{o_1, o_2, o_3, \ldots\}$. Consider the two preference profiles $\succ^*_I$ and $\succ^\circ_I$ shown in Table B.1. Both preference profiles are allowed in both $\mathcal{U}$ and $\mathcal{Q}$. In both preference profiles all agents prefer o_1, o_2, o_3 to the other objects, and i_1, i_2 have equal preferences.

$\succ^*_{i_1}$	$\succ^*_{i_2}$	$\succ^*_j$
o_1	o_1	o_2
o_2	o_2	o_3
o_3	o_3	o_1
$\vdots$	$\vdots$	$\vdots$

(a) $\succ^*_I$

$\succ^\circ_{i_1}$	$\succ^\circ_{i_2}$	$\succ^*_j$
o_2	o_2	o_2
o_1	o_1	o_3
o_3	o_3	o_1
$\vdots$	$\vdots$	$\vdots$

(b) $\succ^\circ_I$

Table B.1: Two preference profiles

Lemma 2. *When $|I| = 3$ and $|O| \geq 3$, in either $\mathcal{U}$ and $\mathcal{Q}$, an ex-post efficient mechanism that satisfies equal treatment of equals is minimally group strategy-proof only if $\rho(\succ^*_I) = \rho(\succ^\circ_I)$.*

Proof. In any Pareto efficient deterministic assignment o_1, o_2, o_3 must be assigned to i_1, i_2, j. Moreover, since in both preference profiles i_1, i_2 prefer o_1 to o_3 while j prefers o_3 to o_1, j must never obtain o_1. Hence, for both $\succ^*_I$ and $\succ^\circ_I$ there are only four Pareto efficient deterministic assignments: $\pi_1, \pi_2, \pi_3, \pi_4$, which are shown in Table B.2.

i_1	i_2	j
o_1	o_2	o_3

(a) π_1

i_1	i_2	j
o_1	o_3	o_2

(b) π_2

i_1	i_2	j
o_2	o_1	o_3

(c) π_3

i_1	i_2	j
o_3	o_1	o_2

(d) π_4

Table B.2: Pareto efficient assignments for both preference profiles

Any ex-post efficient mechanism ρ must assign positive probabilities to a subset of the above four assignments for both preference profiles. Hence, $\rho(\succ^*_I)$ can be denoted by Table B.3. $\rho(\succ^\circ_I)$ can be denoted similarly and omitted.

	i_1	i_2	j
o_1	$\rho(\succ_I^*)(\pi_1) + \rho(\succ_I^*)(\pi_2)$	$\rho(\succ_I^*)(\pi_3) + \rho(\succ_I^*)(\pi_4)$	
o_2	$\rho(\succ_I^*)(\pi_3)$	$\rho(\succ_I^*)(\pi_1)$	$[\rho(\succ_I^*)(\pi_2) + \rho(\succ_I^*)(\pi_4)]$
o_3	$\rho(\succ_I^*)(\pi_4)$	$\rho(\succ_I^*)(\pi_2)$	$[\rho(\succ_I^*)(\pi_1) + \rho(\succ_I^*)(\pi_3)]$

Table B.3: $\rho(\succ_I^*)$

If ρ satisfies equal treatment of equals, then i_1, i_2 obtain the same lotteries in both $\rho(\succ_I^*)$ and $\rho(\succ_I^\circ)$. So for $\succ_I^*$ I have the following equalities:

$$\begin{aligned}
\rho(\succ_I^*)(\pi_1) + \rho(\succ_I^*)(\pi_2) &= \rho(\succ_I^*)(\pi_3) + \rho(\succ_I^*)(\pi_4), && (i_1, i_2 \text{ obtain equal shares of } o_1)\\
\rho(\succ_I^*)(\pi_1) &= \rho(\succ_I^*)(\pi_3), && (i_1, i_2 \text{ obtain equal shares of } o_2)\\
\rho(\succ_I^*)(\pi_2) &= \rho(\succ_I^*)(\pi_4). && (i_1, i_2 \text{ obtain equal shares of } o_3)
\end{aligned}$$

These equalities imply that $\rho(\succ_I^*)(\pi_1) + \rho(\succ_I^*)(\pi_2) = \frac{1}{2}$. So $\rho(\succ_I^*)$ can be characterized by one parameter $\rho(\succ_I^*)(\pi_1)$, which is shown in Table B.4. $\rho(\succ_I^\circ)$ is similarly denoted by Table B.5.

	i_1	i_2	j
o_1	$\frac{1}{2}$	$\frac{1}{2}$	
o_2	$\rho(\succ_I^*)(\pi_1)$	$\rho(\succ_I^*)(\pi_1)$	$[1 - 2\rho(\succ_I^*)(\pi_1)]$
o_3	$\frac{1}{2} - \rho(\succ_I^*)(\pi_1)$	$\frac{1}{2} - \rho(\succ_I^*)(\pi_1)$	$2\rho(\succ_I^*)(\pi_1)$

Table B.4: $\rho(\succ_I^*)$

	i_1	i_2	j
o_1	$\frac{1}{2}$	$\frac{1}{2}$	
o_2	$\rho(\succ_I^\circ)(\pi_1)$	$\rho(\succ_I^\circ)(\pi_1)$	$[1 - 2\rho(\succ_I^\circ)(\pi_1)]$
o_3	$\frac{1}{2} - \rho(\succ_I^\circ)(\pi_1)$	$\frac{1}{2} - \rho(\succ_I^\circ)(\pi_1)$	$2\rho(\succ_I^\circ)(\pi_1)$

Table B.5: $\rho(\succ_I^\circ)$

Since i_1, i_2 always obtain $1/2$ of o_1 in both preference profiles, whenever $\rho(\succ_I^*)(\pi_1) \neq \rho(\succ_I^\circ)(\pi_1)$, the lotteries i_1, i_2 obtain in one preference profile strictly first-order stochastically dominate the lotteries they obtain in another preference profile for both $\succ_{i_1}^*, \succ_{i_2}^*$ and $\succ_{i_1}^\circ, \succ_{i_2}^\circ$. Specifically, if $\rho(\succ_I^*)(\pi_1) > \rho(\succ_I^\circ)(\pi_1)$, i_1, i_2 can strongly group manipulate ρ at $\succ_I^\circ$ by reporting $(\succ_{i_1}^*, \succ_{i_2}^*)$. Symmetrically, if $\rho(\succ_I^*)(\pi_1) < \rho(\succ_I^\circ)(\pi_1)$, i_1, i_2 can strongly manipulate ρ at $\succ_I^*$ by reporting $(\succ_{i_1}^\circ, \succ_{i_2}^\circ)$. So ρ is minimally group strategy-proof only if $\rho(\succ_I^*)(\pi_1) = \rho(\succ_I^\circ)(\pi_1)$, which is equivalent to $\rho(\succ_I^*) = \rho(\succ_I^\circ)$. $\square$

If a mechanism ρ further satisfies equal top-assignment of equal tops and top advantage, then it must be that $\rho(\succ_I^*)(\pi_1) < 1/3$ and $\rho(\succ_I^\circ)(\pi_1) = 1/3$. So ρ is not minimally group strategy-proof.

Lemma 3. *When $|I| = 3$ and $|O| \geq 3$, in either $\mathcal{U}$ or $\mathcal{Q}$, an ex-post efficient assignment that satisfies equal top-assignment of equal tops and weak envy-freeness must satisfy equal treatment of equals.*

Proof. Suppose any two agents i, j report equal preferences and most prefer object o_1. Suppose p is the assignment found by the mechanism. Equal top-assignment of equal tops requires that $p_{i,o_1} = p_{j,o_1}$. If only o is acceptable to i, j, then i, j obviously obtain equal lotteries. If their preference ordering is $o_1 \succ o_2 \succ \emptyset \succ \cdots$, then weak envy-freeness requires that $p_{i,o_2} \geq p_{j,o_2}$ and $p_{i,o_2} \leq p_{j,o_2}$. So i, j still obtain equal lotteries. If their preference ordering is $o_1 \succ o_2 \succ o_3 \succ \cdots \succ \emptyset$, since there are only three agents, ex-post efficiency requires that $\sum_{k=1}^{3} p_{i,o_k} = \sum_{k=1}^{3} p_{j,o_k} = 1$. If $p_{i,o_2} > p_{j,o_2}$, then $p_i \rhd_j p_j$, which violates weak envy-freeness. Hence, $p_{i,o_2} = p_{j,o_2}$, and i, j obtain equal lotteries. So p satisfies equal treatment of equals. $\square$

Hence, in the theorem equal treatment of equals can be replaced by weak envy-freeness.

Proof of Theorem 2

Let $I \equiv \{i_1, i_2, j, k_1, \ldots, k_n\}$ and $O \equiv \{o_1, o_2, \ldots, o_m\}$ with $n \geq 1$ and $m \geq 3$. Consider the two preference profiles in Table B.6 that generalize those in the proof of Theorem 1. In both preference profiles o_1, o_2, o_3 are not acceptable to $k_1, \ldots, k_n$, but are preferred by i_1, i_2, i_3 to all other objects. So $\{i_1, i_2, i_3\}$ and $\{o_1, o_2, o_3\}$ essentially constitute a sub-problem. Then I can repeat previous arguments.

$\succ_{i_1}^*$	$\succ_{i_2}^*$	$\succ_j^*$	$\succ_{k_1}^*$	$\cdots$	$\succ_{k_n}^*$
o_1	o_1	o_2	o_4	$\cdots$	o_4
o_2	o_2	o_3	$\vdots$	$\cdots$	$\vdots$
o_3	o_3	o_1	o_m	$\cdots$	o_m
$\vdots$	$\vdots$	$\vdots$	$\emptyset$	$\cdots$	$\emptyset$

(a) $\succ_I^*$

$\succ_{i_1}^\circ$	$\succ_{i_2}^\circ$	$\succ_j^\circ$	$\succ_{k_1}^*$	$\cdots$	$\succ_{k_n}^*$
o_2	o_2	o_2	o_4	$\cdots$	o_4
o_1	o_1	o_3	$\vdots$	$\cdots$	$\vdots$
o_3	o_3	o_1	o_m	$\cdots$	o_m
$\vdots$	$\vdots$	$\vdots$	$\emptyset$	$\cdots$	$\emptyset$

(b) $\succ_I^\circ$

Table B.6: Two preference profiles

Proof of Theorem 3

Let $I \equiv \{i_1, i_2, \ldots, i_n, j\}$ and $O \equiv \{o_1, o_2, \ldots, o_m\}$ with $n \geq 3$ and $m \geq 3$. Consider the preferences profiles $\succ_I^*$ and $\succ_I^\circ$ in Table B.7. All agents prefer o_1, o_2, o_3 to other objects, and $i_1, \cdots, i_n$ always have equal preferences. As before, in both preference profiles an ex-post efficient mechanism ρ must assign o_1 only to $i_1, \cdots, i_n$. Moreover, if $m \leq n+1$, $o_1, \ldots, o_m$ must be exhausted. If $m > n+1$, $o_1, \ldots, o_{n+1}$ must be exhausted.

