ON NUMERICAL BOUNDARY CONDITIONS FOR THE NAVIER-STOKES EQUATIONS

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1986

(Submitted 27th November, 1985)

Acknowledgments

Firstly I wish to thank my advisor Heinz-Otto Kreiss for his encouragement, guidance and patience through all the phases of this thesis, and for his companionship during many happy days spent hiking in the mountains nearby. I am also grateful to the faculty, students and staff of the Department of Applied Mathematics for the friendliness, helpfulness and excellent education I have experienced here. Special thanks are due to Bill Henshaw for his contributions to the work in this thesis.

Financial support has been generously provided by Caltech from numerous sources — teaching assistantships, an Institute fellowship and research assistantships under Office of Naval Research contract N00014-83-K-0422 and National Science Foundation grants ATM-8201207 and DMS-8312264. Computer time has been provided on the Applied Mathematics Vax 11/750 and the Applied Mathematics IBM 4341. This thesis was typeset in TeX on the IBM 4341 and printed on the HP TeX system.

I wish to express my gratitude to my family for their love and their encouragement in this venture. To my mate Celia great thanks are due for her loving support, tender care and great patience. Finally I thank God for His gracious gifts which have enabled me to accomplish this work.

Abstract

Part I:

We consider the numerical solution of the Navier-Stokes equations governing the unsteady flow of a viscous incompressible fluid. The analysis of numerical approximations to smooth nonlinear problems reduces to the examination of related linearized problems. The well posedness of the linear Navier-Stokes equations and the stability of finite difference approximations are studied by making energy estimates for the initial boundary value problems. Flows with open boundaries (i.e., inflow and outflow) and with solid walls are considered. We analyse boundary conditions of several types involving the velocity components or a combination of the velocity components and the pressure. The properties of these different types of boundary conditions are compared with emphasis on the suppression of undesirable numerical boundary layers for high Reynolds number calculations. The formulation of the Navier-Stokes equations which uses an elliptic equation for the pressure in lieu of the divergence equation for the velocity is shown to be equivalent to the usual formulation if the boundary conditions are treated correctly. The stability of numerical methods which use this formulation is demonstrated.

Part II:

We consider the numerical solution of the stream function vorticity formulation of the two dimensional incompressible Navier-Stokes equations for unsteady flows on a domain with rigid walls. The no-slip boundary conditions on the velocity components at the rigid walls are prescribed. In the stream function vorticity formulation these become two boundary conditions on the stream function and there is no explicit boundary condition on the vorticity. The accuracy of the numerical approximations to the stream function and the vorticity is investigated. The common approach in calculations is to employ second order accurate finite difference approximations for all the space derivatives and the boundary conditions together with a time marching procedure involving iteration at each time step to satisfy the boundary conditions. With such schemes the vorticity may be only first order accurate. Higher order approximations to the no-slip boundary conditions have frequently been used to overcome this problem. A one dimensional initial boundary value problem containing the salient features is proposed and analysed. With the use of this model, the behaviour observed in calculations is explained.

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Part I: Boundary Conditions for the Primitive Variable Formulation

CHAPTER 1

Introduction

The Navier-Stokes equations for a uniform viscous incompressible fluid in two space dimensions can be written in the form

$$\begin{aligned} u_t + uu_x + vu_y + p_x &= \nu \Delta u + f_1, \\ v_t + uv_x + vv_y + p_y &= \nu \Delta v + f_2, \\ u_x + v_y &= 0. \end{aligned} \tag{1.0.1}$$

Here u, v are the velocity components, p is the pressure, and the constant density has been normalized to $\rho = 1$. The external forcing has components f, g and ν is the viscosity coefficient. In addition, initial and boundary conditions are required to complete the specification of the problem. Initial conditions are needed for the velocity components

$$u(x, y, 0) = U^{(0)}(x, y), \qquad v(x, y, 0) = V^{(0)}(x, y). \tag{1.0.2}$$

Boundary conditions can be written in the general form

$$B(u, v, p) = g \tag{1.0.3}$$

for $(x, y) \in \partial\Omega$. These equations describe the flow in some region Ω of R^2 for some time interval $0 \leq t \leq T$.

These equations are important in the modelling of a large number of physical phenomena in fluid dynamics where the effects of compressibility are negligible. Such situations arise widely in scientific and engineering applications, including flows of water and of air at low speeds. Examples are the flow over bodies, flows in pipes and channels, the modelling of turbulence, shear layers, jets, boundary layers, etc. The conditions under which a fluid is almost incompressible are discussed by Batchelor [1967].

The study of the Navier-Stokes equations has traditionally (always) involved using approximate methods of some sort since the number of exact solutions available is very limited. The assumption of incompressibility is itself an approximation. Some other approximations include deriving the boundary layer equations, dropping the viscous terms in parts of the flow and using perturbation methods.

Numerical methods provide yet another powerful tool, and one which can be used in conjunction with other approximation techniques to increase their power. In addition, numerical methods can be applied directly to the complete Navier-Stokes equations in situations where no other approximations are appropriate or to examine the validity of other approximations. We shall confine ourselves here to consideration of approaches to solving the complete equations.

With numerical methods as with any other approximation methods the question must be asked: How does the numerical solution relate to the true solution of the equations? This is the basic question of numerical analysis, and it merits careful consideration. One may ask this question about the results of a particular calculation or about a class of numerical methods for a particular problem. In either case it is desirable to have estimates for the difference between the numerical and the true solution, either *a priori* or *a posteriori* estimates. *A priori* estimates are estimates in terms of the data. In this case the data consist of the forcing functions, the initial conditions, the boundary data and the viscosity coefficient or Reynolds number. *A posteriori* estimates may also be in terms of the computed solution.

The concepts of *convergence*, *stability* and *order of accuracy* of difference approximations form the basis of the theory which has been developed for analysing numerical methods. Another important concept is the *well posedness* of partial differential equations. Equally important is the *principle of linearization* for smooth nonlinear problems, which enables the question of convergence to be reduced to the well posedness of linear problems and the stability of difference approximations.

The first investigations of numerical analysis concentrated on the Cauchy problem, that is, the initial value problem in the absence of boundaries. As the understanding of the properties of approximations to the Cauchy problem became well advanced, attention was directed to the initial boundary value problem. The theory of initial boundary value problems for hyperbolic and parabolic systems of partial differential equations is by now fairly complete. The Navier-Stokes equations share many features in common with hyperbolic and parabolic systems. However because of the incompressibility condition the equations are not covered by the general theory. Incompressibility introduces an elliptic nature into the equations.

It is the aim of the present work to investigate the choice of boundary conditions for the incompressible Navier-Stokes equations. Flows with open boundaries (i.e. inflow and outflow) and with solid walls are considered. Several different types of boundary conditions are studied, involving the velocity components or a combination of the velocity components and the pressure. Previous studies of viscous incompressible flow have only considered either periodic boundary conditions or boundary conditions on the velocity. We are also interested in comparing the properties of the boundary conditions for high Reynolds number calculations.

Synopsis

In the remainder of this chapter we review the framework used to analyse numerical methods for nonlinear initial boundary value problems. We discuss well posedness, stability, the principle of linearization and the reduction to half plane problems. We also consider the pressure formulation of the Navier-Stokes equations which explicitly involves the elliptic nature of the pressure.

In chapter 2 we consider the well posedness of the half plane problem for the linear Navier-Stokes equations. We study four different choices for the boundary conditions. Energy estimates are made for the velocity components and the pressure for each type of boundary condition.

In chapter 3 we investigate the stability of discretization in space. A staggered grid is used for the discretization because it is well suited to making estimates of the same form as in the continuous case. Discrete boundary conditions of each of the four types are chosen such that the semi-discrete velocity components can be estimated.

In chapter 4 we consider the stability of the fully discrete linear Navier-Stokes equations. The emphasis is on how the boundary conditions are effected when time is discretized. Two diffferent time differencing methods are studied, the Crank-Nicholson method and a combination of leap frog for the convective terms and the pressure with Crank-Nicholson for the viscous diffusion terms. In each case appropriate boundary conditions are derived and stability is demonstrated.

1.1 Well Posedness

The well posedness of a problem refers to the dependence of the solution on the data of the problem. A well posed problem is one for which the solution satisfies certain *a priori* estimates.

To discuss these estimates we introduce a general notation for initial boundary value problems. An initial boundary value problem consists of a partial differential equation, initial conditions and boundary conditions

$$\begin{aligned} L\mathbf{u} &= \mathbf{f} && \text{for } (\mathbf{x},t) \text{ in } \Omega \times [0,T], \\ \mathbf{u}(\mathbf{x},0) &= \mathbf{U}^{(0)}(\mathbf{x}) && \text{for } \mathbf{x} \in \Omega, \\ B\mathbf{u} &= \mathbf{g} && \text{for } \mathbf{x} \in \partial\Omega,\ 0 \le t \le T. \end{aligned} \tag{1.1.1}$$

The estimates for well posedness are typically of the form

$$\|\mathbf{u}\|_{\Omega\times[0,T]} + \eta\|\mathbf{u}\|_{\partial\Omega\times[0,T]} \le K\left(\|\mathbf{f}\|_{\Omega\times[0,T]} + \|\mathbf{g}\|_{\partial\Omega\times[0,T]} + \|\mathbf{U}^{(0)}\|_{\Omega}\right) \tag{1.1.2}$$

with appropriately defined L^2 norms over the regions indicated. K is a constant and $\eta = 1$ (strong estimate) or $\eta = 0$ (weak estimate).

These estimates are of importance for a number of reasons. They can be used to prove the existence and uniqueness of solutions to a problem. Questions of existence and uniqueness are clearly important for numerical approximations as one can only hope to accurately approximate a problem which has reasonable solutions. Also the concepts of well posedness for partial differential equations and stability for difference approximations are intimately related — the definitions involved are analogous and the methods of determining well posedness can be applied in their discrete forms to stability calculations.

