The Geometry of Moduli Spaces of Maps from Curves

Thesis by
Seunghee Ye

In Partial Fulfillment of the Requirements for the
degree of
Doctor of Philosophy

Caltech

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2017
Defended May 3, 2017

To Yang

Seunghee Ye
ORCID: 0000-0002-1250-2935

ACKNOWLEDGEMENTS

First, I would like to thank my avdisor, Tom Graber, for his guidance over the past five years. I have learned a lot under his supervision, and his insights helped me get through numerous obstacles that I encountered in my research. I am very grateful for his help and guidance, without which I could not have completed my research.

I would also like to thank Tanvir Ahamed Bhuyain, William Chan, Brian Hwang, Emad Nasrollahpoursamami, and Pablo Solis for helpful discussions and their support. I am also grateful for Craig Sutton for having inspired me to pursue research in mathematics.

Lastly, I would like to thank my parents, my brother, and my wife, Yang, for their love and unending support.

ABSTRACT

A class of moduli spaces that has long been the interest of many algebraic geometers is the class of moduli spaces parametrizing maps from curves to target spaces. Different such moduli spaces have distinct geometry and also invariants associated to them. In this thesis, we will study the geometry of three such moduli spaces, $\widetilde{\mathcal{M}}_{g,n}([pt/\mathbb{C}^\times]), Quot_C(n,d)$, and $\overline{Q}_{0,2}(\mathbb{G}(n,n),d)$. By understanding the global geometry of each moduli space, we will produce a stratification, which plays a central role in proving a result about invariants associated to the space.

In Chapter 1, we study gauge Gromov-Witten invariants, which are the Euler characteristics of admissible classes on $\widetilde{\mathcal{M}}_{g,n}([pt/\mathbb{C}^\times])$, the moduli space of maps from stable curves to $[pt/\mathbb{C}^\times]$. In [1], Frenkel, Teleman, and Tolland show that while $\widetilde{\mathcal{M}}_{g,n}$ is not finite type, thsese gauge Gromov-Witten invariants are well-defined. By using a particular stratification of $\widetilde{\mathcal{M}}_{0,n}$, we prove that when $g = 0$, n-pointed gauge Gromov-Witten invariants can be reconstructed from 3-pointed invariants. This reconstruction theorem provides a concrete way to compute gauge Gromov-Witten invariants, and serves as an alternate proof of well-definedness of the invariants in genus 0 case.

In Chapter 2, we compute the Poincare polynomials of Quot schemes, $Quot_C(n,d)$. We see that by using an appropriate stratification, we can recursively compute the Poincare polynomials of $Quot_C(n,d)$. Moreover, we see that the generating series for the Poincare polynomials is a rational function. As an application, we compute the Poincare polynomials of the moduli spaces of MOP-stable quotients, $\overline{Q}_{0,2}(\mathbb{G}(n,n),d)$. We show that the generating series for these polynomials is also a rational function.

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Chapter 1

RECONSTRUCTION THEOREM IN GAUGE GROMOV-WITTEN THEORY

1.1 Introduction

In [9], Lee defines quantum K-invariants, which are K-theoretic push-forwards to Spec $\mathbb{C}$ of certain vector bundles on $\overline{\mathcal{M}}_{g,n}(X,\beta)$. These quantum K-invariants are shown to satisfy several axioms. Moreover, when $g = 0$ and $X = \mathbb{P}^r$, Lee and Pandharipande prove in [10] that there exist divisor relations in $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,\beta))$ which allow one to reconstruct all quantum K-invariants of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,\beta)$ from quantum K-invariants of $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^r,\beta)$.

In [1], Frenkel, Teleman, and Tolland consider the compactification of the moduli space of maps from curves to a space with automorphisms. They define the moduli stack of Gieseker bundles on stable curves, $\widetilde{\mathcal{M}}_{g,n}$, and showed that there exist well-defined K-theoretic invariants in the case where the target is $[pt/\mathbb{C}^\times]$. Proving well-definedness of these invariants is difficult because the resulting moduli space is complete but not finite type. Their proof of well-definedness of invariants relies on their description of local charts on the moduli stack. While the use of charts allows them to conclude that the invariants are indeed finite, it does not tell us how the invariants can be computed and does not easily generalize to $[X/\mathbb{C}^\times]$ for arbitrary scheme X.

Instead of using local charts, I describe a stratification of $\widetilde{\mathcal{M}}_{g,n}$ by locally closed strata. When $g = 0$, this stratification, along with divisorial relations, allows us to reconstruct n-pointed invariants from lower pointed invariants.

Theorem 1.1. *n-pointed genus 0 gauge Gromov-Witten invariants can be reconstructed from 3-pointed invariants.*

The reconstruction theorem for genus 0 gauge Gromov-Witten invariants not only serves as an alternative proof of the well-definedness of the invariants but also gives an algorithm for computing them.

1.2 Reconstruction in quantum K-theory

In this section, we will define the moduli spaces $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and the resulting quantum K-invariants. We then state the reconstruction theorem for quantum K-invariants when $g = 0$. The background on $\overline{\mathcal{M}}_{g,n}$ follows [2] and the discussion of quantum K-invariants follows [9].

The moduli spaces $\overline{\mathcal{M}}_{g,n}(X,\beta)$

Definition 1.1. *[2] Let X be a scheme and let $\beta \in H_2(X)$. Then, a stable map of class β from a prestable curve $(C, x_1, \ldots, x_n)$ of genus g with n marked points, x_i, is a morphism $f : C \to X$ satisfying the following conditions.*

1. *The homological push-forward of C satisfies $f_*([C]) = \beta$.*

2. *Each irreducible component of C contracted by f is stable. In other words, if E is an irreducible component of C which is contracted by f, then*

$$g(E) + n(E) \geq 3,$$

   where $n(E)$ is the number of nodes and marked points on E.

The moduli space of such maps is denoted by $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

It follows from the definition that stable maps have finite automorphisms. Using this fact, Kontsevich prove the following theorem.

Theorem 1.2. *[8] Let X be a smooth projective scheme over $\mathbb{C}$, and let $\beta \in H_2(X)$. Then, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a proper Deligne-Mumford stack.*

When $X = \operatorname{Spec}\mathbb{C}$, we denote $\overline{\mathcal{M}}_{g,n}(\operatorname{Spec}\mathbb{C}) = \overline{\mathcal{M}}_{g,n}$. There are two classes of morphisms that arise naturally. The first is the class of forgetful morphisms which forget the k-th marked point and stabilize if necessary. We denote the morphism forgetting the k-th marked point by

$$ft_k : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta).$$

We also have the stabilization morphisms which forget the map $f : C \to X$, and stabilize the prestable curve, C, if necessary. This map is denoted by

$$st : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}.$$

Quantum K-invariants

Definition 1.2. *[11] Let X be a scheme. The Grothendieck group of locally free sheaves on X is the quotient of the free abelian group generated by the isomorphism classes of the locally free sheaves on X by the relation $\sum(-1)^i F_i = 0$, whenever $0 \to F_0 \to F_1 \to \cdots F_k \to 0$ is an exact sequence.*

The Grothendieck group of locally free sheaves on X is denoted $K(X)$.

If $f : X \to Y$ is a proper morphism, we define the K-theoretic push-forward homomorphism $f_* : K(X) \to K(Y)$ by

$$f_*([F]) = \sum (-1)^i [R^i f_* F].$$

The K-theoretic push-forward to Spec $\mathbb{C}$ is denoted χ.

Now, we define the quantum K-invariants as the K-theoretic push-forwards of certain K-classes on $\overline{\mathcal{M}}_{g,n}$.

Definition 1.3. *[9] The quantum K-invariants are*

$$\langle \gamma_1, \dots, \gamma_n, F \rangle = \chi(\overline{\mathcal{M}}_{g,n}(X, \beta), O^{vir} \otimes ev^*(\gamma_1 \otimes \cdots \otimes \gamma_n) \otimes st^* F),$$

where $\gamma_1, \dots, \gamma_n \in K(X)$, $F \in K(\overline{\mathcal{M}}_{g,n})$, and O^{vir} is the virtual structure sheaf.

While quantum K-invariants do not satisfy all the axioms of cohomological Gromov-Witten invariants [7], they satisfy seven of them, two of which are the splitting axiom and the string equation.

Proposition 1.1. *[9] Let $g = g_1 + g_2$ and $n = n_1 + n_2$ and let*

$$\Phi : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$$

be the map gluing the last marked point of $\overline{\mathcal{M}}_{g_1,n_1+1}$ with the first marked point of $\overline{\mathcal{M}}_{g_2,n_2+1}$. Then, pulled back quantum K-invariants from $\overline{\mathcal{M}}_{g,n}$ can be written as a sum of products of quantum K-invariants of $\overline{\mathcal{M}}_{g_1,n_1+1}$ and $\overline{\mathcal{M}}_{g_2,n_2+1}$.

Theorem 1.3 (String Equation)**.** *[9] Let $ft : \overline{\mathcal{M}}_{g,n+1}(X, \beta) \to \overline{\mathcal{M}}_{g,n}(X, \beta)$ be the morphism forgetting the last marked point. Let $\mathcal{L}_i$ denote the cotangent line bundle along the i-th marked point. Then, for $g = 0$ we have*

$$\pi_* \left(O^{vir} \left(\prod_{i=1}^{n} \frac{1}{1 - q_i \mathcal{L}_i} \right) \right) = \left(1 + \sum_{i=1}^{n} \frac{q_i}{1 - q_i} \right) \left(O^{vir} \left(\prod_{i=1}^{n} \frac{1}{1 - q_i \mathcal{L}_i} \right) \right),$$

where both sides of the equation are formal series in formal variables q_i.

For $g \geq 1$ we have

$$\pi_*\left(\mathcal{O}^{vir}\frac{1}{1-q\mathcal{H}^{-1}}\prod_{i=1}^{n-1}\frac{1}{1-q_i\mathcal{L}_i}\right) \tag{1.1}$$

$$= \mathcal{O}^{vir}\frac{1}{1-q\mathcal{H}^{-1}}\left[\left(1-\mathcal{H}^{-1}+\sum_{i=1}^{n-1}\frac{q_i}{1-q_i}\right)\left(\prod_{i=1}^{n-1}\frac{1}{1-q_i\mathcal{L}_i}\right)\right], \tag{1.2}$$

where $\mathcal{H} = R^0\pi_*\omega_{C/\overline{M}}$ is the Hodge bundle.

Note that Theorem 1.3 relates $(n+1)$-pointed quantum K-invariants *not* involving $\mathcal{L}_{n+1}$ with n-pointed quantum K-invariants.

Reconstruction of quantum K-invariants

In [10], Lee and Pandharipande prove that two relations hold in $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,\beta))$. These divisor relations, combined with the axioms of quantum K-invariants, show that n-pointed quantum K-invariants can be reconstructed from 1-pointed invariants.

Let $\beta \in H_2(\mathbb{P}^r)$. Let $\beta_1, \beta_2 \in H_2(\mathbb{P}^r)$ such that $\beta_1 + \beta_2 = \beta$. Partition the set $\{1,\ldots,n\}$ into S_1 and its complement $S_2 := S_1^c$. Then, we denote by $D_{S_1,\beta_1|S_2,\beta_2}$ the divisor in $\overline{\mathcal{M}}_{0,n}$ parametrizing reducible curves $C = C_1 \cup C_2$ such that the marked points $p_j \in C_i$ if $j \in S_i$ and the images of C_i are β_i for $i = 1$ and 2. Now, define

$$D_{i,\beta_1|j,\beta_j} = \sum_{i\in S_1|j\in S_2} D_{S_1,\beta_1|S_2,\beta_2}, \quad \text{and} \quad D_{i,j} = \sum_{i\in S_1, j\in S_2, \beta_1+\beta_2=\beta} D_{S_1,\beta_1|S_2,\beta_2}.$$

Denote by $\mathcal{L}_i$ the class in $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,\beta))$ corresponding to the i-th cotangent bundle. Then, we have the following theorem.

Theorem 1.4. *[10] Let $\beta \in H_2(\mathbb{P}^r)$ and let $L \in \mathrm{Pic}(\mathbb{P}^r)$. Then, the following relations hold in $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,\beta))$.*

1. $\mathrm{ev}_i^* L = \mathrm{ev}_j^* L + \langle\beta, L\rangle\mathcal{L}_j - \sum_{\beta_1+\beta_2=\beta}\langle\beta_1, L\rangle D_{i,\beta_1|j,\beta_2}$.

2. $\mathcal{L}_i + \mathcal{L}_j = D_{i|j}$.

With Theorem 1.4, Lee and Pandharipande prove the reconstruction theorem for invariants in both quantum cohomology and quantum K-theory.

Theorem 1.5. *[10]*

1. *Let $R \subset H^*(X)$ be a self-dual subring generated by Chern classes of elements of* $\mathrm{Pic}(X)$. *Suppose*

$$(\tau_{i_1}(\gamma_1), \ldots, \tau_{k_{n-1}}(\gamma_{n-1}), \tau_{k_n}(\xi)) = 0$$

for all n-pointed invariants with $\gamma_i \in R$ and $\xi \in R^\perp$. Then, all n-pointed invariants of classes of R can be reconstructed from 1-point invariants of R.

2. *Let $R \subset K^*(X)$ be a self-dual subring generated by elements of* $\mathrm{Pic}(X)$. *Suppose*

$$(\tau_{i_1}(\gamma_1), \ldots, \tau_{k_{n-1}}(\gamma_{n-1}), \tau_{k_n}(\xi)) = 0$$

for all n-pointed invariants with $\gamma_i \in R$ and $\xi \in R^\perp$. Then, all n-pointed quantum K-invariants of classes of R can be reconstructed from 1-point quantum K-invariants of R.

1.3 Moduli stack of Gieseker bundles

We will now present the moduli stack of Gieseker bundles as defined in [1]. Moduli stack of Gieseker bundles arise when studying the moduli space of maps from prestable curve to spaces with automorphisms such as $[pt/\mathbb{C}^\times]$. We recall the definition of families of prestable marked curves.

Definition 1.4. *$(\pi : C \to B, \{\sigma_i | i \in I\})$ is called a family of prestable marked curves over a base scheme B if*

1. *$\pi : C \to B$ is a flat proper morphism whose fibers are connected curves of genus g with at-worst-nodal singularities, and*

2. *I is an ordered indexing set such that for all i, $\sigma_i : B \to C$ is a section not passing through nodes of fibers, and*

If all rational components of C has at least 3 special points, we say (C, σ_i) is a family of stable marked curves.

We will always assume that any rational component of a fiber of π has at least two special points.

A map from a stable nodal curve C to $[pt/\mathbb{C}^\times]$ is equivalent to a principal $\mathbb{C}^\times$-bundle on C. Such a $\mathbb{C}^\times$ bundle is given by a $\mathbb{C}^\times$-bundle on the normalization of C and identification of the two fibers at the preimages of each of the nodes. Since the space of identifications of the two fibers is isomorphic to $\mathbb{C}^\times$, the moduli stack of principal $\mathbb{C}^\times$-bundles on stable curves fails to be complete.

To make the space complete, we consider all Gieseker bundles on stable curves.

Definition 1.5. *[1] Let (C, σ_i) be a stable marked curve. A Gieseker bundle on (C, σ_i) is a pair $(m, \mathcal{L})$ consisting of*

1. *a morphism $m : (C', \sigma_i') \to (C, \sigma_i)$ such that m is an isomorphism away from preimages of nodes of C, and the preimages of nodes of C are either nodes or a $\mathbb{P}^1$ with two special points; and*

2. *a line bundle $\mathcal{L}$ on C' such that the degree of $\mathcal{L}$ restricted to every unstable $\mathbb{P}^1$ has degree 1. Such unstable rational components of C' are called Gieseker bubbles.*

Then, $\widetilde{\mathcal{M}}_{g,n}$ is defined to be the moduli stack of Gieseker bundles on stable genus g, n-pointed curves.

Definition 1.6. *[1] The stack $\widetilde{\mathcal{M}}_{g,n}$ of Gieseker $\mathbb{C}^\times$-bundles on stable genus g curves with n marked points is a fibered category whose objects are $(X, C, \sigma_i, \mathcal{P})$, where*

1. *X is a test scheme,*

2. *$\pi : C \to X$ is a flat projective family of prestable curves with marked points $\sigma_i : X \to C$, and*

3. *$p : \mathcal{P} \to C$ is a Gieseker bundle on the stabilization of C.*

The morphisms in this category are commutative diagrams

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\tilde{f}} & \mathcal{P}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ C & \xrightarrow{f} & C' \\ {\scriptstyle \sigma_i}\uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi}\downarrow\uparrow{\scriptstyle \sigma_i'} \\ X & \longrightarrow & X' \end{array} \quad ,$$

where $\tilde{f}$ is $\mathbb{C}^\times$ equivariant and the bottom square is Cartesian.

$\widetilde{\mathcal{M}}_{g,n}$ carries several universal families. It has a family of stable curves of genus g with n marked points $\pi : \widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n}$ with $\sigma_i : \widetilde{\mathcal{M}}_{g,n} \to C$, and a Gieseker bundle $p : \mathcal{P}_{g,n} \to \widetilde{C}_{g,n}$. The universal Gieseker bundle defines a map $\varphi : \widetilde{C}_{g,n} \to [pt/\mathbb{C}^\times]$. We define the evaluation maps $\mathrm{ev}_i = \varphi \circ \sigma_i : \widetilde{\mathcal{M}}_{g,n} \to [pt/\mathbb{C}^\times]$.

The moduli space, $\widetilde{\mathcal{M}}_{g,n}$, is a disjoint union of components corresponding to the total degree of the Gieseker bundle. Each of the components of $\widetilde{\mathcal{M}}_{g,n}$ is complete but is not finite type in general. For example, consider the component of the moduli space $\widetilde{\mathcal{M}}_{0,4}$ corresponding to total degree D. There are infinitely many Gieseker bundles over reducible curve with two components, $C = C_1 \cup C_2$, such that the line bundle, $\mathcal{L}$, has degrees d_1 and d_2 over C_1 and C_2 and $d_1 + d_2 = D$. Thus, $\widetilde{\mathcal{M}}_{g,n}$ is not finite type and therefore not proper. However, the following properties hold for $\widetilde{\mathcal{M}}_{g,n}$.

Proposition 1.2. *[1]*

1. *$\widetilde{\mathcal{M}}_{g,n}$ is locally of finite type and locally finitely presented.*
2. *$\widetilde{\mathcal{M}}_{g,n}$ is unobstructed.*

For a prestable curve with a Gieseker bundle $(C, \sigma_i, \mathcal{P})$, we define its topological type to be the pair (γ, d), where γ is the modular graph of C and $d : V(\gamma) \to \mathbb{Z}$ is the degree map. The topological type of Gieseker bundles allow us to stratify $\widetilde{\mathcal{M}}_{g,n}$.

Proposition 1.3. *[1] $\widetilde{\mathcal{M}}_{g,n}$ admits a topological type stratification by locally closed and disjoint substacks $\mathcal{M}_{(\gamma,d)}$ parametrizing all curves with modular graph γ with degree d. Moreover, $\mathcal{M}_{(\gamma,d)}$ are of finite type and finite presentation.*

Moreover, we know which kinds of deformations of curves can occur.

Lemma 1.1. *[1] Let $(C, \sigma_i, \mathcal{P})$ be a $\mathbb{C}^\times$ bundle on a prestable curve having topological type (γ, d). Suppose that we are given a deformation $(C', \sigma_i', \mathcal{P}')$ of $(C, \sigma_i, \mathcal{P})$ over the Spec of a complete discrete valuation ring. The topological type (γ', d') of the generic fiber can be any degree labeled modular graph obtained from (γ, d) by finite combinations of the following elementary operations:*

1. *Resolve a self node: delete a self-edge attached to a vertex v, increasing the genus g_v by 1, leave the multi-degree unchanged.*

2. *Resolve a splitting node: join a pair of adjacent vertices v_1 and v_2 into a single vertex v, having genus $g_v = g_{v_1} + g_{v_2}$ and degree $d_v = d_{v_1} + d_{v_2}$. Delete one edge joining v_1 and v_2, and convert the others to self-edges.*

Moreover, all such modular graphs occur in some deformation.

On $\widetilde{\mathcal{M}}_{g,n}$, there are special K-theory classes that we want to consider.

Definition 1.7. *Let V be a finite dimensional representation of $\mathbb{C}^\times$. Let $\mathcal{L}_i = \sigma_i^* T_\pi$ be the relative tangent sheaf to C at σ_i. Then, we define the following K-theory classes on $\widetilde{\mathcal{M}}_{g,n}$.*

1. *The evaluation bundle is* $\mathrm{ev}_i^*[V] = \sigma_i^* \varphi^* V$.
2. *The descendant bundles are* $\mathrm{ev}_i^*[V] \otimes [\mathcal{L}_i^{\otimes j_i}]$, *where $j_i \in \mathbb{Z}$.*
3. *The Dolbeault index I_V of V is the complex $R\pi_* \varphi^* V$.*
4. *The admissible line bundles $\mathcal{L}$ are $\mathcal{L} \cong (\det R\pi_* \varphi^* \mathbb{C}_1)^{\otimes -q}$, where $q \in \mathbb{Q}_{>0}$.*
5. *An admissible complex is the tensor product of an admissible line bundle with Dolbeaut index, evaluation, and descendant bundles*

$$\alpha = \mathcal{L} \otimes \left(\bigotimes_a R\pi_* \varphi^* V_a \right) \otimes \left(\otimes_i \mathrm{ev}_i^* W_i \otimes \mathcal{L}_i^{n_i} \right).$$

Theorem 1.6. *[1] Let α be an admissible class. Let $F : \widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful morphism forgetting the bundle and stabilizing the curve. Then, the derived push-forward $RF_* \alpha$ is coherent.*

1.4 Outline

Let $\widetilde{\mathcal{M}}_{0,n} := \widetilde{\mathcal{M}}_{0,n}([\mathrm{pt}/\mathbb{C}^\times])$ be the moduli stack of Gieseker stable bundle with the universal curve $\pi_n : \widetilde{C}_{0,n} \to \widetilde{\mathcal{M}}_{0,n}$. We will denote the universal bundle by $\mathcal{P}_{0,n}$.

Let $\alpha = \det(R\pi_* \varphi^* \mathbb{C}_1)^{-q} \otimes \left(\otimes \mathrm{ev}_i^* \mathbb{C}_{\lambda_i} \otimes \mathcal{L}_i^{a_i} \right)$ be an admissible class on $\widetilde{\mathcal{M}}_{0,n}$. We will show that the admissible class α can be reconstructed from finitely many admissible classes of $\widetilde{\mathcal{M}}_{0,3}$. Since $\widetilde{\mathcal{M}}_{0,3} \cong [pt/\mathbb{C}^\times]$, admissible classes on $\widetilde{\mathcal{M}}_{0,3}$ have finite Euler characteristics. The reconstruction proves that for $g = 0$, the invariants are well-defined.

First, we show that one can define an open embedding $\widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n+1}$. If we denote the complement of the image of $\widetilde{C}_{g,n}$ by $\widetilde{\mathcal{Z}}_{g,n+1}$, using the long exact sequence of local cohomologies, we obtain

$$\chi(\widetilde{\mathcal{M}}_{g,n+1}, \alpha) = \chi(\widetilde{C}_{g,n}, \alpha) + \chi_{\widetilde{\mathcal{Z}}_{g,n+1}}(\alpha),$$

provided all the terms above are finite. Now, since $\widetilde{C}_{g,n}$ is the universal curve over $\widetilde{\mathcal{M}}_{g,n}$ we can compute $\chi(\widetilde{C}_{g,n}, \alpha)$ by pushing forward to $\widetilde{\mathcal{M}}_{g,n}$ along the map $\pi_n : \widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n}$.

$$\chi(\widetilde{C}_{g,n}, \alpha) = \chi(\widetilde{\mathcal{M}}_{g,n}, R\pi_{n*}\alpha).$$

Therefore, we have

$$\chi(\widetilde{\mathcal{M}}_{g,n+1}, \alpha) = \chi(\widetilde{\mathcal{M}}_{g,n}, R\pi_{n*}\alpha) + \chi_{\widetilde{\mathcal{Z}}_{g,n+1}}(\alpha).$$

Repeating, we conclude that

$$\chi(\widetilde{\mathcal{M}}_{g,n}, \alpha) = \begin{cases} \chi(\widetilde{\mathcal{M}}_{0,3}, R\pi_*\alpha) + \sum\limits_{4 \le k \le n} \chi_{\widetilde{\mathcal{Z}}_{0,k}}(R\pi_*\alpha) & g = 0 \\ \chi(\widetilde{\mathcal{M}}_{1,1}, R\pi_*\alpha) + \sum\limits_{2 \le k \le n} \chi_{\widetilde{\mathcal{Z}}_{1,k}}(R\pi_*\alpha) & g = 1 \\ \chi(\widetilde{\mathcal{M}}_{g,0}, R\pi_*\alpha) + \sum\limits_{1 \le k \le n} \chi_{\widetilde{\mathcal{Z}}_{g,k}}(R\pi_*\alpha) & g \ge 2 \end{cases},$$

where $\pi : \widetilde{\mathcal{M}}_{g,n} \dashrightarrow \widetilde{\mathcal{M}}_{g,k}, k \le n$ is the composition of $\pi_\ell : \widetilde{C}_{g,\ell} \to \widetilde{\mathcal{M}}_{g,\ell}$ for $k \le \ell \le n-1$.

We then stratify $\widetilde{\mathcal{Z}}_{g,n}$ by countably many locally closed strata. This stratification will have the property that for $g = 0$, we can compute $\chi_{\widetilde{\mathcal{Z}}_{0,n}}(\alpha)$ recursively as a finite sum of products of lower pointed invariants on $\widetilde{\mathcal{M}}_{0,k}$, where $k < n$.

Moreover, the embedding of $\widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n+1}$ will show that for admissible classes, α, on $\widetilde{\mathcal{M}}_{g,n+1}$ that do not involve $\mathrm{ev}_{n+1}^* \mathbb{C}_{\lambda_{n+1}}$ and $\mathcal{L}_{n+1}$, the push-forward of $\alpha|_{\widetilde{C}_{g,n}}$ to $\widetilde{\mathcal{M}}_{g,n}$ is an admissible class on $\widetilde{\mathcal{M}}_{g,n}$.

A divisor relation similar to the relation proven in Theorem 1.4 then reduce the problem of computing admissible classes on $\widetilde{\mathcal{M}}_{0,n+1}$ to computing those that do not involve $\mathrm{ev}_{n+1}^* \mathbb{C}_{\lambda_{n+1}}$ and $\mathcal{L}_{n+1}$. Lastly, understanding the structure of the boundary loci in $\widetilde{C}_{0,n}$ as $\mathbb{A}^s \times (\mathbb{P}^1)^t$ bundles over products of $\widetilde{\mathcal{M}}_{0,n'}$, where $n' < n$, allows us to compute $\chi(\widetilde{\mathcal{M}}_{0,n+1})$ as a finite sum of $\chi(\widetilde{\mathcal{M}}_{0,n'})$ where $n' < n$.

Combining, we will conclude that n-pointed invariants can be computed as a finite sum of products of lower pointed invariants.

1.5 Embedding $\widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n+1}$

In this section, we will define an embedding $\widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n+1}$. Recall that we have a similar embedding for the stable curves. If we let $\overline{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve, we have an embedding $\overline{C}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$. In short, given a point p on a n-pointed stable curve, $(C, p_1, \dots, p_n)$, we can associate to it a $(n+1)$-pointed curve, $(C', p'_1, \dots, p'_{n+1})$, where

1. if $p \in C$ is not a special point, then $C' = C$, $p'_i = p_i$ for all $i = 1, \dots, n$, and $p_{n+1} = p$; or
2. if $p \in C$ is a special point, then $(C', p'_1, \dots, p'_{n+1})$ is the stable curve whose stabilization after forgetting p'_{n+1} is C, with the images of p'_i under the stabilization are p_i for $i = 1, \dots, n$ and p for $i = n+1$. In other words, C' is the stable curve obtained from C by adding a rational component at p with three special points, one of which is p'_{n+1}.

Figure 1.1 show a few examples of the correspondence described above. In the second and third examples in the figure, the components containing p_{n+1} are rational.

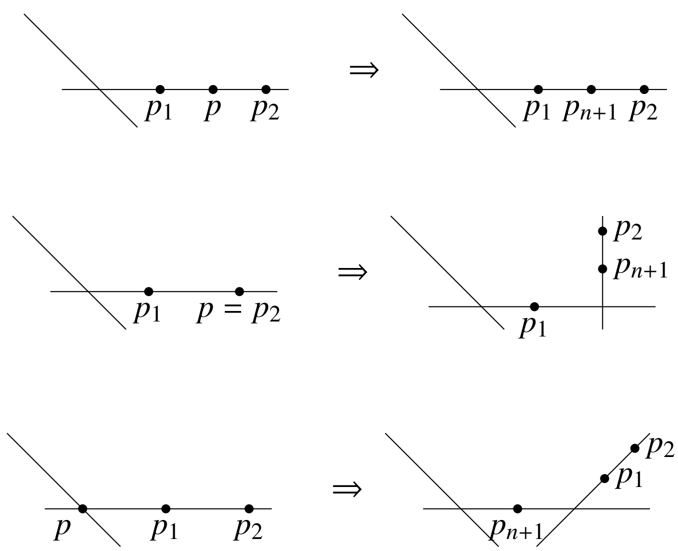


Figure 1.1: Examples of the correspondence $\overline{C}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$.

In other words, we consider a resolution of $\overline{C}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} \overline{C}_{g,n}$ along the subscheme where the diagonal meets the special points to obtain $\overline{C} \to \overline{C}_{g,n}$ such that each fiber is a $(n+1)$-pointed, genus g stable curve. This gives us the desired embedding of $\overline{C}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$.

More precisely, we have the following theorem by Knudsen.