$\succ_{i_1}^*$	$\succ_{i_2}^*$	$\cdots$	$\succ_{i_n}^*$	$\succ_j^*$
o_1	o_1	$\cdots$	o_1	o_2
o_2	o_2	$\cdots$	o_2	o_3
o_3	o_3	$\cdots$	o_3	o_1
o_4	o_4	$\cdots$	o_4	o_4
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
o_m	o_m	$\cdots$	o_m	o_m
$\emptyset$	$\emptyset$	$\cdots$	$\emptyset$	$\emptyset$

(a) $\succ_I^*$

$\succ_{i_1}^\circ$	$\succ_{i_2}^\circ$	$\cdots$	$\succ_{i_n}^\circ$	$\succ_j^*$
o_2	o_2	$\cdots$	o_2	o_2
o_1	o_1	$\cdots$	o_1	o_3
o_3	o_3	$\cdots$	o_3	o_1
o_4	o_4	$\cdots$	o_4	o_4
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
o_m	o_m	$\cdots$	o_m	o_m
$\emptyset$	$\emptyset$	$\cdots$	$\emptyset$	$\emptyset$

(b) $\succ_I^\circ$

Table B.7: Two preference profiles

Since all agents prefer all objects to $\emptyset$, uniform tail-assignment of uniform tails requires that $\rho_{i,\emptyset}(\succ_I^*) = \rho_{i,\emptyset}(\succ_I^*)$ for any distinct i, j. Hence, ex-post efficiency further requires that $\rho_{i,\emptyset}(\succ_I^*) = \max\{1 - \frac{m}{n+1}, 0\}$ for all $i \in I$. For the same reason, $\rho_{i,\emptyset}(\succ_I^\circ) = \max\{1 - \frac{m}{n+1}, 0\}$ for all $i \in I$. Therefore, $\sum_{k=1}^m \rho_{io_k}(\succ_I^*) = \sum_{k=1}^m \rho_{io_k}(\succ_I^\circ) = \min\{\frac{m}{n+1}, 1\}$ for all $i \in I$. Moreover, since all agents prefer o_1, o_2, o_3 to the remaining objects and their preferences over the remaining objects are equal, uniform tail-assignment of uniform tails requires that $\rho_{io_k}(\succ_I^*) = \rho_{io_k}(\succ_I^\circ) = \frac{1}{n+1}$ for all $i \in I$ and all $k = 4, \cdots, \min\{m, n+1\}$.

Equal treatment of equals, equal top-assignment of equal tops, and top advantage further require that for all $i \in J \equiv \{i_1, i_2, \ldots, i_n\}$, $\rho_{i,o_1}(\succ_I^*) = \frac{1}{n}$, $\rho_{i,o_2}(\succ_I^*) < \frac{1}{n+1}$, $\rho_{i,o_1}(\succ_I^\circ) = \frac{1}{n}$, and $\rho_{i,o_2}(\succ_I^\circ) = \frac{1}{n+1}$. So $\rho(\succ_I^*)$ can be denoted by Table B.8 for some $z < \frac{1}{n+1}$, and $\rho(\succ_I^\circ)$ can be denoted by Table B.9. It is easy to see that the lottery obtained by every $i \in J$ in $\rho(\succ_I^\circ)$ strictly first-order stochastically dominates the lottery obtained by i in $\rho(\succ_I^*)$. So J can strongly group manipulate ρ at $\succ_I^*$ by reporting $\succ_J^\circ$. As before, equal treatment of equals can be replaced by weak envy-freeness.

Proof of Theorem 4

Let the two objects be o_1, o_2. I prove the theorem by contradiction. Suppose at some preference profile $\succ_I$, some group $J \subsetneq I$ strongly group manipulates ρ by

	i_1	i_2	$\cdots$	i_n	j
o_1	$\frac{1}{n}$	$\frac{1}{n}$	$\cdots$	$\frac{1}{n}$	
o_2	z	z	$\cdots$	z	$1-nz$
o_3	$\frac{2n-1}{n+1}-z$	$\frac{2n-1}{n+1}-z$	$\cdots$	$\frac{2n-1}{n+1}-z$	$nz-\frac{n-2}{n+1}$
o_4	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\cdots$	$\frac{1}{n+1}$	$\frac{1}{n+1}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$o_{\min\{m,n+1\}}$	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\cdots$	$\frac{1}{n+1}$	$\frac{1}{n+1}$

Table B.8: $\rho(\succ_I^*)$

	i_1	i_2	$\cdots$	i_n	j
o_1	$\frac{1}{n}$	$\frac{1}{n}$	$\cdots$	$\frac{1}{n}$	
o_2	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\cdots$	$\frac{1}{n+1}$	$\frac{1}{n+1}$
o_3	$\frac{2n-1}{n+1}-\frac{1}{n+1}$	$\frac{2n-1}{n+1}-\frac{1}{n+1}$	$\cdots$	$\frac{2n-1}{n+1}-\frac{1}{n+1}$	$\frac{2}{n+1}$
o_4	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\cdots$	$\frac{1}{n+1}$	$\frac{1}{n+1}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$o_{\min\{m,n+1\}}$	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\cdots$	$\frac{1}{n+1}$	$\frac{1}{n+1}$

Table B.9: $\rho(\succ_I^\circ)$

reporting $\succ'_J$. Suppose there are $x > 0$ agents in J and y agents in $I\backslash J$. Among J there are x_1 agents who prefer o_1 to o_2, while among $I\backslash J$ there are y_1 agents who prefer o_1 to o_2. Since there are only two objects, all agents in J must switch their preference ranking of o_1, o_2 in their misreported preferences. So the true and misreported preferences can be denoted as follows:

x_1	$x-x_1$	y_1	$y-y_1$
o_1	o_2	o_1	o_2
o_2	o_1	o_2	o_1

(a) $(\succ_J, \succ_{I\backslash J})$

x_1	$x-x_1$	y_1	$y-y_1$
o_2	o_1	o_1	o_2
o_1	o_2	o_2	o_1

(b) $(\succ'_J, \succ_{I\backslash J})$

Hence, in $\succ_I$ there are $x_1 + y_1$ agents who prefer o_1 to o_2, while in $(\succ'_J, \succ_{I\backslash J})$ there are $x - x_1 + y_1$ agents who prefer o_1 to o_2. Since ρ satisfies equal treatment of equals, $\rho(\succ_I)$ and $\rho(\succ'_J, \succ_{I\backslash J})$ can be denoted as follows:

	x_1+y_1	$(x-x_1)+(y-y_1)$
o_1	a	c
o_2	b	d

(a) $\rho(\succ_I)$

	$(x-x_1)+y_1$	$x_1+(y-y_1)$
o_1	a'	c'
o_2	b'	d'

(b) $\rho(\succ'_J, \succ_{I\backslash J})$

Since ρ satisfies top advantage, if $x_1+y_1 > 0$, then $a > c$; if $(x-x_1)+(y-y_1) > 0$, then $d > b$; if $(x-x_1)+y_1 > 0$, then $a' > c'$; if $x_1+(y-y_1) > 0$, then $d' > b'$.

Since J strongly group manipulates ρ, if $x_1 > 0$, then $c' \geq a$; if $x - x_1 > 0$, then $b' \geq d$.

There are two cases to consider: $(x - x_1) + y_1 > 0$ or $x_1 + (y - y_1) > 0$. Since the two cases are symmetric, I only prove the case of $(x - x_1) + y_1 > 0$. In this case $a' > c'$. There are further two cases to consider.

Case 1: $x_1 > 0$. Then $x_1 + y_1 > 0$, which implies that $a > c$ and $c' \geq a$. Then the total probability of o_1 that assigned in $\rho(\succ'_J, \succ_{I\setminus J})$ is

$$\begin{aligned}
[(x - x_1) + y_1]a' + [x_1 + (y - y_1)]c' &> [(x - x_1) + y_1]c' + [x_1 + (y - y_1)]c' \\
&= (x + y)c' \\
&\geq (x + y)a \\
&\geq (x_1 + y_1)a + [(x - x_1) + (y - y_1)]c.
\end{aligned}$$

$(x_1 + y_1)a + [(x - x_1) + (y - y_1)]c$ is the total probability of o_1 that assigned in $\rho(\succ_I)$. Since ρ is ex-post efficient, o_1 must be exhausted in both $\rho(\succ_I)$ and $\rho(\succ'_J, \succ_{I\setminus J})$. So the above strict inequality is a contradiction.

Case 2: $x_1 = 0$. Then $x - x_1 = x > 0$, which implies that $d > b$ and $b' \geq d$. There are further two subcases to consider.

- If $x_1 + (y - y_1) > 0$, then $d' > b'$. Then the total probability of o_2 that assigned in $\rho(\succ'_J, \succ_{I\setminus J})$ is

  $$\begin{aligned}
  [(x - x_1) + y_1]b' + [x_1 + (y - y_1)]d' &> [(x - x_1) + y_1]b' + [x_1 + (y - y_1)]b' \\
  &= (x + y)b' \\
  &\geq (x + y)d \\
  &\geq (x_1 + y_1)b + [(x - x_1) + (y - y_1)]d,
  \end{aligned}$$

  which, as before, is a contradiction.

- If $x_1 + (y - y_1) = 0$, then $y_1 = y$. Then all agents in J prefer o_2 to o_1, while all agents in $I\setminus J$ prefer o_1 to o_2. So in the misreported preference profile $(\succ'_J, \succ_{I\setminus J})$ all agents prefer o_1 to o_2. Equal treatment of equals implies that $b' = \frac{1}{x+y}$, while top advantage implies that $d > \frac{1}{x+y}$. So J cannot strongly group manipulate ρ, which is a contradiction.

Proof of Theorem 5

As before, suppose a group $J \subsetneq I$ weakly group manipulates a mechanism ρ. Besides the properties in Theorem 4, if ρ further satisfies uniform tail-assignment of uniform tails, then in the proof of Theorem 4 we have $a + b = c + d = a' + b' = d' + c' = \frac{2}{|I|}$.

If some agent in J with the true preference ordering $o_1 \succ o_2$ obtains a different assignment than truth-telling, then weak group manipulation implies that $c' > a$. So the lottery $(c' \cdot o_1, d' \cdot o_2)$ strictly first-order stochastically dominates the lottery $(a \cdot o_1, b \cdot o_2)$ for the x_1 agents in J. Hence, if $x - x_1 = 0$, then J strongly group manipulates ρ, which contradicts the fact that ρ is minimally group strategy-proof. If $x - x_1 > 0$, top advantage requires that $a' > c'$. So the lottery $(b' \cdot o_2, a' \cdot o_1)$ is strictly first-order stochastically dominated by the lottery $(d \cdot o_2, c \cdot o_1)$ for the $x - x_1$ agents in J. So J cannot weakly group manipulate ρ.

