DIFFUSION WITH VARYING DRAG;

THE RUNAWAY PROBLEM

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Abstract

We study the motion of electrons in an ionized plasma of electrons and ions in an external electric field. A probability distribution function describes the electron motion and is a solution of a Fokker-Planck equation. In zero field, the solution approaches an equilibrium Maxwellian. For arbitrarily small field, electrons overcome the diffusive effects and are freely accelerated by the field. This is the electron runaway phenomenon.

We treat the electric field as a small perturbation. We consider various diffusion coefficients for the one-dimensional problem and determine the runaway current as a function of the field strength. Diffusion coefficients, non-zero on a finite interval are examined. Some non-trivial cases of these can be solved exactly in terms of known special functions. The more realistic case where the diffusion coefficient decays with velocity is then considered. To determine the runaway current, the equivalent Schrödinger eigenvalue problem is analyzed. The smallest eigenvalue is shown to be equal to the runaway current. Using asymptotic matching a solution can be constructed which is then used to evaluate the runaway current. The runaway current is exponentially small as a function of field strength. This method is used to extract results from the three-dimensional problem.

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CHAPTER 1

Introduction to the Runaway Problem

1.1 Introduction

We wish to describe the motion of electrons in an ionized plasma consisting of electrons and ions. The mathematical model should yield the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ for the electrons. $f(\mathbf{r}, \mathbf{v}, t)\, d^3\mathbf{r}\, d^3\mathbf{v}$ is the probability of finding an electron with velocity $\mathbf{v}$ at position $\mathbf{r}$ and time t in an infinitesimal volume element $d^3\mathbf{r}\, d^3\mathbf{v}$. The distribution function will be a solution of a Fokker-Planck equation. This differential equation resembles the heat equation with velocity-dependent coefficients. A diffusion tensor represents mathematically the scattering of an electron due to collisions with the ions and other electrons. The diffusion tensor components are such that electrons with large speeds are decelerated by a small amount while those with small speeds are decelerated by a large amount. The solution of the equation is the probability per unit volume of an electron having a specific velocity at a particular time. Without any external forces acting the large-time behavior of the solution will approach an equilibrium state.

The inclusion of a term in the differential equation proportional to the electric field gives rise to the phenomenon of electron *runaway*. This added force makes an equilibrium state impossible, no matter how small the field. The force due to the electric field is enough to overcome the collisional forces, which act to push the distribution of velocities toward an equilibrium, and to allow essentially free acceleration. Hence the name, runaway current. This effect makes the problem

a singular perturbation problem. It is this problem that we address in this paper. Reasonable choices for the diffusion coefficients are taken and the resulting problems analyzed.

1.2 The Fokker-Planck Equation

The Fokker-Planck equation to be studied here results from considering the interaction of ions and electrons under Coulomb forces. In general, the diffusion tensor can be chosen such that collisions between an electron with ions and other electrons are included. The general case leads to an equation with the same form as we consider and so we choose to neglect electron-electron interactions.

We now restrict ourselves to the case of no spatial dependence. Then the function $f(\mathbf{v}, t)$ is defined such that

$$f(\mathbf{v}, t) \, d^3\mathbf{v}$$

is the probability of finding a particle with velocity $\mathbf{v}$ at time t in an infinitesimal volume element $d^3\mathbf{v}$.

The probability distribution $f(\mathbf{v}, t)$ will satisfy the partial differential equation

$$\frac{\partial}{\partial t} f(\mathbf{v}, t) = \nabla \cdot \left(\mathbf{D}(\mathbf{v}) \cdot \left(\nabla f + \frac{m}{k_B T} \mathbf{v} f \right) \right). \qquad (1.2.1)$$

Derivations of such an equation have been done and can be found in, for example, [20] or [21]. $\mathbf{D}(\mathbf{v})$ is the diffusion tensor. The constants m, k_B and T are, respectively, the electron mass, the Boltzmann constant and the temperature.

The diffusion tensor can be defined in the compact form

$$\mathbf{D}(\mathbf{v}) = D_{\parallel}(v)\hat{\mathbf{v}}\hat{\mathbf{v}} + (\mathbf{I} - \hat{\mathbf{v}}\hat{\mathbf{v}}) \, D_{\perp}(v)$$

with $\mathbf{I}$ the identity tensor. The $D_{\parallel}(v)$ part causes energy relaxation due to changes in the velocity magnitude. The $D_{\perp}(v)$ part causes angle relaxation by changes in the velocity direction.

Proposition 1.2.2 $\mathbf{D(v)}$ *is a positive definite tensor for* $D_{\parallel}(v)$ *and* $D_{\perp}(v)$ *positive for all* v.

Proof. The result follows from examining $\mathbf{u}^t \cdot \mathbf{D(v)} \cdot \mathbf{u}$ for $\mathbf{u}$ an arbitrary non-zero vector. $\mathbf{u}$ can be written as a linear combination of the basis vectors in spherical coordinates as

$$\mathbf{u} = u_v \hat{\mathbf{v}} + u_\Omega \hat{\mathbf{\Omega}}.$$

Then

$$\mathbf{u}^t \cdot \mathbf{D(v)} \cdot \mathbf{u} = u_v^2 \hat{\mathbf{v}}^t \cdot \mathbf{D(v)} \cdot \hat{\mathbf{v}} + u_\Omega^2 \hat{\mathbf{\Omega}}^t \cdot \mathbf{D(v)} \cdot \hat{\mathbf{\Omega}}$$

$$= u_v^2 D_{\parallel}(v) + u_\Omega^2 D_{\perp}(v) > 0.$$

∎

$D_{\parallel}(v)$ and $D_{\perp}(v)$ are also taken to decay to zero for large v. An example of typical coefficients is

$$D_{\parallel}(v) = cv^{-3} \int_0^v t^2 e^{-t^2} \, dt$$

$$D_{\perp}(v) = v^{-1} \int_0^v D_{\parallel}(t) \, dt$$

as given in [7]. The thermal velocity v_B is defined by the energy equivalence relation $\frac{1}{2}mv_B^2 = k_B T$. On setting $\frac{\partial f}{\partial t} = 0$, the equilibrium solution is found to be

$$f(\mathbf{v}) = \pi^{-3/2} v_B^{-3} \exp{-v^2/v_B^2},$$

a Maxwellian distribution. The general solution of (1.2.1) will approach this steady state for $t \to \infty$.

A special case of equation (1.2.1) is the Lorentz model. In this case $D_{\parallel}(v)$ is assumed to be zero and $D_{\perp}(v)$ is taken to be a simple function of v. In this model there will be only directional changes in velocity.

In the presence of an external electric field $\mathbf{E}$ (1.2.1) becomes

$$\frac{\partial}{\partial t} f(\mathbf{v}, t) = \nabla \cdot \left(\mathbf{D(v)} \cdot \left(\nabla f + \frac{m}{k_B T} \mathbf{v} f \right) - \frac{q}{m} \mathbf{E} f \right) \qquad (1.2.3)$$

with q the electron charge. Specifically, $\mathbf{E}$ will be taken in the $\hat{\mathbf{z}}$ direction. Equivalently (1.2.3) can be written in the conservation form

$$\frac{\partial}{\partial t} f(\mathbf{v}, t) + \nabla \cdot \mathbf{j}(\mathbf{v}, t) = 0$$

where $\mathbf{j}(\mathbf{v}, t)$ is called the current in velocity space and is given by

$$\mathbf{j}(\mathbf{v}, t) = \frac{q\mathbf{E}}{m} f(\mathbf{v}, t) - \mathbf{D}(\mathbf{v}) \cdot \left(\nabla + \frac{m}{k_B T} \mathbf{v} \right) f(\mathbf{v}, t). \qquad (1.2.4)$$

This current is the interesting quantity when studying the runaway phenomenon. The current $\mathbf{j}(\mathbf{v}, t)$ is a density with units of acceleration per velocity cubed.

Equation (1.2.3) describes test particles of charge q and mass m which are distributed uniformly in space diffusing through a medium that has constant temperature T and is subject to a uniform, static electric field $\mathbf{E}$.

The current $\mathbf{j}(\mathbf{v}, t)$ will be used in describing the runaway phenomenon. When $\mathbf{E} = 0$, the solution to (1.2.3) will approach the equilibrium for $t \to \infty$, and the current will approach zero. For $\mathbf{E} \neq 0$, essentially free acceleration for large enough v will give a non-zero current. Again we expect the large-time behavior of the solution can be used to determine the current. If we consider (1.2.3) after the time dependence has been separated, the spectrum of the resulting operator would give insight into the large-time behavior of $f(\mathbf{v}, t)$. If the spectrum is discrete then the smallest eigenvalue, λ_0 will be of utmost importance. λ_0 as a function of E will then be a measure of the runaway current. This will be seen to be the case in the following chapters.

Also of interest is the current that one might measure in a laboratory due to the motion of the test particles. This is given by

$$\mathbf{J}(t) \equiv n_0 q \int_V \mathbf{v} f(\mathbf{v}, t) \, d^3 \mathbf{v}. \qquad (1.2.5)$$

It is just the first moment of the velocity distribution. The quantity n_0 is the number density of test particles.

Another instructive form is to write (1.2.3) as

$$\frac{\partial}{\partial t} f(\mathbf{v}, t) + \nabla \cdot \Big(\mathbf{F}(\mathbf{v}) f(\mathbf{v}, t) \Big) = \nabla \cdot \Big(\nabla \cdot (\mathbf{D}(\mathbf{v}) f(\mathbf{v}, t)) \Big) \qquad (1.2.6)$$

where $\mathbf{F}(\mathbf{v})$ is the mean force

$$\mathbf{F}(\mathbf{v}) = q\mathbf{E} + m \left(\nabla - 2\mathbf{v} \right) \cdot \mathbf{D}(\mathbf{v}). \qquad (1.2.7)$$

This form follows from (1.2.3) by multiplication by $m\mathbf{v}$ and integration with respect to $\mathbf{v}$ to obtain the form

$$\frac{d}{dt} \int m\mathbf{v} f(\mathbf{v}, t) \, d^3\mathbf{v} = \int \mathbf{F}(\mathbf{v}) f(\mathbf{v}, t) \, d^3\mathbf{v}.$$

For small $|\mathbf{v}|$ the dominant force when γ is small is due to the diffusion and takes the form of a Stokes' drag. Because of the form of $\mathbf{D}(\mathbf{v})$ the electric force becomes dominant for large $|\mathbf{v}|$.

Also of interest in this paper is what will be called the partial high-temperature model. In this model, the $\mathbf{v}f$ term in (1.2.4) is not present. It is high temperature in the sense that the limit of $T \to \infty$ eliminates the $\mathbf{v}f$ term. The appropriate current $\mathbf{j}(\mathbf{v}, t)$ is then

$$\mathbf{j}(\mathbf{v}, t) = \frac{q\mathbf{E}}{m} f(\mathbf{v}, t) - \mathbf{D}(\mathbf{v}) \cdot \nabla f(\mathbf{v}, t). \qquad (1.2.8)$$

The equilibrium in now a constant distribution instead of a Maxwellian. The model is only *partial* in the sense that $\mathbf{D}(\mathbf{v})$ is not altered even though $\mathbf{D}(\mathbf{v})$ is normally temperature dependent.

In this paper, we consider the Fokker-Planck equation (1.2.3) for our model of an ionized plasma in a uniform electric field. The three-dimensional equation presents great difficulty as suggested by previous researchers. A summary of past

work on this problem is given in §1.4. Since the diffusion coefficients vanish for large velocities, the problem is singular. The electric field term is a singular perturbation with the magnitude of **E** being the small parameter. Also, the equation is not separable in any standard coordinate system for **D(v)** non-trivial. For this reason the one-dimensional problem is studied in detail. One approach is given to extending the one-dimensional results to the three-dimensional case. We hope that future research will give more insight into the three-dimensional problem.

The one-dimensional problem to be considered is

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left(D(v)\left(\frac{\partial}{\partial v} + \frac{mv}{k_B T}\right)f(v,t) - \frac{qE}{m}f(v,t)\right).$$

The diffusion coefficient $D(v)$ is taken to be positive, even and vanishingly small for large velocities.

We first consider a diffusion coefficient which is non-zero on a finite interval. In the special case when $D(v)$ is linear or quadratic in v, exact solutions can be given in terms of special functions. For a general $D(v)$ of this form, saddle-point calculations give the runaway current in various time regimes.

We then consider the more realistic case when $D(v)$ decays as a power for large velocities. The one-dimensional equation is analyzed in its equivalent Schrödinger form. In this form all the usual asymptotic ideas about constructing solutions and eigenvalues can be exploited. The runaway current is shown to be proportional to the smallest eigenvalue. This is the perturbation correction to the zero eigenvalue steady-state solution. The smallest eigenvalue is calculated and is exponentially small as a function of the field strength. This calculation is in agreement with those of [19] without making the assumptions used there. We then use the one-dimensional method to estimate the runaway current in the three dimensional problem. We find agreement in the leading behavior with those in the literature.

1.3 Some Properties of the Equation

Some properties of the Fokker-Planck equation will now be examined before proceeding to the analysis of the next chapters.

The variables in equation (1.2.3) or (1.2.6) can be made dimensionless by scaling with the following characteristic quantities. Each velocity component scales with the Boltzmann velocity determined from the relation $\frac{1}{2}mv_B^2 = k_B T$. The scaled velocity will be denoted u. The time scales with τ_0, a characteristic collisional relaxation time controlled by the form of $\mathbf{D}(\mathbf{v})$. The diffusion tensor has units of velocity squared per time and is scaled with $v_B^2\tau_0^{-1}$. The acceleration due to the electric field scales with $v_B\tau_0^{-1}$. We will define the parameter

$$2\gamma \equiv \frac{q\tau_0}{mv_B}|\mathbf{E}|$$

which is the ratio of the electrical to collisional forces. In runaway problems 2γ is taken to be small but in some cases the problem will be examined for $2\gamma \gg 1$.

The following properties follow from the form of the differential equation (1.2.3) or (1.2.6). It is convenient to use the notation

$$Lf(\mathbf{u},t) \equiv \nabla \cdot \Big(\mathbf{D}(\mathbf{u}) \cdot (\nabla f + 2\mathbf{u}f) - 2\gamma\hat{\mathbf{z}}f\Big). \tag{1.3.1}$$

Property 1.3.2 *The integral*

$$\int f(\mathbf{u},t)\, d^3\mathbf{u}$$

is constant for all time, provided $f(\mathbf{u},t)$ falls off faster than u^2. This is a statement of conservation of probability.

Proof. Integrating (1.3.1) over a sphere of radius $\mathbf{u}_0$ gives

$$\frac{d}{dt}\int_{V_0} f(\mathbf{u},t)\, d^3\mathbf{u} = \int_{V_0} Lf(\mathbf{u},t)\, d^3\mathbf{u}.$$

$f(\mathbf{u}, t)$ is assumed to be positive everywhere. Then use Gauss' theorem and the definition of L to integrate to give

$$\frac{d}{dt} \int_{V_0} f(\mathbf{u}, t) \, d^3\mathbf{u} = \int_{S_0} \left(D_\parallel(u) \left(\frac{\partial f}{\partial u} + 2uf \right) - 2\gamma \cos\theta f \right) u^2 \sin\theta \, d\Omega.$$

For the surface integral to vanish the integrand must be smaller than some $\epsilon(\mathbf{u_0})$ with $\epsilon \to 0$ for $|\mathbf{u_0}| \to \infty$. The quantity $u^2 D_\parallel(u) \to 0$ for $u \to \infty$. Then for $f(\mathbf{u}, t)$ falling off faster than u^2 for all Ω the result follows. ∎

Property 1.3.3 *L as defined in (1.3.1) is invariant under the transformation $\gamma \to -\gamma$, and $\mathbf{u} \to -\mathbf{u}$.*

The proof is obvious and just requires making the transformation. This allows one to consider γ to be positive only.

Property 1.3.4 *For the one-dimensional case, L is semi-negative definite with respect to the weight function*

$$w(u) = \exp\left(u^2 - 2\gamma \int_0^u D^{-1}(v) \, dv \right). \tag{1.3.5}$$

Proof. Consider the inner product

$$(f, Lf) = \int_{-\infty}^{\infty} w(u) f(u) L f(u) \, du.$$

Integration by parts using the definition of L gives

$$(f, Lf) = f(u) D(u) \frac{\partial}{\partial u} (wf) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} D(u) w^{-1}(u) \left(\frac{\partial}{\partial u}(wf) \right)^2 du.$$

The integrated terms vanish since $f(u)$ and $D(u)$ vanish at infinity provided the derivative of wf is bounded. The integrand of remaining integral is positive; therefore,

$$(f, Lf) \leq 0.$$

Hence the elements in the spectrum of L are non-positive.

It should be noted that the condition on the integrated terms vanishing at the endpoints is stronger than that for conservation to hold. That is, an $f(\mathbf{u}, t)$ satisfying the requirements of this proof also satisfies those of Property 1.3.2 but the converse is false. ∎

Property 1.3.6 *If $f(\mathbf{u}, t)$ is non-negative initially, then f cannot become negative.*

Proof. Suppose $f(\mathbf{u}, t) = 0$ at a single point $\mathbf{u}_0$ and is positive otherwise, then ∇f is zero there also. It follows from geometric considerations that

$$f(\mathbf{u}, t) = \tfrac{1}{2}(\mathbf{u} - \mathbf{u}_0) \cdot \nabla\nabla f \Big|_{\mathbf{u}_0} \cdot (\mathbf{u} - \mathbf{u}_0) > 0$$

for sufficiently small $|\mathbf{u} - \mathbf{u}_0|$. Therefore, $\nabla\nabla f$ must be positive definite at $\mathbf{u}_0$. Now consider the differential equation (1.2.3) at $\mathbf{u} = \mathbf{u}_0$. It becomes

$$\frac{\partial f}{\partial t}\Big|_{\mathbf{u}_0} = \mathbf{D}(\mathbf{u}_0) : \nabla\nabla f \Big|_{\mathbf{u}_0} > 0$$

recalling that $\mathbf{D}(\mathbf{u})$ is positive definite. Therefore, f must return to being positive at $\mathbf{u}_0$. ∎

Property 1.3.7 *For $\gamma \neq 0$ there is no steady-state integrable solution in the one-dimensional case.*

Proof. Setting $\frac{\partial f}{\partial t} = 0$ and solving (1.2.6) gives

$$f(u) = c_1 w^{-1}(u) \int_0^u w(v) D^{-1}(v)\, dv + c_2 w^{-1}(u)$$

with w(u) as in (1.3.5). When this is expanded for large u one gets

$$f(u) \sim -\frac{c_1}{2\gamma} + \left(c_1 \int_0^\infty w(v) D^{-1}(v)\, dv + c_2 \right) \exp\left(2\gamma \int_0^u D^{-1}(v)\, dv \right) \qquad u \to \infty.$$

Clearly, neither is integrable on $(0, \infty)$ and hence is not a possible probability distribution. ∎

1.4 Historical Background

In this section the history of the electron runaway problem will be given from the early Dreicer papers of 1959 through to the recent papers of 1980. The past work will be briefly described along with some criticisms.

In H. Dreicer's papers [9] through [11], the author examines the Lorentz model by taking $D_\parallel(v) \equiv 0$ and $D_\perp(v) = v^{-1}$ in (1.2.3). He makes two assumptions in tackling the problem. First, the electron velocity distribution function is assumed to have the simple angular dependence

$$f(\mathbf{u}, t) = f_0(u, t) + \mu f_1(u, t)$$

where $\mu = \cos\theta$. θ is the usual spherical coordinate when $\mathbf{E}$ is in the $\hat{z}$ direction. The second assumption is that f_1 is much smaller than f_0 in the usual perturbation sense. The time dependence is removed by assuming a $e^{-\lambda t}$ time dependence. The resulting time-independent problem becomes an eigenvalue problem for λ when a boundary condition is added. This extra condition is that $f(u)$ vanishes at the critical velocity $u_c = u_B \gamma^{-1/2}$. u_B is the thermal velocity. This is the velocity at which electric forces balance those due to diffusion. The lowest eigenvalue is the important quantity. Dreicer in [10] illustrates how it decreases with decreasing field strength but does not get an explicit analytic form for the runaway current. The major problem with this procedure is that the angular dependence in the expansion fails for $u \succeq \gamma^{-1/4}$ when f_1 becomes comparable with f_0.

Gurevich in [12] and [13] considers a quasi-steady approach to runaway problems. In the first [12], he considers the slightly different problem

$$\frac{\partial f}{\partial t} = \frac{1}{u^2} \frac{\partial}{\partial u} \left(\left(u^{-1} + \alpha(u) u^2 \right) \frac{\partial f}{\partial u} + f \right).$$

The choice $\alpha(u) = \alpha_0 u$ gives the problem an accelerating mechanism which does not allow a steady-state solution. The quantity α_0 is assumed to be a small

parameter. This problem is like taking $D_{\parallel}(u) = \alpha(u) + u^{-3}$ and $D_{\perp}(u) = 0$ in (1.2.1). Perturbation theory is then used to calculate the solution in various velocity regimes. In the second paper [13] he considers (1.2.3) with $D_{\parallel}(u) = u^{-3}$ and $D_{\perp}(u) = u^{-1} - \frac{1}{4}u^{-3}$. The runaway rate is found by integrating out velocities perpendicular to the field. The resulting equation is then analyzed. A steady state solution is assumed. Gurevich assumes an expansion in powers of $(1 - \mu)$ of the form

$$f(u, \mu) = C \exp \sum_{n=0}^{\infty} \varphi_n(u)(1 - \mu)^n. \qquad (1.4.1)$$

This expansion is only valid when $u_0 < u < u_c$ and in fact diverges at $u = u_c$. This is to be expected because φ_1 becomes comparable with φ_0. This divergence suggests the calculation is not valid.

Lebedev in [18] improved on Gurevich's work by using two expansions instead of one for $\log f(u, \mu)$. One expansion is valid in the regime $u \ll u_c$ and the other for $u \succeq u_c$. Again the expansion is of the form (1.4.1). A truncation type iteration is used to generate the $\varphi_n(u)$. The procedure is to set φ_{n+1} equal to zero at the nth step and solve for φ_n and so on. This presents difficulties because $\varphi_n(u)$ becomes comparable to $\varphi_{n-1}(u)$ and therefore should not have been neglected in the first place.

M. Kruskal and I. Bernstein consider the Lorentz plasma in [16] and then the non-Lorentz model in [15]. The Lorentz problem with $D_{\perp}(u) = u^{-1}$ was analyzed by considering three regions in velocity space. Two of the regions correspond to where collisional or electrical effects dominate. The third is the connection region. Asymptotic matching is then used to connect the solutions. The electron current density is then found. In [15] the authors begin with the model problem

$$\frac{\partial f}{\partial t} + \frac{q}{m}\mathbf{E} \cdot \nabla f = \nabla \cdot \left(K\mathbf{D}(\mathbf{v}) \cdot \nabla f + \int d^3\mathbf{u}(f'\nabla f - f\nabla f') \cdot \mathbf{D}(\mathbf{u} - \mathbf{u}') \right) \quad (1.4.2)$$

with

$$\mathbf{D}(\mathbf{v}) = v^{-3}(v^2\mathbf{I} - \mathbf{vv}).$$

The integral term in the equation represents the electron-electron scattering. When (1.4.2) is simplified via an expansion of the integral term for large velocity one gets (1.2.3). The static version is then analyzed. It is implicitly assumed that a source of particles is present. One needs to consider five regions in velocity space, each assumed to be independent of the theta coordinate. Scaling in the theta coordinate should be important since one expects some type of boundary layer near $\theta = 0$, the direction of the field. The radial velocity variable is scaled with various powers of the field strength. Expansions of $f(u,\theta)$ or $\log f(u,\theta)$ are then found and asymptotically matched. From three of these solutions, the flux of particles could be calculated. The runaway rate was found to be

$$\gamma^{-3/8} \exp - \left((2\gamma)^{-1} + 2\gamma^{-1/2}\right)$$

to within a constant of order unity. Two questions remain with this procedure: Can the time dependence be dropped and is there a uniform expansion in theta?

E. Lifshitz and L. Pitaevskii in [19] calculate the runaway rate for the one-dimensional case. They assume there is a steady state and use the divergent solutions illustrated in Property 1.3.7 to asymptotically find an expression for the runaway rate. The details of this are shown in Appendix 3. The main difficulty with this approach is neglecting time dependence and using these divergent solutions to find the probability distribution function.

Kulsrud et al. in [14] numerically integrated the Fokker-Planck equation (1.4.2) for electrons. The integration was done on a uniform mesh in the two spherical coordinates, u and θ. The u ranged from 0 to some u_{max} taken to be sufficiently large. The runaway rate was calculated by finding the flux through a sphere in velocity space and dividing by the density. Integrations were done for

several values of field strength and several values of effective ionic charge. The resulting runaway rate agreed with those calculated analytically by Kruskal and Bernstein to within the $O(1)$ term.