Theorem 1.7. *[6] Consider a S-valued point of $\overline{C}_{g,n}$, i.e. an n-pointed stable curve $\pi : X \to S$ with n sections, $\sigma_1, \ldots, \sigma_n$, and an extra section Δ. Let $\mathcal{I}$ be the ideal sheaf of Δ, and define $\mathcal{K}$ on X by the exact sequence*

$$0 \longrightarrow O_X \longrightarrow \mathcal{I}^\vee \oplus O_X(\sigma_1 + \cdots + \sigma_n) \longrightarrow \mathcal{K} \longrightarrow 0\,,$$

where $\delta : O_X \to \mathcal{I}^\vee \oplus O_X(\sigma_1 + \cdots + \sigma_n)$ is the diagonal, $\delta(t) = (t,t)$. Now, let $X^s := \mathrm{Proj}(\mathrm{Sym}\,\mathcal{K})$. Then, $\sigma_1, \ldots, \sigma_n, \Delta$ have unique liftings $\sigma'_1, \ldots, \sigma'_{n+1}$ making X^s into a $(n+1)$-pointed stable curve with $X^s \to X$ a contraction. Moreover, this gives rise to an embedding

$$\overline{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}.$$

We will use a similar strategy to define our embedding of $\widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n+1}$. Let $(C, p_1, \ldots, p_n, \mathcal{P})$ be a Gieseker bundle parametrized by a point of $\widetilde{\mathcal{M}}_{g,n}$ and let $p \in C$. Then, we define a Gieseker bundle on a $(n+1)$-pointed curve, $(C', p'_1, \ldots, p'_{n+1}, \mathcal{P}')$ as follows:

1. if $p \in C$ is not a special point, then $C' = C$, $p'_i = p_i$ for $i = 1, \ldots, n$, $p'_{n+1} = p$, and $\mathcal{P}' = \mathcal{P}$; or
2. if $p \in C$ is a special point, then C' is the curve obtained from C by adding a rational component at p with three special points, one of which is p'_{n+1}. The map, $\varphi : C' \to C$, contracting the component containing p'_{n+1} is an isomorphism away from $p \in C$, and the images of p'_i are p_i for $i = 1, \ldots, n$. Finally, we define $\mathcal{P}' := \varphi^*\mathcal{P}$.

Note that $\mathcal{P}'$ does satisfy the Gieseker condition. We always have a map $\varphi : C' \to C$ which forgets p'_{n+1} and stabilizes the component containing p'_{n+1} if necessary. In both cases, $\mathcal{P}' = \varphi^*\mathcal{P}$ and note that all Gieseker bubbles of C' are preimages of Gieseker bubbles of C[1]. Hence, $\mathcal{P}'$ satisfies the Gieseker conditions since $\mathcal{P}$ satisfies them.

Figure 1.2 shows a few examples of the correspondence described above. The dashed lines in third and fourth figures represent Gieseker bubbles, which are

[1]We are only allowed to add a single stable rational component.

unstable rational components with two nodes over which the line bundle has degree 1. In the first and third examples, the line bundle over the $(n+1)$-pointed curve is the same as the line bundle over the n-pointed curve as the two curves are the same. In second and fourth examples, p collides with a special point on the n-pointed curve and the corresponding $(n+1)$-pointed curve has an extra rational component containing p_{n+1}. In these cases, the line bundle over the $(n+1)$-pointed curve has degree 0 over the component containing p_{n+1}. Over the other components, the line bundle remains "unchanged".

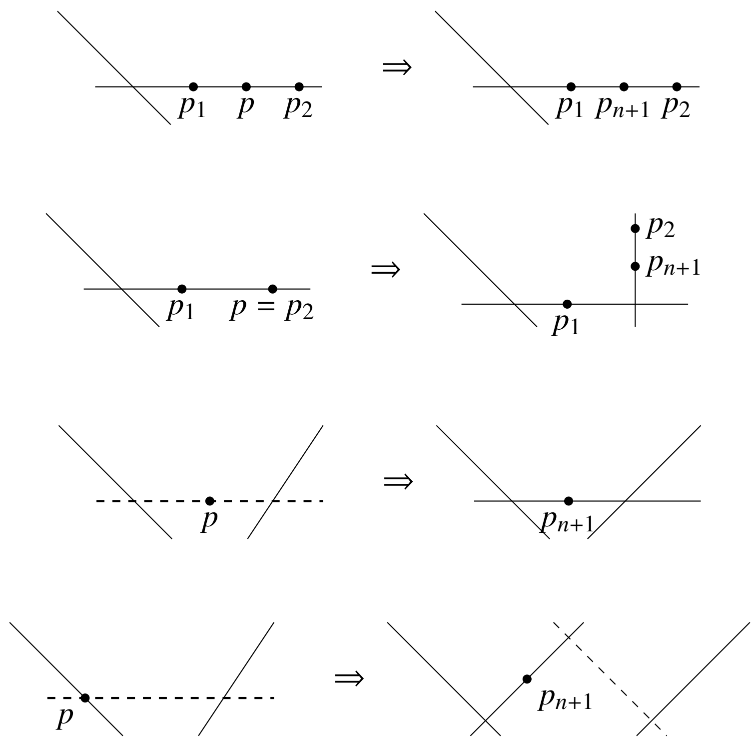


Figure 1.2: Examples of the correspondence $\widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n+1}$

In other words, we can define an embedding $\widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n+1}$ as follows. Let $\pi_n : \widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n}$ be the universal curve over $\widetilde{\mathcal{M}}_{g,n}$. Then, we have sections $\sigma_i : \widetilde{\mathcal{M}}_{g,n} \to \widetilde{C}_{g,n}$ for $i = 1, \ldots, n$. Let $D_i := \sigma_{i*}(\widetilde{\mathcal{M}}_{g,n})$ and let D^{sing} be the locus of singular points of the fibers of π_n.

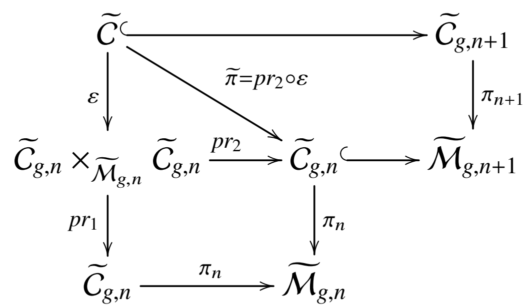
Let $\Delta \subset \widetilde{C}_{g,n} \times_{\widetilde{\mathcal{M}}_{g,n}} \widetilde{C}_{g,n}$ be the diagonal. Then, by using the analogous sheaf, $\mathcal{K}$, defined in Theorem 1.7, we get a contraction

$$\varepsilon : \widetilde{C} := \operatorname{Proj}(\operatorname{Sym} \mathcal{K}) \to \widetilde{C}_{g,n} \times_{\widetilde{\mathcal{M}}_{g,n}} \widetilde{C}_{g,n}.$$

Composing with the projection from the fiber product to $\widetilde{C}_{g,n}$ we get

$$\tilde{\pi} = pr_2 \circ \varepsilon : \widetilde{C} \to \widetilde{C}_{g,n} \times_{\widetilde{\mathcal{M}}_{g,n}} \widetilde{C}_{g,n} \to \widetilde{C}_{g,n},$$

such that the bundle $(pr_1 \circ \varepsilon)^* \mathcal{P}_{g,n}$ is a Gieseker bundle over $\widetilde{C}_{g,n}$. As in Theorem 1.7, the sections, σ_i, have unique lifts, giving us n sections $\widetilde{\sigma}_i : \widetilde{C}_{g,n} \to \widetilde{C}$. The lift of the diagonal, Δ, gives us another section, which we will denote by $\widetilde{\sigma}_{n+1}$. Therefore, $(\widetilde{C}, \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_{n+1}, (pr_1 \circ \varepsilon)^* \mathcal{P}_{g,n})$ is a family of Gieseker bundles over $\widetilde{C}_{g,n}$ with $(n+1)$ sections. Hence, we get a map $\widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n+1}$ which is an open embedding. We have the following diagram:



Note that $\varphi^* \mathbb{C}_1 \cong \widetilde{ev}^*_{n+1} \mathbb{C}_1$, where $\varphi : \widetilde{C}_{g,n} \to [pt/\mathbb{C}^\times]$ and $\widetilde{\mathrm{ev}}_{n+1} : \widetilde{\mathcal{M}}_{g,n+1} \to [pt/\mathbb{C}^\times]$.

Now, we consider the restriction to $\widetilde{C}_{g,n}$ of the determinant bundle and the evaluation bundles on $\widetilde{\mathcal{M}}_{g,n+1}$. In particular, we want to compare these line bundles to the pull-backs of their analogs from $\widetilde{\mathcal{M}}_{g,n}$.

Proposition 1.4. *Let $\pi_n : \widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n}$ be the universal curve, and consider the embedding of $\widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n+1}$ described above. Then the following are true over $\widetilde{C}_{g,n}$.*

1. *For all $i = 1, \ldots, n$, $\pi_n^* \circ \mathrm{ev}_i^* \mathbb{C}_\lambda \cong \widetilde{\mathrm{ev}}_i^* \mathbb{C}_\lambda$, where $\mathrm{ev}_i : \widetilde{\mathcal{M}}_{g,n} \to [pt/\mathbb{C}^\times]$ and $\widetilde{\mathrm{ev}}_i : \widetilde{\mathcal{M}}_{g,n=1} \to [pt/\mathbb{C}^\times]$ are the respective evaluation morphisms.*
2. $\det R\pi_{n+1*} \widetilde{\varphi}^* \mathbb{C}_1 \cong \pi_n^* \det R\pi_{n*} \varphi^* \mathbb{C}_1$, *where $\varphi : \widetilde{C}_{g,n} \to [pt/\mathbb{C}^\times]$ and $\widetilde{\varphi} : \widetilde{C}_{g,n+1} \to [pt/\mathbb{C}^\times]$.*

Proof. First, let's consider the evaluation line bundles. We know

$$\pi_n^* \circ \mathrm{ev}_i^* \mathbb{C}_\lambda \cong \pi_n^* \circ \sigma_i^* \circ \varphi^* \mathbb{C}_\lambda \cong (\sigma_i \circ \pi_n)^* \varphi^* \mathbb{C}_\lambda.$$

By definition, $\tilde{\sigma}_i$ is the lift of σ_i. In other words, $\tilde{\pi} \circ \tilde{\sigma}_i \cong \sigma_i \circ \pi_n$. Hence, we see that $(\sigma_i \circ \pi_n)^* \varphi^* \mathbb{C}_\lambda \cong (\tilde{\pi} \circ \tilde{\sigma}_i)^* \varphi^* \mathbb{C}_\lambda \cong \tilde{\sigma}_i^* (\tilde{\pi}^* \circ \varphi^* \mathbb{C}_\lambda) \cong \widetilde{\mathrm{ev}}_i^* \mathbb{C}_\lambda$. Therefore, the pull-back of the evaluation line bundles on $\widetilde{\mathcal{M}}_{g,n}$ are isomorphic to the evaluation line bundles on $\widetilde{\mathcal{M}}_{g,n+1}$ when restricted to $\widetilde{C}_{g,n}$.

Since $\widetilde{C}_{g,n} \to \widetilde{\mathcal{M}}_{g,n}$ is flat, we know that $\pi_n^*(R\pi_{n*}\mathcal{P}) \cong Rpr_{2*}(pr_1^*\mathcal{P})$ and hence, $\pi_n^*(\det R\pi_{n*}\mathcal{P}) \cong \det Rpr_{2*}(pr_1^*\mathcal{P})$. If $\tilde{\varphi} : \widetilde{C} \to [pt/\mathbb{C}^\times]$, then we know that $\tilde{\varphi}^*\mathbb{C}_1 \cong pr_1^* \circ \varepsilon^* \circ \varphi^*\mathbb{C}_1$. Since $\varepsilon : \widetilde{C} \to \widetilde{C}_{g,n} \times_{\widetilde{\mathcal{M}}_{g,n}} \widetilde{C}_{g,n}$ simply contracts rational curves, we have that $R\varepsilon_*\mathcal{O}_{\widetilde{C}} = \mathcal{O}_{\widetilde{C}_{g,n}\times_{\widetilde{\mathcal{M}}_{g,n}}\widetilde{C}_{g,n}}$. Thus, we conclude that

$$\det Rpr_{2*}(pr_1^*\varphi^*\mathbb{C}_1) \cong \det R(pr_2 \circ \varepsilon)_*\left((\varepsilon \circ pr_1)^* \circ \varphi^*\mathbb{C}_1\right) \cong \det R\tilde{\pi}_*(\tilde{\varphi}^*\mathbb{C}_1).$$

Since $\tilde{\pi}$ is the restriction of π_{n+1} to $\widetilde{C} \to \widetilde{C}_{g,n}$, the pull-back of the determinant line bundle is isomorphic to the restriction of the determinant line bundle. $\square$

Let α be an admissible class on $\widetilde{\mathcal{M}}_{g,n+1}$, which is a class of the form

$$\alpha = (\det R\pi_{n+1*}\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\otimes_i \operatorname{ev}_i^* \mathbb{C}_{\lambda_i} \otimes \mathcal{L}_i^{a_i}\right).$$

We are interested in the push-forward of $\alpha|_{\widetilde{C}_{g,n}}$ to $\widetilde{\mathcal{M}}_{g,n}$. We first recall the projection formula.

Theorem 1.8 (Projection formula)**.** *[4] Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Then, for all i,*

$$R^i f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong R^i f_*(\mathcal{F}) \otimes \mathcal{E}.$$

By the projection formula and the observations above, we have

$$\begin{aligned} R\pi_{n*}\left(\alpha|_{\widetilde{C}_{g,n}}\right) &\cong R\pi_{n*}\left((\det R\pi_{n+1*}\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\bigotimes_{i=1}^{n+1} \operatorname{ev}_i^* \mathbb{C}_{\lambda_i} \otimes \mathcal{L}_i^{a_i}\right)\right) \\ &\cong R\pi_{n*}\left(\pi_n^*(\det R\pi_{n*}\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\bigotimes_{i=1}^{n} \pi_n^* \operatorname{ev}_i^* \mathbb{C}_{\lambda_i} \otimes \mathcal{L}_i^{a_i}\right) \otimes \operatorname{ev}_{n+1}^* \mathbb{C}_{\lambda_{n+1}} \otimes \mathcal{L}_{n+1}^{a_{n+1}}\right) \\ &\cong (\det R\pi_{n*}\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\bigotimes_{i=1}^{n} \operatorname{ev}_i^* \mathbb{C}_{\lambda_i}\right) \otimes R\pi_{n*}\left(\operatorname{ev}_{n+1}^* \mathbb{C}_{\lambda_{n+1}} \otimes \bigotimes_{i=1}^{n+1} \mathcal{L}_i^{a_i}\right). \end{aligned}$$

In particular, if α does not involve $\operatorname{ev}_{n+1}^* \mathbb{C}_{\lambda_{n+1}}$ and $\mathcal{L}_{n+1}^{a_{n+1}}$, we have

$$R\pi_{n*}\left(\alpha|_{\widetilde{C}_{g,n}}\right) = (\det R\pi_{n*}\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\bigotimes_{i=1}^{n} \operatorname{ev}_i^* \mathbb{C}_{\lambda_i}\right) \otimes R\pi_{n*}\left(\bigotimes_{i=1}^{n} \mathcal{L}_i^{a_i}\right).$$

1.6 Stratification of $\widetilde{\mathcal{Z}}_{g,n} = \widetilde{\mathcal{M}}_{g,n} \setminus \widetilde{C}_{g,n-1}$

Now, we will study the complement of the image of $\widetilde{C}_{g,n-1}$ in $\widetilde{\mathcal{M}}_{g,n}$ and define a stratification of the complement by a countably infinite collection of locally closed strata.

Recall that we embedded $\widetilde{C}_{g,n-1}$ in $\widetilde{\mathcal{M}}_{g,n}$ by considering points of the fibers of $\pi_{n-1} : \widetilde{C}_{g,n-1} \to \widetilde{\mathcal{M}}_{g,n-1}$ as the last marked point and attaching an extra rational component at p if necessary. In particular, any n-pointed curve such where p_n lies on a component with more than 4 special points is in the image of $\widetilde{C}_{g,n-1}$. Hence, a point in $\widetilde{\mathcal{M}}_{g,n}$ is not in the image of $\widetilde{C}_{g,n-1}$ only if it parametrizes a Gieseker bundle $(C, p_1, \ldots, p_n, \mathcal{P})$ such that the component containing p_n, call it C', becomes unstable after forgetting p_n.

Thus, C is not in the image of $\widetilde{C}_{g,n-1}$ only if C' is a rational curve containing precisely three special points[2]. Since one of the special points is p_n, C' can have either one or two nodes. If C' has exactly one node, we will call C a curve of type I. If C' has two nodes, $C \setminus C'$ can either have one or two connected components. If $C \setminus C'$ is the disjoint union of two connected components we will say C is a curve of type II. If $C \setminus C'$ is connected, we will say C is of type III.

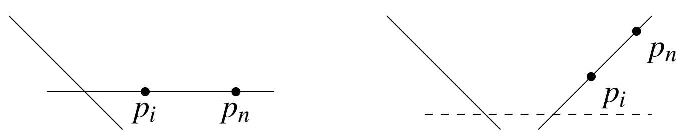


Figure 1.3: Examples of type I curves

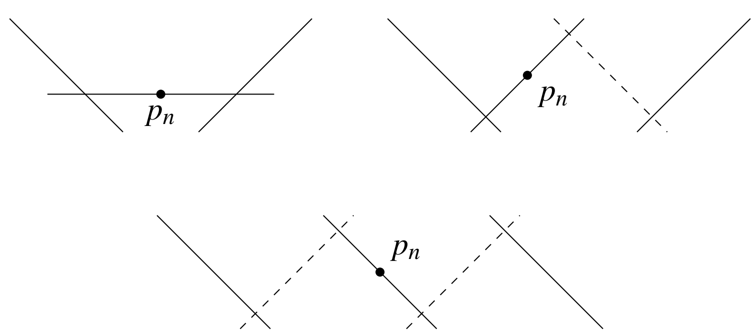


Figure 1.4: Examples of type II curves

Figures 1.3, 1.4, and 1.5 show examples of type I, II, and III curves, respectively. As before, dashed lines represent Gieseker bubbles over which the line bundle has

[2]This is because all rational components have at least two components and only Gieseker bubbles are allowed to have two special points, both of which must be nodes.

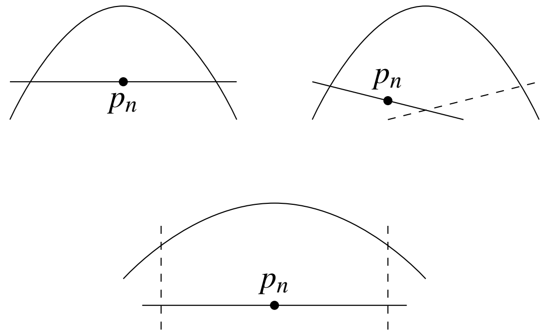


Figure 1.5: Examples of type III curves

degree 1. In all the figures, the component containing p_n is rational. All other connected components of the curves in the figures, along with the restriction of the given line bundle, are lower pointed Gieseker bundles[3].

Note that $\widetilde{\mathcal{Z}}_{g,n}$ is the disjoin union of the strata of type I, II, and III curves. In the subsections that follow, we will stratify subschemes of $\widetilde{\mathcal{Z}}_{g,n}$ of type I, II, and III curves. Also, we will consider the connected component $\widetilde{\mathcal{M}}_{g,n,D} \subset \widetilde{\mathcal{M}}_{g,n}$ parametrizing Gieseker bundles of some fixed total degree D.

1.7 Type I curves

Let $(C, p_1, \ldots, p_n, \mathcal{P}) \in \widetilde{\mathcal{M}}_{g,n}$ be a type I curve. As before, let C' denote the irreducible component of C containing p_n. Then, C' is a rational component containing 2 marked points and a node. Let the marked points be p_i and p_n. Note that there is only one way a type I curve can be in the image of $\widetilde{C}_{g,n-1}$. This happens when we choose the point $p = p_i$ on the fiber as shown in Figure 1.6.

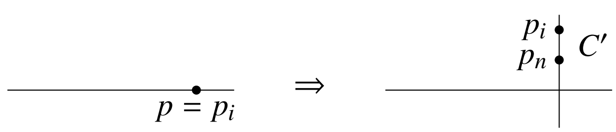


Figure 1.6: Type I curve lying in the image of $\widetilde{C}_{g,n-1}$

If a type I curve is in the image of $\widetilde{C}_{g,n-1}$, then the degree of $\mathcal{P}|_{C'}$ must be 0. Moreover, such curve cannot have a Gieseker bubble attached to C'. Therefore, the type I curves that do not lie in the image of $\widetilde{C}_{g,n-1}$ are the ones such that either $\deg \mathcal{P}|_{C'} \neq 0$ or C' is attached to a Gieseker bubble.

[3]These components can have genus greater than 0.

For $i = 1, \ldots, n-1$, let Z_i^1 be the closed subscheme of $\widetilde{\mathcal{M}}_{g,n}$ whose points parametrize type I curves not in the image of $\widetilde{C}_{g,n-1}$ such that C' contains p_n and p_i. First, note that Z_i^1 is closed in $\widetilde{\mathcal{M}}_{g,n}$: any degeneration of a type I curve is another type I curve; and the degree of $\mathcal{P}|_{C'}$ is locally constant away from the Gieseker bubble.

Now, denote by $W_{i,d}^1$ the locally closed stratum corresponding to the topological types depicted in Figure 1.7, where $(\gamma, D-d)$ is any topological type of genus g Gieseker bundle of degree $D-d$. In other words, $W_{i,d}^1$ is the stratum corresponding

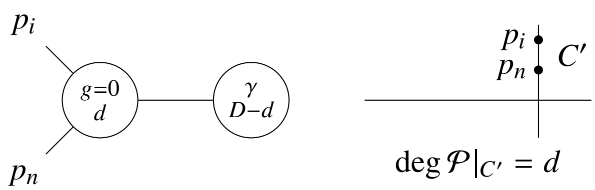


Figure 1.7: Modular graphs and curves of $W_{i,d}^1$

to type I curves such that

1. $\deg \mathcal{P}|_{C'} = d$; and
2. C' is not attached to a Gieseker bubble.

Note that $W_{i,d}^1 \subset \widetilde{C}_{g,n-1}$ if and only if $d = 0$.

Denote by $F_{i,d}^1$ the closed stratum corresponding to the topological types depicted in Figure 1.8, where $(\gamma, D-d-1)$ is any topological type of genus g Gieseker bundle of degree $D-d-1$. In other words, $F_{i,d}^1$ is the stratum corresponding to type I

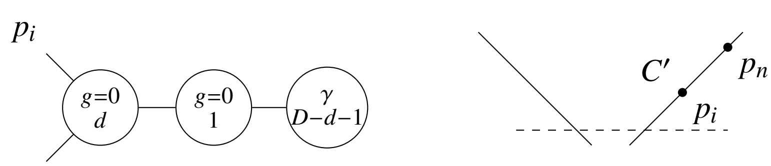


Figure 1.8: Modular graphs and curves of $F_{i,d}^1$

curves such that

1. $\deg \mathcal{P}|_{C'} = d$; and
2. C' is attached to a Gieseker bubble.

Note that $F^1_{i,d} \not\subset \widetilde{C}_{g,n-1}$ for all i and d.

By Lemma 1.1, we see that for each i and d, we have

$$\overline{W^1_{i,d}} = W^1_{i,d} \cup F^1_{i,d} \cup F^1_{i,d-1}.$$

Moreover, all the curves in $\widetilde{\mathcal{Z}}_{g,n}$ that are deformations of curves of $F^1_{i,d}$ are parametrized by points of $F^1_{i,d}$, $W^1_{i,d}$ and $W^1_{i,d+1}$. More precisely, points of $W^1_{i,d}$ parametrize curves obtained from a curve in $F^1_{i,d}$ by smoothing the node on the connecting Gieseker bubble opposite to C'[4]. Figure 1.9 shows such a deformation.

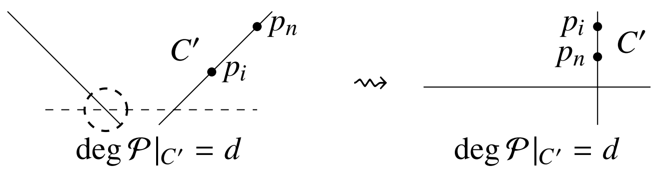


Figure 1.9: Smoothing the node in the dashed circle

Likewise, the points of $W^1_{i,d+1}$ parametrize curves obtained from curves in $F^1_{i,d}$ by smoothing the node on C' as shown in Figure 1.10.

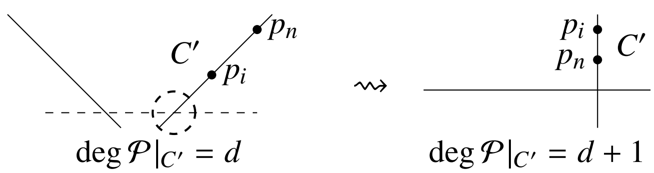


Figure 1.10: Smoothing the node in the dashed circle

We can visualize the stratum of type I curves in the following way:

$$\cdots \rightsquigarrow W^1_{i,d-1} \leftsquigarrow F^1_{i,d-1} \rightsquigarrow W^1_{i,d} \leftsquigarrow F^1_{i,d} \rightsquigarrow W^1_{i,d+1} \leftsquigarrow F^1_{i,d+1} \rightsquigarrow W^1_{i,d+2} \leftsquigarrow \cdots,$$

where $A \rightsquigarrow B$ means A lies in the closure of B.

Now, we define $Z^1_{i,d}$ as follows.

$$Z^1_{i,d} = \begin{cases} W^1_{i,d} \cup F^1_{i,d} & d < 0 \\ W^1_{i,d+1} \cup F^1_{i,d} & d \geq 0 \end{cases}.$$

Keeping in mind $W^1_{i,0} \subset \widetilde{C}_{g,n-1}$, we see that $Z^1_i = \cup Z^1_{i,d}$ gives us the desired stratification of Z^1_i (see Figure 1.11).

[4]We cannot smooth both since such a deformation would result in a curve that lies in the image of $\widetilde{C}_{g,n}$.

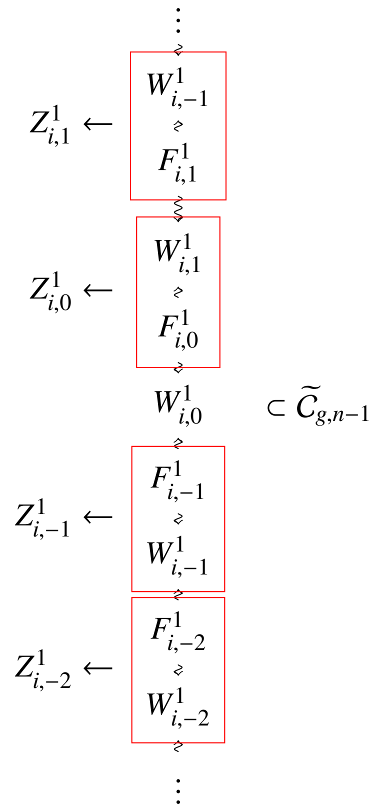


Figure 1.11: Stratification of Z^1_i by $Z^1_{i,d}$

Before we move onto type II curves, we give an alternate way of defining $Z^1_{i,d}$, which will be useful later. Let $U^1_{i,d}$ be the stratum of points parametrizing all curves of $\widetilde{\mathcal{Z}}_{g,n}$ obtained by smoothing nodes of curves in $F^1_{i,d}$. By Lemma 1.1, this is precisely

$$U^1_{i,d} = W^1_{i,d} \cup F^1_{i,d} \cup W^1_{i,d+1}.$$

Note that $\{U^1_{i,d} \mid d \in \mathbb{Z}\}$ is an open cover of Z^1_i.

Then, we can define $Z^1_{i,d}$ as follows.

$$Z^1_{i,d} = \begin{cases} U^1_{i,d} \setminus U^1_{i,d+1} & d < 0 \\ U^1_{i,d} \setminus U^1_{i,d-1} & d \geq 0 \end{cases}.$$

Geometry of $F^1_{i,d}$ and $Z^1_{i,d}$

We defined $F^1_{i,d}$ as the stratum of points parametrizing curves of splitting type $(\{i,n\},\{i,n\}^c)$ where marked points p_i and p_n are on a rational curve connected to a Gieseker bubble. Moreover, the universal bundle has degree d restricted to the component, C', containing p_i and p_n, and degree $e := D - d - 1$ restricted to the component, C_1, containing the other marked points. Hence,

$$F^1_{i,d} \cong \widetilde{\mathcal{M}}^d_{0,3} \times \widetilde{\mathcal{M}}^e_{g,n-1},$$

where we identify the third marked point of $\widetilde{\mathcal{M}}^d_{0,3}$ and the $(n-1)$-st marked point of $\widetilde{\mathcal{M}}^e_{g,n-1}$ as the two nodes on the connecting Gieseker bubble. The marked points of $\widetilde{\mathcal{M}}^d_{0,3}$ are denoted p_i, p_n, and the node p_3. The marked points of $\widetilde{\mathcal{M}}^e_{g,n-1}$ are the points p_j for $j \neq i, n$, and the node p_{n-1}.

Now, we take a closer look at $Z^1_{i,d}$. Proposition 4.15 and Corollary 4.16 of [1] tell us that $Z^1_{i,d}$ is an affine bundle over $F^1_{i,d}$.

Proposition 1.5. *[1]*

1. *For $d \geq 0$, $Z^1_{i,d}$ classifies bundles which arise from $F^1_{i,d}$ by smoothing away the node attaching C' to the connecting Gieseker bubble.*
2. *For $d < 0$, $Z^1_{i,d}$ classifies bundles which arise from $F^1_{i,d}$ by smoothing away the node attaching the connecting Gieseker bubble to the components not containing p_n.*
3. *We have a map $\eta : Z^1_{i,d} \to F^1_{i,d}$ such that η is the structure map of an affine bundle.*

Note that our $F^1_{i,d}$ correspond to those labeled F in [1], and our $Z^1_{i,d}$ correspond to those labeled Z (when $d \geq 0$) and W (when $d < 0$). 1 and 2 of Proposition 1.5 follow directly from the definition of $Z^1_{i,d}$.

While Frenkel, Teleman, and Tolland do not say exactly which affine bundle $\eta : Z^1_{i,d} \to F^1_{i,d}$ is, the proof of Proposition 4.15 in [1] contains more information which leads to the following Proposition:

Proposition 1.6. *The map $\eta : Z^1_{i,d} \to F^1_{i,d}$ from Proposition 1.5 is given by the bundle*

$$\begin{cases} (\mathcal{L}_3^{-1} \otimes \mathcal{P}_3^d) \boxtimes (\mathcal{P}^e_{n-1})^{-1} & d \geq 0 \\ (\mathcal{P}_3^d)^{-1} \boxtimes (\mathcal{L}_{n-1}^{-1} \otimes \mathcal{P}^3_{n-1}) & d < 0 \end{cases},$$

where

1. *$\mathcal{L}_3$ is the cotangent bundle along the third section on $\widetilde{\mathcal{M}}^d_{0,3}$,*

2. $\mathcal{L}_{n-1}$ *is the cotangent bundle along the* $(n-1)$*-st section on* $\widetilde{\mathcal{M}}^e_{g,n-1}$,

3. $\mathcal{P}^d_3$ *is the restriction of the universal bundle along the third section on* $\widetilde{\mathcal{M}}^d_{0,3}$, *and*

4. $\mathcal{P}^e_{n-1}$ *is the restriction of the universal bundle along the* $(n-1)$*-st section on* $\widetilde{\mathcal{M}}^e_{g,n-1}$.

