Periods of Feynman Diagrams

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ABSTRACT

We study differential equations for Feynman amplitudes and show that the corresponding D-module is isomorphic to a GKZ D-modules. We show that the sheaf of solutions to the D-module is isomorphic to a certain relative homology and that the amplitudes are periods of a relative motive. Using these ideas, we develop a method of regularization which specializes to dimensional regularization and analytic regularization.

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Chapter 1

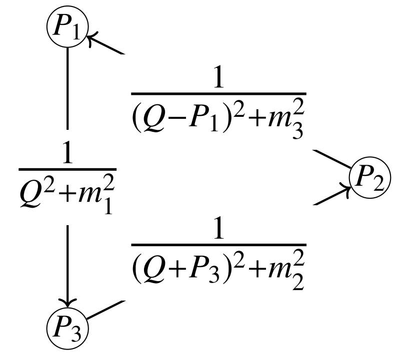
INTRODUCTION

In perturbative quantum field theory the scattering amplitudes, which are the probabilities of physical processes, can be approximated by sums over Feynman diagrams. Feynman diagrams are graphs corresponding to certain integrals. The integral corresponding to each Feynman diagram is a function of parameters called external momenta. The resulting functions are called *amplitudes* of Feynman diagrams. To compute the actual scattering amplitudes one needs to add these functions, and thus understanding the properties of these functions is necessary for both experimental and theoretical physics. In this thesis, we restrict our attention to the case of a scalar field theory. This means that the external momenta are just vectors in $\mathbb{R}^D$, with D the dimension of the theory. Suitable generalizations exist for arbitrary quantum field theories.

A Feynman diagram in D-dimensional scalar field theory is a graph (possibly with multiple edges and self edges) together with an assignment of vectors (momenta) in $\mathbb{R}^D$ to the vertices of the graph such that they add up to zero, and an assignment of positive real numbers (masses) to the edges. To compute the corresponding integral, we first choose an orientation for the edges of the graph and integrate a product of propagators on the space of flows to the graph. By a flow we mean an assignment of vectors in $\mathbb{R}^D$ to edges such that at each vertex sum of incoming vectors minus sum of outgoing vectors is equal to the vector at that vertex. The propagator corresponding to an edge with assigned vector p and positive real number m is defined by the following formula:

$$\frac{1}{p^2 + m^2}.$$

See chapter 2.1 for a detailed explanation. We compute this integral for an example. Consider the following graph on vertices $1, 2, 3$ with momenta P_1, P_2, P_3 and masses m_1, m_2, m_3 for edges $12, 23, 31$, respectively.



The space of flows through this diagram is D dimensional and can be parametrized by vector corresponding to the edge 12 which we denote by Q. Propagators corresponding to edges are shown in the figure above. The corresponding integral is

$$\int_{\mathbb{R}^D} \frac{d^D Q}{\left(Q^2+m_1^2\right)\left((Q+P_3)^2+m_2^2\right)\left((Q-P_1)^2+m_3^2\right)}.$$

Note that we have the condition $P_1 + P_2 + p_3 = 0$ otherwise the space of flows is empty. This integral does not necessary converge (depending on D). Since the early years of quantum field theory physicists have used a different form of this integral which is called the parametric form. The parametric form that we use here was first introduced by Symanzik in [Sym58]. For graph Γ, we denote the set of edges by E, and for each $e \in E$ we define a new variable t_e. The form is defined in section 2.1 as

$$\pi^{D\ell/2} \int_{\mathbb{R}_+^{|E|}} e^{-\sum t_e m_e^2 - \frac{\mathcal{P}_\Gamma}{\Psi_\Gamma}} \frac{1}{\Psi_\Gamma^{D/2}} \prod_e dt_e,$$

where Ψ_Γ and $\mathcal{P}_\Gamma$ are polynomials in t_e's, which are called first and second Symanzik polynomials and are defined in definitions 2.5 and 2.4. ℓ is equal to the number of independent cycles in the graph. For the diagram above we have

$$\begin{aligned}\mathcal{P}_\Gamma &= P_1^2\, t_1 t_2 + P_2^2\, t_2 t_3 + P_3^2\, t_3 t_1\\ \Psi_\Gamma &= t_1 + t_2 + t_3\\ \ell &= 1.\end{aligned}$$

The integral above does not always converge. To extract a meaningful function from the integral, physicists think of D as a complex number and consider the Laurent expansion of the integral in D. As we show in this thesis for values of D in an open subset of the complex plane, the integral is well defined and can be extended to the complex plane as a meromorphic function of D. For the diagram above this can be seen by first integrating radially.

$$\int_{\mathbb{R}^3_+} \exp\left(-\frac{P_1^2 t_1 t_2 + P_2^2 t_2 t_3 + P_3^2 t_3 t_1 + (t_1+t_2+t_3)(m_1 t_1 + m_2 t_2 + m_3 t_3)}{t_1+t_2+t_3}\right) \frac{dt_1 dt_2 dt_3}{(t_1+t_2+t_3)^{D/2}}$$

$$= \Gamma(3-D/2) \int_{\Delta_2} \frac{\left(P_1^2 t_1 t_2 + P_2^2 t_2 t_3 + P_3^2 t_3 t_1 + (t_1+t_2+t_3)(m_1 t_1 + m_2 t_2 + m_3 t_3)\right)^{-3+D/2}}{(t_1+t_2+t_3)^{-3+D}} \Omega,$$

where $\Omega = \sum_i (-1)^i t_i \ dt_1 \wedge \cdots \hat{dt_i} \cdots \wedge dt_n$ and Δ_2 is the standard embedding of 2-simplex in $\mathbb{R}^3$. We see that in dimension 6 the integral has a simple pole in D. An example of a convergent integral is when $D = 4$ and we get

$$\Gamma(1) \int_{\Delta_2} \frac{1}{P_1^2 t_1 t_2 + P_2^2 t_2 t_3 + P_3^2 t_3 t_1 + (t_1+t_2+t_3)(m_1 t_1 + m_2 t_2 + m_3 t_3)} \Omega.$$

The integral converges and is a function of momenta and masses. Note that masses and momenta appear as coefficients of the polynomial in the integrand. This particular example is considered in [DD98]. Authors compute the integral in dimension four as a linear combination of Di-logs.

An interesting type of integral corresponds to primitive log divergent graphs with $D = 4$. A Feynman diagram is called a primitive log divergent in $D = 4$, if for all subgraphs (subset of edges), the number of edges is strictly greater than twice the number of loops in the subgraph. In this case the integral has a simple gamma function pole in D and the interesting function is the coefficient of the pole, which is constant and depends on masses and momenta. In [BK95] authors considered this type of integrals with the degree of the nodes restricted to be less than 4 (ϕ^4 theory). They showed that for all graphs of up to 6 loops one gets a linear combination of multiple zeta values.

Multiple zeta values are a generalization of values of Riemann zeta function at integers and their relation to mixed Tate motives has been studied extensively. In [GM04] authors show that all multiple zeta values appear as a period of mixed Tate motives. In [Bro12] the author shows that any period of a mixed Tate motive over $\mathbb{Z}$ is a linear combination of multiple zeta values. Mixed Tate motives are a quotient of a certain type of varieties with an equivalence relation. Two varieties with the same image in the category of mixed Tate motives have same counting function. All varieties which map to the category of mixed Tate motives have polynomial counting functions, i.e. the number of their points in $\mathbb{F}_q$ are polynomials in q.

In the case of primitive log divergent graphs one gets the integral

$$\int_{\Delta_n} \frac{1}{\Psi_\Gamma(t_1, \ldots, t_n)^2} \Omega.$$

One can think of the integrand as a differential form defined on the complement of zeros of Ψ_Γ and Δ_n as a relative cycle in the pair $(\mathbb{P}^n \setminus V(\Psi_\Gamma), \Sigma)$, where Σ is union of coordinate hyperplanes (see the next paragraph). The work of [BK95] and the relation between mixed Tate motives and counting function led Kontsevich to informally conjecture that zeros of the first Symanzik polynomial (for a general graph) in $\mathbb{P}^n$ have polynomial counting functions. The conjecture was verified for all graphs of up to 12 edges in [Ste98]. But the conjecture was later shown wrong in [BB03]. Authors showed that the counting functions of zeros of second Symanzik polynomial are very general. But the question remained to identify the graphs for which one gets a mixed Tate motive. This direction has been studied in [Sta98], [AM09], [Blo10], and [BS12].

The picture above is not completely accurate since zeros of Ψ_Γ might intersect Σ. Later it was shown in [BEK06] that the integral for a primitive log divergent case is in fact a period in the sense of algebraic geometry, i.e. a pairing between a relative homology class and algebraic differential form both defined over $\mathbb{Q}$ in a pair of varieties defined over $\mathbb{Q}$. To construct this pair, authors use a sequence of blowups along the intersection of coordinate hyperplanes to separate the integration cycle from the the poles of the integrand. They also use techniques from homotopy theory to show that for an infinite class of graphs, the pair is of the mixed Tate type. Later it was shown in [BK10] that for the triangle graph the pair of varieties is of mixed Tate type. Note that the triangle graph is not primitive log divergent.

As we saw in the examples above, one gets interesting functions or numbers out of these integrals either as the coefficients of the poles or when the integral converges. Two important questions that arise here are the following: what is the analytic structure of the integral in D, and what type of functions do we get after removing the poles? We show that the answers to these questions are related. We construct a toric variety and interpret the integral as a pairing between cohomology and homology classes on it. It turns out that for each codimension one component of the boundary of the toric variety, one gets an arithmetic progression of poles in D. These toric varieties appear when we study the differential equations corresponding to the integral as masses and momenta change.

As we have seen in the example, amplitudes of Feynman diagrams are functions of external momenta and masses. We map this space to the vector space constructed by coefficients of the first and second Symanzik polynomial, denoted by $\mathfrak{V}_\Gamma$ so that the amplitude is the pull back of a (multi-valued)function on $\mathfrak{V}_\Gamma$. On this new vector space, we construct a holonomic regular D-module, of which the function we are considering is a solution. As a result, we show that the Feynman amplitude satisfies a holonomic regular system of differential equations.

The differential equations on $\mathfrak{V}_\Gamma$ are a special case of GKZ or A-Hypergeometric system of differential equations introduced in [GZK89] and [GKZ90]. It follows from the results in these references that the corresponding D-module is holonomic and regular. It is well known that these D-modules come from twisted Gauss-Manin connections on toric varieties. Recently it is shown in [Hua+15] that, in the Calabi-Yau case, the relative homology computes the sheaf of solutions. Using results of [AB01] we show that their construction can be generalized to a non-Calabi-Yau case which includes Feynman diagrams.

We construct a toric variety and a family of hypersurfaces in it parametrized by $\mathfrak{V}_\Gamma$. The integral becomes a pairing between a differential form defined on the complement of the hypersurface paired with the positive real points of the toric variety considered as a cycles. We show that the variation of the hypersurface corresponds to the differential equations. Since the construction is explicit we can compute the cohomology using generators and relations.

Using the description of cohomology with generators and relations, we show how one can define the integral for the divergent case as a Laurent expansion. In particular, we prove the following theorem.

Theorem 1.1. *Given a graph Γ with n edges and first Symanzik polynomial Ψ and second Symanzik polynomial Q (including mass terms), the amplitude in dimensional regularization, up to a constant, can be computed by the following integral:*

$$c_0\mathcal{A}(D/2+\epsilon) = \int_{\mathbb{R}^n_+} e^{-Q/\Psi}\frac{1}{\Psi^{D/2+\epsilon}} = \sum_{i\geq -n}\epsilon^i A_i(D/2).$$

The left hand side is meromorphic and poles can be described in the following way. For a 2-connected subgraph $\gamma\subset\Gamma$, let ℓ_γ be the dimension of the first homology of

γ. $|E(\gamma)|$ is the number of edges of γ. $\mathcal{A}(D/2)$ has a pole at $D/2 \in \mathbb{C}$ iff

$$D/2\ \ell_\gamma - |E(\gamma)| \in \mathbb{Z}_{\geq 0}$$

for a 2-connected subgraph γ. A_i's are coefficients of the Laurent expansion of the left hand side. Furthermore, the lowest coefficient, $A_{-n}(D/2)$, at integers comes from a pairing between an algebraic relative cohomology class and a Betti homology class explicitly constructed in section 3.1.

1.1 Structure of the Thesis

In section 2.1 we define the Feynman amplitude for a Feynman diagram and show how one can present it in the parametric form. The new result is that a product of a power of the first Symanzik polynomial and the second Symanzik polynomial is the determinant of a matrix. We also show that the coefficients of the first and second Symanzik polynomials are norms of Plücker coordinates for a Grassmannian naturally defined by the graph.

In section 2.2 we study how the integral changes as we vary the coefficients of the first and second Symanzik polynomials. In the convergent case, we find a set of linear PDEs satisfied by the integral. Using analytic continuation we define the integral for the divergent case but the proof is not constructive. We first show that the analytic continuation exists and, using that, we find the PDEs satisfied by coefficients in the Laurent expansion. This method is based on [BW09]. It turns out that the set of differential equations is a special case of the GKZ differential equations for the convergent case. For the divergent case, the coefficients of the Laurent expansion are solutions to iterated extension of the GKZ differential equations.

In section 3.1 we study the GKZ differential equations as a D-module on a vector space V. Given a polynomial f in n variables with Newton polytope A such that the dimension of V is the number of points in A, and a vector $\beta \in k^{n+1}$, we consider the corresponding GKZ D-module $H_{1\times A}(\beta)$. We construct a projective toric variety $\mathbb{P}_\Sigma$ together with a line bundle on it. The vector space of global sections of the line bundle is isomorphic to V. Let D be the complement of the torus $\mathbb{T}$ in $\mathbb{P}_\Sigma$. Let U be the complement of the zeros of f in $V \times \mathbb{P}_\Sigma$, where V parametrizes the coefficients of the polynomial f. Given $v \in V$ we show that the algebraic relative cohomology of the pair $(U_v, D \cap U_v)$ with the Gauss-Manin connection is isomorphic to $H_A(\beta)$ as a D-module, where U_v is the fiber of U over v. Using the Riemann-Hilbert correspondence, we deduce that the cycle to period map gives us a complete set of solutions.

In section 3.2 we study integrals over positive real points of the toric variety, considered as a chain in relative homology. We show that a condition necessary and sufficient in order for an integral of the type

$$\int_{\mathbb{R}_+^n} \frac{f^{\beta_0}}{t_1^{\beta_1} \dots t_n^{\beta_n}} dt_1 \dots dt_n$$

to converge is that β is semi non-resonant, as defined in Definition 3.15. Using relations in the cohomology ring, we develop a method to define this integral for any value of β by meromorphic continuation. We show that the poles of this function appear in translates of the faces of a cone in $\mathbb{R}^{n+1}$. This cone is the cone over the Newton polytope of f.

In chapter 4 we apply the methods developed in the section 3.2 to amplitudes and we explicitly construct a motive such that its periods give us the amplitude. We completely describe the Newton polytope in this case and show that its facets correspond to the product of subgraphs and quotient graphs. Using regularization methods, we define the ϵ expansion of the amplitude for divergent graphs.

Chapter 2

FEYNMAN DIAGRAMS

2.1 Parametric Representation of Feynman Amplitudes

Feynman diagrams (or Feynman graphs) are one-dimensional simplicial complexes with half edges attached to some of the vertices. These half edges are called *external edges*, while all other one-dimensional simplices are called *internal edges*. In the physics literature, for each external edge, it is common to fix a vector in $\mathbb{R}^D$. These vectors are called *external momenta*. They are subject to a momentum conservation law, given by the requirement that the sum of all external momenta of the graph is zero. Since the amplitude only depends on the sum of the external momenta at each vertex, we can equivalently assign a momentum vector to each vertex of the graph and forget about external edges. Namely, we assign to a vertex the sum of the external momenta of all the external edges attached to that vertex, or zero if there are no external edges at that vertex. So in the following external momenta will always be assigned to vertices.

Let E be the set of edges of the graph and let V be the set of vertices. We have an exact sequence of free $\mathbb{Z}$ modules

$$0 \to H_1(\Gamma, \mathbb{Z}) \xrightarrow{\quad \eta' \quad} \mathbb{Z}^{|E|} \xrightarrow{\qquad} \mathbb{Z}^{|V|-1} \to 0, \tag{2.1}$$

where $H_1(\Gamma, \mathbb{Z})$ is the first homology of the graph Γ with coefficients in $\mathbb{Z}$, that is, the free $\mathbb{Z}$ module generated by loops. The morphism on the right is the boundary map. Note that, to define this map, we need to fix an orientation on the edges of the graph, but the final result is independent of this choice. Taking the tensor product of the sequence above with $\mathbb{R}^D$ gives the exact sequence

$$0 \to H_1(\Gamma, \mathbb{R}^D) \xrightarrow{\quad \eta \quad} \mathbb{R}^{D|E|} \xrightarrow{\quad \beta \quad} \mathbb{R}^{D(|V|-1)} \to 0. \tag{2.2}$$

Note that the choice of external momenta $\{p_v \in \mathbb{R}^D | v \in V, \sum_{v \in V} p_v = 0\}$ is just a choice of a vector a in $\mathbb{R}^{D(|V|-1)}$.

Define $Q_e = P_e^2 + m_e^2 : \mathbb{R}^{|E|D} \to \mathbb{R}$, where P_e^2 is given by first projecting onto the D coordinates corresponding to e and then taking the sum of the squares of these D

coordinates.

Let $\tilde{a} \in \mathbb{R}^{D|E|}$ be a lift of a, under the map β of 2.2.

Definition 2.1. *The amplitude of a Feynman graph Γ with external momenta $a \in \mathbb{R}^{D(|V|-1)}$ is given by the integral*

$$\begin{aligned}\mathcal{A}(\Gamma, a, m_e) : &= \int_{\beta^{-1}(a)} \prod_e \frac{1}{Q_e}(\eta + \tilde{a})_*(d\mu) \\ &= \int_{H_1(\Gamma,\mathbb{R}^D)} (\eta + \tilde{a})^*(\prod_e \frac{1}{Q_e})d\mu,\end{aligned} \tag{2.3}$$

where η is as in 2.2 and $d\mu$ is the tensor product of the standard measure on $\mathbb{R}^D$ with the measure on $H_1(\Gamma,\mathbb{R})$ induced by the morphism $H_1(\Gamma,\mathbb{Z}) \to H_1(\Gamma,\mathbb{R})$. This is the unique positive translation invariant measure on $H_1(\Gamma,\mathbb{R})$, with the property that a basis of $H_1(\Gamma,\mathbb{Z})$ generates a parallelogram of measure 1.

Note that, with this definition, the amplitude is a function on $\mathbb{R}^{D(|V|-1)}$.

The Schwinger trick simply consists of using the identity $\int_0^\infty e^{-ax}dx = \frac{1}{a}$ in order to rewrite the amplitude in “parametric form". For each edge e we introduce a new variable t_e.

Definition 2.2. *A subset of edges $S \subset E$ is called a spanning tree if the subgraph with edges in S is a tree and is maximal in the sense that, if we add any of the remaining edges to it, it will contain a loop. Denote the set of spanning trees by $Span$.*

Definition 2.3. *A subset of edges $C \subset E$ is called a cut if it has the following properties.*

1. *When we remove these edges, the graph becomes a disconnected union of trees.*
2. *The set C is minimal in the sense that, if we add back any edges to the remaining graph, it will either have a loop or become connected.*

Since cuts are minimal, they divide vertices into disjoint sets V_C and V_C^c. For a cut C we denote by P_C the norm of the sum of momenta in either component,

$$P_C = (\sum_{v\in V_C} p_v)^2 = (\sum_{v\in V_C^c} p_v)^2.$$

Denote the set of cuts by Cut.