If some agent in J with the true preference ordering $o_2 \succ o_1$ obtains a different assignment than truth-telling, the proof is similar.

Proof of Theorem 6

Let $I \equiv \{i_1, i_2, \ldots, i_n, j\}$ and $O \equiv \{o_1, o_2\}$ with $n \geq 2$. Consider the following two preference profiles.

$\succ^*_{i_1}$	$\succ^*_{i_2}$	$\cdots$	$\succ^*_{i_n}$	$\succ^*_j$
o_1	o_1	$\cdots$	o_1	o_2
o_2	o_2	$\cdots$	o_2	$\emptyset$
$\emptyset$	$\emptyset$	$\cdots$	$\emptyset$	o_1

(a) $\succ^*_I$

$\succ^\circ_{i_1}$	$\succ^\circ_{i_2}$	$\cdots$	$\succ^\circ_{i_n}$	$\succ^*_j$
o_2	o_2	$\cdots$	o_2	o_2
o_1	o_1	$\cdots$	o_1	$\emptyset$
$\emptyset$	$\emptyset$	$\cdots$	$\emptyset$	o_1

(b) $\succ^\circ_I$

Table B.12: Two preference profiles

As before, at $\succ^*_I$ each $i \in J \equiv \{i_1, \cdots, i_n\}$ obtains $1/n$ of o_1 and less than $1/(n+1)$ of o_2, while at $\succ^\circ_I$ each $i \in J$ obtains $1/n$ of o_1 and $1/(n+1)$ of o_2. So J can strongly group manipulate the mechanism at $\succ^*_I$ by reporting $\succ^\circ_J$.

Proof of Proposition 10

Since PS is envy-free, I only need to prove top advantage. However, it is obvious. So in the following I prove the proposition for RSD.

- Equal top-assignment of equal tops: Suppose any distinct i, j most prefer the same object o. For any ordering of agents in which i obtains o in RSD, there is a symmetric ordering in which i, j switch their positions and j obtains o. Since each ordering is drawn with equal probability, i, j obtain equal probability of o.

- Top advantage: Suppose i most prefers o while j most prefers a different object. For any ordering of agents in which j obtains o, there is a symmetric ordering in which i, j switch their positions and i obtains o. However, for all orderings in which i is ranked first, i obtains o. But for all orderings in which j is ranked first, j does not obtain o. Since each ordering is drawn with equal probability, i obtains higher probability of o than j.

- Uniform tail-assignment of uniform tails: Since $SU(\succ_i, o) = SU(\succ_j, o)$ for all distinct i, j, in any ordering of agents $SU(\succ_i, o)$ must be exhausted before any object in $\hat{O} \backslash SU(\succ_i, o)$ is chosen. So the outcome of RSD can be equivalently found by first running RSD to assign $SU(\succ_i, o)$, then running RSD to assign $\hat{O} \backslash SU(\succ_i, o)$. Since all agents have equal preferences over $\hat{O} \backslash SU(\succ_i, o)$, all agents must obtain equal probability of each object in $\hat{O} \backslash SU(\succ_i, o)$.

Appendix C

APPENDIX TO CHAPTER 3

C.1 Proofs of Theorem 7 and Theorem 8

C.1.1 Proof of Theorem 7

To prove Theorem 7 I first prove a lemma about PS^E and two lemmas about TTC^E.

For any $d \geq 0$, at the beginning of step $d + 1$ in PS^E, if there are no cycles among existing tenants, since every agent in $I(d)$ points to a house and every private endowment in $H(d)$ points to its owner, every agent must be linked through a unique path to a social endowment (as illustrated by Figure C.1), which is either a vacant house or a private endowment whose owner has stopped consuming. Then I prove that for any $h \in H(d)$, $s_h(t_d)$ is equal to the number of agents who are linked to h through some paths. I say an agent i is always linked to himself. Then $s_i(t_d)$ is equal to the number of agents linked to i.

$$\begin{tikzcd} i_3 \arrow[r] & h_1 \arrow[r] & i_1 \arrow[r] & h_2 \arrow[r] & i_2 \arrow[r] & h_3 \\ i_4 \arrow[ur] & & i_5 \arrow[ur] & & i_6 \arrow[ur] & \end{tikzcd}$$

Figure C.1: Illustration of Lemma 4. h_1, h_2 are private endowments of i_1, i_2. h_3 is a social endowment. Hence, $s_{h_1}(t) = 2$, $s_{i_1}(t) = 3$, $s_{h_2}(t) = 4$, $s_{i_2}(t) = 5$, and $s_{h_3}(t) = 6$.

Lemma 4. *For all $h \in H(d)$ and all $i \in I(d)$, $s_h(t_d)$ is equal to the number of agents who are linked to h, and $s_i(t_d)$ is equal to the number of agents linked to i.*

Proof. For any $h \in H(d)$, recall that $A_h(t_d)$ is the set of agents who point to h and $s_h(t_d) = \sum_{i \in A_h(t_d)} s_i(t_d)$.

First, consider all h such that all agents in $A_h(t_d)$ have an eating speed of one, it is obvious that $s_h(t) = |A_h(t_d)|$. $A_h(t_d)$ is also the set of all agents linked to h. If h is a private endowment of some i, then $s_i(t_d)$ is equal to the number of agents in $A_h(t_d)$ plus one. Denote the set of such existing tenants by $X_1(d)$.

Then, consider all h such that all existing tenants in $A_h(t_d)$ are in $X_1(d)$. Given the above result, it is obvious that $s_h(t_d)$ is equal to the number of all agents in

$A_h(t_d)$ plus the number of other agents linked to h through existing tenants in $A_h(t_d)$. Denote the set of existing tenants whose private endowments belong to this case by $X_2(d)$.

Repeating the above argument proves that $s_h(t_d)$ is equal to the number of agents who are linked to h. So $s_i(t_d)$ is equal to the number of agents linked to i. $\square$

With some abuse of notations, I also use $s_h(t_d)$ to denote the set of agents linked to h. Since every agent is linked to a unique social endowment, all agents in $I(d)$ can be partitioned into $\{s_h(t_d)\}_{h\in H_V(d)}$, where $H_V(d)$ is the set of social endowments. So $\cup_{h\in H_V(d)} s_h(t_d) = I(d)$ and $\sum_{h\in H_V(d)} s_h(t_d) = |I(d)|$. I use $H_O(d) := H(d)\backslash H_V(d)$ to denote the set of private endowments.

In TTC^E, for any $d \geq 0$, I define $B_h(d)$ and $w_h(d)$ as the counterparts of $A_h(t_d)$ and $s_h(t_d)$. That is, $B_h(d)$ is the set of agents whose most preferred remaining house is h, and $w_h(d)$ is both the number and the set of agents linked to h through some paths in the graph of TTC^E (see Appendix C.3.2). I still use $H_V(d), H_O(d)$ to denote the set of social endowments and the set of private endowments. Then $\sum_{h\in H_V(d)} w_h(d) = |I(d)|$ and $\cup_{h\in H_V(d)} w_h(d) = I(d)$.

At the beginning of step 1 of TTC^E, if any two pseudo-agents hold same house, the house must be a social endowment and they must hold an equal fraction. In the following I prove two lemmas about TTC^E, which guarantee that the statement is true throughout the procedure of TTC^E.

Lemma 5. *At any step $d+1$ of TTC^E, if there is no existing-tenant cycle, every new pseudo-agent i_h where $i \in I(d)\backslash B_h(d)$ and $h \in H_V(d)$ is involved in $w_h(d)$ selected cycles.*

Proof. All remaining agents in $I(d)$ can be partitioned into $B_h(d)$ and $I(d)\backslash B_h(d)$. $B_h(d)$ can be further partitioned into $B_{Nh}(d)$ and $B_{Eh}(d)$, the set of new agents in $B_h(d)$ and the set of existing tenants in $B_h(d)$. For every existing tenant $j \in B_{Eh}(d)$, there are $w_{\pi(j)}(d)$ agents who are linked to $\pi(j)$ and all of them are further linked to h. So by the definition of $w_h(d)$, $\sum_{j\in B_{Eh}(d)} w_{\pi(j)}(d) + |B_h(d)| = w_h(d)$. For any new pseduo-agent i_h such that $i \in I(d)\backslash B_h(d)$ and $h \in H_V(d)$, i's most preferred remaining house is not h. There are two cases:

Case 1. If i's most preferred remaining house is another $h' \in H_V(d)$, then i_h must point to all pseudo-agents $\{a_{h'}\}_{a\in I(d)\backslash\{i\}}$. Then for every $k_{h'}$ such that $k \in B_h(d)$, i_h

and $k_{h'}$ point to each other and form a cycle of length two. So i_h is involved in a total number of $|B_h(d)|$ of such length-two cycles. All these cycle are selected to trade. Moreover, since for every $j \in B_{Eh}(d)$ there are $w_{\pi(j)}(d)$ agents linked to j, there are $w_{\pi(j)}(d)$ new pseudo-agents who hold h' and are linked to $j_{\pi(j)}$. Since i_h points to all the $w_{\pi(j)}(d)$ new pseudo-agents and $j_{\pi(j)}$ points to i_h, there are $w_{\pi(j)}(d)$ cycles of the form $i_h \to \ell_{h'} \to \cdots \to j_{\pi(j)} \to i_h$, where $\ell_{h'}$ is a typical new pseudo-agent linked to $j_{\pi(j)}$. Note that i_h and $\ell_{h'}$ are the only two new pseudo-agents in the cycle. So all these cycles are selected to trade. Hence the total number of selected cycles that involve i_h is $\sum_{j \in B_{Eh}(d)} w_{\pi(j)}(d) + |B_h(d)| = w_h(d)$.

Case 2. If i's most preferred remaining house is a private endowment $h' \in H_O(d)$, then i_h points only to $g_{h'}$ where g is the owner of h'.

If g's most preferred remaining house is some $h'' \in H_V(d)$ other than h, then g points to all pseudo-agents $\{a_{h''}\}_{a \in I(d) \setminus g}$. As in the first case, for every $k_{h''}$ such that $k \in B_h(d)$, there is a length-three cycle $i_h \to g_{h'} \to k_{h''} \to i_h$ that involves only two new pseudo-agents. All these cycles are selected to trade, and their total number is $|B_h(d)|$. Moreover, for every $j \in B_{Eh}(d)$, there are $w_{\pi(j)}(d)$ new pseudo-agents who hold h'' and are linked to $j_{\pi(j)}$. For every such new pseudo-agent $\ell_{h''}$ there is a cycle $i_h \to g_{h'} \to \ell_{h''} \to \cdots \to j_{\pi(j)} \to i_h$ that involves only two new pseudo-agents i_h and $\ell_{h''}$. So all these cycles are selected to trade. Hence the total number of selected cycles that involve i_h is $\sum_{j \in B_{Eh}(d)} w_{\pi(j)}(d) + |B_h(d)| = w_h(d)$.