Proof. Let $d \geq 0$ and consider curves parametrized by points of $Z^1_{i,d}$. All such curves have splitting type $(\{i,n\},\{i,n\}^c)$. Recall that we denote the component containing p_n by C', and the other component by C_1, where we discard the connecting Gieseker bubble between them if there is one. Let $\mathcal{P}$ denote the universal bundle over curves of $Z^1_{i,d}$. Now, we have two trivializations of $\mathcal{P}$ restricted to the two components C' and C_1, say $t' : \mathcal{P}_{p_n} \to \mathbb{C}^\times$ and $t_1 : \mathcal{P}_{p_k} \to \mathbb{C}^\times$, where $k \neq i, n$. These two trivializations then give us the gluing isomorphism, ι, of the fibers of $\mathcal{P}$ over the node. Now, as proof of Proposition 4.15 in [1] points out, scaling t' to 0, we obtain in the limit a connecting Gieseker bubble with a degree 1 transferred from C'[5]. Hence, this gives rise to a map $\eta : Z^1_{i,d} \to F^1_{i,d}$ for $d \geq 0$. Similarly, scaling t' to ∞ gives a map $Z^1_{i,d} \to F^1_{i,d}$ for $d < 0$.

Moreover, the choices of the trivializations t' and t_1 give us a map between the two fibers of the universal bundles over the nodes on C' and C_1, which are $\mathcal{P}_{p_3}$ and $\mathcal{P}_{p_{n-1}}$, respectively. As Remark 1.12.1 in [1] explains, this map $\mathcal{P}_{p_{n-1}} \to \mathcal{P}_{p_3}$ is given by t'/t_1, and is precisely the gluing isomorphism, ι, over the node attaching C' with C_1 when we have a type I curve with no connecting Gieseker bubble. When $t' = 0$, this map $\mathcal{P}_{p_n} \to \mathcal{P}_{p_3}$ becomes the 0 map and we get a connecting Gieseker bubble, as we saw above. Hence, given a section of $\eta : Z^1_{i,d} \to F^1_{i,d}$, we obtain a morphism $\mathcal{P}|_{\sigma_{n-1}} \to \mathcal{P}|_{\sigma_3}$.

Now, since $F^1_{i,d} \cong \widetilde{\mathcal{M}}^d_{0,3} \times \widetilde{\mathcal{M}}^e_{g,n-1}$, we try to write $\mathcal{P}|_{\sigma_{n-1}}$ and $\mathcal{P}|_{\sigma_3}$ in terms of pull-backs of line bundles over $\widetilde{\mathcal{M}}^d_{0,3}$ and $\widetilde{\mathcal{M}}^e_{g,n-1}$. Let $pr_1 : F^1_{i,d} \to \widetilde{\mathcal{M}}^d_{0,3}$ and $pr_2 : F^1_{i,d} \to \widetilde{\mathcal{M}}^e_{g,n-1}$ be the projection maps. First, $\mathcal{P}|_{\sigma_{n-1}}$ is equal to $pr_2^*\mathcal{P}^e_{n-1}$ by definition. However, $\mathcal{P}|_{\sigma_3}$ is not equal to $pr_1^*\mathcal{P}^d_3$ since $\mathcal{P}^d_3$ is the restriction of the universal bundle over $\widetilde{\mathcal{M}}^d_{0,3}$ to σ_3 and thus, has 1 lower degree than $\mathcal{P}|_{\sigma_3}$: $\deg \mathcal{P}|_{\sigma_3} = d+1$. Recall that the map $\eta : Z^1_{i,d} \to F^1_{i,d}$ inserted a connecting Gieseker bubble by scaling the trivialization, t', to 0 and transferring 1 degree from C' to the bubble. Hence, $\mathcal{P}|_{\sigma_3} \cong pr_1^*(\mathcal{P}^d_3 \otimes \mathcal{L}_3^{-1})$.

[5] See Remark 1.12.1 in [1].

Therefore, sections of $\eta : Z^1_{i,d} \to F^1_{i,d}$ correspond to sections of

$$\mathcal{H}om\left(pr_2^*\mathcal{P}^e_{n-1}, pr_1^*(\mathcal{L}_3^{-1} \otimes \mathcal{P}^d_3)\right) \cong pr_1^*(\mathcal{L}_3^{-1} \otimes \mathcal{P}^d_3) \otimes (pr_2^*\mathcal{P}^e_{n-1})^{-1}.$$

Hence, we conclude that for $d \geq 0$, $\eta : Z^1_{i,d} \to F^1_{i,d}$ is the affine bundle given by

$$(\mathcal{L}_3^{-1} \otimes \mathcal{P}^d_3) \boxtimes (\mathcal{P}^e_{n-1})^{-1}.$$

For $d < 0$, the situation is symmetric. Recall that when $d < 0$, instead of transferring 1 degree to the bubble from C', we transfer it from C_1. The map $\eta : Z^1_{i,d} \to F^1_{i,d}$ is then defined by scaling the trivialization, t', to ∞. By the same argument as in the $d \geq 0$, case we conclude that $\eta : Z^1_{i,d} \to F^1_{i,d}$ is the affine bundle given by

$$(\mathcal{P}^d_3)^{-1} \boxtimes (\mathcal{L}^{-1}_{n-1} \otimes \mathcal{P}^e_{n-1}).$$

□

Another way to show that $Z^1_{i,d}$ is the affine bundle given by $(\mathcal{L}_3^{-1} \otimes \mathcal{P}^d_3) \boxtimes (\mathcal{P}^e_{n-1})^{-1}$ is by considering the formal neighborhood of $F^1_{i,d}$. $Z^1_{i,d}$ corresponds to smoothings of the node attaching C' to the connecting Gieseker bubble, which is the marked point p_3 on $\widetilde{\mathcal{M}}^d_{0,3}$. Smoothing a node is represented by the formal neighborhood given by $T_+ \otimes T_-$ where $T_\pm$ denote the tangent bundles at the node on the two components. In our case, those bundles are $\mathcal{L}_3^{-1}$ from C', and $\mathcal{P}^d_3 \boxtimes (\mathcal{P}^e_{n-1})^{-1}$ from the connecting Gieseker bubble. The tangent bundle at the node on the connecting Gieseker bubble is $\mathcal{P}^d_3 \boxtimes (\mathcal{P}^e_{n-1})^{-1}$ since $\mathcal{O}(1)$ of the Gieseker bubble is glued on the two nodes, p_3 and p_{n-1}, to the fibers $\mathcal{P}_{p_3}$ and $\mathcal{P}_{p_{n-1}}$. Hence, $Z^1_{i,d}$ corresponds to the affine bundle over $F^1_{i,d}$ given by

$$pr_1^*\mathcal{L}_3^{-1} \otimes \left(\mathcal{P}^d_3 \boxtimes (\mathcal{P}^e_{n-1})^{-1}\right) \cong (\mathcal{L}_3^{-1} \otimes \mathcal{P}^d_3) \boxtimes (\mathcal{P}^e_{n-1})^{-1}.$$

1.8 Type II curves

Now, we stratify the stratum of type II curves.

Let $(C, p_1, \ldots, p_n, \mathcal{P}) \in \widetilde{\mathcal{M}}_{g,n}$ be a type II curve and let C' denote the irreducible component of C containing p_n. Since C is a type II curve, C' contains the marked point p_n and two nodes, and $C \setminus C'$ has two connected components. Note that there are precisely three ways for a type II curve to lie in the image of $\widetilde{C}_{g,n-1}$.

1. We choose the node on two stable components as p (Figure 1.12); or

2. we choose a point on a Gieseker bubble as p (Figure 1.13); or
3. we choose the node on a stable component and a Gieseker bubble as p (Figure 1.14).

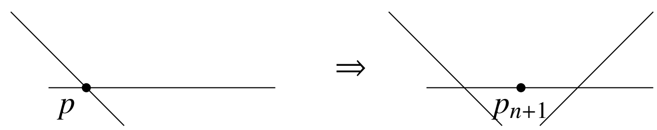


Figure 1.12: Choosing a node on two stable components

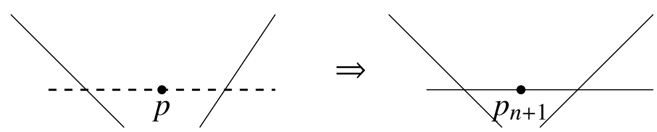


Figure 1.13: Choosing a point on a Gieseker bubble

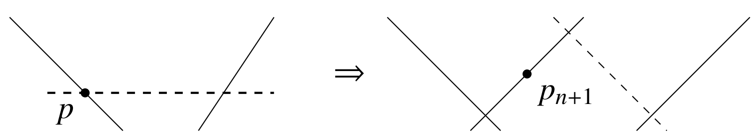


Figure 1.14: Choosing a node on a stable component and a bubble

Therefore, a type II curve is in the image of $\widetilde{C}_{g,n-1}$ if and only if either

1. $\deg \mathcal{P}|_{C'} = 1$ and C' is not connected to a Gieseker bubble; or
2. $\deg \mathcal{P}|_{C'} = 0$ and C' is connected to 0 or 1 Gieseker bubbles.

For a type II curve, $\overline{C \setminus C'}$ has two connected components. For $I \subset [n-1] := \{1, \ldots, n-1\}$ such that $|I|, |I^c| \geq 2$, let Z_I^2 be the closed subscheme of $\widetilde{\mathcal{M}}_{g,n}$ whose points parametrize type II curves not in the image of $\widetilde{C}_{g,n-1}$ such that points $\{p_i \mid i \in I\}$ and $\{p_i \mid i \notin I\}$ are on separate connected components of $\overline{C \setminus C'}$. Without loss of generality, denote by C_1 the curve containing points with indices in I, and C_2 the other connected component.

As we did with type I curves, we will first look at the stratification of Z_I^2 by topological types. We will fix D, the total degree of the Gieseker bundle, and also the splitting $g_1 + g_2 = g$ of the total genus g into genus, g_1, of C_1 and g_2 of C_2. Note that type II curves can have 0, 1, or 2 Gieseker bubbles attached to C'. We will call these strata W^2, Y^2, and F^2, respectively.

Let $d_1, d_2 \in \mathbb{Z}$. We denote by $W^2_{I,(d_1,d_2)}$ the locally closed stratum corresponding to the topological type depicted in Figure 1.15, where (γ_i, d_i) is any topological type of genus g_i Gieseker bundle of degree d_i with marked points of C_i, such that $g = g_1 + g_2$.

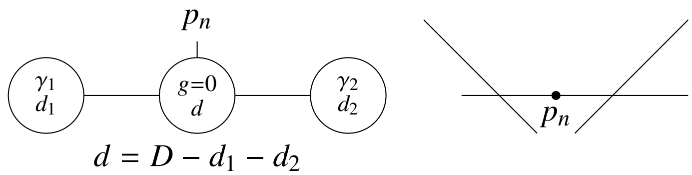


Figure 1.15: Modular graphs and curves of $W^2_{I,(d_1,d_2)}$

In other words, $W^2_{I,(d_1,d_2)}$ is the stratum corresponding to type II curves such that

1. $\deg \mathcal{P}|_{C_i} = d_i$; and
2. C' is not connected to any Gieseker bubbles.

We will denote by

$$W^2_{I,d} := \bigcup_{D-d_1-d_2=d} W^2_{I,(d_1,d_2)}.$$

Note that $W^2_{I,d} \subset \widetilde{C}_{g,n-1}$ if and only if $d = 0$ or 1 by the discussion from the beginning of the section.

The type II curves with a Gieseker bubble connecting C' with C_1 will be denoted $Y^2_{I,(d_1,d_2)}$. The topological type of such curves is shown in Figure 1.16.

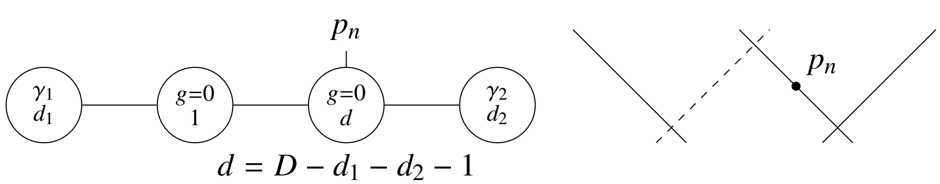


Figure 1.16: Modular graphs and curves of $Y^2_{I,(d_1,d_2)}$

We also define

$$Y^2_{I,d} := \bigcup_{D-d_1-d_2=d} \left(Y^2_{I,(d_1,d_2)} \cup Y^2_{I^c,(d_2,d_1)} \right).$$

Note that $Y^2_{I,d} \subset \widetilde{C}_{g,n-1}$ if and only $d = 1$.

Lastly, we denote by $F^2_{I,(d_1,d_2)}$ the stratum of type II curves with two Gieseker bubbles with the topological type shown in Figure 1.17.

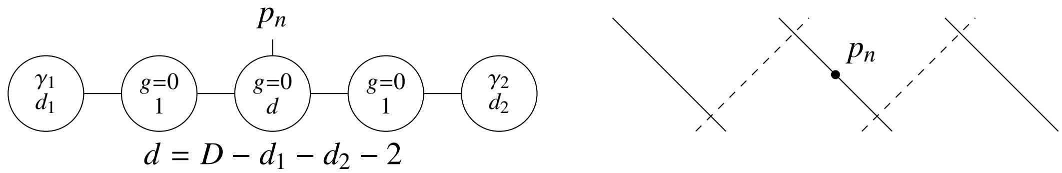


Figure 1.17: Modular graphs and curves of $F^2_{I,(d_1,d_2)}$

Also, define

$$F^2_{I,d} := \bigcup_{D-d_1-d_2=d} F^2_{I,(d_1,d_2)}.$$

Again by Lemma 1.1, we see that $F^2_{I,d}$ and $F^2_{I,(d_1,d_2)}$ are closed in $\widetilde{\mathcal{M}}_{g,n}$. Similarly to the type I case, let $U^2_{I,(d_1,d_2)}$ denote the stratum of points parametrizing all curves of $\widetilde{\mathcal{Z}}_{g,n}$ that are obtained by smoothing 0 or 1 of the nodes on each Gieseker bubble of curves of $F^2_{I,(d_1,d_2)}$ [6].

$$\begin{aligned} U^2_{I,(d_1,d_2)} \quad := \quad & F^2_{I,(d_1,d_2)} \cup Y^2_{I,(d_1,d_2)} \cup Y^2_{I,(d_1,d_2+1)} \cup Y^2_{I^c,(d_2,d_1)} \cup Y^2_{I^c,(d_2,d_1+1)} \\ & \cup W^2_{I,(d_1,d_2)} \cup W^2_{I,(d_1+1,d_2)} \cup W^2_{I,(d_1,d_2+1)} \cup W^2_{I,(d_1+1,d_2+1)}. \end{aligned}$$

Note that $\{U^2_{I,(d_1,d_2)} \mid D - d_1 - d_2 = d\}$ forms an open cover of $Z^2_{I,d}$ in $\widetilde{\mathcal{Z}}_{g,n}$.

Also, the stratum of all smoothings of curves of $F^2_{I,d}$ is

$$\begin{aligned} U^2_{I,d} \quad &= \quad \bigcup_{D-d_1-d_2=d} U^2_{I,(d_1,d_2)} \\ &= \quad F^2_{I,d} \cup Y^2_{I,d} \cup Y^2_{I,d-1} \cup W^2_{I,d} \cup W^2_{I,d-1} \cup W^2_{I,d-2}. \end{aligned}$$

Note that $\{U^2_{I,d} \mid d \in \mathbb{Z}\}$ forms an open cover of Z^2_I.

We can visualize the closure relations of these type II strata using the following infinite two dimensional grid[7] in Figure 1.18[8].

Figure 1.19 shows $U^2_{I,(d_1,d_2)}$ as deformations of curves in $F^2_{I,(d_1,d_2)}$. The dashed lines attached to the nodes on the Gieseker bubbles indicate which deformation happen as we smooth the chosen node.

[6] As we saw in Section 1.7, we cannot smooth both nodes of the same Gieseker bubble since such deformation would result in a curve lying in the image of $\widetilde{C}_{g,n-1}$.

[7] The two dimensions correspond to smoothings of the two connecting Gieseker bubbles. The direction along each axis is determined by which of the two nodes on the bubble is smoothed.

[8] Keep in mind closure relations are transitive.

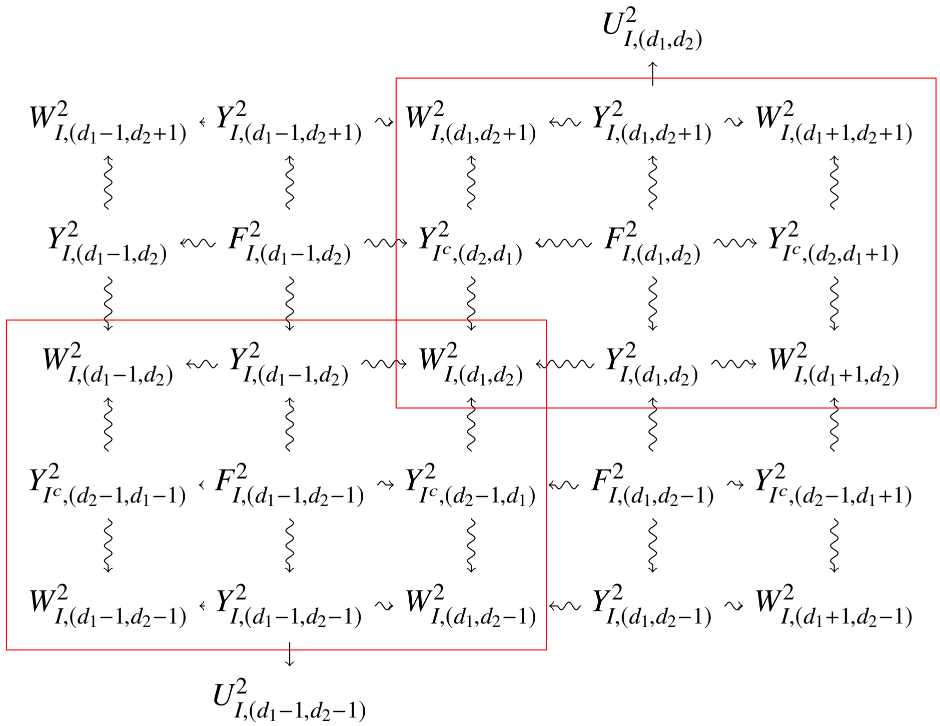


Figure 1.18: Type II strata and their closure relations

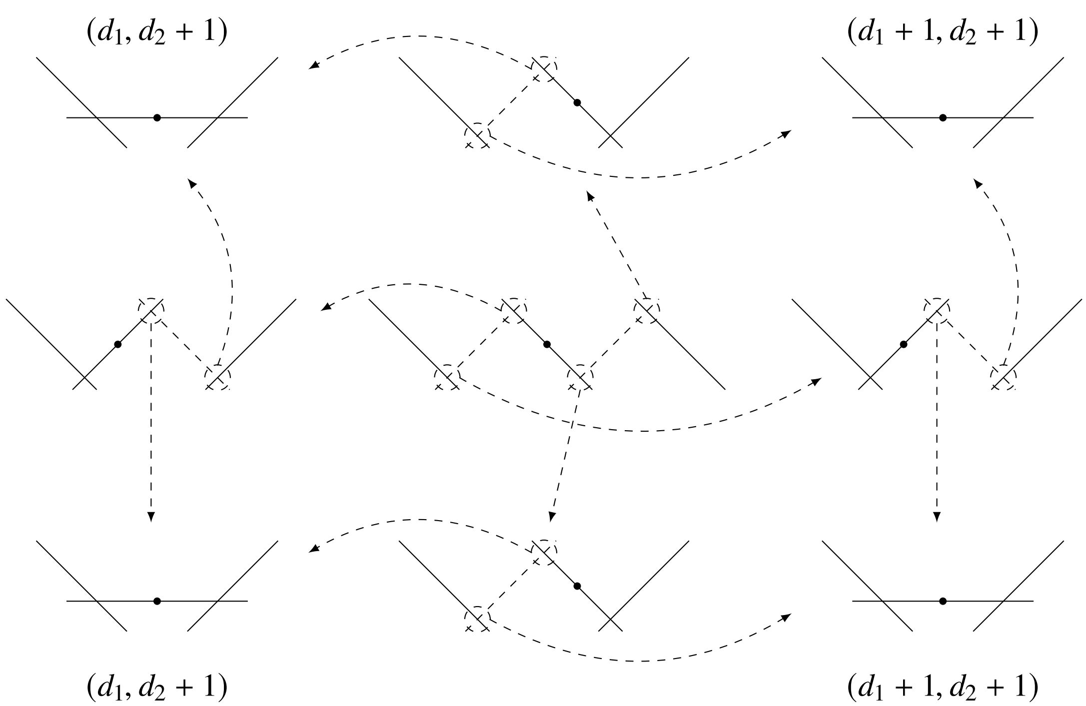


Figure 1.19: Closer look at $U^2_{I,(d_1,d_2)}$

We are finally ready to define our stratification of Z^2_I. Define

$$Z^2_{I,d} := \begin{cases} U^2_{I,d} \setminus U^2_{I,d+1} & d \leq 1 \\ U^2_{I,d} \setminus U^2_{I,d-1} & d \geq 2 \end{cases}.$$

We also define

$$Z^2_{I,(d_1,d_2)} := Z^2_{I,d} \cap U_{I,(d_1,d_2)}.$$

Recalling that $W^2_{I,0}, W^2_{I,1}, Y^2_{I,1} \subset \widetilde{C}_{g,n-1}$, we see that $Z^2_{I,d}$ stratify Z^2_I. Moreover, for each d, $Z^2_{I,d}$ is the disjoint union

$$Z^2_{I,d} = \bigcup_{D-d_1-d_2=d} Z^2_{I,(d_1,d_2)}.$$

Figure 1.20 shows the stratification of Z^2_I by $Z^2_{I,d}$ and $Z^2_{I,(d_1,d_2)}$. All superscripts and subscripts except for the degrees are suppressed for the sake of simplicity.

In Figure 1.20, $d_1, d_2 \in \mathbb{Z}$ are such that $D - d_1 - d_2 = 1$. The strata that lie in the blue shaded region are the ones that are in the image of $\widetilde{C}_{g,n-1}$. The red boxes are the $Z^2_{I,(d',d'')}$, where (d', d'') are the degrees corresponding to the $F_{(d',d'')}$ in the same box. For example, the box labeled (2.2) correspond to $Z^2_{I,(d_1,d_2-1)}$. Moreover, each box labeled (d, k) lie in $Z^2_{I,d}$. For example, boxes (1.1), (1.2), and (1.3), which are $Z^2_{I,(d_1-1,d_2+1)}, Z^2_{I,(d_1,d_2)}, Z^2_{I,(d_1+1,d_2-1)}$, respectively, all lie in $Z^2_{I,1}$. Note that

$$D - (d_1 - 1) - (d_2 + 1) = D - d_1 - d_2 = D - (d_1 + 1) - (d_2 - 1) = 1.$$

Geometry of $F^2_{I,(d_1,d_2)}$ and $Z^2_{I,(d_1,d_2)}$

By the same argument as in the type I case,

$$F^2_{I,d_1,d_2} \cong \widetilde{\mathcal{M}}^{d_1}_{g_1,|I|+1} \times \widetilde{\mathcal{M}}^{d}_{0,3} \times \widetilde{\mathcal{M}}^{d_2}_{g_2,|I^c|+1},$$

where $d = D - d_1 - d_2 - 2$. Also, by Proposition 1.5, we know that there exists a map $\eta : Z^2_{I,(d_1,d_2)} \to F^2_{I,(d_1,d_2)}$, which is the structure map of an affine bundle. From our description of $Z^2_{I,(d_1,d_2)}$, we know that

1. for $d \geq 2$, $Z^2_{I,(d_1,d_2)}$ classifies bundles which arise from $F^2_{I,(d_1,d_2)}$ by smoothing away nodes attaching the connecting Gieseker bubbles to C_1 and C_2; and

2. for $d \leq 1$, $Z^2_{I,(d_1,d_2)}$ classifies bundles which arise from $F^2_{I,(d_1,d_2)}$ by smoothing away nodes attaching the connecting Gieseker bubbles to C'.

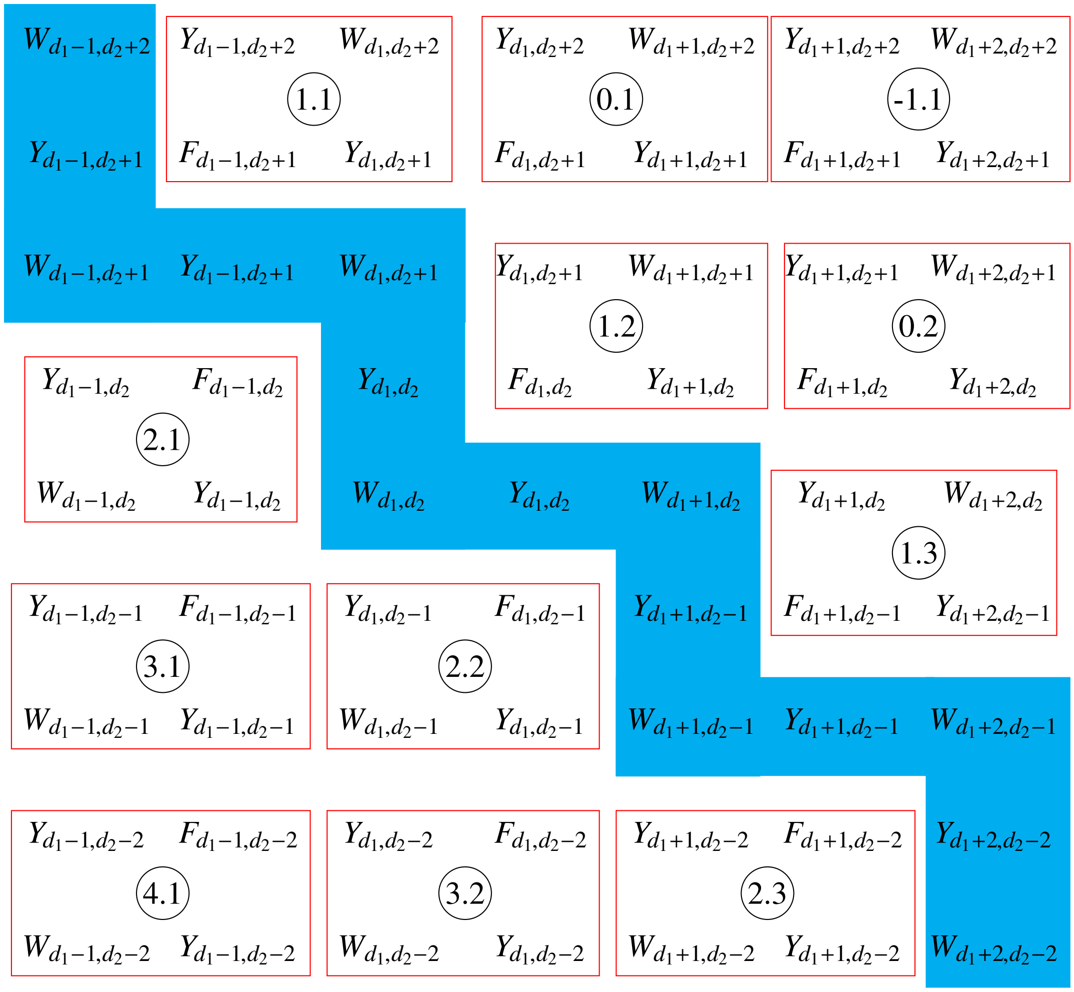


Figure 1.20: Stratification of Z_I^2

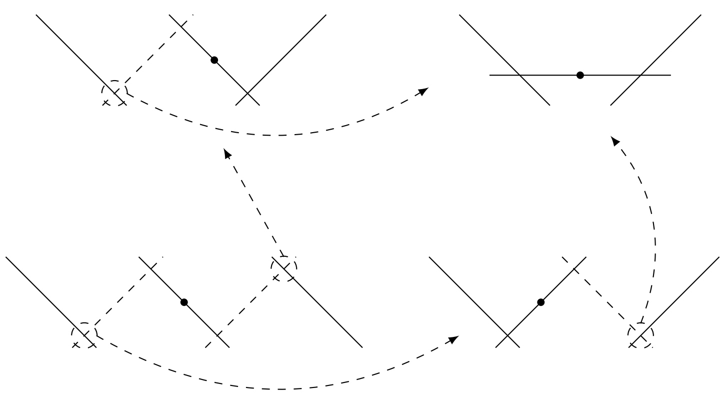


Figure 1.21: $Z^2_{I,(d_1,d_2)}$ when $D - d_1 - d_2 \geq 2$

By the same argument as in the proof of Proposition 1.6, we obtain the following proposition.

Proposition 1.7. *The map* $\eta : Z^2_{I,(d_1,d_2)} \to F^2_{I,(d_1,d_2)}$ *is the structure map of the affine bundle*

$$\begin{cases} \left(\left(\mathcal{L}^{-1}_{|I|+1} \otimes \mathcal{P}^{d_1}_{|I|+1}\right) \boxtimes (\mathcal{P}^d_1)^{-1} \boxtimes O_{\widetilde{M}_2}\right) \oplus \left(O_{\widetilde{M}_1} \boxtimes (\mathcal{P}^d_3)^{-1} \boxtimes \left(\mathcal{L}^{-1}_{|I^c|+1} \otimes \mathcal{P}^{d_2}_{|I^c|+1}\right)\right) & d \geq 2 \\ \left(\left(\mathcal{P}^{d_1}_{|I|+1}\right)^{-1} \boxtimes \left(\mathcal{L}^{-1}_1 \otimes \mathcal{P}^d_1\right) \boxtimes O_{\widetilde{M}_2}\right) \oplus \left(O_{\widetilde{M}_2} \boxtimes \left(\mathcal{L}^{-1}_3 \otimes \mathcal{P}^d_3\right) \boxtimes \left(\mathcal{P}^{d_2}_{|I^c|+1}\right)^{-1}\right) & d \leq 1 \end{cases},$$

where

1. $\mathcal{L}_{|I|+1}$ *is the cotangent bundle along the* $(|I|+1)$*-st section,* $\sigma_{|I|+1}$*, on* $\widetilde{\mathcal{M}}^{d_1}_{g_1,|I|+1}$*;*

2. $\mathcal{P}^{d_1}_{|I|+1}$ *is the restriction to* $\sigma_{|I|+1}$ *of the universal bundle over* $\widetilde{\mathcal{M}}^{d_1}_{g_1,|I|+1}$*;*

3. $\mathcal{L}_1$ *and* $\mathcal{L}_3$ *are the cotangent bundles along* σ_1 *and* σ_3 *on* $\widetilde{\mathcal{M}}^d_{0,3}$*;*

4. $\mathcal{P}^d_1$ *and* $\mathcal{P}^d_3$ *are the restrictions to* σ_1 *and* σ_3 *of the universal bundle over* $\widetilde{\mathcal{M}}^d_{0,3}$*;*

5. $\mathcal{L}_{|I^c|+1}$ *is the cotangent bundle along* $\sigma_{|I^c|+1}$ *on* $\widetilde{\mathcal{M}}^{d_2}_{g_2,|I^c|+1}$*; and*

6. $\mathcal{P}^{d_2}_{|I^c|+1}$ *is the restriction to* $\sigma_{|I^c|+1}$ *of the universal bundle over* $\widetilde{\mathcal{M}}^{d_2}_{g_2,|I^c|+1}$*.*

1.9 Type III curves

A necessary and sufficient condition for a type III curve to be in the image of $\widetilde{C}_{g,n-1}$ is the same as the condition for type II curves. Denote by Z^3 the closed subscheme of $\widetilde{\mathcal{M}}_{g,n}$ whose points parametrize type III curves that do not lie in the image of $\widetilde{C}_{g,n-1}$. For $j \in \mathbb{Z}$ consider the strata F^3_d whose points parametrize type III curves such that

1. $\deg \mathcal{P}|_{C'} = j$; and

2. C' is connected to two Gieseker bubbles.

Then, for $j \geq 0$, we define Z^3_j recursively as

$$Z^3_j = \left(U^3_j \cap Z^3\right) \setminus \bigcup_{0 \leq k \leq j-1} Z^3_k,$$

where $U^3_j = \cup \widetilde{\mathcal{M}}_{\gamma,d}$ is the union running over all (γ, d) such that there exists a modification $f : (\gamma', d') \to (\gamma, d)$ with (γ', d') the modular graph of a bundle in F^3_j.