J. C. Wiley et al. [24] also investigated (1.4.2) numerically. An initial value code was used to solve the two dimensional problem numerically. The time evolution was followed from an initial Maxwellian distribution to the quasi-steady-state solution. The runaway rate is calculated in the same way as in [14]. The results are in agreement with the other numerical calculations.

R. H. Cohen [5] extended the problem (1.4.2) to theories of multiple-ionized, multiple-species plasmas. Of course, this is closer to what an actual plasma is like due to the presence of impurities. The problem is reduced to the simple problem by defining an effective charge. Analytic results are found using Kruskal and Bernstein's method as well as Lebedev's. The results agreed with those found by Kulsrud et al. for multiple ion plasmas.

In this report, we discuss the initial value problem. Instead of assuming there are static solutions as some researchers have, we examine the time dependence of the problem by showing how the eigenvalue spectrum changes for non-zero field. Information about the eigenvalues will give information for characterizing the runaway current. First we considered the case when $D(v)$ is strictly zero outside a finite interval, hoping to see how this case might give insight into the case of a more general $D(v)$ which decays to zero at ∞. This case led only to currents analytic in the field strength, but allowed some exact solutions in terms of known functions. We approach the problem for general $D(v)$ using asymptotic methods similar to those used to calculate the exponentially small imaginary correction to eigenvalues in the radioactive decay problem of quantum mechanics. Throughout our analysis we see the analogy to this quantum mechanical problem. The shift in the zero eigenvalue for zero field case gives to the exponentially small runaway current for

non-zero field. Following the approach used to study the one-dimensional problem, we can then apply this to the three-dimensional problem and get similar results to those of other researchers.

CHAPTER 2

Diffusion Coefficients Non-Zero On a Finite Interval

2.1 Introduction

For a model of test-particle motion in an ionized plasma, we consider a specific class of diffusion coefficients. This choice of coefficient lends itself to analysis. The models are one-dimensional and the diffusion coefficients are non-zero in a finite interval. This choice reduces the problem to solving equations in three regimes. Two of the equations are simple wave equations. The other is an eigenvalue problem on a finite interval. Without completely specifying the diffusion coefficient, much can be said about the runaway current. There are two non-trivial cases when exact solutions can be given in terms of known, special functions.

We start by displaying the problem with its associated equations. We then give the general solution by constructing it from a Green's function. This requires examining the boundary conditions for the eigenvalue problem. The nature of the spectrum is also examined. For the large field limit, general results can be obtained for runaway current using a saddle point integration. A slightly simpler model for two different diffusion coefficients gives solutions in terms of known special functions. One gives Bessel functions and the other Gegenbauer polynomials.

The restriction of having the drag force vanish outside a finite interval limits the physical relevance of the problem. The more general case in which the diffusion coefficient vanishes in the limit of large velocity will be considered in the next chapter.

2.2 Equations for the Model Problem

The Fokker-Planck equation in conservation form is

$$\frac{\partial}{\partial t} f(u,t) + \frac{\partial}{\partial u} j(u,t) = 0 \qquad (2.2.1)$$

where the current, $j(u,t)$, in velocity space is given by

$$j(u,t) = -D(u)\left(\frac{\partial f}{\partial u} + 2uf\right) + 2\gamma f. \qquad (2.2.2)$$

Another convenient notation is to define the operator L by

$$Lf \equiv -\frac{\partial}{\partial u} j\left(f(u,t)\right). \qquad (2.2.3)$$

$f(u,t)$ is thought of as a distribution function so we would like it to satisfy the integral condition

$$\int_{-\infty}^{\infty} f(u,t)\,du < \infty.$$

The distribution function will in general be normalized to unity. It also satisfies all the properties of §1.3.

The diffusion coefficient, $D(u)$, will be chosen to be positive, even, non-zero on a finite interval and decreasing to zero at the end-points. The general case where $D(u) \sim u^{-m}$ as $|u| \to \infty$ will be examined in the next chapter. The diffusion coefficient will be of the form:

$$D(u) = \begin{cases} D_0(\frac{u}{u_0})(1 - \frac{u^2}{u_0^2}), & \text{if } |u| \le u_0; \\ 0, & \text{otherwise.} \end{cases} \qquad (2.2.4)$$

The function $D_0(\cdot)$ in (2.2.4) is an even, positive and smooth function of its argument. A typical diffusion coefficient of this form is shown in Figure (2.1). The
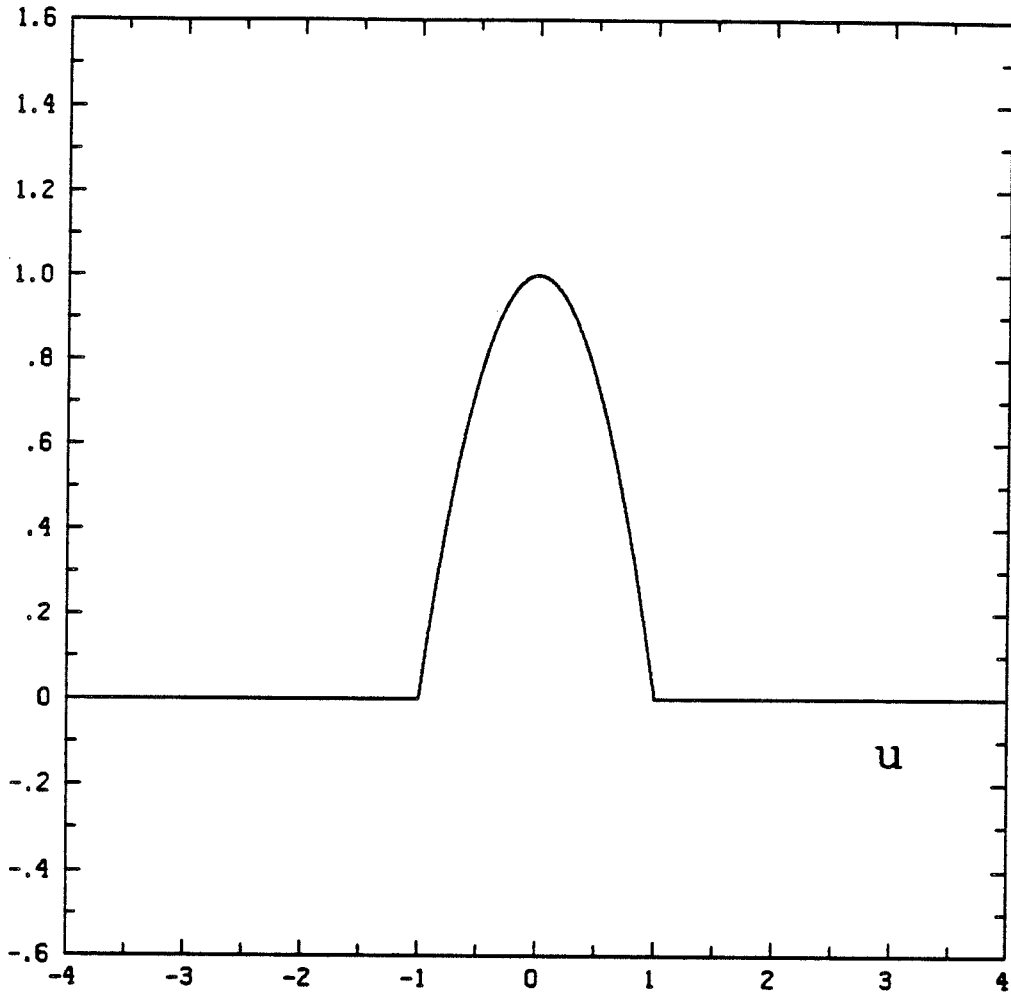
Figure 2.1 A typical diffusion coefficient that vanishes outside $|u_0|$.

quantity $D_0(\pm 1)$ will be denoted by D_0 in subsequent calculations. The drag force at the end-points is $2D_0/v_0$ using the definition (1.2.7).

To solve the kinetic equation on the finite interval $u < |u_0|$ subject to initial and boundary conditions, we look for separable solutions of (2.2.1).The time dependence will be exponential. Thus

$$f(u,t) = e^{-\lambda t}f(u).$$

The resulting ordinary differential equation for $f(u)$ is

$$(L + \lambda)f = \frac{d}{du}D(u)\left(\frac{df}{du} + 2uf\right) - 2\gamma\frac{df}{du} + \lambda f = 0. \tag{2.2.5}$$

The eigenvalues, λ, are positive in this notation. Equation (2.2.5) will be examined later.

When $|u| > u_0$, Equation (2.2.1) reduces to the first-order wave equation

$$\frac{\partial f}{\partial t} + 2\gamma\frac{\partial f}{\partial u} = 0. \tag{2.2.6}$$

This is also considered as an initial value problem. If $u < -u_0$ the wave solution imposes a current at $u = -u_0$ and so is coupled to the solution on $[-u_0, u_0]$. For $u > u_0$ the solution on the finite interval provides a signal at $u = u_0$ for the wave equation in (2.2.6).

2.3 The Green's Function Solution

The general solution for (2.2.1) will now be constructed. Equation (2.2.1) is considered to be an initial value problem in velocity space. That is,

$$f(u, 0) = F_0(u)$$

with $F_0(u)$ prescibed on $(-\infty, \infty)$. The general solution for an arbitrary initial condition can be given using the Green's function for the problem on $[-u_0, u_0]$. For $|u| > u_0$ the wave equation (2.2.6) will be solved directly.

The problem on the finite interval will be examined first. To calculate the general solution it is necessary to define and examine the adjoint equation as well. The companion adjoint equation to (2.2.3) is

$$\frac{\partial}{\partial t}g(u, t) = L^+ g(u, t) \tag{2.3.1}$$

with

$$L^+ g \equiv \left\{ \left(\frac{\partial}{\partial u} - 2u \right) D(u) + 2\gamma \right\} \frac{\partial g}{\partial u}$$

and initial condition $g(u,0) = g_0(u)$. The addition of (2.2.3) and (2.3.1) give the identity

$$gLf - fL^+ g = -\frac{\partial}{\partial u} \left(g(u,t) j \left(f(u,t) \right) + f(u,t) D(u) \frac{\partial g}{\partial u} \right). \tag{2.3.2}$$

The Laplace transformed versions of (2.2.1) and (2.3.1) are

$$(s - L)\tilde{f}(u,s) = f_0(u) \tag{2.3.3a}$$

$$(s - L^+)\tilde{g}(u,s) = g_0(u). \tag{2.3.3b}$$

We will now consider the problem (2.3.3) on the interval $[-u_0, u_0]$. Boundary conditions for the two endpoints will be needed. Since the current in velocity space is of particular interest, we would like it to be bounded and continuous at the end-points. If $\frac{\partial f}{\partial u}$ is bounded, the definition of the current in (2.2.2) along with the choice of $D(u)$ implies $j(\pm u_0, t) = 2\gamma f(\pm u_0, t)$. Since 2γ is a constant, $f(\pm u_0, t)$ should be bounded and continuous at the end-points.

Consider the homogeneous version of (2.3.3). For $D(u)$ of the form (2.2.4), the end-points are regular singular points. The indicial exponents for (2.3.3a) are $(0, -\rho)$ at $u = u_0$ and $(0, \rho)$ at $u = -u_0$. The adjoint problem has indicial exponents $(0, \rho)$ at $u = u_0$ and $(0, -\rho)$ at $u = -u_0$. The quantity ρ is $\frac{\gamma u_0}{D_0}$ which is the ratio of the force due to the electric field to the drag force.

The boundary conditions for $\tilde{f}(u,s)$ and $\tilde{g}(u,s)$ can be determined directly from the indicial exponents. For the solution $\tilde{f}(u,s)$ to be bounded for $u \to u_0$ the negative exponent is rejected. If we consider the problem for zero imposed current at $u = -u_0$ then we reject the $O(1)$ solution valid near $u = -u_0$. Otherwise, it would describe a spontaneous current even when the initial condition was

identically zero for $u < -u_0$. Therefore, we take indicial exponent ρ at $u = -u_0$. Then the solution and the current vanish at $u = -u_0$.

If we denote the solution that satisfies the boundary condition at $u = -u_0$ as $h_0(u,s)$ and the solution that satisfies the boundary condition at $u = u_0$ as $h_1(u,s)$, then the Green's function is

$$\tilde{G}(u,u';s) = \frac{h_0(u_<,s)h_1(u_>,s)}{D(u)W(h_0,h_1)}. \tag{2.3.4}$$

$W(h_0,h_1)$ is the Wronskian of the two solutions. The general solution for $\tilde{f}(u,s)$ is then

$$\tilde{f}(u,s) = \int_{-u_0}^{u_0} \tilde{G}(u,u';s)f_0(u')\,du'.$$

It should be noted that $\tilde{f}(-u_0,s) = 0$ since $\tilde{G}(-u_0,u';s) = 0$. So this solution holds only when there is no imposed current at $u = -u_0$. To consider the general case the adjoint problem has to be analyzed.

Solutions of the adjoint equation can be connected to those of the original equation using a corollary of the identity (2.3.2). Using the Laplace transformed equations of (2.3.3) with (2.3.2) one gets the identity

$$\int_{-u_0}^{u_0} \left(\tilde{g}(u,s)f_0(u) - \tilde{f}(u,s)g_0(u) \right)\,du =$$

$$\left(\tilde{g}(u,s)j\left(\tilde{f}(u,s)\right) + \tilde{f}(u,s)D(u)\frac{\partial}{\partial u}\tilde{g}(u,s) \right)\Bigg|_{-u_0}^{u_0}. \tag{2.3.5}$$

This holds for any pair of solutions of (2.3.3). If we take $\tilde{f}(u,s)$ to be the Green's function (2.3.4) in (2.3.5), then

$$\tilde{g}(u,s) - \int_{-u_0}^{u_0} \tilde{G}(u,u';s)g_0(u)\,du = \left(\tilde{g}(u,s)j(\tilde{f}(u,s)) + \tilde{f}(u,s)D(u)\frac{\partial}{\partial u}\tilde{g}(u,s) \right)\Bigg|_{-u_0}^{u_0}. \tag{2.3.6}$$

For boundary conditions of the adjoint equation we require that $\tilde{g}(u,s)$ vanish at $u = u_0$ and be $O(1)$ at $u = -u_0$. This corresponds to indicial exponents ρ and

0, respectively. Then the integrated terms vanish at $\pm u_0$. If $g_0(u) = \delta(u - w)$ for the forcing term of (2.3.3b), then the solution is the Green's function $\tilde{G}^+(u', w; s)$ of the adjoint problem. Equation (2.3.6) then becomes

$$\tilde{G}^+(u', w; s) - \tilde{G}(w, u'; s) = 0. \qquad (2.3.7)$$

Hence the Green's functions are just transposes of each other.

To solve the general problem for an imposed current at $u = -u_0$ and an arbitrary initial $f_0(u)$, take $g_0(u) = \delta(u - w)$ and $\tilde{g}(u, s) = \tilde{G}^+(u, w; s)$ in the identity (2.3.5). This gives

$$\tilde{f}(u, s) = \int_{-u_0}^{u_0} G(u, u'; s) f_0(u') \, du' + G(u, -u_0; s) j\left(\tilde{f}(-u_0, s)\right)$$

with the property (2.3.7) being used. Laplace inversion gives the result

$$f(u, t) = \int_{-u_0}^{u_0} G(u, u'; t) f_0(u') \, du' + \int_0^t G(u, -u_0; t - \tau) j\left(f(-u_0, \tau)\right) \, d\tau.$$

Using the definition of $j(f)$ in (2.2.2) this simplifies to

$$f(u, t) = \int_{-u_0}^{u_0} G(u, u'; t) f_0(u') \, du' + 2\gamma \int_0^t G(u, -u_0; t - \tau) f(-u_0, \tau) \, d\tau. \qquad (2.3.8)$$

The time evolution of the distribution is due to two effects. One is due to the initial condition and the other due to the imposed current at the endpoint $u = -u_0$.

To complete the picture for the general solution, the regions with $|u| > u_0$ have to be considered. Then $f(-u_0, t)$ can be given in (2.3.8). For $u < -u_0$ the wave equation (2.2.6) is to be solved subject to an initial condition. The solution is

$$f(u, t) = f_0(u - 2\gamma t) \qquad u < -u_0.$$

Then $f(-u_0, t) = f_0(-u_0 - 2\gamma t)$ gives the imposed distribution function in (2.3.8). For $u > u_0$ the solution in (2.3.8) evaluated at $u = u_0$ provides the signal for the wave. The solution is then

$$f(u, t) = \begin{cases} f(u_0, t - \frac{u - u_0}{2\gamma}), & \text{if } u - u_0 \leq 2\gamma t; \\ f_0(u - 2\gamma t), & \text{otherwise.} \end{cases}$$

This completes the solution for the general case.

For the remainder of the discussion in this chapter we will restrict $f_0(u)$ to be non-zero only on $[-u_0, u_0]$. Then for $u < -u_0$, $f(u,t) \equiv 0$ for all time. This is the case of zero-imposed current. This simplifies (2.3.8) to

$$f(u,t) = \int_{-u_0}^{u_0} G(u,u';t) f_0(u') \, du'.$$

The solution for $u > u_0$ is then due to the signal only and simplifies to

$$f(u,t) = \begin{cases} f\left(u_0, t - \frac{u - u_0}{2\gamma}\right), & \text{if } u - u_0 \le 2\gamma t; \\ 0, & \text{otherwise.} \end{cases}$$

2.4 Other Properties of the Equation

Another approach to specifying the boundary conditions for (2.2.5) is to consider the Weyl theory for the cases of limit circle and limit point. When (2.2.5) is multiplied by

$$w(u) = \exp\left(u^2 - 2\gamma \int_0^u \frac{dv}{D(v)}\right) \tag{2.4.1}$$

the equation (2.2.5) can be brought to self-adjoint form

$$\frac{d}{du}\left(D(u)w(u)\frac{df}{du}\right) + \left(\frac{d}{du}\left(2uD(u)\right) + \lambda\right)w(u)f(u) = 0 \tag{2.4.2}$$

with $w(u)$ the weight function. We look for square integrable solutions with

$$\int_{-u_0}^{u_0} w(u) f^2(u) \, du < \infty. \tag{2.4.3}$$

The weight function has the following behaviors near the end-points:

$$w(u) \sim \left(1 + \frac{u}{u_0}\right)^{-\rho} \qquad u \to -u_0$$

$$w(u) \sim \left(1 - \frac{u}{u_0}\right)^{\rho} \qquad u \to u_0.$$

For $0 < \rho < 1$, all solutions of (2.2.5) are square integrable with respect to the weight function $w(u)$. This is the limit circle case and an additional condition

will be needed to specify the solution. This extra condition is the final boundary condition that is needed for the right end-point. It should also be noted that the more physical requirement that $f(u)$ be integrable, that is, satisfy

$$\int_{-u_0}^{u_0} f(u)\,du < \infty,$$

is also satisfied for any of the solutions when $0 < \rho < 1$. But those solutions with

$$f \sim \left(1 - \frac{u}{u_0}\right)^{-\rho} \qquad u \to u_0$$

may not be wanted because $f(u)$ is unbounded at $u = u_0$.

The final condition follows from the definiteness of the operator L. To determine the condition, the proof of Property (1.3.4) is followed but now integration is on the interval $(-u_0, u_0)$. Using (1.3.4) one now gets

$$\lambda \int_{-u_0}^{u_0} wf^2\,du = wfj \Big|_{-u_0}^{u_0} + \int_{-u_0}^{u_0} w^{-1}(u)D(u)\left(\frac{d}{du}(wf)\right)^2 du. \qquad (2.4.4)$$

Because the operator L is semi-positive definite, the integrated terms must vanish at $\pm u_0$. It follows from the properties of $w(u)$ and $j(u,t)$ that the solutions of (2.2.5) should have the following behaviors,

$$f(u) \sim \left(1 + \frac{u}{u_0}\right)^{\rho} \qquad u \to -u_0 \qquad (2.4.5a)$$

$$f(u) = O(1) \qquad u \to u_0. \qquad (2.4.5b)$$

When $\rho \geq 1$ the limit point case is in effect since only one of the two solutions at $u = u_0$ satisfies the square integrability condition (2.4.3). This case requires that

$$f(u) = O(1) \qquad u \to u_0.$$

This corresponds to indicial exponent 0.

The special case when $\rho = N$, a positive integer, doesn't present a problem as the Frobenius theory guarantees a solution of the Frobenius form for the larger of

the two indicial exponents. Hence the conditions of (2.4.5) apply in this case as well.

The problem (2.2.5) on $[-u_0, u_0]$ is now set with boundary conditions at the end-points specified.

Using the self-adjoint form of the equation (2.4.2), one can show by the usual multiply and subtract method that the eigenfunctions associated with different eigenvalues are orthogonal with respect to the weight function.

Additional information about the solution can be found by examining the equivalent Schrödinger equation for the problem. The choice of D(u) determines the features of the potential. Potentials for diffusion coefficients of the form (2.2.4) all have basically the same shape as that shown in the Figure (2.2). The transformation of dependent variables by

$$h(u) = \sqrt{D(u)w(u)}f(u)$$

brings (2.4.2) into the equation:

$$\frac{d^2h}{du^2} + \left(\frac{\lambda}{D(u)} - q^{-1/2}\frac{d^2}{du^2}q^{1/2} \right) h = 0$$

with $q(u) = D^{1/2}(u)w^{-1/2}(u)$. Further simplification to eliminate the multiplying factor from the λ term can be achieved with the transformation:

$$\frac{dx}{du} = D^{-1/2}(u) \qquad\qquad (2.4.6a)$$

$$H(x) = D^{-1/4}(u)h(u), \qquad\qquad (2.4.6b)$$

to give the resulting Schrödinger equation

$$\frac{d^2H}{dx^2} + \left(\lambda - \hat{q}^{-1/2}\frac{d^2}{dx^2}\hat{q}^{1/2} \right) H(x) = 0$$
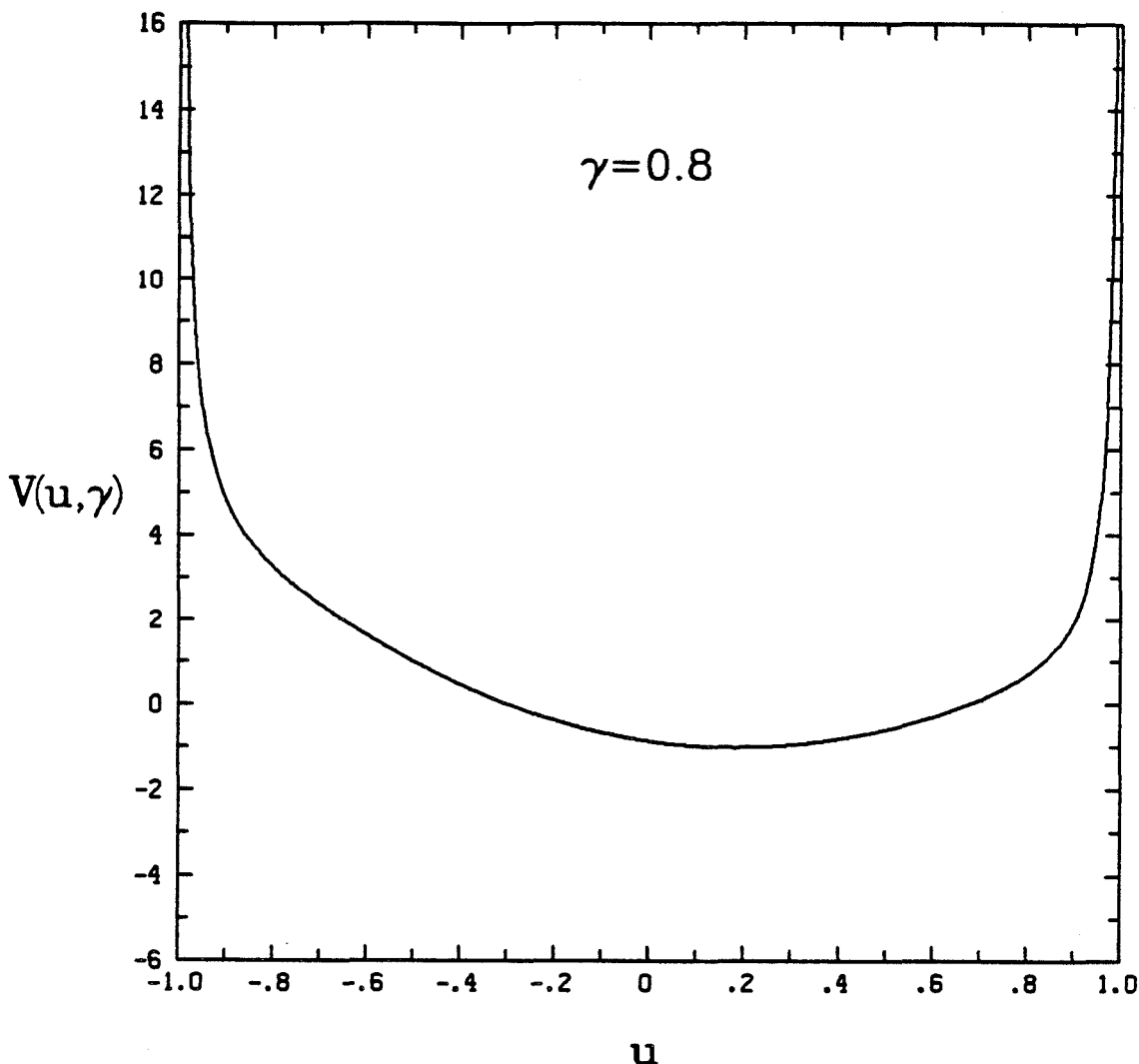
Figure 2.2 The equivalent potential for $D(u)$ of the form (2.2.4).

with $\hat{q}(x) = D^{-1/2}(u)q(u)$.