Definition 2.4. *The first Symanzik (or Kirchhoff) polynomial of a Feynman graph is given by*

$$\Psi_\Gamma(t_1, ..., t_{|E|}) := \sum_{S \in Span} \prod_{e \notin S} t_e.$$

Definition 2.5. *The second Symanzik polynomial of a Feynman graph is given by*

$$\mathcal{P}_\Gamma(t_1, ..., t_{|E|}, P_C) := \sum_{C \in Cut} P_C \prod_{e \in C} t_e.$$

Given a Feynman graph Γ, we enumerate edges by $1, ..., |E|$. We define

$$T := \mathrm{d}iag(\sqrt{t_1}, \sqrt{t_2}, ..., \sqrt{t_{|E|}}) \otimes Id_{D\times D}, \quad \text{and } T_{red} := \mathrm{d}iag(\sqrt{t_1}, ..., \sqrt{t_{|E|}}). \tag{2.4}$$

Let $\vec{P}$ be a vector in $\mathbb{R}^{D\times|E|}$, where the coordinates are ordered in the same way as the variables t_i. Note that, for each edge, we have D coefficients. Let H denote the image of $H_1(\Gamma, \mathbb{R})$ in $\mathbb{R}^{|E|}$.

Lemma 2.6. *The measure $d\nu$ on $T\beta^{-1}(a)$ induced by the standard measure on $\mathbb{R}^{D|E|}$ satisfies*

$$d\nu = \Psi_\Gamma(t_1, ..., t_{|E|})^{D/2}(T\eta + T\tilde{a})_*(d\mu), \tag{2.5}$$

with Ψ_Γ the Kirchhoff polynomial, T defined as in 2.4, and η as in 2.2.

Proof. Since $T\eta + T\tilde{a}$ is linear, $(T\eta + T\tilde{a})_*(d\mu)$ is a constant multiple of the measure on $T\beta^{-1}(a)$ induced from $\mathbb{R}^{D|E|}$. In order to compute this constant, we can compare the volume of the image of the standard cube in these two measures. We have $(T\eta + T\tilde{a})_*(d\mu)\,(T\eta(\mathrm{C}ube) + T\tilde{a}) = d\mu(\mathrm{C}ube) = 1$, by definition. On the other hand, we have

$$d\nu\,(T\eta(\mathrm{C}ube) + T\tilde{a}) = d\nu\,(T\eta(\mathrm{C}ube)),$$

since the standard measure on $\mathbb{R}^{D|E|}$ is translation invariant.

We choose bases $\{v_1, ...v_\ell\}$ for $H_1(\Gamma, \mathbb{Z})$ and $\{A_1, ..., A_D\}$ for $\mathbb{R}^D$. With η as in 2.2, η' as in 2.1, and $T_{\mathrm{r}ed}$ as in 2.4, the volume of the image of the standard cube is then given by

$$\sqrt{\det\left(T\eta(v_i \otimes a_j) \cdot T\eta(v_k \otimes a_l)\right)} = \left(\det(T_{red}\eta' v_i \cdot T_{red}\eta' v_k)\right)^{D/2},$$

since the volume form corresponding to a metric g is $det(g)^{1/2}$. Consider $T_{\mathrm{r}ed}\eta' : H_1(\Gamma, \mathbb{R}) \to \mathbb{R}^{|E|}$. We have

$$\wedge^\ell T_{\mathrm{r}ed}\eta' : \wedge^\ell H_1(\Gamma, \mathbb{R}) \to \wedge^\ell \mathbb{R}^{|E|},$$

where ℓ is the dimension of $H_1(\Gamma, \mathbb{R})$. The determinant above is the norm square of $\wedge^{\ell} T_{\text{red}}\eta'(v_1 \wedge v_2 \wedge \ldots \wedge v_\ell)$ in the induced metric; thus it can be computed as the sum of the squares of the coefficients in an orthonormal basis. Let $\{w_1, \ldots, w_{|E|}\}$ be the standard basis for $\mathbb{R}^{|E|}$. Then $\{\wedge_{i\in I} w_i\}_{|I|=\ell, I\subset\{1,..,|E|\}}$ is an orthonormal basis for $\wedge^{\ell}\mathbb{R}^{|E|}$. We have

$$\begin{aligned}\wedge^{\ell} T_{\text{red}}\eta'(v_1 \wedge v_2 \wedge \ldots \wedge v_\ell) &= (T_{\text{red}}\eta'(v_1) \wedge T_{\text{red}}\eta'(v_2) \wedge \ldots \wedge T_{\text{red}}\eta'(v_\ell)) \\ &= (\sum_{j=1}^{|E|} \eta'_{1j}\sqrt{t_j}w_j \wedge \sum_{j=1}^{|E|} \eta'_{2j}\sqrt{t_j}w_j \wedge \ldots \wedge \sum_{j=1}^{|E|} \eta'_{\ell j}\sqrt{t_j}w_j)).\end{aligned}$$

Note that, since $\eta'_{i,j} = \pm 1$, the coefficient of the term $w_{i_1} \wedge w_{i_2} \wedge \ldots \wedge w_{i_\ell}$ is either zero or equal to $\pm \prod_k \sqrt{t_{i_k}}$. On the other hand, the coefficient of $w_{i_1} \wedge w_{i_2} \wedge \ldots \wedge w_{i_\ell}$ is nonzero iff the orthogonal projection onto the subspace $W = span(w_{i_1}, w_{i_2}, \ldots, w_{i_\ell})$ is an isomorphism when we restrict it to the image $T_{red}\eta'$. Since T' fixes the coordinate subspaces, this map is an isomorphim iff $\text{Im}(\eta') \cap W = 0$, and that happens iff the subgraph with edges $i_1, \ldots, i_\ell$ does not have a loop, which means it is the complement of a spanning tree. To summarize, we have

$$\big(\det(T_{red}\eta' v_i \cdot T_{red}\eta' v_k)\big)^{D/2} = \big(\sum_{S\in Span}\prod_{e\notin S} t_e\big)^{D/2}.$$

□

Proposition 2.7. *When the integral 2.3 converges, it is equal to*

$$\int_{\mathbb{R}_+^{|E|}} e^{-\sum_e t_e m_e^2} \prod_{e\in E} dt_e \frac{1}{\Psi_\Gamma(t_1, \ldots, t_{|E|})^{D/2}} \int_{T\beta^{-1}(a)} e^{-\vec{P}.\vec{P}} d\nu,$$

where $d\nu$ is the measure on $T\beta^{-1}(a)$ induced by the standard measure on $\mathbb{R}^{D|E|}$.

Proof. Using the Schwinger trick $\int_0^\infty e^{-ax}dx = \frac{1}{a}$ we write

$$\int_{\beta^{-1}(a)} \prod_e \frac{1}{Q_e}(\eta + \tilde{a})_*(d\mu) = \int_{\beta^{-1}(a)} \left(\int_{\mathbb{R}_+^{|E|}} e^{-\sum_e t_e Q_e} \prod_{e\in E} dt_e\right)(\eta + \tilde{a})_*(d\mu).$$

By the definition of Q_e, this is equal to

$$\int_{\beta^{-1}(a)} \left((\int_{\mathbb{R}_+^{|E|}} e^{-\sum_e t_e m_e^2 - \sum_e t_e P_e^2} \prod_{e\in E} dt_e\right)(\eta + \tilde{a})_*(d\mu).$$

In vector form, with T as in 2.4, this can be written equivalently as

$$\int_{\beta^{-1}(a)} \left(\int_{\mathbb{R}_+^{|E|}} e^{-\sum_e t_e m_e^2 - T\vec{P}\cdot T\vec{P}} \prod_{e\in E} dt_e\right)(\eta + \tilde{a})_*(d\mu).$$

Since all functions are positive, convergence is the same as absolute convergence and we can switch integrals. This gives

$$\int_{\mathbb{R}_+^{|E|}} e^{-\sum_e t_e m_e^2} \prod_{e\in E} dt_e \left(\int_{\beta^{-1}(a)} e^{-T\vec{P}\cdot T\vec{P}} (\eta + \tilde{a})_*(d\mu) \right).$$

Using the fact that $(\eta + \tilde{a})^*(e^{-T\vec{P}\cdot T\vec{P}}) = (T\eta + T\tilde{a})^*(e^{-\vec{P}\cdot\vec{P}})$, we rewrite the above as

$$\int_{\mathbb{R}_+^{|E|}} e^{-\sum_e t_e m_e^2} \prod_{e\in E} dt_e \left(\int_{T\beta^{-1}(a)} e^{-\vec{P}\cdot\vec{P}} (T\eta + T\tilde{a})_*(d\mu) \right).$$

Then applying the result of Lemma 2.6 we obtain

$$\int_{\mathbb{R}_+^{|E|}} e^{-\sum_e t_e m_e^2} \prod_{e\in E} dt_e \left(\int_{T\beta^{-1}(a)} e^{-\vec{P}\cdot\vec{P}} \frac{d\nu}{\Psi_\Gamma(t_1, \ldots, t_{|E|})^{D/2}} \right).$$

□

A standard computation shows the following simple facts.

Lemma 2.8. *Let H be a d-dimensional affine linear subspace in $\mathbb{R}^n$ and let L be the distance of the affine subspace H from the origin. Then the integral of a Gaussian function on H with the induced measure is equal to $\pi^{d/2} e^{-L^2}$.*

Lemma 2.9. *Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$. One can compute the volume squared of the parallelogram generated by these vectors in the induced metric on the subspace they generate, in the form*

$$|v_1 \wedge v_2 \wedge \ldots \wedge v_k|^2 = \det(v_i \cdot v_j).$$

The next statement then follows easily.

Lemma 2.10. *Suppose given a vector subspace V of $\mathbb{R}^n$ with a basis $v_1, \ldots, v_k$, and a vector $a \in \mathbb{R}^n$. Let $P(w_1, \ldots, w_m)$ denote the parallelogram generated by $w_1, \ldots, w_m$. The distance of an affine subspace $a + V$ from the origin is equal to*

$$\frac{Vol(P(a, v_1, \ldots, v_k))}{Vol(P(v_1, \ldots, v_k))} = \frac{|a \wedge v_1 \wedge \ldots \wedge v_k|}{|v_1 \wedge \ldots \wedge v_k|}.$$

Proof. Volume is defined by the metric, and thus the distance times $Vol(P(v_1, \ldots, v_k))$ is the volume of $P(a, v_1, \ldots, v_k)$. □

Proposition 2.11. *The Gaussian integral of $e^{-\vec{P}\cdot\vec{P}}$, with respect to the measure $d\nu$ on $T\beta^{-1}(a)$ defined as above, is given by*

$$\int_{T\beta^{-1}(a)} e^{-\vec{P}\cdot\vec{P}} d\nu = \pi^{D\ell/2} e^{-\mathcal{P}_\Gamma(t)/\Psi_\Gamma(t,a)},$$

where Ψ_Γ and $\mathcal{P}_\Gamma$ are the two Symanzik polynomials.

Proof. By lemma 2.8, it is enough to show that the distance squared of $T\beta^{-1}(a)$ from the origin is

$$\mathcal{P}_\Gamma(t)/\Psi_\Gamma(t,a).$$

Let $v_1, v_2, ..., v_\ell$ be a basis of $H_1(\Gamma, \mathbb{Z})$. We denote the image of these vectors in $H_1(\Gamma, \mathbb{Z}) \otimes \mathbb{R}$ by the same notation. Note that the affine subspace over which we are integrating the Gaussian is parallel to the space generated by $T\eta\big(v_1 \otimes e_1, v_1 \otimes e_2, ..., v_1 \otimes e_D, v_2 \otimes e_1, ..., v_2 \otimes e_D, ..., v_\ell \otimes e_D\big)$, where $\{e_1, ..., e_D\}$ is the standard basis for $\mathbb{R}^D$. As we have shown before we then have

$$T\eta(v_1 \otimes e_1) \wedge ... \wedge T\eta(v_\ell \otimes e_D) =$$
$$\left(\sum_{S\in Span} (\pm \prod_{e\notin S} \sqrt{t_e}) \wedge_{e\notin S} w_e \otimes e_1\right) \wedge \left(\sum_{S\in Span} (\pm \prod_{e\notin S} \sqrt{t_e}) \wedge_{e\notin S} w_e \otimes e_2\right) \wedge$$
$$... \wedge \left(\sum_{S\in Span} (\pm \prod_{e\notin S} \sqrt{t_e}) \wedge_{e\notin S} w_e \otimes e_D\right),$$

where $\{w_e\}_{e\in E}$ is a basis for $\mathbb{R}^{|E|}$. We can write $\tilde{a} = \sum_{e,i} P_{e,i} w_e \otimes e_i$. Then, for $i \neq j$, we see that the vector

$$T(w_e \otimes e_i) \wedge T\eta(v_1 \otimes e_1) \wedge ... \wedge T\eta(v_\ell \otimes e_D)$$

is orthogonal to

$$T(w_{e'} \otimes e_j) \wedge T\eta(v_1 \otimes e_1) \wedge ... \wedge T\eta(v_\ell \otimes e_D).$$

One can see this from the expansion in the standard basis: all terms in the first expression have $\ell + 1$ terms with e_i, while the second one has ℓ terms with e_i. Thus, one can compute the norm squared of

$$T(\sum_e P_{e,i} w_e \otimes e_i) \wedge T\eta(v_1 \otimes e_1) \wedge ... \wedge T\eta(v_\ell \otimes e_D)$$

for different i's and add them up to get the squared norm of

$$T(\tilde{a}) \wedge T\eta(v_1 \otimes e_1) \wedge ... \wedge T\eta(v_\ell \otimes e_D).$$

The vector above is in $\wedge^{D\ell+1}\mathbb{R}^{|E|}\otimes\mathbb{R}^D$. We can identify $\mathbb{R}^{|E|}\otimes\mathbb{R}^D$ with D copies of $\mathbb{R}^{|E|}$. The norm squared of this vector is equal to the volume squared of the parallelogram generated by the vectors. Since we have $\ell+1$ vectors in one of the copies and ℓ vectors in the other copies, we can compute the volume of each of them and multiply them together. For $j \neq i$ we have $v_1\otimes e_j, \ldots, v_\ell \otimes e_j$, all of which have the same volume squared, equal to Ψ. To compute the volume of the copy with $\ell+1$ vectors, it is enough to compute

$$T_{\mathrm{red}}(\sum_e P_{e,i} w_e) \wedge T_{\mathrm{red}}\eta'(v_1) \wedge \ldots \wedge T_{\mathrm{red}}\eta'(v_\ell)$$

$$= \left(\sum_e P_{e,i}\sqrt{t_e}\, w_e\right) \wedge \left(\sum_{S\in Span} (\pm\prod_{e\in s}\sqrt{t_e}) \wedge_{e\in s} w_e\right).$$

The terms that appear in the coefficients in the standard basis

$$\{w_{e_{i_1}} \wedge \ldots \wedge w_{e_{i_{\ell+1}}}\}_{i_1<\ldots<i_{\ell+1}}$$

are sums of $P_{e,i}$. Nonzero terms correspond to $\ell+1$ edges that are a complement of a spanning tree plus one extra edge. Note that, if we remove these edges from the graph, it becomes disconnected and, if we add any of these edges to the graph, it becomes a spanning tree. So the term $w_{e_{i_1}} \wedge \ldots \wedge w_{e_{i_{\ell+1}}}$ appears $\ell+1$ times, once of each of its edges. Thus, the coefficient is $\prod_j \sqrt{t_{e_j}} \sum_j P_{e_j,i}$. Since $P_{e_j,i}$ is a lift of a, this sum is equal to the sum of momenta in one of the connected components of the graph. We get the norm as

$$\mathcal{P}_\Gamma(t, P_C) = \sum_{C\in Cut} P_C \prod_{e\in C} t_e.$$

The original norm squared we wanted to compute is then

$$\mathcal{P}_\Gamma \Psi_\Gamma^{D-1}$$

and, by Lemma 2.10, the distance squared is

$$\frac{\mathcal{P}_\Gamma \Psi_\Gamma^{D-1}}{\Psi_\Gamma^D} = \frac{\mathcal{P}_\Gamma}{\Psi_\Gamma}.$$

□

The following result then follows from Proposition 2.11 and Proposition 2.7.

Proposition 2.12. *When the integral 2.3 converges, it is equal to*

$$\mathcal{A}(\Gamma, a) = \pi^{D\ell/2} \int_{\mathbb{R}_+^{|E|}} e^{-\sum t_e m_e^2 - \frac{\mathcal{P}_\Gamma}{\Psi_\Gamma}} \frac{1}{\Psi_\Gamma^{D/2}} \prod_e dt_e$$

Remark 2.13. *The coefficients of the second Symanzik polynomials are always positive. According to the computation in Proposition 2.11, they correspond to squared norms of certain differential forms. We will use this property in chapter 4.*

2.2 Amplituded As a Function of Momenta

The amplitude is defined as an integral which depends on the external momenta. This integral does not always converge. For some graphs that are called ultraviolet divergent, the integral diverges for any value of external momenta, while for some graphs the divergences happen only for special values of the external momenta. The most widely used method in physics for treating these divergences is called dimensional regularization. Within this method, a regularization of divergent integrals is achieved by formally computing the integral for D a complex variable in a neighborhood of the integer spacetime dimension in the complex plane. For a detailed explanation of this method see [CM08]. In this chapter, we define the integral for any D and find differential equations satisfied by it.

The integral depends on a parameter in $\mathbb{C}^{D(|V|-1)}$ and masses of edges. We map this vector space into the vector space $\mathfrak{V}_\Gamma$, which parametrizes the coefficients of the first and second Symanzik polynomials so that it agrees with the amplitude on the image. We generalize the integral to an integral which has $\mathfrak{V}_\Gamma$ as its parameter space. Note that all coefficients in the second Symanzik polynomial are equal to 1. We consider general coefficients for these terms and look at the integral as we vary them. Over $\mathfrak{V}_\Gamma$ the differential equation satisfied by the new integral is geometric in nature and can be solved using series. One can identify $\mathfrak{V}_\Gamma$ with the parameter space of a family of hypersurfaces in toric varieties. The value of the integral for integer D is a period of a relative motive defined by the complement of this hypersurface. From now on we consider the following polynomials:

$$\Psi_\Gamma(t, P_S) := \sum_{S \in Span} P_S \prod_{e \notin S} t_e$$

$$\mathcal{P}_\Gamma(t, P_C) := \sum_{C \in Cut} P_C \prod_{e \in C} t_e$$

$$Q_\Gamma(t, P_C, m_e) = \mathcal{P}_\Gamma(t, P_C) + (\sum_e t_e m_e^2) \sum_{S \in Span} \prod_{e \notin S} t_e = \sum_T Q_T t^T \qquad (2.6)$$

Here T ranges over all monomials appearing in Q_Γ and by t^T we mean the monomial corresponding to T. The amplitude is

$$\mathcal{A}(\Gamma, a, m_e) = \pi^{D\ell/2} \int_{\mathbb{R}_+^{|E|}} e^{-Q_\Gamma/\Psi_\Gamma} \frac{1}{\Psi_\Gamma^{D/2}} \prod_e dt_e,$$

where coefficients of Ψ_Γ are 1 and coefficients of Q_Γ come from equation (2.6) and are sums of masses and momentum variables P_C.