Similarly for $j \leq -1$, we define Z_j^3 recursively as

$$Z_j^3 = \left(U_j^3 \cap Z^3\right) \setminus \bigcup_{j+1 \leq k \leq -1} Z_k^3.$$

By the same argument we see that Z_j^3 is locally closed for all $j \in \mathbb{Z}$ and that

$$\chi_{Z^3}(\mathcal{F}) = \sum_{j \in \mathbb{Z}} \chi_{Z_j^3}(\mathcal{F}).$$

For $j \geq 0$, Z_j^3 parametrize all type III curves that are obtained from curves of F_j^3 by smoothing the nodes on C'. For $j \leq -1$, Z_j^3 parametrize all type III curves that are obtained from curves of F_j^3 by smoothing the nodes on the two connecting Gieseker bubbles that do not lie on C'.

1.10 Cohomology over $\widetilde{\mathcal{Z}}_{g,n}$

Recall that we would like to compute $\chi(\widetilde{\mathcal{M}}_{g,n}, \alpha)$, where α is an admissible bundle on $\widetilde{\mathcal{M}}_{g,n}$. In order to compute $\chi(\widetilde{\mathcal{M}}_{g,n}, \alpha)$, we use the stratification of $\widetilde{\mathcal{M}}_{g,n}$ as a union of $\widetilde{\mathcal{C}}_{g,n}$, Z_i^1, Z_I^2, and Z^3. First, we recall the definition of cohomology with support on a locally closed subscheme.

Definition 1.8. *[5] Let X be a topological space and let $Z \subset X$ be a locally closed subset. Define the sections of $\mathcal{F}$ with support in Z as*

$$\Gamma_Z(X, \mathcal{F}) := \{s \in \mathcal{F}(X) \mid Supp(s) \subset Z\}.$$

Then, Γ_Z is left exact but not necessarily exact. We define the right derived functors of Γ_Z to be the local cohomology groups with support with Z,

$$H_Z^i(X, \mathcal{F}) := R^i\Gamma_Z(X, \mathcal{F}).$$

Local cohomologies satisfy several properties.

Proposition 1.8. *[5] Let Z be a locally closed subset of X. Suppose $Z \subset Y \subset X$. Then,*

$$H_Z^i(X, \mathcal{F}) = H_Z^i(Y, \mathcal{F}|_Y),$$

for all i and for all sheaves $\mathcal{F}$ on X.

Using Proposition 1.8, we will simply denote $H_Z^i(\mathcal{F}) := H_Z^i(X, \mathcal{F})$.

Proposition 1.9. *Let $Z \subset X$ be an open subset. Then,*

$$H^i_Z(\mathcal{F}) = H^i(Z, \mathcal{F}),$$

for all i and for all sheaves $\mathcal{F}$ on X.

When $Z \subset X$ is a locally closed subset, there is an associated long exact sequence of cohomologies.

Lemma 1.2. *[5] Let X be a topological space and let $Z \subset X$ be a locally closed subset. Let $Z' \subset Z$ be closed in Z and let $Z'' := Z \setminus Z'$. Then, we have the following long exact sequence of local cohomologies for any abelian sheaf $\mathcal{F}$ on X:*

$$0 \to H^0_{Z'}(\mathcal{F}) \to H^0_Z(\mathcal{F}) \to H^0_{Z''}(\mathcal{F}) \to H^1_{Z'}(\mathcal{F}) \to H^1_Z(\mathcal{F}) \to H^1_{Z''}(\mathcal{F}) \to \cdots .$$

Corollary 1.1. *Let X be a topological space and let $Z \subset X$ be a locally closed subset. Let $Y = X \setminus Z$ and let $\mathcal{F}$ be a sheaf on X. Then, we have the following long exact sequence of local cohomologies:*

$$0 \to H^0_Z(\mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(Y, \mathcal{F}) \to H^1_Z(\mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(Y, \mathcal{F}) \to \cdots .$$

Proof. The long exact sequence is the one associated to the triple $Z, Y \subset X$ from Lemma 1.2, where the local cohomologies with support on open subsets of X are replaced with regular cohomologies using Proposition 1.9. □

Note that $\widetilde{\mathcal{C}}_{g,n-1}$ is open in $\widetilde{\mathcal{M}}_{g,n}$ and thus, $\widetilde{\mathcal{Z}}_{g,n}$ is closed. By Corollary 1.1, for any sheaf $\mathcal{F}$ on $\widetilde{\mathcal{M}}_{g,n}$, we have the following long exact sequence of local cohomologies

$$0 \to H^0_{\widetilde{\mathcal{Z}}_{g,n}}(\mathcal{F}) \to H^0(\mathcal{F}) \to H^0(\widetilde{\mathcal{C}}_{g,n-1}, \mathcal{F}) \to H^1_{\widetilde{\mathcal{Z}}_{g,n}}(\mathcal{F}) \to H^1(\mathcal{F}) \to H^1(\widetilde{\mathcal{C}}_{g,n-1}, \mathcal{F}) \to \cdots ,$$

where $H^i(\mathcal{F}) := H^i(\widetilde{\mathcal{M}}_{g,n}, \mathcal{F})$. Hence, if all the following terms are well-defined, we have

$$\chi(\widetilde{\mathcal{M}}_{g,n}, \mathcal{F}) = \chi(\widetilde{\mathcal{C}}_{g,n-1}, \mathcal{F}) + \chi_{\widetilde{\mathcal{Z}}_{g,n}}(\mathcal{F}).$$

In following sections, we will show that when $g = 0$, the terms are indeed well-defined and that the equation above gives us a formula for computing n-pointed invariants, $\chi(\widetilde{\mathcal{M}}_{0,n}, \alpha)$, from lower pointed invariants, $\chi(\widetilde{\mathcal{M}}_{0,m}, \alpha)$, where $m < n$.

Note that the strata Z^1, Z^2, and Z^3 are a pairwise disjoint collection of locally closed strata. Hence, by using Lemma 1.2 on $Z^1, Z^2, Z^3 \subset \widetilde{\mathcal{Z}}_{g,n}$, we get

$$\chi_{\widetilde{\mathcal{Z}}_{g,n}}(\mathcal{F}) = \chi_{Z^1}(\mathcal{F}) + \chi_{Z^2}(\mathcal{F}) + \chi_{Z^3}(\mathcal{F}).$$

Since $Z^1 = \coprod_{i=1}^{n-1} Z_i^1$, where Z_i^1 are pairwise disjoint, we conclude that

$$\chi_{Z^1}(\mathcal{F}) = \sum_{i=1}^{n-1} \chi_{Z_i^1}(\mathcal{F}).$$

Also, recall from Section 1.7 that for all i, $Z_{i,0}^1$ is open in Z_i^1. And $Z_{i,\pm 1}^1$ is open in $Z_i^1 \setminus Z_{i,0}^1$. Similarly, for all $d \geq 0$, $Z_{i,\pm d}^1$ is open in $Z_i^1 \setminus \left(\bigcup_{|k|<d} Z_{i,d}^1\right)$. Therefore, by the long exact sequence of local cohomologies, we conclude that if $\chi_{Z_{i,j}^1}(\mathcal{F})$ and $\sum_d \chi_{Z_{i,d}^1}(\mathcal{F})$ are well defined for a sheaf $\mathcal{F}$, then

$$\chi_{Z_i^1}(\mathcal{F}) = \sum_{d \in \mathbb{Z}} \chi_{Z_{i,d}^1}(\mathcal{F}),$$

for all $i \in [n-1]$. Combining the two results, we get

$$\chi_{Z^1}(\mathcal{F}) = \sum_{i \in [n-1]} \sum_{d \in \mathbb{Z}} \chi_{Z_{i,d}^1}(\mathcal{F}).$$

Similarly, we know from Section 1.8 that $Z^2 = \coprod_I Z_I^2$, where the union is over all subsets $I \subset [n-1]$ such that $2 \leq I \leq n-3$. Moreover, we also know that Z_I^2 can be stratified by $Z_{I,d}^2$. We know that $Z_{I,0}^2$ is open in Z_I^2, and for $d \geq 0$, $Z_{i,\pm d}^2$ is open in $Z_i^2 \setminus \left(\bigcup_{|k|<d} Z_{i,d}^2\right)$. Hence, by Lemma 1.2, we have

$$\chi_{Z_i^1}(\mathcal{F}) = \sum_{d \in \mathbb{Z}} \chi_{Z_{i,d}^1}(\mathcal{F}).$$

Finally, for each I and d, $Z_{I,d}^2$ is a disjoint union of $Z_{I,(d_1,d_2)}^2$, where we run over all $d_1, d_2 \in \mathbb{Z}$ such that $D - d_1 - d_2 = d$. Therefore, we get,

$$\chi_{Z_I^2}(\mathcal{F}) = \sum_{d_1, d_2 \in \mathbb{Z}} \chi_{Z_{I,d_1,d_2}^2}(\mathcal{F}).$$

Since this holds true for all $I \subset [n-1]$, we conclude that

$$\chi_{Z^2}(\mathcal{F}) = \sum_{\substack{I \subset [n-1] \\ 2 \leq |I| \leq n-3}} \sum_{d_1, d_2 \in \mathbb{Z}} \chi_{Z_{I,d_1,d_2}^2}(\mathcal{F}).$$

Combining the results above, we obtain

$$\chi_{\widetilde{\mathcal{Z}}_{g,n}}(\mathcal{F}) = \left(\sum_{i\in[n-]} \chi_{Z_i^1}(\mathcal{F})\right) + \left(\sum_{\substack{I\subset[n-1]\\ 2\leq|I|\leq n-3}} \chi_{Z_I^2}(\mathcal{F})\right) + \chi_{Z^3}(\mathcal{F}) \tag{1.3}$$

$$= \left(\sum_{i\in[n-1]}\sum_{d\in\mathbb{Z}} \chi_{Z_{i,d}^1}(\mathcal{F})\right) \tag{1.4}$$

$$+\left(\sum_{\substack{I\subset[n-1]\\ 2\leq|I|\leq n-3}}\sum_{d_1,d_2\in\mathbb{Z}} \chi_{Z_{I,d_1,d_2}^2}(\mathcal{F})\right) + \chi_{Z^3}(\mathcal{F}). \tag{1.5}$$

In particular, when $g = 0$, there are no Type III curves[9]. Hence, $Z^3 = \emptyset$ and thus,

$$\chi_{\widetilde{\mathcal{Z}}_{0,n}}(\mathcal{F}) = \left(\sum_{i\in[n-1]}\sum_{d\in\mathbb{Z}} \chi_{Z_{i,d}^1}(\mathcal{F})\right) + \left(\sum_{\substack{I\subset[n-1]\\ 2\leq|I|\leq n-3}}\sum_{d_1,d_2\in\mathbb{Z}} \chi_{Z_{I,d_1,d_2}^2}(\mathcal{F})\right). \tag{1.6}$$

1.11 Towards finiteness of $\chi(\widetilde{\mathcal{M}}_{0,n}, \alpha)$

Recall that we have

$$\chi(\widetilde{\mathcal{M}}_{0,n}, \alpha) = \chi(\widetilde{\mathcal{M}}_{0,n-1}, R\pi_*\alpha) + \chi_{\widetilde{\mathcal{Z}}_{0,n}}(\alpha).$$

In Section 1.10, we showed that the second term, $\chi_{\widetilde{\mathcal{Z}}_{0,n}}(\alpha)$, can be written as a countable sum of local Euler characteristics $\chi_Z(\alpha)$, where Z is locally closed.

We wish to show that when $g = 0$, n-pointed gauge Gromov-Witten invariants can be reconstructed from lower pointed invariants and thus, are well defined. In this section, we will show that all but finitely many terms of the sum in the equation 1.6 vanish. Moreover, the nonvanishing terms can be written as a sum of products of $\chi(\widetilde{\mathcal{M}}_{0,n'}, \alpha')$, where $n' < n$ and α' is an admissible class on $\widetilde{\mathcal{M}}_{0,n'}$.

Vanishing of $\chi_{Z_{i,d}^1}(\alpha)$ for $d \ll 0$ and $d \gg 0$

First, fix $i \in [n-1]$. We will show that $\chi_{Z_{i,d}^1}(\alpha)$ vanishes for all but finitely many $d \in \mathbb{Z}$.

Suppose $d \geq 0$ and let $e = D-d-1$. Recall from Proposition 1.6 that $F_{i,d}^1 \cong \widetilde{\mathcal{M}}_{0,3}^d \times \widetilde{\mathcal{M}}_{0,n-1}^e$, and that $Z_{i,d}^1$ is an affine bundle over $F_{i,d}^1$ given by $(\mathcal{L}_3^{-1} \otimes \mathcal{P}_3^d) \boxtimes (\mathcal{P}_{n-1}^e)^{-1}$,

[9] Since all type III curves have genus greater than 0.

where $\mathcal{L}_3$ is the cotangent bundle along σ_3 in $\widetilde{\mathcal{M}}^d_{0,3}$, $\mathcal{P}^d_3$ is the restriction of the universal bundle to σ_3, and $\mathcal{P}^e_{n-1}$ is the restriction of the universal bundle to σ_{n-1} in $\widetilde{\mathcal{M}}^e_{0,n-1}$. Also, recall that the normal bundle $N_{Z^1_{i,d}}$ is given by $(\mathcal{P}^d_3)^{-1} \boxtimes (\mathcal{L}^{-1}_{n-1} \otimes \mathcal{P}^e_{n-1})$. Now, using the filtration spectral sequence, we obtain

$$\chi_{Z^1_{i,d}}(\alpha) \quad = \quad \chi\left(F^1_{i,d}, \alpha \otimes \det N_{Z^1_{i,d}} \otimes \operatorname{Sym} N_{Z^1_{i,d}} \otimes \operatorname{Sym} N^{-1}_{F^1_{i,d}/Z^1_{i,d}}\right).$$

Consider the $\mathbb{C}^\times$-action on $F^1_{i,d}$ given by the global scaling of the universal bundle on C', i.e. the pull back of the global $\mathbb{C}^\times$ action on $\widetilde{\mathcal{M}}^d_{0,3}$. The weight of this $\mathbb{C}^\times$ action is 1 on $\mathcal{P}^d_3$ and 0 on $\mathcal{L}^{-1}_3, \mathcal{L}^{-1}_{n-1}$, and $\mathcal{P}^e_{n-1}$. Since $\det N_{Z^1_{i,d}} \cong (\mathcal{P}^d_3)^{-1} \boxtimes (\mathcal{L}^{-1}_{n-1} \otimes \mathcal{P}^e_{n-1})$, $\mathbb{C}^\times$ action has weight -1 on it. Hence, the $\mathbb{C}^\times$ action has negative weights on the components of $\operatorname{Sym} N_{Z^1_{i,d}}$. Lastly,

$$\operatorname{Sym} N^{-1}_{F^1_{i,d}/Z^1_{i,d}} \cong \operatorname{Sym}\left(\left(\mathcal{L}_3 \otimes (\mathcal{P}^d_3)^{-1}\right) \boxtimes \mathcal{P}^e_{n-1}\right).$$

Therefore, we see that $\mathbb{C}^\times$ action has negative weights on all the components of $\operatorname{Sym} N^{-1}_{F^1_{i,d}/Z^1_{i,d}}$. Hence, the weights of the chosen $\mathbb{C}^\times$ action on all the components of $\det N_{Z^1_{i,d}} \otimes \operatorname{Sym} N_{Z^1_{i,d}} \otimes \operatorname{Sym} N^{-1}_{F^1_{i,d}/Z^1_{i,d}}$ are negative.

We have left to compute the weight of the $\mathbb{C}^\times$-action on α. Since α is an admissible class, $\alpha \cong (\det R\pi_*\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\bigotimes_j \operatorname{ev}^*_j \mathbb{C}_{\lambda_j} \otimes \mathcal{L}^{a_j}_j\right)$. However, for all $j \in [n]$, the weight of our chosen $\mathbb{C}^\times$ action on $\operatorname{ev}^*_j \mathbb{C}_{\lambda_j} \otimes \mathcal{L}^{a_j}_j$ is independent of d.

Lemma 1.3. *[1] For the chosen $\mathbb{C}^\times$-action,*

1. *$\mathbb{C}^\times$ acts on $\operatorname{ev}^*_j \mathbb{C}_{\lambda_j}$ with weight λ_j, if p_j is parametrized by a point of $\widetilde{\mathcal{M}}_{0,3}$.*
2. *$\mathbb{C}^\times$ acts on $\operatorname{ev}^*_j \mathbb{C}_{\lambda_j}$ with weight 0, if p_j is not parametrized by a point of $\widetilde{\mathcal{M}}_{0,3}$.*
3. *$\mathbb{C}^\times$ acts on $\mathcal{L}_j$ with weight 0.*

Finally, we want to compute the weight of the $\mathbb{C}^\times$ action on $\det R\pi_*\varphi^*\mathbb{C}_1$. However, the action is simply global rescaling of the universal degree d line bundle on $\widetilde{\mathcal{M}}_{0,3}$. Hence, the weight of the $\mathbb{C}^\times$ action $\det R\pi_*\varphi^*\mathbb{C}_1$ is $d+1$, and its weight on $(\det R\pi_*\varphi^*\mathbb{C}_1)^{-q}$ is $-q(d+1)$ where $q > 0$.

We see that the weight of the $\mathbb{C}^\times$-action on α over $F^1_{i,d}$ is a linear function of d with slope $-q < 0$. Since all the components of $\det N_{Z^1_{i,d}} \otimes \operatorname{Sym} N_{Z^1_{i,d}} \otimes \operatorname{Sym} N^{-1}_{F^1_{i,d}/Z^1_{i,d}}$

have negative weights with respect to our chosen $\mathbb{C}^\times$ action, we see that for $d \gg 0$, $\mathbb{C}^\times$ acts with negative weights on all components of

$$(\det R\pi_*\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\bigotimes_j \mathrm{ev}_j^* \mathbb{C}_{\lambda_j} \otimes \mathcal{L}_j^{a_j}\right) \otimes \det N_{Z^1_{i,d}} \otimes \mathrm{Sym}\, N_{Z^1_{i,d}} \otimes \mathrm{Sym}\, N^{-1}_{F^1_{i,d}/Z^1_{i,d}}.$$

In particular, there do not exist sections that are invariant with respect to the chosen $\mathbb{C}^\times$-action. Therefore, we conclude that for $d \gg 0$, $\chi_{Z^1_{i,d}}(R\pi_*\alpha) = 0$.

Moreover, note that α over $F^1_{i,d} \cong \widetilde{\mathcal{M}}_{0,3} \times \widetilde{\mathcal{M}}_{0,n-1}$ can be written as a box sum of a line bundle over $\widetilde{\mathcal{M}}_{0,3}$ and a line bundle over $\widetilde{\mathcal{M}}_{0,n-1}$. Recall that the isomorphism $\widetilde{\mathcal{M}}_{0,3} \times \widetilde{\mathcal{M}}_{0,n-1} \cong F^1_{i,d}$ is defined by associating the rational component of C containing points p_i and p_n with the corresponding point on $\widetilde{\mathcal{M}}_{0,3}$, and associating the other (possibly reducible) component containing points p_j, $j \neq i, n$ with the corresponding point on $\widetilde{\mathcal{M}}_{0,n-1}$.

Using the isomorphism, we see that $\bigotimes_j \mathrm{ev}_j^* \mathbb{C}_{\lambda_j} \otimes \mathcal{L}_j^{a_j}$ can be written as

$$\left(\mathrm{ev}_i^* \mathbb{C}_{\lambda_i} \otimes \mathcal{L}_i^{a_i} \otimes \mathrm{ev}_n^* \mathbb{C}_{\lambda_n} \otimes \mathcal{L}_n^{a_n}\right) \boxtimes \left(\otimes_{j \neq i,n} \mathrm{ev}_j^* \mathbb{C}_{\lambda_j} \otimes \mathcal{L}_j^{a_j}\right),$$

where the first term is a line bundle over $\widetilde{\mathcal{M}}_{0,3}$, and the second term is a line bundle over $\widetilde{\mathcal{M}}_{0,n-1}$.

Now, we have the following lemma.

Lemma 1.4. *We have an isomorphism*

$$\det R\pi_*\varphi^*\mathbb{C}_1 \cong \det R\pi_*\varphi_1^*\mathbb{C}_1 \boxtimes \det R\pi_*\varphi_2^*\mathbb{C}_1,$$

where $\varphi_i : \widetilde{\mathcal{M}}_{0,3} \to [pt/\mathbb{C}^\times]$ *and* $\varphi_2 : \widetilde{\mathcal{M}}_{0,n-1} \to [pt/\mathbb{C}^\times]$.

Proof. Before we prove the isomorphism globally, we will first verify the isomorphism fiber-wise over points of $F^1_{i,d}$ to get a clear picture of why the two bundles are isomorphic. The global picture, and the proof of the isomorphism are almost identical. Let $p \in F^1_{i,d}$ parametrizing a reducible curve $C = C' \cup B \cup C''$, where B is the Gieseker bubble connecting the component C' containing p_i and p_n with the component C'' containing the rest of the marked points. Let $\mathcal{P}$ be the restriction of the universal line bundle $\mathcal{P}_{0,n}$ to C. Note that $B \cong \mathbb{P}^1$ and that $\mathcal{P}|_B \cong \mathcal{O}(1)$. Consider the morphism $C' \coprod B \coprod C'' \to C$ given by normalizing the two nodes on B. This morphism gives a short exact sequence of sheaves on C.

$$0 \to \mathcal{P} \to \mathcal{P}' \oplus i_*\mathcal{P}|_B \oplus \mathcal{P}'' \to \mathcal{P}|_p \oplus \mathcal{P}|_q \to 0,$$

where $\mathcal{P}' := i_*\mathcal{P}|_{C'}$, $\mathcal{P}'' := i_*\mathcal{P}|_{C''}$, and p and q are the two nodes on B. Now, consider the associated long exact sequence of cohomology groups:

$$0 \to H^0(\mathcal{P}) \to H^0(\mathcal{P}') \oplus H^0(\mathbb{P}^1, \mathcal{O}(1)) \oplus H^0(\mathcal{P}'') \to \mathcal{P}|_p \oplus \mathcal{P}|_q$$
$$\to H^1(\mathcal{P}') \oplus H^1(\mathcal{P}'') \to H^1(\mathcal{P}) \to 0.$$

However, $H^0(\mathbb{P}^1, \mathcal{O}(1))$ is canonically isomorphic to $\mathcal{O}(1)|_0 \oplus \mathcal{O}(1)|_\infty$. In particular, we have $H^0(\mathbb{P}^1, \mathcal{O}(1)) \cong \mathcal{P}|_p \oplus \mathcal{P}|_q$. Therefore, we conclude that $H^0(\mathcal{P}) \cong H^0(\mathcal{P}') \oplus H^0(\mathcal{P}'')$ and $H^1(\mathcal{P}) \cong H^1(\mathcal{P}') \oplus H^1(\mathcal{P}'')$. Hence, we conclude that

$$R\pi_*\mathcal{P} \cong R\pi_*\mathcal{P}' \oplus R\pi_*\mathcal{P}''.$$

Taking determinants of both sides, we obtain

$$\det(R\pi_*\mathcal{P}) \cong \det(R\pi_*\mathcal{P}') \otimes \det(R\pi_*\mathcal{P}'').$$

Since $\mathcal{P}' = i_*\mathcal{P}|_{C'}$ and $\mathcal{P}'' = i_*\mathcal{P}|_{C''}$, we can write

$$\det(R\pi_*\mathcal{P}) \cong \det(R\pi_*\mathcal{P}|_{C'}) \boxtimes \det(R\pi_*\mathcal{P}|_{C''}).$$

The isomorphism above holds for all points of $F^1_{i,d}$ and moreover, hold globally over $F^1_{i,d} \cong \widetilde{\mathcal{M}}^d_{0,3} \times \widetilde{\mathcal{M}}^e_{0,n-1}$. Let $\widetilde{C}_{i,d}$ be the restriction of the universal curve, $\widetilde{C}_{0,n}$, over $F^1_{i,d}$. Let $\mathcal{B}$ be the family of Gieseker bubbles over $F^1_{i,d}$. Then, we see that $\widetilde{C}_{i,d}$ is the union of three connected components, $\widetilde{C}_{0,3}$, $\mathcal{B}$, and $\widetilde{C}_{0,n-1}$, where $\mathcal{B}$ is connected to $\widetilde{C}_{0,3}$ and $\widetilde{C}_{0,n-1}$ along the last marked sections σ_3 and σ_{n-1}, respectively. Letting $\mathcal{P}'$ and $\mathcal{P}''$ denote the restrictions of $\mathcal{P}$ to $\widetilde{C}_{0,3}$ and $\widetilde{C}_{0,n-1}$, and letting $\sigma' = \widetilde{C}_{0,3} \cap \mathcal{B}$, and $\sigma'' = \widetilde{C}_{0,n-1} \cap \mathcal{B}$, we have the following short exact sequence:

$$0 \to \mathcal{P} \to \mathcal{P}' \oplus i_*\mathcal{P}|_{\mathcal{B}} \oplus \mathcal{P}'' \to \mathcal{P}|_{\sigma'} \oplus \mathcal{P}|_{\sigma''} \to 0.$$

Since $\mathcal{B}$ is the family of Gieseker bubbles over $F^1_{i,d}$, it is a $\mathbb{P}^1$ bundle over $F^1_{i,d}$ and $\mathcal{P}|_{\mathcal{B}} \cong \mathcal{O}_{\mathcal{B}}(1)$. Pushing forward along $\pi : \widetilde{C} \to F^1_{i,d}$, we obtain the following long exact sequence:

$$0 \to \pi_*\mathcal{P} \to \pi_*\mathcal{P}' \oplus \pi_*\mathcal{O}_{\mathcal{B}}(1) \oplus \pi_*\mathcal{P}'' \to \mathcal{P}|_p \oplus \mathcal{P}|_q$$
$$\to R^1\pi_*\mathcal{P}' \oplus R^1\pi_*\mathcal{P}'' \to R^1\pi_*\mathcal{P} \to 0.$$

Since $\pi : \mathcal{B} \to F^1_{i,d}$ is a $\mathbb{P}^1$-bundle, $\pi_*\mathcal{O}_{\mathcal{B}}(1) \cong \mathcal{P}|_{\sigma'} \oplus \mathcal{P}|_{\sigma''}$. Hence, we obtain $R\pi_*\mathcal{P} \cong R\pi_*\mathcal{P}' \oplus R\pi_*\mathcal{P}''$, and thus, $\det(R\pi_*\mathcal{P}) \cong \det(R\pi_*\mathcal{P}') \otimes \det(R\pi_*\mathcal{P}'')$. Finally, noting that $\mathcal{P}' = \mathcal{P}|_{\widetilde{C}_{0,3}}$ and $\mathcal{P}'' = \mathcal{P}|_{\widetilde{C}_{0,n-1}}$, we conclude that over $F^1_{i,d}$ we have

$$\det(R\pi_*\varphi^*\mathbb{C}_1) \cong \det(R\pi_*\varphi_1^*\mathbb{C}_1) \boxtimes \det(R\pi_*\varphi_2^*\mathbb{C}_1).$$

□

Combining both results, we get the following proposition.