If g's most preferred remaining house is h, then i_h is involved in only one cycle: $i_h \to g_{h'} \to i_h$ in which i_h is the only new pseudo-agent. Then I regard the cycle as $w_h(d)$ cycles.

If g's most preferred remaining house is in $H_O(d)$, then g must be linked to a house in $H_V(d)$. If the house is not h, then as proved before, i_h is involved in $w_h(d)$ selected cycles. If the house is h, then i_h is involved in only one cycle looking like $i_h \to g_{h'} \to \cdots \to j'_{\pi(j')} \to i_h$, where i_h is the only new pseudo-agent. Then I also regard the cycle as $w_h(d)$ cycles. $\square$

In the above proof, when i_h is the only new pseudo-agent in a cycle, I regard the cycle as $w_h(d)$ cycles. Now if a new pseudo-agent i_h most prefers h and points to himself, I also regard the self-cycle as $w_h(d)$ cycles. These cycles may be traded in one step in the my definition of TTC^E. Trading these cycles in multiple steps does not change the assignment of TTC^E, but as shown later, this trick simplifies the proof of Theorem 7 by making TTC^E equivalent to PS^E step by step.

By Lemma 5, the set of new pseudo-agents $\{i_h\}_{h \in H_V(d)}$ are involved in a total number of $\sum_{h \in H_V(d)} w_h(d) = |I(d)|$ selected cycles. So I have the following corollary.

Corollary 11. *At any step $d+1$ of TTC^E, if there is no existing-tenant cycle, then every new agent $i \in I_N(d)$ is involved in $|I(d)|$ selected cycles.*

Remember that every existing tenant is further represented by a pseudo-tenant, so I have the following lemma.

Lemma 6. *At any step $d+1$ of TTC^E, if there is no existing-tenant cycle, then every pseudo-tenant $i_{\pi(i)}$ where $i \in I_E(d)$ is involved in $|I(d)| \cdot w_{\pi(i)}(d)$ selected cycles.*

Proof. For any $i \in I_E(d)$, if i's most preferred remaining house is some $h \in H_V(d)$, then $i_{\pi(i)}$ points to all pseudo-agents $\{a_h\}_{a \in I(d) \setminus \{i\}}$. For every agent $j \in w_{\pi(i)}(d)$, j_h is linked to $i_{\pi(i)}$. So there is a cycle $i_{\pi(i)} \rightarrow j_h \rightarrow \cdots \rightarrow i_{\pi(i)}$ that involves only one new pseudo-agent j_h, and the cycle is regarded as $w_h(d)$ selected cycles. Every new pseudo-agent $j_{h'}$ such that $h' \in H_V(d) \setminus \{h\}$ is also linked to $i_{\pi(i)}$. By Lemma 5, $j_{h'}$ is involved in $w_{h'}(d)$ selected cycles. All these cycles also involve $i_{\pi(i)}$. By Corollary 11, for every $j \in w_{\pi(i)}(d)$, $\{j_{h'}\}_{h' \in H_V(d)}$ is involved in $|I(d)|$ selected cycles. So $i_{\pi(i)}$ is involved in $|I(d)| \cdot w_{\pi(i)}(d)$ selected cycles.

If i's most preferred remaining house is in $H_O(d)$, then $i_{\pi(i)}$ must be linked to a pseudo-tenant whose most preferred remaining house is in $H_V(d)$. Then a similar argument as before proves that each of the $|I(d)|$ selected cycles that involves every $j \in w_{\pi(i)}(d)$ must also involve $i_{\pi(i)}$. So $i_{\pi(i)}$ is involved in $|I(d)| \cdot w_{\pi(i)}(d)$ selected cycles. □

So every existing tenant $i \in I_E(d)$ is involved in $|I(d)| + |I(d)| \cdot w_{\pi(i)}(d) = |I(d)| \cdot [w_{\pi(i)}(d) + 1]$ selected cycles.

Corollary 12. *At any step $d+1$ of TTC^E, if there is no existing-tenant cycle, then every existing tenant $i \in I_E(d)$ is involved in $|I(d)| \cdot [w_{\pi(i)}(d) + 1]$ selected cycles.*

Proof of Theorem 7:

Recall that in Lemma 5 and 6 I use the *trick* that each cycle in TTC^E that involves only one new pseudo-agent i_h at step $d+1$ is regarded as $w_h(d)$ cycles. Now I use another *trick* that if there are existing-tenants cycles at any step $d+1$ of TTC^E, I trade existing-tenants cycles at step $d+1$, and trade the remaining cycles at step

$d+2$. Both tricks do not change the assignment of TTC^E, but they will make TTC^E equivalent to PS^E step by step.

1. I prove the equivalence result for step $d = 1$.

   At the beginning of step 1, agents and their pseudo-agents point to same houses in PS^E and TTC^E respectively. So $A_h(t_0) = B_h(0)$ and $s_h(t_0) = w_h(0)$ for all $h \in H(0)$. There are two cases to consider.

   1.a If there are existing-tenant cycles in PS^E, there must be same existing-tenant cycles in TTC^E. Denote the set of these cycles by $C(1)$. Both mechanisms trade them immediately. Let $TQ_c^P(d)$ and $TQ_c^T(d)$ be the trading quotas of cycle c at step d of PS^E and at step d of TTC^E respectively. It is obvious that $TQ_c^P(1) = TQ_c^T(1)$ for every cycle $c \in C(1)$. Let $TQ_h^P(d)$ and $TQ_h^T(d)$ be the fraction of house h that are traded at step d of PS^E and at step d of TTC^E respectively. Then $TQ_h^P(1) = TQ_h^T(1)$ for every house $h \in H_O(0)$. So the two mechanisms are equivalent at step 1.

   1.b If there are no existing-tenant cycles in PS^E, step 1 of PS^E must end with a house being exhausted. Denote the house by $h^P(1)$, then $s_{h^P(1)}(t_0) \geq s_h(t_0)$ for any $h \in H(0)$. Let $EPS(d)$ (eating per speed) be the fraction of houses an agent of speed one eats at step d. Then $EPS(1) = 1/s_{h^P(1)}(t_0)$. So every new agent $i \in I_N(0)$ eats $EPS(1)$ of his most preferred house, and every existing tenant $j \in I_E(0)$ eats $s_j(t_0) \cdot EPS(1)$ of his most preferred house.

   In TTC^E, by Lemmas 5 and 6, every new pseudo-agent i_h holds $1/|I(0)|$ of $h \in H_V(0)$ and is involved in $w_h(0)$ selected cycles, while every pseudo-tenant $i_{h'}$ holds $h' \in H_O(0)$ and is involved in $|I(0)| \cdot w_{\pi(i)}(0)$ selected cycles. Since all selected cycles are traded with a common quota, step 1 of TTC^E must end with a house being exhausted, which is the solution to the following problem:

   $$\min_h\{\min_{h\in H_V(0)} \frac{1/|I(0)|}{w_h(0)}, \min_{h\in H_O(0)} \frac{1}{|I(0)|w_h(0)}\} \Leftrightarrow \min_{h\in H(0)} \frac{1}{|I(0)|w_h(0)}.$$

   Since $s_h(t_0) = w_h(0)$, the solution to the above problem must be $h^P(1)$. Let $TPC(d)$ (trading per cycle) be the common trading quota at step d of TTC^E. Then $TPC(1) = 1/[|I(0)|w_{h^P(1)}(0)] = 1/[|I(0)|s_{h^P(1)}(t_0)]$.

So $|I(0)|TPC(1) = EPS(1)$. Hence, every new agent $i \in I_N(0)$ obtains $|I(0)|TPC(1) = EPS(1)$ of his most preferred house, and every existing tenant $j \in I_E(0)$ obtains $|I(0)|[w_{\pi(j)}(0) + 1]TPC(1) = s_j(t_0)EPS(1)$ of his most preferred house. So the two mechanisms are equivalent at step 1.

2. Suppose at every step $d \leq k$ for some $k \geq 1$, the two mechanisms are equivalent, and $|I(d-1)|TPC(d) = EPS(d)$, $TQ_h^P(d) = TQ_h^T(d)$ for every $h \in H_O(d-1)$ and $s_{h'}(t_{d-1}) = w_{h'}(d-1)$ for every $h' \in H(d-1)$. Then I prove that the two mechanisms are still equivalent at step $k+1$. There are three cases to consider.

   2.a If there are existing-tenant cycles in PS^E, since PS^E is equivalent to TTC^E in previous steps, same cycles must exist at step $k+1$ of TTC^E. Denote the set of cycles by $C(k+1)$. Both mechanisms trade these cycles immediately. Since every agent has the same residual demand and every house has the same remainder in both mechanisms, $TQ_c^P(k+1) = TQ_c^T(k+1)$ for every $c \in C(k+1)$ and $TQ_h^P(k+1) = TQ_h^T(k+1)$ for every $h \in H_O(k)$. So every agent i in $I(k)$ must obtain the same fraction of the same house in both mechanisms, and the two mechanisms are equivalent at step $k+1$.

   2.b If there are no existing-tenant cycles in PS^E and step $k+1$ ends with some house $h^P(k+1)$ being exhausted, then I divide the previous k steps into two sets: the set $\alpha(k)$ of steps at which there are existing-tenant cycles and the set $\beta(k)$ of steps at which there are no existing-tenant cycles. Then

$$EPS(k+1) = \min_{h \in H(k)} \frac{1 - \sum_{d \in \alpha(k)} TQ_h^P(d) - \sum_{d \in \beta(k)} s_h(t_{d-1})EPS(d)}{s_h(t_k)}, \tag{C.1}$$

   and $EPS(k+1) \leq \frac{r_i^P(k)}{s_i(t_k)}$ for every $i \in I(k)$ where $r_i^P(k)$ is agent i's residual demand. By definition, $h^P(k+1)$ is a solution to the above problem.