Existence and uniqueness questions for the Navier-Stokes equations are treated by Ladyzhenskaya [1969]. Her treatment applies to flows on a bounded domain with homogeneous boundary conditions on the velocity. In two space dimensions a unique smooth solution is shown to exist for all time. I am not aware of any existence and uniqueness results for open boundaries. In three space dimensions the estimates only suffice to prove the existence of the solution for a finite time T which depends on the initial data and the external forces. The existence of the solution in three dimensions for all time remains an open qusestion.

The general theory of well posedness for linear systems of hyperbolic initial boundary value problems was elucidated in the works of Kreiss [1970] and Majda and Osher [1975] and for incompletely parabolic systems by Strikwerda [1977]. The theory applies to the initial boundary value problem with variable coefficients and lower order terms on a general domain Ω with smooth boundary. The results of these papers reduce the question of well posedness to the well posedness of families of half plane problems with constant coefficients. This reduction is sometimes called *freezing coefficients* or *localization.*

It is the mathematical theory of pseudo-differential operators which undergirds the freezing of coefficients. The application of this theory involves the construction of a resolvent function for each of the frozen coefficient problems. It is these constructions which are demonstrated in the referenced works for the problems studied. The existing theory covers hyperbolic problems with a smooth non-characteristic boundary (Kreiss [1970]) or a uniformly characteristic boundary (Majda and Osher [1975]), and the extensions of Strikwerda [1977] cover viscous diffusion terms of the type which appear in the Navier-Stokes equations. The theory has some limitations which should be noted. The existing theory does not apply at nonuniformly characteristic points on the boundary or at points where the boundary is not smooth. We are interested in flows on regions with open boundaries such as flow through channels. In these applications the boundary can be divided into walls (which are uniformly characteristic) and sections of inflow and outflow. Wherever a wall meets an inflow or outflow section of the boundary or along an open boundary where the flow reverses direction the general theory does not apply. For the incompressible Navier-Stokes equations no resolvent construction has been made and so the incomplete justification of the technique of freezing coefficients in this application should be borne in mind.

The two methods for establishing estimates for well posedness are the *energy method* and the *normal mode analysis.* The classical energy method involves taking inner products, integrating by parts and bounding the terms which result.

For the energy method the freezing of coefficients is only a convenient device for simplifying the calculations required. If desired, the variable coefficients and lower order terms can be retained and demonstrated to be dominated by the principal terms. The limitation with the energy method is that it gives sufficient conditions for well posedness but it does not give necessary conditions. The Kreiss normal mode analysis for initial boundary value problems involves estimating the solution obtained by taking the Laplace transform in time. The normal mode analysis is applied to constant coefficient half plane problems and hence relies critically on the technique of freezing coefficients for its extension to more general problems. For the classes of problems where freezing coefficients is justified, the normal mode analysis gives necessary and sufficient conditions for well posedness.

Aspects of initial boundary value problems in fluid mechanics are studied by Oliger and Sundström [1978], Gustafsson and Sundström [1978] and Gustafsson and Kreiss [1983].

The Euler equations of inviscid compressible flow are a hyperbolic system, and with the usual viscous terms added they form an incompletely parabolic system of two parabolic momentum equations coupled to a hyperbolic equation for the pressure. The incompressible Navier-Stokes equations are the limiting equations which are obtained from the equations of viscous compressible flow in the limit as the Mach number of the flow tends to zero. These systems are quasilinear with the nonlinearity entering through the convective terms.

The relation between compressible and incompressible flow has received attention. The incompressible limiting process has been studied in the inviscid case by Kreiss et al. (Kreiss [1980], Browning and Kreiss [1982], Gustafsson and Kreiss [1983]), including the effects of boundaries, and in the inviscid case with periodic boundary conditions by Klainerman and Majda [1981]. Basically the limit is well behaved provided the fast sound waves are suppressed by proper initialization and choice of boundary conditions. The presence of viscous terms in the momentum equations improves the energy estimates for this limit and otherwise does not alter the analysis except with regard to the boundary conditions.

The incompressible equations are no longer hyperbolic or incompletely parabolic. The two characteristic or subcharacteristic speeds associated with the sound waves become infinite in this limit. The mass conservation equation, which in the compressible equations provides an evolution equation for the pressure, reduces to the divergence equation which does not include the pressure and involves only derivatives in space. For this reason the incompressible equations are sometimes described as time singular.

1.2 Stability of Difference Approximations

The concept of *stability* for difference approximations is the discrete analogue of well posedness for differential equations. A difference approximation is stable if it satisfies an estimate of the form (1.1.2) with appropriate discrete norms.

The general theory of stability of finite difference approximations for hyperbolic initial boundary value problems is given by Gustafsson, Kreiss and Sundström [1972]. Extensions of this theory have been made by Michelson [1983], who develops the theory for more than one space dimension, and by Strikwerda [1980], who gives the theory for the method of lines (i.e. finite difference approximation in space with time remaining continuous).

For the equations of fluid dynamics there have been studies made for inviscid compressible flow (i.e., the Euler equations of gas dynamics) by Gustafsson and Oliger [1982] and for.inviscid incompressible flow by Gustafsson and Kreiss [1983] and Guerra and Gustafsson [preprint].

For the incompressible Navier-Stokes equations Chorin [1969] showed the convergence of a finite difference scheme (the Projection method) for the Cauchy problem. Porsching [1978] demonstrated the convergence of a modification of the MAC method for the linearized Navier-Stokes equations with boundary conditions on the velocity. Porsching's results show convergence for the velocity components but they say nothing about the behaviour of the pressure field.

1.3 Principle of Linearization

The principle of linearization is fundamental to the numerical analysis of nonlinear problems with smooth solutions. The relation between stability, consistency and convergence for nonlinear problems is not so trivial as for linear problems. For linear problems

$$\text{consistency} \ + \ \text{stability} \ \Rightarrow \ \text{convergence},$$

which is often referred to as the Lax equivalence theorem.

The *principle of linearization* for nonlinear problems with smooth solutions states that the stability of linearized problems, together with consistency, is enough to ensure convergence. This is the case because the error of the discrete solution has an asymptotic expansion in terms of the smooth solution of the nonlinear problem and the solutions of appropriate linearized variational problems. The asymptotic expansion is valid if the remainder term in the expansion has the right bound, and this is so provided the linearization of the difference approximation is stable. The error tends to zero as the approximation is refined (i.e., the approximation is convergent) provided the terms in the expansion are bounded. This is the case if the linearized variational problems are well posed.

Consider the nonlinear problem

$$\mathbf{F}\,\mathbf{u} = 0 \tag{1.3.1}$$

where $\mathbf{F}$ is a differential operator and $\mathbf{u}$ is in an appropriate space of functions. The Navier-Stokes equations including their initial and boundary conditions are of this form. Assume that the operator $\mathbf{F}$ is a smooth function of $\mathbf{u}$. Let $\mathbf{A}(\mathbf{w})$ denote the Jacobian of $\mathbf{F}$ with respect to $\mathbf{u}$ at $\mathbf{u} = \mathbf{w}$. Assume that the solution $\mathbf{u}$ of (1.3.1) exists and is smooth. The linearized variational problem of (1.3.1) is

$$\mathbf{A}(\mathbf{u})\mathbf{u}' = 0. \tag{1.3.2}$$

For the Navier-Stokes equations let $(\overline{u}, \overline{v}, \overline{p})$ be the smooth solution. Then the Jacobian is the linearized operator obtained by letting $(u, v, p) = (\overline{u}, \overline{v}, \overline{p}) + (u', v', p')$ in (1.0.1) and linearizing in the primed variables. Variations of the forcing functions can also be taken. The resulting linearized variational problem is

$$\begin{aligned} u'_t + \overline{u}u'_x + \overline{v}u'_y + \overline{u}_x u' + \overline{v}_y u' + p'_x &= \nu \Delta u' + f'_1, \\ v'_t + \overline{u}v'_x + \overline{v}v'_y + \overline{u}_x v' + \overline{v}_y v' + p'_y &= \nu \Delta v' + f'_2, \\ u'_x + v'_y &= 0 \end{aligned} \tag{1.3.3}$$

together with variational forms of the initial and boundary conditions. This system is called the linearized Navier-Stokes equations.

A finite difference approximation for (1.3.1) can be written as

$$\mathbf{F_h v_h} = 0 \tag{1.3.4}$$

where $\mathbf{F_h}$ is a finite difference operator and $\mathbf{v_h}$ is in an appropriate space of discrete functions. Let $\mathbf{A_h}(\mathbf{w_h})$ denote the Jacobian of $\mathbf{F_h}$ with respect to $\mathbf{v_h}$ at $\mathbf{v_h} = \mathbf{w_h}$.

The *truncation error* of the difference approximation is

$$\mathbf{r_h} := \mathbf{F_h u_h} \tag{1.3.5}$$

where $\mathbf{u_h}$ denotes the restriction of $\mathbf{u}$ to the grid. The difference approximation is *consistent* if $\|\mathbf{r_h}\| \to 0$ as $\mathbf{h} \to 0$ where $\mathbf{h}$ is a measure of the grid spacing or refinement of the approximation and an appropriate norm is used. For finite differences the truncation error can usually be expressed in terms of $\mathbf{u}$ and its derivatives evaluated on the grid. We assume that the truncation error can be expanded as

$$\mathbf{r} = \mathbf{h}^q \Psi = \mathbf{h}^q \Psi_q + \mathbf{h}^{q+1} \Psi_{q+1} + \ldots + \mathbf{h}^p \Psi_p + \mathbf{h}^{p+1} \mathbf{R}_{p+1}. \tag{1.3.6}$$