Proposition 1.10. *Let $i \in [n-1]$ and $d \in \mathbb{Z}$. Let $F^1_{i,d}$ be the stratum of type I curves as defined in Section 1.6. Then, $F^1_{i,d} \cong \widetilde{\mathcal{M}}_{0,3} \times \widetilde{\mathcal{M}}_{0,n-1}$. Moreover, the isomorphism of spaces gives rise to the following isomorphism of line bundles:*

$$(\det R\pi_*\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\bigotimes_j \mathrm{ev}_j^*\,\mathbb{C}_{\lambda_j} \otimes \mathcal{L}_j^{a_j}\right) \cong \left((\det R\pi_*\varphi^*\mathbb{C}_1)^{-q} \otimes \mathrm{ev}_i^*\,\mathbb{C}_{\lambda_i} \otimes \mathcal{L}_i^{a_i} \otimes \mathrm{ev}_n^*\,\mathbb{C}_{\lambda_n} \otimes \mathcal{L}_n^{a_n}\right)$$
$$\boxtimes \left((\det R\pi_*\varphi^*\mathbb{C}_1)^{-q} \otimes (\otimes_{j\neq i,n}\,\mathrm{ev}_j^*\,\mathbb{C}_{\lambda_j} \otimes \mathcal{L}_j^{a_j}\right).$$

In particular, for any admissible class, α over $F^1_{i,d}$, we conclude that there exist admissible classes α_1 and α_2 on $\widetilde{\mathcal{M}}_{0,3}$ and $\widetilde{\mathcal{M}}_{0,n-1}$, respectively such that

$$\alpha \cong \alpha_1 \boxtimes \alpha_2.$$

Proof. The proposition follows directly from the discussion above. Note that the analysis is independent of our choice of $i \in [n-1]$ and $d \in \mathbb{Z}$. □

Now, suppose $d < 0$. We want to show that for $d \ll 0$, $\chi_{Z^1_{i,d}}(R\pi_*\alpha) = 0$. Again, recall that $F^1_{i,d} \cong \widetilde{\mathcal{M}}_{0,3} \times \widetilde{\mathcal{M}}_{g,n-1}$, the normal bundle to $Z^1_{i,d}$ in $\widetilde{\mathcal{M}}_{g,n}$ is isomorphic to $(\mathcal{L}_3^{-1} \otimes \mathcal{P}_3) \boxtimes \mathcal{P}_{n-1}^{-1}$, and $Z^1_{i,d}$ is an affine bundle over $F^1_{i,d}$ given by $(\mathcal{L}_{n-1}^{-1} \otimes \mathcal{P}_{n-1}) \boxtimes \mathcal{P}_3^{-1}$. Consider the pullback of the $\mathbb{C}^\times$ action on $\widetilde{\mathcal{M}}_{0,3}$. We already showed that the weight of this action on α over $F^1_{i,d}$ is given by a linear polynomial with slope $-q < 0$. However, we have

$$\det N_{Z^1_{i,d}} \cong (\mathcal{L}_3^{-1} \otimes \mathcal{P}_3^d) \boxtimes (\mathcal{P}_{n-1}^e)^{-1},$$
$$\mathrm{Sym}\, N_{Z^1_{i,d}} \cong \left(\mathrm{Sym}\,\mathcal{L}_3^{-1} \otimes \mathrm{Sym}\,\mathcal{P}_3^d\right) \boxtimes \mathrm{Sym}(\mathcal{P}_{n-1}^e)^{-1},$$
$$\mathrm{Sym}\, N^{-1}_{F^1_{i,d}/Z^1_{i,d}} \cong \mathrm{Sym}(\mathcal{P}_3^d)^{-1} \boxtimes \left(\mathrm{Sym}\,\mathcal{L}_{n-1}^{-1} \otimes \mathrm{Sym}\,\mathcal{P}_{n-1}^e\right).$$

Recall that, the weight of our chosen $\mathbb{C}^\times$ action is 1 on $\mathcal{P}_3^d$ and 0 on $\mathcal{L}_3^{-1}, \mathcal{L}_{n-1}^{-1}$, and $\mathcal{P}_{n-1}^e$. Therefore, $\mathbb{C}^\times$ acts with positive weight on all the components of the vector bundles above. Hence, for $d \ll 0$, the weight of the chosen $\mathbb{C}^\times$-action on all components of $\det N_{Z^1_{i,d}} \otimes \mathrm{Sym}\, N_{Z^1_{i,d}} \otimes \mathrm{Sym}\, N^{-1}_{F^1_{i,d}/Z^1_{i,d}}$ must be positive. Hence, $\alpha \otimes \det N_{Z^1_{i,d}} \otimes \mathrm{Sym}\, N_{Z^1_{i,d}} \otimes \mathrm{Sym}\, N^{-1}_{F^1_{i,d}/Z^1_{i,d}}$ does not have any invariant sections under the chosen $\mathbb{C}^\times$ action. Consequently, $\chi_{Z^1_{i,d}}(\alpha) = 0$ for $d \ll 0$. Moreover, by Proposition 1.10, we know that there exist admissible classes α_1 and α_2 over $\widetilde{\mathcal{M}}_{0,3}$ and $\widetilde{\mathcal{M}}_{0,n-1}$, respectively, such that over $F^1_{i,d}$ we have

$$\alpha \cong \alpha_1 \boxplus \alpha_2.$$

Since $\chi_{Z^1_{i,d}}(\alpha) = 0$ for $d \gg 0$ and $d \ll 0$, we conclude that $I_i = \{d \in \mathbb{Z} | \chi_{Z^1_{i,d}}(\alpha) \neq 0\}$ is finite. Then, Proposition 1.10 implies that for all d, there exist admissible classes $\alpha_{1,d}$ and $\alpha_{2,d}$ on $\widetilde{\mathcal{M}}_{0,3}$ and $\widetilde{\mathcal{M}}_{0,n-1}$, respectively such that

$$\chi_{Z^1_{i,d}}(\alpha) = \sum_{d \in I_i} \chi(\widetilde{\mathcal{M}}_{0,3}, \alpha_{1,d}) \chi(\widetilde{\mathcal{M}}_{0,n-1}, \alpha_{2,d}).$$

Vanishing of $\chi_{Z^2_{I,(d_1,d_2)}}(\alpha)$ for all but finitely many pairs (d_1, d_2)

Now, we show that $\chi_{Z^2_{I,(d_1,d_2)}}(\alpha) = 0$ for all but finitely many pairs $(d_1, d_2) \in \mathbb{Z}^2$.

Fix $I \subset [n-1]$ such that $2 \leq |I| \leq n-3$ and $d_1, d_2 \in \mathbb{Z}$. Let $d := D - d_1 - d_2 - 2$ and suppose $d \geq 2$. Then, we saw from Section 1.8 that

$$F^2_{I,(d_1,d_2)} \cong \widetilde{\mathcal{M}}^{d_1}_{0,|I|+1} \times \widetilde{\mathcal{M}}^{d}_{0,3} \times \widetilde{\mathcal{M}}^{d_2}_{0,|I^c|+1}.$$

For simplicity of notation, we write $F := F^2_{I,(d_1,d_2)}$, $Z := Z^2_{I,(d_1,d_2)}$, $\widetilde{M}_1 := \widetilde{\mathcal{M}}^{d_1}_{0,|I|+1}$ and $\widetilde{M}_2 := \widetilde{\mathcal{M}}^{d_2}_{0,|I^c|+1}$. Recall from Proposition 1.7 that the formal neighborhood of F is isomorphic to

$$\mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \mathcal{N}_3 \oplus \mathcal{N}_4,$$

where

$$\begin{aligned}
\mathcal{N}_1 &= \left(\mathcal{L}^{-1}_{|I|+1} \otimes \mathcal{P}^{d_1}_{|I|+1}\right) \boxtimes (\mathcal{P}^d_1)^{-1} \boxtimes \mathcal{O}_{\widetilde{M}_2},\\
\mathcal{N}_2 &= (\mathcal{P}^{d_1}_{|I|+1})^{-1} \boxtimes \left(\mathcal{L}^{-1}_1 \otimes \mathcal{P}^d_1\right) \boxtimes \mathcal{O}_{\widetilde{M}_2},\\
\mathcal{N}_3 &= \mathcal{O}_{\widetilde{M}_1} \boxtimes \left(\mathcal{L}^{-1}_3 \otimes \mathcal{P}^d_3\right) \boxtimes (\mathcal{P}^{d_2}_{|I|^c+1})^{-1},\\
\mathcal{N}_4 &= \mathcal{O}_{\widetilde{M}_1} \boxtimes (\mathcal{P}^d_3)^{-1} \boxtimes \left(\mathcal{L}^{-1}_{|I^c|+1} \otimes \mathcal{P}^{d_2}_{|I^c|+1}\right).
\end{aligned}$$

Since $d \geq 2$, Z parametrize type II curves obtained by smoothing nodes attached to C_1 and C_2. Hence, the normal bundle to Z in $\widetilde{\mathcal{M}}_{g,n}$ is isomorphic to $\mathcal{N}_2 \oplus \mathcal{N}_3$ and Z is the affine bundle to F given by $\mathcal{N}_1 \oplus \mathcal{N}_4$.

Consider the pullback of the global $\mathbb{C}^\times$-action on $\widetilde{M}_1$. This $\mathbb{C}^\times$-action has weight 1 on $\mathcal{P}_{|I|+1}$ and 0 on all other line bundles appearing in $\mathcal{N}_i$. Hence, $\mathbb{C}^\times$-action has weight 1 on $\mathcal{N}_1$, -1 on $\mathcal{N}_2$, and 0 on $\mathcal{N}_3$ and $\mathcal{N}_4$.

The weight of this particular $\mathbb{C}^\times$-action on α is again given by a linear polynomial in d_1 with slope $-q < 0$. As before, the weights of this $\mathbb{C}^\times$-action on $\mathrm{ev}_j^* \mathbb{C}_{\lambda_j}$ and on $\mathcal{L}_j^{a_j}$ are independent of d_1 for all $j = 1, \ldots, n$ by Lemma 1.3.

Moreover, the following lemma allows us to compute the weights of our chosen $\mathbb{C}^\times$-action on $(\det R\pi_* \varphi^* \mathbb{C}_1)^{-q}$.

Lemma 1.5. *We have*

$$\det R\pi_*\varphi^*\mathbb{C}_1 \cong \det R\pi_*\varphi_1^*\mathbb{C}_1 \boxtimes \det R\pi_*\varphi_2^*\mathbb{C}_1 \boxtimes \det R\pi_*\varphi_3^*\mathbb{C}_1,$$

where $\varphi_1 : \widetilde{\mathcal{M}}_1 \to [pt/\mathbb{C}^\times]$, $\varphi_2 : \widetilde{\mathcal{M}}_2 \to [pt/\mathbb{C}^\times]$, *and* $\varphi_3 : \widetilde{\mathcal{M}}_{0,3} \to [pt/\mathbb{C}^\times]$.

Proof. The proof is similar to the proof of Lemma 1.4. Every point of F parametrizes a type II curve, C, with two connecting Gieseker bubbles attached to C', the component containing p_n. In other words, we have, $C = C_1 \cup B_1 \cup C' \cup B_2 \cup C_2$, where B_1 and B_2 are connecting Gieseker bubbles. Moreover, C_1 contains the marked points with indices in I, and C_2 contains the marked points with indices in I^c.

Let $\mathcal{C}$ be the restriction of the universal curve, $\widetilde{\mathcal{C}}_{0,n}$, over $F = F^2_{I,(d_1,d_2)}$. Let $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}'$ be the families of curves over F containing the sections σ_i with $i \in I$, $i \in I^c$, and $i = n$ respectively. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ denote the two families of Gieseker bubbles, and let $S_i = \mathcal{C}_i \cap \mathcal{B}_i$, $T_i = \mathcal{C} \cap \mathcal{B}_i$.

By normalizing S_i and T_i, we obtain a morphism

$$\mathcal{C}_1 \coprod \mathcal{B}_1 \coprod \mathcal{C}' \coprod \mathcal{B}_2 \coprod \mathcal{C}_2 \to \mathcal{C}.$$

This morphism gives rise to the following short exact sequence

$$0 \to \mathcal{P} \to \mathcal{P}_1 \oplus \mathcal{Q}_1 \oplus \mathcal{P}' \oplus \mathcal{Q}_2 \oplus \mathcal{P}_2 \to \mathcal{P}|_{S_1} \oplus \mathcal{P}|_{T_1} \oplus \mathcal{P}|_{T_2} \oplus \mathcal{P}|_{S_2} \to 0,$$

where $\mathcal{P}$ is the restriction of the universal bundle to $\mathcal{C}$, $\mathcal{P}_j = i_*\mathcal{P}|_{\mathcal{C}_j}, \mathcal{P}' = i_*\mathcal{P}|_{\mathcal{C}'}$, and $\mathcal{Q}_j = i_*\mathcal{P}|_{\mathcal{B}_j}$. Since $\mathcal{B}_1$ and $\mathcal{B}_2$ are Gieseker bubbles, we know that $\mathcal{B}_j \to F$ is a $\mathbb{P}^1$-bundle and $\mathcal{Q}_j \cong \mathcal{O}_{\mathcal{B}_j}(1)$. Therefore, we have

$$\pi_*\mathcal{Q}_j \cong \mathcal{P}|_{S_j} \oplus \mathcal{P}|_{T_j}.$$

Moreover, $R^p\pi_*\mathcal{Q}_j = 0$ for all $p > 0$.

The short exact sequence above gives rise to the following long exact sequence:

$$0 \to \pi_*\mathcal{P} \to \pi_*\mathcal{P}_1 \oplus \pi_*\mathcal{Q}_1 \oplus \pi_*\mathcal{P}' \oplus \pi_*\mathcal{Q}_2 \oplus \pi_*\mathcal{P}_2$$

$$\to \mathcal{P}|_{S_1} \oplus \mathcal{P}|_{T_1} \oplus \mathcal{P}|_{T_2} \oplus \mathcal{P}|_{S_2} \to R^1\pi_*\mathcal{P} \to R^1\pi_*\mathcal{P}_1 \oplus R^1\pi_*\mathcal{P}' \oplus R^1\pi_*\mathcal{P}_2 \to 0.$$

Since $\pi_*\mathcal{Q}_j \cong \mathcal{P}|_{S_j} \oplus \mathcal{P}|_{T_j}$, we conclude that

$$\pi_*\mathcal{P} \cong \pi_*\mathcal{P}_1 \oplus \pi_*\mathcal{P}' \oplus \pi_*\mathcal{P}_2,$$

and

$$R^1\pi_*\mathcal{P} \cong R^1\pi_*\mathcal{P}_1 \oplus R^1\pi_*\mathcal{P}' \oplus R^1\pi_*\mathcal{P}_2.$$

Since fibers of π are one dimensional, higher derived push-forwards vanish. Therefore, we have

$$R\pi_*\mathcal{P} \cong R\pi_*\mathcal{P}_1 \oplus R\pi_*\mathcal{P}' \oplus R\pi_*\mathcal{P}_2.$$

Now, note that C_1, C', and C_2 are identified with the universal curves over $\widetilde{\mathcal{M}}_{0,|I|+1}$, $\widetilde{\mathcal{M}}_{0,3}$, and $\widetilde{\mathcal{M}}_{0,|I^c|+1}$ under the isomorphism $F \cong \widetilde{\mathcal{M}}_{0,|I|+1} \times \widetilde{\mathcal{M}}_{0,3} \times \widetilde{\mathcal{M}}_{0,|I^c|+1}$. Hence, above isomorphism can be written as

$$R\pi_*\varphi^*\mathbb{C}_1 \cong R\pi_*\varphi_1^*\mathbb{C}_1 \boxplus R\pi_*\varphi_2\mathbb{C}_1 \boxplus R\pi_*\varphi_3\mathbb{C}_1.$$

Taking determinants of both sides, we get the desired isomorphism. □

The weight of the chosen $\mathbb{C}^\times$-action on $\det R\pi_*\varphi_1^*\mathbb{C}_1$ is $d_1 + 1$. Since the chosen $\mathbb{C}^\times$-action is trivial over the other components C_2 and C_3, the weights are 0 on $\det R\pi_*\varphi_2^*\mathbb{C}_1$ and $\det R\pi_*\varphi_3^*\mathbb{C}_1$. Hence, the $\mathbb{C}^\times$-action has weight $-q(d_1 + 1)$ on $(\det R\pi_*\varphi^*\mathbb{C}_1)^{-q}$.

Now, note that

$$\det N_Z \otimes \operatorname{Sym} N_Z \otimes \operatorname{Sym} N_{F/Z}^{-1} \cong \det(\mathcal{N}_2 \oplus \mathcal{N}_3) \otimes \operatorname{Sym}(\mathcal{N}_2 \oplus \mathcal{N}_3) \otimes \operatorname{Sym}(\mathcal{N}_1^{-1} \oplus \mathcal{N}_4^{-1}).$$

Recalling that $\mathbb{C}^\times$-action has weight 1 on $\mathcal{N}_1$, -1 on $\mathcal{N}_2$, and 0 on $\mathcal{N}_3$ and $\mathcal{N}_4$, we see that the $\mathbb{C}^\times$-action has negative weights on all components of the vector bundle above. Hence, for $d_1 \gg 0$, the $\mathbb{C}^\times$-action has negative weights on all components of the vector bundle $\alpha \otimes \det N_Z \otimes \operatorname{Sym} N_Z \otimes \operatorname{Sym} N_{F/Z}^{-1}$. We conclude that $\chi_{Z^2_{I,(d_1,d_2)}}(\alpha) = 0$ for $d_1 \gg 0$.

By a symmetric argument, we conclude that for $d_1 \ll 0$, $\chi_{Z^2_{I,(d_1,d_2)}}(R\pi_*\alpha) = 0$. Also, note that the choice of d_1 was made without loss of generality. Therefore, we can conclude that $\chi_{Z^2_{I,(d_1,d_2)}} = 0$ for $d_2 \gg 0$ and $d_2 \ll 0$ as well.

Therefore, for all but finitely many (d_1, d_2), $\chi_{Z^2_{I,(d_1,d_2)}}(R\pi_*\alpha) = 0$ and thus, $\chi_{Z^2_I}$ is finite for all I.

Combining the results, we obtain the following proposition similar to Proposition 1.10.

Proposition 1.11. *Let $I \subset [n-1]$ with $2 \le |I| \le n-3$ and let $(d_1, d_2) \in \mathbb{Z}^2$. Let $F^2_{I,(d_1,d_2)}$ be the stratum of type II curves as defined in Section 1.6. Then,*

$$F^2_{I,(d_1,d_2)} \cong \widetilde{\mathcal{M}}^{d_1}_{0,|I|+1} \times \widetilde{\mathcal{M}}^{d}_{0,3} \times \widetilde{\mathcal{M}}^{d_2}_{0,|I^c|+1}.$$

Moreover, the isomorphism of spaces gives rise to the following isomorphism of line bundles:

$$
\begin{aligned}
(\det R\pi_*\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\bigotimes_i \mathrm{ev}_i^* \mathbb{C}_{\lambda_i} \otimes \mathcal{L}_i^{a_i}\right) \;\cong\; & \left((\det R\pi_*\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\bigotimes_{i\in I} \mathrm{ev}_i^* \mathbb{C}_{\lambda_i} \otimes \mathcal{L}_i^{a_i}\right)\right) \\
& \boxtimes \left((\det R\pi_*\varphi^*\mathbb{C}_1)^{-q} \otimes \mathrm{ev}_n^* \mathbb{C}_{\lambda_n} \otimes \mathcal{L}_n^{a_n}\right) \\
& \boxtimes \left((\det R\pi_*\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\bigotimes_{i\in I^c} \mathrm{ev}_i^* \mathbb{C}_{\lambda_i} \otimes \mathcal{L}_i^{a_i}\right)\right).
\end{aligned}
$$

In particular, for any admissible class, α, over $F^2_{I,(d_1,d_2)}$, there exist admissible classes α_1, α_2, and α_3 on $\widetilde{\mathcal{M}}_{0,|I|+1}$, $\widetilde{\mathcal{M}}_{0,3}$, and $\widetilde{\mathcal{M}}_{0,|I^c|+1}$, respectively, such that

$$\alpha \cong \alpha_1 \boxtimes \alpha_2 \boxtimes \alpha_3.$$

Propositions 1.10, 1.11, and the analysis in this section give us the following important result.

Proposition 1.12. *Let α be an admissible class on $\widetilde{\mathcal{M}}_{0,n}$. Let $\widetilde{\mathcal{Z}}_{0,n} \subset \widetilde{\mathcal{M}}_{0,n}$ be the complement of the image of the universal curve, $\widetilde{C}_{0,n-1}$, over $\widetilde{\mathcal{M}}_{0,n-1}$. Then, $\chi_{\widetilde{\mathcal{Z}}_{0,n}}(\alpha)$ can be written as a finite sum of products of $\chi(\widetilde{\mathcal{M}}_{0,n'}, \alpha')$, where α' is an admissible class on $\widetilde{\mathcal{M}}_{0,n'}$ with $n' < n$.*

1.12 String equation and divisor relations on $\widetilde{\mathcal{M}}_{0,n}$

Section 1.11 made the first step towards reconstructing gauge Gromov-Witten invariants by showing that the local cohomology of an admissible class over the complement of $\widetilde{C}_{0,n-1}$ in $\widetilde{\mathcal{M}}_{0,n}$ can be computed as a finite sum of fewer pointed invariants. Recall that

$$\chi(\widetilde{\mathcal{M}}_{0,n}, \alpha) = \chi(\widetilde{C}_{0,n-1}, \alpha) + \chi_{\widetilde{\mathcal{Z}}_{0,n}}(\alpha).$$

If we can show that $\chi(\widetilde{C}_{0,n-1}, \alpha)$ can be reconstructed from fewer pointed invariants, we will have proven the reconstruction theorem for gauge Gromov-Witten invariants.

Showing that $\chi(\widetilde{C}_{0,n}, \alpha)$ can be reconstructed from fewer pointed gauge Gromov-Witten invariants is similar to the proof of the reconstruction theorem for quantum K-invariants from [10]. The open stratum, $\widetilde{C}_{0,n} \subset \widetilde{\mathcal{M}}_{0,n+1}$, is the universal curve over $\widetilde{\mathcal{M}}_{0,n}$ and thus, we have $\pi_n : \widetilde{C}_{0,n} \to \widetilde{\mathcal{M}}_{0,n}$.

In Section 1.5, we showed that the determinant line bundles and the evaluation line bundles at points other than the $(n+1)$-st point restricted to $\widetilde{C}_{0,n}$ can all be written as pull-backs of corresponding line bundles on $\widetilde{\mathcal{M}}_{0,n}$. In this section, we will prove relations concerning the $(n+1)$-st evaluation bundle and the cotangent line bundles $\mathcal{L}_i$.

String equation

First, we study the relation between the cotangent line bundles. More precisely, how do the line bundles $\mathcal{L}_i|_{\widetilde{C}_{0,n}}$ and $\pi_n^*\ell_i$ differ, where $\mathcal{L}_i$ and ℓ_i are the cotangent line bundles for $\widetilde{\mathcal{M}}_{0,n+1}$ and $\widetilde{\mathcal{M}}_{0,n}$, respectively, and $\pi_n : \widetilde{C}_{0,n} \to \widetilde{\mathcal{M}}_{0,n}$?

We recall the string equation for $\overline{\mathcal{M}}_{0,n}$, which answer the analogous question in the regular Gromov-Witten theory.

Theorem 1.9. *[14] Let $\pi : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ be the forgetful morphism forgetting the last marked point. Let L_i and ℓ_i be the cotangent bundles along the i-th marked point on $\overline{\mathcal{M}}_{0,n+1}$ and $\overline{\mathcal{M}}_{0,n}$ respectively. Then, we have the equality*

$$L_i \cong \pi^*\ell_i \otimes \mathcal{O}(D_i),$$

where D_i is the divisor whose generic point parametrizes a curve with two components. One of the components contains i-th and the last marked point, and the other contains the rest.

A similar relation exist for the moduli stack of Gieseker bundles.

Proposition 1.13. *Let $\mathcal{L}_i$ be the i-th relative cotangent bundle on $\widetilde{\mathcal{M}}_{0,n+1}$ and let ℓ_i be the i-th relative cotangent bundle on $\widetilde{\mathcal{M}}_{0,n}$. Recall that we have an embedding $\widetilde{C}_{0,n} \to \widetilde{\mathcal{M}}_{0,n+1}$. Let D_i be the divisor on $\widetilde{\mathcal{M}}_{0,n+1}$ whose generic curve has two components, one of which contains the i-th and the $(n+1)$-st marked points and the other component contains the rest. Then, we have the relation,*

$$\mathcal{L}_i|_{\widetilde{C}_{0,n}} \cong \pi_n^*\ell_i \otimes \mathcal{O}(D_i)|_{\widetilde{C}_{0,n}},$$

Proof. Note that for all i, $\mathcal{L}_i \cong F^*L_i$ where $F : \widetilde{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n}$ is the forgetful morphism forgetting the line bundle. Moreover, since $\overline{\mathcal{M}}_{0,n+1}$ is the universal curve over $\overline{\mathcal{M}}_{0,n}$, we have the following diagram:

$$\begin{array}{ccc} \widetilde{C}_{0,n} & \xrightarrow{F} & \overline{\mathcal{M}}_{0,n+1} \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi} \\ \widetilde{\mathcal{M}}_{0,n} & \xrightarrow{F} & \overline{\mathcal{M}}_{0,n} \end{array}$$

where $F : \widetilde{C}_{0,n} \to \overline{\mathcal{M}}_{0,n+1}$ factors through $\widetilde{C}_{0,n} \to \widetilde{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n+1}$. Hence, the relation above still holds true for $\widetilde{\mathcal{M}}_{0,n}$. More precisely, let $\mathcal{L}_i$ and ℓ_i be the cotangent bundles along i-th marked point on $\widetilde{\mathcal{M}}_{0,n+1}$ and $\widetilde{\mathcal{M}}_{0,n}$, respectively. Since the cotangent bundles are simply pullbacks of the cotangent bundles over $\widetilde{\mathcal{M}}_{0,n}$, we conclude that over $\widetilde{C}_{0,n}$, we have the relation

$$\mathcal{L}_i|_{\widetilde{C}_{0,n}} \cong \pi_n^* \ell_i \otimes O(D_i)|_{\widetilde{C}_{0,n}},$$

where D_i the divisor on $\widetilde{\mathcal{M}}_{0,n+1}$ whose generic curve has two components, one of which contains the i-th and the last marked point and the other contains the rest. $\square$

Divisor relations

We now look at the bundles involving the $(n+1)$-st marked point, namely $\mathrm{ev}_{n+1}^* \mathbb{C}_\lambda$ and $\mathcal{L}_{n+1}$. Since the relations between the restriction and pull-backs for the evaluation bundles and the cotangent bundles are well understood for all other points, we will prove relations between the evaluation bundles and cotangent bundles at different marked points. Once we understand the relations between the evaluation bundles and cotangent bundles at different marked points, we will be able to reduce any admissible line bundle to one which does not involve bundles at the $(n+1)$-st marked point.

We first recall two divisor relations that hold in $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, \beta))$.

Theorem 1.10. *[10] Let $D_{S_1,\beta_1|S_2,\beta_2}$ denote the divisor in $\overline{\mathcal{M}}_{0,n}$ parametrizing maps whose domain $C = C_1 \cup C_2$ is reducible such that the marked points $p_k \in S_i$ lies on C_i, and the class of the image of C_i is β_i with $\beta_1 + \beta_2 = \beta$. Now, we define*

$$D_{i,\beta_1|j,\beta_2} = \sum_{i \in S_1, j \in S_2} D_{S_1,\beta_1|S_2,\beta_2}$$

$$D_{i|j} = \sum_{i \in S_1, j \in S_2, \beta_1+\beta_2=\beta} D_{S_1,\beta_1|S_2,\beta_2}$$

Let $L \in \mathrm{Pic}(\mathbb{P}^r)$ and $i \neq j$. Then the following relations hold in $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, \beta))$.

1. $\mathrm{ev}_i^* L = ev_j^* L + \langle \beta, L \rangle \mathcal{L}_j - \sum_{\beta_1+\beta_2=\beta} \langle \beta_1, L \rangle D_{i,\beta_1|j,\beta_2}$,
2. $\mathcal{L}_i + \mathcal{L}_j = D_{i|j}$.

Similar relations hold for the moduli stack of Gieseker bundles. Let $\widetilde{\mathcal{M}}_{0,n,E} \subset \widetilde{\mathcal{M}}_{0,n}$ be the connected component whose points parametrize bundles of total degree

$E \in Z$. Let $\mathcal{L}_i$ denote the i-th cotangent bundle. Also, let $D_{i|j}$ denote the divisor whose generic point is a reducible curve where the i-th and j-th marked points are on distinct components. Finally, let $D_{i,d|j,E-d}$ be the divisor whose generic point parametrizes a reducible curve, say $C \cup C'$ such that $p_i \in C, p_j \in C'$ and the degree of the line bundle restricted to C and C' are d and $E - d$, respectively. Then, we have the following proposition.

Proposition 1.14. *Let $\lambda \in \mathbb{Z}$ and let $i \neq j = 1, \ldots, n$. Then, the following relations hold in* $\mathrm{Pic}(\widetilde{\mathcal{M}}_{0,n,E})$.

1. $ev_i^*\mathbb{C}_\lambda = ev_j^*\mathbb{C}_\lambda + \lambda E \mathcal{L}_j - \sum_{d \in \mathbb{Z}} \lambda d D_{i,d|j,E-d}$
2. $\mathcal{L}_i + \mathcal{L}_j = D_{i|j}$

Proof. Proof of the second relation follows directly from the fact that $\mathcal{L}_i$ on $\widetilde{\mathcal{M}}_{0,n}$ is the pull back of the corresponding cotangent line bundle on $\overline{\mathcal{M}}_{0,n}$. Since the relation holds in $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$, we conclude that the relation holds in $\mathrm{Pic}(\widetilde{\mathcal{M}}_{0,n,E})$ simply by pulling back the relation from $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$.

The proof of the first relation is almost identical to the proof for the case of $\overline{M}_{0,n}$ in [10].