The change of independent variables in (2.4.6a) determines x as a monotonic, increasing and odd function of u which maps the interval $[-u_0, u_0]$ onto $[-x_0, x_0]$. This can be seen by noting that the derivative x_u is strictly positive on $[-u_0, u_0]$ and changes sign under reflection of u. The potential can be written in terms of $D(u)$ and its derivatives as

$$V(x(u)) = \frac{1}{4}D_{uu} - \frac{1}{16}\frac{D_u^2}{D} - (uD)_u + u^2 D - 2\gamma u + \gamma^2 D^{-1}. \qquad (2.4.7)$$

Figure (2.2) shows a typical case. As noted in the figure or by examining (2.4.7), one sees that the potential becomes unbounded at the end-points and hence by [22] the spectrum is an infinity of discrete eigenvalues.

If the current in (2.2.2) is defined without the $2uf$ term the potential takes the simpler form

$$V(x(u)) = \frac{1}{4}D_{uu} - \frac{1}{16}\frac{D_u^2}{D} + \gamma^2 D^{-1}.$$

In this case $V(x)$ is an even function of x and the solutions $H_{\lambda_n}(x)$ will alternate with either even or odd eigenfunctions of x. The eigenfunctions of the original problem, $f(u)$, will also have parity to within a common multiplying function; that is,

$$f_{\lambda_n}(u) = g(u)F_{\lambda_n}(u) \tag{2.4.8}$$

with $F_{\lambda_n}(u)$ having parity. This simpler case holds in the examples of §2.6. It corresponds to a partial high-temperature version of the problem.

2.5 Runaway Current

When the electric field γ is small an initial distribution will evolve in time in the following way. On a time scale, $t = O(1)$, the distribution will approach a Maxwellian distribution which is the equilibrium solution when $\gamma = 0$. But since this is not an equilibrium solution for $\gamma \neq 0$, the solution will decay on $[-u_0, u_0]$. The particles then propagate out from u_0 on a time scale $t = O(\gamma^{-1})$ and are accelerated to unbounded speeds with their distribution being a travelling wave.

The eigenvalues of the problem (2.2.5) are all $O(1)$ except the smallest which is $O(\gamma)$. The field dependence of λ_0 can be shown by integrating (2.2.5) when $\lambda = \lambda_0$ and $f(u) = f_0(u)$, the corresponding eigenfunction, from $-u_0$ to u_0. The calculation gives

$$\lambda_0 \int_{-u_0}^{u_0} f_0(u)\, du = j(u_0) = 2\gamma f_0(u_0).$$

A first approximation to $f_0(u)$ is to take $f_0(u) = M(u)$, the Maxwellian distribution which is the eigenfunction for $\lambda_0 = 0$. Therefore,

$$\lambda_0 \sim 2\gamma \frac{M(u_0)}{\int_{-u_0}^{u_0} M(u)\, du}. \tag{2.5.1}$$

So λ_0 is $O(\gamma)$ and this appears to be independent of the shape of this cut-off $D(u)$. Also the coefficient of γ in (2.5.1) vanishes for $u_0 \to \infty$, suggesting that something interesting occurs when $D(u) \to 0$ for $u \to \infty$.

The results of this calculation allow one to calculate the laboratory current $J(t)$ as defined in (1.2.5). After the exponentially decaying transients have died away, the resulting current is due to $f(u,t)$ for $u > u_0$. In this region particles experience a constant acceleration 2γ, so one expects the current to be proportional to $2\gamma t$.

To show this we calculate the integral

$$\int_{u_0}^{u(t)} u f\big(\tfrac{1}{2\gamma}(u(t) - u)\big)\, du \qquad u(t) = u_0 + 2\gamma t, \tag{2.5.2}$$

which is the dominant contribution to the current as defined in (1.2.5). It is sufficient to let

$$f(u,t) = c_0 f_0(u) e^{-\lambda_0 t} \tag{2.5.3}$$

as this is the mode that gives the largest contribution to (2.5.2). The solution $f_0(u)$ is assumed normalized on $[-1,1]$ and c_0 is its projection onto the initial distribution. Integration of (2.5.2) using (2.5.3) gives

$$J(t) \sim n_0 q c_0 f_0(u_0) \frac{(2\gamma)^2}{\lambda_0} t. \tag{2.5.4}$$

Then use the fact that $\lambda_0 = 2\gamma f_0(u_0)$ to get the desired result that $J(t) \sim 2 n_0 q c_0 \gamma t$.

The current pulse in velocity space satisfies a simple conservation law. Integration of the differential equation (2.2.1) with respect to u over $[-u_0, u_0]$ and then with respect to time gives

$$\int_0^\infty j(u_0, t)\, dt = \int_{-u_0}^{u_0} f(u,0)\, du.$$

If initially the distribution is zero outside the interval $[-u_0, u_0]$, and its integral is normalized to unity, then

$$\int_0^\infty j(u_0, t)\, dt = 1.$$

This is a statement that the total current at $u = u_0$ over all time is constant.

2.6 The Large Field Limit for the General Problem

The general class of diffusion coefficients with $D(u)$ given by (2.2.4) can be examined in some detail. For large electric field ($\gamma \gg 1$), asymptotic analysis using a WKB expansion can be done. In this limiting case the effects due to diffusion which relax the solutions to a Maxwellian are small compared to the electric field effect. From these approximations one can find expressions for the shape of the current pulse $j(u_0, t)$. These expressions are given in terms of the properties of the diffusion coefficient, e.g., the area of $D(u)$ and the drag force at the right endpoint. Expressions for the current are found by finding the Green's function for the original Fokker-Planck equation in the limit of large field.

We look for solutions of the Laplace transformed equation (2.2.1) of the form

$$\tilde{f}(u; \epsilon; \sigma) = \exp(-\frac{1}{\epsilon}\varphi(u; \epsilon; \sigma)) \tag{2.6.1a}$$

$$\varphi(u; \epsilon; \sigma) = \sum_{n=0}^\infty \varphi(u; \sigma)\epsilon^n \tag{2.6.1b}$$

with small parameter $\epsilon = (2\gamma)^{-1}$ and σ is the scaled Laplace transform variable defined by $\sigma = \epsilon^2 s$ (since we're interested in the slow time scale) of the electric field. The initial distribution will be a delta function at the origin; $f(u, 0) = \delta(u)$.

We construct the Green's function for (2.2.5) in terms of the solutions of the homogeneous problem. Substitution of (2.6.1) into (2.2.5) gives

$$(L - \epsilon^{-2}\sigma)\tilde{f}(u; \epsilon; \sigma) = 0$$

and grouping of powers of ϵ give the following two equations

$$D(u)\left(\frac{d\varphi_0}{du}\right)^2 + \frac{d\varphi_0}{du} - \sigma = 0 \tag{2.6.2a}$$

$$\left(1 + 2D(u)\frac{d\varphi_0}{du}\right)\frac{d\varphi_1}{du} = \frac{d}{du}\left(D(u)\frac{d\varphi_0}{du}\right) + 2uD(u)\frac{d\varphi_0}{du}. \tag{2.6.2b}$$

Equations (2.6.2a) and (2.6.2b) determine φ_0 and φ_1, respectively. Solving (2.6.2) gives the solutions

$$\varphi_0(u;\epsilon;\sigma) = \int_0^u \frac{2\sigma}{\sqrt{1 + 4D(v)\sigma} + 1}\, dv \tag{2.6.3a}$$

$$\varphi_1(u;\epsilon;\sigma) = \tfrac{1}{4}\log\frac{1 + 4D(u)\sigma}{1 + 4D(0)\sigma} + \int_0^u v\left(1 - \frac{1}{\sqrt{1 + 4D(v)\sigma}}\right)\, dv. \tag{2.6.3b}$$

The distribution function $\tilde{f}(u;\epsilon;\sigma)$ at $u = u_0$ is then found to be

$$\tilde{f}(u_0;\epsilon;\sigma) = \epsilon A(u_0,\sigma)\exp\left(-\frac{1}{\epsilon}\int_0^{u_0}\frac{2\sigma}{\sqrt{1 + 4D(v)\sigma} + 1}\, dv\right) \tag{2.6.4}$$

where

$$A(u_0,\sigma) = (1 + 4D(0)\sigma)^{-1/4}\exp\left(-\int_0^{u_0} v\left(1 - \frac{1}{\sqrt{1 + 4D(v)\sigma}}\right)\, dv\right).$$

The approximate solution in (2.6.4) satisfies $\tilde{j}(u_0,0) = 2\gamma\tilde{f}(u_0;\epsilon;0) = 1$, the conservation relation for the Laplace transform of $j(u_0,t)$.

The current can be written as a Laplace inversion along a Bromwich contour as

$$j(u_0,t) = \frac{1}{2\epsilon^2\pi i}\int_C A(u_0,\sigma)\exp\frac{1}{\epsilon}\left(\sigma\tau - \varphi_0(\sigma)\right)\, d\sigma$$

with $\varphi_0(\sigma) \equiv \varphi_0(u_0;\epsilon;\sigma)$ as in (2.6.3a) and $t \equiv \epsilon\tau$. The dominant contribution to $j(u_0,t)$ for large field can be found by a saddle point calculation. The saddle point $\sigma_0 = \sigma(\tau)$ is found from the solution of

$$\tau = \varphi_0'(\sigma_0) = \int_0^{u_0}\frac{dv}{\sqrt{1 + 4D(v)\sigma_0}}. \tag{2.6.5}$$

For σ_0 real and positive the integrand is bounded by 1 and hence a solution exists for $0 < \tau \leq \tau_0$ with $\tau_0 = u_0$. As τ increases, σ_0 becomes negative until the denominator vanishes at $4D(0)\sigma = -1$. Any further analysis of the saddle point would require examining the function (2.6.5) on another sheet of the Riemann surface. The second derivative is negative on the interval $0 \leq \tau \leq \tau_0$ since

$$\varphi'' = -\int_0^{u_0} \frac{2D(v)\, dv}{(1 + 4D(v)\sigma)^{3/2}}.$$

This has positive integrand on $[0, u_0]$. The integration is along a contour parallel to the imaginary axis and through the saddle point and gives the following result:

$$j(u_0, \tau) \sim (2\gamma)^{3/2} \frac{A(u_0, \sigma_0)}{\sqrt{2\pi |\varphi_0''(\sigma_0)|}} \exp \frac{1}{\epsilon} (\sigma_0 \tau - \varphi_0(\sigma_0)). \qquad (2.6.6)$$

The τ dependence is from the saddle point $\sigma_0 = \sigma_0(\tau)$.

We shall examine two limiting cases of this saddle-point formula. First we consider (2.6.6) for $\tau \simeq \tau_0$. In that case σ_0 is small, so (2.6.5) can be approximated by

$$\tau = \int_0^{u_0} (1 - 2\sigma_0 D(v))\, dv + O(\sigma_0^2). \qquad (2.6.7)$$

Therefore,

$$\tau - \tau_0 = -2\sigma_0 D_1 + O(\sigma_0^2),$$

where the constant

$$D_1 \equiv \int_0^{u_0} D(u)\, du.$$

Evaluation of $\varphi_0(\sigma)$ at σ_0 determined from (2.6.7) gives

$$\varphi_0(\sigma_0) = \sigma_0 \tau_0 - \sigma_0^2 D_1 + O(\sigma_0^3)$$

when (2.6.3a) is evaluated. The saddle-point formula (2.6.6) gives the current

$$j(u_0, \tau) = \frac{2\gamma}{\sqrt{4\pi\epsilon D_1}} \exp -\frac{(\tau - \tau_0)^2}{4\epsilon D_1}$$

which is a Gaussian about $\tau = \tau_0$. This expression can be written in the more transparent form

$$j(u_0, t) = \frac{2\gamma}{\sqrt{4\pi\epsilon D_1}} \exp -\frac{\left(t - \frac{u_0}{2\gamma}\right)^2}{4\epsilon^3 D_1}, \tag{2.6.8}$$

which shows that the current peaks at $t = \frac{u_0}{2\gamma}$ with a very narrow width of $\sqrt{4\epsilon^3 D_1}$. This peak occurs in the time required for a particle to accelerate in a constant field of strength 2γ to a speed u_0.

To calculate the small time behavior requires knowing the expansion of $\varphi_0'(\sigma)$ in (2.6.5) for large σ. This requires more subtle analysis than in the previous case since $\varphi_0'(\sigma)$ cannot be expanded uniformly for large σ, since $D(u)$ is small for $u \to u_0$. This *boundary layer* within an integral requires the type of analysis as illustrated in, for example [2]. The calculation requires splitting the integral at an intermediate point and then using appropriate expansions of the integrand valid in each interval. The final result is found by asymptotic matching with the intermediate point dropping out. It is convenient to find $\varphi_0(\sigma)$ as $\sigma \to \infty$ since this is needed for the saddle-point formula and then $\varphi_0'(\sigma)$ can be found by differentiation. The details of this calculation will be left to Appendix 1. The result of this calculation is

$$\varphi_0(\sigma) \sim D_* \sigma^{-1/2} - \frac{1}{2\alpha}\log\alpha\sigma - \frac{1}{\alpha} - \frac{1}{2}J + O(\sigma^{-3/2}) \qquad \sigma \to \infty \tag{2.6.9}$$

with

$$D_* \equiv \int_0^{u_0} D^{-1/2}(u)\, du,$$

$$\alpha \equiv -D'(u_0)$$

$$J \equiv \int_0^{u_0} \left(\frac{1}{D(u)} + \frac{1}{\alpha(u - u_0)}\right) du.$$

To determine the saddle point for given τ small, differentiate (2.6.9) with respect to σ and solve (2.6.5) perturbatively. This gives the saddle point

$$\sigma_0 \sim \left(\frac{D_*}{2\tau}\right)^2 - \frac{1}{\alpha\tau} + O(1) \qquad \tau \to 0.$$

Using the formula for the saddle point, (2.6.6), gives the resulting current

$$j(u_0, t) \sim \frac{(2\gamma)^{3/2}}{\sqrt{\pi D_* \sqrt{D(0)}}} \left(\frac{D_*}{4\gamma t}\right)^{1+2\gamma/\alpha} \exp\left(-\frac{D_*^2}{4t} + \Lambda\right) \qquad (2.6.10)$$

with

$$\Lambda \equiv \frac{\gamma}{\alpha}\log\alpha - \frac{2\gamma}{\alpha} + \gamma J - \tfrac{1}{2}u_0^2 = constant$$

for $\tau \to 0$. The form of $j(u_0, t)$ is typical of what would be expected for a diffusion process.

For the partial high-temperature version of the problem the only change is in the form of Λ. The $\frac{1}{2}u_0^2$ term which is due to the Maxwellian equilibrium is not present.

2.7 Examples of Exact Solutions

There are two examples which can be solved exactly for the partial high temperature version of the problem. For this version, the $2uf$ term is dropped from (2.2.2). The boundary conditions of §2.3 still apply. The solutions $f_n(u,t) = e^{-\lambda_n t}f_n(u)$ are of the form $f_n(u) = g(u)F_n(u)$ with $F_n(u)$ either even or odd. This follows because (2.4.8) applies. It is convenient to scale the velocity variable by $u = u_0 x$ so that $|x| \le 1$. Then $D(u) = D_0(x)(1 - x^2)$. Also in the scaling of $D(u)$ and the definition of 2γ, v_B is replaced by u_0. The resulting equation is

$$\frac{\partial}{\partial t}f(u,t) = \frac{\partial}{\partial x}\left(D_0(x)(1 - x^2)\frac{\partial f}{\partial u} - 2\gamma f\right).$$

In the first example $D_0(x)$ is taken to be unity which leads to Gegenbauer polynomials. The second has $D_0(x) = 2(1 + |x|)^{-1}$ which has Bessel functions for solutions. The general solutions are exhibited. From the Green's function representation, saddle-point integration gives short and intermediate time behavior for the current.

Example 1: $D_0(x) = 1$

The Fokker-Planck equation in this case is

$$\frac{d}{dx}(1 - x^2)\frac{df}{dx} - 2\gamma\frac{df}{dx} + \lambda f = 0 \qquad |x| \leq 1. \qquad (2.7.1)$$

A change of independent variables $y = \frac{1}{2}(1 + x)$ gives the hypergeometric equation in standard form:

$$y(1 - y)\frac{d^2 f}{dy^2} + (1 - 2y - \gamma)\frac{df}{dy} + \lambda f = 0. \qquad (2.7.2)$$

The eigenvalues can be found by finding the Green's function for (2.7.2) with its boundary conditions, and locating the poles and calculating their associated residues which are the eigenvalues and eigenfunctions, respectively. They can also be found in a simple way by first making the change of dependent variable

$$f(x) = (1 + x)^\gamma h(x)$$

in (2.7.1). Then $h(x)$ satisfies the equation

$$(1 - x^2)\frac{d^2 h}{dx^2} - (1 + 2\gamma)x\frac{dh}{dx} + \lambda h = 0.$$

This can be recognized with the help of [1] to be the Gegenbauer equation when

$$\lambda_n = (n + \gamma)(n + \gamma + 1)$$

and the eigenfunctions are

$$f_n(x) = (1 + x)^\gamma C_n^{(\gamma + \frac{1}{2})}(x) \qquad (2.7.3)$$

with the $C_n^{(\alpha)}$ being the Gegenbauer polynomial of order n. The solutions in (2.7.3) clearly satisfy the boundary conditions of §2.3 at the end-points.

As predicted in §2.3 the solutions have parity to within a fixed function since the $C_n^{(\alpha)}(x)$ alternate being even and odd in x. Of passing interest is that when

$\gamma = 0$ the solutions reduce to Legendre polynomials and when $\gamma = \frac{1}{2}$ the solutions are to within a fixed function the Chebyshev polynomials of the second kind. The Gegenbauer polynomials are complete and orthogonal on $[-1, 1]$ with a weight function of $(1 - x^2)^\gamma$. The general solution is given by

$$f(x, t) = (1 + x)^\gamma \sum_{n=0}^{\infty} a_n C_n^{(\gamma + \frac{1}{2})}(x) e^{-\lambda_n t} \tag{2.7.4}$$

with the coefficients determined from

$$a_n \theta_n = \int_{-1}^{1} (1 - x)^\gamma C_n^{(\gamma + \frac{1}{2})}(x) F(x) \, dx$$

where $F(x)$ is the given initial distribution. The normalization constant is given by

$$\theta_n = \frac{\pi 2^{-2\gamma} \Gamma(n + 1 + 2\gamma)}{n! \left(n + \gamma + \frac{1}{2}\right) \left(\Gamma(\gamma + \frac{1}{2})\right)^2}.$$

The current is given by

$$j(x, t) = 2\gamma (1 + x)^\gamma \sum_{n=0}^{\infty} a_n C_n^{(\gamma + \frac{1}{2})}(x) e^{-\lambda_n t} \tag{2.7.5}$$

and at $x = 1$ is

$$j(1, t) = \gamma 2^{\gamma + 1} \sum_{n=0}^{\infty} a_n C_n^{(\gamma + \frac{1}{2})}(1) e^{-\lambda_n t}$$

with

$$C_n^{(\gamma + \frac{1}{2})}(1) = \frac{\Gamma(n + 2\gamma + 1)}{n! \, \Gamma(2\gamma + 1)}.$$

The series for the distribution function in (2.7.4) shows explicitly the results of the general discussion. The distribution function does not reach a final steady state. The modes with $n > 0$ decay on a fast time scale since $\lambda_n = O(1)$ and the lowest mode with $n = 0$ decays on the slow time scale since $\lambda_0 = \gamma(\gamma + 1) = O(\gamma)$. The important contribution to the leakage current is due to this lowest mode and is given by

$$j(1, t) \sim \gamma 2^{\gamma + 1} a_0 e^{-\gamma(\gamma + 1)t} \qquad t \to \infty$$

where the fact that $C_0^{(\alpha)}(x) = 1$ has been used. The laboratory current computed from (1.2.5) for the lowest mode is

$$J(t) \sim \frac{4\gamma t}{\gamma + 1} + \gamma e^{-\lambda_0 t}$$

with the dominant contribution for large time being

$$J(t) \sim \frac{4\gamma t}{\gamma + 1} + o(t) \qquad t \to \infty.$$

The laboratory current for large times is due only to particles moving in a constant field and agrees with the general result (2.5.4).

With initial condition $F(x) = \delta(x)$ the odd modes drop from the sum (2.7.4) and solutions have some interesting properties. The current in (2.7.5) takes the form

$$j(1,t) = \gamma 2^{\gamma+1} \sum_{n=0}^{\infty} \theta_{2n} C_{2n}^{(\gamma+\frac{1}{2})}(0) C_{2n}^{(\gamma+\frac{1}{2})}(1) e^{-\lambda_{2n} t}. \qquad (2.7.6)$$

Using the identity

$$C_{2n}^{(\gamma+\frac{1}{2})}(0) = \frac{(-)^n \Gamma(\gamma + n + \frac{1}{2})}{n! \, \Gamma(\gamma + \frac{1}{2})}$$

(2.7.6) simplifies to

$$j(1,t) = \frac{\gamma 2^{\gamma+2}}{\sqrt{\pi}\Gamma(\gamma+1)} e^{t/4} \sum_{n=0}^{\infty} \frac{(-)^n}{n!}(n+\beta)\Gamma(n+2\beta) e^{-4(n+\beta)^2 t} \qquad (2.7.7)$$

where $\beta \equiv \frac{1}{2}(\gamma + \frac{1}{2})$.

It is interesting to note that (2.7.7) resembles the type of expansion that appears in the theory of theta functions. When $\gamma = \frac{1}{2}$ the sum is just the derivative of ϑ_1 in the notation of Whittaker and Watson [23]. That is, when $\gamma = \frac{1}{2}$ the current is

$$j(1,t) = \pi^{-1} 2^{3/2} e^{t/4} \left(\frac{d}{dy} \vartheta_1(y, e^{-4t}) \right) \Bigg|_{y=0}. \qquad (2.7.8)$$

Using the Jacobi transformation for theta functions we can get an expansion for the current which gives the small time behavior. The transformation for the derivative of ϑ_1 is

$$\left(\frac{d}{dy}\vartheta_1(y, e^{-4t})\right)\bigg|_{y=0} = \left(\frac{\pi}{4t}\right)^{3/2}\left(\frac{d}{dy}\vartheta_1(y, e^{-\frac{\pi^2}{4t}})\right)\bigg|_{y=0}. \qquad (2.7.9)$$

One gets the following expansion

$$j(1,t) = \sqrt{\pi}(2t)^{-3/2}e^{t/4}\sum_{n=0}^{\infty}(-)^n(2n+1)\exp -\frac{(n+\frac{1}{2})^2\pi^2}{4t}$$

when (2.7.9) is applied to (2.7.8). The dominant term for small time is

$$j(1,t) \sim \sqrt{\pi}(2t)^{-3/2}e^{-\pi^2/16t} \qquad t \to 0^+.$$

This confirms the general theory that was shown in §2.6 concerning the small time behavior. For this choice of diffusion coefficient, $D_* = \frac{\pi}{2}$, $D(0) = 1$ and $\alpha = 2$ in (2.6.10).

The current can also be written in terms of the ϑ_1 function for β any integer or half-integer. For these β the current can also be written as a finite sum of the time derivatives of the ϑ_1 function. This follows from the properties of the gamma function. This method becomes cumbersome for obtaining representations of the solution useful for small time. It is best to write the Green's function for the problem and manipulate that to get the short time behavior as well as intermediate times.