The normalized error is defined by

$$\mathbf{e_h} := (\mathbf{u_h} - \mathbf{v_h}) / \mathbf{h}^q; \tag{1.3.7}$$

i.e.,

$$\mathbf{u_h} = \mathbf{v_h} + \mathbf{h}^q \mathbf{e_h}.$$

Then

$$\begin{aligned} 0 &= \mathbf{F_h v_h} = \mathbf{F_h}(\mathbf{u_h} - \mathbf{h}^q \mathbf{e_h}) \\ &= \mathbf{F_h u_h} - \mathbf{h}^q \mathbf{A_h}(\mathbf{u_h}) \mathbf{e_h} - \Phi(\mathbf{u_h}; \mathbf{h}^q \mathbf{e_h}) \end{aligned} \tag{1.3.8}$$

where the remainder is a nonlinear function given by

$$\begin{aligned} \Phi(\mathbf{u_h}; \mathbf{h}^q \mathbf{e_h}) &= \int_0^1 \left[\mathbf{A_h}(\mathbf{u_h} - \tau \mathbf{h}^q \mathbf{e_h}) - \mathbf{A_h}(\mathbf{u_h}) \right] \mathbf{h}^q \mathbf{e_h} \, d\tau \\ &= \int_0^1 \int_0^1 d^2 \mathbf{F_h}(\mathbf{u_h} - \sigma \tau \mathbf{h}^q \mathbf{e_h}; -\tau \mathbf{h}^q \mathbf{e_h}, \mathbf{h}^q \mathbf{e_h}) \, d\sigma \, d\tau \\ &= -h^{2q} \int_0^1 \int_0^1 \tau d^2 \mathbf{F_h}(\mathbf{u_h} - \sigma \tau \mathbf{h}^q \mathbf{e_h}; \mathbf{e_h}, \mathbf{e_h}) \, d\sigma \, d\tau \end{aligned} \tag{1.3.9}$$

where $d^2\mathbf{F_h}$ is the second Fréchet derivative of $\mathbf{F_h}$. The remainder term can be estimated by

$$\| \Phi(\mathbf{u_h}; \mathbf{h}^q \mathbf{e_h}) \| \leq \text{const}\, \mathbf{h}^{2q} \, \| \mathbf{e_h} \|^2$$

for $\mathbf{h}$ sufficiently small provided $\mathbf{F_h}$ is a C^2 function. Scale the remainder by letting

$$\Phi(\mathbf{u_h}; \mathbf{h}^q \mathbf{e_h}) = \mathbf{h}^{2q} \mathbf{H}(\mathbf{u_h}, \mathbf{h}^q; \mathbf{e_h}) = \mathbf{h}^{2q} \mathbf{H}(\mathbf{e_h}).$$

The error equation (1.3.8) can be written as

$$\mathbf{A_h}(\mathbf{u_h}) \mathbf{e_h} + \mathbf{h}^q \mathbf{H}(\mathbf{u_h}, \mathbf{h}^q; \mathbf{e_h}) = \Psi_\mathbf{h}. \tag{1.3.10}$$

It is possible to expand the error in terms of solutions to continuous linear problems. The first term is $\mathbf{w}^{(q)}$ given by

$$\mathbf{A}(\mathbf{u})\mathbf{w}^{(q)} = \Psi_q. \tag{1.3.11}$$

The linearization of the discrete approximation is also a q^{th} order approximation to the linearized continuous problem; i.e.,

$$\mathbf{A_h}(\mathbf{u_h})\mathbf{w_h}^{(q)} = \left[\mathbf{A}(\mathbf{u})\mathbf{w}^{(q)}\right]_{\mathbf{h}} + O(\mathbf{h}^q) = \Psi_{q\mathbf{h}} + \mathbf{h}^q\widetilde{\Psi}_{\mathbf{h}}$$

where $\widetilde{\Psi}$ is a function of $\mathbf{u}$ and $\mathbf{w}^{(q)}$. The remaining part of the error can be written as

$$\mathbf{e_h} = \mathbf{w_h}^{(q)} + \mathbf{h}\mathbf{e_h}^{(q+1)}.$$

The equation for $\mathbf{e_h}^{(q+1)}$ is found as follows:

$$\begin{aligned}
0 = \mathbf{F_h}\mathbf{v_h} &= \mathbf{F_h}(\mathbf{u_h} - \mathbf{h}^q\mathbf{w_h}^{(q)} - \mathbf{h}^{q+1}\mathbf{e_h}^{(q+1)}) \\
&= \mathbf{F_h}(\mathbf{u_h} - \mathbf{h}^q\mathbf{w_h}^{(q)}) - \mathbf{h}^{q+1}\mathbf{A_h}(\mathbf{u_h} - \mathbf{h}^q\mathbf{w_h}^{(q)})\mathbf{e_h}^{(q+1)} \\
&\quad + \Phi(\mathbf{u_h} - \mathbf{h}^q\mathbf{w_h}^{(q)}; \mathbf{h}^{q+1}\mathbf{e_h}^{(q+1)}) \\
&= \mathbf{F_h}\mathbf{u_h} - \mathbf{h}^q\mathbf{A_h}(\mathbf{u_h})\mathbf{w_h}^{(q)} - \Phi(\mathbf{u_h}; \mathbf{h}^q\mathbf{w_h}^{(q)}) \\
&\quad - \mathbf{h}^{q+1}\mathbf{A_h}(\mathbf{u_h} - \mathbf{h}^q\mathbf{w_h}^{(q)})\mathbf{e_h}^{(q+1)} - \Phi(\mathbf{u_h} - \mathbf{h}^q\mathbf{w_h}^{(q)}; \mathbf{h}^{q+1}\mathbf{e_h}^{(q+1)}),
\end{aligned}$$

which can be written as

$$\begin{aligned}
\mathbf{A_h}(\mathbf{u_h} - \mathbf{h}^q\mathbf{w_h}^{(q)})\mathbf{e_h}^{(q+1)} &+ \mathbf{h}^{q+1}\mathbf{H}(\mathbf{u_h} - \mathbf{h}^q\mathbf{w_h}^{(q)}, \mathbf{h}^{q+1}; \mathbf{e_h}^{(q+1)}) \\
&= \frac{1}{\mathbf{h}}\left[\Psi_{\mathbf{h}} - \mathbf{A_h}(\mathbf{u_h})\mathbf{w_h}^{(q)}\right] - \mathbf{h}^{q-1}\mathbf{H}(\mathbf{u_h}, \mathbf{h}^q; \mathbf{w_h}^{(q)}).
\end{aligned}$$

Hence

$$\mathbf{A_h}(\mathbf{u_h} - \mathbf{h}^q\mathbf{w_h}^{(q)})\mathbf{e_h}^{(q+1)} + \mathbf{h}^{q+1}\mathbf{H}^{(q+1)}(\mathbf{e_h}^{(q+1)}) = \Psi_{\mathbf{h}}^{(q+1)}, \tag{1.3.12}$$

where $\mathbf{H}^{(q+1)}$ is a nonlinear function of $\mathbf{e_h}^{(q+1)}$ and both $\mathbf{H}^{(q+1)}$ and $\Psi^{(q+1)}$ depend on $\mathbf{u}$ and $\mathbf{w}^{(q)}$.

This process can be continued to produce an asymptotic expansion

$$\mathbf{u_h} = \mathbf{v_h} + \mathbf{h}^q\mathbf{w_h}^{(q)} + \mathbf{h}^{q+1}\mathbf{w_h}^{(q+1)} + \ldots + \mathbf{h}^p\mathbf{w_h}^{(p)} + \mathbf{h}^{p+1}\mathbf{e_h}^{(p+1)}, \tag{1.3.13}$$

where the $\mathbf{w}^{(j)}$ are solutions of the continuous linearized equations

$$\begin{aligned} \mathbf{A}(\mathbf{u}^{(j-1)})\mathbf{w}^{(j)} &= \Psi_q^{(j)} \\ \mathbf{u}^{(j)} := \mathbf{u} - \mathbf{h}^q\mathbf{w}^{(q)} &- \mathbf{h}^{q+1}\mathbf{w}^{(q+1)} - \mathbf{h}^j\mathbf{w}^{(j)} \end{aligned} \tag{1.3.14}$$

with different forcing functions $\Psi^{(j)}$. The remainder $\mathbf{e}_{\mathbf{h}}^{(p+1)}$ satisfies the nonlinear difference equation

$$\mathbf{A}_{\mathbf{h}}(\mathbf{u}_{\mathbf{h}}^{(p)})\mathbf{e}_{\mathbf{h}}^{(p+1)} + \mathbf{h}^{p+1}\mathbf{H}^{(p+1)}(\mathbf{e}_{\mathbf{h}}^{(p+1)}) = \Psi - \mathbf{h}^{(p+1)}. \tag{1.3.15}$$

Now assume that the linearized difference approximation is *stable* in a neighbourhood of $\mathbf{u}_{\mathbf{h}}$; i.e., there exist constants K_1, ρ_1 such that for any $\mathbf{z}$ the solution of

$$\mathbf{A}_{\mathbf{h}}(\tilde{\mathbf{u}}_{\mathbf{h}})\mathbf{w}_h = \mathbf{z}$$

satisfies

$$\|\mathbf{w}_{\mathbf{h}}\| \leq K_1\|\mathbf{z}\| \qquad \text{for } \|\mathbf{u} - \tilde{\mathbf{u}}\| \leq \rho_1.$$

Assume also that $\mathbf{H}^{(p+1)}(\mathbf{w}_{\mathbf{h}})$ is locally *Lipschitz continuous*; i.e., there exist constants K_2, ρ_2 such that

$$\|\mathbf{H}^{(p+1)}(\mathbf{w}_{\mathbf{h}}) - \mathbf{H}^{(p+1)}(\tilde{\mathbf{w}}_{\mathbf{h}})\| \leq K_2\|\mathbf{w}_{\mathbf{h}} - \tilde{\mathbf{w}}_{\mathbf{h}}\| \qquad \text{for } \|\mathbf{w} - \tilde{\mathbf{w}}\| \leq \rho_2.$$

Then the iteration

$$\begin{aligned} \tilde{\mathbf{e}}^{(0)} &= 0, \\ \mathbf{A}_{\mathbf{h}}(\mathbf{u}_{\mathbf{h}}^{(p)})\tilde{\mathbf{e}}_{\mathbf{h}}^{(n+1)} + \mathbf{h}^{p+1}\mathbf{H}^{(p+1)}(\tilde{\mathbf{e}}_{\mathbf{h}}^{(n)}) &= \Psi^{(p+1)} \end{aligned} \tag{1.3.16}$$

converges to the solution $\mathbf{e}^{(p+1)}$ of (1.3.15) and

$$\|\mathbf{e}^{(p+1)}\| \leq \text{const}\, \|\Psi^{(p+1)}\|,$$

provided $\mathbf{h}$ is sufficiently small.