To show that the first relation is true, it suffices to show that the relation is true when restricted to curves B on $M := \widetilde{M}_{0,n,E}$ which intersect boundary divisors transversely. Given, $B \to M$, we have the following cartesian diagram:

$$\begin{array}{ccccc} S & \longrightarrow & \widetilde{C} & \xrightarrow{\varphi} & [pt/\mathbb{C}^\times] \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} & & \\ B & \longrightarrow & M & & \end{array}$$

By pulling back sections $\sigma_i, \sigma_j : M \to \widetilde{C}$, we get two sections of $S \to B$, say s_i and s_j. Note that for $k = i, j$ we have

$$\langle B, \mathrm{ev}_k^* \mathbb{C}_\lambda \rangle = \langle s_k, \varphi^* \mathbb{C}_\lambda \rangle,$$

$$\langle B, \mathcal{L}_k \rangle = -\langle s_k, s_k \rangle.$$

Since S is the fiber product $B \times_M \widetilde{C}$, we know that S is a $\mathbb{P}^1$-bundle over B blown up at points where B intersects the boundary divisors of M. Since boundary divisors whose points parametrize reducible curves with more than 2 components have codimension strictly greater than 1, we can assume B meets only the boundary divisors whose points parametrize reducible curves with 2 components. Therefore, S is a $\mathbb{P}^1$ bundle, P, over B blown up at points such that each reducible fiber is a union of two (-1)-curves. Let $P = \mathbb{P}(V)$, where V is a rank two vector bundle over B. Let $Sing \subset B$ denote the points where B intersects a boundary divisor of M. Then,

$$\mathrm{Pic}(S) \cong \mathrm{Pic}(P) \oplus \left(\oplus_{b\in Sing}\mathbb{Z}_b\right) \cong \mathrm{Pic}(B) \oplus \mathbb{Z} \oplus \left(\oplus_{b\in Sing}\mathbb{Z}_b\right).$$

In particular, every line bundle on S can be written as $\pi^*\mathcal{L} \otimes \mathcal{O}(d)\left(-\sum_{b\in Sing} e_b E_b\right)$ where E_b is the exceptional divisor over $b \in Sing$. Write the line bundles corresponding to divisors s_i and s_j as $\pi^* L_i \otimes \mathcal{O}(1)\left(-\sum \delta^i_b E_b\right)$ and $\pi^* L_j \otimes \mathcal{O}(1)\left(-\sum \delta^j_b E_b\right)$, where $\delta^i_b = 1$ if s_i intersects E_b and 0 otherwise, and similarly for δ^j_b. We can also write $\varphi^*\mathbb{C}_\lambda$ as $\pi^* L \otimes \mathcal{O}(\lambda E)\left(-\sum_b \lambda d_b E_b\right)$ where d_b is the degree of the universal bundle restricted to E_b. Then, intersection theory on S gives:

$$\begin{aligned}
\langle s_i, \varphi^*\mathbb{C}_\lambda\rangle &= \deg L + \lambda E \deg L_i + \lambda E c_1(V) - \sum \lambda d_b \delta^i_b,\\
\langle s_j, \varphi^*\mathbb{C}_\lambda\rangle &= \deg L + \lambda E \deg L_j + \lambda E c_1(V) - \sum \lambda d_b \delta^j_b,\\
\langle s_i, s_i\rangle &= 2\deg L_i + c_1(V) - \sum \delta^i_b,\\
\langle s_j, s_j\rangle &= 2\deg L_j + c_1(V) - \sum \delta^j_b.
\end{aligned}$$

Moreover, since s_i and s_j never intersect, we have

$$\langle s_i, s_j\rangle = 0 = \deg L_i + \deg L_j + c_1(V) - \sum \delta^i_b \delta^j_b.$$

Let $Sing(i)$ be the points $b \in B$ such that s_i intersects E_b but s_j does not, and let $Sing(j)$ be the points such that s_j intersects E_b but s_i does not. Then, we have

$$\langle s_i, \varphi^*\mathbb{C}_\lambda\rangle - \langle s_j, \varphi^*\mathbb{C}_\lambda\rangle + \lambda E\langle s_j, s_j\rangle = \lambda E(\deg L_i - \deg L_j) - \sum_{b\in Sing(i)} \lambda d_b + \sum_{b\in Sing(j)} \lambda d_b + \lambda E\langle s_j, s_j\rangle$$

$$\begin{aligned}
&= \lambda E\left(-\deg L_j - c_1(V) + \sum \delta^i_b\delta^j_b - \deg L_j\right) - \sum_{b\in Sing(i)} \lambda d_b + \sum_{b\in Sing(j)} \lambda d_b + \lambda E\langle s_j, s_j\rangle\\
&= \lambda E \sum(\delta^i_b\delta^j_b - \delta^j_b) - \sum_{b\in Sing(i)} \lambda d_b + \sum_{b\in Sing(j)} \lambda d_b\\
&= -\sum_{b\in Sing(i)} \lambda d_b + \lambda \sum_{b\in Sing(j)} (E - d_b)\\
&= -\lambda\left(\sum_{b\in Sing(i)} d_b + \sum_{b\in Sing(j)} (E - d_b)\right).
\end{aligned}$$

Therefore, we conclude that

$$\langle s_i, \varphi^*\mathbb{C}_\lambda\rangle = \langle s_j, \varphi^*\mathbb{C}_\lambda\rangle - \lambda E\langle s_j, s_j\rangle - \lambda \sum_{b\in Sing(i)} d_b - \lambda \sum_{b\in Sing(j)} (E - d_b).$$

Hence,

$$\langle B, \mathrm{ev}_i^* \mathbb{C}_\lambda\rangle = \langle B, \mathrm{ev}_j^* \mathbb{C}_\lambda\rangle + \lambda E\langle B, \mathcal{L}_j\rangle - \lambda\left(\sum_{b\in Sing(i)} d_b + \sum_{b\in Sing(j)} (E - d_b)\right).$$

Since this holds for all B meeting boundary divisors transversely, we conclude that in $\mathrm{Pic}(M)$ we have the relation

$$\mathrm{ev}_i^* \mathbb{C}_\lambda = \mathrm{ev}_j^* \mathbb{C}_\lambda + \lambda E\mathcal{L}_j - \sum_{d\in\mathbb{Z}} \lambda d D_{i,d|j,E-d}.$$

□

1.13 Reduction to boundary loci

Using the results of Section 1.12 we can reduce computing the Euler characteristic of an admissible class, $\chi(\widetilde{C}_{0,n}, \alpha)$, where

$$\alpha = (\det R\pi_*\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\bigotimes_{i=1}^{n+1} \mathrm{ev}_i^* \mathbb{C}_{\lambda_i} \otimes \mathcal{L}_i^{a_i}\right),$$

to computing the Euler characteristic, $\chi(\widetilde{C}_{0,n}, \alpha')$, where

$$\begin{aligned}\alpha' \quad = \quad & (\det R\pi_*\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\bigotimes_{i=1}^{n} \mathrm{ev}_i^* \mathbb{C}_{\lambda_i} \otimes \pi_n^*\ell_i^{a_i} \otimes \mathcal{O}(a_iD_i)|_{\widetilde{C}_{0,n}}\right) \\ & \otimes \left(\mathrm{ev}_1^* \mathbb{C}_{\lambda_{n+1}} \otimes \mathcal{L}_1^{\lambda_{n+1}E} \otimes \left(\bigotimes_{d\in\mathbb{Z}} \mathcal{O}(-\lambda_{n+1} d D_{n+1,d|1,E-d})\right)\right) \otimes \left(\mathcal{L}_1^{-1} \otimes \mathcal{O}(D_{i|j})\right)^{a_{n+1}},\end{aligned}$$

with $E \in \mathbb{Z}$ such that the weight of the $\mathbb{C}^\times$-action on $\widetilde{\mathcal{M}}_{0,n+1}$ is 0. Hence, it suffices to compute $\chi(\widetilde{C}_{0,n}, \alpha')$ for a class, α', on $\widetilde{C}_{0,n}$ of the form

$$\alpha' = (\det R\pi_*\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\bigotimes_{i=1}^{n} \mathrm{ev}_i^* \mathbb{C}_{\lambda_i} \otimes \pi_n^*\ell_i^{a_i}\right) \otimes \left(\bigotimes_{j\in J} \mathcal{O}(d_jB_j)\right),$$

where $\{B_j \mid j \in J\}$ is a countable set of boundary divisors on $\widetilde{C}_{0,n}$ that are either D_i, $D_{i|n+1}$, or $D_{i,d|n+1,E-d}$ for some $i \in [n]$ and $d_j \in \mathbb{Z}$.

General results from K-theory

We now recall a few results from K-theory to address the $\mathcal{O}(d_jB_j)$ terms.

Proposition 1.15. *Let X be a scheme and let $D \subset X$ be a divisor. Then, we have the following relation in K-theory for all $n \in \mathbb{Z}_{\geq 0}$:*

$$\mathcal{O}(-nD) = \mathcal{O}_X - \mathcal{O}_{nD}.$$

Proof. The relation follows immediately from the exact sequence

$$0 \to \mathcal{O}(-nD) \to \mathcal{O}_X \to \mathcal{O}_{nD} \to 0.$$

□

Lemma 1.6. *[9] Let $D = \cup_1^k D_i$ be a divisor with normal crossing, such that D_i are smooth. Then,*

$$0 \to \mathcal{O}_D \to \sum_i \mathcal{O}_{D_i} \to \sum_{i<j} \mathcal{O}_{D_i \cap D_j} \to \cdots \to \mathcal{O}_{\cap D_i} \to 0$$

is an exact sequence.

Proof. We reproduce the proof of the lemma from [9] to deduce a slightly more general corollary.

The relation is equivalent to the exactness of the following sequence at the origin

$$0 \to \mathcal{O}/(x_1 \cdots, x_k) \to \sum_i \mathcal{O}/(x_i) \to \cdots \to \mathcal{O}/(x_1 \cdots x_k) \to 0.$$

For $k = 2$, we have the sequence

$$0 \to \mathcal{O}/(x_1x_2) \to \mathcal{O}/(x_1) \oplus \mathcal{O}/(x_2) \to \mathcal{O}/(x_1, x_2) \to 0,$$

which is exact. By the inclusion-exclusion principle, the case of $k \geq 3$ is proven. Exactness of the sequence away from the origin is a simple induction since we have fewer divisors meeting transversely. □

Corollary 1.2. *Let $D_1, \ldots, D_k$ be smooth divisors, meeting in normal crossings. Then,*

$$0 \to \mathcal{O}_D \to \sum_i \mathcal{O}_{n_iD_i} \to \sum_{i<j} \mathcal{O}_{n_iD_i \cap n_jD_j} \to \cdots \to \mathcal{O}_{\cap n_iD_i} \to 0$$

is an exact sequence.

Proof. The proof is identical to the proof of Lemma 1.6, where instead of x_i we have $x_i^{n_i}$. □

Stratification of the boundary loci

Using the results stated above, we can reduce the original problem further to computing $\chi(\Sigma, \alpha)$, where Σ is a boundary locus and α is the restriction of an admissible bundle, possibly tensored with normal bundle to other boundary loci. More precisely, $\Sigma \subset \widetilde{C}_{0,n} \subset \widetilde{\mathcal{M}}_{0,n+1}$ is the intersection of boundary divisors $D_i, D_{i|j}$, and $D_{i,d|j,E-d}$. And α is the restriction to Σ of

$$(\det R\pi_*\varphi^*\mathbb{C}_1)^{-q} \otimes \left(\otimes_{i=1}^n \operatorname{ev}_i^n \mathbb{C}_{\lambda_i} \otimes \pi_n^*\ell_i\right) \otimes \alpha',$$

where α' the tensor product of normal bundles to other boundary divisors.

Let Σ be the intersection of boundary divisors of type $D_i, D_{i|j}$, and $D_{i,d|j,E-d}$. Then, the boundary divisors that contain Σ prescribe the splitting type of curves parametrized by Σ. In other words, Σ is the locus whose generic point parametrizes curves of certain splitting type[10]. Now, by Proposition 1.3, we can further stratify Σ as the disjoint union of strata

$$\Sigma = \bigcup_{(\gamma,d)} \mathcal{M}_{(\gamma,d)},$$

where $\mathcal{M}_{(\gamma,d)}$ is the stratum whose generic point parametrizes a curve with the topological type (γ, d) satisfying the prescribed splitting type. However, since Σ is closed in $\widetilde{C}_{0,n}$, we know that for each $\mathcal{M}_{(\gamma,d)}$, $\overline{\mathcal{M}_{(\gamma,d)}} \cap \widetilde{C}_{0,n} \subset \Sigma$. Therefore, we have

$$\Sigma = \bigcup_{(\gamma,d)} \overline{\mathcal{M}_{(\gamma,d)}} \cap \widetilde{C}_{0,n}.$$

To simplify notation, we will write $\mathcal{N}_{(\gamma,d)} := \overline{\mathcal{M}_{(\gamma,d)}} \cap \widetilde{C}_{0,n}$. Note that $\mathcal{N}_{(\gamma,d)}$ are not mutually disjoint in general. In fact, we have already seen strata of this form when stratifying $\widetilde{\mathcal{Z}}_{g,n}$. Type I, II, III curves all corresponded to a particular splitting type and we stratified each by countably many disjoint locally closed subsets, $\Sigma = \cup_{i\in\mathbb{Z}}\Sigma_i$, such that Σ_i is closed in $\Sigma\backslash\cup_{0\leq j<i}\Sigma_j$ for $i \geq 0$ and in $\Sigma\backslash\cup_{i<j\leq 0}\Sigma_j$ for $i < 0$. Moreover, for each j, Σ_j contained a closed subset (which we denoted by F^1, F^2, and F^3) over which Σ_j was an affine bundle.

We now stratify Σ using the same strategy as the one employed in Section 1.6. Since each $\mathcal{N}_{(\gamma,d)}$ is closed in Σ, and there are at most countably many such $\mathcal{N}_{(\gamma,d)}$, any enumeration of the $\mathcal{N}_{(\gamma,d)}$, say $\Sigma'_i = \mathcal{N}_{(\gamma_i,d_i)}$ for $i \in \mathbb{Z}_{\geq 0}$ allows us to define

[10]Possibly with a prescribed degree splitting

$\Sigma_i := \Sigma'_i \setminus \left(\cup_{0\le j<i}\Sigma_j\right)$. Then, we see that Σ_i are disjoint sets such that $\cup\Sigma_i = \Sigma$, and Σ_i is closed in $\Sigma \setminus \left(\cup_{0\le j<i}\Sigma_j\right)$.

First, we show that we can cover Σ with $\mathcal{N}_{(\gamma,d)}$, where γ is a modular graph with no Gieseker bubbles.

Proposition 1.16. *We can find a collection of* (γ, d) *such that*

1. $\Sigma = \bigcup \mathcal{N}_{(\gamma,d)}$,
2. *For all* $(\gamma, d) \neq (\gamma', d')$, $\mathcal{N}_{(\gamma,d)} \not\subset \mathcal{N}_{(\gamma',d')}$, *and*
3. (γ, d) *contains no Gieseker bubbles.*

Proof. The idea of the proof is very simple. Σ corresponds to a splitting type with possible degree splitting. However, since Gieseker bubbles do not affect the splitting type, if a topological type (γ, d) containing Gieseker bubble is in Σ, we can contract those bubbles and still stay in Σ.

Suppose (γ, d) contains a Gieseker bubble. Then by Lemma 1.1, we know that there exist deformations from (γ, d) to (γ', d'), where (γ', d') contains no Gieseker bubbles. These deformations are precisely the ones where we contract all the Gieseker bubbles and transfer the degrees to the adjacent components. In particular, $\mathcal{M}_{(\gamma,d)} \subset \overline{\mathcal{M}_{(\gamma',d')}}$. Now, note that if (γ, d) satisfies the splitting type prescribed by the boundary locus, Σ, at least one of the topological types (γ', d') obtained by contracting the Gieseker bubbles must also satisfy the splitting type of Σ.

In fact, there exists (γ', d') such that $\mathcal{M}_{(\gamma',d')} \subset \Sigma \cap \widetilde{C}_{0,n}$. Suppose there does not exist deformations of (γ, d) in $\widetilde{C}_{0,n}$. That means all deformations (γ', d') of (γ, d) are such that either $\mathcal{M}_{(\gamma',d')} \subset Z^1$ or Z^2 as defined in Section 1.6. However, that means that $\mathcal{M}_{(\gamma,d)}$ must in fact be one of F^1 or F^2. In particular, $\mathcal{M}_{(\gamma,d)} \not\subset \widetilde{C}_{0,n}$, which is a contradiction.

Therefore, we can find $\mathcal{M}_{(\gamma',d')} \subset \Sigma \cap \widetilde{C}_{0,n}$ such that $\mathcal{M}_{(\gamma,d)} \subset \overline{\mathcal{M}_{(\gamma',d')}}$. Hence, even after discarding all (γ, d) that contain Gieseker bubbles, we still have

$$\Sigma = \bigcup \left(\overline{\mathcal{M}_{(\gamma,d)}} \cap \widetilde{C}_{0,n}\right) = \bigcup \mathcal{N}_{(\gamma,d)}.$$

Now, we discard all $\mathcal{N}_{(\gamma,d)}$ for which there exists a topological type, (γ', d'), with $\mathcal{N}_{(\gamma,d)} \subset \mathcal{N}_{(\gamma',d')}$. Then, we obtain a collection of topological types (γ, d) satisfying the conditions of the proposition. $\square$

From now on, we will always assume that we have a collection of topological types, (γ, d), satisfying the conditions in Proposition 1.16.

Note that Σ is the intersection of boundary divisors specifying splitting types with possible degree condition. Hence, we can write

$$\Sigma = \bigcup_{d} \mathcal{N}_{(\gamma,d)},$$

where

1. γ is determined by Σ and is the same for all $\mathcal{N}_{(\gamma,d)}$,
2. γ contains no Gieseker bubbles, and
3. d satisfies degree conditions determined by Σ of the type $d(v_i) = d_i$ for a subset of the vertices $v_i \in V(\gamma)$ [11].

Now, we define a stratification of Σ into locally closed strata. Choose an arbitrary $\Sigma'_0 := \mathcal{N}_{(\gamma,d_0)}$. Now, consider the set

$$\Gamma_0 = \{(\gamma, d) \mid \exists! v_i \in V(\gamma) \text{ such that } d_0(v_i) - d(v_i) = \pm 1 \text{ and } d_0(v_{i+1}) - d(v_{i+1}) = \mp 1\}.$$

In other words, Γ_0 is the set of topological types obtained from (γ, d_0) by transferring a single degree from one of the vertices to an adjacent vertex. If there exist $(\gamma, d) \in \Gamma_0$ such that $\mathcal{N}_{(\gamma,d)} \subset \Sigma$, we let them be $\Sigma'_i = \mathcal{N}_{(\gamma,d_i)}$ for $i = 1, \ldots, n_0$. If there does not exist such (γ, d), we let Σ'_1 be an arbitrary $\mathcal{N}_{(\gamma,d_1)}$ distinct from Σ'_0. Now, define

$$\Gamma_1 = \{(\gamma, d) \mid \exists! v_i \in V(\gamma) \text{ such that } d_1(v_i) - d(v_i) = \pm 1 \text{ and } d_1(v_{i+1}) - d(v_{i+1}) = \mp 1\}.$$

Similarly as before, we let $\Sigma'_{n_0+1}, \ldots, \Sigma'_{n_0+n_1}$ be those $\mathcal{N}_{(\gamma,d')}$ such that $(\gamma, d') \in \Gamma_1$. Note that $\Gamma_0 \cap \Gamma_1$ is not empty in general. Hence, we only choose those $\mathcal{N}_{(\gamma,d')}$ that have not been chosen before.

Since $V(\gamma)$ is finite, so are the sets Γ_i and thus, each step in the enumeration process above is finite. Hence, we get an enumeration of the $\mathcal{N}_{(\gamma,d)}$ as $\Sigma'_i = \mathcal{N}_{(\gamma,d_i)}, i \in \mathbb{Z}_{\geq 0}$ by repeating the process above. Now, we recursively define

$$\Sigma_i := \Sigma'_i \setminus \left(\bigcup_{0 \leq j < i} \Sigma_j \right).$$

As before, the strata Σ_i have the property that

[11] Σ might not impose any degree condition.

1. $\Sigma = \bigcup_i \Sigma_i$, and

2. Σ_i is closed in $\Sigma \setminus \left(\bigcup_{0 \le j < i} \Sigma_j \right)$.

By repeatedly applying the long exact sequence from Lemma 1.2, we conclude that

$$\chi(\Sigma, \mathcal{F}) = \sum_i \chi_{\Sigma_i}(\mathcal{F})$$

for any sheaf $\mathcal{F}$ on Σ. In particular, computation of $\chi(\Sigma, \alpha)$ can be reduced to computing $\chi_{\Sigma_i}(\alpha)$, where α is the restriction to Σ of an admissible class.

1.14 Admissible classes on Σ_i

Before computing $\chi_{\Sigma_i}(\alpha)$, we first look at the geometry of Σ_i.

Proposition 1.17. *Let (γ, d) be a topological type containing no Gieseker bubbles and let $r + 1 = |V(\gamma)|$. Let γ' be a modular graph obtained from γ by inserting a Gieseker bubble between every pair of adjacent vertices. Let d' be a degree map on γ' such that there exists a deformation $(\gamma', d') \to (\gamma, d)$. Let Γ be the collection of all topological types (γ'', d'') such that the deformation above factors through (γ'', d'')[12]:*

$$(\gamma', d') \to (\gamma'', d'') \to (\gamma, d).$$

In other words, (γ'', d'') is a topological type obtained by contracting Gieseker bubbles of (γ', d') in a way that does not violate the degree map d. Then, $\bigcup_{(\gamma'',d'')\in\Gamma} \mathcal{M}_{(\gamma'',d'')}$ is an $\mathbb{A}^r$-bundle over $\mathcal{M}_{(\gamma',d')}$[13].

More precisely, let $\mathcal{B}_1, \ldots, \mathcal{B}_r$ be the r families of Gieseker bubbles over $\mathcal{M}_{(\gamma',d')}$. Now let p_i and q_i be the sections representing the loci of the two nodes on $\mathcal{B}_i$, where p_i is the locus of nodes that are getting smoothed away. Then, there exists a map

$$\eta : \bigcup_{(\gamma'',d'')\in\Gamma} \mathcal{M}_{(\gamma'',d'')} \to \mathcal{M}_{(\gamma',d')},$$

which is the structure map of the $\mathbb{A}^r$-bundle

$$\bigoplus \left((\mathcal{L}_i^{-1} \otimes \mathcal{P}_i) \boxtimes Q_i^{-1} \right),$$

where $\mathcal{L}_i^{-1}$ and $\mathcal{P}_i$ are the cotangent bundle and restriction of the universal bundle along p_i, and Q_i is the restriction of the universal bundle to q_i[14].

[12] Note that $(\gamma, d), (\gamma', d') \in \Gamma$.

[13] The proof of Proposition 6.2 in [1] briefly mentions that Z is an affine bundle over F. Z and F of Proposition 6.2 are precisely $\cup\mathcal{M}_{(\gamma'',d'')}$ and $\mathcal{M}_{(\gamma',d')}$.

[14] Both $\mathcal{P}_i$ and Q_i of the appropriate degree.

Moreover, $\overline{\mathcal{M}_{(\gamma,d)}}$ *is a* $(\mathbb{P}^1)^r$*-bundle over* $\mathcal{M}_{(\gamma',d')}$.

Proof. The deformation $(\gamma', d') \to (\gamma, d)$ is one where you contract the r connecting Gieseker bubbles. More precisely, for each connecting Gieseker bubble, the deformation smooths one of the two nodes that lie on each bubble. Moreover, the node that is being smoothed is prescribed by the degree maps d' and d. Now, note that $(\gamma'', d'') \in \Gamma$ are precisely the topological types obtained by smoothing a prescribed subset of the nodes. Hence, $\cup_\Gamma \mathcal{M}_{(\gamma'',d'')}$ is the set of points parametrizing deformations of curves of $\mathcal{M}_{(\gamma',d')}$ by smoothing r prescribed nodes. Therefore, by Proposition 1.5, we conclude that $\cup_\Gamma \mathcal{M}_{(\gamma'',d'')}$ is an $\mathbb{A}^r$-bundle over $\mathcal{M}_{(\gamma',d')}$.

In fact, the same argument as in Sections 1.11 and 1.20, $\mathcal{M}_{(\gamma',d')} \cong \widetilde{\mathcal{M}}_{0,n_1} \times \cdots \times \widetilde{\mathcal{M}}_{0,n_{r+1}}$, where $(n_1, \ldots, n_{r+1})$ is the splitting type of the modular graph γ'. Moreover, by the argument of the proofs of Propositions 1.6 and 1.7, we conclude that $\cup_\Gamma \mathcal{M}_{(\gamma'',d'')}$ is the total space of the $\mathbb{A}^r$-bundle on $\mathcal{M}_{(\gamma',d')}$

$$\bigoplus \left((\mathcal{L}_i^{-1} \otimes \mathcal{P}_i) \boxtimes Q_i^{-1} \right),$$

where $\mathcal{L}_i^{-1}$ and $\mathcal{P}_i$ are the cotangent bundle and restriction of the universal bundle along p_i, and Q_i is the restriction of the universal bundle to q_i. Note that we must choose the universal bundle of the appropriate degree for $\mathcal{P}_i$ and Q_i. More precisely, the degree of $\mathcal{P}_i$ is d_i, where d_i is the degree of the corresponding vertex prescribed by our topological type (γ', d').

Now, $\overline{\mathcal{M}_{(\gamma,d)}}$ is simply the stratum of points parametrizing all smoothings of the curves of $\mathcal{M}_{(\gamma',d')}$. However, this is precisely the $(\mathbb{P}^1)^r$-bundle over $\mathcal{M}_{(\gamma',d')}$ defined by

$$\prod_{i=1}^{r} \operatorname{Proj} \left(\left((\mathcal{L}_i^{-1} \otimes \mathcal{P}_i) \boxtimes Q_i^{-1} \right) \oplus O \right).$$

□

Corollary 1.3. Σ_i *is a* $\mathbb{A}^s \times (\mathbb{P}^1)^t$*-bundle over* $\mathcal{M}_{(\gamma',d')}$*, where* $|V(\gamma)| = r+1 \geq s+t+1$ *and* (γ', d') *is a topological type satisfying the following conditions.*

1. (γ', d') *is obtained from* (γ, d) *by inserting Gieseker bubbles between every pair of adjacent vertices such that there exists a deformation* $(\gamma', d') \to (\gamma, d)$*, and*

2. $\mathcal{M}_{(\gamma',d')} \subset \Sigma_i$.

Moreover, $\mathcal{M}_{(\gamma',d')}$ *is isomorphic to a product*

$$\mathcal{M}_{(\gamma',d')} \cong \widetilde{\mathcal{M}}_{0,n_1} \times \cdots \times \widetilde{\mathcal{M}}_{0,n_{r+1}},$$

where $(n_1, \ldots, n_{r+1})$ *is the splitting type of* (γ, d)*.*

Proof. Using the notation from Proposition 1.16, suppose $\Sigma_i' = \mathcal{N}_{(\gamma,d)}$. Suppose $|V(\gamma)| = r+1$. Then, we can insert at most r connecting Gieseker bubbles. Suppose k of the connecting Gieseker bubbles are allowed in Σ_i i.e. there exists a topological type (γ', d') with one of those k connecting Gieseker bubbles inserted such that $\mathcal{M}_{(\gamma',d')} \subset \Sigma_i'$. Note that for each of the k connecting Gieseker bubbles, there are two choices for the degree splitting. Since the universal bundle must have degree 1 on the Gieseker bubble, we must decrease the degree of one of the adjacent components by 1. Suppose for s of the allowed connecting Gieseker bubbles, only one such degree splitting is allowed in Σ_i. Then, for $t := k - s$ of the allowed connecting Gieseker bubbles, both degree splittings are allowed in Σ_i.

Now, we claim that for any degeneration (γ', d') of (γ, d) obtained by a combination of inserting any of the allowed Gieseker bubbles with the degree splitting, $\mathcal{M}_{(\gamma',d')} \subset \Sigma_i$. In other words, let (γ_1, d_1) and (γ_2, d_2) be topological types obtained from (γ, d) by inserting distinct bubbles B_i and decreasing the degree on the components C_i[15]. Then, if (γ_3, d_3) is obtained from (γ, d) by inserting both bubbles B_1 and B_2, and decreasing the degree of the components C_1 and C_2, $\mathcal{M}_{(\gamma_3,d_3)} \subset \Sigma_i$. However, this is true by the construction of Σ_i. In the enumeration process, we successively looked at *neighboring* topological types obtained by transferring a single degree from a component to one of the adjacent components. The transfer of degrees is done precisely through the connecting Gieseker bubble and thus, if two transfers of degrees via two distinct bubbles are each allowed in Σ_i, the degeneration by both operations must also be allowed in Σ_i.

Therefore,

$$\mathcal{M}_{(\gamma',d')} \cong \widetilde{\mathcal{M}}_{0,n_1} \times \cdots \times \widetilde{\mathcal{M}}_{0,n_{r+1}},$$

and Σ_i is an $\mathbb{A}^s \times (\mathbb{P}^1)^t$-bundle over $\mathcal{M}_{(\gamma',d')}$. Moreover, the normal bundle $N_{\Sigma_i/\widetilde{C}_{0,n}}$ restricted to $\mathcal{M}_{(\gamma',d')}$, and $N_{\mathcal{M}_{(\gamma',d')}/\Sigma_i}$ are isomorphic to the tensor product of the pulled back cotangent line bundles along the marked points of $\widetilde{\mathcal{M}}_{0,n_k}$ playing the roles of the nodes on the connecting Gieseker bubbles. $\square$

[15] C_1 and C_2 are not necessarily distinct.

Now, we show that admissible classes are well-defined over the boundary loci.

Proposition 1.18. *Let $\Sigma = \cup \Sigma_i$ be as above and let α be an admissible class. Then, $\chi_{\Sigma_i}(\alpha)$ is zero for all but finitely many i. Moreover, $\chi_{\Sigma_i}(\alpha)$ can be written as a product of lower pointed gauge Gromov-Witten invariants.*

Proof. We saw in Corollary 1.3 that Σ_i is a $\mathbb{A}^s \times (\mathbb{P}^1)^t$-bundle over $\mathcal{M}_{(\gamma',d')}$. Let $\widetilde{\Sigma_i}$ be the $(\mathbb{P}^1)^t$-bundle over $\mathcal{M}_{(\gamma',d')}$ such that Σ_i is an $\mathbb{A}^s$-bundle over $\widetilde{\Sigma_i}$. By the same argument as in the proof of in Proposition 1.12 of Section 1.11, we see that

$$\chi_{\Sigma_i}(\alpha) = \chi(\widetilde{\Sigma_i}, \alpha \otimes \det N_{\Sigma_i/\widetilde{C}_{0,n}} \otimes \operatorname{Sym} N_{\Sigma_i/\widetilde{C}_{0,n}} \otimes \operatorname{Sym} N^{-1}_{\widetilde{\Sigma}_i/\Sigma_i}).$$

Again by Proposition 6.2 of [1], we conclude that $\chi_{\Sigma_i}(\alpha)$ vanish for all but finitely many i[16].

Now, we know precisely which $(\mathbb{P}^1)^t$-bundle $\widetilde{\Sigma_i}$ is over $\mathcal{M}_{(\gamma',d')}$. Namely,

$$\widetilde{\Sigma_i} \cong \prod \operatorname{Proj}_j \left(\left((\mathcal{L}_{i_j}^{-1} \otimes \mathcal{P}_{i_j}) \boxtimes Q_{i_j}^{-1} \right) \oplus O \right),$$

where $\mathcal{L}_{i_j}$ is the cotangent bundle along one of the two nodes of the connecting Gieseker bubble, B_{i_j}, for which the degree transfer to either adjacent component is allowed in Σ_i, $\mathcal{P}_{i_j}$ is the restriction of the universal bundle on the same node, Q_{i_j} is the restriction of the universal bundle on the other node.

Let $\pi : \widetilde{\Sigma_i} \to \mathcal{M}_{(\gamma',d')}$. Since $\widetilde{\Sigma_i}$ is a $(\mathbb{P}^1)^t$-bundle over $\mathcal{M}_{(\gamma',d')}$, we know that any line bundle on $\widetilde{\Sigma_i}$ is isomorphic to $\pi^* L(e_1, \ldots, e_t)$. Moreover, since each of the $\mathbb{P}^1$ corresponds to transferring a single degree from one component to an adjacent component, the only $\mathbb{C}^\times$-weights that change are the weights on $\det R\pi_* \varphi^* \mathbb{C}_1$ over the $\widetilde{\mathcal{M}}_{0,n_i}$ corresponding to the affected components. In particular, we know that $e_i = \pm 1$ since the degree on the affected components change by exactly 1[17].

We compute $\chi(\widetilde{\Sigma_i}, \pi^* L(e_1, \ldots, e_t))$ in t steps, in each of which we project down along a $\mathbb{P}^1$ fiber. Hence, we reduce to the case where $\widetilde{\Sigma_i}$ is a $\mathbb{P}^1$-bundle over $\widetilde{\Sigma_i}'$ and the line bundle in question is $\pi^* L(e)$. By the projection formula, Theorem 1.8, we know that

$$R^i \pi_* (\mathcal{F} \otimes \pi^* \mathcal{E}) \cong R^i \pi_* (\mathcal{F}) \otimes \mathcal{E}.$$

[16] As in Section 1.11, the $\mathbb{C}^\times$-weights on the components will all be negative or positive for all but finitely many degree splittings on $V(\gamma)$.