The Laplace transformed equation is (2.7.1) with $\lambda = -s$ with the Green's function in terms of hypergeometric functions. The Green's function is given by

$$g(y, y'; s) = \frac{g_1(y_<; s)g_2(y_>; s)}{w(s)}$$

where $g_1(y; s)$ satisfies the left end-point boundary condition and $g_2(y; s)$ the right. The appropriate solutions to (2.7.1) are

$$g_1(y; s) = y^\gamma(1-y)^\gamma F(a, 1-a; 1+\gamma; y)$$

$$g_2(y; s) = F(a, 1 - a; 1 + \gamma; 1 - y)$$

with $s = a(1 - a)$ and $y = \frac{1}{2}(1 + x)$.

The Wronskian $w(s)$ is found to be

$$w(s) = \frac{(\Gamma(1 + \gamma))^2}{\Gamma(\gamma + a)\Gamma(\gamma + 1 - a)}.$$

The zeros of $w(s)$ are simple as expected for a discrete spectrum. They are determined from the location of the poles of the gamma function, that is, when

$$a = -n - \gamma$$

with n a non-negative integer which corresponds to

$$s = -(n + \gamma)(n + \gamma + 1).$$

The residues of the Green's function at these poles give the relation

$$F(-n - \gamma, n + \gamma + 1; 1 + \gamma; 1 - y) = (1 + x)^\gamma C_n^{(\gamma + \frac{1}{2})}(x).$$

A simple Laplace inversion will reproduce (2.7.4).

The current is given by the integral

$$\tilde{j}(1; s) = 2\gamma \int_0^1 F_0(y') g(y, y'; s) \, dy'$$

and if $F_0(y) = \delta(y - \frac{1}{2})$ the current is given by

$$\tilde{j}(1; s) = \frac{\gamma 2^{\gamma - 1}}{\sqrt{\pi}\Gamma(1 + \gamma)} \Gamma(\beta + i\sqrt{s'})\Gamma(\beta - i\sqrt{s'}) \tag{2.7.10}$$

with $s' = \frac{1}{4}(s - \frac{1}{4})$. The fact that probability is conserved requires that $\tilde{j}(1; 0) = 1$. In the context of (2.7.10) this is just a restatement of the duplication formula for the gamma function .

The case when $\beta = \frac{1}{2}$ gives the simplification of (2.7.10) to

$$\tilde{j}(1; s) = \pi \operatorname{sech} \pi \sqrt{s'},$$

which is the Laplace transform of the derivative of the ϑ_1 function. The short time expansion can be found by expanding (2.7.10) for large s. Using the asymptotic relation

$$\Gamma(\beta + i\sqrt{s'})\Gamma(\beta - i\sqrt{s'}) \sim 2\pi(s')^{\beta - \frac{1}{2}}e^{-\pi\sqrt{s'}} \qquad s \to \infty,$$

then (2.7.10) can be inverted to give

$$j(t) \sim \sqrt{\frac{8}{\pi}}\frac{1}{\Gamma(\gamma)}\left(\frac{\pi}{4t}\right)^{\gamma+1}e^{-\pi^2/16t} \qquad t \to 0^+. \tag{2.7.11}$$

(2.7.11) gives the short time behavior on both the slow and the fast time scales. It exhibits a rise for short time, then reaches a maximum and then decreases. (2.7.11) agrees exactly in form with the general expression (2.6.10).

The behavior of the current at longer times can be found from a saddle-point calculation for large field. First, scale the Laplace transform variable by $s' = \beta^2\sigma$. Then (2.7.10) becomes

$$\tilde{j}(1;\sigma) = \frac{\gamma 2^{\gamma-1}}{\sqrt{\pi}\Gamma(1+\gamma)}\Gamma(\beta\phi_+)\Gamma(\beta\phi_-) \tag{2.7.12}$$

with $\phi_\pm = 1 \pm i\sqrt{\sigma}$. Applying Stirling's asymptotic formula for the gamma function gives

$$\Gamma(\beta\phi_+)\Gamma(\beta\phi_-) \sim \frac{(\Gamma(\beta))^2}{\sqrt{1+\sigma}}\exp(\beta\phi + O(\beta^{-1})) \qquad \beta \to \infty$$

where $\phi = \phi_+ \log\phi_+ + \phi_- \log\phi_-$. Since we are interested in $t \sim (2\gamma)^{-1}$ we introduce a scaling of the time variable

$$\tau = 4\beta t.$$

The Laplace inversion of (2.7.12) for large β is then

$$j(t) \sim \frac{4\beta^2}{2\pi i}\int_C \frac{e^{\beta(\sigma\tau+\phi(\sigma))}}{\sqrt{1+\sigma}}\,d\sigma \qquad \beta \to \infty.$$

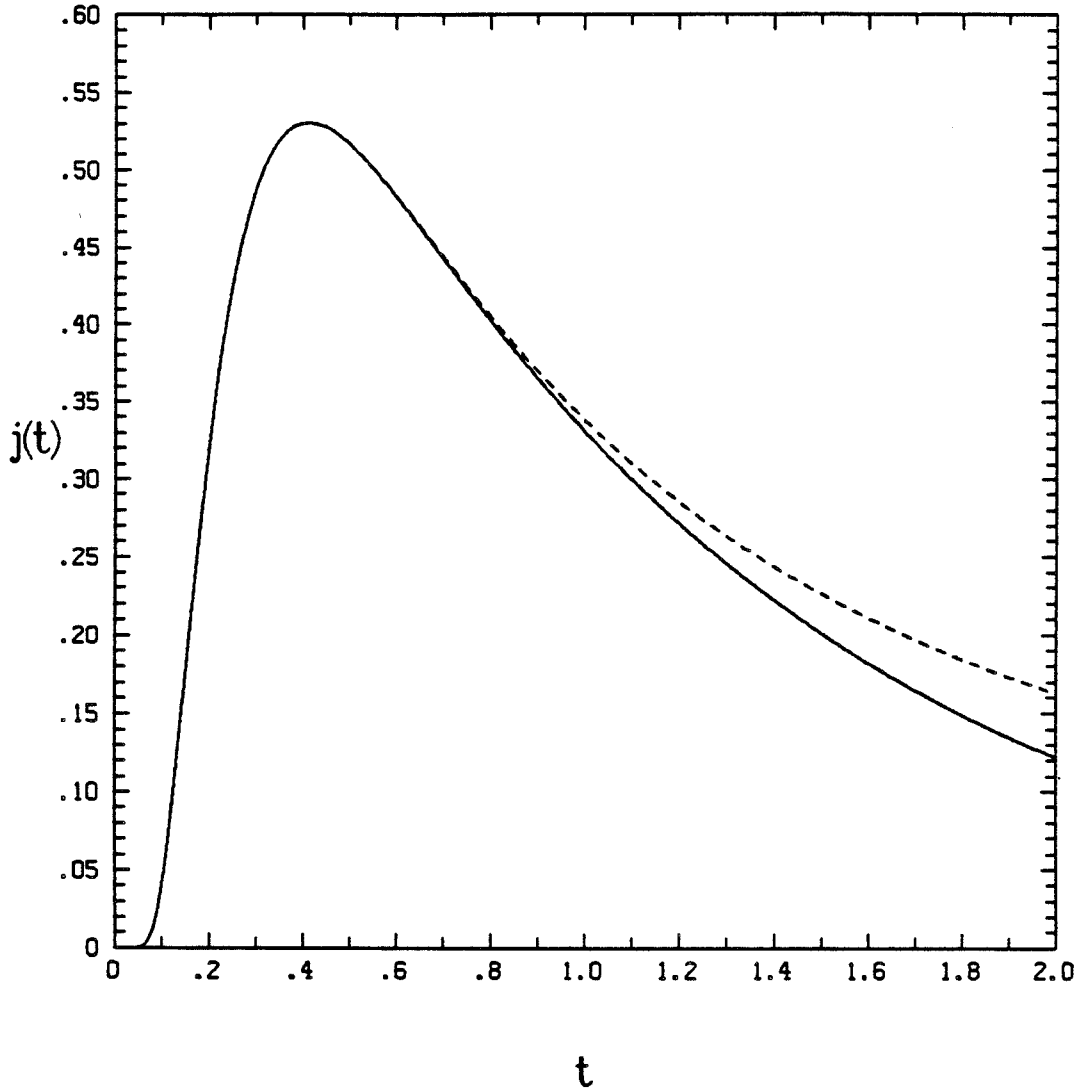j(t)

t

Figure 2.3 Comparison of the asymptotics and the exact current for short time. The dashed line is the asymptotic expression.

The saddle point $\sigma_0(\tau)$ is determined by solving

$$\tau = -\frac{d\phi}{d\sigma} = \frac{1}{2i\sqrt{\sigma}} \log \frac{1 + i\sqrt{\sigma}}{1 - i\sqrt{\sigma}} \tag{2.7.13}$$

or equivalently

$$\tau = \frac{1}{\sqrt{\sigma}} \tan^{-1}\sqrt{\sigma}.$$

Equation (2.7.13) has real solutions for τ real and maps $\tau \in [0, \infty)$ onto $\sigma_0 \in [-1, \infty)$. The contribution to the current from the saddle point is

$$j(t) = 2\gamma \sqrt{\frac{\beta}{2\pi \phi''(\sigma_0)}} \frac{\exp\left(\beta(\sigma_0 \tau + \phi(\sigma_0))\right)}{\sqrt{1 + \sigma_0}}. \qquad (2.7.14)$$

This gives excellent representation of the current for short times, intermediate and long times. The saddle points in these three cases are:

$$\sigma_0 \sim \left(\frac{\pi}{2\tau}\right)^2 - 2\tau^{-1} + O(1) \qquad \tau \to 0^+ \qquad (2.7.15a)$$

$$\sigma_0 \sim 3(\tau - 1) - \tfrac{27}{5}(\tau - 1)^2 + O((\tau - 1)^3) \qquad 0 < \tau - 1 \ll 1 \qquad (2.7.15b)$$

$$\sigma_0 \sim -1 + 4e^{-2\tau} \qquad \tau \to \infty. \qquad (2.7.15c)$$

When the saddle-point formula (2.7.14) is applied to (2.7.15a), the short time of (2.7.11) is obtained. Figure (2.3) shows how well the asymptotics agree with the exact current of (2.7.7). When (2.7.14) is evaluated for (2.7.15b) we get a Gaussian of the form

$$j(t) = 2\gamma \sqrt{\frac{3\beta}{2\pi}} e^{-3\beta(1-\tau)^2/2} \qquad |\tau - 1| < 1.$$

This is precisely (2.6.8) with $D_1 = \tfrac{2}{3}$ and $\epsilon = (4\beta)^{-1}$. Finally, the saddle-point formula when applied to (2.7.15c) gives the long time exponential fall-off due to the lowest eigenvalue.

Example 2: $D_0(x) = 2(1 + |x|)^{-1}$

Now the Fokker-Planck equation is reduced to Bessel's equation. Because one of the coefficients in the differential equation

$$\frac{d}{dx}\left(\frac{2}{1 + |x|}\frac{df}{dx}\right) - 2\gamma\frac{df}{dx} + \lambda f = 0$$

is an absolute value function, the derivative of a solution is required to be continuous at the origin.

For $x > 0$ the solution satisfying the boundary condition at $x = 1$ is

$$f(x) = (1 - x)^{-\gamma/2} J_\gamma(\sqrt{2\lambda(1-x)}) \qquad 0 \le x \le 1 \qquad (2.7.16)$$

and for $x < 0$ the solution satisfying the boundary condition at $x = -1$ is

$$f(x) = (1 + x)^{\gamma/2} J_\gamma(\sqrt{2\lambda(1+x)}) \qquad -1 \le x \le 0. \qquad (2.7.17)$$

Matching the derivatives of (2.7.16) and (2.7.17) gives the eigenvalue condition

$$J_\gamma'(\sqrt{2\lambda}) = 0; \qquad (2.7.18)$$

thus the eigenvalues are

$$\lambda_n = \tfrac{1}{2}(r_\gamma^n)^2 \qquad (2.7.19)$$

where r_γ^n is the nth zero of $J_\gamma'(x)$. The eigenfunctions are

$$f_n(x) = \begin{cases} (1 - x)^{-\gamma/2} J_\gamma(\sqrt{2\lambda_n(1-x)}) & 0 \le x \le 1 \\ (1 + x)^{\gamma/2} J_\gamma(\sqrt{2\lambda_n(1+x)}) & -1 \le x \le 0, \end{cases} \qquad (2.7.20)$$

which are orthogonal with respect to the weight function

$$w(x) = \begin{cases} (1 - x)^\gamma & 0 \le x \le 1 \\ (1 + x)^\gamma & -1 \le x \le 0, \end{cases} \cdot$$

The normalization is given by

$$\int_{-1}^{1} w(x) f_n^2(x)\, dx = 2(1 - \tfrac{\gamma^2}{2\lambda_n}) J_\gamma^2(\sqrt{2\lambda_n}).$$

Again the general solution is given by a series and in this case is

$$f_n(x) = \sum_{n=0}^{\infty} a_n f_n(x) e^{-\lambda_n t}$$

with $f_n(x)$ as in (2.7.20) and the λ_n as in (2.7.19). The lowest mode gives the description of the runaway current for $\gamma \ll 0$. An approximation for λ_0 can be found by solving (2.7.18) approximately. Since $r_\gamma^0 \to 0$ for $\gamma \to 0$ we can find r_γ^0 by expanding $J_\gamma'(x)$ for small argument. The expansion is

$$J_\gamma'(x) = 0 = \frac{\gamma 2^{-\gamma} x^{\gamma-1}}{\Gamma(\gamma+1)} - \frac{(\gamma+2)2^{-\gamma-2} x^{\gamma+1}}{\Gamma(\gamma+2)} + O(x^{\gamma+3})$$

and when set equal to zero implies that

$$x^2 = 2\gamma + o(x^2).$$

Then use the definition of x to obtain

$$\lambda_0 = \gamma + O(\gamma^2).$$

This confirms the prediction of §2.5.

The Green's function for the problem is given by

$$g(y, y'; s) = \frac{g_1(y_<; s) g_2(y_>; s)}{w(s)}$$

with

$$g_1(y; s) = (1 - x)^{-\gamma/2} J_\gamma\left(\sqrt{-2s(1 - x)}\right)$$

$$g_2(y; s) = (1 + x)^{\gamma/2} J_\gamma\left(\sqrt{-2s(1 + x)}\right)$$

and the Wronskian is

$$w(s) = \tfrac{1}{2}\sqrt{-2s} J_\gamma\left(\sqrt{-2s}\right) J_\gamma'\left(\sqrt{-2s}\right).$$

The Wronskian when set equal to zero contains the eigenvalue condition (2.7.18) as well as the extra condition $J_\gamma(\sqrt{-2s}) = 0$. This extra condition only leads to solutions whose derivative at the origin is discontinuous and so need not be considered.

As before with the previous example, consider a delta function for the initial distribution function. The current is given by

$$j(1; s) = 2\gamma g(0, 1; s) \tag{2.7.21}$$

with

$$g(0, 1; s) = \frac{\eta^{\gamma-1}}{2^{\gamma+1}\Gamma(1 + \gamma) J_\gamma'(\eta)} \tag{2.7.22}$$

and $\eta \equiv \sqrt{-2s}$. Conservation of probability holds since $j(1;0) = 1$.

The short time behavior is found by considering the large s behavior of (2.7.22), which is

$$g(0,1;s) \sim \sqrt{\frac{\pi}{2}} \frac{\xi^{\gamma-\frac{1}{2}}e^{-\xi}}{2^\gamma \Gamma(1+\gamma)} \qquad s \to \infty \qquad (2.7.23)$$

with $\xi \equiv \sqrt{2s}$. The Bessel function relation

$$J_\gamma(\eta) = e^{\gamma\pi i/2} I_\gamma(\xi)$$

has been used in calculating the large s behavior of (2.7.22). Equation (2.7.23) is then inverted to give

$$j(t) \sim \frac{2}{\Gamma(\gamma)}(2t)^{-\gamma-1}e^{-1/2t} \qquad t \to 0^+. \qquad (2.7.24)$$

Again there is agreement with the general result (2.6.10).

To consider the behavior of the current for large electric field, $\gamma \gg 1$, rewrite $\tilde{j}(1;s)$ in (2.7.21) using the scaled Laplace transform variable $s = \gamma^2\sigma$ and $u = \sqrt{-2\sigma}$. The equations (2.7.21) and (2.7.22) become

$$\tilde{j}(1;\sigma) = \left(\frac{\gamma}{2}\right)^\gamma \frac{u^{\gamma-1}}{\Gamma(1+\gamma)J'_\gamma(\gamma u)}. \qquad (2.7.25)$$

To express (2.7.25) asymptotically for large γ, write the Bessel function in terms of the Airy function of the first kind using [1]

$$J'_\gamma(\gamma u) \sim -\frac{2}{u\gamma^{2/3}} \left(\frac{1-u^2}{4\varsigma}\right)^{1/4} Ai'(\gamma^{2/3}\varsigma) \qquad \gamma \to \infty \qquad (2.7.26)$$

where the function $\varsigma(u)$ is defined by

$$\tfrac{2}{3}\varsigma^{3/2} = \log\left(\frac{1+\sqrt{1-u^2}}{u}\right) - \sqrt{1-u^2}.$$

Since we are interested in the case where σ is real and ς is bounded away from 0 we can further expand the Airy function in the usual way using

$$Ai'(x) \sim -\frac{1}{2\sqrt{\pi}}x^{1/4}e^{-2x^{3/2}/3} \qquad t \to \infty \qquad |\arg x| < \pi. \qquad (2.7.27)$$

Then using (2.7.27) in (2.7.26) gives

$$J'_\gamma(\gamma u) \sim \frac{1}{\sqrt{2\pi\gamma}} u^{-1}(1-u^2)^{1/4} e^{-2\gamma s^{3/2}/3} \qquad \gamma \to \infty.$$

The current $j(1;t)$ is then

$$j(1;t) = \left(\frac{\gamma}{2}\right)^\gamma \frac{\gamma^{5/2}}{\sqrt{2\pi} i \Gamma(1+\gamma)} \int_C h^{-1/2}(u) \exp\gamma\left(\gamma\sigma t + \log(1+h(u)) - h(u)\right) d\sigma$$

$$(2.7.28)$$

with C the usual Bromwich contour and $h(u) = \sqrt{1-u^2}$. The change of variables

$$y = 1 + 2\sigma$$

$$\tau = 2\gamma t$$

allows one to write (2.7.28) in the simpler form

$$j(1;\tau) = \left(\frac{\gamma}{2}\right)^\gamma \frac{\gamma^{5/2}}{2\sqrt{2\pi} i \Gamma(1+\gamma)} \int_C y^{-1/4} \exp\gamma\left(\tau(y-1)/4 + \varphi(y)\right) dy \quad (2.7.29)$$

with $\varphi(y) = \log(1+\sqrt{y}) - \sqrt{y}$.

The saddle point $y_0(\tau)$ for (2.7.29) is determined by solving

$$\tau = -4\varphi'(y)$$

or equivalently

$$(1+\sqrt{y})\tau = 2 \tag{2.7.30}$$

When (2.7.29) is evaluated at this saddle point one gets

$$j(1,\tau) \sim \frac{\gamma}{2\Gamma(\gamma)\sqrt{|\varphi''(y_0)|}} \left(\frac{\gamma}{2}\right)^\gamma y_0^{-1/4} \exp\gamma\left(\tau(y_0-1)/4 + \varphi(y_0)\right) \qquad \gamma \to \infty.$$

$$(2.7.31)$$

When $\tau \ll 1$, (2.7.31) reduces to

$$j(1,t) \sim \frac{2}{\Gamma(\gamma)} (2t)^{-\gamma-1} e^{-1/2t} \qquad t \to 0^+ \tag{2.7.32}$$

which surprisingly agrees with (2.7.24) as well as the general theory of §2.6. For this choice of $D(v)$, $D_* = \sqrt{2}$, $D(0) = 2$ and $\alpha = 2$. The maximum value of this current in (2.7.32) occurs at $2(\gamma + 1) = t$ and since $\gamma \gg 1$, this corresponds to $\tau = 1 + O(\gamma^{-1})$. Thus (2.7.32) holds for small time as well as times on the t scale through $O(\gamma^{-1})$ as (2.7.32) shows an increase to a peak and then a decrease.

To find the behavior for $\tau = O(1)$, let $\epsilon = \tau - 1$ and then use the saddle-point formula (2.7.30) to evaluate (2.7.31). This gives the following result

$$j(1,\tau) = \sqrt{\frac{\gamma}{2\pi}} \exp -\tfrac{\gamma}{2}(\tau - 1)^2$$

which is precisely (2.6.8) with $D_1 = 1$.

In this example it is interesting to note that there are simplifications when the field is half-integral. This is analogous to the previous example when $j(t)$ can be written in terms of the theta function and its derivatives. When γ is half-integral the Bessel function can be written as a finite sum of trigonometric functions and powers. The eigenvalue condition is just a trigonometric equation, but in this case the eigenvalues do not have a simple closed form.

This completes the analysis for those examples which have exact solutions in terms of known special functions.

CHAPTER 3

General Diffusion Coefficient

3.1 Introduction

Here, the runaway problem is considered for the more general case when the diffusion coefficient is such that for large velocities it decays to zero. The runaway current is calculated as a function of the field strength. It is found to be non-analytic in the field strength, in comparison to the cut-off problems where it is analytic.

In the analysis, first the time dependence of the Fokker-Planck equation is separated out. The resulting ordinary differential equation is then transformed via changes of independent and dependent variables into a Schrödinger equation which is more amenable to analysis. The properties of the original Fokker-Planck equation are now contained in the potential for this Schrödinger equation. The potential depends explicitly on the diffusion coefficient, its derivatives and the electric field parameter. The fact that the perturbation due to the field is of the singular variety is seen immediately from the form of the potential. From examining this potential using standard theorems for the spectrum of an operator, one can deduce how the spectrum changes for the electric field increasing from zero. The spectrum for zero field is a continuum plus a single discrete value at zero. This zero eigenvalue corresponds to the steady-state solution. For an arbitrarily small field the spectrum is an infinity of discrete values with zero not included. There is no steady state.

The smallest eigenvalue, which is shifted from the zero eigenvalue of the unperturbed problem, characterizes the runaway. It is shown to be exponentially small as a function of the electric field. This is analogous to the problem in quantum mechanics of radioactive decay where eigenvalues are shifted by an exponentially small imaginary part. The associated eigenfunction can be considered a quasi-steady state since it will linger for a time proportional to the reciprocal of the smallest eigenvalue. This will be very large for small field.

To calculate this function the Schrödinger equation is solved for small field in two regions of velocity space. In one region a uniformly valid solution for this eigenfunction is obtained using Langer's method of mapping one potential onto a simpler one for which the solution is known. In the other region the fact that the perturbation terms are small compared to the unperturbed potential is used to find a solution. To zeroth order, the solution is the steady-state solution for zero field. These two solutions are then matched in an overlap region. Using this calculated eigenfunction an expression for the smallest eigenvalue as a function of field is found. As expected it is exponentially small. It has the standard form of a transmission through a barrier due to tunnelling in quantum mechanics. The explicit dependence of the smallest eigenvalue on the electric field can be found by expanding the resulting integral asymptotically for small field strength. These calculations are shown in Appendix 2. This method of calculating the runaway current is also applied to the three-dimensional problem.

3.2 The Equivalent Schrödinger Problem

For the one-dimensional problem, the distribution function, $f(u,t)$, satisfies the Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial u}\left(D(u)\left(\frac{\partial f}{\partial u} + 2uf\right) - 2\gamma f\right) \qquad (3.2.1)$$

for $t > 0$ and $-\infty < u < \infty$. An initial condition

$$f(u, 0) = f_0(u)$$

is specified with $f_0(u)$ non-negative and integrable. The solution will satisfy

$$\int_{-\infty}^{\infty} f(u, t)\, du < \infty \tag{3.2.2}$$

for all time t. This is just the conservation relation (1.3.2). The diffusion coefficent has the property that

$$\lim_{|u| \to \infty} D(u) = 0.$$

A typical diffusion coefficient is shown in Figure (3.1). Without loss of generality one can take the electric field parameter, γ, to be positive which was noted in proposition (1.3.3).