1.4 Pressure Formulation

One way to view the incompressible Navier-Stokes equations is as a pair of evolution equations for the velocity with the pressure adjusting itself continuously to ensure that the divergence of the velocity field remains zero. From this point of view it is useful to derive an additional elliptic equation for the pressure. Taking the divergence of the momentum equations in (1.0.1) gives

$$\Delta p + R + \delta_t + u\delta_x + v\delta_y = \nu\Delta\delta \tag{1.4.1}$$

where

$$R := u_x^2 + 2u_y v_x + v_x^2 - (f_{1x} + f_{2y}) \tag{1.4.2}$$

and

$$\delta := u_x + v_y. \tag{1.4.3}$$

Using the divergence equation this equation simplifies to

$$\Delta p + R = 0, \tag{1.4.4}$$

which is called the elliptic equation for the pressure.

Now consider the system of equations consisting of the evolution equations for the velocity components coupled with the elliptic equation for the pressure. We shall call this system the *pressure formulation* of the incompressible Navier-Stokes equations. The usual formulation (1.0.1) of momentum equations and divergence equation will be called the *divergence formulation.*

Our motivation for studying this question is an interest in methods for the numerical solution of the unsteady Navier-Stokes equations. We consider methods which are based on the pressure formulation such as the MAC method of Harlow and Welch [1965]. In these methods advancement from one time level to the next is achieved by splitting into two steps. The first step is to advance the velocities by considering the pressure terms as forcing (i.e. explicitly) in the momentum equations using an explicit or implicit scheme. The second step is to solve the elliptic equation for the pressure at the new time level using the new velocities to calculate the terms in R.

The first question which we address is the following: Under what conditions is the pressure formulation of the incompressible Navier-Stokes equations equivalent to the divergence formulation? To pose this question properly, initial and boundary conditions must also be specified. Assume that these are such that the divergence formulation has a unique smooth solution for $(x, y) \in \Omega$, $0 \leq t \leq T$.

We shall see that the pressure formulation requires an additional boundary condition in order to be specified properly. This is to be expected since the pressure formulation was obtained by differentiating in space and so the order of the equations is higher. The extra boundary condition is required for the divergence — otherwise the divergence satisfies an evolution equation with no boundary conditions and hence is not uniquely determined. It is also clear that the extra boundary condition must be chosen to ensure that the divergence is identically zero, and that this is both necessary and sufficient for the pressure formulation and the divergence formulation to be equivalent.

The equation for the divergence in the pressure formulation is found by taking the divergence of the momentum equations and making use of the pressure equation. The result is

$$\delta_t + u\delta_x + v\delta_y = \nu \Delta \delta. \tag{1.4.5}$$

This equation can be used to obtain an energy estimate for δ,

$$\begin{aligned} \frac{d}{dt}\|\delta\|^2 + 2\nu\|\nabla\delta\|^2 &= -\|\delta\|^2 + (\delta, \delta^2) \\ &\quad + \int_{\partial\Omega} \left[u_n \delta^2 - 2\nu\delta \frac{\partial \delta}{\partial n} \right] ds. \end{aligned} \tag{1.4.6}$$

Suppose the initial data are divergence free, and the boundary terms are non-positive; then it follows that $\|\delta\| \equiv 0$; i.e., $\delta(x, y, t) = 0$ almost everywhere in Ω for all time.

From this we can derive pointwise boundary conditions on the divergence by requiring that

$$H(\delta, \frac{\partial \delta}{\partial n}) = u_n \delta^2 - 2\nu\delta \frac{\partial \delta}{\partial n} \leq 0 \qquad (1.4.7)$$

for $(x, y) \in \partial\Omega$, where u_n is the inward normal component of the velocity on the boundary and $\partial/\partial n$ is the normal derivative. H is a quadratic form in δ, $\partial\delta/\partial n$. The simplest type of boundary condition which will assure that (1.4.7) is satisfied is the Dirichlet condition $\delta = 0$. At an outflow or solid wall boundary (i.e., where $u_n \leq 0$) the Neumann boundary condition $\partial\delta/\partial n = 0$ will also suffice. More general mixed boundary conditions involving a linear combination of δ and $\partial\delta/\partial n$ are also possible.

The conclusion is that the two formulations of the Navier-Stokes equations are equivalent provided the boundary conditions are treated correctly.

This equivalence between the two formulations provides justification for numerical methods which use the pressure formulation. By the principle of linearization it implies that difference approximations for the pressure formulation which are consistent and linearly stable are also convergent to the divergence free solution of the Navier-Stokes equations. We shall investigate well posedness and stability questions for the linearized Navier-Stokes — both the divergence formulation and the pressure formulation — in subsequent chapters.

1.5 Numerical Methods

Numerical methods for the Navier-Stokes equations are discussed at length in Peyret and Taylor [1983] and Roache [1972]. For the incompressible Navier-Stokes equations the considerations centre around the special nature of the constraint on the velocity field posed by the divergence equation.

One way to guarantee zero divergence is to use the stream function vorticity formulation. Aspects of numerical methods for the stream function vorticity equations are discussed elsewhere in this thesis.

For the primitive variable formulation it is necessary to devise methods to advance the solution in time whilst preserving zero divergence from one time step to the next. This can be achieved by a splitting process using the elliptic nature of the pressure. The splitting can be constructed in more than one way. The best known methods are the MAC method (Harlow and Welch [1965]) and the projection method (Chorin [1968]) and derivatives of these. Peyret and Taylor [1983] discuss the relationship between the two methods — the two methods are essentially equivalent.

In the projection method the first step is to advance the velocity taking into account the convective and the viscous terms with the pressure terms omitted. The second step is to use the pressure to project the velocity onto the subspace of divergence free velocity fields. This is done by solving an elliptic equation for the pressure and then calculating the velocity.

In the MAC method the first step is to calculate the pressure by solving an elliptic equation. The forcing in the equation for the pressure is calculated in such a way that the divergence of the velocity field is zero at the next time step. The second step is to advance the velocity field.

Different variants of these methods can be devised by changing the schemes used to advance the velocity components and by treating the boundary conditions in various ways.

Other approaches to solving the incompressible Navier-Stokes equations involve adding a small amount of compressibility. This may be an artificial compressibility (see Peyret and Taylor [1983] for references) or the equations of slightly compressible flow can be solved as discussed by Guerra and Gustafsson [preprints] for the incompressible Euler equations.

CHAPTER 2

Linear Navier-Stokes Equations

2.1 Introduction

The linear incompressible Navier-Stokes equations in two space dimensions are

$$\begin{aligned} u_t + cu_x + du_y + p_x &= \nu \Delta u + f_1, \\ v_t + cv_x + dv_y + p_y &= \nu \Delta v + f_2, \\ u_x + v_y &= 0. \end{aligned} \tag{2.1.1}$$

Here u, v are the velocity components and p is the pressure. The coefficients c and d of the convective terms are taken to be smooth bounded functions of x, y and t. The external forcing has components f_1, f_2 and ν is the viscosity coefficient. Consider this problem on the strip $x \geq 0$, $0 \leq y \leq 1$ with periodic boundary conditions in y and $t \geq 0$. In addition, initial and boundary conditions are required to complete the specification of the problem. Initial conditions are needed for the velocity components

$$u(x,y,0) = U^{(0)}(x,y), \qquad v(x,y,0) = V^{(0)}(x,y). \tag{2.1.2}$$

Boundary conditions are needed at $x = 0$ and $x = \infty$. We assume that u and v smoothly and rapidly approach zero as $x \to \infty$ and that p approaches some constant which can also be taken as zero without loss of generality. These boundary conditions are chosen since it is the local influence of the boundary at $x = 0$

which is of interest to us. The boundary conditions at $x = 0$ are linear and can be written in the general form

$$B(u, v, p) = g. \tag{2.1.3}$$

We shall discuss the specific forms of the boundary conditions in due course.

The pressure formulation of the linear Navier-Stokes equations consists of the momentum equations with the pressure equation

$$\Delta p + R = 0 \tag{2.1.4}$$

where

$$R := c_x u_x + d_x u_y + c_y v_x + d_y v_y - (f_{1x} + f_{2y}). \tag{2.1.5}$$

Different forms of R are also possible. The pressure equation is obtained by taking the divergence of the momentum equations and eliminating terms involving the divergence $u_x + v_y$ and its derivatives. Any function of $u_x + v_y$ can be added to R. In numerical methods this may be a desirable thing to achieve a damping effect on the non-zero divergence.

The boundary conditions which we study are of several types. All of them are local in the sense that they involve the dependent variables at a single point on the boundary. They are linear since we have linearized about a smooth solution of the Navier-Stokes equations. The boundary conditions are applied on the boundary $x = 0,\ 0 \leq y < 1$. The sign of c determines whether the boundary is inflow $(c > 0)$, outflow $(c < 0)$ or a solid wall $(c = 0)$. There are four types of boundary conditions which we categorize as follows.

I.	Velocity	$u = 0$	$v = 0$	$u_x = 0$	$(\delta = 0)$	(2.1.6)
II.	Inflow	$p + \gamma u = 0$	$v = 0$	$u_x = 0$	$(\delta = 0)$	(2.1.7)
III.	Outflow	$p + \gamma u - \nu u_x = 0$	$v_x = 0$	$u_{xx} = 0$	$(\delta_x = 0)$	(2.1.8)

IV. Outflow $\quad p = 0 \quad v_x = 0 \quad u_x = 0 \qquad$ (2.1.9)

where γ is a parameter which we shall take to be constant. These boundary conditions are written down with the pressure formulation in mind; there is some redundancy if they are used for the divergence formulation.