   In TTC^E every pseudo-agent i_h points to the same house as i does in PS^E. I prove that step $k+1$ of TTC^E must end with $h^P(k+1)$ being exhausted. Since PS^E and TTC^E are equivalent step by step before step $k+1$, the previous k steps of TTC^E can also be partitioned into $\alpha(k)$ and $\beta(k)$. So every remaining agent in $I(k)$ holds an equal fraction

$[1-\sum_{d\in\alpha(k)} TQ_h^T(d)-\sum_{d\in\beta(k)} |I(d-1)|w_h(d-1)TPC(d)]/|I(k)|$ of every $h \in H_V(k)$.[1]

By induction assumption, $|I(d-1)|TPC(d) = EPS(d)$, $TQ_h^P(d) = TQ_h^T(d)$ for every $h \in H_O(d-1)$, and $s_{h'}(t_{d-1}) = w_{h'}(d-1)$ for every $h' \in H(d-1)$. So equation (C.1) implies that $h^P(k+1)$ is also a solution to the following problem:

$$TPC(k+1) = \min_{h\in H(k)} \frac{1-\sum_{d\in\alpha(k)} TQ_h^T(d)-\sum_{d\in\beta(k)} |I(d-1)|w_h(d-1)TPC(d)}{|I(k)|w_h(k)}, \tag{C.2}$$

Since $r_i^T(k) = r_i^P(k)$, $TPC(k+1) \leq \frac{r_i^T(k)}{|I(k)|[w_{\pi(i)}(k)+1]}$ for every $i \in I(k)$ where $r_i^T(k)$ is agent i's residual demand. So step $k+1$ of TTC^E ends with $h^P(k+1)$ being exhausted and $|I(k)|TPC(k+1) = EPS(k+1)$. Hence, every new agent $i \in I_N(k)$ obtains $|I(k)|TPC(k+1) = EPS(k+1)$ of his most preferred house, and every existing tenant $j \in I_E(k)$ obtains $|I(k)|[w_{\pi(j)}(k)+1]TPC(k+1) = s_j(t_k)EPS(k+1)$ of his most preferred house. So the two mechanisms are equivalent at step $k+1$.

2.c If there are no existing-tenant cycles in PS^E and step $k+1$ ends with some existing tenant, denoted by $i^P(k+1)$, being full and leaving the algorithm, I divide the previous k steps into $\alpha(k)$ and $\beta(k)$ as before. Then by definition,

$$\begin{aligned} EPS(k+1) &= \frac{r^P_{i^P(k+1)}(k)}{s_{i^P(k+1)}(t_k)} \\ &\leq \min\{\min_{h\in H(k)} \frac{1-\sum_{d\in\alpha(k)} TQ_h^P(d)-\sum_{d\in\beta(k)} s_h(t_{d-1})EPS(d)}{s_h(t_k)}, \min_{i\in I(k)} \frac{r_i^P(k)}{s_i(t_k)}\}. \end{aligned}$$

By the induction assumption, in TTC^E I have

$$\begin{aligned} TPC(k+1) &= \frac{r^T_{i^P(k+1)}(k)}{|I(k)|[w_{\pi(i^P(k+1))}(k)+1]} \\ &\leq \min\{\min_{h\in H(k)} \frac{1-\sum_{d\in\alpha(k)} TQ_h^T(d)-\sum_{d\in\beta(k)} |I(d-1)|w_h(d-1)TPC(d)}{|I(k)|w_h(k)}, \\ &\quad \min_{i\in I(k)} \frac{r_i^T(k)}{|I(k)|[w_{\pi(i)}(k)+1]}\}. \end{aligned}$$

[1]Recall that when an agent leaves the algorithm, his remaining endowments are uniformly assigned to remaining agents.

So step $k+1$ of TTC^E also ends with $i^P(k+1)$ being full and leaving the algorithm. Moreover, since $|I(k)|TPC(k+1) = EPS(k+1)$, the two mechanisms are still equivalent at step $k+1$.

C.1.2 Proof of Theorem 8

Given any problem, I first define finite large problems and the limit problem. Then I characterize the procedure of PS^E in finite problems, and prove that PS^E is equivalent to random YRMH-IGYT in the limit problem. In the on-line appendix I provide the characterization of random YRMH-IGYT in finite problems, and prove that the assignments of the two mechanisms in finite problems converge to their assignments in the limit problem. Then I finish my proof. The techniques here are almost same with those of Che and Kojima (2010), so detailed explanations are omitted.

Given a problem $m = \{I, H, \pi, \succsim_I\}$, for every $\ell \in \mathbb{N}\backslash\{0\}$, an ℓ-problem is $m^\ell = (I^\ell, H^\ell, \pi^\ell, (\gamma_i)_{i\in I^\ell})$ in which every non-null house in H and every existing tenant in I have ℓ copies. The endowment function is extended accordingly such that a copy of an existing tenant j owns a copy of $\pi(j)$.[2] Each agent i has a preference type γ_i, which is a one-to-one mapping from H to $\{1, ..., |H|\}$ such that $\gamma_i(h) < \gamma_i(h')$ if and only if $h \succ_i h'$. The set of all preference types is denoted by Γ. The set of new agents I^ℓ_N is partitioned into $\{I^\ell_{N\gamma}\}_{\gamma\in\Gamma}$ where $I^\ell_{N\gamma}$ is the set of new agents with preference type γ. Let $a^\ell_{N\gamma} := \frac{|I^\ell_{N\gamma}|}{\ell}$ be the per-unit number of new agents with preference type γ. The set of new agents grows in the size of the problem under the requirement that there exists $a^\infty_{N\gamma} \in \mathbb{R}_+$ such that $a^\ell_{N\gamma} \to a^\infty_{N\gamma}$ for every preference type γ as $\ell \to +\infty$. I define $\{I^\ell_{E\gamma}\}_{\gamma\in\Gamma}$ similarly. Obviously, $\frac{|I^\ell_{E\gamma}|}{\ell} = |I^1_{E\gamma}|$ for any $\ell \in \mathbb{N} \cup \{\infty\}$ and γ.

For any set of houses $H' \subseteq H$, let $Ch_\gamma(H')$ be the set of houses in H' that most preferred by the preference type $\gamma \in \Gamma$. When the set of remaining agent is $I^{\ell\prime}$ and the set of remaining houses is $H^{\ell\prime}$, let $S^\ell_i(H^{\ell\prime}, I^{\ell\prime}) := S^\ell_{\pi^\ell(i)}(H^{\ell\prime}, I^{\ell\prime})+1$ be the per-unit eating speed of agent i; let $S^\ell_h(H^{\ell\prime}, I^{\ell\prime}) := \sum_{\gamma\in\Gamma:h\in Ch_\gamma(H')}\{a^\ell_{N\gamma} + \sum_{i\in I^{\ell\prime}_{E\gamma}} S^\ell_i(H^{\ell\prime}, I^{\ell\prime})\}$ be the per-unit speed at which house $h \in H^{\ell\prime}$ is being eaten. Then whenever $H^\ell(d-1)$, $I^\ell(d-1)$, t^ℓ_{d-1}, $\{r^\ell_h(d-1)\}_{h\in H^\ell}$ and $\{r^\ell_i(v-1)\}_{i\in I^\ell}$ are given, the step d of PS^E in any ℓ-problem can be characterized by the following equations:

[2] In PS, any two agents are homogeneous if they have equal preferences. But in PS^E, any two existing tenants are homogeneous only if they have equal preferences and equal private endowments. So I replicate existing tenants in large problems.

(a) If there are no existing-tenant cycles, define

(a.1) $t_h^\ell(d) := \sup\{t \in [0,1] \mid r_h^\ell(d-1) - S_h^\ell(H^\ell(d-1), I^\ell(d-1))(t - t_{d-1}^\ell) > 0\}$ for all $h \in H^\ell(d-1)$;

(a.2) $t_i^\ell(d) := \sup\{t \in [0,1] \mid r_i^\ell(d-1) - S_i^\ell(H^\ell(d-1), I^\ell(d-1))(t - t_{d-1}^\ell) > 0\}$ for all $i \in I^\ell(d-1)$;

(a.3) $t_d^\ell := \min\{\min_{h \in H^\ell(d-1)} t_h^\ell(d), \min_{i \in I^\ell(d-1)} t_i^\ell(d)\}$;

(a.4) $H^\ell(d) := H^\ell(d-1) \backslash \{h \in H^\ell(d-1) \mid t_h^\ell(d) = t_d^\ell\}$;

(a.5) $I^\ell(d) := I^\ell(d-1) \backslash \{i \in I^\ell(d-1) \mid t_i^\ell(d) = t_d^\ell\}$;

(a.6) $r_h^\ell(d) := r_h^\ell(d-1) - S_h^\ell(H^\ell(d-1), I^\ell(d-1))(t_d^\ell - t_{d-1}^\ell)$;

(a.7) $r_i^\ell(d) := r_i^\ell(d-1) - S_i^\ell(H^\ell(d-1), I^\ell(d-1))(t_d^\ell - t_{d-1}^\ell)$.

Here, $t_h^\ell(d)$ is time that h is exhausted; $t_i^\ell(d)$ is the time that i is full. Step d ends with a house being exhausted or an agent being full, depending on which happens earlier.

(b) Otherwise, denote the set of existing-tenant cycles by $C^\ell(d)$. For each $c \in C^\ell(d)$, denote the set of existing tenants and the set of houses involved in c by $c(I)$ and $c(H)$ respectively. Then define

(b.1) $TC(c) = \min\{\min_{h \in c(H)} r_h^\ell(d-1), \min_{i \in c(I)} r_i^\ell(d-1)\}$ for each $c \in C^\ell(d)$;

(b.2) $r_h^\ell(d) = r_h^\ell(d-1) - TC(c)$ if there exists $c \in C^\ell(d)$ such that $h \in c(H)$. Otherwise $r_h^\ell(d) = r_h^\ell(d-1)$;

(b.3) $r_i^\ell(d) = r_i^\ell(d-1) - TC(c)$ if there exists $c \in C^\ell(d)$ such that $i \in c(I)$. Otherwise $r_i^\ell(d) = r_i^\ell(d-1)$;

(b.4) $H^\ell(d) = H^\ell(d-1) \backslash \{h \in H^\ell(d-1) \mid r_h^\ell(d) = 0\}$;

(b.5) $I^\ell(d) = I^\ell(d-1) \backslash \{i \in I^\ell(d-1) \mid r_i^\ell(d) = 0\}$;

(b.6) $t_d^\ell = t_{d-1}^\ell$, and $t_a^\ell(d) = t^\ell(d)$ for $a \in \{h \in H^\ell(d-1) \mid r_h^\ell(d) = 0\} \cup \{i \in I^\ell(d-1) \mid r_i^\ell(d) = 0\}$.

In random YRMH-IGYT, let every agent draw a lottery number uniformly and independently from $[0,1]$. Let agents choose their most preferred houses among remaining ones according to the increasing ordering of their lotteries numbers. I say that a step of random YRMH-IGYT ends at $\hat{t} \in [0,1]$ if in expectation a house is exhausted or an agent is full when some agent with a lottery number $\hat{t}$ chooses his most preferred house. By this interpretation I can track the procedure of random YRMH-IGYT by discrete steps. Then I define $\hat{H}^{\infty}(d)$, $\hat{I}^{\infty}(d)$, $\hat{t}^{\infty}_{d}$, $\{\hat{r}^{\infty}_{h}(d)\}_{h\in H^{\infty}}$ and $\{\hat{r}^{\infty}_{i}(d)\}_{i\in I^{\infty}}$ similarly as I do for PS^{E}. Now I want to prove that random YRMH-IGYT in the limit problem m^{∞} can be characterized by equations (a.1)-(a.7) and (b.1)-(b.6) by setting $\ell = \infty$. This implies that PS^{E} and random YRMH-IGYT are equivalent in the limit problem.