Definition 2.14. *(Parameter Space) Let $\mathfrak{V}_\Gamma$ denote the parameter space for Q_T and P_S. It is a complex vector space of dimension equal to the number of monomials in Q_Γ plus $|Span|$.*

Definition 2.15. *(Generalized Amplitude I) Given $(c_1, c_2, \vec{v}) \in \mathbb{C}^{n+2}$, let*

$$I(c_1, c_2, Q_T, P_S, \vec{v}) := \int_{\mathbb{R}_+^{|E|}} e^{-Q_\Gamma/\Psi_\Gamma} \frac{Q_\Gamma^{c_1}}{\Psi_\Gamma^{c_2}} t^{\vec{v}} \prod_e dt_e, \qquad (2.7)$$

where $t^{\vec{v}}$ means $t_1^{v_1} t_2^{v_2} \ldots t_n^{v_n}$ and $n = |E|$.

Remark 2.16. *Note that this integral is not well defined for all values of the parameters and is a (multi-valued) function on a dense domain in $\mathbb{C}^{n+2} \times \mathfrak{V}_\Gamma$. To define this (multi-valued) function first we define it for some open subset and then we take the analytic continuation.*

Lemma 2.17. *When the generalized amplitude I converges we have:*

$$I(c_1, c_2, Q_T, P_S, \vec{v}) = \Gamma(n + |\vec{v}| + c_1(\ell+1) - c_2\ell) \int_{\Delta_{n-1}} \frac{Q_\Gamma^{-n-|\vec{v}|+(c_2-c_1)\ell}}{\Psi_\Gamma^{-n-|\vec{v}|+(c_2-c_1)(\ell+1)}} t^{\vec{v}} \Omega,$$

where $|\vec{v}| = \sum_i v_i$,

$$\Omega = \sum_{i=1}^n (-1)^{i+1} t_i \, dt_1 \wedge \ldots \wedge \hat{dt_i} \wedge \ldots \wedge dt_n,$$

and Δ_{n-1} is the standard $n-1$ simplex embedded in $\mathbb{R}^n$.

Proof. To show this we parametrize $\mathbb{R}_+^{|E|}$ with $\Delta_{n-1} \times \mathbb{R}_+$. Consider the map ϕ : $\Delta_{n-1} \times \mathbb{R}_+ \to \mathbb{R}_+^{|E|}$ given by

$$\phi(x, s) = sx.$$

Note that

$$\phi^*(dt_1 dt_2 ... dt_n) = \Omega|_{\Delta_{n-1}} s^{n-1} ds,$$

$$Q_\Gamma(sx) = s^{\ell+1} Q_\Gamma(x),$$

$$\Psi_\Gamma(sx) = s^\ell \Psi_\Gamma(x)$$

and $(sx)^{\vec{v}} = s^{|\vec{v}|} x^{\vec{v}}$. Pulling back the integrand to $\Delta_{n-1} \times \mathbb{R}_+$ we have:

$$I(c_1, c_2, Q_T, P_S, \vec{v}) = \int_{\Delta_{n-1}\times\mathbb{R}_+} e^{-sQ_\Gamma(x)/\Psi_\Gamma(x)} \frac{s^{c_1(\ell+1)} Q_\Gamma^{c_1}(x)}{s^{c_2\ell} \Psi_\Gamma^{c_2}(x)} s^{|\vec{v}|} x^{\vec{v}} \Omega s^{n-1} ds \tag{2.8}$$

$$= \int_{\Delta_{n-1}} \Omega \frac{Q_\Gamma^{c_1}(x)}{\Psi_\Gamma^{c_2}(x)} x^{\vec{v}} \int_{\mathbb{R}_+} s^{c_1(\ell+1) - c_2\ell + |\vec{v}| + n - 1} e^{-sQ_\Gamma(x)/\Psi_\Gamma(x)} ds. \tag{2.9}$$

The fact that $\int_{\mathbb{R}_+} e^{-s\lambda} s^x ds = \lambda^{-x-1}\Gamma(x+1)$ then implies the lemma. □

Definition 2.18. *(Generalized Amplitude II) Given $(c, d, \vec{v}) \in \mathbb{C}^{n+2}$, let*

$$J(c, d, Q_T, P_S, \vec{v}) := \int_{\Delta_{n-1}} \frac{Q_\Gamma^c}{\Psi_\Gamma^d} t^{\vec{v}} \Omega. \tag{2.10}$$

Remark 2.19. *By Lemma 2.17, the generalized amplitudes I and II of Definitions 2.15 and 2.18 are related by*

$$I(c_1, c_2, Q_T, P_S, \vec{v}) = \Gamma(n + |\vec{v}| + c_1(\ell+1) - c_2\ell) \times J(-n - |\vec{v}| + (c_2 - c_1)\ell, -n - |\vec{v}| + (c_2 - c_1)(\ell+1), Q_T, P_S, \vec{v}). \tag{2.11}$$

Note that this is a function of P_S and Q_T. For $S \in Span$, let $\vec{S}$ be the vector in $\mathbb{Z}^n$ with 1 for edges that are not in S and zero in the other coefficients. For a monomial $T = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$, let $\vec{T}$ be the vector $(\alpha_1, \cdots, \alpha_n)$. We have

$$\frac{\partial}{\partial Q_T} I(c_1, c_2, Q_T, P_S, \vec{v}) = c_1 I(c_1 - 1, c_2, Q_T, P_S, \vec{v} + \vec{T}) - I(c_1, c_2 + 1, Q_T, P_S, \vec{v} + \vec{T}) \tag{2.12}$$

$$\frac{\partial}{\partial P_S} I(c_1, c_2, Q_T, P_S, \vec{v}) = I(c_1 + 1, c_2 + 2, Q_T, P_S, \vec{v} + \vec{S}) - c_2 I(c_1, c_2 + 1, Q_T, P_S, \vec{v} + \vec{S}). \tag{2.13}$$

Let $A \subset \mathbb{Z}^{n+1}$ be the set containing the following points. For each monomial T in Q_Γ, consider the lattice point $(0, \vec{T})$ and, for any Spanning tree S, consider the lattice point $(1, \vec{S})$.

Denote the subset of A of lattice points corresponding to spanning trees by A_S and the subset of lattice points that correspond to monomials by $A_C = A \setminus A_S$. For $a = (1, \vec{S}) \in A_S$, P_a refers to P_S and for $a = (0, \vec{T}) \in A_C$, P_a refers to Q_T.

Since Q_Γ is of degree $\ell + 1$ and Ψ_Γ is of degree ℓ, all the lattice points lie on the affine hyperplane where the sum of the coordinates is $\ell + 1$. Let $\phi : \mathbb{Z}^{n+1} \to \mathbb{Z}$ be the function that computes the sum of the coordinates and let $p_0 : \mathbb{Z}^{n+1} \to \mathbb{Z}$ be the projection onto the first coordinate and $p_1 : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ the projection onto the last n coordinates. For any integer relation $\sum_{a\in A} n_a \vec{a} = 0$ among lattice points in the set A, we have

$$0 = \phi(0) = \phi\left(\sum_{a\in A} n_a \vec{a}\right) = \sum_{a\in A} n_a \phi(\vec{a}) = (\ell+1)\sum_{a\in A} n_a$$

$$0 = p_0(0) = p_0\left(\sum_{a\in A} n_a \vec{a}\right) = \sum_{a\in A} n_a p_0(\vec{a}) = \sum_{a\in A_S} n_a.$$

Combining these two we have

$$\begin{cases} \sum_{a\in A_S, n_a>0} n_a = \sum_{a\in A_S, n_a<0} -n_a \\ \sum_{a\in A_C, n_a>0} n_a = \sum_{a\in A_C, n_a<0} -n_a. \end{cases} \tag{2.14}$$

For the relation $(n_a)_{a\in A}$, we consider the following differential operator:

$$\prod_{a:n_a>0} \left(\frac{\partial}{\partial P_a}\right)^{n_a} - \prod_{a:n_a<0} \left(\frac{\partial}{\partial P_a}\right)^{-n_a}.$$

Proposition 2.20. *Let A and P_a be as above. For any $\mathbb{Z}$-linear relation $\sum_{a\in A} n_a \vec{a} = 0$ we have*

$$\left(\prod_{a:n_a>0} \left(\frac{\partial}{\partial P_a}\right)^{n_a} - \prod_{a:n_a<0} \left(\frac{\partial}{\partial P_a}\right)^{-n_a}\right) J(c, d, P_a, \vec{v}) = 0$$

$$\left(\prod_{a:n_a>0} \left(\frac{\partial}{\partial P_a}\right)^{n_a} - \prod_{a:n_a<0} \left(\frac{\partial}{\partial P_a}\right)^{-n_a}\right) I(c_1, c_2, P_a, \vec{v}) = 0.$$

Proof. Consider the set $Z = \{(c_1, c_2) + \mathbb{Z}^2\} \subset \mathbb{C}^2$. Each time we apply a derivation 2.12 or 2.13 to $I(c_1, c_2, Q_T, P_S, \vec{v})$, we get a weighted sum of two $I(c_1', c_2', Q_T, P_S, \vec{v} + p_1(a))$ where $(c_1', c_2') \in Z$. If we apply the positive part of the differential operator above, we get a weighted sum of $I(c_1', c_2', Q_T, P_S, \vec{v} + p_1(\sum_{a\in A, a>0} n_a a))$. If we apply the negative part of differential operator above, we get a weighted sum of $I(c_1', c_2', Q_T, P_S, \vec{v} + p_1(\sum_{a\in A, a<0} -n_a a))$. To show that the generalized amplitude goes to zero under this differential operator, it is enough to show that the weights are the same for positive and negative parts. Note that 2.14 implies that we apply each type (derivation with respect to Q_T or P_S) of derivation the same number of times on both sides. Therefore it is enough to show that $\delta_{(c_1,c_2)} \mapsto c_1\delta_{(c_1-1,c_2)} - \delta_{(c_1,c_2+1)}$ commutes with $\delta_{(c_1,c_2)} \mapsto \delta_{(c_1+1,c_2+2)} - c_2\delta_{(c_1,c_2+1)}$, which can be verified by direct inspection. A similar argument works for $J(c, d, P_a, \vec{v})$.

□

Proposition 2.21. *For each $i = 0, ..., n$ and for $a \in A$, let a_i be the i-th coefficient of a. Assume that the generalized amplitude I converges. We have*

$$\left(\sum_{a\in A} a_i P_a \frac{\partial}{\partial P_a}\right) I(c_1, c_2, P_a, \vec{v}) = \begin{cases} i \neq 0 & (-1 - \vec{v}_i) I(c_1, c_2, P_a, \vec{v}) \\ i = 0 & ((c_1 - c_2)(\ell + 1) + \sum_i \vec{v}_i + n) I(c_1, c_2, P_a, \vec{v}). \end{cases}$$

Proof. For $i \neq 0$ consider the action $(\alpha, P_a) \mapsto \alpha^{a_i} P_a$ of $\mathbb{G}_m$ on the P_a's. We want to see how the integral changes under this action. We have

$$\int_{\mathbb{R}_+^{|E|}} e^{-\frac{Q_\Gamma(\alpha^{T_i} Q_T, t_1, ..., t_n)}{\Psi_\Gamma(\alpha^{S_i} P_S, t_1, ..., t_n)}} \frac{Q_\Gamma^{c_1}(\alpha^{T_i} Q_T, t_1, ..., t_n)}{\Psi_\Gamma^{c_2}(\alpha^{S_i} P_S, t_1, ..., t_n)} t^{\vec{v}} dt_1 ... dt_n =$$
$$\int_{\mathbb{R}_+^{|E|}} e^{-\frac{Q_\Gamma(Q_T, t_1, ..., \alpha t_i, ..., t_n)}{\Psi_\Gamma(P_S, t_1, ..., \alpha t_i, ..., t_n)}} \frac{Q_\Gamma^{c_1}(Q_T, t_1, ..., \alpha t_i, ..., t_n)}{\Psi_\Gamma^{c_2}(P_S, t_1, ..., \alpha t_i, ..., t_n)} t^{\vec{v}} dt_1 ... dt_n =$$
$$\int_{\mathbb{R}_+^{|E|}} e^{-\frac{Q_\Gamma}{\Psi_\Gamma}} \frac{Q_\Gamma^{c_1}}{\Psi_\Gamma^{c_2}} \frac{1}{\alpha^{\vec{v}_i + 1}} t_1^{\vec{v}_1} ... (\alpha t_i)^{\vec{v}_i} ... t_n^{\vec{v}_n} dt_1 ... d(\alpha t_i) ... dt_n.$$

The last line is valid for multiplication by α real and positive which does not change the integration cycle hence it is also valid for all α. As a result we have

$$I(c_1, c_2, \alpha^{a_i} P_a, \vec{v}) = \alpha^{-1-\vec{v}_i} I(c_1, c_2, P_a, \vec{v}).$$

Taking the derivative with respect to α, evaluated at $\alpha = 1$, we have

$$\left(\sum_{a\in A} a_i P_a \frac{\partial}{\partial P_a}\right) I(c_1, c_2, P_a, \vec{v}) = \frac{\partial}{\partial \alpha} I(c_1, c_2, \alpha^{a_i} P_a, \vec{v})|_{\alpha=1}$$
$$= (-1 - \vec{v}_i) I(c_1, c_2, P_a, \vec{v}).$$

The other case we consider is when $i = 0$. Note that a_0 is nonzero iff a corresponds to a spanning tree. We scale all terms by $(\alpha, P_S) \mapsto \alpha^{a_0} P_S$ and we obtain

$$\int_{\mathbb{R}_+^{|E|}} e^{-\frac{Q_\Gamma(Q_T,t_1,\dots,t_n)}{\Psi_\Gamma(\alpha P_S,t_1,\dots,t_n)}} \frac{Q_\Gamma^{c_1}(Q_T,t_1,\dots,t_n)}{\Psi_\Gamma^{c_2}(\alpha P_S,t_1,\dots,t_n)} t^{\vec{v}} dt_1 \dots dt_n =$$
$$\int_{\mathbb{R}_+^{|E|}} e^{-\frac{Q_\Gamma(Q_T,t_1,\dots,t_n)}{\alpha\Psi_\Gamma(P_S,t_1,\dots,t_n)}} \frac{Q_\Gamma^{c_1}(Q_T,t_1,\dots,t_n)}{\alpha^{c_2}\Psi_\Gamma^{c_2}(P_S,t_1,\dots,t_n)} t^{\vec{v}}.dt_1 \dots dt_n = .$$

After setting $s_i = t_i/\alpha$ we obtain

$$\int_{\mathbb{R}_+^{|E|}} e^{-\frac{Q_\Gamma(Q_T,s_1,\dots,s_n)}{\Psi_\Gamma(P_S,s_1,\dots,s_n)}} \frac{\alpha^{c_1(\ell+1)-c_2\ell} Q_\Gamma^{c_1}(Q_T,s_1,\dots,s_n)}{\alpha^{c_2}\Psi_\Gamma^{c_2}(P_S,s_1,\dots,s_n)} \alpha^{\sum_i \vec{v}_i} s^{\vec{v}} \alpha^n ds_1 \dots ds_n =$$
$$\alpha^{(c_1-c_2)(\ell+1)+\sum_i \vec{v}_i+n} \int_{\mathbb{R}_+^{|E|}} e^{-\frac{Q_\Gamma(Q_T,s_1,\dots,s_n)}{\Psi_\Gamma(P_S,s_1,\dots,s_n)}} \frac{Q_\Gamma^{c_1}(Q_T,s_1,\dots,s_n)}{\Psi_\Gamma^{c_2}(P_S,s_1,\dots,s_n)} s^{\vec{v}} ds_1 \dots ds_n.$$

As a result we have

$$I(c_1, c_2, \alpha^{a_0} P_a, \vec{v}) = \alpha^{(c_1-c_2)(\ell+1)+\sum_i \vec{v}_i+n} I(c_1, c_2, P_a, \vec{v}).$$

Taking the derivative with respect to α, evaluated at $\alpha = 1$, we then have

$$\left(\sum_{a\in A} a_0 P_a \frac{\partial}{\partial P_a}\right) I(c_1, c_2, P_a, \vec{v}) = \frac{\partial}{\partial \alpha} I(c_1, c_2, \alpha^{a_0} P_a, \vec{v})|_{\alpha=1}$$
$$= ((c_1 - c_2)(\ell + 1) + \sum_i \vec{v}_i + n) I(c_1, c_2, P_a, \vec{v}).$$

□

Proposition 2.22. *For each $i = 0, \dots, n$ and for $a \in A$, let a_i be the i-th coefficient of a. Assume that c is positive, d is negative and that all coefficients of $\vec{v}$ are positive. We have*

$$\left(\sum_{a\in A} a_i P_a \frac{\partial}{\partial P_a}\right) J(c, d, P_a, \vec{v})$$
$$= \begin{cases} i \neq 0 & (-v_i - 1) J(c, d, P_a, \vec{v}) + (c(\ell+1) - d\ell + |v| + n) J(c, d, P_a, \vec{v} + e_i) \\ i = 0 & -d J(c, d, P_a, \vec{v}). \end{cases}$$

Proof. The case $i = 0$ can be dealt with easily by applying the scaling argument of the previous lemma. For other values of i, the scaling argument does not work, since scaling changes the integration cycle. Let $\theta = dt_1 \wedge dt_2 \dots \wedge dt_n$ and let v be the vector field $\sum t_i \frac{\partial}{\partial t_i}$. Then it is not hard to see that $\Omega = \iota_v \theta$. Since $v - \frac{\partial}{\partial t_i}$ is tangent to Δ_{n-1}, we have $\iota_v\theta|_{\Delta_{n-1}} = \iota_{\frac{\partial}{\partial t_i}}\theta|_{\Delta_{n-1}}$. We then obtain

$$\begin{aligned}
\frac{\partial}{\partial \alpha}|_{\alpha=1} J(c,d,\alpha P_a,\vec{v}) &= \frac{\partial}{\partial \alpha}|_{\alpha=1} \int_{\Delta_{n-1}} \frac{Q_\Gamma^c(\alpha P_a)}{\Psi_\Gamma^d(\alpha P_a)} t^{\vec{v}}\Omega \\
&= \frac{\partial}{\partial \alpha}|_{\alpha=1} \int_{\Delta_{n-1}} \alpha^* \left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\right) \alpha^{-v_i}\Omega \\
&= \frac{\partial}{\partial \alpha}|_{\alpha=1} \int_{\Delta_{n-1}} \alpha^* \left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\Omega\right) \alpha^{-v_i-1} \\
&= (-v_i - 1) J(c,d,P_a,\vec{v}) + \int_{\Delta_{n-1}} \frac{\partial}{\partial \alpha}|_{\alpha=1}\ \alpha^* \left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\Omega\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \alpha}|_{\alpha=1}\ \alpha^* \left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\Omega\right) &= \mathcal{L}_{t_i \frac{\partial}{\partial t_i}} \left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\Omega\right) \\
&= (\iota_{t_i \frac{\partial}{\partial t_i}} \circ d + d \circ \iota_{t_i \frac{\partial}{\partial t_i}}) \left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\Omega\right) \\
&= (\iota_{t_i \frac{\partial}{\partial t_i}} \circ d) \circ \iota_v \left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\theta\right) + d\left(\iota_{t_i \frac{\partial}{\partial t_i}} \frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\Omega\right) \\
&= \iota_{t_i \frac{\partial}{\partial t_i}} \mathcal{L}_v \left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\theta\right) + d\left(\iota_{t_i \frac{\partial}{\partial t_i}} \frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\Omega\right) \\
&= deg\left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\theta\right) \iota_{t_i \frac{\partial}{\partial t_i}} \left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\theta\right) + d\left(\iota_{t_i \frac{\partial}{\partial t_i}} \frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\Omega\right).
\end{aligned}$$

Let Σ be the union of the coordinate hyperplanes. We have

$$\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\Omega|_\Sigma = 0,$$

and hence

$$\iota_{t_i \frac{\partial}{\partial t_i}} (\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)} t^{\vec{v}}\Omega)|_\Sigma = 0.$$

Here we are using the fact that the vector field $t_i \frac{\partial}{\partial t_i}$ is tangent to Σ, so that we can first restrict to Σ and then perform the contraction. Since the boundary of Δ_{n-1} lies on Σ, the integral of the second term vanishes.