These are the homogeneous forms of the boundary conditions. In fact, the process of linearization leads to inhomogeneous linear boundary conditions. It is convenient for the classical energy method to remove the inhomogeneity in the boundary conditions at the outset. This is achieved by subtracting off a function satisfying the inhomogeneous boundary conditions and thereby the boundary conditions become homogeneous and the effect of the inhomogeneity appears as an extra contribution to the forcing terms. For example, consider the inhomogeneous form of the velocity boundary conditions

$$u(0,y,t) = U^{(1)}(y,t), \qquad v(0,y,t) = V^{(1)}(y,t), \qquad u_x(0,y,t) = -V_y^{(1)}(y,t). \tag{2.1.10}$$

The boundary conditions can be made homogeneous by subtracting off a smooth function which satisfies the inhomogeneous boundary conditions (2.1.10) at $x = 0$, dies off sufficiently fast as $x \to \infty$, and has zero divergence in the interior. Such a velocity field can be constructed in infinitely many ways. Let $\psi(x)$ be a C^∞ function on $[0,\infty)$ with $\psi(0) = 0$, $\psi_x(0) = 1$, $\psi_{xx}(0) = 0$ and $\psi(x) \equiv 0$ for $x \geq 1$. Then the velocity field

$$\begin{aligned} u^{(1)}(x,y,t) &:= U^{(1)}(y,t)\,\psi_x(x) - V_y^{(1)}(y,t)\,\psi(x), \\ v^{(1)}(x,y,t) &:= V^{(1)}(y,t)\,\psi_x(x) - \int_0^y U^{(1)}(y,t)\,dy\,\psi_{xx}(x) \end{aligned} \tag{2.1.11}$$

suffices. Another suitable choice for ψ with exponential type decay would be $\psi(x) = x(x+1)e^{-x}$. So we assume that the boundary conditions are homogeneous.

We define the usual inner product and norm by

$$(f,g) := \int_0^\infty \int_0^1 \overline{f}(x,y)g(x,y)\,dy\,dx, \qquad \|f\|^2 := (f,f)$$

where the bar denotes complex conjugate in this instance. Also we define

$$\mathbf{w} := \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathbf{f} := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

with

$$\|\mathbf{w}\|^2 := \|u\|^2 + \|v\|^2,$$

$$\|\nabla \mathbf{w}\|^2 := \|u_x\|^2 + \|u_y\|^2 + \|v_x\|^2 + \|v_y\|^2.$$

We shall also use norms for functions of one variable only, either x or y, which we define like those above and write as $\|\cdot\|_x$ and $\|\cdot\|_y$, respectively.

The aim is to establish the well posedness of the linearized Navier-Stokes equations with each set of boundary conditions. We derive estimates for the velocity components and the pressure in terms of the data, and also estimates for derivatives of the flow variables as well. It is important to obtain estimates for as many derivatives as possible. The principle of linearization provides error estimates for difference approximations to the nonlinear Navier-Stokes equations. In that argument the error estimates depend on bounds on a number of derivatives of the solutions of the linearized Navier-Stokes equations. Also the estimates for u, v alone give no information about the pressure field.

When it comes to making estimates of derivatives we assume that the data are smooth in all the independent variables x, y and t. Estimates for any number of y derivatives require no work since the equations can be differentiated in the y direction. The form of the equations for the y derivatives are the same and so the same type of estimates hold. Similarly, estimates for t derivatives are obtained. The only complication in this case is that smoothness of the initial data alone does not ensure that the initial time derivatives will be bounded and smooth. The determination of the initial time derivatives involves the initial conditions and the differential equations. At the boundary there are compatibility conditions between the initial and boundary data which are needed to ensure boundedness of time derivatives. The question of compatibility is taken up in detail elsewhere in this thesis for the stream function vorticity equations. In the present context the conditions for compatibility and their consequences have not been investigated.

On the other hand, it is not possible to obtain estimates for the x derivatives in the same way since the boundary conditions cannot be differentiated in the x direction (i.e., away from the boundary). Manipulation of the differential equations is required to express higher x derivatives in terms of lower derivatives. This is where the work is.

We are especially interested in choosing numerical boundary conditions which are suitable for high Reynolds number flows. This is the limit of small viscosity coefficient (i.e., ν small) in (2.1.1). Estimates which depend inversely on ν indicate the possibility that boundary layers will be present in the numerical solution even in situations where the true solution is smooth. This is undesirable and so we endeavour to obtain estimates independent of ν wherever possible.

In incompressible flows the boundary can be divided into walls and open boundaries where the fluid may flow into or out of the region. At a wall the no-slip boundary condition is a natural one on physical grounds due to viscous effects. In general this leads to the occurence of a boundary layer adjacent to the wall where the interior flow matches to this boundary condition. However, the situation is different at inflow and outflow boundaries. Since these are artificial rather than physical boundaries, there is no reason to expect a boundary layer to be present. That is, one can consider the flow as part of a flow on a larger domain.

Provided the artificial boundary does not coincide with an internal boundary layer the flow in the neighbourhood of the open boundary should be locally smooth.

Of the four types of boundary conditions only type I involves only the velocity components; the other types also use the pressure. For the primitive variable formulation this is not an important distinction *per se*. However, for the stream function vorticity formulation, the pressure does not appear in the equations and so it is not clear how a boundary condition on the pressure could be used.

The first type of boundary conditions consists of specifying the velocity components. These are the boundary conditions used in most of the incompressible flow calculations which I have seen reported, and also in the theoretical studies. More general forms of boundary conditions are commonly used for calculating compressible flows and in numerical meteorology. Reference Gustafsson and Kreiss [1983], Oliger and Sundström [1978], Gustafsson and Oliger [1982], Gustafsson and Sundström [1978].

For the type I boundary conditions we show how to derive estimates for u, v, p and any number of derivatives. The limitation is that the estimates for derivatives rely on the viscous terms and hence they become weak and allow boundary layers for high Reynolds numbers. The dependence on ν becomes stronger for higher derivatives. The estimates are valid for any type of flow at the boundary — walls ($c = 0$), inflow ($c > 0$) or outflow ($c < 0$).

The interest in studying the other types of boundary conditions is in obtaining better estimates; i.e., more estimates independent of ν.

The type II boundary conditions satisfy better estimates for an inflow boundary — estimates for all derivatives can be generated independent of ν. The type II boundary conditions are related to the type I boundary conditions; the boundary condition $p + \gamma u = 0$ becomes $u = 0$ in the limit $\gamma \to \infty$. The same estimates which are made for the type I boundary conditions for wall, inflow and outflow boundaries go through also for the type II boundary conditions.

The question of outflow boundary conditions is more delicate. This is because the number of outflow boundary conditions required for viscous incompressible flow is more than the number for inviscid flow. We expect that there are no boundary conditions which remove the possibility of an outflow boundary layer altogether. However, the possible boundary layer can be made weak by choosing *soft* boundary conditions.

High Reynolds number flow is a singular perturbation of inviscid flow. From the study of singular perturbation problems it is well known that boundary layers are weaker if boundary conditions on derivatives are used (*soft* boundary conditions) rather than boundary conditions on a function itself (*hard* boundary conditions). The outflow boundary conditions of types III and IV make use of this. Some investigators recommend that in computations a fine grid be used near the outflow boundary in order to resolve the possible outflow boundary layer. This is a boundary layer not in the true solution but in the computed solution. We feel that this should not be necessary if appropriate soft boundary conditions are used since any possible boundary layer should be weak.

The type III boundary conditions are a modification of the type II boundary conditions. The condition $v = 0$ is softened to $v_x = 0$, and the divergence boundary condition $\delta = 0$ is softened to $\delta_x = 0$. The νu_x term in the mixed boundary condition is needed to account for one of the viscous boundary terms in the velocity estimate. The sequence of estimates for the type III follows closely those for the type II boundary conditions; the estimates are valid only for an outflow boundary where $c < 0$.

The type IV outflow boundary conditions are different; they impose no explicit constraint on the divergence or its normal derivative. The sequence of estimates starts with a combined estimate for the velocity components and the gradient of the pressure instead of first estimating the divergence. An estimate for the divergence on the boundary is then obtained as a corollary. With this type of boundary condition there is the possibility that the divergence may not remain zero even if the initial conditions are divergence free.

To summarize these differences we write down the variables which can be estimated independent of ν in the order in which the estimates are made.

I.	II.	III.	IV.
Velocity	Inflow	Outflow	Outflow
$u = 0$	$p + \gamma u = 0$	$p + \gamma u - \nu u_x = 0$	$p = 0$
$v = 0$	$v = 0$	$v_x = 0$	$v_x = 0$
$u_x = 0$	$u_x = 0$	$u_{xx} = 0$	$u_x = 0$
	$(c > 0)$	$(c < 0)$	$(c < 0)$
δ	δ	$\delta,\ \delta\|_{x=0}$	u, v
u, v	u, v	u, v	$p_x,\ p_y$
u_x	$u\|_{x=0},\ p\|_{x=0}$	$u\|_{x=0},\ v\|_{x=0}$	$u\|_{x=0},\ v\|_{x=0}$
p_x	u_x	u_x	$\delta\|_{x=0}$
	p_x, p_y	p_x, p_y	δ
	$e^{-\alpha x} p$	$p\|_{x=0}$	u_x
	v_x	$e^{-\alpha x} p$	$e^{-\alpha x} p$
	p_{xx}	v_x	v_x
	u_{xx}	p_{xx}	p_{xx}
	v_{xx}	u_{xx}	
	etc.		

Table 2.1

2.2 Velocity Boundary Conditions

In this section we make energy estimates for the solution of the linear Navier-Stokes equations (2.1.1) with boundary conditions on the velocity components

$$u(0,y,t)=0, \qquad v(0,y,t)=0, \qquad u_x(0,y,t)=0 \tag{2.2.1}$$

which we have called type I boundary conditions.

We consider both the divergence formulation and the pressure formulation. The condition $u_x=0$ is not an independent boundary condition in the divergence formulation since it is a consequence of the divergence equation.

We first consider the divergence formulation. The energy estimate for the velocity is obtained by taking the inner product of the velocity components with the momentum equations and integrating by parts.