Specifically, suppose $H^{\infty}(d-1) = \hat{H}^{\infty}(d-1)$, $I^{\infty}(d-1) = \hat{I}^{\infty}(d-1)$, $t^{\infty}_{d-1} = \hat{t}^{\infty}_{d-1}$, $\{r^{\infty}_{h}(d-1)\}_{h\in H^{\infty}} = \{\hat{r}^{\infty}_{h}(d-1)\}_{h\in \hat{H}^{\infty}}$ and $\{r^{\infty}_{i}(d-1)\}_{i\in I^{\infty}} = \{\hat{r}^{\infty}_{i}(d-1)\}_{i\in \hat{I}^{\infty}}$, which are true for $d = 1$, then I prove that they are still true for step d. There are two cases to consider for step d of random YRMH-IGYT:

(1) If there are no existing-tenant cycles, then for any $h \in \hat{H}^{\infty}(d-1)$, by the proof of Theorem 7, $S^{\infty}_{h}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1))$ is the mass of agents who either most prefer h, or most prefer the private endowments of some agents who are linked to h. As long as one of these agents has the chance to consume, h must be consumed. By the weak law of large numbers (see Che and Kojima 2010), the proportion of these agents who can draw a lottery number between $\hat{t}^{\infty}_{d-1}$ and any $t > \hat{t}^{\infty}_{d-1}$ is exactly $S^{\infty}_{h}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1))(t - \hat{t}^{\infty}_{d-1})$. So the lottery number $\hat{t}^{\infty}_{h}(d)$, which is the expected "time" that h is exhausted, is exactly characterized by equation (a.1).

A new agent $i \in \hat{I}^{\infty}_{N}(d-1)$ can choose his most preferred house in $\hat{H}^{\infty}(d-1)$ only if he draws the smallest lottery number among all agents in $\hat{I}^{\infty}(d-1)$. Since the lottery number is drawn from a uniform distribution, i can obtain $t - \hat{t}^{\infty}_{d-1}$ of his most preferred house between $\hat{t}^{\infty}_{d-1}$ and any $t > \hat{t}^{\infty}_{d-1}$. So the lottery number at which i is full is characterized by equation (a.2) by setting $S^{\infty}_{i}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1)) = 1$.

An existing tenant $j \in \hat{I}^{\infty}_{E}(d-1)$ can choose his most preferred house either if he draws the smallest lottery number among $\hat{I}^{\infty}(d-1)$, or if he is involved in trading chains, which are triggered by a mass $S^{\infty}_{\pi^{\infty}(j)}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1))$ of other agents who draw the smallest lottery number. By the weak law of large numbers, the chance that i can choose his most preferred house between $\hat{t}^{\infty}_{d-1}$ and any $t > \hat{t}^{\infty}_{d-1}$ is $\left[S^{\infty}_{\pi^{\infty}(j)}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1)) + 1\right](t - \hat{t}^{\infty}_{d-1})$. Since $S^{\infty}_{j}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1)) = S^{\infty}_{\pi^{\infty}(j)}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1)) + 1$, j can obtain $S^{\infty}_{j}(\hat{H}^{\infty}(d-1), \hat{I}^{\infty}(d-1))(t - \hat{t}^{\infty}_{d-1})$ of his most preferred house. So the lottery number $\hat{t}^{\infty}_{j}(d)$ at which j is full is also characterized by equation (a.2). So I can use the remaining equations (a.3)-(a.7) to characterize random YRMH-IGYT.

(2) If there are existing-tenant cycles, all the cycles must also appear at step d

of PS^E. I trade these cycles immediately in random YRMH-IGYT. So equations (b.1)-(b.6) also characterize random YRMH-IGYT.

If I denote the assignments found by the two mechanisms in m^∞ by $PS^E(m^\infty)$ and $\text{RYI}(m^\infty)$ respectively, then the above arguments prove that $||PS^E(m^\infty) - \text{RYI}(m^\infty)|| = 0$.[3] In the on-line appendix I prove that $||PS^E(m^\ell) - PS^E(m^\infty)|| \to 0$ and $||\text{RYI}(m^\ell) - \text{RYI}(m^\infty)|| \to 0$ as $\ell \to \infty$. So $||\text{RYI}(m^\ell) - PS^E(m^\ell)|| \to 0$ as $\ell \to \infty$.

C.2 Proofs of Propositions 10-13

Proof of Proposition 11

For any simultaneous eating algorithm that satisfies conditions (1) and (2), since any two new agents i, i' always have same endowments, it is obvious that $s_i(t) = s_{i'}(t)$ for all $t \in [0, 1]$. Then there are two cases to consider:

Case 1. If there are no cycles among existing tenants, then I normalize the eating speed of every new agent to one. That is, $\sum_{h \in E_i(t)} (s_h(t)/|O_h(t)|) = 1$ for all $i \in I_N$ and all $t \in [0, 1]$. So for every existing tenant $j \in I_E$, $s_j(t) = \sum_{h \in E_j(t)} (s_h(t)/|O_h(t)|) = s_{\pi(j)}(t) + \sum_{h \in E_i(t)} (s_h(t)/|O_h(t)|) = s_{\pi(j)}(t) + 1$ where i is any new agent.

Case 2. If there are cycles among existing tenants, without loss of generality, I let a typical cycle be $\pi(j_1) \to j_1 \to \pi(j_2) \to j_2 \to \cdots \to \pi(j_n) \to j_n \to \pi(j_1)$, where every $j_o (o = 1, \ldots, n)$ is an existing tenant. Then "you request my house - I get your speed" requires that $s_{j_1}(t) \leq s_{j_2}(t) \leq \cdots \leq s_{j_n}(t) \leq s_{j_1}(t)$. Hence, $s_{j_1}(t) = s_{j_2}(t) = \cdots = s_{j_n}(t)$. However, $s_{j_2}(t) = s_{\pi(j_2)}(t) + \sum_{h \in E_i(t)} (s_h(t)/|O_h(t)|) \geq s_{j_1}(t) + s_i(t)$ where i is any new agent. So it must be that $s_i(t) = 0$ for any new agent i. For any existing tenant j who is not involved in any cycle, it must be that $s_{\pi(j)} = 0$. So $s_j(t) = s_{\pi(j)}(t) + s_i(t) = 0$. Therefore, only existing tenants in cycles can have positive eating speeds. This is equivalent to trading the cycles immediately.

Proof of Proposition 12

PS^E is ordinal efficient since it can be seen as a simultaneous eating algorithm. It is obviously individually rational for new agents. It is individually rational for existing tenants because private endowments are exhausted never earlier than their owners are full. Since all new agents have equal eating speeds, there is no envy among them.

Proof of Proposition 13

[3] $||PS^E(m^\infty) - \text{RYI}(m^\infty)|| = \sup_{i \in I^\infty, h \in H^\infty} |PS^E(m^\infty)_{ih} - \text{RYI}(m^\infty)_{ih}|$ where $PS^E(m^\infty)_{ih}$ and $\text{RYI}(m^\infty)_{ih}$ are the probabilities that i obtains h in $PS^E(m^\infty)$ and $\text{RYI}(m^\infty)$ respectively.

For any $h \in H$ and any $i \in I$, suppose i reports a strict preference relation $\succsim_i' \in \mathcal{R}$ such that $U(\succsim_i', h) = U(\succsim_i, h)$ and $\succsim_i' |_{U(\succsim_i', h)} = \succsim_i |_{U(\succsim_i, h)}$. It is easy to see that changes happen in the procedure of PS^E only after all houses in $U(\succsim_i, h)$ are exhausted. So the assignments of all houses in $U(\succsim_i, h)$ do not change, and PS^E is boundedly invariant.

Suppose $\succsim_i'$ is a dropping strategy of $\succsim_i$ and $U(\succsim_i, \pi(i)) \backslash U(\succsim_i', \pi(i)) = \{h_1, h_2, \ldots, h_\ell\}$. If i does not obtain any fraction of the houses in $\{h_1, h_2, \ldots, h_\ell\}$ by reporting $\succsim_i$, then the outcome of PS^E does not change no matter i reports $\succsim_i$ or $\succsim_i'$. So without loss of generality, I assume that h_k is the best house in $\{h_1, h_2, \ldots, h_\ell\}$ that i obtains a positive fraction by reporting $\succsim_i$ but is dropped in $\succsim_i'$. Since PS^E is boundedly invariant, the fraction of any house better than h_k that i obtains does not change if he reports $\succsim_i'$. However, by dropping h_k agent i must lose a positive fraction of h_k. So the lottery i obtains by reporting $\succsim_i'$ cannot first-order stochastically dominate the lottery i obtains by reporting $\succsim_i$. So PS^E is weakly dropping-strategy-proof.

If $\succsim_i'$ is a truncation strategy of $\succsim_i$, then there exists some $h \succsim_i \pi(i)$ such that $\succsim_i' |_{U(\succsim_i', \pi(i))} = \succsim_i |_{U(\succsim_i, h)}$. As before, the fraction of every house better than h that i obtains does not change if he reports $\succsim_i'$. But by reporting $\succsim_i'$ he fills his remaining demand by $\pi(i)$, while by reporting $\succsim_i$ he may obtain a positive fraction of some house strictly better than $\pi(i)$. So the lottery obtained by reporting $\succsim_i$ first-order stochastically dominates the lottery obtained by reporting $\succsim_i'$. So PS^E is truncation-strategy-proof.

Proof of Proposition 14

Individual rationality, *new-agent envy-freeness* and *bounded invariance* hold in the same way as before. In the following I prove the remaining properties.

Suppose the random assignment PS^E under weak preferences finds for some problem is not ordinally efficient. Then there must exist $k \geq 2$ agents $i_1, i_2, \ldots, i_k$, and k houses in their consumption profiles $h_1^{i_1}, h_2^{i_2}, \ldots, h_k^{i_k}$ such that if the agents trade their consumptions as the following cycle, none of them is worse off and some is strictly better off:

$$i_1 \to h_2^{i_2} \to i_2 \to h_3^{i_3} \to i_3 \to \cdots \to h_k^{i_k} \to i_k \to h_1^{i_1} \to i_1.$$

Here $h_o^{i_o} (o = 1, \ldots, k)$ is the consumption of i_o and is obtained by i_{o-1} after trading the cycle. Without loss of generality, I assume that i_1 is strictly better off while others are as good as before. That is, i_1 strictly prefers h_2 to h_1, while $i_o (o = 2, \ldots, k)$

is indifferent between h_o and h_{o+1}. Then in the procedure of PS^E under weak preferences, when i_1 starts consuming h_1 at some step d, either h_1 is not exhausted, or h_1 has been exhausted but some agent labels his consumption of h_1 as available, and at the same time h_2 have been exhausted and no agent labels his consumption of h_2 as available. Then since i_k is indifferent between h_1 and h_k, at step d either h_k is not exhausted, or it has been exhausted but i_k labels his consumption of h_k as available. By the same argument, since i_{k-1} is indifferent between h_k and h_{k-1}, at step d either h_{k-1} is not exhausted, or it has been exhausted but i_{k-1} labels his consumption of h_{k-1} as available. Repeating this argument we reach the conclusion that at step d, either h_2 is not exhausted, or it has been exhausted but i_2 label his consumption of h_2 as available. However, this contradicts an earlier argument. So PS^E under weak preferences is ordinally efficient.