Using $\iota_v\theta|_{\Delta_{n-1}} = \iota_{\frac{\partial}{\partial t_i}}\theta|_{\Delta_{n-1}}$, we can compute the integral of the first term as

$$\begin{aligned}\int_{\Delta_{n-1}} \frac{\partial}{\partial\alpha}|_{\alpha=1}\ \alpha^*\left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)}t^{\vec{v}}\Omega\right) &= \int_{\Delta_{n-1}} deg\left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)}t^{\vec{v}}\theta\right) t_i\iota_{\frac{\partial}{\partial t_i}}\left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)}t^{\vec{v}}\theta\right)\\ &= deg\left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)}t^{\vec{v}}\theta\right)\int_{\Delta_{n-1}}\left(\frac{Q_\Gamma^c(P_a)}{\Psi_\Gamma^d(P_a)}t^{\vec{v}+e_i}\Omega\right)\\ &= (c(\ell+1)-d\ell+|v|+n)J(c,d,P_a,\vec{v}+e_i).\end{aligned}$$

Summing up, we obtain

$$\frac{\partial}{\partial\alpha}|_{\alpha=1}J(c,d,\alpha P_a,\vec{v}) = (-v_i-1)J(c,d,P_a,\vec{v})+(c(\ell+1)-d\ell+|v|+n)J(c,d,P_a,\vec{v}+e_i).$$

□

Lemma 2.23. *The generalized amplitude $J(c,d,P_a,\vec{v})$, which is holomorphic for $\Re(c) > 0$, $\Re(d) < 0$ and $\Re(v) > 0$, has an analytic continuation which is meromorphic on $\mathbb{C}^{n+2}$.*

Proof. This basically follows from resolution of singularity and the following fact. Let $P_i(x)$ be polynomials in n variables which are bounded away from zero on the hypercube $[0,1]^n$. One needs to show that the integral

$$\int_{[0,1]^n} P_1(t)^c P_2(t)^d t_1^{a_1}t_2^{a_2}\cdots t_n^{a_n}dt_1\cdots dt_n,$$

which is defined and holomorphic in $\{\Re(a_i) > 0\}$, has an analytic continuation to $\mathbb{C}^{n+2}$. We prove this by induction on n. The base case is the observation that C^cD^d has analytic continuation for C and D nonzero, which is clearly true. Note that we have

$$\begin{aligned}&\int_{[0,1]^n\cap t_1=\{0,1\}}(-1)^{t_1+1}\left(P_1(t)^cP_2(t)^dt_1^{a_1+1}t_2^{a_2}\cdots t_n^{a_n}\right)dt_2\cdots dt_n\\ &= \int_{[0,1]^n}\frac{\partial}{\partial t_1}\left(P_1(t)^cP_2(t)^dt_1^{a_1+1}t_2^{a_2}\cdots t_n^{a_n}\right)dt_1\cdots dt_n\\ &= \int_{[0,1]^n}\frac{\partial}{\partial t_1}\left(P_1(t)^cP_2(t)^d\right)t_1^{a_1+1}t_2^{a_2}\cdots t_n^{a_n}dt_1\cdots dt_n\\ &+ \int_{[0,1]^n}P_1(t)^cP_2(t)^d(a_1+1)t_1^{a_1}t_2^{a_2}\cdots t_n^{a_n}dt_1\cdots dt_n\\ &= \int_{[0,1]^n}\left(cP_1(t)^{c-1}\frac{\partial P_1}{\partial t_1}P_2(t)^d + dP_1(t)^c\frac{\partial P_2}{\partial t_1}P_2(t)^{d-1}\right)t_1^{a_1+1}t_2^{a_2}\cdots t_n^{a_n}dt_1\cdots dt_n\\ &+ \int_{[0,1]^n}P_1(t)^cP_2(t)^d(a_1+1)t_1^{a_1}t_2^{a_2}\cdots t_n^{a_n}dt_1\cdots dt_n.\end{aligned}$$

Assume we have the analytic continuation for the region $\Re(a_i) > m_i$. Note that we also have it for $m_i = 0$, since the P_i's are nonzero on the hypercube. From the computation above we have

$$\begin{aligned}
&\int_{[0,1]^n} P_1(t)^c P_2(t)^d t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n} dt_1 \cdots dt_n \\
&= \frac{1}{a_1+1} \int_{[0,1]^n \cap t_1 = \{0,1\}} (-1)^{t_1+1} \left(P_1(t)^c P_2(t)^d t_1^{a_1+1} t_2^{a_2} \cdots t_n^{a_n} \right) dt_2 \cdots dt_n \\
&+ \frac{1}{a_1+1} \int_{[0,1]^n} c P_1(t)^{c-1} P_2(t)^d \frac{\partial P_1}{\partial t_1} t_1^{a_1+1} t_2^{a_2} \cdots t_n^{a_n} dt_1 \cdots dt_n \\
&+ \frac{1}{a_1+1} \int_{[0,1]^n} d P_1(t)^c P_2(t)^{d-1} \frac{\partial P_2}{\partial t_1} t_1^{a_1+1} t_2^{a_2} \cdots t_n^{a_n} dt_1 \cdots dt_n.
\end{aligned}$$

By induction, the first term is meromorphic. The second and third terms have analytic continuation to the region $\Re(a_1) + 1 > m_1$ and $\Re(a_i) \geq m_i$. Thus, we have analytic continuation to the region $\Re(a_1) > m_1 - 1$. We can continue this for all m_i and prove it for any value of a_i.

Now, using a special case of resolution of singularities, we can rewrite any integral over Δ_{n-1} as a sum of integrals of the form above. This is the same argument that is used in [BW09] Theorem 2, so we do not repeat it explicitly here. □

By Remark 2.11, we can define $I(P_a)$ using the corresponding value of $J(P_a)$, i.e.

$$I(c_1, c_2, Q_T, P_S, \vec{v}) = CJ(-n - |\vec{v}| + (c_2 - c_1)\ell, -n - |\vec{v}| + (c_2 - c_1)(\ell + 1), Q_T, P_S, \vec{v}),$$

where C is constant. Therefore we see that the second term in the case $i \neq 0$ of proposition 2.22 vanishes.

$$\begin{aligned}
c &= -n - |\vec{v}| + (c_2 - c_1)\ell \\
d &= -n - |\vec{v}| + (c_2 - c_1)(\ell + 1) \\
c(\ell + 1) &- d\ell + |\vec{v}| + n = 0
\end{aligned}$$

Theorem 2.24. *Let $w_0 = (c_1, c_2, \vec{v})$ and $w = (x, y, \vec{u})$ be any vectors in $\mathbb{C}^{n+2}$. One can pull back $I(P_a, w_0 + \epsilon w)$ to a neighborhood of $\epsilon = 0$ and take the Laurent expansion. Assume that the Laurent expansion has the following form:*

$$I(P_a, w_0 + \epsilon w) = \sum_{i \geq -n} \epsilon^i I_i(P_a).$$

Then we have

$$\left(\prod_{a:n_a>0}(\frac{\partial}{\partial P_a})^{n_a}-\prod_{a:n_a<0}(\frac{\partial}{\partial P_a})^{-n_a}\right)I_i(P_a)=0$$

and

$$\left(\sum_{a\in A}a_0P_a\frac{\partial}{\partial P_a}\right)I_i(P_a)=((\ell+1,-\ell-1,1,\cdots,1)\cdot w_0+n)I_i(P_a)$$
$$+(\ell+1,-\ell-1,1,\cdots,1)\cdot w\ I_{i-1}(P_a)$$

and for $k\neq 0$

$$\left(\sum_{a\in A}a_kP_a\frac{\partial}{\partial P_a}\right)I_i(P_a)=(-1-e_k\cdot\vec{v})I_i(P_a)-e_k\cdot\vec{u}\ I_{i-1}(P_a).$$

Proof. Our definition of the integral is by analytic continuation of $J(c,d,P_a,\vec{v})$. Note that for any differential operator L, $L(J)$ is meromorphic. Since the differential equation is satisfied for an open subset of $\mathbb{C}^{n+2}$ ($\Re(c)>0$,$\Re(d)<0$ and $\Re(\vec{v})>0$), it is valid for all values of c,d and $\vec{v}$. Note that $I(P_a,w_0+\epsilon w)$ for all values of ϵ satisfies the first equation since J has the same property. By the calculation above we see that the second term in the case $k\neq 0$ vanishes and we have

$$\left(\sum_{a\in A}a_kP_a\frac{\partial}{\partial P_a}\right)I(P_a,w_0+\epsilon w)=(-1-e_k.(\vec{v}+\epsilon\vec{u}))\ I(P_a,w_0+\epsilon w)$$
$$=(-1-e_k.\vec{v})I(P_a,w_0+\epsilon w)-\epsilon\ e_k.\vec{u}\ I(P_a,w_0+\epsilon w).$$

In the case $k=0$ we have

$$\left(\sum_{a\in A}a_0P_a\frac{\partial}{\partial P_a}\right)I(P_a,w_0+\epsilon w)$$
$$=(n+|\vec{v}+\epsilon\vec{u}|-(c_2+\epsilon y-c_1-\epsilon x)(\ell+1))\ I(P_a,w_0+\epsilon w)$$
$$=((\ell+1,-\ell-1,1,\cdots,1).w_0+n)I(P_a,w_0+\epsilon w)$$
$$+\epsilon\ (\ell+1,-\ell-1,1,\cdots,1).w\ I(P_a,w_0+\epsilon w).$$

The theorem follows from expanding $I(P_a,w_0+\epsilon w)$ and comparing terms with different powers of ϵ.

□

Remark 2.25. *In standard dimensional regularization for the amplitude, assuming that we take the expansion with respect to* ϵ *in* $D/2+\epsilon$*, i.e.*

$$\sum_{i\geq -n} \epsilon^i I_i(P_a) = \int_{\mathbb{R}_+^{|E|}} e^{-Q_\Gamma/\Psi_\Gamma} \frac{1}{\Psi_\Gamma^{D/2+\epsilon}},$$

we have $w_0 = (0, D/2, \vec{0})$ *and* $w = (0, 1, \vec{0})$*, and thus the amplitude satisfies the differential equations*

$$\left(\prod_{a:n_a>0} (\frac{\partial}{\partial P_a})^{n_a} - \prod_{a:n_a<0} (\frac{\partial}{\partial P_a})^{-n_a}\right) I_i(P_a) = 0$$

$$\left(\sum_{a\in A} a_0 P_a \frac{\partial}{\partial P_a}\right) I_i(P_a) = (n - (\ell+1)D/2) I_i(P_a) - (\ell+1) I_{i-1}(P_a)$$

and for $k \neq 0$

$$\left(\sum_{a\in A} a_k P_a \frac{\partial}{\partial P_a}\right) I_i(P_a) = -I_i(P_a)\,.$$

In particular the lowest coefficient satisfies the so called GKZ hypergeometric differential equation, which we consider in the next chapter.

Chapter 3

GKZ A-HYPERGEOMETRIC DIFFERENTIAL EQUATIONS

3.1 Geometric Origin of GKZ D-modules

Definition 3.1. *Given $\mathbb{Z}^n$, the n-dimensional lattice, we fix a basis and denote an element as n-tuple of integers. We define ϕ_i as the map from $\mathbb{Z}^n$ to $\mathbb{Z}$, which gives us the i-th coordinate in the fixed basis.*

Let $A = \{a_1, \cdots, a_N\} \subset \mathbb{Z}^n$ be a set of lattice points such that they all lie in the hyperplane $\phi_1 = 1$ and generate the lattice as a $\mathbb{Z}$ module. For a tuple of integers $r = (n_a : a \in A)$ consider the relation among the points of A of the form

$$\sum_{a\in A} n_a a = 0.$$

Denote the set of relations by R. For each $r \in R$, we consider a corresponding differential operator

$$\Box_r := \prod_{\substack{a\in A\\ n_a>0}} \left(\frac{\partial}{\partial p_a}\right)^{n_a} - \prod_{\substack{a\in A\\ n_a<0}} \left(\frac{\partial}{\partial p_a}\right)^{-n_a} \tag{3.1}$$

and for $i = 1, \cdots, n$ we define

$$Z_i := \sum_{a\in A} \phi_i(a) p_a \frac{\partial}{\partial p_a}\,. \tag{3.2}$$

On $V = \mathbb{C}^N$, with coordinates $p_1, ..., p_N$, consider the differential equations

$$\Box_r \phi = 0\,.$$

For $\beta = (\beta_1, ..., \beta_n) \in k^n \subset \mathbb{C}^n$, consider the differential equations

$$(Z_i - \beta_i)\phi = 0\,.$$

We want to find solutions to these differential equations. We denote by W the Weyl algebra

$$W = k[p_a, \frac{\partial}{\partial p_a} : a \in A]/([\frac{\partial}{\partial p_a}, p_b] = \delta^a_b, [p_a, p_b] = 0, [\frac{\partial}{\partial p_b}, \frac{\partial}{\partial p_a}] = 0),$$

where k is a sub-field of $\mathbb{C}$. Then the GKZ left W-module is defined by

$$H_A(\beta) = W/\sum_i W(Z_i - \beta_i) + \sum_r W \,\Box_r \,.$$

Remark 3.2. *One can consider $H_A(\beta)$ as a D_V-module, i.e. as a sheaf of modules over the sheaf of differential operators on V. W is the ring of global sections of D_V and $H_A(\beta)$ is the space of global section of the corresponding sheaf.*

This set of differential equations and the corresponding W-module was considered by Gelfand, Kapranov and Zelevinsky in [GKZ90], [GZK89]. For a complete discussion of results in this direction see [GKZ08], [SST00] and the references there. We prove the relevant results and state the theorems that we need. Their main results can be summarized in the following theorem from [Cat06].

Theorem 3.3. *(GKZ) Let $H_A(\beta)$ be a GKZ hypergeometric system.*

(1) *$H_A(\beta)$ is always holonomic.*

(2) *The singular locus of $H_A(\beta)$ is independent of $\beta \in \mathbb{C}^n$ and agrees with the zero locus of the principal A-determinant $E_A(x)$ defined in chapter* 10 *of [GKZ08].*

(3) *For arbitrary A and generic β, the holonomic rank of $H_A(\beta)$ equals the normalized volume of the convex hull of A, $vol(conv(A))$.*

(4) *For arbitrary A and β, $rank(H_A(\beta)) \geq vol(conv(A))$.*

(5) *Given A, $rank(H_A(\beta)) = vol(conv(A))$ for all $\beta \in \mathbb{C}^n$ if and only if the toric ideal I_A is Cohen-Macaulay.*

Proposition 3.4. *Let $t^a = t_1^{\phi_1(a)} t_2^{\phi_2(a)} \cdots t_n^{\phi_n(a)}$. Consider the ring*

$$\mathfrak{R} = k[p_a, t^a : a \in A]$$

with an action of W given by

$$(p_a, P) \mapsto p_a P$$
$$(\frac{\partial}{\partial p_a}, P) \mapsto \frac{\partial}{\partial p_a} P + t^a P.$$

Moreover, consider the map

$$\Psi : W \longrightarrow \Re$$
$$\Psi(p^I \prod_{a\in A} \left(\frac{\partial}{\partial p_a}\right)^{m_a}) = p^I t^{\sum m_a a}.$$

Let

$$f = \sum_{a\in A} P_a t^a$$
$$Y_i = t_i \frac{\partial f}{\partial t_i} + t_i \frac{\partial}{\partial t_i} - \beta_i$$

Then we have:

(1) Ψ *is surjective*

(2) $ker(\Psi) = W \sum_a \Box_a$*, with* $\Box_a$ *as in (3.1)*

(3) $image(W(Z_i - \beta_i)) = Y_i \Re$ *, with* Z_i *as in (3.2)*

(4) Ψ *gives an isomorphism between* $H_A(\beta)$ *and* $\Re / \sum_i Y_i \Re$*.*

Proof. The first two statements follow from the definition. Note that Ψ is $k[p_a : a \in A]$ linear and Y_i acts $k[p_a : a \in A]$ linearly. Thus, to check (3) it is enough to compute the image of

$$\prod_{a\in A} \left(\frac{\partial}{\partial p_a}\right)^{m_a} (Z_i - \beta_i).$$

With ϕ_i as Definition 3.1, we have

$$\begin{aligned}
\Psi(\prod_{a\in A} \left(\frac{\partial}{\partial p_a}\right)^{m_a} (Z_i - \beta_i)) &= \Psi(\prod_{a\in A} \left(\frac{\partial}{\partial p_a}\right)^{m_a} \sum_{b\in A} \phi_i(b) p_b \frac{\partial}{\partial p_b} - \beta_i \prod_{a\in A} \left(\frac{\partial}{\partial p_a}\right)^{m_a}) \\
&= \sum_{b\in A} \phi_i(b) p_b \Psi(\frac{\partial}{\partial p_b} \prod_{a\in A} \left(\frac{\partial}{\partial p_a}\right)^{m_a}) \\
&+ \sum_{b\in A} \phi_i(b) m_b \Psi(\prod_{a\in A} \left(\frac{\partial}{\partial p_a}\right)^{m_a}) - \beta_i \Psi(\prod_{a\in A} \left(\frac{\partial}{\partial p_a}\right)^{m_a}) \\
&= \sum_{b\in A} \phi_i(b) p_b t^b t^{\sum m_a a} + \sum_{b\in A} \phi_i(b) m_b t^{\sum m_a a} - \beta_i t^{\sum m_a a} \\
&= (\sum_{a\in A} \phi_i(a) p_a t^a + \phi_i(\sum_{a\in A} a m_a) - \beta_i) t^{\sum m_a a} \\
&= Y_i t^{\sum m_a a}.
\end{aligned}$$

To check (4) we need to show that, for $P \in GKZ$, we have

$$\Psi(\frac{\partial}{\partial p_b}P) = \frac{\partial}{\partial p_b}\Psi(P) + t^b\Psi(P)\,.$$

If P has the form $p^I \prod_{a\in A}\left(\frac{\partial}{\partial p_a}\right)^{m_a}$, we have

$$\begin{aligned}\Psi(\frac{\partial}{\partial p_b}P) &= \Psi(\left(\frac{\partial}{\partial p_b}p^I\right)\prod_{a\in A}\left(\frac{\partial}{\partial p_a}\right)^{m_a} + p^I\prod_{a\in A}\left(\frac{\partial}{\partial p_a}\right)^{m_a}\frac{\partial}{\partial p_b}) \\ &= \frac{\partial}{\partial p_b}\Psi(P) + t^b\Psi(P)\,.\end{aligned}$$

□

This construction makes GKZ a quotient of $\mathfrak{R}$. Note that it is not a $\mathfrak{R}$-module since the action of Y_i does not commute with multiplication by t^a. However, it is a $k[p_a : a \in A]$ module. We want to understand the structure as a $k[p_a : a \in A]$-module.