Lemma 2.2.1 *Energy estimate for u, v. Let $\alpha>0$ be arbitrary. Then*

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2+\nu\|\nabla\mathbf{w}\|^2\le\frac{\alpha}{2}\|\mathbf{w}\|^2+\frac{1}{2\alpha}\|\mathbf{f}\|^2 \tag{2.2.2}$$

and

$$\begin{aligned}\|\mathbf{w}\|^2&\le\|\mathbf{W}^{(0)}\|^2e^{\alpha t}+\frac{1}{\alpha}\int_0^t e^{\alpha(t-t')}\|\mathbf{f}(\cdot,\cdot,t')\|^2\,dt'\\&\le\|\mathbf{W}^{(0)}\|^2e^{\alpha t}+\theta(\alpha,t)\max_{0\le t'\le t}\|\mathbf{f}(\cdot,\cdot,t')\|^2\end{aligned} \tag{2.2.3}$$

where $\theta(\alpha,t):=(e^{\alpha t}-1)/\alpha$.

Proof. Taking the inner product of the velocity components with the momentum equations gives

$$\begin{aligned}&(u,u_t+cu_x+du_y+p_x-\nu\Delta u-f_1)\\&+(v,v_t+cv_x+dv_y+p_y-\nu\Delta v-f_2)=0.\end{aligned}$$

Consider each of the terms in turn. The time derivatives are

$$(u,u_t)+(v,v_t)=\frac{1}{2}\frac{d}{dt}\left(\|u\|^2+\|v\|^2\right)=\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2.$$

The first of the convective terms is

$$(u, c\,u_x) = (u, (cu)_x) - (u, c_x\,u) = \int_0^1 c\,u^2|_{x=0}^{\infty}\,dy - (u_x, c\,u) - (u, c_x\,u),$$

so

$$(u, c\,u_x) = -\int_0^1 \frac{c}{2}\,u^2|_{x=0}\,dy - \frac{1}{2}(u, c_x\,u).$$

Taking the convective terms together and moving them to the right hand side

$$\begin{aligned} -\{(u, cu_x + du_y) + (v, cv_x + dv_y)\} &= \int_0^1 \frac{c}{2}(u^2 + v^2)|_{x=0}\,dy \\ &\quad + \frac{1}{2}(u, c_x u + d_y u) + \frac{1}{2}(v, c_x v + d_y v) \\ &= \int_0^1 \frac{c}{2}(u^2 + v^2)|_{x=0}\,dy \end{aligned}$$

since the divergence of the coefficients (i.e., $c_x + d_y$) is assumed to be zero.

The pressure terms are

$$-\{(u, p_x) + (v, p_y)\} = \int_0^1 u\,p|_{x=0}\,dy + (u_x + v_y, p).$$

The diffusion terms are

$$(u, \nu\,\Delta u) = -\nu \int_0^1 u\,u_x|_{x=0}\,dy - \nu\left(\|u_x\|^2 + \|u_y\|^2\right),$$

so

$$(u, \nu\,\Delta u) + (v, \nu\,\Delta v) = -\nu \int_0^1 (u\,u_x + v\,v_x)|_{x=0}\,dy - \nu\|\nabla \mathbf{w}\|^2.$$

The forcing terms are

$$(u, f_1) + (v, f_2) = (\mathbf{w}, \mathbf{f}),$$

and

$$|(\mathbf{w}, \mathbf{f})| \le \|\mathbf{w}\|\,\|\mathbf{f}\| \le \frac{\alpha}{2}\|\mathbf{w}\|^2 + \frac{1}{2\alpha}\|\mathbf{f}\|^2.$$

Putting these results together

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \nu\|\nabla \mathbf{w}\|^2 &= (u_x + v_y, p) + (\mathbf{w}, \mathbf{f}) \\ &\quad + \int_0^1 G(y,t)\,dy \end{aligned} \tag{2.2.4}$$

where the boundary integral terms are

$$G(y,t) := \left[\frac{c}{2}(u^2+v^2) + up - \nu(uu_x + vv_x)\right] |_{x=0}.$$

The divergence term vanishes in the divergence formulation and the boundary terms are annihilated by the boundary conditions.

We assume that estimates for any number of y and t derivatives follow immediately with no further work. We write out the form of the estimates corresponding to Lemma Lemma 2.2.1; likewise, y and t derivatives can be taken for all of the estimates made below. On the other hand, it is not possible to obtain estimates for the x derivatives in the same way since the boundary conditions cannot be differentiated in the x direction (i.e., away from the boundary).

Lemma 2.2.2 *Energy estimates for y and t derivatives of the velocity. For any k and l*

$$\frac{d}{dt}\left\|\frac{\partial^{k+l}\mathbf{w}}{\partial y^k \partial t^l}\right\|^2 + 2\nu\left\|\nabla\frac{\partial^{k+l}\mathbf{w}}{\partial y^k \partial t^l}\right\|^2 \leq \text{const} \sum_{i+j\leq k+l}\left\{\left\|\frac{\partial^{i+j}\mathbf{w}}{\partial y^i \partial t^j}\right\|^2 + \frac{1}{\alpha}\left\|\frac{\partial^{i+j}\mathbf{f}}{\partial y^i \partial t^j}\right\|^2\right\} \quad (2.2.5)$$

where const *depends on the maximum norm of a number of derivatives of c and d.*

The energy estimate for u_x follows from the divergence equation. The energy estimate for v_x is obtained by using the viscous terms in (2.2.2); however this estimate depends on the Reynolds number.

Lemma 2.2.3 *Energy estimates for u_x, v_x.*

$$\|u_x\| = \|v_y\| \leq \|\mathbf{w}_y\| \quad (2.2.6)$$

$$\nu\|v_x\|^2 \leq \text{const}\left(\|\mathbf{w}\|^2 + \|\mathbf{w}_t\|^2 + \|\mathbf{f}\|^2\right) \quad (2.2.7)$$

Lemma 2.2.4 *Energy estimates for u_{xx} and p_x.*

$$\|u_{xx}\|^2 = \|v_{xy}\|^2 = \frac{\text{const}}{\nu}\left(\|\mathbf{w}_y\|^2 + \|\mathbf{w}_{ty}\|^2 + \|\mathbf{f}_y\|^2\right) \tag{2.2.8}$$

$$\|p_x\| \leq \|u_t\| + \|c\,u_x\| + \|d\,u_y\| + \nu\left(\|u_{xx}\| + \|u_{yy}\|\right) + \|f_1\| \tag{2.2.9}$$

Proof. These estimates follow from the x derivative of the divergence equation and from the first momentum equation, respectively.

To obtain estimates for v_{xx}, u_{xxx} and higher x derivatives of the velocity components we need first to estimate the pressure. The elliptic equation for the pressure is (2.1.4) and the boundary conditions at $x = 0$ are obtained from the first momentum equation at the boundary

$$p_x = r := \nu u_{xx} + f_1. \tag{2.2.10}$$

At $x = \infty$ we have the boundary condition $p_x = 0$. The pressure is determined up to a constant which is fixed by adding the constraint $p = 0$ at $x = \infty$.

In order to estimate $\|p\|$ and $\|p_y\|$ we shall make use of the estimates we have already for $\|R\|$ and $\|p_x\|$. We note that at this point we do not have an estimate for $u_{xx}|_{x=0}$ or $v_{xx}|_{x=0}$ and so we cannot estimate p in terms of its boundary data.

The estimates are obtained by writing p as a Fourier series in y and estimating the coefficients. We write

$$p(x,y,t) = \sum_{\omega=-\infty}^{\infty} \widehat{p}(x,\omega,t)e^{2\pi i\omega y}$$

and similarly for the other variables. The Fourier coefficients $\widehat{p}$ are given by

$$\widehat{p}(x,\omega,t) = \int_0^1 p(x,y,t)e^{2\pi i\omega y}\,dy$$

and Parseval's equality is

$$\|p\|^2 = \sum_{\omega} \|\widehat{p}\|_x^2$$

where $\|.\|_x$ is the norm over x.

The Fourier coefficients of p satisfy the equations

$$\widehat{p}_{xx} - 4\pi^2\omega^2\widehat{p} + \widehat{R} = 0, \qquad \widehat{p}_x|_{x=0} = \widehat{r} \tag{2.2.11}$$

for each frequency ω.

Lemma 2.2.5 *Estimate for p_y.*

$$\|p_y\| \leq \text{const}(\|p_x\| + \|R\|) \tag{2.2.12}$$

Proof. By Parseval's equation for p_y

$$\|p_y\|^2 = \sum_{\omega \neq 0} 4\pi^2\omega^2\|\widehat{p}\|_x^2.$$

To estimate the Fourier coefficients $\widehat{p}$ for $\omega \neq 0$ we decompose $\widehat{p}$ as

$$\widehat{p} = \widehat{p}^{(1)} + \widehat{p}^{(2)}$$

where

$$\widehat{p}_{xx}^{(1)} - 4\pi^2\omega^2\widehat{p}^{(1)} + \widehat{R} = 0, \qquad \widehat{p}_x^{(1)}|_{x=0} = 0$$

and

$$\widehat{p}_{xx}^{(2)} - 4\pi^2\omega^2\widehat{p}^{(2)} = 0, \qquad \widehat{p}_x^{(2)}|_{x=0} = \widehat{r}.$$

Then $\widehat{p}^{(1)}$ and $\widehat{p}_x^{(1)}$ can be estimated in terms of $\widehat{R}$, and $\widehat{p}^{(2)}$ can be estimated in terms of $\widehat{p}_x$ and $\widehat{p}_x^{(1)}$. Taking the inner product over x and integrating by parts gives the estimate for $\widehat{p}^{(1)}$

$$\|\widehat{p}_x^{(1)}\|_x^2 + 4\pi^2\omega^2\|\widehat{p}^{(1)}\|_x^2 = (\widehat{p}^{(1)}, \widehat{R})_x \leq \frac{1}{2}\left(\|\widehat{p}^{(1)}\|_x^2 + \|\widehat{R}\|_x^2\right).$$

So

$$(4\pi^2\omega^2 - \frac{1}{2})\|\widehat{p}^{(1)}\|_x^2 + \|\widehat{p}_x^{(1)}\|_x^2 \leq \frac{1}{2}\|\widehat{R}\|_x^2.$$

To estimate $\widehat{p}^{(2)}$ for $\omega \neq 0$ we express the Fourier coefficients as

$$\widehat{p}^{(2)} = -\frac{\widehat{r}}{2\pi|\omega|}\, e^{-2\pi|\omega|x},$$

so

$$4\pi^2\omega^2\|\widehat{p}^{(2)}\|_x^2 = \|\widehat{p}_x^{(2)}\|_x^2 \leq 2\left(\|\widehat{p}_x\|_x^2 + \|\widehat{p}_x^{(1)}\|_x^2\right) \leq 2\|\widehat{p}_x\|_x^2 + \|\widehat{R}_x\|_x^2.$$

The estimate for $\|p_y\|$ follows from combining these two results.