For the analysis it is convenient to rewrite (3.2.1) as a Schrödinger equation. To remove the time dependence we seek separable solutions of the form

$$f(u, t) = e^{-\lambda t} f(u)$$

where λ is positive since the operator is semi-negative definite. Equation (3.2.1) is then the homogeneous, ordinary differential equation

$$\frac{d}{du}\left(D(u)\left(\frac{df}{du} + 2uf \right) - 2\gamma f \right) + \lambda f = 0. \tag{3.2.3}$$

When this equation (3.2.3) is written as a Schrödinger equation, the potential is determined by the particular choice of diffusion coefficent, $D(u)$, and depends explicitly on the small parameter γ. When the electric field is increased from zero one can see how the spectrum of the operator in (3.2.3) changes by examining the behavior of the potential. To make the correspondence to a Schrödinger equation, let

$$h(u) \equiv D^{1/2}(u) \exp\left(\tfrac{1}{2}u^2 - \gamma \int_0^u D^{-1}(v)\, dv \right) f(u). \tag{3.2.4}$$
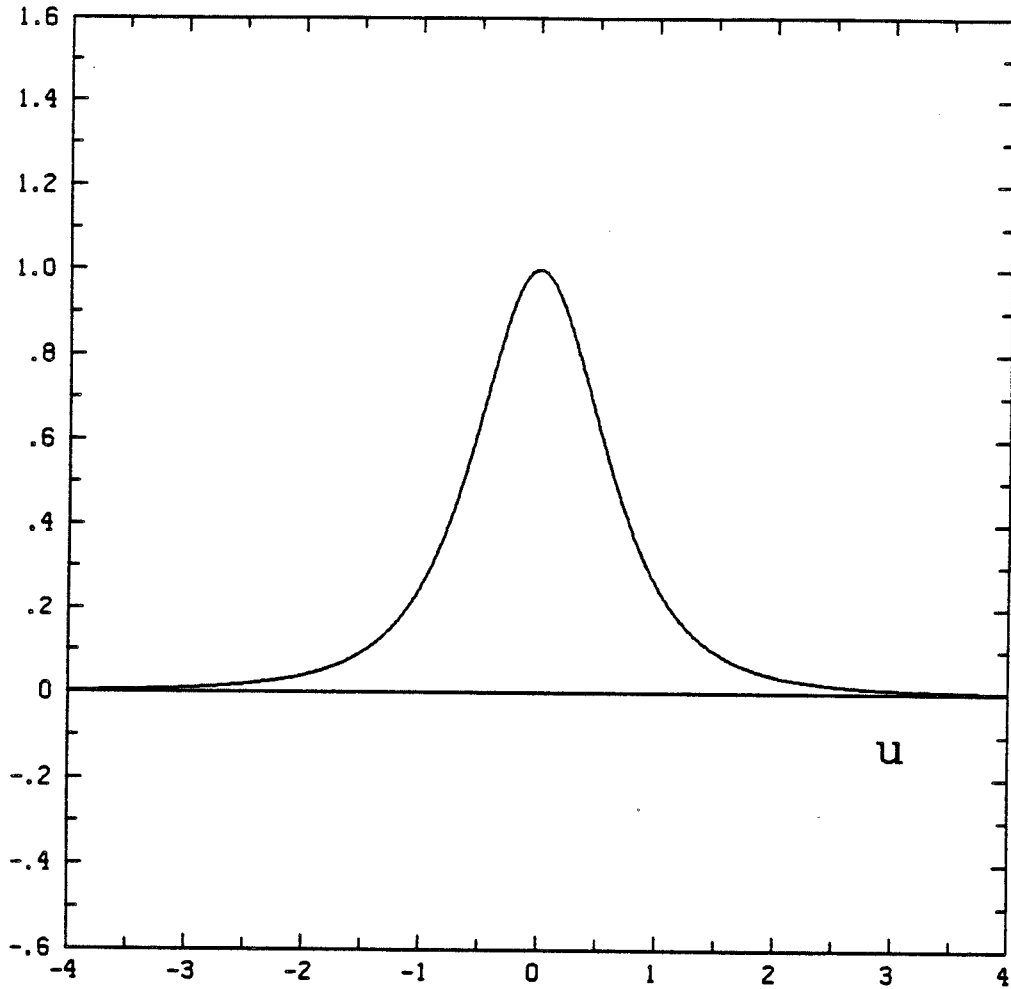
Figure 3.1 A typical diffusion coefficient.

Then (3.2.3) takes the form

$$\frac{d^2h}{du^2} + \left(\frac{\lambda}{D(u)} - q^{-1}\frac{d^2q}{du^2}\right)h(u) = 0 \qquad (3.2.5)$$

where

$$q(u;\gamma) = D^{1/2}(u)\exp\left(-\tfrac{1}{2}u^2 + \gamma\int_0^u D^{-1}(v)\,dv\right).$$

The equation (3.2.5) can be put in standard Schrödinger form by a further change of variables to *multiply* out the $D(u)$. This is accomplished by a Schwarz

transformation via the change of independent variable

$$\frac{dx}{du} = D^{-1/2}(u) \qquad (3.2.6)$$

and dependent variable

$$H(x) = \left(\frac{dx}{du}\right)^{1/2} h(u).$$

Equation (3.2.5) then becomes the Schrödinger equation

$$\frac{d^2 H}{dx^2} + \left(\lambda - \hat{q}^{-1}\frac{d^2\hat{q}}{dx^2}\right) H(x) = 0, \qquad (3.2.7)$$

where

$$\hat{q}(x;\gamma) = \frac{dx}{du} q\left(u(x);\gamma\right).$$

The equivalent potential in (3.2.7) is given by

$$V(x;\gamma) = \hat{q}^{-1}\frac{d^2}{dx^2}\hat{q}(x;\gamma).$$

The change of independent variables defined by (3.2.6) is well defined every-where and is differentiable on $u \in (-\infty, \infty)$ with

$$x(u) = \int_0^u D^{-1/2}(v)\, dv. \qquad (3.2.8)$$

Proposition 3.2.9 *The transformation $x(u)$ is a monotonic, odd function of u mapping the interval $u \in (-\infty, \infty)$ onto $x \in (-\infty, \infty)$.*

Proof. Since $D(u)$ is positive everywhere, $x(u)$ is strictly increasing. $D(u)$ is an even function of u, so $x(u)$ is odd in u. (The positive square root has been assumed in the integrand.) Also since $D^{-1/2}(u)$ becomes unbounded for large u, $x(u)$ must also become unbounded. ∎

The proposition shows that the transformation is no more than a stretching of the u coordinates.

It is more illustrative to write the potential in the old coordinates u remembering that u is a function of x as in (3.2.8). The potential is

$$\tilde{V}(u) \equiv V(x(u); \gamma) = \tfrac{1}{4}D_{uu} - \tfrac{1}{16}(D_u)^2 D^{-1} - (uD)_u + u^2 D - 2\gamma u + \gamma^2 D^{-1}, \quad (3.2.10)$$

with the u subscripts denoting differentiation with respect to u. The first two terms, $\tfrac{1}{4}D_{uu}$ and $\tfrac{1}{16}(D_u)^2 D^{-1}$, are due to diffusion effects only. The $(uD)_u$ and $u^2 D$ terms are due to the thermal effects (the $2uD(u)$ term in (3.2.1)). The remaining terms, $2\gamma u$ and $\gamma^2 D^{-1}$, are due to the perturbation of the electric field. The first of these is a *thermal-field* term and the other a *diffusion-field* term. The thermal terms were not present in the cut-off model as examined in the examples of §2.6 and shown in Figure (2.1).

It is also convenient notation to write the potential as

$$V(x) \equiv V_0(x) + \gamma V_1(x; \gamma).$$

The figures show a typical potential when

$$D(u) = \frac{1}{(1 + u^2)^2}.$$

Then the potential is given by

$$\tilde{V}(u; \gamma) = \frac{u^2 - 1}{(1 + u^2)^2} + \frac{4u^2 - 1}{(1 + u^2)^3} + \frac{5u^2}{(1 + u^2)^4} - 2\gamma u + \gamma^2(1 + u^2)^2.$$

The cases of zero electric field and non-zero electric field are shown in Figures (3.2) and (3.3), respectively. The plot is of $\tilde{V}(u; \gamma)$. The potential $V(x; \gamma)$ will have the same general shape as $\tilde{V}(u; \gamma)$ but with the u coordinate axis appropriately
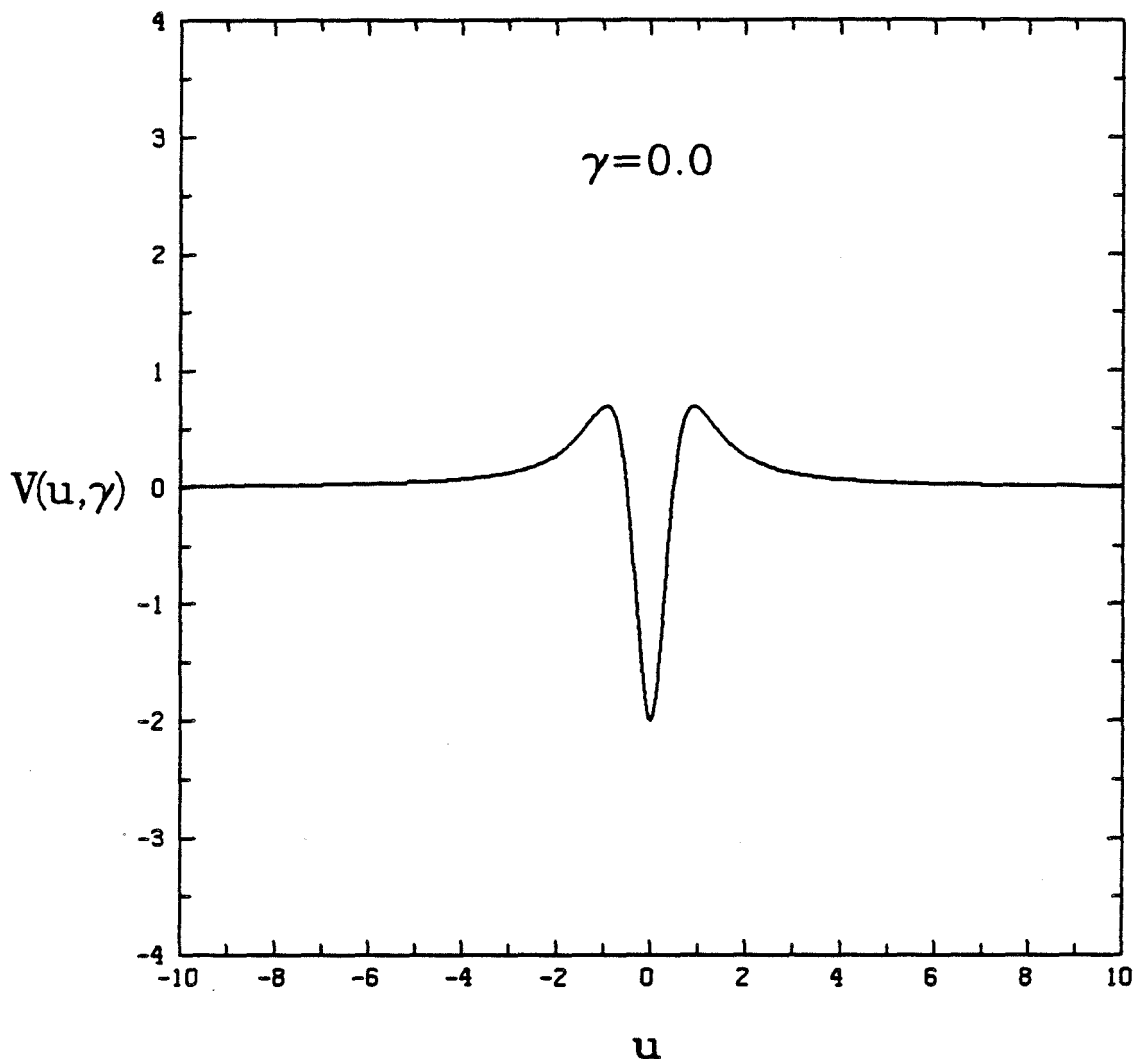
Figure 3.2 Equivalent potential for $\gamma = 0$.

stretched. The distinguishing feature of Figure (3.3) as compared to Figure (3.2) is the behavior of the potential for large u.

For later calculations, the critical points of the potential will have to be examined. In this regard we note two points on the u axis on the graphs of Figures (3.2) and (3.3). One occurs at the intersection of the potential and the u axis and is denoted u_0. The other, denoted u_1, occurs when the potential for non-zero field reaches a minimum in value. The potential reaches its minimum when the diffusion and thermal terms of the equivalent potential just balance the electric
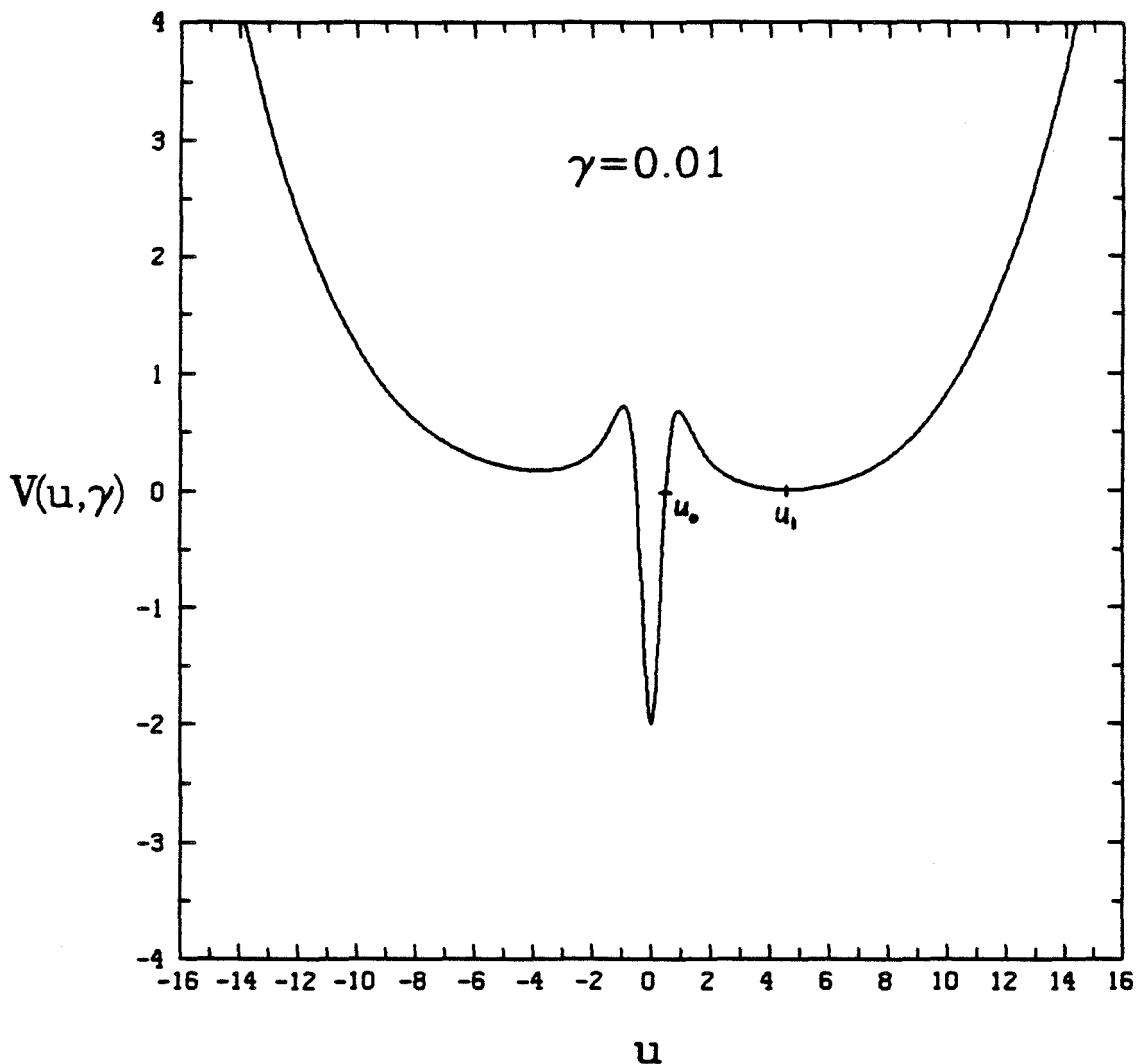
Figure 3.3 Equivalent potential for $\gamma \neq 0$.

field perturbation terms. The value of this minimum as a function of γ will be needed to determine if there are turning points in its vicinity. Equivalently x_0 and x_1 are the corresponds points for $V(x; \gamma)$.

The point u_0 or equivalently x_0 is determined from

$$V(x; \gamma) = 0$$

and is given by

$$x_0 = c_0 + \gamma c_1 + O(\gamma^2) \tag{3.2.12}$$

where c_0 and c_1 are constants dependent on the explicit form of $D(u)$. The corrections to the unperturbed x_0 are of order γ because the perturbation $|\gamma V_1| \ll 1$ in a neighborhood of c_0 for $\gamma \ll 1$.

The point x_1 or u_1 is determined from

$$\frac{d}{dx} V(x;\gamma) = \frac{d\tilde{V}}{du}\frac{du}{dx} = 0.$$

For γ decreasing to zero, the point x_1 moves out to infinity. This follows from the continuity of $V(x;\gamma)$ as a function of γ and from the fact that the minimum does not exist when $\gamma = 0$. The details of x_1 as a function of γ are given in the next section.

It can be noted from (3.2.10) by examining each term and using the evenness of $D(u)$ that

$$V(x;0) = V(-x;0),$$

but for non-zero γ

$$V(x;\gamma) \neq V(-x;\gamma).$$

The term $2\gamma u(x)$ is the only term in (3.2.10) that is an odd function of x and it is also responsible for the simple invariance property that was noted in Property (1.3.3).

The spectrum for this Schrödinger eigenvalue problem can be found by examining (3.2.10) or the figures and then applying the theorems of for example, Titchmarsh [22]. When $\gamma = 0$, $V(x;0) \to 0$ for $|x| \to \infty$ and for $V(x;0)$ integrable on $(-\infty, \infty)$, the spectrum is continuous in $(0, \infty)$ with a possible discrete spectrum in $(-\infty, 0]$. Because the operator in (3.2.3) is semi-positive definite, negative discrete values must be ruled out. In fact, $\lambda = 0$ is the only element of the discrete spectrum and corresponds to the Maxwellian solution. This problem was studied in detail in Corngold [6] and [7]. When $\gamma \neq 0$, $V(x;\gamma) \to \infty$ for $|x| \to \infty$, so

the spectrum is an infinity of discrete values. The elements of the discrete spectrum are strictly greater than zero. Zero cannot be included since there are no steady-state solutions as was noted already in §1.3.

It should be noted that this analysis of the spectrum only holds in the case where $D(u)$ decays to zero faster than u^{-2}. This guarantees that the $u^2D(u)$ term in (3.2.10) goes to zero for large values of u. When $D(u)$ does decay as u^{-2}, the spectrum is changed for $\gamma = 0$ with continuous spectrum now $(1, \infty)$.

The nature of the perturbation can be seen as the spectrum for the operator (3.2.3) or (3.2.7) changes dramatically from the $\gamma = 0$ case to one of arbitrarily small γ. The spectrum has changed from continuous plus a single discrete state to an infinity of discrete states. This can be expected since $V_1(x; \gamma)$ becomes unbounded for large x while $V_0(x)$ decays to zero no matter how small γ is. Thus the perturbation is a singular one.

Since we are seeking solutions of the Schrödinger equation corresponding to the discrete eigenvalues, the condition on the solutions is that

$$\int_{-\infty}^{\infty} H^2(x)\, dx < \infty. \tag{3.2.12}$$

Equivalently when the potential is an even function, we could seek solutions that are square integrable on $[0, \infty)$ and satisfy a boundary condition at 0. We will show later that solutions of (3.2.7) satisfying this boundary condition also give the correct asymptotic behavior for $f(u)$ when transformed back. From (3.2.3) the asymptotic behavior of $f(u)$ for large, positive u can be found. The leading term is

$$f(u) \sim \exp\frac{\lambda u}{2\gamma} \qquad u \to +\infty. \tag{3.2.13}$$

Note that for $\lambda > 0$ solutions with this asymptotic behavior are not integrable on (u, ∞). This asymptotic behavior also follows from an intuitive understanding of the Fokker-Planck equation (3.2.1). Consider (3.2.1) for fixed γ. Then there is a

velocity u such that, loosely speaking, the $2\gamma f$ term dominates the $D(u)$ terms. The equation is then essentially a wave equation with solutions behaving like

$$f(u,t) \sim \exp\lambda\left(u/2\gamma - t\right).$$

A few remarks should be made concerning a representation of the solution as an infinite sum. One expects to write the solution as

$$f(u,t) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n t} f_n(u).$$

The $f_n(u)$ are the transformed eigenfunctions of the Schrödinger problem. The a_n's are determined from the initial condition. Usually one finds that the lowest mode describes the solution uniformly in u for $t \to \infty$. This is not the case here. For a given large time, the contribution from the terms $n \neq 0$ is comparable to the term $n = 0$ for sufficiently large u. This is due to the $f_i(u)$'s having the asymptotic behavior of (3.2.13). Therefore, for increasing u, more and more terms of the sum are needed to accurately represent the solution. Physically, this also makes sense. Since electrons are uniformly accelerated by the field when diffusion effects are small, the distribution function becomes more weighted at large velocites as time increases. We do not have uniform convergence to $f_0(u)$ for arbitrarily large time.

This non-uniformity is illustrated when one considers whether the conservation principle holds. Since the spectrum is discrete with each eigenvalue's magnitude strictly greater than zero one would expect that the conservation property might be violated. The exponential time dependence of $f(u,t)$ seems to imply this. This anomaly is resolved by noting that interchanging integration with the limit $t \to \infty$ in (3.2.2) is not valid. The $t \to \infty$ limit cannot be interchanged with the $u \to \infty$ limit of the integration.

For fixed α and β the integral decays to zero with

$$\int_\alpha^\beta f(u,t)\,du \to 0 \qquad t \to \infty.$$

But for a fixed, arbitrarily large t_0, the integral relation

$$\int_{-\infty}^{\infty} f(u, t_0)\, du = 1$$

holds if the initial distribution function was normalized. This can be expected because for large u the equation (3.2.1) looks like a wave equation with profile moving off to the right. It would not be accurate to replace $f(u,t)$ in these integrals by the lowest mode due to the aforementioned non-uniformity.

3.3 Calculation of Smallest Eigenvalue with Its Solution

In this section an approximate eigenfunction will be constructed along with its corresponding eigenvalue for the smallest eigenvalue. The smallest eigenvalue, which will be seen later, is exponentially small and requires special care to find it. It characterizes the runaway current.

This dominating effect can be shown if one integrates (3.2.3) for $\lambda = \lambda_0$ with respect to u from $-\infty$ to u. Then

$$\lambda_0 \int_{-\infty}^{u} f_0(u', t)\, du' = \int_{-\infty}^{u} \frac{\partial}{\partial u} j\left(f_0(u)\right)\, du' = j\left(f_0(u)\right) - j\left(f_0(-\infty)\right).$$

$f_0(u)$ is the eigenfunction for λ_0. The smallest eigenvalue is then given by

$$\lambda_0 = \frac{j\left(f_0(u)\right) - j\left(f_0(-\infty)\right)}{\int_{-\infty}^{u} f_0(u')\, du'} \tag{3.3.1}$$

for arbitrary u. Evaluation of this quotient at a particular u will give λ_0. This corresponds roughly to the $j(1,t)$ of Chapter 2. Therefore, it is important to find λ_0 as a function of γ. The expression for λ_0 is analogous to the expression for the imaginary correction to the eigenvalues of the anharmonic oscillator given in [8]. In both cases we have a quotient of a current to a probability integral.

We expect $\lambda_0(\gamma)$ to be non-analytic at $\gamma = 0$ because any power series expansion of the form

$$\lambda_0(\gamma) = \sum_{n=0}^{\infty} a_n \gamma^n \qquad (3.3.2)$$

is doomed to fail. If it did converge for positive γ, then it will converge in a disc about the origin in the complex γ plane. This would imply that the perturbation due to the field in non-singular. This is a contradiction and is seen by examining the following slightly simpler problem. If we consider the simpler potential that occurs for the partial high temperature model we can see what happens. In this model the $2\gamma u$ term in not present in the potential and the perturbation only enters in the $\gamma^2 D^{-1}(u)$ term. Now apply the spectral theorems of [22] to see how the spectrum changes with γ in the complex plane. When γ is real the spectrum is purely discrete. When γ is purely imaginary, the specrum is continuous with spectrum $(-\infty, \infty)$. Therefore (3.3.2) couldn't possibly converge in a disc. So at most, any expansion of the form (3.3.2) would be asymptotic for $\gamma \to 0$. Hence the standard method of expanding the solution, $H(x)$, by a complete set of solutions of the unperturbed problem would fail.