Proposition 3.5. *Assume $-\beta_1 + n$ is nonzero in k for all $n \in \mathbb{Z}_{\geq 0}$. We have an isomorphism of $k[p_a : a \in A]$ modules*

$$\mathfrak{R}/Y_1\mathfrak{R} \cong k[p_a : a \in A][t^a/f] \cong k[p_a : a \in A][t^a]/\left(\sum_{a\in A} p_a t^a = 1\right) =: \hat{\mathfrak{R}}\,. \quad (3.3)$$

Let ϕ_1 be the first coordinate as Definition 3.1. The isomorphism is given by

$$\mathfrak{F} : \mathfrak{R} \to \mathfrak{R}$$

$$\mathfrak{F}(t^J) = (-1)^{\phi_1(J)}\gamma(\phi_1(j))\,t^J,$$

where $\gamma(n) = -\beta_1(-\beta_1 + 1)\cdots(-\beta_1 + n - 1)$ and $\gamma(0) = \gamma(1) = -\beta_1$.

Proof. Note that we have $\gamma(n+1)/\gamma(n) = -\beta_1 + n$ and since all points of A have

$\phi_1 = 1$, we have $t_1 \frac{\partial f}{\partial t_1} = f$

$$\begin{aligned}
\mathfrak{F}(Y_1 t^J) &= \mathfrak{F}(t_1 \frac{\partial f}{\partial t_1} t^J + t_1 \frac{\partial}{\partial t_1} t^J - \beta_1 t^J) \\
&= \mathfrak{F}(\sum_{a \in A} p_a t^{J+a} + \phi_1(J) t^J - \beta_1 t^J) \\
&= \sum_{a \in A} p_a t^a (-1)^{\phi_1(J)+1} \gamma(\phi_1(J) + 1)\, t^J \\
&+ \phi_1(J)(-1)^{\phi_1(J)} \gamma(\phi_1(J))\, t^J - \beta_1 (-1)^{\phi_1(J)} \gamma(\phi_1(J))\, t^J \\
&= \left(-f \frac{\gamma(\phi_1(J)+1)}{\gamma(\phi_1(J))} + \phi_1(J) - \beta_1\right) (-1)^{\phi_1(J)} \gamma(\phi_1(J)) t^J \\
&= (-\beta_1 + \phi_1(J))(1 - f)(-1)^{\phi_1(J)} \gamma(\phi_1(J)) t^J \\
&= (1 - f)(-\beta_1 + \phi_1(J)) \mathfrak{F}(t^J).
\end{aligned}$$

By definition $\mathfrak{F}$ is surjective. The equations above shows that the image of $Y_1 \mathfrak{R}$ is the ideal generated by $1 - f$ and we have the isomorphism. □

Proposition 3.6. *Assume $-\beta_1 + n$ is nonzero in k, for all $n \in \mathbb{Z}_{\geq 0}$. Define $\tilde{Y}_i$ by*

$$\tilde{Y}_i : \hat{\mathfrak{R}} \to \hat{\mathfrak{R}}$$

$$\tilde{Y}_i = \left(t_i \frac{\partial}{\partial t_i} - \beta_i - t_i \frac{\partial f}{\partial t_i}(-\beta_1 + t_1 \frac{\partial}{\partial t_1})\right) .$$

We have

$$H_A(\beta) = \hat{\mathfrak{R}} / \sum_{i=2}^{n} \tilde{Y}_i \hat{\mathfrak{R}} .$$

Proof. We need to find the image of $Y_i \mathfrak{R}$ under $\mathfrak{F}$. We have $\mathfrak{F}(Y_i t^J) =$

$$\begin{aligned}
&\mathfrak{F}(t_i \frac{\partial f}{\partial t_i} t^J + t_i \frac{\partial}{\partial t_i} t^J - \beta_i t^J) \\
&= (-1)^{\phi_1(J)+1} \gamma(\phi_1(J) + 1) t_i \frac{\partial f}{\partial t_i} t^J + (-1)^{\phi_1(J)} \gamma(\phi_1(J)) t_i \frac{\partial}{\partial t_i} t^J - \beta_i (-1)^{\phi_1(J)} \gamma(\phi_1(J)) t^J \\
&= \left(-(-\beta_1 + \phi_1(J)) t_i \frac{\partial f}{\partial t_i} + t_i \frac{\partial}{\partial t_i} - \beta_i\right) (-1)^{\phi_1(J)} \gamma(\phi_1(J)) t^J \\
&= \left(-t_i \frac{\partial f}{\partial t_i}(-\beta_1 + t_1 \frac{\partial}{\partial t_1}) + t_i \frac{\partial}{\partial t_i} - \beta_i\right) (-1)^{\phi_1(J)} \gamma(\phi_1(J)) t^J .
\end{aligned}$$

Thus, we have that the image is

$$\left(t_i \frac{\partial}{\partial t_i} - \beta_i - t_i \frac{\partial f}{\partial t_i}(-\beta_1 + t_1 \frac{\partial}{\partial t_1})\right) \mathfrak{R}.$$

□

To show that the equations come from geometry we observe that the ring $\mathfrak{R}$ is the coordinate ring of an affine toric variety. Let $\mathbb{N}A$ be the semigroup generated by the set A as a sub-semigroup of $\mathbb{Z}^n$. By definition, $\mathfrak{R}$ is the semigroup algebra $k[p_a : a \in A][\mathbb{N}A]$. It is well known that Abelian semigroup algebras are local models for toric varieties.

Definition 3.7. *We denote the semigroup above by Σ. It is an Abelian semigroup, which is generated by the set A and 0. We denote the semigroup algebra by $S_\Sigma = k[t^a : a \in A]$ and the corresponding toric variety by $X_\Sigma = \mathbf{spec}[S_\Sigma]$. ϕ_1 induces a grading on Σ. We denote the projective toric variety $\mathbf{Proj}(S_\Sigma, \phi_1)$ by $\mathbb{P}_\Sigma$ and the corresponding line bundle by $\mathcal{O}(1)$.*

Remark 3.8. *Toric varieties defined by semigroup algebras are not necessarily normal, while all toric varieties defined by a rational polyhedral fan are normal. It turns out that a spectrum of an Abelian semigroup ring is normal iff the semigroup is saturated, i.e. iff all $\mathbb{Z}^n$ points in the real cone generated by the semigroup are in the semigroup. (see [Hoc72])*

Note that the k-vector space of global sections of $\mathcal{O}(1)$ is canonically isomorphic to V, since degree one elements in the semigroup are a basis for V. We consider f as the universal section of $\mathcal{O}(1)$. Let $Y = \mathbb{V}(f)$ be the codimension one subvariety of the zeros of f in $V \times \mathbb{P}_\Sigma$ and let U be the complement.

Lemma 3.9. *We have isomorphisms*

$$\begin{aligned} \boldsymbol{spec}(\mathfrak{R}) &= V \times X_\Sigma \\ \boldsymbol{spec}(\hat{\mathfrak{R}}) &= U = V \times \mathbb{P}_\Sigma \setminus Y \end{aligned}$$

Proof. The first part follows from the definition of X_Σ as the spectrum of $k[t^a : a \in A]$. For the second part, note that, since $V \times \mathbb{P}_\Sigma \setminus Y$ is an affine chart in $V \times \mathbb{P}_\Sigma$, its coordinate ring can be described by degree zero elements in $\mathfrak{R}[1/f]$. This agrees with the definition of the $\hat{\mathfrak{R}}$ as in (3.3). □

We state some well known facts from the theory of toric varieties.

Definition 3.10. *Let P_A (respectively, $\bar{P}_A$) be the convex hull of the points in A (respectively, $A \cup \{0\}$), as a subset of $\mathbb{R}^n$. P_A has the structure of $n-1$ dimensional polytope and $\bar{P}_A$ has the structure of n dimensional polytope.*

By a k-dimensional face of a m-dimensional polytope P we mean points in P that lie on a hyperplane, with k dimensional span (as an affine subspace) such that all points of P are on the same side of the hyperplane. There is exactly one m-dimensional face. Codimension one faces are called *facets*.

Lemma 3.11. *Faces of the P_A (respectively, $\bar{P}_A$) are in one to one correspondence with torus orbits in $\mathbb{P}_\Sigma$ (respectively, X_Σ).*

P_A is a subset of $\mathbb{R}^n$, where the the first coordinate is 1. Thus, we can find hyperplanes defining faces that pass through the origin. Such a hyperplane can be defined as the set of $x \in \mathbb{R}^n$ with $\langle w, x\rangle = 0$, for a vector w in $\check{\mathbb{R}^n}$. In the case where points are all integer points, we can choose w to have integer coefficients as well. The following definition gives an algebraic description of the torus orbits.

Definition 3.12. *Let $w \in \mathbb{Z}^n$ be a lattice point with the property $\langle w, a\rangle \geq 0$ for all $a \in \Sigma$ and denote by Σ_w^c the set of elements of Σ that have nonzero inner product with w, i.e. $\langle w, a\rangle > 0$, for $a \in \Sigma_w^c$. Moreover, denote by Σ_w the set of elements with zero inner product , i.e. $\langle w, a\rangle = 0$ for $a \in \Sigma_w$. Note that the sub k-vector space of S_Σ generated by Σ_w^c is a graded ideal in S_Σ. We denote this ideal by I_w. The quotient ring, which we denote by S_w, is isomorphic to $k[\Sigma_w]$. We denote by $\mathbb{P}_w = \mathbf{Proj}(S_w, \Phi_1)$ the corresponding projective toric variety.*

The projective toric varieties defined above are not necessarily smooth. Assuming that the toric variety X is smooth, we have the following exact sequence of coherent sheaves:

$$0 \to \Omega_X^1 \to \Omega^1(logD) \to \bigoplus_{\{Facet\}} O_{\mathbb{P}_w} \to 0,$$

where D is the union of codimension one orbits and the second map is the residue map.

For a non-smooth toric variety $\Omega^1(logD)$ is well defined. See [Ish87]. Note that the complement of D is a $n-1$ torus. By definition its coordinate ring is the ring of degree zero elements in the graded ring $k[\Sigma - \Sigma] \simeq k[\mathbb{Z}^n]$. Here by $\Sigma - \Sigma$,we mean differences of the elements of Σ. It is equal to $\mathbb{Z}^n$ since Σ generates $\mathbb{Z}^n$ as a $\mathbb{Z}$-module. Thus, it is isomorphic to $k[\mathbb{Z}^n/\mathbb{Z}]$. We denote $\mathbb{Z}^n/\mathbb{Z}$ by $\tilde{M}$. An element $\tilde{m} \in \tilde{M}$ gives us a rational function on $\mathbb{P}_\Sigma$, then $dlog(\tilde{m})$ is an element of $\Omega_X^1(logD)$, which we denote by same notation $\tilde{m}$. It turns out that $\Omega^1(logD)$ is always free and we have an isomorphism of coherent sheaves

$$\Omega^1(logD) \simeq \tilde{M} \otimes_{\mathbb{Z}} O_{\mathbb{P}_\Sigma}.$$

We define the free sheaf of algebraic differential forms with logarithmic poles along D to be

$$\bigoplus_{i=0}^{n-1} \Omega^i(logD) \simeq \bigoplus_{i=0}^{n-1} \bigwedge^i \tilde{M} \otimes_{\mathbb{Z}} O_{\mathbb{P}_\Sigma}.$$

We have a diagram of varieties

$$\begin{array}{ccccc} V \times \mathbb{T} \setminus Y_0 & \overset{j}{\hookrightarrow} & U & \hookrightarrow & V \times \mathbb{P}_\Sigma \\ \downarrow p & & \downarrow p & & \downarrow p \\ V & \xrightarrow{id} & V & \xrightarrow{id} & V, \end{array} \tag{3.4}$$

where $Y_0 = Y \cap V \times \mathbb{T}$ and $\mathbb{T} = \mathbb{P}_\Sigma \setminus D$. We consider the sheaf of relative differential forms with logarithmic poles along D on $V \times \mathbb{P}_\Sigma$. Since p is the projection on the first factor, we have

$$\Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD) = \bigwedge^\bullet \tilde{M} \otimes_{\mathbb{Z}} O_{V\times\mathbb{P}_\Sigma}.$$

Sections of this sheaf on U are

$$\Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)(U) = \bigwedge^\bullet \tilde{M} \otimes_{\mathbb{Z}} \hat{\mathfrak{R}}.$$

This graded sheaf comes with a differential

$$d : \Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD) \to \Omega^{\bullet+1}_{V\times\mathbb{P}_\Sigma/V}(logD),$$

which makes it a complex of sheaves. For each element $a \in \Sigma \subset M$, there is a corresponding $\tilde{a} \in \tilde{M}$. Note that the restriction of d to U acts as

$$d(m \otimes (t^I/f^{\phi_1(I)})) = (\tilde{I} \wedge m) \otimes (t^I/f^{\phi_1(I)}) - \phi_1(I) \sum_{a\in A} \left((\tilde{a} \wedge m) \otimes (p_a t^{I+a}/f^{\phi_1(I)+1}) \right).$$

Note that this is well defined since we have $d(m \otimes (ft^I/f^{\phi_1(I)+1})) =$

$$
\begin{aligned}
& d(m \otimes (\sum_{a\in A} p_a t^{a+I}/f^{\phi_1(I)+1})) \\
&= \sum_{a\in A}((\tilde{a}+\tilde{I})\wedge m)\otimes(p_a t^{a+I}/f^{\phi_1(I)+1}) \\
&- \sum_{a\in A} p_a(1+\phi_1(I))\sum_{b\in A}\left((\tilde{b}\wedge m)\otimes(p_b t^{I+a+b}/f^{\phi_1(I)+2})\right) \\
&= (\tilde{I}\wedge m)\otimes(\sum_{a\in A} p_a t^a t^I/f^{\phi_1(I)+1}) + \sum_{a\in A}(\tilde{a}\wedge m)\otimes(p_a t^{a+I}/f^{\phi_1(I)+1}) \\
&- (1+\phi_1(I))\sum_{b\in A}\left((\tilde{b}\wedge m)\otimes(\sum_{a\in A} p_a t^a p_b t^{I+b}/f^{\phi_1(I)+2})\right) \\
&= (\tilde{I}\wedge m)\otimes(t^I/f^{\phi_1(I)}) - \phi_1(I)\sum_{b\in A}\left((\tilde{b}\wedge m)\otimes(p_b t^{I+b}/f^{\phi_1(I)+1})\right) \\
&= d(m\otimes(t^I/f^{\phi_1(I)}))\,.
\end{aligned}
$$

Moreover, it is not hard to check that d^2 is zero. Note that this differential is the standard definition of d coming from derivation below on rational functions

$$\left(t_i\frac{\partial}{\partial t_i}\right)\frac{t^J}{f^{\phi_1(J)}} = \frac{t_i\frac{\partial}{\partial t_i}t^J}{f^{\phi_1(J)}} - \frac{\phi_1(J)t_i\frac{\partial f}{\partial t_i}t^J}{f^{\phi_1(J)+1}}\,.$$

We can change the differential to

$$\nabla_\beta = d + \beta_1 df/f\wedge - \sum_{i=2}^n \beta_i\frac{dt_i}{t_i}\wedge = d + \beta_1\left(\sum_{a\in A}\tilde{a}\otimes p_a t^a/f\right)\wedge - \sum_{i=2}^n\beta_i\frac{dt_i}{t_i}\wedge,$$

and we still have a complex, since we have

$$\nabla_\beta = \frac{f^{\beta_1}}{t_2^{\beta_2}\cdots t_n^{\beta_n}}\circ d\circ\frac{t_2^{\beta_2}\cdots t_n^{\beta_n}}{f^{\beta_1}}\,. \tag{3.5}$$

Proposition 3.13. *Consider the complex $(\Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)(U),\nabla_\beta)$, whose terms are free $\hat{\mathfrak{R}}$-modules and with $\mathcal{O}_V$ linear differential ∇_β. The $n-1$ hyper-cohomology of this complex is isomorphic to $H_A(\beta)$, i.e.*

$$R^{n-1}p_*(\Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U,\nabla_\beta) = H_A(\beta)\,.$$

By $R^{n-1}p_*$ we mean the $(n-1)$-th derived functor of p_*. Everything is in Zariski topology.

Proof. Since p is affine, hyper-cohomology agrees with cohomology of the complex. Choose the standard basis $e_2, e_3, \cdots, e_n$ for $\tilde{M}$, which gives us coordinates $t_2, \cdots, t_n$ on $\mathbb{T}$. We have

$$\bigwedge^{n-2} \tilde{M} \otimes_{\mathbb{Z}} \hat{\mathfrak{R}} = \oplus_{i=2}^{n} \frac{dt_2}{t_2} \cdots \frac{\hat{dt_i}}{t_i} \cdots \frac{dt_n}{t_n} \otimes \hat{\mathfrak{R}},$$

and with ϕ_i as Definition 3.1, we have

$$\begin{aligned}
&\left(d + \beta_1 df/f \wedge - \sum_{i=2}^{n} \beta_i \frac{dt_i}{t_i} \wedge\right) \frac{dt_2}{t_2} \cdots \frac{\hat{dt_i}}{t_i} \cdots \frac{dt_n}{t_n} \otimes t^I / f^{\phi_1(I)} \\
&= (\tilde{I} \wedge \frac{dt_2}{t_2} \cdots \frac{\hat{dt_i}}{t_i} \cdots \frac{dt_n}{t_n}) \otimes (t^I / f^{\phi_1(I)}) \\
&- \phi_1(I) \sum_{a \in A} \left((\tilde{a} \wedge \frac{dt_2}{t_2} \cdots \frac{\hat{dt_i}}{t_i} \cdots \frac{dt_n}{t_n}) \otimes (p_a t^{I+a} / f^{\phi_1(I)+1}) \right) \\
&+ \beta_1 \left(\sum_{a \in A} \tilde{a} \otimes p_a t^a / f \right) \wedge \frac{dt_2}{t_2} \cdots \frac{\hat{dt_i}}{t_i} \cdots \frac{dt_n}{t_n} \otimes t^I / f^{\phi_1(I)} \\
&- \frac{dt_2}{t_2} \cdots \frac{dt_n}{t_n} \beta_i \otimes t^I / f^{\phi_1(I)} \\
&= \frac{dt_2}{t_2} \cdots \frac{dt_n}{t_n} \otimes \left(-\beta_i + \phi_i(I) - (\phi_1(I) - \beta_1) \sum_{a \in A} \phi_i(\tilde{a}) p_a t^a / f \right) t^I / f^{\phi_1(I)} \\
&= \frac{dt_2}{t_2} \cdots \frac{dt_n}{t_n} \otimes \left(t_i \frac{\partial}{\partial t_i} - \beta_i - t_i \frac{\partial f}{\partial t_i} (-\beta_1 + t_1 \frac{\partial}{\partial t_1}) \right) t^I / f^{\phi_1(I)} .
\end{aligned}$$

Identifying top forms with $\hat{\mathfrak{R}}$, we see that the top cohomology is exactly $H_A(\beta)$ by Proposition 3.6. ◻

Definition 3.14. *Given an element w of the dual of $\tilde{M}$ we have the contraction map*

$$\iota_w : \bigwedge^{\bullet} \tilde{M} \to \bigwedge^{\bullet - 1} \tilde{M}$$

$$\iota_w(a_1 \wedge \cdots \wedge a_k) = \sum_{i=1}^{k} (-1)^i \langle w, a_i \rangle a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_k .$$

We extend it $\hat{\mathfrak{R}}$-linearly to $\hat{\mathfrak{R}} \otimes \bigwedge^{\bullet} \tilde{M}$.