Lemma 2.2.6 *Estimate for p.*

$$\|p\|^2 \leq \|\widehat{p}_0\|^2 + \text{const}\left(\|p_x\|^2 + \|R\|^2\right),$$

where

$$\widehat{p}_0 := \widehat{p}|_{\omega=0} = \widehat{p}(x,0,t).$$

Proof.

$$\|p\|^2 = \sum_{\omega} \|\widehat{p}\|_x^2 = \|\widehat{p}_0\|_x^2 + \sum_{\omega\neq 0} \|\widehat{p}\|_x^2$$

and the sum can be estimated as in the previous lemma.

The zero coefficient of $\widehat{p}$ is

$$\widehat{p}_0 = -\int_x^\infty \int_{x'}^\infty \widehat{R}(x'',0,t)\,dx''\,dx'$$

which satisfies the boundary condition at $x=0$ since

$$\widehat{p}_{0x}|_{x=0} = \int_0^\infty \widehat{R}(x,0,t)\,dx = \int_0^1\int_0^\infty R\,dx\,dy = \\ -\int_0^1 [cu_x + du_y - f_1]\,|_{x=0}\,dy = \int_0^1 f_1|_{x=0}\,dy,$$

and

$$\widehat{r}|_{\omega=0} = \int_0^1 [\nu u_{xx} + f_1]\,|_{x=0}\,dy = \int_0^1 f_1|_{x=0}\,dy,$$

since $u_{xx} = -v_{xy}$.

At this point we have estimates for u, v, p, u_x, v_x, p_x and u_{xx} and by assumption also for any y and t derivatives of these. This process can be continued to give estimates for higher x derivatives. However in the process the dependence on the Reynolds number becomes stronger still. We include here a couple more estimates.

Lemma 2.2.7 *Estimates for v_{xx}, p_{xx}, u_{xxx}.*

$$\nu^2 \|v_{xx}\| \leq \nu \|v_t\| + \nu \left(\|d\, v_y\| + \|p_y\| + \nu \|v_{yy}\| + \|f_2\| \right) \tag{2.2.13}$$

$$\|p_{xx}\| \leq \|p_{yy}\| + \|R\| \tag{2.2.14}$$

$$\|u_{xxx}\| = \|v_{xxy}\| \tag{2.2.15}$$

Lemma 2.2.8 *Estimates for p_{xxx}.*

$$\|p_{xxx}\| \leq \|p_{xyy}\| + \|R_x\| \tag{2.2.16}$$

where

$$R_x = \frac{\partial}{\partial x} R(u_x, u_y, v_x, v_y) = R_x(u_{xx}, u_{xy}, v_{xx}, v_{xy})$$

$$\|R_x\| \leq \text{const} \left(\|u_{xx}\| + \|u_{xy}\| + \|v_{xx}\| + \|v_{xy}\| \right).$$

2.2.1 Estimates for Pressure Formulation

We now consider estimates for the pressure formulation. In passing from the original formulation to the pressure formulation the equations have been differentiated in space. We shall see that a boundary condition for the divergence is needed in addition to those for the velocity components. The boundary condition on the divergence is

$$\delta(0, y, t) = 0,$$

since the boundary condition $v = 0$ can be differentiated along the boundary $x = 0$ to give $v_y = 0$ and the extra boundary condition in (2.2.1) is $u_x = 0$.

In order to obtain estimates for the velocity components and the pressure in this formulation we look first at the divergence. The divergence satisfies

$$\delta_t + c\, \delta_x + d\, \delta_y = \nu \Delta \delta + \nabla . \mathrm{f}, \tag{2.2.17}$$

which is obtained by taking the divergence of the momentum equations and subtracting off the terms in the pressure equation. Other terms may appear in this equation if the function R is chosen differently.

The energy estimate for the divergence is obtained using integration by parts in the same way as the estimates for the velocity components above. Integration by parts gives

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 + \nu\|\nabla\delta\|^2 = (\delta, \nabla.\mathbf{f}) + \int_0^1 \left[\frac{c}{2}\delta^2 - \nu\delta\delta_x\right]\Big|_{x=0} dy. \tag{2.2.18}$$

The boundary condition for the divergence is needed to bound the boundary terms.

Lemma 2.2.9 *Energy estimate for δ. Let $\alpha > 0$ be arbitrary. Then*

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 + \nu\|\nabla\delta\|^2 = (\delta, \nabla.\mathbf{f}) \le \frac{\alpha}{2}\|\delta\|^2 + \frac{1}{2\alpha}\|\nabla.\mathbf{f}\|^2 \tag{2.2.19}$$

$$\|\delta\|^2 \le \|\nabla.\mathbf{W}^{(0)}\|^2 e^{\alpha t} + \frac{1}{\alpha}\int_0^t e^{\alpha(t-t')}\|\nabla.\mathbf{f}(\cdot,\cdot,t')\|^2 dt' \tag{2.2.20}$$

If $\nabla.\mathbf{f} \equiv 0$ then

$$\|\delta\| \le \|\nabla.\mathbf{W}^{(0)}\|. \tag{2.2.21}$$

If we assume that the divergence is zero initially and the forcing is divergence free, then it follows from this estimate that the divergence remains identically zero for all time. That is, the divergence equation is satisfied and so all the estimates above for the velocity and the pressure hold.

In the case where the divergence is not initially zero the estimate of Lemma Lemma 2.2.1 remains valid; however, the non-zero divergence enters the estimates for the velocity components. The divergence term brings the pressure into the estimate (2.2.4) for the velocity in a way which seems difficult to handle. To resolve this difficulty, we first subtract out the divergent part of the velocity field and then the remainder can be estimated as above.

Since we have an estimate for the divergence δ, the divergent part of the velocity field can be estimated. Choose

$$\widetilde{u}(x,y,t) = -\int_x^\infty \delta(x',y,t)\,dx', \qquad \widetilde{v}(x,y,t) \equiv 0.$$

Then the velocity field $(\widetilde{u},\widetilde{v})$ satisfies

$$\widetilde{u}_x + \widetilde{v}_y = \delta$$

and $(\widetilde{u},\widetilde{v}) \to 0$ as $x \to \infty$. Now let

$$u = \widetilde{u} + u', \qquad v = \widetilde{v} + v'.$$

Then (u',v') satisfies the same momentum equations as (u,v) with a different forcing $\mathbf{f}'$ depending on $\mathbf{f}$, $\widetilde{u}$ and $\widetilde{v}$. In addition,

$$\delta' = u'_x + v'_y = 0$$

is satisfied everywhere and u', v' vanish at $x = \infty$. The boundary conditions at $x = 0$ are

$$u'(0,y,t) = -\widetilde{u}(0,y,t) = \int_0^\infty \delta(x,y,t)\,dx, \qquad v'(0,y,t) = 0.$$

The boundary conditions can be made homogeneous as described in Section 2.1. Then the form of the resulting system for (u',v',p) is the same as (2.1.1) for (u,v,p) — the linear Navier-Stokes equations with zero divergence and homogeneous boundary conditions. Hence, all the estimates carry over to this case.

2.3 Inflow Boundary Conditions

We obtain estimates for the linear Navier-Stokes equations which do not depend on the Reynolds number. That is, we obtain estimates for the velocity components and the pressure which remain good as $\nu \to 0$. We consider boundary conditions which involve the pressure as well as the velocity. The boundary conditions which we consider are

$$p + \gamma u = 0, \qquad v = 0, \qquad u_x = 0. \tag{2.3.1}$$

These boundary conditions (type II) are suitable for an inflow boundary, where $c > 0$. We consider the pressure formulation, and the estimates apply to the divergence formulation also.

The energy estimate for the divergence is obtained in the usual way. The boundary terms vanish since the boundary conditions guarantee that the divergence vanishes on the boundary.

Lemma 2.3.1 *Energy estimate for divergence.*

$$\frac{d}{dt}\|\delta\|^2 + 2\nu\|\nabla\delta\|^2 \leq C\|\delta\|^2 + |(\delta, \nabla.\mathbf{f})| \tag{2.3.2}$$

where $C = (|c_x|_\infty + |d_y|_\infty)/2$.

We assume for simplicity that the initial velocity is divergence free and hence by this estimate the divergence remains identically zero for all time. Otherwise, the divergent part can be subtracted off as in Section 2.2.

Estimates for any number of y and t derivatives are assumed throughout.

The energy estimate for the velocity is obtained in the same way. We obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \nu\|\nabla\mathbf{w}\|^2 = (u_x + v_y, p) + (\mathbf{w}, \mathbf{f})\\ + \int_0^1 G(y,t)\,dy,\end{aligned} \tag{2.3.3}$$

where the boundary integral terms are

$$\begin{aligned} G(y,t) &:= \left[\frac{c}{2}(u^2 + v^2) + up - \nu(uu_x + vv_x)\right]\Big|_{x=0} \\ &= -\left[(\gamma - \frac{c}{2})u^2\right]\Big|_{x=0}. \end{aligned} \tag{2.3.4}$$

The contribution of the boundary term to the estimate has the right sign provided

$$\gamma \geq \frac{c^*}{2}, \qquad c^* := \max_{0 \leq y < 1} c(0,y,t). \tag{2.3.5}$$

Furthermore, if this is a strict inequality, then the contribution of the boundary terms is dissipative.