Suppose $\succsim_i'$ is a dropping strategy of $\succsim_i$ and $U(\succsim_i, \pi(i)) \backslash U(\succsim_i', \pi(i)) = \{h_1, h_2, \ldots, h_\ell\}$. Let h_k be one of the best houses in $\{h_1, h_2, \ldots, h_\ell\}$ that i obtains a positive fraction by reporting $\succsim_i$ but is dropped in $\succsim_i'$. Let $H(h_k)$ and $H'(h_k)$ be the set of houses indifferent with h_k in $\succsim_i$ and $\succsim_i'$ respectively. If $H'(h_k) = \emptyset$, then it is obvious that the lottery i obtains by reporting $\succsim_i'$ cannot first-order stochastically dominate the lottery i obtains by reporting $\succsim_i$. If $H'(h_k) \neq \emptyset$, then it must be that $H'(h_k) \subseteq H(h_k)$. Since PS^E under weak preferences is boundedly invariant, i obtains the same fraction of any house better than h_k by reporting either $\succsim_i$ or $\succsim_i'$. But $H'(h_k) \subseteq H(h_k)$ implies that the total fraction of the houses in $H'(h_k)$ that i obtains by reporting $\succsim_i'$ cannot exceed the total fraction of the houses in $H(h_k)$ that i obtains by reporting $\succsim_i$. By repeating the argument for the remaining houses in $\{h_1, h_2, \ldots, h_\ell\}$, I can prove weak dropping-strategy-proofness. Truncation-strategy-proofness can be proved very similarly.

C.3 PS^{IR}, TTC^E

C.3.1 The Procedure of PS^{IR} in Example 2

Step 1: At $t = 0$, every agent eats his most preferred house with speed one. Since the set of acceptable houses for i_1, i_2 is $\{h_1, h_2, h_3\}$, when more than one unit in $\{h_1, h_2, h_3\}$ are eaten by agents other than i_1, i_2, the individual rationality of i_1, i_2 is violated. So i_3, i_4, i_5, i_6 are blocked from eating h_1, h_2, h_3 after $t = 1/4$. So at $t = 1/4$, the problem is broken into two sub-problems: $m_1 = \{\{1/2h_1, 1/2h_2, 1/2h_3\}, \{i_1, i_2\}\}$ and $m_2 = \{\{h_4, h_5, h_6\}, \{i_3, i_4, i_5, i_6\}\}$.

Step 2: For sub-problem m_1, at $t = 1/4$, i_1, i_2 eat h_2, h_3 respectively with speed one. Then at $t = 1/2$, i_1 obtains another $1/4h_2$ and i_2 obtains another $1/4h_3$. So i_2's residual demand is $1/2$, while his acceptable remaining houses are $\{1/4h_2, 1/4h_3\}$. So if i_1 continues eating h_2 after $t = 1/2$, i_2's individual rationality will be violated. So at $t = 1/2$, m_1 is broken into two sub-problems: $m_{11} = \{\{1/4h_2, 1/4h_3\}, \{i_2\}\}$ and $m_{12} = \{\{1/2h_1\}, \{i_1\}\}$.

Step 3: For each of m_{11}, m_{12}, m_2, the procedure of PS^{IR} coincides with that of PS.

The above procedure is summarized as:

time	i_1	i_2	i_3	i_4	i_5	i_6
$1/4$	$1/4h_2$	$1/4h_3$	$1/4h_1$	$1/4h_2$	$1/4h_1$	$1/4h_3$
$+1/4$	$1/4h_2$	$1/4h_3$	$1/4h_5$	$1/4h_6$	$1/4h_6$	$1/4h_4$
$+1/4$	$1/4h_1$	$1/4h_3$	$1/4h_5$	$1/4h_6$	$1/4h_6$	$1/4h_4$
$+1/4$	$1/4h_1$	$1/4h_2$	$1/4h_5$	$1/4h_5$	$1/4h_4$	$1/4h_4$

It is easy to see that PS^{IR} is not boundedly invariant: although at $t = 1/4$ agents i_1, i_2 have not revealed their preferences over houses other than h_2, h_3 in the above procedure, PS^{IR} predicts that the set of acceptable houses for them is $\{h_1, h_2, h_3\}$ and blocks i_3, i_4, i_5, i_6 from eating h_1, h_2, h_3 after $t = 1/4$.

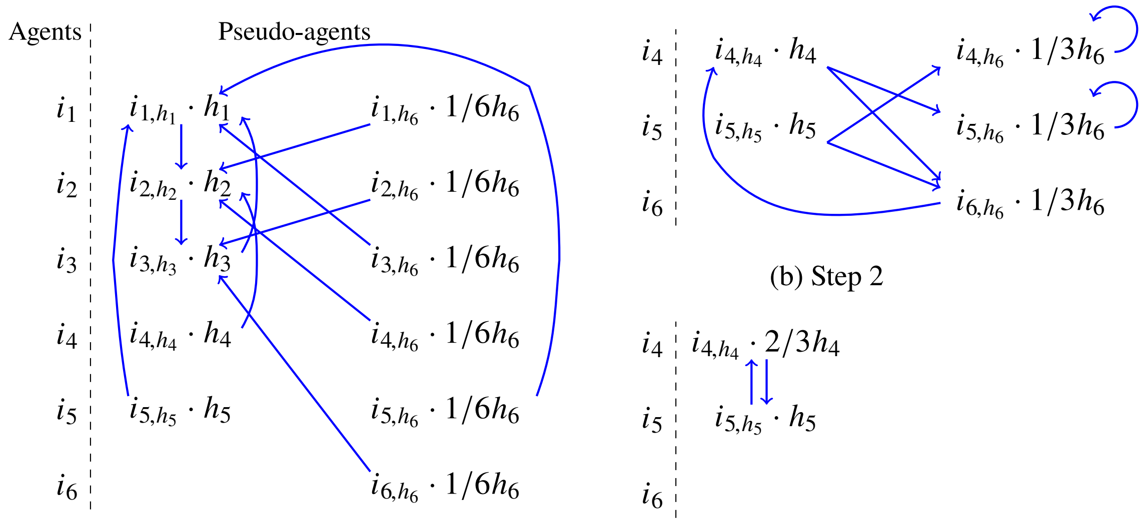
C.3.2 The Procedure of TTC^E in Example 2

Initialization: Uniformly assign the probabilities of h_6 to all agents. So each agent's endowment contains $1/6h_6$ and his private endowment, if any.

Step 1: There is one existing-tenant cycle: $i_{1,h_1} \longrightarrow i_{2,h_2} \longrightarrow i_{3,h_3} \longrightarrow i_{1,h_1}$. After trading the cycle, i_1 gets h_2, i_2 gets h_3 and i_3 gets h_1. They are full and leave the algorithm. Their remaining endowments, $3 \times 1/6h_6 = 1/2h_6$, are uniformly assigned to remaining agents. Step 1 ends.

Step 2: There are two self-cycles: i_{4,h_6} and i_{5,h_6} pointing to themselves. Since each of them holds $1/3h_6$, the trading quota of both cycles is $1/3$. There is also a feasible new-agent cycle: $i_{4,h_4} \longrightarrow i_{6,h_6} \longrightarrow i_{4,h_4}$, which involves one new pseudo-agent i_{6,h_6}. The trading quota of this cycle is also $1/3$. After trading these cycles, i_4 gets $2/3h_6$, i_5 gets $1/3h_6$, and i_6 gets $1/3h_4$. h_6 is exhausted. Step 2 ends.

Step 3: There is one existing-tenant cycle: $i_{4,h_4} \longrightarrow i_{5,h_5} \longrightarrow i_{4,h_4}$. After trading the cycle with quota $1/3$, i_4 gets $1/3h_5$ and i_5 gets $1/3h_4$. Then i_4 is full and



(a) Step 1

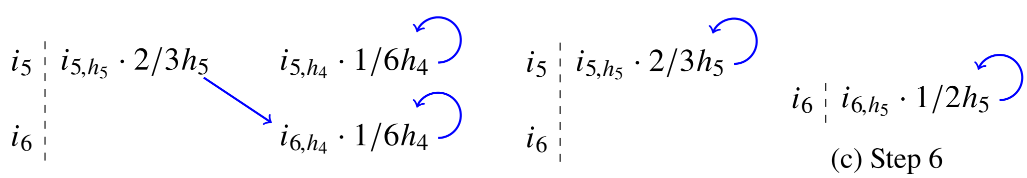
(b) Step 2

(c) Step 3

Figure C.2: Steps 1, 2, and 3

leaves the algorithm. His remaining endowment $1/3h_4$ is uniformly assigned to i_5 and i_6. Step 3 ends.

Step 4: There are two self-cycles: i_{5,h_4} and i_{6,h_4} pointing to themselves. Trade these cycles with quota $1/6$. Step 4 ends.



(a) Step 4

(b) Step 5

(c) Step 6

Figure C.3: Steps 4, 5, and 6

Step 5: There is one self-cycle: i_{5,h_5} pointing to himself. After trading the cycle with quota $1/6$, i_5 obtains $1/6h_5$ and leaves the algorithm. His remaining endowment $1/2h_5$ is assigned to i_6. Step 5 ends.

Step 6: There is one self-cycle: i_{6,h_5} pointing to himself. After trading the cycle with quota $1/2$, i_6 gets $1/2h_5$ and leaves the algorithm. The algorithm stops.

BIBLIOGRAPHY

Abdulkadiroğlu, Atila, Yeon-Koo Che, and Yosuke Yasuda. 2011. "Resolving Conflicting Preferences in School Choice: The "Boston Mechanism" Reconsidered". *American Economic Review* 101 (1): 399–410.

Abdulkadiroğlu, Atila, and Tayfun Sönmez. 1999. "House allocation with existing tenants". *Journal of Economic Theory* 88 (2): 233–260.

– . 1998. "Random serial dictatorship and the core from random endowments in house allocation problems". *Econometrica*: 689–701.

– . 2003. "School choice: A mechanism design approach". *The American Economic Review* 93 (3): 729–747.

Abdulkadiroğlu, Atila, et al. 2005. "The Boston public school match". *American Economic Review, Papers and Proceedings*: 368–371.