Definition 3.15. *Let $w_1, \ldots, w_m$ be elements of $\mathbb{Z}^n$ defining the facets of P_A. An element $\beta \in \mathbb{C}^n$ is called non-resonant (respectively, semi non-resonant) if $\langle w_i, -\beta + \mathbb{Z}^n \rangle \neq 0$ (respectively, if $\langle w_i, -\beta + \Sigma \rangle \neq 0$), for all i.*

Proposition 3.16. *Let w correspond to a face of P_A, the polytope defined in Definition 3.10, and let I_w, $\mathbb{P}_w$, and S_w be as in Definition 3.12. Let $i_w : U_w \to U$ be the fiber product*

$$\begin{array}{ccc} U_w & \xrightarrow{i_w} & U \\ \downarrow p & & \downarrow \\ V \times \mathbb{P}_w & \longrightarrow & V \times \mathbb{P}_\Sigma, \end{array}$$

which is the inclusion into U of the intersection of the boundary components corresponding to w and U. This inclusion is given by the ideal sheaf I_w. Furthermore, assume β is semi non-resonant. Then the inclusion

$$(I_w^n \Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U, \nabla_\beta) \xrightarrow{q.i.s} (\Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U, \nabla_\beta)$$

is a quasi-isomorphism.

Proof. First note that U_w is affine. By induction it is enough to show that

$$(I_w^n \Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U, \nabla_\beta) \hookrightarrow (I_w^{n-1} \Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U, \nabla_\beta)$$

is a quasi-isomorphism, or equivalently that the cokernel

$$\left(\frac{I_w^{n-1} \Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U}{I_w^n \Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U}, \nabla_\beta\right)$$

is quasi-isomorphic to zero.

Let $t^J \otimes \alpha$ be a section on $I_w^{n-1} \Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U$. We show that a multiple of $\iota_{\tilde{w}} t^J \otimes \alpha$ is ∇_β-primitive of $t^J \otimes \alpha$, where $\tilde{w} = (w_2, \cdots, w_n)$. We have $\nabla_\beta \iota_{\tilde{w}} t^J \otimes \alpha =$

$$t^J \otimes J \wedge \iota_{\tilde{w}}\alpha + (-\phi_1(J) + \beta_1) \sum_{a\in A} p_a t^{a+J} \otimes a \wedge \iota_{\tilde{w}}\alpha - \sum \beta_i t^J \otimes e_i \wedge \iota_{\tilde{w}}\alpha =$$

$$- t^J \otimes \iota_{\tilde{w}} J \wedge \alpha - (-\phi_1(J) + \beta_1) \sum_{a\in A} p_a t^{a+J} \otimes \iota_{\tilde{w}} a \wedge \alpha + \sum \beta_i t^J \otimes \iota_{\tilde{w}} e_i \wedge \alpha +$$

$$t^J \otimes (\iota_{\tilde{w}} J) \wedge \alpha + (-\phi_1(J) + \beta_1) \sum_{a\in A} p_a t^{a+J} \otimes (\iota_{\tilde{w}} a) \wedge \alpha - \sum \beta_i t^J \otimes (\iota_{\tilde{w}} e_i) \wedge \alpha =$$

$$- \iota_{\tilde{w}} \nabla_\beta t^J \otimes \alpha + \left(t^J \langle (0,\tilde{w}), J\rangle + (-\phi_1(J) + \beta_1) \sum_{a\in A} P_a t^{a+J} \langle (0,\tilde{w}), a\rangle - t^J \sum \beta_i \phi_i(w) \right) \otimes \alpha \,.$$

Note that

$$\begin{aligned} \langle (0,\tilde{w}), J\rangle &= \langle w, J\rangle - w_1 \phi_1(J) \\ \langle (0,\tilde{w}), a\rangle &= \langle w, a\rangle - w_1 \\ \sum_{a\in A} P_a t^{a+J} &= t^J \,. \end{aligned}$$

Using these equalities we can rewrite the equation as

$$\nabla_\beta \iota_{\tilde{w}} t^J \otimes \alpha = -\iota_{\tilde{w}} \nabla_\beta t^J \otimes \alpha + \left(t^J \langle w, J\rangle + (-\phi_1(J) + \beta_1) \sum_{a\in A} P_a t^{a+J} \langle w, a\rangle - t^J \langle w, \beta\rangle \right) \otimes \alpha \,.$$

Note that $\langle w, J\rangle \in \mathbb{Z}_{\geq 0}$, and terms that appear in the first sum are all zero, since if $a \in S_w$ we have $\langle w, a\rangle = 0$ and if $a \in S_w^c$ we have $t^{a+J} \in I_w^n \Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U$. If $t^J \otimes \alpha$ is closed, we see that

$$\langle w, J\rangle t^J \otimes \alpha - t^J \sum \beta_i \langle w, e_i\rangle \otimes \alpha = (\langle w, J\rangle - \langle w, \beta\rangle) t^J \otimes \alpha = c\; t^J \otimes \alpha$$

is exact, where c is a nonzero constant, by the semi non-resonance assumption. □

In [AB01], authors construct a category of complexes of sheaves. De Rham complexes live in this category. Given an open embedding $j : X \to Y$ with X smooth and a D-module M on X, they define a $Rj_!$ functor on this category. If Y is smooth they show

$$DR(j_! M) = Rj_! DR(M),$$

where $j_!$ is the left adjoint of $j^!$ operator on holonomic D-modules as in [Bor+87]. See appendix D of [AB01].

Corollary 3.17. *Let $j : V \times \mathbb{T} \setminus Y_0 \to U$ be the inclusion of the complement of the torus boundary as in (3.4). Assuming that β is semi non-resonant, we have*

$$Rj_! j^*(\Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U, \nabla_\beta) \xrightarrow{q.i.s} (\Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U, \nabla_\beta)\,.$$

Here we think of complexes as objects of the derived category of sheaves of Abelian groups. The functor j^* is the pull back functor and $Rj_!$ is as in Definition D.2.14 of [AB01].

Proof. To compute $Rj_!$ we need to take an extension of $j^*(\Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U, \nabla_\beta)$ to U and take the limit by the ideal defining the complement. We already have an extension and we have shown that the powers of the ideal defining the boundary do not change the cohomology. Since both varieties are affine, the limit is $(\Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U, \nabla_\beta)$. □

Lemma 3.18. *Consider $O_{V\times\mathbb{T}}$ as a D-module and let $DR_{V\times\mathbb{T}/V}(O_{V\times\mathbb{T}})$ be its relative de Rham complex. Assume β has integer coefficients. Then we have a quasi-isomorphism*

$$j^*(\Omega^\bullet_{V\times\mathbb{P}_\Sigma/V}(logD)|_U, \nabla_\beta) \xrightarrow{q.i.s} (\Omega^\bullet_{V\times\mathbb{T}/V}|_{V\times\mathbb{T}\setminus Y_0}, d) = DR_{V\times\mathbb{T}\setminus Y_0/V}(O_{V\times\mathbb{T}\setminus Y_0})\,.$$

Proof. This simply follows from the fact that the t_i's are invertible on $\mathbb{T}$. The isomorphism is given by twisting the differential by

$$\frac{f^{\beta_1}}{t_2^{\beta_2}\dots t_n^{\beta_n}}$$

and its inverse as in (3.5). Note that, since A generates $\mathbb{Z}^n$ as a $\mathbb{Z}$-module, multiplication by the rational function above is an isomorphism. □

Theorem 3.19. *Assume β has integer coefficients, is semi non-resonant and β_1 is negative. Then we have*

$$R^{n-1}p_*(Rj_!DR_{V\times\mathbb{T}\setminus Y_0/V}(\mathcal{O}_{V\times\mathbb{T}\setminus Y_0})) = H_A(\beta)\,.$$

Proof. Proposition 3.13 and Lemma 3.18 together with Corollary 3.17 imply this. □

For the following theorem, we assume that the toric variety is normal, which is equivalent to Σ being saturated.

Theorem 3.20. *Assume β has integer coefficients, is semi non-resonant and β_1 is negative. Assume the semigroup Σ is saturated. Let U_v be the fiber of p over $v \in V$ and let D be the boundary divisor, i.e. the complement of $V \times \mathbb{T}$. We have*

$$H^0(Sol(H_A(\beta)))_v := Hom_{D_V}(H_A(\beta), \mathcal{O}^{an}_{V,v}) = H_{n-1}(U_v, U_v \cap D)\,.$$

Proof. If $\mathbb{P}_\Sigma$ was smooth, we could consider D-modules on U. Note that taking the relative de Rham complex commutes with $j_!$ by , i.e. we have

$$Rj_!DR_{V\times\mathbb{T}\setminus Y_0/V}(\mathcal{O}_{V\times\mathbb{T}\setminus Y_0}) = DR_{U/V}(j_!\mathcal{O}_{V\times\mathbb{T}\setminus Y_0}).$$

We have p_+ functor of [Bor+87]. By definition of p_+ for projections, $R^{n-1}p_* \circ DR_{U/V} = H^0p_+$. This implies

$$H_A(\beta) = H^0p_+(j_!\mathcal{O}_{V\times\mathbb{T}\setminus Y_0})$$

as quasi-coherent sheaves. The fact that connections agree follows from direct computation. The rest of the proof is the same as in [Hua+15]. The idea is to use the Riemann-Hilbert correspondence and the fact that $Sol \circ j_! = j_* \circ Sol$ from section 15 of [Bor+87] and the sheaf theoretic definition of relative homology from lemma 3.4 of [Hua+15]. Furthermore, the isomorphism is given by the cycle to period map defined in [HLZ13].

$X = \mathbb{P}_\Sigma$ is normal since Σ is saturated. By [Ish87], there exists an equivariant resolution of singularities $g : X' \to X$, such that X' is smooth and $Rg_*\mathcal{O}_{X'} = \mathcal{O}_X$. We have a diagram of fiber products of the form

$$\begin{array}{ccccc} V \times \mathbb{T} \setminus Y_0 & \xhookrightarrow{j'} & U' & \hookrightarrow & V \times X' \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle id \times g} \\ V \times \mathbb{T} \setminus Y_0 & \xhookrightarrow{j} & U & \hookrightarrow & V \times X. \end{array}$$

By the same computations in Proposition 3.16, one can show that the de Rham complex for $j'_!\mathcal{O}$ can be computed using the de Rham complex with logarithmic poles with twisted differential. The twisted de Rham complex is again a complex of free $\mathcal{O}_{X'}$ modules. Since $Rg'_*\mathcal{O}_{U'} = \mathcal{O}_U$ we have

$$Rg'_* DR_{U'/V}(j'_!\mathcal{O}_{V\times\mathbb{T}\setminus Y_0}) = Rg'_*(\Omega^\bullet_{V\times X'/V} log(D)|_{U'}, \nabla_\beta) = (\Omega^\bullet_{V\times X/V} log(D)|_U, \nabla_\beta)\,.$$

Thus, we have

$$\begin{aligned} H_A(\beta) &= R^{n-1}p_*(\Omega^\bullet_{V\times X/V} log(D)|_U, \nabla_\beta) \\ &= R^{n-1}p_* Rg'_* DR_{U'/V}(j'_!\mathcal{O}_{V\times\mathbb{T}\setminus Y_0}) \\ &= R^{n-1}(p\circ g')_* DR_{U'/V}(j'_!\mathcal{O}_{V\times\mathbb{T}\setminus Y_0}) = H^0(p\circ g')_+(j'_!\mathcal{O}_{V\times\mathbb{T}\setminus Y_0})\,. \end{aligned}$$

Applying the Sol functor we get

$$Sol(j'_!\mathcal{O}_{V\times\mathbb{T}\setminus Y_0}) = Rj'_*\mathbb{C}_{V\times\mathbb{T}\setminus Y_0}\,.$$

Moreover, using $Sol(p\circ g')_+ = R(p\circ g')_! Sol[n-1]$ from section 14 of [Bor+87] , we have

$$Sol(H_A(\beta)) = {}^pR^0 R(p\circ g')_! Rj'_*\mathbb{C}_{V\times\mathbb{T}\setminus Y_0}[n-1] = {}^pR^0 Rp_! \circ Rg'_! \circ Rj'_*\mathbb{C}_{V\times\mathbb{T}\setminus Y_0}[n-1]\,.$$

However, g' is proper, therefore $Rg'_! = Rg'_*$ and we have

$$Rg'_* \circ Rj'_* = R(g'\circ j')_* = Rj_*\,.$$

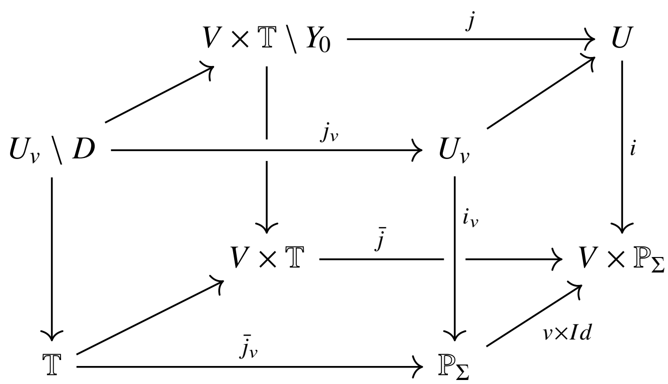
Moreover, we have

$$Sol(H_A(\beta)) = {}^pR^0 Rp_! \circ Rj_*\mathbb{C}_{V\times\mathbb{T}\setminus Y_0}[n-1]\,.$$

Thus, the sheaf of classical solution is

$$H^0(Sol(H_A(\beta))) = R^{n-1}p_! \circ Rj_*\mathbb{C}_{V\times\mathbb{T}\setminus Y_0}\,.$$

Note that $p_!$ commutes with taking fiber, by the compact support base change theorem. We want to find the restriction of $Rj_*\mathbb{C}_{V\times\mathbb{T}\setminus Y_0}$ to the fiber U_v of p over a point $v \in V$. Denote the inclusion of $V\times\mathbb{T}$ into $V\times\mathbb{P}_\Sigma$ by $\bar{j}$ and denote the map from U to $V\times\mathbb{P}_\Sigma$ by i. We have a diagram of varieties.



Since Rj_* is local on the target, we have $Rj_*\mathbb{C}_{V\times\mathbb{T}\setminus Y_0} = i^*R\bar{j}_*\mathbb{C}_{V\times\mathbb{T}}$. Therefore

$$Rj_*\mathbb{C}_{V\times\mathbb{T}\setminus Y_0}|_{U_v} = i^*R\bar{j}_*\mathbb{C}_{V\times\mathbb{T}}|_{U_v} = i_v^*(v\times Id)^*R\bar{j}_*\mathbb{C}_{V\times\mathbb{T}}\,.$$

Since $\bar{j}$ does not depend on the V, we have $(v\times Id)^*R\bar{j}_*\mathbb{C}_{V\times\mathbb{T}} = R\bar{j}_{v*}\mathbb{C}_{\mathbb{T}}$. From the square in the front, we see that $i_v^*R\bar{j}_{v*}\mathbb{C}_{\mathbb{T}} = Rj_{v*}\mathbb{C}_{U_v\setminus D}$. Thus, j_* commutes with restriction to a fiber.

$$Rj_*\mathbb{C}_{V\times\mathbb{T}\setminus Y_0}|_{U_v} = Rj_{v*}\mathbb{C}_{U_v\setminus D}$$

By sheaf theoretic definition of relative homology we deduce

$$Hom_{D_V}(H_A(\beta), O^{an}_{V,v}) = (R^{n-1}p_! \circ Rj_*\mathbb{C}_{V\times\mathbb{T}\setminus Y_0})_v = H_{n-1}(U_v, U_v\cap D)\,.$$

□

We showed that, for semi-nonresonant integer β, the sheaf of classical solutions to $H_A(\beta)$ is isomorphic to a relative homology. To find the isomorphism one needs to follow the proof of the Riemann-Hilbert correspondence which can be found in [HLZ13]. Assume that $v \in V$ is point and consider a relative chain $\delta_v \in H_{n-1}(U_v, U_v\cap D)$. We can extend this cycle to an analytic neighborhood of v in V. Let $\phi(v)$ be the function defined in the neighborhood of v by

$$\phi(v) = \int_{\delta_v} \frac{f^{\beta_1}}{t^{\tilde{\beta}}}\frac{dt_2}{t_2}\dots\frac{dt_n}{t_n}\,.$$

Some computations similar to what we had in Proposition 2.22 show that this function satisfies $H_A(\beta)$. In fact the isomorphism in the Riemann-Hilbert correspondence comes from this morphism. We do not use this fact in the general form, since in our case the cycle δ is always the positive real numbers. For more precise formulation see [HLZ13].

3.2 Regularization

To have a better notation, we change the dimension from n to $n+1$. Assume $A \subset \mathbb{Z}^{n+1}$ such that all points have first coordinate equal to 1. We denote the semigroup generated by A by Σ and we assume $\Sigma - \Sigma = \mathbb{Z}^{n+1}$. As before, let P_A be the convex hull of points in A and let C_A be the cone consisting of rays originating from zero and passing through P_A. For $(\beta_0, \ldots, \beta_n) = \beta \in \mathbb{C}^{n+1}$, we can consider the corresponding differential equations and integral forms of the solutions.

Assume all P_a's are positive i.e. v is a positive real point of V. We claim that the closure of $\mathbb{R}_+^n$ is a relative chain for the pair $(U_v, U_v \cap D)$. To check this, it is enough to show that f does not vanish on the closure of $\mathbb{R}_+^n$. One can check that closure of $\mathbb{R}_+^n$ is homeomorphic to the polytope P_A by moment map. The restriction of f to each torus orbit corresponding to a face defined by w is

$$\sum_{\langle w,a\rangle=0} P_a t^{\tilde{a}} .$$

Since P_a are positive and $t^{\tilde{a}}$ are positive, we see that f does not vanish. Assuming β is semi non-resonant and integer, it follows that ϕ is well defined and the integral

$$\int_{\mathbb{R}_+^n} \frac{t^{\tilde{\alpha}}}{f^{\alpha_0}} \frac{dt_1}{t_1} \ldots \frac{dt_n}{t_n}$$

is convergent, where $\alpha = -\beta$ and $\alpha = (\alpha_0, \tilde{\alpha})$. We prove this fact for more general values of β.

Lemma 3.21. *Assume the P_a's are positive real numbers indexed by $a \in A$. The integral*

$$\mathcal{I} := \int_{\mathbb{R}_+^n} \frac{t^{\tilde{\alpha}}}{(\sum_{a\in A} P_a t^{\tilde{a}})^{\alpha_0}} \frac{dt_1}{t_1} \ldots \frac{dt_n}{t_n} \tag{3.6}$$

converges for $\alpha = (\alpha_0, \tilde{\alpha})$ iff $\Re(\alpha)$ is in the interior of C_A.