[17] The sign of e_i depends on the particular (γ', d') we choose and whether the degree is transferred from or to C_i.

Hence,

$$R^i\pi_*(\pi^* L(e)) = L \otimes R^i\pi_*(\mathcal{O}(e)).$$

If $e = -1$, then we know that $R^i\pi_*(\mathcal{O}(-1)) = 0$ for all i. If $e = 1$, we know $R^i\pi_*(\mathcal{O}(1)) = 0$ for all $i \neq 0$. Moreover, $R^0\pi_*(\mathcal{O}(1)) \cong \mathcal{O}_0 \oplus \mathcal{O}_\infty$, where $\mathcal{O}_0$ and $\mathcal{O}_\infty$ are the structure sheaves of the zero and the infinity sections, respectively.

Hence, by pushing down to $\mathcal{M}_{(\gamma',d')} \cong \prod \widetilde{\mathcal{M}}_{0,n_i}$, we can compute $\chi(\Sigma_i, \alpha)$ as $\chi(\mathcal{M}_{(\gamma',d')}, \alpha')$, where α' is α restricted to $\mathcal{M}_{(\gamma',d')}$ possibly tensored by cotangent bundles along marked points of $\widetilde{\mathcal{M}}_{0,n_i}$. However, since $\mathcal{M}_{(\gamma',d')}$ is a product of the $\widetilde{\mathcal{M}}_{0,n_i}$ and α' is a tensor product of bundles associated with an admissible class, we conclude that

$$\chi(\mathcal{M}_{(\gamma',d')}, \alpha') = \prod \chi(\widetilde{\mathcal{M}}_{0,n_i}, \alpha_i),$$

where α_i is an admissible class on $\widetilde{\mathcal{M}}_{0,n_i}$. □

1.15 Proof of the reconstruction theorem

We are finally ready to prove the main theorem, Theorem 1.1. As mentioned before, the reconstruction theorem for genus 0 gauge Gromov-Witten invariants not only provides an alternate proof of well-definedness of genus 0 gauge Gromov-Witten invariants but also gives an explicit algorithm for computing the invariants from pointed invariants.

Proof of Theorem 1.1. Let α be an admissible class on $\widetilde{\mathcal{M}}_{0,n}$. We want to show that

$$\chi(\widetilde{\mathcal{M}}_{0,n}, \alpha) = \sum \left(\prod \chi(\widetilde{\mathcal{M}}_{0,n'}, \alpha') \right),$$

where the the right hand side is a finite sum of finite products of $\chi(\widetilde{\mathcal{M}}_{0,n'}, \alpha')$ with $n' < n$ and α' admissible on $\widetilde{\mathcal{M}}_{0,n'}$.

Consider the embedding $\widetilde{C}_{0,n-1} \to \widetilde{\mathcal{M}}_{0,n}$ defined in Section 1.5. As before, we define $\widetilde{\mathcal{Z}}_{0,n}$ to be the complement of $\widetilde{C}_{0,n-1}$ in $\widetilde{\mathcal{M}}_{0,n}$. By Lemma 1.2, we know that

$$\chi(\widetilde{\mathcal{M}}_{0,n}, \alpha) = \chi(\widetilde{C}_{0,n-1}, \alpha) + \chi_{\widetilde{\mathcal{Z}}_{0,n}}(\alpha),$$

if all the terms above are well-defined. By Proposition 1.12, we know that $\chi_{\widetilde{\mathcal{Z}}_{0,n}}(\alpha)$ can be written as a finite sum of finite products of lower pointed gauge Gromov-Witten invariants. Hence, it suffices to show that $\chi(\widetilde{C}_{0,n-1}, \alpha)$ can be written as a finite sum of finite products of lower pointed invariants.

Let $\pi : \widetilde{C}_{0,n-1} \to \widetilde{\mathcal{M}}_{0,n-1}$ be the map associated to the universal curve over $\widetilde{\mathcal{M}}_{0,n-1}$. By Propositions 1.13 and 1.14, the class of α is the same as the class of

$$\alpha' = (\det R\pi_* \varphi^* \mathbb{C}_1)^{-q} \otimes \left(\bigotimes_{i=1}^{n} \mathrm{ev}_i^* \mathbb{C}_{\lambda_i} \otimes \pi^* \ell_i^{a_i} \right) \otimes \left(\bigotimes \mathcal{O}(d_j B_j) \right),$$

Now, by Proposition 1.15 and Lemma 1.6, we conclude that $\chi(\widetilde{C}_{0,n-1}, \alpha)$ can be written as a sum

$$\chi(\widetilde{C}_{0,n-1}, \alpha) = \chi(\widetilde{C}_{0,n-1}, \alpha') + \sum \chi(\Sigma, \alpha_\Sigma),$$

where

1. α' is of the form $(\det R\pi_* \varphi^* \mathbb{C}_1)^{-q} \otimes \left(\bigotimes_{i=1}^{n-1} \mathrm{ev}_i^* \mathbb{C}_{\lambda_i} \otimes \pi^* \ell_i^{a_i} \right)$,
2. the sum is over a countable collection of boundary loci, Σ, each of which is an intersection of boundary divisors of the type D_i, $D_{i|j}$, and $D_{i,d|j,E-d}$ as defined in Proposition 1.14, and
3. α_Σ is the restriction to Σ of an admissible bundle on $\widetilde{\mathcal{M}}_{0,n}$, possibly tensored with normal bundles to other boundary divisors.

By the projection formula, Theorem 1.8, and Proposition 1.4, we see that $\chi(\widetilde{C}_{0,n-1}, \alpha')$ is equal to $\chi(\widetilde{\mathcal{M}}_{0,n-1}, \alpha'')$ where α'' is an admissible class on $\widetilde{\mathcal{M}}_{0,n-1}$.

Finally, by Proposition 1.18, we conclude that all but finitely many terms $\chi(\Sigma, \alpha_\Sigma)$ vanish. Moreover, each $\chi(\Sigma, \alpha_\Sigma)$ can be written as a finite product of $\chi(\widetilde{\mathcal{M}}_{0,n'}, \beta)$, where $n' < n$ and β is an admissible class on $\widetilde{\mathcal{M}}_{0,n'}$.

Therefore, $\chi(\widetilde{\mathcal{M}}_{0,n}, \alpha)$ can be written as the finite sum of products of lower pointed invariants. □

Corollary 1.4. *Genus 0 gauge Gromov-Witten invariants are well-defined.*

Proof. Since $\widetilde{\mathcal{M}}_{0,3} \cong [pt/\mathbb{C}^\times]$, $\chi(\widetilde{\mathcal{M}}_{0,3}, \alpha)$ is finite for any admissible class α on $\widetilde{\mathcal{M}}_{0,3}$. By Theorem 1.1, higher pointed invariants can be reconstructed from the 3-pointed invariants. Hence, we conclude that genus 0 gauge Gromov-Witten invariants are well-defined. □

1.16 Future directions

The proof of the reconstruction theorem leads to several questions. First, can we generalize the stratification of $\widetilde{\mathcal{M}}_{0,n}$ to $\widetilde{\mathcal{M}}_{0,n}([X/\mathbb{C}^\times])$? In [1], Frenkel, Teleman, and Tolland define the moduli space $\widetilde{\mathcal{M}}_{0,n}([X/\mathbb{C}^\times])$ and suggest existence of invariants. The stratification of $\widetilde{\mathcal{M}}_{0,n}$ is canonical and allows one to recursively compute invariants of $\widetilde{\mathcal{M}}_{0,n}$ from invariants of $\widetilde{\mathcal{M}}_{0,3}$. If we can stratify $\widetilde{\mathcal{M}}_{0,n}([X/\mathbb{C}^\times])$ in a similar way, we would then reduce the proof of well-definedness of the invariants to the case of $\widetilde{\mathcal{M}}_{0,3}([X/\mathbb{C}^\times])$. Since Frenkel, Teleman, and Tolland prove in [1] that $\widetilde{\mathcal{M}}_{g,n}([X/\mathbb{C}^\times]) \to \widetilde{\mathcal{M}}_{g,n}$ is proper, and since $\widetilde{\mathcal{M}}_{0,3} \cong [pt/\mathbb{C}^\times]$, we know that invariants of $\widetilde{\mathcal{M}}_{0,3}([X/\mathbb{C}^\times])$ are well-defined. Thus, reducing well-definedness of invariants to the case of $\widetilde{\mathcal{M}}_{0,3}([X/\mathbb{C}^\times])$ would prove existence of gauge Gromov-Witten invariants for arbitrary $[X/\mathbb{C}^\times]$.

Similarly, the stratification could be used to study the finiteness of invariants for $\widetilde{\mathcal{M}}_{g,n}$ for $g > 0$. The stratification used in the proof of reconstruction theorem breaks the proof of finiteness down to the finiteness of invariants for $\widetilde{\mathcal{M}}_{1,1}$ and $\widetilde{\mathcal{M}}_{g,0}$ for $g \geq 2$, and a study of the invariants over boundary divisors. A closer look at the boundary divisors might provide an alternative proof of well-definedness of gauge Gromov-Witten invariants for higher genus.

Another topic of interest is the S_n-equivariant K-theory of $\widetilde{\mathcal{M}}_{g,n}$, where S_n acts by permuting the marked points. Understanding the S_n-equivariant K-theory of $\widetilde{\mathcal{M}}_{g,n}$ is important because for higher genus, the boundary strata of $\widetilde{\mathcal{M}}_{g,n}$ are naturally quotients of products of strata under an action of S_n. Hence, to study the K-theory of $\widetilde{\mathcal{M}}_{g,n}$, we must understand the S_n-equivariant K-theory of lower genus, lower pointed spaces. Moreover, in [3], Givental studies the permutation equivariant K-theory of $\overline{M}_{0,n}$. It would be interesting to compare the S_n-equivariant K-theory of $\overline{M}_{0,n}$ with that of $\widetilde{\mathcal{M}}_{0,n}$.

Lastly, one could try to generalize gauge Gromov-Witten theory to cases where the target space is $[X/G]$ with an arbitrary group G. We would first need to define Gieseker G-bundles over stable curves, generalizing the definition of Gieseker $\mathbb{C}^\times$-bundles. Then, we can study the moduli space of Gieseker G-bundles over stable curves, $\widetilde{\mathcal{M}}_{g,n}([pt/G])$, and define gauge Gromov-Witten invariants. Once we establish a gauge Gromov-Witten theory for $[pt/G]$, one can then generalize the theory to the case of arbitrary $[X/G]$ and study whether there exist well-defined invariants.

In [12], Solis constructs toric varieties, a special case of which recovers the local model for $\widetilde{\mathcal{M}}_{0,4}([pt/\mathbb{C}^\times])$. It would be interesting to see if the higher dimensional toric varieties Solis constructs can admit a modular interpretation as local models of $\widetilde{\mathcal{M}}_{g,n}([pt/T])$, where T is a torus of arbitrary rank. These toric varieties could give us an idea as to what the appropriate definition of Gieseker T-bundles on stable curves should be. Moreover, they exhibit many similarities with $\widetilde{\mathcal{M}}_{g,n}([pt/\mathbb{C}^\times])$ and I plan to study whether a similar stratification exists.

Once we have a gauge Gromov-Witten theory for $[X/G]$, we can study its relation to the Gromov-Witten theory of GIT quotients. We do not impose stability conditions in the sense of geometric invariant theory to maps to $[X/G]$, which is the reason that the resulting moduli stack fails to be proper. In [1], Frenkel, Teleman, and Tolland conjecture that the Gromov-Witten invariants of the GIT quotients can be recovered from the gauged Gromov-Witten invariants by applying the Chern character to certain limits of the gauged invariants. The case of smooth curves and G-bundles was proven in [13].

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Chapter 2

GENERATING SERIES FOR THE POINCARE POLYNOMIALS OF QUOT SCHEMES AND $\overline{Q}_{0,2}(\mathbb{G}(n,n),d)$

2.1 Introduction

The moduli space of stable quotients was defined in [8] and can be interpreted as a compactification of the moduli space of curves to $\mathbb{G}(r,n)$ using the philosophy of Grothendieck Quot schemes. The space of stable quotients, $\overline{Q}_{g,m}(\mathbb{G}(n,r),d)$, is different from the Kontsevich stable maps compactification, $\overline{\mathcal{M}}_{g,m}(\mathbb{G}(n,r),d)$, and provides an alternate method of compactification. One interesting case is when $r=n$, when $\mathbb{G}(n,n)$ is simply a point. Hence, the space of stable maps, $\overline{\mathcal{M}}_{g,m}(\mathbb{G}(n,n),d)$ is equal to $\overline{\mathcal{M}}_{g,m}$. In particular, $\overline{\mathcal{M}}_{0,2}(\mathbb{G}(n,n),d)$ is empty for all n and d. However, $\overline{Q}_{0,2}(\mathbb{G}(n,n),d)$ is nonempty whenever $n\geq 1$ and $d\geq 0$.

We study the spaces $\overline{Q}_{0,2}(\mathbb{G}(n,n),d)$ as we vary n and d. In particular, we compute their Poincare polynomials. We first compute the Poincare polynomials of Quot schemes, $Quot_C(n,d)$, parametrizing torsion quotient sheaves of $O_C^{\oplus n}$ over projective curves. Then, we use the result to find the Poincare polynomials of $\overline{Q}_{0,2}(\mathbb{G}(n,n),d)$. In fact, we will show

Theorem 2.1. *1. Let C be a projective curve and let $Q_{C,n,d}(t)$ denote the Poincare polynomial of $Quot_C(n,d)$. Then, the generating series for $Q_{C,n,d}(t)$ is a rational function. More precisely, let*

$$Q_{C,n}(t,x):=\sum_d Q_{C,n,d}(t)x^d\in\mathbb{Z}[[x,t]]$$

be the generating series for $Q_{C,n,d}(t)$. Then, we have

$$Q_{C,n}(t,x)=\frac{[(1+tx)(1+t^3x)\cdots(1+t^{2n-1}x)]^{2g}}{(1-x)(1-t^{2n}x)[(1-t^2x)(1-t^4x)\cdots(1-t^{2n-2}x)]^2}.$$

2. Let $\overline{Q}_{n,d}(t)$ be the Poincare polynomial of $\overline{Q}_{0,2}(\mathbb{G}(n,n),d)$. Then, the generating series for $\overline{Q}_{n,d}(t)$ is a rational function. More precisely, let

$$\overline{Q}_n(t,x):=\sum_d \overline{Q}_{n,d}(t)x^d\in\mathbb{Z}[[t,x]]$$

be the generating series for $\overline{Q}_{n,d}(t)$. Then, we have

$$\overline{Q}_n(t,x)=\frac{(1-t^{2n}x)(t^2-1)}{t^2-t^{2n+2}x+x-1}.$$

2.2 Grothendieck ring of varieties and the Poincare polynomial

In this section, we introduce the Grothendieck ring of varieties and the Poincare polynomial of a topological space. The Grothendieck ring of varieties was first mentioned by Grothendieck in his correspondences with Serre, published in [1], and will be useful in computing the Poincare polynomials of Quot schemes that we examine later.

Definition 2.1. *Consider the free abelian group generated by the isomorphism classes of $\mathbb{C}$-varieties, $\mathbb{Z}[Var/\mathbb{C}]$. Then, the Grothendieck ring of varieties over $\mathbb{C}$ is the quotient of $\mathbb{Z}[Var/\mathbb{C}]$ by the relations*

$$[X] = [X \setminus Z] + [Z],$$

where $Z \subset X$ is a closed subscheme. The multiplication is defined by

$$[X] \cdot [Y] = [X \times Y].$$

The Grothendieck ring of $\mathbb{C}$-varieties is denoted $K(Var/\mathbb{C})$.

The multiplicative inverse in $K(Var/\mathbb{C})$ is the class of $\operatorname{Spec}\mathbb{C}$ and will be denoted by 1. Another important element of $K(Var/\mathbb{C})$ is the class of $\mathbb{A}^1$, which will be denoted $\mathbb{L}$.

The following relations can easily be derived from the definition of $K(Var/\mathbb{C})$ and their proofs can be found in [6].

Proposition 2.1. *[6] The following relations hold in $K(Var/\mathbb{C})$.*

1. *Let $X = \coprod X_i$ where X_i are mutually disjoint locally closed subschemes of X. Then,*
$$[X] = \sum_i [X_i].$$
2. *Suppose $Z \to X$ is a Zariski fibration with fibers isomorphic to Y. Then,*
$$[Z] = [X] \cdot [Y].$$

Knowing the class of X in $K(Var/\mathbb{C})$ is helpful in computing its Poincare polynomial, which we define now.

Definition 2.2. *Let X be a compact topological space and let b_i be its i-th Betti number for $i = 0, \ldots, 2\dim X$. Then, the Poincare polynomial of X is the polynomial*

$$P_X(t) = \sum b_i t^i \in \mathbb{Z}[t].$$

We now have the following proposition that says the Poincare polynomial can be extended to all of $K(Var/\mathbb{C})$.

Proposition 2.2. *[6] There exists a well-defined map $P : K(Var/\mathbb{C}) \to \mathbb{Z}[t]$ such that for any smooth projective scheme, X, $P_X(t) := P([X])(t)$ is equal to its Poincare polynomial. Moreover, P is a ring homomorphism. In particular,*

1. *If $[X] = [Y]$, then $P_X(t) = P_Y(t)$;*
2. *If $Y \subset X$ is a closed subscheme, then*

$$P_X(t) = P_Y(t) + P_{X\setminus Y}(t); \text{ and}$$

3. *For any X, Y,*

$$P_{X\times Y}(t) = P_X(t)P_Y(t).$$

We call $P([X])$ the virtual Poincare polynomial *of X.*

Let's compute the class and the Poincare polynomial for a few varieties. Recall that $\mathbb{L} := [\mathbb{A}^1]$. Since $\mathbb{A}^n$ can be viewed as an $\mathbb{A}^1$-bundle over $\mathbb{A}^{n-1}$, we see that

$$[\mathbb{A}^n] = \mathbb{L}^n.$$

And since we can stratify $\mathbb{P}^n$ as

$$\mathbb{P}^n = \cup_1^n \mathbb{A}^k,$$

in $K(Var/\mathbb{C})$ we have the following relation:

$$[\mathbb{P}^n] = 1 + \mathbb{L} + \mathbb{L}^2 + \cdots + \mathbb{L}^n.$$

Lastly, since $\mathbb{A}^1 = \mathbb{C}^\times \cup pt$ and $pt = \operatorname{Spec}\mathbb{C}$ plays the role of the multiplicative identity in $K(Var/\mathbb{C})$, we have

$$[\mathbb{C}^\times] = \mathbb{L} - 1.$$

Now, we compute the Poincare polynomials for $\mathbb{A}^n, \mathbb{P}^n$, and $\mathbb{C}^\times$. From [5], we know that for $\mathbb{A}^1$ the virtual Betti numbers are $b_0 = 0, b_1 = 0$, and $b_2 = 1$. In general, the virtual Betti numbers for $\mathbb{A}^n$ are

$$b_i = \begin{cases} 1 & i = 2n \\ 0 & i \neq 2n \end{cases}.$$

Hence, the Poincare polynomials for $\mathbb{A}^n$ are

$$P_{\mathbb{A}^n}(t) = t^{2n}.$$

Note that we have

$$P_{\mathbb{A}^n}(t) = (P_{\mathbb{A}^1}(t))^n,$$

as we could have seen using Proposition 2.2 and the relation $[\mathbb{A}^n] = \mathbb{L}^n$. For $\mathbb{P}^n$, the stratification $\mathbb{P}^n = \cup \mathbb{A}^k$ and Proposition 2.2 give us

$$P_{\mathbb{P}^n}(t) = \sum_{k=0}^{n} t^k.$$

Again, note that the answer matches with the one computed using the Betti numbers of $\mathbb{P}^n$ [5]. Finally, since $\mathbb{C}^\times = \mathbb{A}^1 \setminus \operatorname{Spec} \mathbb{C}$, we see that

$$P_{\mathbb{C}^\times}(t) = t^2 - 1.$$

2.3 Grothendieck Quot schemes

We first give the definition of Quot schemes following [3].

Definition 2.3. *[3] Let X be a finite type S-scheme and let $\mathcal{F}$ be a coherent sheaf. Let T be a S-scheme and consider the following diagram:*

$$\begin{array}{ccc} X \times_S T & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}.$$

Then, $Quot_{X/\mathcal{F}/S}$ is the functor whose objects of $Quot_{X/\mathcal{F}/S}(T)$ are surjections

$$\pi^*\mathcal{F} \twoheadrightarrow \mathcal{E},$$

where $\mathcal{E}$ is a coherent sheaf on $X \times_S T$ flat over T.

If $X \to S$ is projective, we can decompose $Quot_{X/\mathcal{F}/S}$ further. Fix a relatively very ample line bundle $\mathcal{L}$ on X over S. Then, we have

$$Quot_{X/\mathcal{F}/S} = \coprod_{P \in \mathbb{Q}[\lambda]} Quot^P_{X/\mathcal{F}/S},$$

where $Quot^P_{X/\mathcal{F}/S}$ parametrizes coherent quotients, $\mathcal{E}$, with Hilbert polynomial P.

Grothendieck proved that under certain conditions, the functors $Quot^P_{X/\mathcal{F}/S}$ are representable.

Theorem 2.2. *[3] Let S be Noetherian, $X \to S$ projective with $\mathcal{L}$ a relatively very ample line bundle. Then, for any coherent sheaf $\mathcal{F}$ on X and any polynomial $P \in \mathbb{Q}[\lambda]$, the functor $Quot^P_{X/\mathcal{F}/S}$ is represented by a projective S-scheme $Quot^P_{X/\mathcal{F}/S}$ called a Quot scheme.*

We are interested in the Quot schemes over projective curves C. In particular, we are interested in the Quot schemes over C parametrizing surjections

$$\mathcal{O}_C^{\oplus n} \twoheadrightarrow \mathcal{E},$$

where $\mathcal{E}$ is a torsion sheaf on C. In other words, we are interested in Quot schemes, $Quot^P_{C/\mathcal{O}_C^{\oplus n}/\operatorname{Spec}\mathbb{C}}$, where $P(\lambda) = d$ is a constant polynomial. To simplify notations we will write

$$Quot_C(n, d) := Quot^d_{C/\mathcal{O}_C^{\oplus n}/\operatorname{Spec}\mathbb{C}}.$$

For $p \in C$, we have a subscheme of $Quot_C(n, d)$ parametrizing torsion sheaves $\mathcal{E}$ supported only on p. We will denote this *punctual* Quot scheme by $Quot_p(n, d)$.

2.4 Moduli space of stable quotients

Now we introduce the moduli space of stable quotients defined in [8].

Let $\mathbb{G}(r, n)$ denote the Grassmanian parametrizing r-dimensional subspaces of $\mathbb{C}^n$. As Toda explains in [9], the moduli space of stable quotients can be viewed as a way of compactifying the space of smooth curves to $\mathbb{G}(r, n)$. For a curve, C, defining a map $C \to \mathbb{G}(r, n)$ is equivalent to a surjection $\mathcal{O}_C^{\oplus n} \twoheadrightarrow Q$, where Q is a locally free sheaf of rank $(n - r)$ on C. The moduli space of such maps, where we fix the underlying curve C, is not compact and there are two natural ways to compactify it.

1. Kontsevich stable maps compactification as defined in Definition 1.1; and

2. Grothendieck's Quot scheme compactification described in Section 2.3.

If we consider the moduli space of maps to $\mathbb{G}(r,n)$ while varying the underlying curve, C, the Kontsevich stable maps compactification gives rise to $\overline{\mathcal{M}}_{g,n}(\mathbb{G}(r,n),d)$. The Quot scheme compactification gives rise to the moduli space of stable quotients.

Definition 2.4. *[8] A stable quotient is a collection $(C, p_1, \ldots, p_m, \mathcal{O}_C^{\oplus n} \twoheadrightarrow Q)$ of a m-pointed nodal curve, C, and a quotient sheaf Q on C such that*

1. *Q is locally free near the marked points and the nodes. In particular* $\det Q$ *is well-defined, and*

2. *The $\mathbb{R}$-line bundle*

$$\omega_C(p_1 + \cdots + p_m) \otimes (\det Q)^{\otimes \varepsilon}$$

   is ample for all $\varepsilon > 0$.

The moduli space of stable quotients, $(C, \mathcal{P}_1, \ldots, p_m, \mathcal{O}_C^{\oplus n} \twoheadrightarrow Q)$, where C has genus g, Q is locally free of rank $(n-r)$ with $\deg Q = d$ is denoted $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$.

Theorem 2.3. *[8] $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$ is a separated and proper Deligne-Mumford stack of finite type over $\mathbb{C}$ with a perfect obstruction theory.*

One interesting phenomenon regarding stable quotients is when $n = r$. In this case, $\mathbb{G}(n,n) = \operatorname{Spec}\mathbb{C}$ and thus, $\overline{\mathcal{M}}_{g,m}(\mathbb{G}(n,n),d)$ is non-empty only if $d = 0$ and we have $\overline{\mathcal{M}}_{g,m}(\mathbb{G}(n,n),0) = \overline{\mathcal{M}}_{g,m}$. However, $\overline{Q}_{g,m}(\mathbb{G}(n,n),d)$ parametrize all stable quotients $(C, p_1, \ldots, p_m, \mathcal{O}_C^{\oplus n} \twoheadrightarrow Q)$, where Q is a torsion sheaf of length d on C, giving us more interesting spaces.

Lemma 2 of [8] computes the Poincare polynomial of $\overline{Q}_{0,2}(\mathbb{G}(1,1),d)$.

Lemma 2.1. *[8] Let P_d be the Poincare polynomial of $\overline{Q}_{0,2}(\mathbb{G}(1,1),d)$. Then,*

$$P_d(t) = (1+t^2)^{d-1},$$

for all $d > 0$.

In Section 2.7, we will compute the Poincare polynomials for $\overline{Q}_{0,2}(\mathbb{G}(n,n),d)$ for all $n, d > 0$.

2.5 Poincare polynomials of punctual Quot schemes

We first find the Poincare polynomials of punctual Quot schemes, $Quot_p(n,d)$. Recall that $Quot_p(n,d)$ is a subscheme of $Quot_C(n,d)$ parametrizing surjections $O_C^{\oplus n} \twoheadrightarrow Q$, where Q is a torsion sheaf of length d supported on $p \in C$.

We will compute the Poincare polynomial of $Quot_p(n,d)$ by examining its class in $K(Var/\mathbb{C})$. To do that, we first prove the following lemma.

Lemma 2.2. *Let $Quot_p(n,d)$ be as before where $n,d \geq 1$. Then, we have the following relation in $K(Var/\mathbb{C})$:*

$$[Quot_p(n,d)] = \sum_{d' \leq d} \mathbb{L}^{d'}[Quot_p(n-1,d')].$$

Proof. The points of $Quot_p(n,d)$ parametrize surjections $O_C^{\oplus n} \twoheadrightarrow Q$ where Q is a torsion sheaf of length d supported on p. Equivalently, they parametrize injections $S \to O_C^{\oplus n}$ such that the quotient is a torsion sheaf of length d supported on p. Since p is a smooth point of C, its formal neighborhood is $\operatorname{Spec} \mathbb{C}[[x]]$. By looking at the formal neighborhood of p, we see that the points of $Quot_p(n,d)$ correspond to the matrices representing an injection, $\mathbb{C}[[x]]^{\oplus n} \to \mathbb{C}[[x]]^{\oplus n}$, up to right multiplication by an element of $GL_n(\mathbb{C}[[x]])$ such that the image ideal in $\mathbb{C}[[x]]^{\oplus n}$ has colength d. In other words, points of $Quot_p(n,d)$ parametrize colength d ideals of $\mathbb{C}[[x]]^{\oplus n}$[1].

We first consider the $Quot_p(2,d)$ case before considering $Quot_p(n,d)$ for general n. Let

$$M = \begin{pmatrix} f & g \\ h & k \end{pmatrix} \in Quot_p(2,d).$$

Note that right multiplication by elements of $GL_2(\mathbb{C}[[x]])$ are simply the column operations. Hence, we can add, subtract, swap columns, and multiply an entire column by a unit in $\mathbb{C}[[x]]$ i.e. a non-multiple of x. By applying appropriate column operations, we can reduce M to the matrices of the form

$$M \Rightarrow \begin{pmatrix} f & 0 \\ g & k \end{pmatrix} \Rightarrow \begin{pmatrix} f & 0 \\ g & x^{d'} \end{pmatrix}.$$

Now, by multiplying the first column by an appropriate unit in $\mathbb{C}[[x]]$, we can further reduce M to a matrix of the form

$$\begin{pmatrix} x^{d''} & 0 \\ g & x^{d'} \end{pmatrix}.$$

[1] Right multiplication by an element of $GL_n(\mathbb{C}[[x]])$ tells us that we are free to choose the generators of that ideal.

Note that since the matrix represents an ideal of colength d, we must have $d'+d''=d$.

Finally, by subtracting an appropriate multiple of the second column from the first column, we can further reduce g to a polynomial, $\sum_0^{d'-1} a_i x^i$, of degree less than d'. Hence, for each element M of $Quot_p(2,d)$, there exists a unique matrix of the form,

$$\begin{pmatrix} x^{d''} & 0 \\ \sum_0^{d'-1} a_i x^i & x^{d'} \end{pmatrix},$$

where $d'+d''=d$, that represents M. We saw that every element has such a representation, and two distinct matrices of the form above cannot differ by a series of column operations. We will call d' the length of the "sub-quotient" restricted to the second factor of O. More precisely, consider the injection $i:\mathbb{C}[[x]]\to\mathbb{C}[[x]]^{\oplus 2}$ into the second factor. Now consider the following cartesian square

$$\begin{array}{ccc} \mathbb{C}[[x]] & \xrightarrow{M'} & \mathbb{C}[[x]] \\ \downarrow & & \downarrow i \\ \mathbb{C}[[x]]^{\oplus 2} & \xrightarrow{M} & \mathbb{C}[[x]]^{\oplus 2} \end{array} .$$

Then, $M':\mathbb{C}[[x]]\to\mathbb{C}[[x]]$ is precisely the ideal $(x^{d'})$.