Agarwal, Nikhil, and Paulo Somaini. 2014. "Demand Analysis using Strategic Reports: An application to a school choice mechanism". *National Bureau of Economic Research*.

Arad, Ayala, and Ariel Rubinstein. 2012. "The 11–20 money request game: a level-k reasoning study". *The American Economic Review* 102 (7): 3561–3573.

Ashlagi, Itai, and Yannai A Gonczarowski. 2015. "No stable matching mechanism is obviously strategy-proof". *working paper*.

Ashlagi, Itai, Yash Kanoria, and Jacob D Leshno. 2015. "Unbalanced Random Matching Markets: The Stark Effect of Competition". *Journal of Political Economy, forthcoming*.

Athanassoglou, Stergios, and Jay Sethuraman. 2011. "House allocation with fractional endowments". *International Journal of Game Theory* 40 (3): 481–513.

Bade, Sophie. 2016a. "Fairness and group-strategyproofness clash in assignment problems". *Journal of Economic Theory* 165:257–262.

– . 2016b. "Pareto-Optimal Matching Allocation Mechanisms for Boundedly Rational Agents". *Social Choice and Welfare, forthcoming*.

– . 2016c. "Random serial dictatorship: the one and only". *working paper*.

Balbuzanov, Ivan. 2014. "Short Trading Cycles: Kidney Exchange with Strict Ordinal Preferences". *working paper*.

Basteck, Christian, and Marco Mantovani. 2016. "Cognitive Ability and Games of School Choice". *working Paper*.

Bogomolnaia, Anna. 2015. "Random assignment: redefining the serial rule". *Journal of Economic Theory* 158:308–318.

Bogomolnaia, Anna, and Eun Jeong Heo. 2012. “Probabilistic assignment of objects: Characterizing the serial rule”. *Journal of Economic Theory* 147 (5): 2072–2082.

Bogomolnaia, Anna, and Hervé Moulin. 2001. “A new solution to the random assignment problem”. *Journal of Economic theory* 100 (2): 295–328.

Budish, Eric, et al. 2013. “Designing random allocation mechanisms: Theory and applications”. *The American Economic Review* 103 (2): 585–623.

Calsamiglia, Caterina, Chao Fu, and Maia Güell. 2015. “Structural Estimation of a Model of School Choices: the Boston Mechanism vs. Its Alternatives”. *working paper*.

Calsamiglia, Caterina, Guillaume Haeringer, and Flip Klijn. 2010. “Constrained school choice: An experimental study”. *The American Economic Review*: 1860–1874.

Camerer, Colin F, Teck-Hua Ho, and Juin-Kuan Chong. 2004. “A cognitive hierarchy model of games”. *The Quarterly Journal of Economics*: 861–898.

Castillo, Marco, and Ahrash Dianat. 2016. “Truncation strategies in two-sided matching markets: Theory and experiment”. *Games and Economic Behavior* 98:180–196.

Che, Yeon-Koo, and Fuhito Kojima. 2010. “Asymptotic equivalence of probabilistic serial and random priority mechanisms”. *Econometrica* 78 (5): 1625–1672.

Chen, Yan, and Tayfun Sönmez. 2006. “School choice: an experimental study”. *Journal of Economic theory* 127 (1): 202–231.

Coles, Peter, and Ran Shorrer. 2014. “Optimal truncation in matching markets”. *Games and Economic Behavior* 87:591–615.

Costa-Gomes, Miguel A, and Vincent P Crawford. 2006. “Cognition and behavior in two-person guessing games: An experimental study”. *The American economic review* 96 (5): 1737–1768.

Costa-Gomes, Miguel, Vincent P Crawford, and Bruno Broseta. 2001. “Cognition and behavior in normal-form games: An experimental study”. *Econometrica* 69 (5): 1193–1235.

Crawford, Vincent P, and Nagore Iriberri. 2007a. “Fatal attraction: Salience, naivete, and sophistication in experimental “hide-and-see” games”. *The American Economic Review* 97 (5): 1731–1750.

– . 2007b. “Level-k Auctions: Can a Nonequilibrium Model of Strategic Thinking Explain the Winner’s Curse and Overbidding in Private-Value Auctions?” *Econometrica* 75 (6): 1721–1770.

Dogan, Battal, Serhat Dogan, and Kemal Yildiz. 2016. “A New Ex-Ante Efficiency Criterion and Implications for the Probabilistic Serial Mechanism”. *Available at SSRN 2777970*.

Dur, Umut, Robert Hammond, and Thayer Morrill. 2015. "Identifying the Harm of Manipulable School-Choice Mechanisms". *working paper*.

Ehlers, Lars. 2008. "Truncation strategies in matching markets". *Mathematics of Operations Research* 33 (2): 327–335.

Ekici, Özgün, and Onur Kesten. 2015. "An equilibrium analysis of the probabilistic serial mechanism". *International Journal of Game Theory*: 1–20.

Erdil, Aytek. 2014. "Strategy-proof stochastic assignment". *Journal of Economic Theory* 151:146–162.

Ergin, Haluk, and Tayfun Sönmez. 2006. "Games of school choice under the Boston mechanism". *Journal of Public Economics* 90 (1): 215–237.

Featherstone, Clayton R, and Muriel Niederle. 2014. "Improving on Strategy-proof School Choice Mechanisms: An Experimental Investigation". *working paper*.

Gale, David, and Lloyd S Shapley. 1962. "College admissions and the stability of marriage". *American mathematical monthly*: 9–15.

Haeringer, Guillaume, and Flip Klijn. 2009. "Constrained school choice". *Journal of Economic Theory* 144 (5): 1921–1947.

Hashimoto, Tadashi, et al. 2014. "Two axiomatic approaches to the probabilistic serial mechanism". *Theoretical Economics* 9 (1): 253–277.

He, Yinghua. 2014. "Gaming the boston school choice mechanism in beijing". *Manuscript, Toulouse School of Economics*.

Heo, Eun Jeong. 2014. "Probabilistic assignment problem with multi-unit demands: A generalization of the serial rule and its characterization". *Journal of Mathematical Economics* 54:40–47.

Ho, Teck-Hua, Colin Camerer, and Keith Weigelt. 1998. "Iterated dominance and iterated best response in experimental "p-beauty contest"". *The American Economic Review* 88 (4): 947–969.

Hugh-Jones, David, Morimitsu Kurino, and Christoph Vanberg. 2014. "An experimental study on the incentives of the probabilistic serial mechanism". *Games and Economic Behavior* 87:367–380.

Hylland, Aanund, and Richard Zeckhauser. 1979. "The efficient allocation of individuals to positions". *Journal of Political Economy*: 293–314.

Jaramillo, Paula, and Vikram Manjunath. 2012. "The difference indifference makes in strategy-proof allocation of objects". *Journal of Economic Theory* 147 (5): 1913–1946.

Katta, Akshay-Kumar, and Jay Sethuraman. 2006. "A solution to the random assignment problem on the full preference domain". *Journal of Economic Theory* 131 (1): 231–250.

Kesten, Onur. 2009. “Why do popular mechanisms lack efficiency in random environments?” *Journal of Economic Theory* 144 (5): 2209–2226.

Kesten, Onur, and M Utku Ünver. 2015. “A theory of school-choice lotteries”. *Theoretical Economics* 10 (2): 543–595.

Kojima, Fuhito. 2009. “Random assignment of multiple indivisible objects”. *Mathematical Social Sciences* 57 (1): 134–142.

Kojima, Fuhito, and M Utku Ünver. 2014. “The “Boston” school-choice mechanism: an axiomatic approach”. *Economic Theory* 55 (3): 515–544.

Li, Shengwu. 2015. “Obviously strategy-proof mechanisms”. *working paper*.

Liu, Qingmin, and Marek Pycia. 2013. “Ordinal efficiency, fairness, and incentives in large markets”. *working paper*.

Martini, Giorgio. 2016. “Strategy-proof and fair assignment is wasteful”. *Games and Economic Behavior* 98:172–179.

Nagel, Rosemarie. 1995. “Unraveling in guessing games: An experimental study”. *The American Economic Review*: 1313–1326.

Nesterov, Alexander. 2016. “Fairness and Efficiency in Strategy-proof Object Allocation Mechanisms”. *working paper*.

Pápai, Szilvia. 2000. “Strategyproof assignment by hierarchical exchange”. *Econometrica* 68 (6): 1403–1433.

Pathak, Parag A. 2016. “What Really Matters in Designing School Choice Mechanisms”. *Preparation for 11th Econometric Society World Congress Plenary Talk*.

Pathak, Parag A, and Tayfun Sönmez. 2008. “Leveling the playing field: Sincere and sophisticated players in the Boston mechanism”. *The American Economic Review* 98 (4): 1636–1652.

– . 2013. “School Admissions Reform in Chicago and England: Comparing Mechanisms by Their Vulnerability to Manipulation”. *American Economic Review* 103 (1): 80–106.

Pycia, Marek, and Peter Troyan. 2016. “Obvious dominance and random priority”. *working paper*.

Pycia, Marek, and M Utku Ünver. 2017. “Incentive compatible allocation and exchange of discrete resources”. *Theoretical Economics* 12 (1): 287–329.

Rees-Jones, Alex. 2015. “Suboptimal behavior in strategy-proof mechanisms: Evidence from the residency match”. *working paper*.

Roth, Alvin E. 1986. “On the allocation of residents to rural hospitals: a general property of two-sided matching markets”. *Econometrica: Journal of the Econometric Society*: 425–427.

Roth, Alvin E, and Uriel G Rothblum. 1999. "Truncation strategies in matching markets–in search of advice for participants". *Econometrica* 67 (1): 21–43.

Sethuraman, Jay. 2001. "A New Solution to the House Allocation Problem with Existing Tenants". *unpublished mimeo*.

Shapley, Lloyd, and Herbert Scarf. 1974. "On cores and indivisibility". *Journal of Mathematical Economics* 1 (1): 23–37.

Stahl, Dale O, and Paul W Wilson. 1994. "Experimental evidence on players' models of other players". *Journal of Economic Behavior & Organization* 25 (3): 309–327.

– . 1995. "On players' models of other players: Theory and experimental evidence". *Games and Economic Behavior* 10 (1): 218–254.

Svensson, Lars-Gunnar. 1999. "Strategy-proof allocation of indivisible goods". *Social Choice and Welfare* 16 (4): 557–567.

Troyan, Peter. 2012. "Comparing school choice mechanisms by interim and ex-ante welfare". *Games and Economic Behavior* 75 (2): 936–947.

Yılmaz, Özgür. 2009. "Random assignment under weak preferences". *Games and Economic Behavior* 66 (1): 546–558.

– . 2010. "The probabilistic serial mechanism with private endowments". *Games and Economic Behavior* 69 (2): 475–491.

Zhou, Lin. 1990. "On a conjecture by Gale about one-sided matching problems". *Journal of Economic Theory* 52 (1): 123–135.