Proof. To show this, we reparametrize $\mathbb{R}_+^n$ by $\mathbb{R}^n$, with the map $t_i = \exp(x_i)$, which gives

$$\int_{\mathbb{R}^n} \frac{\exp\langle\tilde{\alpha}, x\rangle}{(\sum_{a\in A} P_a \exp\langle\tilde{a}, x\rangle)^{\alpha_0}} dx_1 \dots dx_n \,. \tag{3.7}$$

For any $a \in A$, we have

$$\left|\frac{\exp\langle\tilde{\alpha}, x\rangle}{(\sum_{a\in A} P_a \exp\langle\tilde{a}, x\rangle)^{\alpha_0}}\right| \leq \frac{\exp\langle\Re(\tilde{\alpha}), x\rangle}{(P_a \exp\langle\tilde{a}, x\rangle)^{\Re(\alpha_0)}} \leq C \exp\langle\Re(\tilde{\alpha} - \alpha_0\tilde{a}), x\rangle,$$

where C is a constant that only depends on the P_a's. Taking the minimum on A we can rewrite the inequality as

$$\int_{\mathbb{R}^n} \frac{\exp\langle\tilde{\alpha}, x\rangle}{(\sum_{a\in A} P_a \exp\langle\tilde{a}, x\rangle)^{\alpha_0}} dx_1 \dots dx_n \leq C' \int_{\mathbb{R}^n} \exp\left(\min_a\langle\Re(\tilde{\alpha} - \alpha_0\tilde{a}), x\rangle\right) dx_1 \dots dx_n \,.$$

Note that $\Re(\alpha)$ is in the interior of the cone iff $\Re(\tilde{\alpha})$ is in the interior of the convex hull of $\{\Re(\alpha_0)\tilde{a}\}_{a\in A}$. For a fixed x, there exists at least one a such that

$$\langle\Re(\tilde{\alpha} - \alpha_0\tilde{a}), x\rangle = \langle\Re(\tilde{\alpha}), x\rangle - \langle\Re(\alpha_0)\tilde{a}, x\rangle < 0,$$

otherwise all point of $\Re(\alpha_0)\tilde{a}$ would be in the half-space $\langle \cdot, x\rangle \geq \langle\tilde{\alpha}, x\rangle$ and $\Re(\tilde{\alpha})$ would be on the boundary, which would contradict the fact that $\tilde{\alpha}$ is in the interior of the convex hull. To show that the integral converges, we use radial coordinates and we write

$$\mathcal{I} \leq C' \int_{S^{n-1}\times\mathbb{R}_+} \exp\left(r \, \min_a\langle\Re(\tilde{\alpha} - \alpha_0\tilde{a}), \frac{x}{|x|}\rangle\right) r^{n-1} dr \, d\Omega \,.$$

We showed that, for any x, $\min_a\langle\Re(\tilde{\alpha} - \alpha_0\tilde{a}), x\rangle$ is negative. Let $-\varepsilon$ be the supremum of this function on the sphere of radius one, which is negative by compactness of the sphere. Substituting, we get

$$\mathcal{I} \leq C' \int_{S^{n-1}\times\mathbb{R}_+} r^{n-1} e^{-\varepsilon r} dr \, d\Omega \leq C' \int_{S^{n-1}} \frac{\Gamma(n)}{\varepsilon^n} d\Omega < \infty \,.$$

□

We want to find relations among integrals with different α. Let $K(\alpha, P_a)$ be the integral

$$K(\alpha, P_a) = \int_{\mathbb{R}_+^n} \frac{t^{\tilde{\alpha}}}{(\sum_{a\in A} P_a t^{\tilde{a}})^{\alpha_0}} \frac{dt_1}{t_1} \dots \frac{dt_n}{t_n} \,. \tag{3.8}$$

Lemma 3.22. *Assume $\Re(\alpha)$ is in the interior of the cone. Let $w \in \mathbb{Z}^{n+1}$. We have the equality*

$$\langle w, \alpha\rangle K(\alpha, P_a) = \alpha_0 \sum_{a\in A} \langle w, a\rangle P_a K(\alpha + a, P_a) \,.$$

Proof. Both sides are linear in w, hence it is enough to check this for $w = e_i$. Note that, for $w = e_0$, this equality is trivial. For $i \neq 0$ and $w = e_i$, we use exponential change of variable as in equation (3.7). Consider the differential form on $\mathbb{R}^n$ given by

$$\theta = \frac{\exp\langle\tilde{\alpha}, x\rangle}{(\sum_{a\in A} P_a \exp\langle\tilde{a}, x\rangle)^{\alpha_0}} dx_1 \ldots \hat{dx_i} \ldots dx_n \, .$$

A basic computation shows that we have

$$d\theta = \left(\alpha_i \frac{\exp\langle\tilde{\alpha}, x\rangle}{(\sum_{a\in A} P_a \exp\langle\tilde{a}, x\rangle)^{\alpha_0}} - \alpha_0 \frac{\sum_{a\in A} P_a a_i \exp\langle\tilde{\alpha} + a, x\rangle}{(\sum_{a\in A} P_a \exp\langle\tilde{a}, x\rangle)^{\alpha_0+1}}\right) dx_1 \ldots dx_n \, .$$

Assuming that $K(\alpha, P_a)$ converges implies that $\Re(\alpha)$ is the interior of the cone generated by A. Since $\Re(\alpha)$ is in the interior of the cone, $\Re(\alpha) + a$ is also in the interior of the cone and all terms in $d\theta$ have absolutely convergent integrals. Therefore, we can integrate $d\theta$ in the interior of ball of radius r and take the limit as r goes to infinity. By Stokes' theorem the integral of $d\theta$ over a ball of radius r is equal to integral of θ on a sphere of radius r. By the same computations as in the previous lemma, we can see that θ has exponential decay, while the volume of the sphere grows polynomially. This implies that the limit is zero. Note that the integral of $d\theta$ is equal to

$$\langle e_i, \alpha\rangle K(\alpha, P_a) - \alpha_0 \sum_{a\in A} \langle e_i, a\rangle P_a K(\alpha + a, P_a),$$

which implies the statement.

□

Theorem 3.23. *Let $K(\alpha, P_a)$ be as in (3.8), defined for $\Re(\alpha)$ in the interior of the cone generated by A. Then $K(\alpha, P_a)$ has meromorphic continuation to $\mathbb{C}^{n+1}$, with poles along the $-\Sigma$ translates of the hyperplanes defining the facets of the convex hull of A, or equivalently the semi-resonant $-\alpha$'s. Furthermore, we have the identity in Lemma 3.22.*

Proof. First note that the integral is absolutely convergent and that the integrand is analytic in α, which implies that the integral is holomorphic for $\Re(\alpha)$ in the interior of the cone, i.e. for $\langle w_i, \Re(\alpha)\rangle > 0$. We use induction to show the statement for the sets $\langle w_i, \Re(\alpha)\rangle > m_i$, which cover $\mathbb{C}^{n+1}$ for $m_i \in \mathbb{Z}$. The statement is true for $\vec{m} = (0, \ldots, 0)$. Assume it is true for $\vec{m} = (m_1, \ldots, m_{n+1})$. By Lemma 3.22 we have

$$\langle w_i, \alpha\rangle K(\alpha, P_a) = \alpha_0 \sum_{a\in A} \langle w_i, a\rangle P_a K(\alpha + a, P_a) \, .$$

Note that $\langle w, a\rangle = 0$ for a in the facet defined by w_i. Thus, $\langle w_i, \alpha\rangle K(\alpha, P_a)$ can be expressed as a linear combination of $K(b, P_a)$, where, for all b, we have $\langle w_i, \Re(b)\rangle \geq m_i + 1$ and $\langle w_j, \Re(b)\rangle \geq m_j$ for $j \neq i$. We can define the integral for $\langle w_i, \Re(\alpha)\rangle > m_i - 1$ and $\langle w_j, \Re(\alpha)\rangle > m_j$ by dividing both sides by $\langle w_i, \alpha\rangle$. This means that there is pole when $\langle w_i, \alpha\rangle$ is zero. By repeating this operation, we get poles on the $-\Sigma$ translates of $\langle w_i, \alpha\rangle = 0$. Note that the equation above is analytic in α and is valid for $\Re(\alpha)$ in the interior of the cone, hence it is valid everywhere. □

Remark 3.24. *Note that in the identity of Lemma 3.22, the sum on the right is multiplied by α_0. If we start from an α with $\alpha_0 \in \mathbb{Z}_{\leq 0}$, we eventually multiply be zero, since all points in the interior of the cone have $\alpha_0 > 0$. Thus, $K(\alpha, P_a)$ has degree one zero along hyperplanes $\alpha_0 \in \mathbb{Z}_{\leq 0}$.*

We know that, for positive real P_a's, if $K(\alpha, P_a)$ converges, then it is a solution to $H_A(\beta)$. We claim that this is true for the analytic continuation of $K(\alpha, P_a)$ as well. From the identity of Lemma 3.22, we know that for α with real part in the interior of the positive cone, the left hand side is a solution to $H_A(-\alpha)$. Note that both differential equations and relations are analytic in α and P_a, therefore they are valid for the analytic continuation of $K(\alpha, P_a)$. As a result we have that $K(\alpha, P_a)$ is a solution to $H_A(-\alpha)$ when $-\alpha$ is non semi-resonant. For a resonant $-\alpha$ we can find a vector $\vec{u} \in \mathbb{C}^{n+1}$ such that $-(\alpha + \epsilon\vec{u})$ is semi non-resonant for small enough ϵ. We can take the Laurent expansion in the $\vec{u}$ direction and we obtain

$$K(\alpha + \epsilon\vec{u}, P_a) = \sum_{i=-k}^{\infty} \epsilon^i K_i^{\vec{u}}(\alpha, P_a)\,. \tag{3.9}$$

As in the previous chapter, we denote the set of integer relations among points of A by R. For $r \in R$, we have a corresponding box differential operator $\Box_r$ as in (3.1) and

$$Z_w = \sum_{a \in A} \langle w, a\rangle P_a \frac{\partial}{\partial P_a}\,. \tag{3.10}$$

Note that Z_i in (3.2) is Z_{e_i} in 3.10. In this notation, a solution ϕ to $H_A(\beta)$ is equivalent to a function satisfying

$$\Box_r \phi = 0 \qquad Z_w \phi = \langle w, \beta\rangle \phi\,.$$

By the same computation of Theorem 2.24, we arrive to the following proposition.

Proposition 3.25. *For $w \in \mathbb{C}^{n+1}$, we have*

$$\Box_r K_i^{\vec{u}}(\alpha, P_a) = 0 \qquad Z_w K_i^{\vec{u}}(\alpha, P_a) = \langle w, -\alpha\rangle K_i^{\vec{u}}(\alpha, P_a) + \langle w, -\vec{u}\rangle K_{i-1}^{\vec{u}}(\alpha, P_a)\,.$$

Proof. This follows from expanding both sides of (3.9). □

In particular we see that the lowest coefficient gives us a solution to $H_A(-\alpha)$.

Remark 3.26. *All the calculations we have done here can be done for any chain δ replacing the positive real points. In fact, Lemma 3.22 is valid for the integral over any chain, and the rest of the calculation is exactly the same. In this way we can construct a set of solutions for resonant β.*

Chapter 4

AMPLITUDES AND REGULARIZATION

As we showed in chapter 2.2, amplitudes satisfy certain differential equations. From a Feynman diagram, we constructed a subset of $\mathbb{Z}^{n+1}$ in the following way. For each monomial t^S in the first Symanzik polynomial Ψ_Γ of the graph, we consider the point $(1, \vec{S}) \in \mathbb{Z}^{n+1}$ and, for each monomial t^T in Q_Γ, we consider the lattice point $(0, \vec{T})$. We denote this set of lattice points by A. In general A does not generate the lattice $\mathbb{Z}^{n+1}$ but it generates a sublattice of dimension n. To see this, note that $(1, 0, 0, ...0, -1, 0, ..., 0)$, where -1 is in the i-th place, is in the sublattice generated by A. This term is $(1, \vec{S}) - (0, \vec{T})$, where the monomial T is the product of $m_i^2 t_i$ and t^S. On the other hand, all points of A lie on the hyperplane $\sum_{i=0}^n a_i = \ell + 1$, where ℓ is the number of loops in Γ. Thus, the sublattice generated by A in $\mathbb{Z}^{n+1}$ is the set of lattice points x, such that $\ell + 1 | \sum x_i$. Denote this sublattice by L and let $r : L \longrightarrow \mathbb{Z}^{n+1}$ be the function defined by $r_0(x) = (x_0 + x_1 + ... + x_n)/(\ell + 1)$ and $r_i(x) = x_i$, for $1 \leq i \leq n$. It is easy to see that r is invertible and that the determinant is $1/(\ell + 1)$, hence the image of the standard cube has volume 1 and $r(A)$ generates the lattice. We replace A by $r(A)$. Note that all points of $r(A)$ have first coordinate equal to 1. By Proposition 2.21 we know that $I(c_1, c_2, P_a, \vec{v})$ satisfies $H_A(\beta)$, where

$$\beta = ((c_1 - c_2)(\ell + 1) + \sum_i v_i + n, -1 - v_1, \dots, -1 - v_n)\,.$$

Thus, I satisfies $H_{r(A)}(r(\beta))$ and we have

$$r(\beta) = (c_1 - c_2, -1 - v_1, \dots, -1 - v_2)\,.$$

For the rest of this chapter we replace A by $r(A)$ and β by $r(\beta)$. Note that, for the original amplitude $\mathcal{A}(\Gamma, P_a)$, we have $c_1 = 0$, $c_2 = D/2$ and $v = \vec{0}$, and thus the corresponding vector β is

$$\beta = (-D/2, -1, \dots, -1)\,.$$

Assuming the normality condition and the semi non-resonant condition in Theorem 3.20, all solutions of $H_A(\beta)$ come from integrals. Thus, $\mathcal{A}(\Gamma, P_a)$ is equal to

$$\int_\delta \frac{t_1 \dots t_n}{(\sum_{a \in A} P_a t^{\tilde{a}})^{D/2}} \frac{dt_1}{t_1} \dots \frac{dt_n}{t_n},$$

for a relative chain δ. In fact, using the projective version of the integral, we can show that it is equal to $K(-\beta, P_a)$, up to multiplication by a rational number.

Lemma 4.1. *Feynman's parametric integral formula gives*

$$\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^\infty \frac{\lambda^{a-1} d\lambda}{[\lambda A + B]^{a+b}}\,.$$

Proof. See Chapter 8 of [KS01]. □

Corollary 4.2. *Assume $K(\alpha, P_a)$ is convergent, i.e. α is in the interior of the cone generated by A. Then we have the equality*

$$K(\alpha, P_a) = \frac{\Gamma(-|\tilde{\alpha}| + \alpha_0(\ell+1))}{\Gamma(\alpha_0)} I(0, \alpha_0, P_a, \tilde{\alpha} - (1, \dots, 1))\,.$$

Proof. By Lemma 4.1 we have

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} \frac{t^{\tilde{\alpha}}}{(Q_\Gamma + \Psi_\Gamma)^{\alpha_0}} \\
=& \int_{\mathbb{R}_+ \times \Delta_{n-1}} \frac{r^{|\tilde{\alpha}|+n-1} t^{\tilde{\alpha}}}{\left(r^{\ell+1} Q_\Gamma + r^\ell \Psi_\Gamma\right)^{\alpha_0}} dr\Omega \\
=& \int_{\mathbb{R}_+ \times \Delta_{n-1}} \frac{r^{|\tilde{\alpha}|+n-1-\alpha_0 \ell} t^{\tilde{\alpha}}}{(r Q_\Gamma + \Psi_\Gamma)^{\alpha_0}} dr\Omega \\
=& \frac{\Gamma(-n - |\tilde{\alpha}| + \alpha_0(\ell+1))\Gamma(|\tilde{\alpha}| + n - \alpha_0 \ell)}{\Gamma(\alpha_0)} \int_{\Delta_{n-1}} \frac{Q_\Gamma^{-n-|\tilde{\alpha}|+\alpha_0 \ell}}{\Psi_\Gamma^{-n-|\tilde{\alpha}|+\alpha_0(\ell+1)}} t^{\tilde{\alpha}} \Omega \\
=& \frac{\Gamma(-n - |\tilde{\alpha}| + \alpha_0(\ell+1))\Gamma(|\tilde{\alpha}| + n - \alpha_0 \ell)}{\Gamma(\alpha_0)} J(-n - |\tilde{\alpha}| + \alpha_0 \ell, -n - |\tilde{\alpha}| + \alpha_0(\ell+1), P_a, \tilde{\alpha})\,.
\end{aligned}$$

By the definition of $K(\alpha, P_a)$ and the equality (2.11), we have $K(\alpha, P_a) =$

$$\begin{aligned}
&\frac{\Gamma(-|\tilde{\alpha}| + \alpha_0(\ell+1))\Gamma(|\tilde{\alpha}| - \alpha_0 \ell)}{\Gamma(\alpha_0)} J(-|\tilde{\alpha}| + \alpha_0 \ell, -|\tilde{\alpha}| + \alpha_0(\ell+1), P_a, \tilde{\alpha} - (1, \dots, 1)) \\
=& \frac{\Gamma(-|\tilde{\alpha}| + \alpha_0(\ell+1))}{\Gamma(\alpha_0)} I(0, \alpha_0, P_a, \tilde{\alpha} - (1, \dots, 1))\,.
\end{aligned}$$

□

By corollary above and the fact that $\mathcal{A}(\Gamma, P_a) = \pi^{D\ell/2} I(0, D/2, P_a, \vec{0})$, we can compute the amplitude from K as

$$\mathcal{A}(\Gamma, P_a) = \pi^{D\ell/2} \frac{\Gamma(D/2)}{\Gamma(-n + D/2(\ell+1))} K((D/2, 1, \dots, 1), P_a)\,. \tag{4.1}$$

Note that the Gamma function never vanishes. As a result, when $K(\alpha, P_a)$ converges, we can define the amplitude. In fact, we can define the amplitude by this equation.

For the rest of this chapter we study $K(\alpha, P_a)$, and the structure of its poles. To do that, we first find the defining inequalities for P_A. We need a few definitions and notation.

Definition 4.3. *For a graph Γ with n edges, we denote by $SP_\Gamma \subset \mathbb{R}^n$ the polytope constructed by the incidence vectors of the complements of the spanning trees of Γ, i.e. it is the convex hull of $\{\vec{S}\}$, where S corresponds to a monomial t^S in the first Symanzik polynomial of Γ. We denote by P_Γ the polytope constructed from the terms in the first and second Symanzik polynomials (including mass terms). Note that we have $P_\Gamma = SP_\Gamma + E_n$, where E_n is the convex hull of $\{0, e_1, \ldots, e_n\}$ and plus is the Minkowski sum. For a subgraph $\gamma \subset \Gamma$, we have natural inclusions $P_\gamma \subset P_\Gamma$ and $SP_\gamma \subset SP_\gamma$.*

Definition 4.4. *For a subgraph $\gamma \subset \Gamma$, let 1_γ be the incidence vector of γ in $\mathbb{Z}^n$, i.e. the vector where the coefficient corresponding to an edge $e \in \Gamma$ is 1 if $e \in \gamma$ and is zero otherwise. We denote by ℓ_γ the number of loops in γ, i.e. the dimension of the first homology of γ.*

Lemma 4.5. *Given two subgraph γ_1 and γ_2, we have*

$$\ell_{\gamma_1} + \ell_{\gamma_2} \leq \ell_{\gamma_1 \cap \gamma_2} + \ell_{\gamma_1 \cup \gamma_2}\,.$$

Proof. By Mayer-Vietoris for the pair γ_1 and γ_2, we have the exact sequence

$$0 = H_2(\gamma_1 \cup \gamma_2) \to H_1(\gamma_1 \cap \gamma_2) \to H_1(\gamma_1) \oplus H_1(\gamma_2) \to ker\,(H_1(\gamma_1 \cup \gamma_2) \to H_0(\gamma_1 \cap \gamma_2)) \to 0\,.$$

Counting dimensions implies the inequality. □

We define another polytope using inequalities and we will show it is the same polytope we considered before. Spanning tree polytopes (SP_Γ) and in general matroid polytopes have been studied in combinatorial optimization theory, see [Cho89], [Edm71] and [Ful71]. The polytope P_Γ is similar to these polytopes and we can translate some of the results in combinatorial optimization to our setting.

Definition 4.6. *For a graph Γ, let P'_Γ be the subset of $\mathbb{R}^n$ defined by the inequalities*

$$\langle 1_\gamma, \tilde{x} \rangle \geq \ell_\gamma \qquad \langle 1_\Gamma, \tilde{x} \rangle \leq \ell_\Gamma + 1,$$

where the first inequality is valid for all subgraphs.