We can expect $\lambda_0(\gamma)$ to have some type of singularity at $\gamma = 0$. A pole can be ruled out because λ_0 goes to zero with the field strength γ. Thus, $\lambda_0(\gamma)$ will have an algebraic or essential singularity at the origin. A rough idea of what can be expected for $\lambda_0(\gamma)$ can be found by considering a quantum mechanical analog. The potential in (3.2.10) is the quantum mechanical problem of a particle in a well onto which barriers have been added at a very large distance from the well. Recall that the barrier's position in (3.2.10) occurs when the perturbation becomes important and thus is large for small γ. Various potentials which mimic the potential of interest lead to Schrödinger equations which can be solved exactly in terms of elementary functions. A particularly simple example can illustrate this. Consider

the potential that consists of a delta function with negative strength at the origin with infinite barriers located at $x = \pm\epsilon^{-1}$ with $\epsilon \ll 1$. This is illustrated in Figure (3.4). There is a single discrete eigenvalue plus a continuous spectrum when $\epsilon = 0$. For $\epsilon \neq 0$ the spectrum is all discrete with the exact eigenvalue condition known in terms of elementary functions. Hence, the analogy to the problem of interest. When this condition is solved approximately when $\epsilon \ll 1$, the perturbation of the smallest eigenvalue is found to be proportional to $e^{-k/\epsilon}$ with k a constant. So we see that the correction is exponentially small. This shift is related to the transmission coefficient of the potential barrier with the constant k related to the *area* of the barrier. We then will expect this also to be the case for the more general potential in (3.2.10) with $\lambda_0(\gamma)$ being exponentially small. This is by no means a rigorous proof that the smallest eigenvalue is shifted by an exponentially small amount but this will be shown to be the case when the calculations of this section unfold.

This problem is also analogous to the problem of radioactive decay in quantum mechanics. The purely imaginary correction to a bound state eigenvalue is proportional to this transmission coefficient, with the lifetime of a state being inversely proportional to the eigenvalue shift. The anharmonic oscillator with potential $x^2 + \epsilon x^4$ is an example of such a problem. It has been analyzed in Bender and Wu, [3] and [4], and Corngold, Harrell and Simon [8] and as well in other references. In each of the references the exponentially small imaginary correction is calculated differently. In this runaway problem we then expect that the transmission due to the bump between x_0 and x_1 will be important. We can think of this in terms of particles exiting from the collision region on a time scale of the *lifetime* defined by some transmission coefficient and then becoming accelerated only by the electric field as it enters the region where this is the dominant effect.
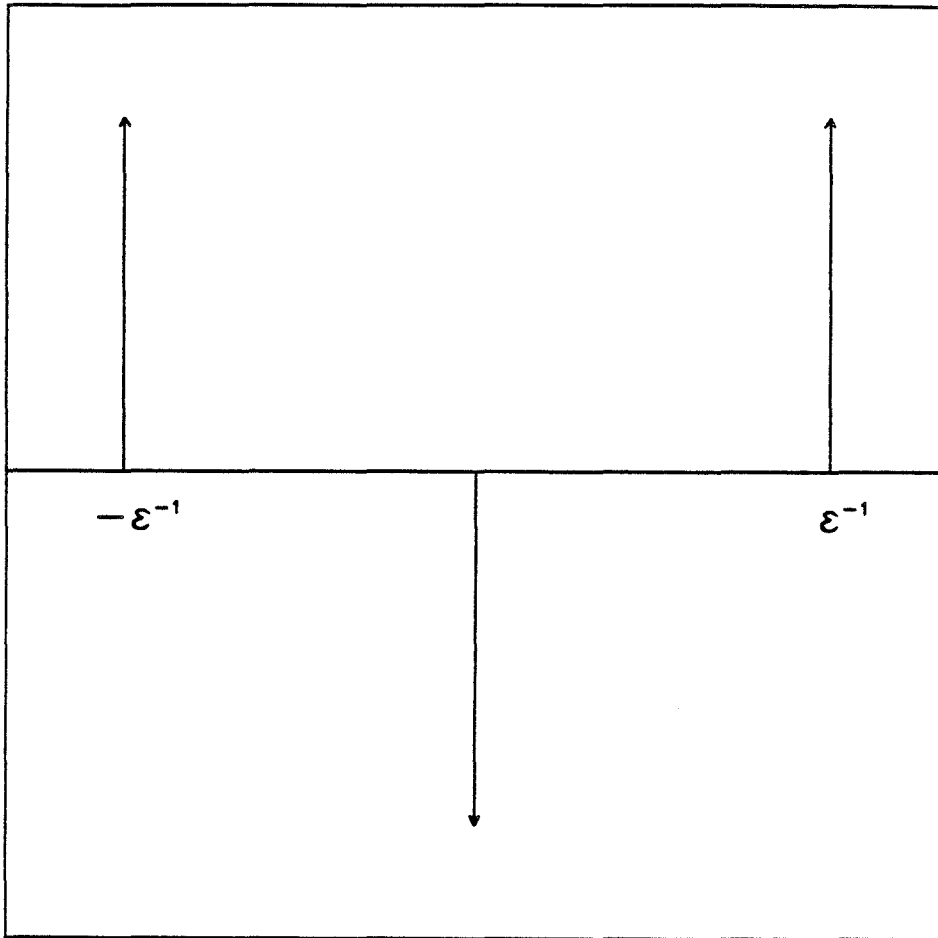
Figure 3.4 The delta function potential.

The standard method used to calculate approximate eigenfunctions requires matching WKB solutions, each valid in a certain region, asymptotically, in an overlap region. Bender and Wu's analysis of the anharmonic oscillator in [4] is typical of this procedure. The domain is divided into regions corresponding to being close and far from the turning points. Then WKB solutions are matched in overlap regions. The correction to the eigenvalue was also found through this matching. A similar approach can be done here but due to the shape of the potential (Figure (3.3)) it would require examining five different regions for positive

x. Matching between regions would then give one information for obtaining a formula for the exponentially small correction. (It is enough to consider positive x with similar analysis applying for negative values.) Corngold, Harrell and Simon [8] showed that the analysis of the anharmonic oscillator can also be done by using an idea of R. Langer [17]. This simplifies the calculations of [4]. Langer's approach will be used here. Langer showed that one could get uniform asymptotic approximations to the solution of a Schrödinger equation with a potential having a turning point of order ν. The idea is to map one Schrödinger equation into another via a change of variables for which the solution is known exactly. The potential in the new coordinates will be referred to as the comparison potential. Of course, the gross features of the potential like the number of turning points should be conserved under such a transformation. The Schwarz transformation is such that it maps one Schrödinger equation into another. With the use of this idea it will be possible to write down two expressions instead of five for the solution in the regime $x > 0$. When these two solutions are matched we have an approximate eigenfunction. One expression will be constructed via the Schwarz mapping. The other will be found by exploiting the smallness of γ and λ_0. The extension of this approximation for $u < 0$ can easily be found. This will be given near the end of this section.

3.3.1 Turning Points

To use the Langer procedure we need to know the location of the turning points of the potential. We now examine the potential $V(x; \gamma)$ in more detail.

The turning points for the problem are determined by solving

$$V\left(x_0(\gamma); \gamma\right) = \lambda_0(\gamma). \tag{3.3.3}$$

We will now show that there is only one turning point for $x > 0$.

To a first approximation, the turning point is the solution of (3.3.3) when $\gamma = 0$. Therefore, the turning point is the same constant c_0 of (3.2.11) to $O(\gamma)$ since it satisfies

$$V(x_0; 0) = 0.$$

Since the perturbation terms in (3.2.10) are $O(\gamma)$ for $x \ll x_1$, the next order correction to the turning point is

$$x_0 = c_0 + c_1\gamma + O(\gamma^2). \tag{3.3.4}$$

The constant c_1 can be determined by substituting (3.3.4) into (3.3.3), expanding in powers of γ and setting the coefficient of the γ power equal to zero. The actual numerical values of c_0 and c_1 are not needed in further calculations since all that is really important is the γ dependence of the turning point.

Letting $\Delta = V(x; \gamma) - \lambda$, the sign of Δ will determine whether there is a turning point near x_1 or not. x_1 is the location of the minimum of $V(x; \gamma)$. To determine the sign of Δ we need to know the value of $V(x_1; \gamma)$. It will then be neccessary to calculate x_1 as a function of γ. The details will also be needed in later calculations involving the comparison potential. To do this the asymptotic behavior of the diffusion coefficient will have to be specified. Assume $D(u)$ has the following asymptotic behavior for large u:

$$D(u) \sim u^{-m} + \sum_{n=1}^{\infty} a_n u^{-m-n} \qquad u \to +\infty \tag{3.3.5}$$

with $m > 2$. The point u_1 (or x_1) is determined from

$$\frac{d\tilde{V}}{du} = \frac{1}{4}D_{uuu} - \frac{1}{8}D^{-1}D_u D_{uu} + \frac{1}{16}(D^{-1}D_u)^2 D_u$$

$$-(uD)_{uu} + (u^2 D)_u - 2\gamma - \gamma^2 D^{-2}D_u = 0. \tag{3.3.6}$$

We are interested in the point $x_1 \gg x_0$, so we consider (3.3.6) asymptotically for large u using (3.3.5). Equation (3.3.6) then takes the form

$$\frac{d\tilde{V}}{du} \sim \gamma^2 u^{m-1}(m - a_1(m-1)u^{-1} + (a_1^2 - a_2)u^{-2} + O(u^{-3})) - 2\gamma$$

$$-u^{1-m}((m-2) + a_1(m-1)u^{-1} + m(m+a_2-1)u^{-2} + O(u^{-3})). \qquad (3.3.7)$$

To solve (3.3.7) when set equal to zero, we solve perturbatively, exploiting the smallness of γ. First, it can be noted that a first approximation to the solution of (3.3.6) is

$$u_1 = \gamma^{-1/(m-1)}, \qquad (3.3.8)$$

or in x coordinates

$$x_1 = \frac{2}{m+2}\gamma^{-(m+2)/2(m-1)}. \qquad (3.3.9)$$

This approximation is found by scaling u with a power of γ to find the dominant balance when the perturbation terms of the potential balance the other terms. Then we let

$$\epsilon u_1 = \tilde{u} = \sum_{n=0}^{\infty} \alpha_n \epsilon^n \qquad (3.3.10)$$

where $\epsilon \equiv \gamma^{1/(m-1)}$, and substitute (3.3.10) into (3.3.7) and solve for the u_n by gathering powers of ϵ. The first three α_i are needed in later calculations and are

$$\alpha_0 = 1 \qquad (3.3.11a)$$

$$\alpha_1 = \frac{a_1}{m-1} \qquad (3.3.11b)$$

$$\alpha_2 = \frac{m^2 + (2a_2 - a_1^2 - 1)m + 2a_2}{2(m-1)^2}. \qquad (3.3.11c)$$

Now we can evaluate $V(x_1; \gamma)$. The asymptotic expansion for $\tilde{V}(u; \gamma)$ for $u \to \infty$ is given by

$$\tilde{V}(u; \gamma) \sim \gamma^2 u^m \left(1 - a_1 u^{-1} + (a_1^2 - a_2)u^{-2} + O(u^{-3})\right) - 2\gamma u$$

$$+u^{2-m} \left(1 + a_1 u^{-1} + (m - 1 + a_2)u^{-2} + O(u^{-3})\right) \qquad (3.3.12)$$

and the similar expansion for the second derivative of $V(u; \gamma)$ is

$$\frac{d^2 \tilde{V}}{du^2} \sim (m-1)(m-2)u^{-m} + m(m-1)\gamma^2 u^{m-2} + O(u^{-m-1}) + O(\gamma^2 u^{m-3}). \quad (3.3.13)$$

Evaluation of (3.3.12) at $u = u_1$ in (3.3.10) and (3.3.11) gives

$$\tilde{V}(u_1; \epsilon) \sim (m - 1)\epsilon^m + O(\epsilon^{m+1}) \qquad (3.3.14)$$

and the second derivative at the same point is

$$\frac{d^2}{du^2}\tilde{V}(u_1; \epsilon) \sim 2(m - 1)^2 \epsilon^m + O(\epsilon^{m+1}) \qquad (3.3.15)$$

when (3.3.13) is evaluated. For future calculations the second derivative with respect to x coordinates will be needed so it will noted here. Using

$$\frac{d^2 V}{dx^2} = D(u(x))\frac{d^2 \tilde{V}}{du^2} + \frac{1}{2}\frac{dD}{du}\frac{d\tilde{V}}{du}$$

and then using (3.3.15) gives the result that

$$\frac{d^2}{dx^2}V(x_1; \epsilon) \sim 2(m - 1)^2 \epsilon^{2m} + O(\epsilon^{2m+1}).$$

The minimum value of the potential is seen to vanish with ϵ and hence γ as shown in (3.3.14). Also from (3.3.8) the point u_1 increases to infinity with decreasing γ.

It can now be noted that for $\lambda_0 = o(\gamma^2)$, which will be the case for λ_0 exponentially small, $\Delta > 0$ for $x > x_0$. This holds because the ratio of λ to $V(x_1; \epsilon)$

vanishes with γ. Therefore, no turning points occur for $x > x_0$. The only turning point for positive x is at x_0.

Now that we know the location of the turning point, we need to consider what comparison potential the known potential should be mapped onto. At first thought, we might expect a mapping of the interval $[0, \infty)$ onto a linear potential would be adequate because this corresponds to a single turning point. Even though such a mapping conserves the number of turning points, this method is not satisfactory. This method gives a good representation of the solution for $x > x_0$, but for $0 < x < x_0$, the solution is not as good as is possible. It is much better for calculating λ_0 to use the $\gamma = 0$ solution for $0 < x < x_0$. The fact that the $\gamma = 0$ solution can be used will be seen later. Thus the calculation of the eigenfunction will require two regions.

In standard WKB analysis one would examine the following five regions: $0 \leq u \ll u_0, |u - u_0| = O(1), u_0 < u < u_1, |u - u_1| = O(1)$ and $u \gg u_1$. The region $|u - u_0| = O(1)$ is at the turning point and the region $|u - u_1| = O(1)$ is near the point where perturbation terms balance the unperturbed potential. The solutions for each of the five regions would then be matched in an overlap region to construct an approximation. This procedure has been done as well to check the following calculations.

Our procedure will be to use Langer's mapping idea to cover the outer three regions, $u \gg 1$, and then match onto an solution which will be good on the two inner regions, $0 < u \ll \epsilon^{-1}$. To find a solution for the inner two regions we can use the fact that the perturbation and the eigenvalue λ_0 are small in those regions. Hence the number of regions has been reduced from five to two. The regions of interest are shown in Figure (3.5).

Because x_1 corresponds to a minimum point of the potential, the potential is locally quadratic about $x = x_1$. Hence a mapping onto a shifted quadratic is a
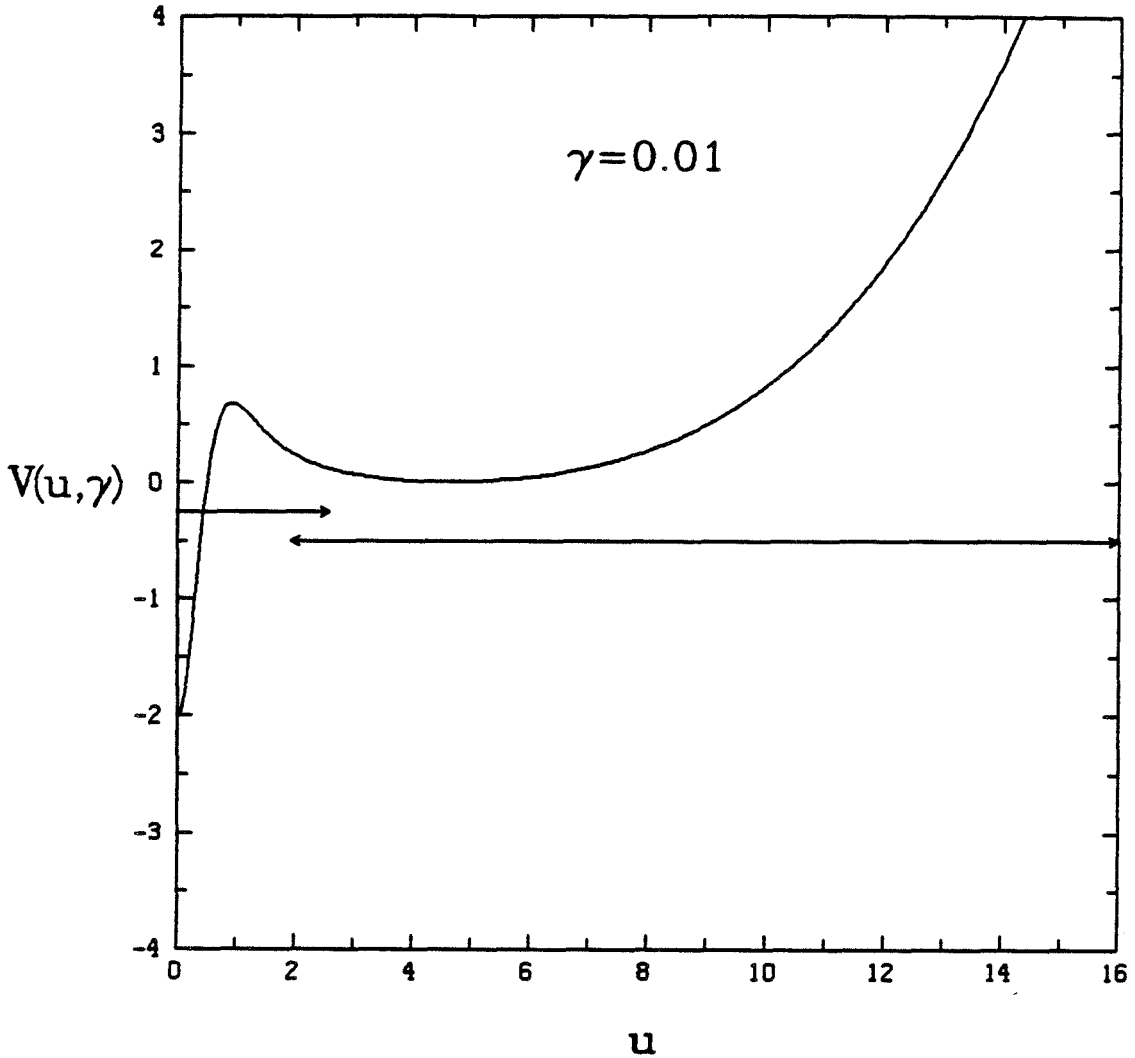
Figure 3.5 The regions of interest for calculating solution.

reasonable choice for the comparison potential. Parabolic cylinder functions, the solution of such a Schrödinger equation, can then to used for the solution.

3.3.2 Solution in the Outer Region

In the region $x \gg 1$ the equation of interest is

$$\frac{d^2 H}{dx^2} - Q(x; \epsilon)H = 0 \qquad x \gg 1 \tag{3.3.16}$$

where $Q(x; \epsilon) \equiv V(x; \epsilon) - \lambda_0$. $V(x; \epsilon)$ dominates λ_0 for small ϵ, so $Q(x; \epsilon)$ is strictly positive on the domain of interest. Because the minimum point is field-dependent we will normalize it to unity by the scaling $x = x_1 y$, where $x_1 = \nu^{-1}$ with $\nu \equiv \frac{1}{2}(m + 2)\epsilon^{(m+2)/2} \ll 1$ from (3.3.9). Then (3.3.16) is

$$\nu^2 \frac{d^2 H}{dy^2} - \hat{Q}(y; \epsilon)H(y) = 0 \qquad (3.3.17)$$

with $\hat{Q}(y; \epsilon) = Q(x_1 y; \epsilon)$.

To motivate the choice of mapping function, note that by expanding $\hat{Q}(y; \epsilon)$ about $y = 1$ in a Taylor series

$$\nu^{-2}\hat{Q}(y; \epsilon) = \frac{4(m - 1)}{(m + 2)^2}\epsilon^{-2} + \frac{16(m - 1)^2}{(m + 2)^4}\epsilon^{-4}(y - 1)^2 + O(\epsilon^{-4}(y - 1)^3). \quad (3.3.18)$$

Therefore, for $|y - 1| \ll \epsilon^{4/3}$, the approximate equation is

$$\frac{d^2 H}{dy^2} - \left(\frac{4(m - 1)}{(m + 2)^2}\epsilon^{-2} + \frac{16(m - 1)^2}{(m + 2)^4}\epsilon^{-4}(y - 1)^2 \right) H = 0$$

with solutions in term of the parabolic cylinder functions

$$H(y; \epsilon) = c_1 D_{-1}(a(y - 1)) + c_2 \exp \frac{a^2(y - 1)^2}{4} \qquad (3.3.19)$$

with the constant a given by

$$a \equiv \frac{2\sqrt{2m - 2}}{m + 2}\epsilon^{-1}.$$

This argument suggests taking the comparison potential to be the quadratic in (3.3.18). The transformation will then be the identity near $y = 1$ and a stretching as y moves away from 1. The solution in the new coordinates will be the same parabolic cylinder functions of (3.3.19).

The transformation

$$\eta = \eta(y)$$

$$\hat{H}(\eta) = \left(\frac{d\eta}{dy} \right)^{1/2} H(y; \epsilon)$$

takes the equation (3.3.17) into the a new Schrödinger equation given by

$$\frac{d^2 \hat{H}}{d\eta^2} - (\eta_y)^{-2} (\nu^{-2} \hat{Q}(y; \epsilon) - (\eta_y)^{1/2} \frac{d^2}{dy^2} (\eta_y)^{-1/2}) \hat{H}(\eta) = 0. \qquad (3.3.20)$$

Equation (3.3.20) will be a parabolic cylinder equation with the identification:

$$\frac{1}{4} \nu^2 (\eta_y)^2 (\eta^2 + 2) = \hat{Q}(y; \epsilon) - \nu^2 (\eta_y)^{1/2} \frac{d^2}{dy^2} (\eta_y)^{-1/2}. \qquad (3.3.21)$$

Note that the point $\eta = 0$ maps to $y = 1$.

In general (3.3.21) is a non-linear equation in η_y and to make it tractable the last term on the right, the so-called Schwarzian derivative, will be neglected. This is reasonable because the comparison potential does not deviate much from the original potential $\hat{Q}(y; \epsilon)$ in analytic behavior. This implies that η_y will be approximately constant. Also the Schwarzian derivative is approximately independent of the absolute magnitude of η_y by its very form. It will be shown in §3.4 that the Schwarzian derivative is uniformly bounded for all η and y and also uniformly small due to multiplication by ν^2. After dropping this term (3.3.21) can be solved exactly to give η as an implicit function of y. The solution is

$$\frac{1}{2} \eta \sqrt{\eta^2 + 2} + \log(\eta + \sqrt{\eta^2 + 2}) = \frac{2}{\nu} \int_1^y \hat{Q}^{1/2}(y'; \epsilon) \, dy'. \qquad (3.3.22)$$

The solution in these new coordinates is

$$\hat{H}(\eta) = c_1 D_{-1}(\eta) + c_2 \exp \frac{1}{4} \eta^2.$$

The second solution can be eliminated because it grows exponentially for $\eta \to \infty$ which violates the boundary condition (3.2.12), and the solution to (3.3.17) is

$$H(\eta(y)) = K(\eta_y)^{-1/2} D_{-1}(\eta) \qquad (3.3.23)$$

with the constant K to be determined later when matching.

We can immediately note that when η is near 0, then $y \approx 1$ and the solution reduces to (3.3.19) as expected.

It will now be shown that (3.3.23) is equivalent to the standard WKB expressions one might use. First, expand (3.3.22) for large positive η to get

$$\eta^2 + 2\log\eta + O(1) = \tfrac{4}{\nu}\int_1^y \hat{Q}^{1/2}(y';\epsilon)\,dy', \quad \eta \gg 1 \qquad (3.3.24)$$

and when (3.3.24) is differentiated with respect to y gives

$$\eta_y = 2(\nu\eta)^{-1}\hat{Q}^{1/2}(y) + O(\eta^{-1}), \quad \eta \gg 1. \qquad (3.3.25)$$

Now use the asymptotics of the parabolic cylinder function for large η to write (3.3.23) as

$$H(\eta) \sim K(\eta_y)^{-1/2}\eta^{-1}e^{-\eta^2/4}, \qquad (3.3.26)$$

and then use the results of (3.3.24) and (3.3.25) to write (3.3.26) as

$$H(\eta) \sim K(\tfrac{\nu}{2})^{1/2}\hat{Q}^{-1/4}(y)\exp{-\tfrac{1}{\nu}\int_1^y \hat{Q}^{1/2}(y')\,dy'} \qquad (3.3.27)$$

which is the standard WKB formula for $\eta \to \infty$.