Now, we can stratify $Quot_p(2,d)$ by the value of d' in the (2,2)-entry of the matrix M in the reduced form. Alternatively, this stratification can be described as follows. Let $i:O_C\to O_C^{\oplus 2}$ be the second factor. Then, given $(S\to O_C^{\oplus 2}\twoheadrightarrow Q)\in Quot_p(2,d)$, we can consider the pullback $S'=S\times_{O_C^{\oplus 2}} O_C$ from the following cartesian square:

$$\begin{array}{ccccc} 0 & \longrightarrow & S' & \longrightarrow & O_C \\ & & \downarrow & & \downarrow i \\ 0 & \longrightarrow & S & \longrightarrow & O_C^{\oplus 2} \end{array} .$$

Then, we get a new exact sequence $S'\to O_C\twoheadrightarrow Q'$. Since Q is a torsion sheaf supported on p, Q' must also be a torsion sheaf supported on p of length $d'\leq d$. Then, $(S'\to O_C\twoheadrightarrow Q')$ is an element of $Quot_p(1,d')$. This gives us a map $Quot_p(2,d)\to\coprod_{d'\leq d} Quot_p(1,d')$, which we can use to stratify $Quot_p(2,d)$.

By the analysis above, we see that over $Quot_p(1,d')$, the fibers are isomorphic to $\mathbb{A}^{d'}$[2]. The points of the stratum corresponding to d' are uniquely determined by

[2]More precisely, the fibers are isomorphic to $Quot_p(1,d-d')\times\mathbb{A}^{d'}$ since we get to choose $x^{d-d'}$ in the (1,1)-entry and a polynomial of degree at most d' in the (2,1)-entry. However, since $Quot_p(1,d-d')$ is just a point, the fibers are isomorphic to $\mathbb{A}^{d'}$.

matrices of the form

$$\begin{pmatrix} x^{d-d'} & 0 \\ \sum_0^{d'-1} a_i x^i & x^{d'} \end{pmatrix}.$$

In other words, these points are uniquely determined by the choice of coefficients $a_0, \ldots, a_{d'-1} \in \mathbb{C}$. Therefore, we get the desired recursion

$$Quot_p(2, d) = \sum_{d' \le d} \mathbb{L}^{d'} Quot_p(1, d').$$

Now, we proceed by induction on n. Again, we take a matrix $M \in Quot_p(n, d)$. By column operation, we can reduce M to the form

$$M \Rightarrow \begin{pmatrix} x^{d_1} & 0 & 0 & \cdots & 0 \\ * & * & * & \cdots & * \end{pmatrix},$$

where * are the rest of the columns. In other words, we have a block matrix

$$\begin{pmatrix} x^{d_1} & 0 \\ * & M' \end{pmatrix},$$

where M' is a $(n-1) \times (n-1)$ matrix with a nontrivial determinant. Hence, $M' \in Quot_p(n-1, d')$ for some d'. However, since $M \in Quot_p(n, d)$ we must have $d' = d - d_1$.

Now, by induction or by further column operations, we can reduce M to an lower triangular matrix whose diagonal entries are all powers of x, say x^{d_r}. And by further column operations we can assume $*$'s are polynomials of degree at most $d_r - 1$, where r is the corresponding column number:

$$\begin{pmatrix} x^{d_1} & 0 & 0 & \cdots & 0 \\ * & x^{d_2} & 0 & \cdots & 0 \\ * & * & x^{d_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & * & \cdots & x^{d_n} \end{pmatrix},$$

where $\sum d_i = d$. Analogous to the $Quot_p(2, d)$ case, we get a map $Quot_p(n, d) \to \coprod_{d' \le d} Quot_p(n-1, d')$ by restricting to the last $(n-1)$ factors of $O^{\oplus n}$, which is precisely represented by M' from above[3]. Fixing $d' = d_2 + \cdots + d_n$, the fibers of the above map over $Quot_p(n-1, d')$ are again isomorphic to $\mathbb{A}^{d'}$ coming from the

[3] As before, this is equivalent to taking the fiber product of the kernel, $S \to O_C^{\oplus n} \twoheadrightarrow Q$, with the injection of the last $(n-1)$ factors, $i : O_C{}^{\oplus} n - 1 \to O_C$.

choice of polynomials $\sum_0^{d_r-1} a_i x^i$ for all $r = 2, \ldots, n$ on the first column. Therefore, we get the desired relation in $K(Var/\mathbb{C})$:

$$[Quot_p(n,d)] = \sum_{d' \leq d} \mathbb{L}^{d'} [Quot_p(n-1,d')].$$

□

With the recursion in Lemma 2.2, we can compute the class of $Quot_p(n,d)$ in $K(Var/\mathbb{C})$.

Proposition 2.3. *For all $n \geq 1$ and $d \geq 0$, the class of $Quot_p(n,d)$ in $K(Var/\mathbb{C})$ can be written in the following form*[4]

$$[Quot_p(n,d)] = \frac{(\mathbb{L}^n - 1)(\mathbb{L}^{n+1} - 1) \cdots (\mathbb{L}^{n+d-1} - 1)}{(\mathbb{L} - 1)(\mathbb{L}^2 - 1) \cdots (\mathbb{L}^d - 1)}. \qquad (2.1)$$

Proof. First, note that when $n = 1$, $Quot_p(1,d) \cong \operatorname{Spec} \mathbb{C}$. Points of $Quot_p(1,d)$ parametrize colength d ideals in $\mathbb{C}[[x]]$. However, since $\mathbb{C}[[x]]$ is a PID, (x^d) is the unique ideal of colength d in $\mathbb{C}[[x]]$. Hence, $Quot_p(1,d) \cong \operatorname{Spec} \mathbb{C}$.

Now, the boundary conditions and the recursive relation in Lemma 2.2 uniquely determine the class of $Quot_p(n,d)$ in $K(Var/\mathbb{C})$. Hence, it suffices to show that equation 2.1 satisfy the conditions.

For $n = 1$, equation 2.1 becomes

$$[Quot_p(1,d)] = \frac{(\mathbb{L} - 1)(\mathbb{L}^2 - 1) \cdots (\mathbb{L}^d - 1)}{(\mathbb{L} - 1)(\mathbb{L}^2 - 1) \cdots (\mathbb{L}^d - 1)} = 1.$$

Hence, we only have left to show that the equation satisfies the recursion. Suppose the equation 2.1 holds for all $n' < n$ and all d'. We want to show that the recursion in Lemma 2.2 gives us equation 2.1. By the recursion we have

$$[Quot_p(n,d)] = \sum_{d' \leq d} \mathbb{L}^{d'} [Quot_p(n-1,d')] \qquad (2.2)$$

$$= \sum_{d' \leq d} \mathbb{L}^{d'} \frac{(\mathbb{L}^{n-1} - 1)(\mathbb{L}^n - 1) \cdots (\mathbb{L}^{n+d'-2} - 1)}{(\mathbb{L} - 1)(\mathbb{L}^2 - 1) \cdots (\mathbb{L}^{d'} - 1)}. \qquad (2.3)$$

However, (2.3) is equal to

$$\frac{(\mathbb{L}^n - 1) \cdots (\mathbb{L}^{n+d-1} - 1)}{(\mathbb{L} - 1) \cdots (\mathbb{L}^d - 1)} \sum_{d' \leq d} \mathbb{L}^{d'} (\mathbb{L}^{n-1} - 1) \frac{(\mathbb{L}^{d'+1} - 1) \cdots (\mathbb{L}^d - 1)}{(\mathbb{L}^{n+d'-1} - 1) \cdots (\mathbb{L}^{n+d-1} - 1)}.$$

[4] When $d = 0$, the formula gives $[Quot_p(n,0)] = 1$.

Hence, it suffices to show that

$$\sum_{d'\leq d} \mathbb{L}^{d'}(\mathbb{L}^{n-1}-1)\frac{(\mathbb{L}^{d'+1}-1)\cdots(\mathbb{L}^{d}-1)}{(\mathbb{L}^{n+d'-1}-1)\cdots(\mathbb{L}^{n+d-1}-1)} = 1.$$

We will prove this by induction on d. For $d = 0$, we have

$$(\mathbb{L}^{n-1}-1)\frac{1}{\mathbb{L}^{n-1}-1} = 1.$$

Now let $d \geq 1$ and assume the equality holds for all $d' < d$. Then, we get

$$\begin{aligned}
&\frac{\mathbb{L}^{d}(\mathbb{L}^{n-1}-1)}{\mathbb{L}^{n+d-1}-1} + \sum_{d'\leq d-1} \mathbb{L}^{d'}(\mathbb{L}^{n-1}-1)\frac{(\mathbb{L}^{d'+1}-1)\cdots(\mathbb{L}^{d}-1)}{(\mathbb{L}^{n+d'-1}-1)\cdots(\mathbb{L}^{n+d-1}-1)}\\
= &\frac{\mathbb{L}^{d}(\mathbb{L}^{n-1}-1)}{\mathbb{L}^{n+d-1}-1} + \frac{\mathbb{L}^{d}-1}{\mathbb{L}^{n+d-1}-1}\sum_{d'\leq d-1} \frac{\mathbb{L}^{d'}(\mathbb{L}^{n-1}-1)(\mathbb{L}^{d'+1}-1)\cdots(\mathbb{L}^{d-1}-1)}{(\mathbb{L}^{n+d'-1}-1)\cdots(\mathbb{L}^{n+d-2}-1)}\\
= &\frac{\mathbb{L}^{d}(\mathbb{L}^{n-1}-1)}{\mathbb{L}^{n+d-1}-1} + \frac{\mathbb{L}^{d}-1}{\mathbb{L}^{n+d-1}-1}\\
= &\frac{\mathbb{L}^{d}(\mathbb{L}^{n-1}-1)+\mathbb{L}^{d}-1}{\mathbb{L}^{n+d-1}-1}\\
= &\frac{\mathbb{L}^{n+d-1}-1}{\mathbb{L}^{n+d-1}-1}\\
= &\ 1.
\end{aligned}$$

Hence equation 2.1 satisfies the recursive relation from Lemma 2.2 and the initial conditions. □

In particular, we have

$$[Quot_p(n,1)] = \frac{\mathbb{L}^n-1}{\mathbb{L}-1} = \sum_{0}^{n-1}\mathbb{L}^k = [\mathbb{P}^{n-1}],$$

for all n.[5] Moreover, by Proposition 2.2, we obtain the following corollary.

Corollary 2.1. *Let $P_{n,d}(t)$ denote the Poincare polynomial of $Quot_p(n,d)$. Then, we have*

$$P_{n,d}(t) = \frac{(t^{2n}-1)(t^{2n+2}-1)\cdots(t^{2n+2d-2}-1)}{(t^2-1)(t^4-1)\cdots(t^{2d}-1)}.$$

[5] In fact, $Quot_p(n,1) \cong \mathbb{P}^{n-1}$ for all n.

2.6 Generating series for Poincare polynomials of $Quot_C(n,d)$

In this section, we will find the Poincare polynomial of $Quot_C(n,d)$. Then, we will show that the generating series for the Poincare polynomials of $Quot_C(n,d)$ is, in fact, a rational function.

We first start by examining the class of $[Quot_C(n,d)]$ in $K(Var/\mathbb{C})$ using the same stratification as in the proof of Lemma 2.2.

Lemma 2.3. *Let C be a smooth curve over $\mathbb{C}$, and let $n, d \geq 1$. Then, we have the following relation in $K(Var/\mathbb{C})$:*

$$[Quot_C(n,d)] = \sum_{(d_1,\dots,d_n)\in P(d)} \prod_{i=1}^{n} [\mathrm{Hilb}_C^{d_i}]\mathbb{L}^{id_i},$$

where $\mathrm{Hilb}_C^{d_i} \cong Quot_C(1,d_i)$ is the Hilbert scheme over C of colength d_i ideal sheaves, and $P(d) = \{(d_1,\dots,d_n) \in \mathbb{Z}^n \mid d_i \geq 0, \sum d_i = d\}$ is the set of all ordered partitions of d.

Proof. Recall that points of $Quot_C(n,d)$ parametrize exact sequence $(S \to \mathcal{O}_C^{\oplus n} \twoheadrightarrow Q)$ such that Q is a torsion sheaf of length d. Let $i : \mathcal{O}_C^{\oplus(n-1)} \to \mathcal{O}_C^{\oplus n}$ be the inclusion of the last $(n-1)$ factors of $\mathcal{O}_C$. Now, consider the following cartesian diagram:

$$\begin{array}{ccc} S' & \longrightarrow & \mathcal{O}_C^{\oplus(n-1)} \\ \downarrow & & \downarrow i \\ S & \longrightarrow & \mathcal{O}_C^{\oplus n} \end{array}.$$

The diagram above gives us another exact sequence $(S' \to \mathcal{O}_C^{\oplus(n-1)} \twoheadrightarrow Q')$, where Q is a torsion sheaf of length $d' \leq d$. In other words, we have $(S' \to \mathcal{O}_C^{\oplus(n-1)} \twoheadrightarrow Q') \in Quot_C(n-1,d')$.

Therefore, we have a map $Quot_C(n,d) \to \coprod_{d' \leq d} Quot_C(n-1,d')$. Now, the fibers over $Quot_C(n-1,d')$ can be further stratified. Consider the expanded diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S' & \longrightarrow & \mathcal{O}_C^{\oplus(n-1)} & \longrightarrow & Q' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow i & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & \mathcal{O}_C^{\oplus n} & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S'' & \longrightarrow & \mathcal{O}_C & \longrightarrow & Q'' & \longrightarrow & 0 \end{array}.$$

obtained from the cartesian diagram above. Now, $(S'' \to O_C \to Q'') \in Quot_C(1, d-d')$ and thus we can stratify the fibers of $Quot_C(n, d)$ over $Quot_C(n-1, d')$ by where each point maps to $Quot_C(1, d-d')$. Now, the fibers over $Quot_C(1, d-d')$ are isomorphic to $\mathbb{A}^{d'}$ by exactly the same argument as in the proof of Lemma 2.2.

Note that $Quot_C(1, d)$ is isomorphic to Hilb_C^d, the Hilbert scheme parametrizing ideal sheaves on C of colength d, by definition. Hence, by Proposition 2.2, we conclude that the following relation holds in $K(Var/\mathbb{C})$:

$$[Quot_C(n, d)] = \sum_{d' \le d} [Quot_C(n-1, d')][\mathrm{Hilb}_C^{d-d'}]\mathbb{L}^{d'}.$$

By induction on n, we get

$$\begin{aligned}
[Quot_C(n, d)] &= \sum_{d' \le d} [Quot_C(n-1, d')][\mathrm{Hilb}_C^{d-d'}]\mathbb{L}^{d'} \\
&= \sum_{(d_1,\ldots,d_n)\in P(d)} [\mathrm{Hilb}_C^{d_1}]\mathbb{L}^{d_2+\cdots+d_n}[\mathrm{Hilb}_C^{d_2}]\mathbb{L}^{d_3+\cdots+d_n}\cdots[\mathrm{Hilb}_C^{d_n}] \\
&= \sum_{(d_1,\ldots,d_n)\in P(d)} [\mathrm{Hilb}_C^{d_1}][\mathrm{Hilb}_C^{d_2}]\mathbb{L}^{d_2}[\mathrm{Hilb}_C^{d_3}]\mathbb{L}^{2d_3}\cdots[\mathrm{Hilb}_C^{d_n}]\mathbb{L}^{(n-1)d_n} \\
&= \sum_{(d_1,\ldots,d_n)\in P(d)} \prod_{i=1}^{n} [\mathrm{Hilb}_C^{d_i}]\mathbb{L}^{id_i},
\end{aligned}$$

as desired. □

Now, let $Q_{C,n,d}(t)$ denote the Poincare polynomial of $Quot_C(n, d)$. We will show that the generating series for $Q_{C,n,d}(t)$ is a rational function.

Theorem 2.4. *Let C be a projective curve and let $Q_{C,n,d}(t)$ denote the Poincare polynomial of $Quot_C(n, d)$. Then, the generating series for $Q_{C,n,d}(t)$ is a rational function. More precisely, let*

$$Q_{C,n}(t, x) := \sum_d Q_{C,n,d}(t)x^d \in \mathbb{Z}[[x, t]]$$

be the generating series for $Q_{C,n,d}(t)$. Then, we can write

$$Q_{C,n}(t, x) = \frac{[(1+tx)(1+t^3x)\cdots(1+t^{2n-1}x)]^{2g}}{(1-x)(1-t^{2n}x)[(1-t^2x)(1-t^4x)\cdots(1-t^{2n-2}x)]^2}.$$

Proof. First, we find the generating series for the classes of $Quot_C(n,d)$ in the power series Grothendieck ring, $K(Var/\mathbb{C})[[x]]$, using Lemma 2.3.

$$\begin{aligned}
\sum_d [Quot_C(n,d)]x^d &= \sum_d \left(\sum_{(d_1,\ldots,d_n)\in P(d)} [\mathrm{Hilb}_C^{d_1}][\mathrm{Hilb}_C^{d_2}]\mathbb{L}^{d_2} \cdots [\mathrm{Hilb}_C^{d_n}]\mathbb{L}^{(n-1)d_n} x^d \right) \\
&= \sum_d \left(\sum_{(d_1,\ldots,d_n)\in P(d)} [\mathrm{Hilb}_C^{d_1}]x^{d_1}[\mathrm{Hilb}_C^{d_2}]\mathbb{L}^{d_2}x^{d_2} \cdots [\mathrm{Hilb}_C^{d_n}]\mathbb{L}^{(n-1)d_n} x^{d_n} \right) \\
&= \left(\sum_d [\mathrm{Hilb}_C^d]x^d\right)\left(\sum_d [\mathrm{Hilb}_C^d]\mathbb{L}^d x^d\right)\cdots\left(\sum_d [\mathrm{Hilb}_C^d]\mathbb{L}^{(n-1)d} x^d\right) \\
&= \prod_{i=0}^{n-1}\left(\sum_d [\mathrm{Hilb}_C^d]\mathbb{L}^{id} x^d\right).
\end{aligned}$$

Since C is a projective curve, we know by [7] that the generating series for the Poincare polynomials, $Q_{C,1,d}(t)$, of Hilb_C^d is

$$Q_{C,1}(t,x) = \sum_d Q_{C,1,d}(t)x^d = \frac{(1+tx)^{2g}}{(1-x)(1-t^2x)},$$

where g is the genus of C. Hence, we have

$$\sum_d Q_{C,1,d}(t)t^{2kd}x^d = \frac{(1+t^{2k+1}x)^{2g}}{(1-t^{2k}x)(1-t^{2k+2}x)}.$$

By Lemma 2.3 and Proposition 2.2, we know that

$$Q_{C,n,d}(t) = \sum_{(d_1,\ldots,d_n)\in P(d)} Q_{C,1,d_1}(t)Q_{C,1,d_2}(t)t^{2d_2}Q_{C,1,d_3}(t)t^{4d_3}\cdots Q_{C,1,d_n}(t)t^{2(n-1)d_n}.$$

Now, we are ready to find the generating series $Q_n(t,x)$. We use the same manipulation as we did in finding the generating series for the classes of $Quot_C(n,d)$ in $K(Var/\mathbb{C})[[x]]$ to find $Q_{C,n}(t,x)$ in $\mathbb{Z}[[x,t]]$.

$$\begin{aligned}
Q_{C,n}(t,x) &= \sum_d Q_{C,n,d}(t)x^d \\
&= \left(\sum_d Q_{C,1,d}(t)x^d\right)\left(\sum_d Q_{C,1,d}(t)t^{2d}x^d\right)\cdots\left(\sum_d Q_{C,1,d}(t)t^{2(n-1)d}x^d\right) \\
&= \frac{(1+tx)^{2g}}{(1-x)(1-t^2x)}\left(\frac{(1+t^3x)^{2g}}{(1-t^2x)(1-t^4x)}\right)\cdots\left(\frac{(1+t^{2n-1}x)^{2g}}{(1-t^{2n-2}x)(1-t^{2n}x)}\right) \\
&= \frac{[(1+tx)(1+t^3x)\cdots(1+t^{2n-1}x)]^{2g}}{(1-x)(1-t^{2n}x)[(1-t^2x)(1-t^4x)\cdots(1-t^{2n-2}x)]^2}.
\end{aligned}$$

Therefore, we conclude that the generating series, $Q_{C,n}(t,x)$, is in fact a rational function. □

When, $C = \mathbb{P}^1$ we get a particularly nice expression for the generating series for $Quot_{\mathbb{P}^1}(n,d)$ in the power series Grothendieck ring.

Corollary 2.2. *The generating series for* $[Quot_{\mathbb{P}^1}(n,d)]$ *in* $K(Var/\mathbb{C})[[x]]$ *is a rational function in x. More precisely, we have*

$$\sum_d Quot_{\mathbb{P}^1}(n,d)x^d = \frac{1}{(1-x)(1-\mathbb{L}x)^2\cdots(1-\mathbb{L}^{n-1}x)^2(1-\mathbb{L}^n x)}.$$

Proof. From the proof of Theorem 2.4, we saw

$$\sum_d Quot_{\mathbb{P}^1}(n,d)x^d = \prod_{i=0}^{n-1}\left(\sum_d [\mathrm{Hilb}^d_{\mathbb{P}^1}]\mathbb{L}^{id}x^d\right).$$

We know that $\mathrm{Hilb}^d_{\mathbb{P}^1} \cong \mathbb{P}^d$ [6]. Since $[\mathbb{P}^d] = \sum_{i=0}^d \mathbb{L}^i$, we have

$$\sum_d [\mathrm{Hilb}^d_{\mathbb{P}^1}]x^d = \sum_d \left(\sum_{i=0}^d \mathbb{L}^i\right)x^d = \frac{1}{(1-x)(1-\mathbb{L}x)}.$$

Similarly, for general k, we have

$$\sum_d [\mathrm{Hilb}^d_{\mathbb{P}^1}]\mathbb{L}^{kd}x^d = \sum_d \left(\sum_{i=0}^d \mathbb{L}^i\right)\mathbb{L}^{kd}x^d = \frac{1}{(1-\mathbb{L}^k x)(1-\mathbb{L}^{k+1}x)}.$$

Hence, we have

$$\begin{aligned}
\sum_d Quot_{\mathbb{P}^1}(n,d)x^d &= \prod_{i=0}^{n-1}\left(\sum_d [\mathrm{Hilb}^d_{\mathbb{P}^1}]\mathbb{L}^{id}x^d\right)\\
&= \prod_{i=0}^{n-1}\left(\frac{1}{(1-\mathbb{L}^i x)(1-\mathbb{L}^{i+1}x)}\right)\\
&= \frac{1}{(1-x)(1-\mathbb{L}x)^2\cdots(1-\mathbb{L}^{n-1}x)^2(1-\mathbb{L}^n x)}.
\end{aligned}$$

□

2.7 The generating series for Poincare polynomials of $\overline{Q}_{0,2}(\mathbb{G}(n,n),d)$

Lemma 2 of [8] computes the Poincare polynomial of $\overline{Q}_{0,2}(\mathbb{G}(1,1),d)$ for $d > 0$.

Lemma 2.4. *[8] Let* $P_d(t)$ *be the Poincare polynomial of* $\overline{Q}_{0,2}(\mathbb{G}(1,1),d)$*, where* $d > 0$*. Then,*

$$P_d(t) = (1+t^2)^{d-1}.$$

[6] See for example [4].

In the proof of this lemma, they show that $\overline{Q}_{0,2}(\mathbb{G}(n,n),d)$ can be stratified as follows.

Lemma 2.5. *[8] $\overline{Q}_{0,2}(\mathbb{G}(n,n),d)$ can be written as a disjoint union of quasi-projective strata, $S_{(d_1,\ldots,d_k)}$, indexed by ordered partitions, $(d_1,\ldots,d_k)$, of d [7]. Moreover, for each ordered partition, $(d_1,\ldots,d_k)$, we have*

$$S_{(d_1,\ldots,d_k)} \cong \prod_{i=1}^{k} Quot_{\mathbb{C}^\times}(n,d_i)/\mathbb{C}^\times.$$

In fact, $S_{(d_1,\ldots,d_k)} \subset \overline{Q}_{0,2}(\mathbb{G}(n,n),d)$ is the stratum of points parametrizing (C,p_1,p_2,Q), where C is a nodal curve with k rational components, $C_1,\ldots,C_k$, such that Q has length d_i on C_i. Note that since there are only two marked points on C, p_1 and p_2 must lie on the two extremal components of C. Moreover, all components of C are rational with two special points and thus, by the stability condition, Q must have positive length on each C_i.

To simplify notation, we will write

$$Q_{n,d}(t) := Q_{\mathbb{C}^\times,n,d},$$
$$Q_n(t,x) := Q_{\mathbb{C}^\times,n}(t,x) = \sum_d Q_{\mathbb{C}^\times,n,d}(t)x^d,$$
$$\overline{Q}_{n,d}(t) := P_{\overline{Q}_{0,2}(\mathbb{G}(n,n),d)}(t),$$
$$S_{n,\overline{d}}(t) := P_{S_{n,\overline{d}}}(t),$$

where $S_{n,\overline{d}} \subset \overline{Q}_{0,2}(\mathbb{G}(n,n),d)$ is the stratum associated to the ordered partition $\overline{d} = (d_1,\ldots,d_k)$.

Now, we try to find the generating function for the Poincare polynomials of $\overline{Q}_{0,2}(\mathbb{G}(n,n),d)$ with n fixed.

Theorem 2.5. *Let*

$$\overline{Q}_n(t,x) := \sum_d \overline{Q}_{n,d}(t)x^d \in \mathbb{Z}[[t,x]]$$

be the generating series for the Poincare polynomials of $\overline{Q}_{n,d}(\mathbb{G}(n,n),d)$. Then, $\overline{Q}_n(t,x)$ is a rational function in t and x. More precisely, we have

$$\overline{Q}_n(t,x) = \frac{(1-t^{2n}x)(t^2-1)}{t^2-t^{2n+2}x+x-1}.$$

[7] i.e. tuples $(d_1,\ldots,d_k)$ such that $d_i > 0$ for all i and $\sum d_i = d$.

Proof. We first compute $Q_n(t,x)$. The computation is similar to the computation in the proof of Theorem 2.4. MacDonald's generating function for $Q_{1,d}(t)$ does not hold in our case since $\mathbb{C}^\times$ is not projective.

From the proof of Lemma 2 in [8], we know that the Poincare polynomial of $\mathrm{Hilb}^d(\mathbb{C}^\times)$ is

$$Q_{1,d}(t) = t^{2d} - t^{2d-2}.$$

Hence, we get

$$\begin{aligned} Q_1(t,x) &= \sum_{d\geq 0} Q_{1,d}(t)x^d \\ &= 1 + \sum_{d>0}(t^{2d} - t^{2d-2})x^d \\ &= 1 + (t^2-1)\sum_{d>0} t^{2d-2}x^d \\ &= 1 + (t^2-1)x\sum_{d\geq 0} t^{2d}x^d \\ &= 1 + \frac{x(t^2-1)}{1-t^2x} \\ &= \frac{1-x}{1-t^2x}. \end{aligned}$$

It follows that for general k we have

$$\sum_d P(\mathrm{Hilb}^d(\mathbb{C}^\times))(t)t^{2kd}x^d = \frac{1-t^{2k}x}{1-t^{2k+2}x}.$$

Now, by Lemma 2.3, we obtain

$$\begin{aligned} Q_n(x,t) &= \left(\sum_d Q_{1,d}(t)x^d\right)\left(\sum_d Q_{1,d}(t)t^{2d}x^d\right)\cdots\left(\sum_d Q_{1,d}(t)t^{2(n-1)d}x^d\right) \\ &= \frac{1-x}{1-t^2x}\cdot\frac{1-t^2x}{1-t^4x}\cdots\frac{1-t^{2n-2}x}{1-t^{2n}x} \\ &= \frac{1-x}{1-t^{2n}x}. \end{aligned}$$

Now, we compute $\overline{Q}_{n,d}(t,x)$. By Lemma 2.5 we have

$$[\overline{Q}_{0,2}(\mathbb{G}(n,n),d)] = \sum_{\overline{d}\in P(d)} [S_{n,\overline{d}}],$$

where $P(d) = \{(d_1,\ldots,d_k) \mid d_i > 0 \forall i \text{ and } \sum d_i = d\}$ is the set of all ordered partitions of d. Therefore,

$$\overline{Q}_{n,d}(t) = \sum_{\overline{d}\in P(d)} S_{n,\overline{d}}(t).$$

Recall that

$$S_{n,(d_1,\ldots,d_k)} \cong \prod_{i=1}^{k} Quot_{\mathbb{C}^\times}(n, d_i)/\mathbb{C}^\times.$$

From [2], we know that the Poincare polynomial of $Quot_{\mathbb{C}^\times}(n, d_i)/\mathbb{C}^\times$ is equal to

$$\frac{Q_{n,d_i}(t)}{t^2 - 1}.$$

Hence,

$$S_{n,(d_1,\ldots,d_k)}(t) = \prod_{i=1}^{k} \frac{Q_{n,d_i}(t)}{t^2 - 1}.$$

We are now ready to compute $\overline{Q}_n(t, x)$:

$$\begin{aligned}
\overline{Q}_n(t, x) &= \sum_{d\geq 0} \overline{Q}_{n,d}(t)x^d \\
&= \sum_{d\geq 0}\left(\sum_{\overline{d}\in P(d)} S_{n,\overline{d}}(t)x^d\right) \\
&= \sum_{k\geq 0}\left(\sum_{d_1,\ldots,d_k\geq 1} S_{n,(d_1,\ldots,d_k)}(t)x^{d_1+\cdots+d_k}\right) \\
&= \sum_{k\geq 0}\left(\sum_{d_1,\ldots,d_k\geq 1}\left(\prod_{i=1}^{k}\frac{Q_{n,d_i}(t)}{t^2-1}\right)x^{d_1+\cdots+d_k}\right) \\
&= \sum_{k\geq 0}\left(\sum_{d_1,\ldots,d_k\geq 1}\left(\prod_{i=1}^{k}\frac{Q_{n,d_i}(t)x^{d_i}}{t^2-1}\right)\right) \\
&= \sum_{k\geq 0}\left(\left(\sum_{d_1\geq 1}\frac{Q_{n,d_1}(t)x^{d_1}}{(t^2-1)}\right)\cdots\left(\sum_{d_k\geq 1}\frac{Q_{n,d_k}(t)x^{d_k}}{(t^2-1)}\right)\right) \\
&= \sum_{k\geq 0}\left(\frac{Q_n(x,t)-1}{t^2-1}\right)^k \\
&= \frac{1}{1-(Q_n(x,t)-1)\left(t^2-1\right)^{-1}} \\
&= \left(1-\frac{x(t^{2n}-1)}{(1-t^{2n}x)(t^2-1)}\right)^{-1} \\
&= \frac{(1-t^{2n}x)(t^2-1)}{(1-t^{2n}x)(t^2-1)-x(t^{2n}-1)} \\
&= \frac{(1-t^{2n}x)(t^2-1)}{t^2-t^{2n+2}x+x-1}.
\end{aligned}$$

Hence, $\overline{Q}_n(t, x)$ is a rational function in t and x. □

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