Assume that the graph γ in the first inequality above is a single edge. In this case, the inequality implies that all coefficients are positive. The second inequality implies the sum of the coefficients is bounded. Thus, these equations define a bounded set and P'_Γ is a polytope. Any face of the polytope $P = P'_\Gamma$ is defined by setting some of the inequalities to equalities. Assume we have a set of equalities $\langle 1_\gamma, \tilde{x}\rangle = \ell_\gamma$ for $\gamma \in \mathcal{F}$, where $\mathcal{F}$ is a family of subgraphs.

Lemma 4.7. *Given a point $\tilde{x}$ in P, let $\mathcal{F}$ be the set*

$$\mathcal{F} = \{\gamma \subset \Gamma : \langle 1_\gamma, \tilde{x}\rangle = \ell_\gamma\}.$$

Then $\mathcal{F}$ is closed under intersection and union of its elements.

Proof. We have

$$\langle 1_{\gamma_1\cap\gamma_2}, \tilde{x}\rangle \geq \ell_{\gamma_1\cap\gamma_2}$$

and

$$\langle 1_{\gamma_1\cup\gamma_2}, \tilde{x}\rangle \geq \ell_{\gamma_1\cup\gamma_2},$$

and we have $1_{\gamma_1} + 1_{\gamma_2} = 1_{\gamma_1\cap\gamma_2} + 1_{\gamma_1\cup\gamma_2}$. Combining this with the inequality of Lemma 4.5, we have

$$\ell_{\gamma_1} + \ell_{\gamma_2} = \langle 1_{\gamma_1}, \tilde{x}\rangle + \langle 1_{\gamma_1}, \tilde{x}\rangle \geq \ell_{\gamma_1\cap\gamma_2} + \ell_{\gamma_1\cup\gamma_2} \geq \ell_{\gamma_1} + \ell_{\gamma_2},$$

and hence both inequalities are equalities. This shows that $\mathcal{F}$ is closed under intersection and union of its elements. □

By the previous lemma, the defining equations of faces (coming from subgraphs) can be chosen to be closed under intersection and union. By a chain of subgraphs we mean a family C of subgraphs such that, for $\gamma_1, \gamma_2 \in C$, we have either $\gamma_1 \subset \gamma_2$ or $\gamma_2 \subset \gamma_1$.

Lemma 4.8. *Let $\mathcal{F}$ be the set of equalities corresponding to subgraphs and let $P_\mathcal{F}$ be the corresponding face. Let $C \subset \mathcal{F}$ be a maximal chain in $\mathcal{F}$. The family of linear equations $\{\langle 1_\gamma, \tilde{x}\rangle = \ell_\gamma : \gamma \in \mathcal{F}\}$ is equivalent to $\{\langle 1_\gamma, \tilde{x}\rangle = \ell_\gamma : \gamma \in C\}$, i.e. we have $P_\mathcal{F} = P_C$.*

Proof. Given a subgraph γ, by a chain violation we mean a subgraph $c \in C$ such that neither $c \subset \gamma$ nor $\gamma \subset c$. Since C is maximal, for any subgraph γ in $\mathcal{F} \setminus C$ there exists $c \in C$ such that it is a chain violation for γ and $P_{C\cup\{c\}} \neq P_C$, otherwise the statement follows. Among all of the subgraphs choose the one with minimal number of chain violations. By Lemma 4.7, we have $c \cap \gamma \in \mathcal{F}$ and $c \cup \gamma \in \mathcal{F}$. For all $\tilde{x} \in P_{\mathcal{F}}$ we have

$$\langle 1_\gamma, \tilde{x}\rangle = \ell_\gamma \qquad \langle 1_c, \tilde{x}\rangle = \ell_c \qquad \langle 1_{\gamma\cap c}, \tilde{x}\rangle = \ell_{\gamma\cap c} \qquad \langle 1_{\gamma\cup c}, \tilde{x}\rangle = \ell_{\gamma\cup c}\,.$$

These four linear equations are dependent, i.e. the sum of the first two is equal to the sum of the last two. The first equation is not satisfied by all points of P_C, so we either have $P_{C\cup\{\gamma\cap c\}} \neq P_C$ or $P_{C\cup\{\gamma\cup c\}} \neq P_C$. Replacing c by $\gamma \cap c$ or $\gamma \cup c$ decreases the number of chain violations, which contradicts the maximality condition. □

Proposition 4.9. *Two polytope are equal, i.e. $P_\Gamma = P'_\Gamma$. Furthermore let C_Γ be the cone over P_Γ in $\mathbb{R}^{n+1}$ and let*

$$\vec{\gamma} = (-\ell_\gamma, 1_\gamma).$$

Then C_Γ is given by the equalities

$$\langle \vec{\gamma}, x\rangle \geq 0$$

for all $\gamma \subset \Gamma$, and

$$\langle \vec{\Gamma} - e_0, x\rangle \leq 0\,.$$

Proof. Note that the inequalities define a cone since 0 satisfies all the equations. The intersection of this cone with the hyperplane $\langle x, e_0\rangle = 1$ is P'_Γ, which is bounded. Since the intersection is bounded, the cone defined by the inequalities is the cone over the intersection. As a result it is enough to show that the intersection is equal to P_Γ. The first inequality for a subgraph is satisfied by all points of A, since an intersection of a spanning tree with a subgraph is a spanning forest of the subgraph, hence number of edges in its complement is greater than the number of loops. The second inequality is trivial. P'_Γ contains all the extreme points of P_Γ, and thus it contains P_Γ. First we find the integer points of P'_Γ. Any edge $e \in E$ is a subgraph and the corresponding first inequality implies $x_i \geq 0$. Thus, integer points of P'_Γ have the form

$$(a_1, \ldots, a_n),$$

where $a_i \in \mathbb{Z}_{\geq 0}$. For any loop $\gamma \in \Gamma$ we have

$$\sum_{e\in\gamma} a_e \geq 1.$$

Thus, for at least one $e \in \gamma$, we have $a_e \neq 0$. If we remove this edge from the graph, the remaining graph has $\ell_\Gamma - 1$ loops and we can find another loop that does not have e in it. This in turn implies that another $a_{e'}$ is non-zero. Iterating this procedure, we get at least ℓ_Γ many nonzero a_i's, which are chosen from different loops. The complement of the edges corresponding to these a_i's is a spanning tree. The second inequality implies $\sum_i a_i \leq \ell_\Gamma + 1$, hence $(a_1, \dots, a_n)$ corresponds to the incidence vector of a complement of a spanning tree, or e_i plus the incidence vector of a complement of a spanning tree. The integer points of P'_Γ are exactly the integer points of P_Γ. To finish the proof we need to show that the extreme points of P'_Γ are integers.

Let $\tilde{x}$ be an extreme point of P'_Γ. Then $\tilde{x}$ is the unique solution to a set of linear equations corresponding to defining equations of P'_Γ. There are two cases. The first case occurs when the second equation is not used and $\tilde{x}$ is defined by $\langle 1_\gamma, \tilde{x}\rangle = \ell_\gamma$ for $\gamma \in \mathcal{F}$. By Lemma 4.8, we can replace $\mathcal{F}$ by a chain of subgraphs C. Since the solution is unique, we need n many equalities and we have

$$\gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_n = \Gamma \qquad C = \{\gamma_1, \dots, \gamma_n\}.$$

Thus, $\gamma_i \setminus \gamma_{i-1}$ has only one edge. We denote this edge by e_i. By the equalities we see that

$$x_i = \ell_{\gamma_i} - \ell_{\gamma_{i-1}} \in \{0, 1\},$$

which implies we have an integer point. The set of extreme points we obtain in this way corresponds to a spanning trees. The second case is when we have the equality $\langle 1_\Gamma, \tilde{x}\rangle = \ell_\Gamma + 1$. By the same argument, we see that $x_i \in \{0, 1\}$, for all i except one of them. Since these add up to an integer, the other one has to be an integer too. The set of extreme points we obtain in this way corresponds to the set of monomials in the second Symanzik polynomial. □

A subgraph is called 2-connected, if it is connected and remains connected after removing any vertex. Note that single edges are 2-connected. We have found a set of inequalities that define P_Γ, but this set is not minimal. We just need the equations defining the facets of P_Γ. Any facet is defined by a single equation, so we need to find subgraphs for which the equality defines a facet.

Lemma 4.10. *For a 2-connected graph γ with no self-loops, SP_γ is $|E(\gamma)| - 1$ dimensional.*

Proof. Note that all integer points have the form 1_S, where S is complement of a spanning tree. Thus, all points of SP_γ lie on the hyperplane where the sum of coefficients is ℓ_γ. This implies that SP_γ is at most $|E(\gamma)|-1$ dimensional. We show that $e_i - e_j$ can be constructed from differences of points of SP_γ.

Let e be an edge of γ and let C be a loop in γ which contain e. This exists, since γ is 2-connected. Let T be a spanning tree that contains all edges of C except e. Note that such a spanning tree exists, since any tree can be extended to a spanning tree. Let e' be another edge of C. We claim that $T \cup e \setminus e'$ is a spanning tree. The reason is that $T \cup e$ has only one loop C, and removing any edge from it makes it a tree. Let $S = T^c$ and $S' = (T \cup e \setminus e')^c$. Then $1_S - 1_{S'}$ is $1_e - 1'_e$, hence, for any two edges $e,\ e'$ in the same loop, $1_e - 1_{e'}$ can be computed as a difference of points in SP_γ. Since γ is 2-connected and does not have self loops, we can compute $1_e - 1_{e'}$ for any pair of edges, and we find that SP_γ is of codimension one. □

For a self loop e in Γ, SP_e is just a point, which indeed is $|E(e)|-1 = 0$ dimensional. Note that P_Γ is always n dimensional, since it is equal to $SP_\Gamma + E_n$, where E_n is n dimensional.

Lemma 4.11. *For a subgraph $\gamma \subset \Gamma$, $P_\Gamma \cap \{\tilde{x} : \langle 1_\gamma, \tilde{x}\rangle = \ell_\gamma\}$ is $SP_\gamma \times P_{\Gamma//\gamma}$, where $\Gamma//\gamma$ is constructed from contracting connected components of γ to points.*

Proof. It is enough to find extreme points of $P_\Gamma \cap \{\tilde{x} : \langle 1_\gamma, \tilde{x}\rangle = \ell_\gamma\}$, since the equation defines a face of P_Γ. The equation $\langle \tilde{x}, 1_\gamma\rangle = \ell_\gamma$ for an extreme point $(a_e : e \in \Gamma)$, implies the vector $(a_e : e \in \gamma)$ is equal to 1_S, where S is the complement of a spanning tree in γ. A monomial t^T can be written as $t^S t^{T-S}$, where S has coefficients in γ and $T - S$ corresponds to edges in $\Gamma - \gamma$ that are the edges in the contracted graph. Note that t^{T-S} always corresponds to a monomial for the first or second Symanzik polynomial of $\Gamma//\gamma$. On the other hand $S + S'$, where S comes from a spanning tree in γ and S' comes from a monomial in $\Gamma//\gamma$, is a monomial in the first or second Symanzik polynomial of Γ. Thus, we can identify the corresponding face P_Γ with $SP_\gamma \times P_{\Gamma//\gamma}$ □

Theorem 4.12. *For a graph Γ, the polytope P_Γ is given by the inequality*

$$\langle 1_\Gamma, \tilde{x}\rangle \leq \ell_\Gamma + 1$$

and inequalities

$$\langle \gamma, \tilde{x}\rangle \geq \ell_\gamma$$

indexed by 2-connected subgraphs without self-loops $\gamma \subset \Gamma$, as well as inequalities

$$\langle 1_e, \tilde{x} \rangle = x_e \geq 1$$

indexed by self loops $e \in \Gamma$. Replacing any inequality corresponding to a subgraph or self-loop γ with equality, defines a facet of P_Γ that is equal to $SP_\gamma \times P_{\Gamma//\gamma}$.

Proof. By Proposition 4.9, we know that these equation define the polytope. We just need to find the ones that define facets, i.e. are codimension one. By Lemma 4.11 the codimension is equal to the codimension of SP_γ. The latter is equal to one for self loops and for 2-connected subgraphs without self-loops, by Lemma 4.10. On the other hand, if we have a subgraph which is not 2-connected, then the complement of a spanning tree has to have ℓ_{γ_i} many edges in γ_i, where γ_i are 2-connected components of γ. Thus, SP_γ is at least of codimension 2. For a subgraph that is not a self loop but that contains a self loop e, we have two linear equalities for points of SP_γ, i.e. $\langle 1_\gamma, \tilde{x} \rangle = \ell_\gamma$ and $x_e = 1$. This makes SP_γ at least of codimension 2. □

For a Feynman diagram Γ, the vector α is equal to

$$(D/2, 1, \ldots, 1)\,.$$

To find the inequalities defining the interior of the cone C_A, we need to replace inequalities with strict inequalities. Applying these to a vector α, we find the necessary and sufficient condition for convergence of $K(\alpha, P_a)$, namely

$$D/2(\ell + 1) > |E(\Gamma)|\,.$$

If we have self-loops, then the inequality

$$D/2 = D\ell/2 < |E| = 1$$

is not satisfied and the integral diverges. For all 2-connected subgraphs without self-loops γ we have

$$D\ell/2 < |E(\gamma)|\,.$$

Note that for single edges this equality is satisfied, hence it is enough to check this for 2-connected subgraphs that are not single edges, i.e. for the so called $1PI$ subgraphs.

Lemma 4.13. *Let Σ be the semigroup generated by A in $\mathbb{Z}^{n+1}$. Then Σ is saturated.*

Proof. Assume $a = (a_0, \tilde{a}) \in \mathbb{Z}^{n+1}$ is an integer point in C_A. We want to find $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in A^k$ such that $a = \sum_i \alpha_i$. Assume

$$\langle \vec{\Gamma}, a \rangle > 0\,.$$

We claim that there exists $i \in \{1, \ldots, n\}$, such that $a - e_i$ is in C_A. Assume this is not the case. Then, for each i, one of the inequalities is not satisfied by $a - e_i$. Note that we have

$$\langle \vec{\Gamma} - e_0, a - e_i \rangle = \langle \vec{\Gamma} - e_0, a \rangle - 1 \leq 0\,.$$

Thus, for each i, there exists a subgraph γ_i such that

$$\langle \vec{\gamma}_i, a - e_i \rangle < 0\,.$$

Since a has integer coefficients and a is in C_A, we must have

$$\langle \vec{\gamma}_i, a \rangle = 0,$$

which is equivalent to

$$\langle 1_{\gamma_i}, \tilde{a}/a_0 \rangle = \ell_{\gamma_i}\,.$$

By Lemma 4.5 we have the same equality for a union of γ_i's, i.e.

$$\langle 1_\Gamma, \tilde{a}/a_0 \rangle = \ell_\Gamma\,.$$

This contradicts our assumption. As a result, we can write a as $b + c$, where b is an integer point in C_A that satisfies $\langle \vec{\Gamma}, b \rangle = 0$ and c has positive integer coordinates with $c_0 = 0$. Note that

$$0 \geq \langle \vec{\Gamma} - e_0, b + c \rangle = -b_0 + \sum_i c_i,$$

which implies $b_0 \geq \sum_i c_i$. Assume we can write $b = \sum_i \alpha'_i$ with α'_i in Σ satisfying $\langle \vec{\Gamma}, \alpha'_i \rangle = 0$. Since $b_0 \geq \sum_i c_i$, we can distribute $\sum_i c_i$ many e_i's among the α'_i and define $\alpha_i = \alpha'_i + e_{j_i}$. To finish the proof, we need to show that the semigroup generated by the points of A on the facet $\langle \vec{\Gamma}, x \rangle = 0$ is saturated. This has been shown in [Whi77]. □

Corollary 4.14. *$\mathbb{P}_\Sigma$ is projectively normal, i.e. $k[\Sigma]$ is integrally closed. By [Hoc72] the toric ideal has the Cohen-Macaulay property.*

Theorem 4.15. *For a graph Γ, with the properties that*

$$D/2(\ell_\Gamma + 1) > |E(\Gamma)| \quad \textit{and} \quad D\ell_\gamma/2 < |E(\gamma)|,$$

and for all $1PI$ *subgraphs* γ*, the amplitude is equal to*

$$K((D/2, 1, \ldots, 1), P_a),$$

up to multiplication by rational numbers and powers of π*. This is a period of the motive* $H^n(U_v, U_v \cap D)$*. Furthermore, for any graph* Γ*, the lowest coefficient of the* ϵ*-expansion of the amplitude agrees with the lowest coefficient of the* ϵ*-expansion of* $K((D/2 + \epsilon, 1, \ldots, 1))$*, up to multiplication by rational numbers and powers of* π*. This can be computed as a linear combination*

$$\sum p_i(P_a) K(\alpha_i, P_a),$$

where the p_i*'s are polynomials in the* P_a*'s with rational coefficients and all the* $K(\alpha_i, P_a)$*'s are convergent and are periods of the motive* $H^n(U_v, U_v \cap D)$*.*

Proof. Note that the value of the gamma function at positive integers is an integer. The first part then follows from the Relation (4.1) between $K(\alpha, P_a)$ and $\mathcal{A}(\Gamma, P_a)$. For the second part, note that, using the relations in Lemma 3.22, we can replace a divergent K with a linear combination of convergent integrals, with polynomial coefficients in P_a. The poles arise by diving by terms $\langle w, \alpha \rangle$, which have a rational residue in the variable ϵ. Moreover, we have division by the gamma function, which has rational residue at negative integers. Note that all the resulting convergent integrals are periods of $H^n(U_v, U_v \cap D)$ by Theorem 3.20. □

Proof of Theorem 1.1. We have

$$c_0 \mathcal{A}(D/2) := I(0, D/2, P_a, \vec{0}) = \frac{\Gamma(\alpha_0)}{\Gamma(-\langle \vec{\Gamma} - e_0, \alpha \rangle)} K(\alpha, P_a),$$

where $\alpha = (D/2, 1, \cdots, 1)$. By Remark 3.24, zeros of $K(\alpha, P_a)$ cancel poles of the gamma function in the numerator. By Theorem 3.23, poles of $K(\alpha, P_a)$ appear on semi non-resonant $-\alpha$'s. To find the semi non-resonant locus, we need to find the $-\Sigma$ translates of the facets. By description of the facets in Theorem 4.12, $-\alpha$ is semi non-resonant iff

$$\langle \vec{\Gamma} - e_0, \alpha \rangle \in \mathbb{Z}_{\geq 0}$$

or

$$\langle \vec{\gamma}, \alpha \rangle \in \mathbb{Z}_{\leq 0}$$

for a 2-connected subgraph γ. However, the poles of the gamma function at negative integers cancel the poles coming from the first equation. The second equation for $\alpha = (D/2, 1, \cdots, 1)$ is equivalent to

$$-D/2\ell_\gamma + |E(\gamma)| \in \mathbb{Z}_{\leq 0}.$$

Thus, the first part of the theorem follows. The second part is a special case of Theorem 4.15. □

Remark 4.16. *To find the ϵ expansion of the integral, using the relations in Lemma 3.22, we can replace the differential form*

$$\frac{t^{\tilde{\alpha}}}{(\sum_a P_a t^a)^{\alpha_0+\epsilon}} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}$$

with differential forms that have logarithmic poles along the boundary, since division by zero appears when we want to replace α on a facet with a linear combination of points that are not on the facet. The boundary components correspond to the products of subgraphs and quotient graphs. Based on this observation, we think there is a relation between the Connes-Kreimer renormalization and limiting mixed Hodge structures of [BK08].

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