Then similarly, for large negative η (3.3.22) becomes

$$\eta^2 + 2\log(-\eta) + O(1) = \tfrac{4}{\nu}\int_y^1 \hat{Q}^{1/2}(y';\epsilon)\,dy', \quad \eta \ll -1 \qquad (3.3.28)$$

and its derivative is

$$\eta_y = 2(-\nu\eta)^{-1}\hat{Q}^{1/2}(y) + O(\eta^{-1}), \quad \eta \ll -1. \qquad (3.3.29)$$

Then the asymptotic expansion of (3.3.23) for large negative η gives

$$H(\eta) \sim (2\pi)^{1/2}K(\eta_y)^{-1/2}e^{\eta^2/4}, \qquad (3.3.30)$$

and then use (3.3.28) and (3.3.29) to write (3.3.30) as

$$H(\eta) \sim (\nu\pi)^{1/2}K\hat{Q}^{-1/4}(y)\exp{\tfrac{1}{\nu}\int_y^1 \hat{Q}^{1/2}(y')\,dy'} \qquad (3.3.31)$$

which is in WKB form for $\eta \to -\infty$. It will be convenient for later matching to write the integrals in (3.3.31) as

$$\int_y^1 \hat{Q}^{1/2}(y')\,dy' = \int_{y_0}^1 \hat{Q}^{1/2}(y')\,dy' - \int_{y_0}^y \hat{Q}^{1/2}(y')\,dy' \qquad (3.3.32)$$

where y_0 is the turning point in the y coordinates.

The solution (3.3.23) will now be written in a form that is suitable for matching to the inner region. The solution will be approximated in the region $1 \ll u \ll \epsilon^{-1}$. When $u \ll \epsilon^{-1}$ the potential can be approximated by

$$\tilde{Q}(u;\gamma) = \tilde{V}_0(u) + O(\gamma)$$

since the next term in the expansion of $V(u;\gamma)$ is $2\gamma u$. Therefore, when the asymptotic expansion of (3.3.12) is used when $\gamma = 0$, one gets

$$\tilde{V}_0(u) \sim u^{2-m}\left(1 + O(u^{-1})\right)$$

and then

$$\tilde{Q}^{-1/4}(u) \sim u^{(m-2)/4}\left(1 + O(u^{-1})\right) \qquad (3.3.33)$$

which holds in the region $1 \ll u \ll \epsilon^{-1}$. In the integrals of (3.3.32) change coordinates back to u via the relations $x = x_1 y$ and (3.2.6) to get

$$\int_{x_0}^x Q^{1/2}(x';\gamma)\,dx' = \int_{u_0}^u \tilde{Q}^{1/2}(u';\gamma)D^{-1/2}(u')\,du'. \qquad (3.3.34)$$

The integrand of (3.3.34) when expanded for $1 \ll u \ll \epsilon^{-1}$ gives

$$\frac{\tilde{Q}(u;0)}{D(u)} \sim u^2 + m - 1 + O(u^{-2}).$$

Then the integral of (3.3.34) is

$$\int_{u_0}^u \left(\frac{\tilde{Q}(u';0)}{D(u')}\right)^{1/2}\,du' \sim \tfrac{1}{2}u^2 + \tfrac{1}{4}(m-1) + \tfrac{1}{2}(m-1)\log u - c_m \qquad (3.3.35)$$

where

$$c_m \equiv \int_0^{u_0} \sqrt{u^2 + m - 1} \, du + \tfrac{1}{2}(m-1) \log \left(\frac{\sqrt{m-1}}{2} \right) \tag{3.3.36}$$

which is valid for $u \gg (m-1)^{1/2}$. The overlap region is then $(m-1)^{1/2} \ll u \ll \epsilon^{-1}$. Using the results of (3.3.33), (3.3.35) and (3.3.36) in (3.3.31) give the result

$$H(u) \sim (\nu\pi)^{1/2} K \kappa_m^{-1} u^{-m/4} \exp \left(-\frac{u^2}{2} + \frac{1}{\nu} \int_{y_0}^1 \hat{Q}^{1/2} \, dy \right) \tag{3.3.37}$$

with

$$\kappa_m \equiv \exp \left(-c_m + \tfrac{1}{4}(m-1) \right).$$

3.3.3 Solution in the Inner Region

In the inner region where $0 < u \ll \epsilon^{-1}$, the $V_0(x)$ part of the potential is much larger than $\gamma V_1(x)$. That is, $|\gamma V_1(x)| \ll 1$ while $V_0(x) = O(1)$ on the region of interest. So we will be able to neglect γV_1 in the differential equation. A perturbation expansion about $\gamma = 0$ of the form

$$H(x; \gamma) = \sum_{n=0}^{\infty} \gamma^n H_n(x)$$

can be used to find the solution. To zeroth order $H_0(x)$ satisfies

$$\frac{d^2 H_0}{dx^2} - V_0(x) H_0(x) = 0. \tag{3.3.38}$$

The λ_0 term has been neglected since $\lambda_0 = o(\gamma^n)$ for all n. The potential $V_0(x)$ is an even function; therefore, the solution to (3.3.38) satisfies the boundary condition $H'(0) = 0$. This equation is just the unperturbed problem for $\gamma = 0$. The solution is the Maxwellian in the transformed coordinates. The solution is

$$H\left(x(u) \right) = \pi^{-1/2} D^{1/4}(u) e^{-u^2/2} \tag{3.3.39}$$

in u coordinates. Its integral on $(-\infty, \infty)$ has been normalized to unity. This solution is a decaying exponential so it will match the outer solution in (3.3.23).

Further corrections to $H(u)$ will be $O(\gamma)$ and will account for the asymmetry of the potential. Therefore, we are justified in only considering the interval $u > 0$.

To match with the outer region expand (3.3.39) for large u. The large u behavior of (3.3.39) is

$$H(u) \sim \pi^{-1/2}u^{-m/4}e^{-u^2/2}.$$

(3.3.40)

The first term in the asymptotic expansion of $D(u)$ has been used here. Equation (3.3.40) has the correct form for matching with (3.3.37). Comparing (3.3.37) with (3.3.40) requires that

$$K(\gamma) = \pi^{-1}\nu^{-1/2}\kappa_m \exp -\frac{1}{\nu}\int_{y_0}^{1} \hat{Q}^{1/2}(y;\epsilon)\,dy.$$

(3.3.41)

The overlap region for the two solutions is $(m-1)^{1/2} \ll u \ll \epsilon^{-1}$. We note that a transmission integral is already present in this constant, suggesting an exponentially small value for λ_0.

In summary, the approximate eigenfunction for the smallest eigenvalue is

$$H(u) = \pi^{-1/2}D^{1/4}(u)e^{-u^2/2} \qquad u \ll \epsilon^{-1}$$

(3.3.42a)

$$H(\eta(y)) = K(\eta_y)^{-1/2}D_{-1}(a\eta) \qquad u \gg 1$$

(3.3.42b)

with $y = y(u)$ determined via (3.2.6) and $\eta = \eta(y)$ determined via (3.3.22) and the constant K from (3.3.41).

We will now sketch the details of the solution for $u < 0$. Equation (3.3.42a) still holds for a certain regime of negative u. The only turning point of the potential $\tilde{Q}$ for negative u is at

$$-\tilde{u}_0 = -c_0 + c_1\gamma + O(\gamma^2) < 0.$$

The approximate solution for $u < -\tilde{u}_0$ is

$$H(u) = K_2\hat{Q}^{-1/4}(y)\exp -\frac{1}{\nu}\int_{y}^{-\tilde{y}_0} \hat{Q}^{1/2}(y')\,dy'.$$

This solution is non-singular since $\hat{Q}(y)$ does not vanish on this region. Matching this with (3.3.42a) to first order gives

$$K_2 = \pi^{-1/2} \tilde{\kappa}_m.$$

$\tilde{\kappa}_m$ is the same functionally as before with u_0 replaced by $\tilde{u}_0$. The overlap region for the matching is as before, $(m-1)^{1/2} \ll -u \ll \epsilon^{-1}$. When the solution for $u < -\tilde{u}_0$ is transformed back to $f(u)$ one gets

$$f(u) = K_2 \hat{Q}^{-1/4}(y) D^{-1/4}(u) \exp\left(-\frac{1}{\nu} \int_y^{-\tilde{y}_0} \hat{Q}^{1/2}(y')\, dy' + \gamma \int_0^u D^{-1}(v)\, dv - \frac{1}{2} u^2\right).$$
$$(3.3.43)$$

The asymptotic behavior of (3.3.43) is

$$f(u) \sim C \exp\left(2\gamma \int_0^u D^{-1}(v)\, dv - u^2\right) \qquad u \to -\infty \qquad (3.3.44)$$

with C a constant. Since $D(u)$ is even in u, the solution decays exponentially for $u \to -\infty$.

Now we can immediately write down the solution for $f_0(u)$ via the transformations (3.2.4) and (3.2.6). The distribution function corresponding to the lowest eigenvalue is

$$f_0(u) = \pi^{-1/2} \exp\left(-u^2 + \gamma \int_0^u D^{-1}(v)\, dv\right) \qquad (3.3.45a)$$

$$f_0(u) = K(\gamma) D^{-1/4}(u) (\eta_\nu)^{-1/2} D_{-1}(a\eta) \exp\left(-\frac{1}{2} u^2 + \gamma \int_0^u D^{-1}(v)\, dv\right). \quad (3.3.45b)$$

Equation (3.3.45a) is valid for $0 < u \ll u_1$ and (3.3.45b) for $u_0 < u$. Note that for $\gamma \to 0$, (3.3.45) reduces to the exact Maxwellian solution because $\lambda_0(\gamma) \to 0$ and $u_1 \to \infty$ making (3.3.45a) valid for all $u > 0$. It should now be checked that (3.3.45) does, in fact, give the correct asymptotic behavior of (3.2.13). To check this, use the asymptotic relation (3.3.27) derived earlier which is

$$H(\eta) \sim K\left(\frac{\nu}{2}\right)^{1/2} \hat{Q}^{-1/4}(y) \exp -\frac{1}{\nu} \int_1^y \hat{Q}^{1/2}(y')\, dy'.$$

Substitute for $\hat{Q}(y)$ and evaluate this for $y \gg 1$. One gets

$$H(\eta) \sim K(\gamma)(\tfrac{\nu}{2})^{1/2}\gamma^{-1/2}D^{1/4}\exp\left(-\gamma\int_0^u D^{-1}(v)\,dv + \frac{1}{2}u^2 + \frac{\lambda_0 u}{2\gamma}\right)$$

for $y \to \infty$. Therefore, the proper terms in (3.3.45b) cancel and the asymptotic behavior of (3.2.13) is found.

3.3.4 Calculation of λ_0

Recall that the smallest eigenvalue is given by the relation (3.3.1)

$$\lambda_0 = \frac{j(u) - j(-\infty)}{\int_{-\infty}^u f_0(u;\epsilon)\,du}.$$

The current is evaluated using the eigenfunction (3.3.45a) constructed. The simplification $j(-\infty) = 0$ follows from the asymptotic behavior of $f_0(u)$ given in (3.3.44). Because (3.3.1) is an exact relationship, we are free to choose the point u at which to evaluate it. Clearly, we should take $u > u_0$ since the solution for $u < u_0$ is the Maxwellian to zeroth order which gives zero current. This would be too crude an estimate. The choice that simplifies the calculation is the point $u = u_1 = \epsilon^{-1}$. Then we will evaluate the outer part of $f_0(u)$ at ϵ^{-1}.

Substituting the outer solution (3.3.45a) into the expression for the current

$$j(u) = 2\gamma f(u) - D(u)\left(\frac{\partial f}{\partial u} + 2uf\right)$$

gives on evaluation at $u = \epsilon^{-1}$

$$j(\epsilon^{-1}) = \left(\tfrac{1}{4}D_u(\epsilon^{-1}) + D^{1/2}(\epsilon^{-1})\frac{\nu\eta_{yy}(0)}{2\eta_y(0)} - \nu\eta_y(0)D(\epsilon^{-1})\frac{D'_{-1}(0)}{D_{-1}(0)}\right)f_0(\epsilon^{-1}).$$

$$(3.3.46)$$

The constant ν is as before. Simplification of (3.3.46) gives

$$j(\epsilon^{-1}) = 2\sqrt{\frac{m-1}{\pi}}\epsilon^m f_0(\epsilon^{-1})\left(1 + O(\epsilon)\right) \tag{3.3.47}$$

with

$$f_0(\epsilon^{-1}) = K\epsilon^{-m/4}\eta_y^{-1/2}(0)D_{-1}(0)\exp\left(-\tfrac{1}{2}\epsilon^{-2} + \epsilon^{m-1}\int_0^{\epsilon^{-1}} D^{-1}(u)\,du\right). \quad (3.3.48)$$

The denominator in (3.3.1) is

$$\int_{-\infty}^{\epsilon^{-1}} f_0(u;\epsilon)\,du = 1 + O(\epsilon).$$

This follows from the properties of $f_0(u;\epsilon)$. For $\epsilon \to 0$ (no electric field), $f_0(u;0) = M(u)$. The zeroth order approximation is found just by setting $\epsilon = 0$ and the result follows.

The smallest eigenvalue is then

$$\lambda_0 \sim j(\epsilon^{-1}) \qquad \epsilon \to 0.$$

It is exponentially small as claimed. The constant K in λ_0 has the standard form of a tunnelling integral which occurs in numerous quantum mechanical problems where transmission through a barrier is important. K can be evaluated asymptotically for $0 < \gamma \ll 1$. The analysis requires subtle treatment since the integrand cannot be expanded uniformly in γ. The result of this calculation for a diffusion coefficient of a form that behaves as in (3.3.5) is

$$K(\epsilon) \sim \pi^{-1}\nu^{-1/2}\kappa_m\epsilon^m \exp\left\{-\tfrac{m-1}{2(m+1)}\epsilon^{-2} - \tfrac{a_1}{m}\epsilon^{-1}\right\} \quad (3.3.49)$$

for $\epsilon \to 0$. The details of this calculation are left to Appendix 2. Combining (3.3.49) with (3.3.47) and (3.3.48) gives the relation

$$\lambda_0(\epsilon) \sim 2^{1/4}(m-1)^{1/4}\pi^{-1}\kappa_m\epsilon^m \exp\left\{-\tfrac{m-1}{m+1}\epsilon^{-2} - \tfrac{2a_1}{m}\epsilon^{-1}\right\}$$

for the smallest eigenvalue. Using $\gamma = \epsilon^{m-1}$ we can write the eigenvalue in terms of the field parameter as

$$\lambda_0(\gamma) \sim 2^{1/4}(m-1)^{1/4}\pi^{-1}\kappa_m\gamma^{m/(m-1)}\exp\left\{-\tfrac{m-1}{m+1}\gamma^{-2/(m-1)} - \tfrac{2a_1}{m}\gamma^{-1/(m-1)}\right\}$$

for $\gamma \to 0$.

Comparisons with the current calculation done by Lifshitz and Pitaevskii in [19] and displayed in Appendix 3 show the only difference is the multiplying constant. This could be expected considering the two different approaches. In [19], the authors assume a steady-state distribution is possible and proceed to construct a solution based on that assumption.

3.4 Error Estimates

In §3.3 it was noted that in computing the change of coordinates for the mapping to the simpler potential the Schwarzian derivative term

$$(\eta_y)^{1/2}\frac{d^2}{dy^2}(\eta_y)^{-1/2} \tag{3.4.1}$$

in (3.3.21) was neglected. It will now be shown that this was a reasonable action to take, as its effect is small.

We will show that the Schwarzian derivative is uniformly bounded for $y > y_0$ or $\eta > \eta_0$ with $\eta_0 = \eta(y_0)$. And since it is multiplied by a small parameter in the differential equation for $\eta(y)$, it is uniformly small and hence reasonable to neglect.

To show this we substitute the solution

$$\tfrac{1}{2}\eta\sqrt{\eta^2+2} + \log(\eta + \sqrt{\eta^2+2}) = \tfrac{2}{\nu}\int_1^y \hat{Q}^{1/2}(y';\epsilon)\,dy'$$

of the linearized equation (3.3.21) into (3.4.1). The notation

$$g(\eta) \equiv \tfrac{1}{4}\nu^2(\eta^2+2)$$

will be used for the comparison potential. Letting $\eta_y = \hat{Q}^{1/2}(y; \epsilon)g^{-1/2}(\eta)$ in (3.4.1) gives the result

$$(\eta_y)^{1/2} \frac{d^2}{dy^2}(\eta_y)^{-1/2} = \frac{5}{16}\left(\frac{\hat{Q}_y}{\hat{Q}}\right)^2 - \frac{1}{4}\frac{\hat{Q}_{yy}}{\hat{Q}} + (\eta_y)^2\left(\frac{1}{4}\frac{g_{\eta\eta}}{g} - \frac{5}{16}\left(\frac{g_\eta}{g}\right)^2\right).$$

We will examine each term and show that it is uniformly bounded on $[y_0, \infty)$.

First, note that each term $g_{\eta\eta}g^{-1}$ and $(g_\eta g^{-1})^2$, is a rational smooth function of η. These terms go to zero for large η and the denominators have no real zeros. Hence the terms are bounded by some constant M_1 for all η. The choice of M_1 is independent of η and hence a uniform bound on $[\eta_0, \infty)$.

Since $(\eta_y)^2 = \hat{Q}(y; \epsilon)g^{-1}(\eta)$ is also a rational function with no poles for real η, then if η_y falls off for large y, it is bounded uniformly. To determine the large y behavior of η_y solve the equation

$$\frac{1}{4}\nu^2\eta^2\eta_y^2 = \gamma^2 D^{-1}\left(u(y)\right).$$

This is just equation (3.3.21) explicitly for large η and y. The solution of this is

$$\eta = Cy^{(m+1)/(m+2)}$$

with C a constant independent of γ. Hence $\eta_y \sim y^{-1/(m+2)}$ for $y \to \infty$. Therefore, η_y^2 is also uniformly bounded.

The same argument applies to the final two terms, $\hat{Q}_y^2\hat{Q}^{-2}$ and $\hat{Q}_{yy}\hat{Q}^{-1}$. They are both rational functions of y with no real poles. And because $\hat{Q}$ grows as a power of y, both behave as y^{-2} for large y. Therefore, they too are uniformly bounded by some constant M_2 on $[y_0, \infty)$.

If we take $M = max(M_1, M_2)$, then this gives a bound on the Schwarzian derivative. Multiplication by ν^2 gives $\nu^2(\gamma)M$ for the bound. We can now see

that the ratio of the Schwarzian derivative term to the retained term can be made arbitrarily small for a sufficiently small field, γ. The ratio is

$$\left| \frac{\nu^2 M}{\hat{Q}(y; \epsilon)} \right| < \frac{\epsilon^2 M}{(m-1)}.$$

This argument gives the justification for neglecting the Schwarzian derivative in solving the equation (3.3.21).

3.5 Three-Dimensional Problem

The three-dimensional problem can be treated in the manner of §3.2 through §3.4. A single approximation is made to reduce the problem to a one-dimensional equation. The equivalent potential in the Schrödinger description is found and an identical analysis is done. Calculation of the lowest eigenvalue gives a description of the runaway current in three dimensions.

The three-dimensional problem is

$$\frac{\partial}{\partial t} f(\mathbf{u}, t) + \nabla \cdot \mathbf{j}(\mathbf{u}, t) = 0 \tag{3.5.1}$$

with

$$\mathbf{j}(\mathbf{u}, t) = \Big(2\gamma \hat{z} - \mathbf{D}(\mathbf{u}) \cdot (\nabla + 2\mathbf{u}) \Big) f(\mathbf{u}, t). \tag{3.5.2}$$

The gradients are with respect to $\mathbf{u}$ coordinates. The calculation will be done in spherical coordinates. The distribution function is assumed to be independent of the φ coordinate, which reduces the problem to two dimensions. Equation (3.5.1) in spherical coordinates is

$$\frac{\partial f}{\partial t} + \frac{1}{u^2} \frac{\partial}{\partial u} \left(u^2 j_u \right) + \frac{1}{u \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \, j_\theta \right) = 0.$$

Multiplication by $\sin\theta$ and integration over the range of θ gives

$$\frac{\partial}{\partial t} \int_0^\pi f(\mathbf{u}, t) \sin\theta \, d\theta + \frac{1}{u^2} \frac{\partial}{\partial u} u^2 \int_0^\pi j_u(\mathbf{u}, t) \sin\theta \, d\theta + \frac{1}{u} \int_0^\pi \frac{\partial}{\partial \theta} \left(j_\theta(\mathbf{u}, t) \sin\theta \right) \, d\theta = 0.$$

$$\tag{3.5.3}$$

For $j_\theta(\mathbf{u}, t)$ bounded at 0 and π, the last of the three integrals vanishes. Using (3.5.2) and the properties of the diffusion tensor $\mathbf{D}(\mathbf{u})$, the radial part of $\mathbf{j}(\mathbf{u}, t)$ is

$$j_u(\mathbf{u}, t) = 2\gamma \cos\theta f(\mathbf{u}, t) - D_\parallel(u) \left(\frac{\partial}{\partial u} + 2u\right) f(\mathbf{u}, t).$$

Defining

$$f_0(\mathbf{u}, t) \equiv \int_{-1}^{1} f(u, \eta, t)\, d\eta$$

with $\eta = \cos\theta$ and using the radial component in (3.5.3) gives the equation

$$\frac{\partial f_0}{\partial t} + \frac{1}{u^2}\frac{\partial}{\partial u} u^2 \left\{ 2\gamma \int_{-1}^{1} \eta f(u, \eta, t)\, d\eta - D_\parallel(u) \left(\frac{\partial}{\partial u} + 2u\right) f_0(u, t) \right\}. \qquad (3.5.4)$$

So far the result is exact. The approximation is to take $\eta = 1$ in the integrand of (3.5.4). This is a reasonable approximation for large time because the distribution function will become concentrated along the direction of the electric field, i.e., about $\eta = 1$. This approach is due to Lifshitz and Pitaevskii in [19].

Equation (3.5.4) becomes with this simplification the one-dimensional problem

$$\frac{\partial f_0}{\partial t} = \frac{1}{u^2}\frac{\partial}{\partial u} u^2 \left(D_\parallel(u) \left(\frac{\partial f_0}{\partial u} + 2u f_0\right) - 2\gamma f_0 \right)$$

on the interval $[0, \infty)$. In this one-dimensional form we can apply all the techniques used in this chapter to evaluate the runaway current and the smallest eigenvalue. The equivalent potential corresponding to this three-dimensional problem is slightly different but the analysis will still apply. In following the derivation of (3.3.1) the smallest eigenvalue is now given by

$$\lambda_0 = \frac{j(u)}{\int_0^u u^2 f_0(u)\, du} \qquad (3.5.5)$$

with

$$j(u) \equiv u^2 \left\{ 2\gamma f_0(u) - D(u) \left(\frac{\partial f_0}{\partial u} + 2u f_0\right) \right\}.$$

Since $j(0) = 0$, the other boundary term vanishes in the derivation of (3.5.5).

After the time dependence is separated out, the radial equation can be transformed into an equivalent Schrödinger equation. The transformation of independent variable defined by

$$\frac{dx}{du} = D_\parallel^{-1/2}(u)$$

and dependent variable

$$H(x) = u^2 D_\parallel^{1/2}(u)\hat{q}^{-1}(x;\gamma)f(u)$$

gives the equation

$$\frac{d^2H}{dx^2} + \left(\lambda - \hat{q}^{-1}\frac{d^2\hat{q}}{dx^2}\right)H(x) = 0.$$

The quantity $\hat{q}(x;\gamma)$ is given by

$$\hat{q}(x;\gamma) = uD_\parallel^{1/4}(u)\exp\left(-\tfrac{1}{2}u^2 + \gamma\int_0^u D_\parallel^{-1}(v)\,dv\right).$$

The equivalent potential is given in terms of $D_\parallel(u)$ and is

$$V(u;\gamma) = \tfrac{1}{4}D_{\parallel uu} - \tfrac{1}{16}(D_{\parallel u})^2(D_\parallel)^{-1} + (u^2-3)D_\parallel - (D_{\parallel u}+2\gamma)(u-u^{-1}) + \gamma^2(D_\parallel)^{-1}.$$

The u subscripts denote differentiation with respect to u. This potential is almost identical to the potential (3.2.10) examined earlier in the chapter. The point where the minimum of $V(x;\gamma)$ occurs is at $u_1 = \epsilon^{-1}$ as before.

The exact same analysis goes through in calculating the eigenfunction and eigenvalue of the lowest mode. An inner and an outer solution are found and matched. The outer solution, valid for $u > u_0$, with u_0 the turning point, is found by mapping the potential onto the simple quadratic

$$g(\eta) = \tfrac{1}{4}\nu^2(\eta^2 + 2).$$

The inner solution is just the transformed Maxwellian solution. The solutions will be analogs of equations (3.3.42) and (3.3.45) with $D(u)$ replaced by $D_\parallel(u)$. The solution is

$$H(u) = \sqrt{\frac{2}{\pi}}\,uD_\parallel^{1/4}(u)e^{-u^2/2} \qquad 0 < u \ll \epsilon^{-1} \tag{3.5.6a}$$

$$H(\eta(y)) = K(\eta_y)^{-1/2}D_{-1}(a\eta) \qquad u \gg 1 \tag{3.5.6b}$$

with $y = y(u)$ and $\eta = \eta(y)$ as before. To match (3.5.6a) to (3.5.6b) we expand the first for $u \gg 1$ and the second for $u_0 < u \ll \epsilon^{-1}$. Expanding (3.5.6) accordingly gives

$$H(u) \sim \sqrt{\frac{2}{\pi}}\, u^{1-m/4} e^{-u^2/2}$$

$$H(u) \sim (\nu\pi)^{1/2} K \kappa_m^{-1} u^{1-m/4} \exp\left(-\frac{u^2}{2} + \frac{1}{\nu}\int_{y_0}^1 \hat{Q}^{1/2}\, dy\right).$$

Matching requires that

$$K = \pi^{-1}\sqrt{\frac{2}{\nu}}\,\kappa_m \exp -\frac{1}{\nu}\int_{y_0}^1 \hat{Q}^{1/2}\, dy.$$

Now (3.5.5) can be applied to the solution. The choice of point to evaluate (3.5.5) is again at $u = \epsilon^{-1}$. The smallest eigenvalue is then given by

$$\lambda_0(\epsilon) = 2\sqrt{\frac{m-1}{\pi}}\,\epsilon^{m-2} f_0(\epsilon^{-1})$$

with

$$f_0(\epsilon^{-1}) = K(\epsilon) D^{-1/4}(\epsilon^{-1})\eta_y^{-1/2}(0) D_{-1}(0) \exp\left(-\frac{1}{2}\epsilon^{-2} + \epsilon^{m-1}\int_0^{\epsilon^{-1}} D_\parallel^{-1}(u)\, du\right).$$

$K(\epsilon)$ is calculated in the same way as is done in the previous example. The final result is

$$\boxed{\lambda_0(\epsilon) \sim 2^{1/4}(m-1)^{1/4}\pi^{-1}\kappa_m \epsilon^{m-3} \exp\left\{-\frac{m-1}{m+1}\epsilon^{-2} - \frac{2a_1}{m}\epsilon^{-1}\right\}}$$

for $\epsilon \to 0$. Again $\gamma = \epsilon^{m-1}$. Of particular interest is the case $m = 3$. This is the case of references [15] and [24], e.g.. Then the result is

$$\lambda_0(\gamma) \sim 2\kappa_3\pi^{-1} \exp\left\{-\frac{1}{2}\gamma^{-1} - \frac{2a_1}{3}\gamma^{-1/2}\right\}.$$

The exponential part is in agreement with the work of Kruskal and Bernstein [15] whose result is

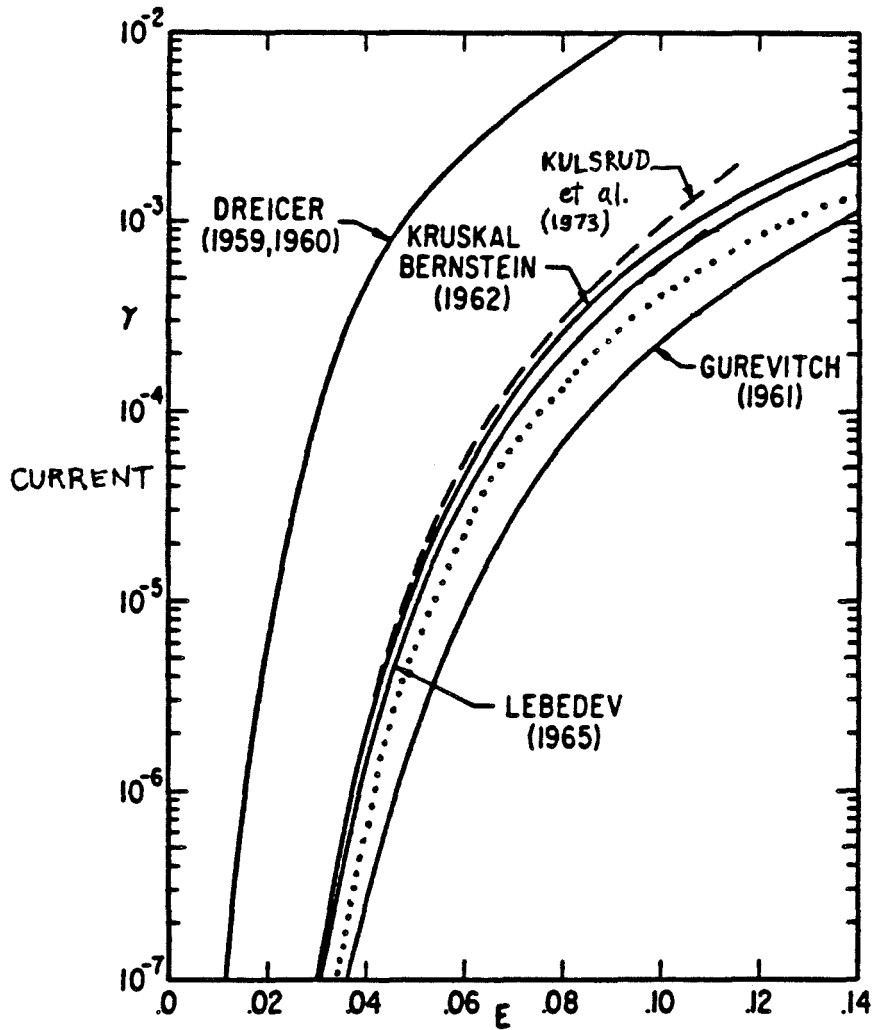$$\lambda_0(\gamma) \sim C\gamma^{-3/8} \exp\left\{-\frac{1}{2}\gamma^{-1} + 2\gamma^{-1/2}\right\}.$$

Figure 3.6 The current as a function of $\gamma = 2E$. The dotted line is $\lambda_0(\gamma)$.

The multiplying power of γ does not match. Further analysis might give a further correction to the power term in the expression for $\lambda_0(\epsilon)$. Figure (3.6) compares our result, the dotted line, and the results of other researchers.

In summary, we have shown the connection between the runaway current and the spectral properties of the Fokker-Planck equation. The smallest eigenvalue, exponentially small as a function of field strength, characterizes the current.

APPENDIX 1

Details of an Asymptotic Expansion

In this appendix a derivation of the integral in result (2.6.9) of Chapter 2 will be given. The integral to be analyzed for large σ is

$$\varphi(\sigma) = 2\sigma \int_0^1 \frac{dv}{1 + \sqrt{1 + 4\sigma D(v)}}. \qquad (A1.1.1)$$

A scaling of integration variable onto the interval $[0, 1]$ is done first to simplify the calculation. A multiplication by v_0 is then needed for the general case. One cannot simply expand the integrand of (A1.1.1) for large σ as the quantity $\sigma D(v)$ is not uniformly large on $[0, 1]$. This is due to $D(v)$ vanishing at $v = 1$.

For the analysis let $\epsilon = (4\sigma)^{-1} \ll 1$ and $x = 1 - v$ and then (A1.1.1) becomes

$$\varphi(\sigma) = \sqrt{\sigma} \int_0^1 \frac{dx}{(\tilde{D}(x) + \epsilon)^{1/2} + \epsilon^{1/2}} \qquad (A1.1.2)$$

with $\tilde{D}(x) = D(1 - x)$. Now split the integral in (A1.1.2) into two parts, one integration over an interval where $\sigma \tilde{D}(x)$ is uniformly small and the other where it is not. This gives

$$\varphi(\sigma) = \sqrt{\sigma} \int_0^\delta \frac{dx}{(\tilde{D}(x) + \epsilon)^{1/2} + \epsilon^{1/2}} + \sqrt{\sigma} \int_\delta^1 \frac{dx}{(\tilde{D}(x) + \epsilon)^{1/2} + \epsilon^{1/2}} \qquad (A1.1.3)$$

where $\delta = \delta(\epsilon)$ and $\epsilon \prec \delta(\epsilon) \prec 1$. Each of the two integrals will be considered separately and then asymptotic matching will be used to eliminate any dependence on the matching parameter δ.

For the second integral in (A1.1.3), $\epsilon \tilde{D}^{-1}$ is uniformly small on $[\delta, 1]$ so one can integrate term by term after expanding the integral in powers of ϵ. The second integral is then

$$\int_\delta^1 \frac{dx}{\tilde{D}^{1/2}(x)} - \epsilon^{1/2} \int_\delta^1 \frac{dx}{\tilde{D}(x)} + \tfrac{1}{2}\epsilon \int_\delta^1 \frac{dx}{\tilde{D}^{3/2}(x)} + O(\epsilon^{3/2}).$$

Further simplification can be done by extending the limits of integration and using the property that $\tilde{D}(x)$ vanishes linearly at $x = 1$. This gives the following:

$$\int_\delta^1 \frac{dx}{\sqrt{\tilde{D}(x)}} = \int_0^1 \frac{dx}{\sqrt{\tilde{D}(x)}} - 2\alpha^{-1/2}\delta^{1/2},$$

$$\int_\delta^1 \frac{dx}{\tilde{D}(x)} = -\tfrac{1}{\alpha}\log\delta + \int_0^1 \frac{\alpha x - \tilde{D}(x)}{\alpha x \tilde{D}(x)}\, dx + O(\delta),$$

$$\int_\delta^1 \frac{dx}{\tilde{D}^{3/2}(x)} = O(\delta^{-1/2}).$$

Adding these three results gives

$$D_* + \alpha^{-1}\epsilon^{1/2}\log\delta - \epsilon^{1/2}J + -2\alpha^{-1/2}\delta^{1/2} + O(\epsilon^{1/2}\delta) + O(\epsilon^{1/2}\delta^{-1/2}) \qquad (A1.1.4)$$

where D_*, α and J are as defined in §2.6.

To evaluate the first integral in (A1.1.3) requires rewriting the integrand in a form that can be expanded. On the interval $[0, \delta]$, $\tilde{D}(x)$ can be expanded in its Taylor series about 0. The denominator of the integrand is then

$$\left(\epsilon + \tilde{D}(x)\right)^{1/2} + \epsilon^{1/2} = \epsilon^{1/2} + (\epsilon + \alpha x)^{1/2}\left(1 + O(x)\right)^{1/2}.$$

Further expansion gives

$$\left(\epsilon + \tilde{D}(x)\right)^{1/2} + \epsilon^{1/2} = \left(\epsilon^{1/2} + (\epsilon + \alpha x)^{1/2}\right)\left(1 + \beta x \psi(x; \epsilon) + O(x^2)\right)$$

with $\psi(x; \epsilon)$ given by

$$\psi(x; \epsilon) = \frac{(\epsilon + \alpha x)^{1/2}}{\epsilon^{1/2} + (\epsilon + \alpha x)^{1/2}}$$

and β a constant. The integral then becomes

$$\int_0^\delta \frac{dx}{\epsilon^{1/2} + (\epsilon + \alpha x)^{1/2}} - \beta \int_0^\delta \frac{x\psi(x;\epsilon)\,dx}{\epsilon^{1/2} + (\epsilon + \alpha x)^{1/2}} + \cdots$$

where the ellipses denote higher order terms in ϵ. In the first of these two integrals

let $t^2 = \epsilon + \alpha x$ to get

$$\frac{2}{\alpha} \int_{\epsilon^{1/2}}^{(\epsilon+\alpha\delta)^{1/2}} \frac{dt}{t(t + \epsilon^{1/2})} \sim \frac{\epsilon^{1/2}}{\alpha} \log \frac{4\epsilon}{\alpha\delta} - \frac{2\epsilon^{1/2}}{\alpha} + \frac{2\delta^{1/2}}{\alpha^{1/2}} + O(\epsilon\delta^{-1/2}) + O(\epsilon^{3/2}\delta^{-1}). \quad (A1.1.5)$$

The change of variables $\epsilon y^2 = \epsilon + \alpha x$ in the second of the two integrals gives

$$\epsilon^{3/2} \int_1^{(1+\alpha\delta\epsilon^{-1})^{1/2}} \frac{y^2(y-1)}{y+1}\,dy = O(\delta^{3/2}) + O(\epsilon^{3/2}) + O(\epsilon^{3/2}\log\epsilon\delta^{-1}).$$

This shows that the second integral of the two does not make a contribution to

the order we need.

Now we combine the results of (A1.1.4) and (A1.1.5) to get

$$D_* + \frac{\epsilon^{1/2}}{\alpha} \log \frac{4\epsilon}{\alpha} - \epsilon^{1/2}\left(\frac{2}{\alpha} - J\right) + O(\epsilon^{1/2}\delta) + O(\epsilon^{1/2}\delta^{-1/2}) + O(\epsilon\delta^{-1/2}) + O(\epsilon^{3/2}\delta^{-1})$$

$$(A1.1.6)$$

for $\delta \to 0^+$ and $\epsilon/\delta \to 0^+$. It can be noted that the $\log\delta$ and $\delta^{1/2}$ terms cancelled

on addition as expected. To assure that the error terms are smaller than those

retained, $\delta(\epsilon)$ is restricted such that $\epsilon^{3/2} \prec \delta(\epsilon) \prec 1$.

For the final result mutiply (A1.1.6) by $\sigma^{1/2}$ and use the definition of ϵ to get

the asymptotic expansion

$$\varphi_0(\sigma) \sim D_*\sigma^{-1/2} - \tfrac{1}{2\alpha}\log\alpha\sigma - \tfrac{1}{\alpha} - \tfrac{1}{2}J + O(\sigma^{-3/2}) \qquad \sigma \to \infty.$$

APPENDIX 2

Asymptotic Expansion for Calculating Smallest Eigenvalue

In this appendix, some of the details for showing the behavior of the smallest eigenvalue as a function of the electric field parameter are given. The calculation shown is the evaluation of the constant K of (3.3.52). The constant K contains an integral which depends on the small parameter γ. Its integrand cannot be expanded in a power series in γ and so asymptotic matching is used to evaluate the integral.

The constant K is given by

$$K(\epsilon) = \pi^{-1} \nu^{-1/2} \kappa_m \exp \left\{ -\frac{1}{\nu} \int_{y_0}^1 (\hat{V}(y; \epsilon) - \lambda_0)^{1/2} dy \right\}$$

with $\nu(\epsilon) = \frac{1}{2}(m+2) \epsilon^{(m+2)/2}$. A first approximation to the solution of this equation is

$$K(\epsilon) = (\pi \nu)^{-1/2} \kappa_m \exp \left\{ -\frac{1}{\nu} \int_{y_0}^1 \hat{V}^{1/2}(y; \epsilon) \, dy \right\}. \qquad (A2.1.1)$$

The error made is exponentially small in the quantity λ_0.

The change of variables defined by

$$\frac{\epsilon}{\nu(\epsilon)} \frac{dy}{dt} = D^{-1/2}(\epsilon^{1/2} t)$$

simplifies the analysis of the integral in (A2.1.1). This is just a scaling of the velocity variable u with $\epsilon = \gamma^{1/(m-1)}$. Then the argument of the exponential in (A2.1.1) becomes

$$\epsilon^{-2} \int_{t_0}^{t_1} \left(\epsilon^2 \tilde{V}(\epsilon^{-1} t) D^{-1}(\epsilon^{-1} t) \right)^{1/2} dt \qquad (A2.1.2)$$

with the limits of integration being $t_0 = \epsilon c_0 + O(\epsilon^m)$ and $t_1 = 1 - \alpha_1 \epsilon$. These limits follow from §3.3 where they were derived. The potential $V(\epsilon^{-1}t)$ cannot be expanded uniformly in its argument due to the singular nature of the perturbation. Near the point t_0 the argument $\epsilon^{-1}t$ can no longer be considered to be large. Because of this the integral is split in two at some intermediate point $\delta(\epsilon)$ which vanishes as $\epsilon \to 0$. The point $\delta(\epsilon)$ is such that $\epsilon \prec \delta(\epsilon) \prec 1$. Thus (A2.1.2) becomes

$$\epsilon^{-2} \int_{t_0}^{\delta} \left(\epsilon^2 \tilde{V}(\epsilon^{-1}t) D^{-1}(\epsilon^{-1}t) \right)^{1/2} dt + \epsilon^{-2} \int_{\delta}^{t_1} \left(\epsilon^2 \tilde{V}(\epsilon^{-1}t) D^{-1}(\epsilon^{-1}t) \right)^{1/2} dt. \quad (A2.1.3)$$

Asymptotic matching between these two integrals will be used to remove any dependence on the matching parameter $\delta(\epsilon)$.

The second of these integrals will be considered first. As the argument of V and D^{-1} are large on the interval $[\delta, t_1]$ for some range of $\delta(\epsilon)$, the integrand can be expanded using the asymptotics of $V(u)$ and $D(u)$. From (3.3.8), the asymptotic expansion of $D(u)$, and (3.2.17), the definition of $V(u)$, one gets

$$\epsilon^2 \tilde{V}(\epsilon^{-1}t) D^{-1}(\epsilon^{-1}t) \sim t^2(1 - t^{m-1})^2 + 2a_1 \epsilon t^m(1 - t^{m-1})$$

$$+ \epsilon^2 \left(m - 1 + a_1^2 t^{m-1} + (2a_2 - 3a_1^2)t^{m-1}(1 - t^{m-1}) \right) + O(\epsilon^3)$$

as $\epsilon^{-1}t \to \infty$. The square root of this is then taken and a power series in ϵ results which is

$$\int_{\delta}^{t_1} \left(\epsilon^2 \tilde{V}(\epsilon^{-1}t) D^{-1}(\epsilon^{-1}t) \right)^{1/2} dt \sim \int_{\delta}^{t_1} t(1 - t^{m-1}) \, dt + a_1 \epsilon \int_{\delta}^{t_1} t^{m-1} \, dt$$

$$+ \epsilon^2 \left((a_2 - a_1^2) \int_{\delta}^{t_1} t^{m-2} \, dt + \frac{m-1}{2} \int_{\delta}^{t_1} \left(\frac{1}{t} + \frac{t^{m-2}}{1 - t^{m-1}} \right) dt \right) + O(\epsilon^3), \quad (A2.1.4)$$

all of which are simple integrals. In the first integral let

$$\int_{\delta}^{t_1} t(1 - t^{m-1}) \, dt = \left\{ \int_0^1 - \int_0^{\delta} - \int_{t_1}^1 \right\} t(1 - t^{m-1}) \, dt$$

and then evaluation of the integrals in (A2.1.4) gives

$$\int_{\delta}^{t_1} \left(\epsilon^2 \tilde{V}(\epsilon^{-1}t) D^{-1}(\epsilon^{-1}t)\right)^{1/2} dt \sim \frac{m-1}{2(m+1)} - \frac{1}{2}\delta^2 + \frac{a_1}{m}\epsilon - \frac{1}{2}\epsilon^2 \log \epsilon - \frac{m-1}{2}\epsilon^2 \log \delta$$

$$+ O(\epsilon^2) + O(\delta^{m+1}) + O(\epsilon^2 \delta) + O(\epsilon^2 \delta^{m-1}). \qquad (A2.1.5)$$

To evaluate the first integral in (A2.1.3) asymptotically for small ϵ, let $w = \epsilon^{-1}t$. Then the integral becomes

$$\int_{\delta}^{t_1} \left(\epsilon^2 \tilde{V}(\epsilon^{-1}t) D^{-1}(\epsilon^{-1}t)\right)^{1/2} dt = \epsilon^2 \int_{c_0}^{\delta/\epsilon} \left(\tilde{V}(w) D^{-1}(w)\right)^{1/2} dw. \qquad (A2.1.6)$$

The integrand has to be expanded near t_0 in the t coordinates or c_0 in the w coordinates. In this domain the perturbation part of $V(w)$ is of $O(\gamma)$ or in terms of ϵ, $O(\epsilon^{(m-1)/2})$. To evaluate (A2.1.6) define the function

$$I(x) = \int_{c_0}^{x} (\tilde{V}(w) D^{-1}(w))^{-1/2} dw$$

where $x = \delta/\epsilon \gg 1$. Then an asymptotic expansion for $I'(x)$ for $x \to \infty$ can be found by using the properties of $V(w)$ and $D(w)$ from (3.2.17) and (3.3.8). This gives

$$I'(x) \sim x + \frac{1}{2}(m-1)x^{-1} + O(x^{-2}) + O(\epsilon^{(m-1)/2}) \qquad x \to \infty.$$

On integration and using $x = \delta/\epsilon$ this becomes

$$I(x) \sim \frac{\delta^2}{2\epsilon^2} + \frac{1}{2}(m-1) \log \delta\epsilon^{-1} + O(\epsilon\delta^{-1}) + O(\epsilon^{(m-1)/2}). \qquad (A2.1.7)$$

When (A2.1.5) is added to ϵ^2 times (A2.1.7), the result is that

$$\int_{t_0}^{t_1} \left(\epsilon^2 \tilde{V}(\epsilon^{-1}t) D^{-1}(\epsilon^{-1}t)\right)^{1/2} dt \sim \frac{m-1}{2(m+1)} + \frac{a_1}{m}\epsilon - \frac{m}{2}\epsilon^2 \log \epsilon$$

$$+ O(\epsilon^2) + O(\epsilon^2 \delta) + O(\epsilon\delta^{m-1}) + O(\epsilon^2 \delta^{-1}) + O(\delta^{m+1})$$

for $\delta \to 0^+$ and $\epsilon/\delta \to 0^+$. It can be noted that the $\log \delta$ and δ^2 terms cancelled. To ensure that the error terms are smaller than those retained, $\delta(\epsilon)$ must be such that $\epsilon \prec \delta \prec \epsilon^{1/2}$.

The constant K is then given by

$$K(\epsilon) \sim \pi^{-1} \nu^{-1/2} \kappa_m \epsilon^{m/2} \exp \left\{ -\frac{m-1}{2(m+1)} \epsilon^{-2} - \frac{a_1}{m} \epsilon^{-1} \right\}$$

for $\epsilon \to 0$.

APPENDIX 3

Lifshitz and Pitaevskii Calculation Of Runaway Rate

It is instructive to consider the following approach to finding the runaway current rate. The calculation is due to E. M. Lifshitz and L. P. Pitaevskii and appears in [19].

The assumption made in the calculation is that there is a steady-state solution. In the Fokker-Planck equation (3.2.1), $\frac{\partial f}{\partial t}$ is set equal to zero. The resulting equation

$$-\frac{dj}{du} = \frac{d}{du}\left(D(u)\left(\frac{df}{du} + 2uf \right) - 2\gamma f \right) = 0 \qquad (A3.1.1)$$

is then solved. This assumption implies j is constant for all time, but we know that is not the case for very large time. Solving (A3.1.1) gives

$$f(u) = w^{-1}(u)\left(c_1 \int_0^u D^{-1}(v)w(v)\,dv + c_2 \right) \qquad (A3.1.2)$$

with

$$w(u) = \exp\left(u^2 - 2\gamma \int_0^u D^{-1}(v)\,dv \right).$$

These two solutions are divergent for large u due to the presence of the $F(u)$ term. The constant value of j is $-c_1$. Without loss of generality we take $f(0) = 1$ since the solution $w^{-1}(u)$ corresponds to zero current. This implies $c_2 = 1$.

The constant c_1 is determined by requiring that the expressions in the parentheses in (A3.1.2) vanish to leading order for $u \to \infty$. Therefore,

$$j^{-1} = \int_0^\infty D^{-1}(u)w(u)\,du. \qquad (A3.1.3)$$

Equation (A3.1.2) still will be non-integrable on $[0, \infty)$ since this choice of c_1 removes only part of the large u behavior. This approach is inconsistent since it is not possible for any choice of constants to make this steady state solution integrable.

A simple saddle point calculation for (A3.1.3) is done for $\gamma \to 0$. The saddle point occurs at $u = \gamma^{-1/(m-1)}$ which is the same as where the minimum point of the potential occurs in the calculations of Chapter 3. The resulting calculation gives

$$j \sim \sqrt{\frac{m-1}{\pi}} \gamma^{m/(m-1)} \exp -\tfrac{m-1}{m+1} \gamma^{-2/(m-1)}.$$

This gives the identical power and exponential factor as in the calculation of Chapter 3 for the one-dimensional problem.

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