Lemma 2.3.2 *Energy estimate for u, v.*

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \nu\|\nabla\mathbf{w}\|^2 + \int_0^1 (\gamma - \frac{c}{2})u^2|_{x=0}\, dy \leq C\|\mathbf{w}\|^2 + |(\mathbf{w},\mathbf{f})| \tag{2.3.6}$$

Lemma 2.3.3 *Energy estimates for u and p on the boundary.*

$$\int_0^1 u^2|_{x=0}\, dy \leq \frac{1}{\gamma - \frac{c^*}{2}} \left(\|\mathbf{w}\|^2 + \|\mathbf{w}_t\|^2 + \|\mathbf{f}\|^2\right) \tag{2.3.7}$$

$$\int_0^1 p^2|_{x=0}\, dy \leq \frac{\gamma^2}{\gamma - \frac{c^*}{2}} \left(\|\mathbf{w}\|^2 + \|\mathbf{w}_t\|^2 + \|\mathbf{f}\|^2\right) \tag{2.3.8}$$

Proof. Equation (2.3.3) can be written as

$$\begin{aligned} \int_0^1 (\gamma - \frac{c}{2})u^2|_{x=0}\, dy &= (\mathbf{w},\mathbf{f}) - \nu\|\nabla\mathbf{w}\|^2 - (\mathbf{w},\mathbf{w}_t) \\ &\leq \|\mathbf{w}\|^2 + \frac{1}{2}\left(\|\mathbf{w}_t\|^2 + \|\mathbf{f}\|^2\right). \end{aligned}$$

This proves the first estimate and the second estimate follows from $p = -\gamma u$.

From these estimates we can decide what is a good choice for γ. The parameter γ appears in the estimates for both u and p on the boundary. The estimate for u improves as γ increases without bound. The estimate for p deteriorates for large γ and for this estimate the optimum value is $\gamma = c^*$. This is the value of $\gamma > c^*/2$ which minimizes $\gamma^2/(\gamma - c^*/2)$. If $c^* < 0$ on the boundary, then the optimum value is $\gamma = 0$. Just by looking at these two estimates it appears that for large values of γ the u velocity would be smooth but the pressure may be rough on the boundary. In the limit as γ tends to infinity the boundary condition $u = 0$ is obtained, and for this boundary condition (type I) the estimate for p on the boundary is lost.

Lemma 2.3.4 *Energy estimate for u_x.*

$$\|u_x\| = \|v_y\| \tag{2.3.9}$$

To obtain an estimate for v_x which does not depend on ν, we need first to estimate the pressure. Estimates for the pressure are obtained from the pressure equation.

To get the desired estimates it is necessary to eliminate the boundary terms which arise. This is done by subtracting off a function which is equal to the pressure on the boundary and dies off smoothly into the interior, such as

$$\widetilde{p}(x,y,t) = p(0,y,t)e^{-x}. \tag{2.3.10}$$

This allows the boundary condition for the pressure to be taken as $p = 0$ and it adds extra terms to the forcing functions. The forcing functions are replaced by

$$f_1 + e^{-x}p(0,y,t), \qquad f_2 - e^{-x}p_y(0,y,t),$$

respectively.

Lemma 2.3.5 *Energy estimate for p_x, p_y.*

$$\|\nabla p\|^2 \le \text{const}\left(\int_0^1 (p^2 + p_y^2)\big|_{x=0}\,dy + \|\mathbf{w}\|^2 + \|\mathbf{f}\|^2\right) \tag{2.3.11}$$

Proof. We write the forcing function R in the form

$$\begin{aligned} R &= c_x u_x + c_y v_x + d_x u_y + d_y v_y - (f_{1x} + f_{2y}) \\ &= (c_x u + c_y v - f_1)_x + (d_x u + d_y v - f_2)_y, \end{aligned}$$

assuming that the coefficient functions are divergence free; i.e., $c_x + d_y$ is identically zero. Taking the inner product of p with the pressure equation and integrating by parts gives

$$\begin{aligned} \|\nabla p\|^2 &= (p, R) - \int_0^1 p\,p_x\big|_{x=0}\,dy \\ &= -(p_x, c_x u + c_y v - f_1) - (p_y, d_x u + d_y v - f_2) \\ &\quad - \int_0^1 p\,(p_x + c_x u + c_y v - f_1)\big|_{x=0}\,dy. \end{aligned} \tag{2.3.12}$$

At this point the boundary terms are eliminated by the boundary condition $p = 0$. The extra terms which arise are $\|\nabla\widetilde{p}\|^2$ and the extra contributions to the forcing functions. These account for the other terms in the estimate.

Lemma 2.3.6 *Estimate for p. Let $\alpha > 0$ be arbitrary. Then*

$$\|e^{-\alpha x}p\|^2 \leq \text{const} \int_0^1 p^2|_{x=0}\, dy + \frac{\text{const}}{\alpha^2}\left(\|\nabla p\|^2 + (1+\frac{1}{\alpha})\|\mathbf{w}\|^2\right). \tag{2.3.13}$$

Proof. Take the inner product of $e^{-2\alpha x}p$ with the pressure equation. Then

$$\cdot \quad (e^{-2\alpha x}p, R) = -(e^{-2\alpha x}p, \Delta p).$$

Now integrate by parts. The left side gives

$$(e^{-2\alpha x}p, R) = -(e^{-2\alpha x}p_y, d_x u + d_y v) - (e^{-2\alpha x}(p_x - 2\alpha p), c_x u + c_y v),$$

so

$$|(e^{-2\alpha x}p, R)| \leq \|\nabla p\|^2 + \alpha\beta\,\|e^{-\alpha x}p\|^2 + \text{const}\,(1+\frac{1}{\beta})\,(\|u\|^2 + \|v\|^2).$$

The right hand side gives

$$\begin{aligned} -(e^{-2\alpha x}p, \Delta p) &= (e^{-2\alpha x}p_y, p_y) + (e^{-2\alpha x}p_x, p_x) - 2\alpha(e^{-2\alpha x}p, p_x) \\ &= \|e^{-\alpha x}\nabla p\|^2 - 2\alpha^2\|e^{-\alpha x}p\|^2, \end{aligned}$$

since

$$(e^{-2\alpha x}p, p_x) = -(e^{-2\alpha x}p_x, p) + 2\alpha\,(e^{-2\alpha x}p, p).$$

Hence

$$2\alpha^2\|e^{-\alpha x}p\|^2 = \|e^{-\alpha x}\nabla p\|^2 - (e^{-2\alpha x}p, R);$$

thus

$$(2\alpha^2 - \alpha\beta)\,\|e^{-\alpha x}p\|^2 \leq \text{const}\left(\|\nabla p\|^2 + (1+\frac{1}{\beta})\|\mathbf{w}\|^2\right).$$

Take $\beta = \alpha$ and the result follows. The boundary integral term comes from $\widetilde{p}$.

Next we estimate v_x. At this point we have estimates for the L^2 norms of all the terms in the v momentum equation except v_x and v_{xx}. So we can write

$$c v_x - \nu v_{xx} = H \tag{2.3.14}$$

where

$$H := \nu v_{yy} + f_2 - (v_t + d v_y + p_y) \tag{2.3.15}$$

is bounded in terms of the data. In addition, we know that $v = 0$ at $x = 0$ and we assume that v_x and v_{xx} both tend to zero as $x \to \infty$. Now define

$$c^{**} := \min_{\substack{x \geq 0 \\ 0 \leq y \leq 1}} c(0, y, t).$$

Taking the inner product of v_x with (2.3.14) gives

$$\begin{aligned} c^{**} \|v_x\|^2 &\leq (v_x, c v_x) = (v_x, H) + \nu (v_x, v_{xx}) \\ &\leq \frac{c^{**}}{2} \|v_x\|^2 + \frac{1}{2c^{**}} \|H\|^2 - \frac{\nu}{2} \int_0^1 v_x^2|_{x=0}\, dy, \end{aligned}$$

provided $c^{**} > 0$. Thus we have shown the following.

Lemma 2.3.7 *Estimate for v_x. If $c^{**} > 0$ then*

$$c^{**} \|v_x\|^2 + \nu \int_0^1 v_x^2|_{x=0}\, dy \leq \frac{1}{c^{**}} \|H\|^2 \tag{2.3.16}$$

and

$$\|v_x\| \leq \frac{1}{c^{**}} \left(\|v_t\| + \|d\, v_y\| + \|p_y\| + \nu \|v_{yy}\| + \|f_1\| \right). \tag{2.3.17}$$

If $c^{**} \leq 0$, then $x = 0$ is not an inflow boundary. In this case the estimate for v_x breaks down and it is necessary to fall back to weaker estimates depending on the Reynolds number. This is a real breakdown and not just an artifact of the estimates. A boundary layer in the v velocity is expected to be present at the outflow boundary if these boundary conditions are used. We shall see below that this boundary layer is weaker if v_x is specified at outflow. For this reason the current choice of boundary conditions is not recommended at an outflow boundary.

At this point for inflow we have estimates for u, v, p and u_x, v_x, p_x and all their y and t derivatives. Estimates for higher x derivatives are obtained by continuing this process as follows. R can be estimated now that v_x has been estimated. p_{xx} can be estimated in terms of R and p_{yy}. An estimate for δ_x is obtained in the same way as the estimate for v_x. Then u_{xx} follows, v_{xx} follows, et cetera.

2.4 Outflow Boundary Conditions

Next we consider boundary conditions suitable for an outflow boundary. We consider boundary conditions which involve the pressure as well as the velocity. The boundary conditions which we consider are of two types:

$$p + \gamma u - \nu u_x = 0, \qquad v_x = 0, \qquad u_{xx} = 0 \tag{2.4.1}$$

and

$$p = 0, \qquad v_x = 0, \qquad u_x = 0. \tag{2.4.2}$$

These are type III and type IV of those boundary conditions which we introduced above.

2.4.1 Type III Boundary Conditions

We consider first the boundary conditions of type III. The sequence of estimates follows closely those for the inflow boundary conditions (type II). The energy estimate for the divergence is obtained in the usual way.

The boundary terms have the right sign provided

$$c^* := \max_{0 \le y \le 1} c(0, y, t) \le 0, \tag{2.4.3}$$

since the boundary conditions guarantee that the normal derivative of the divergence vanishes on the boundary; i.e., $\delta_x = (u_x + v_y)_x = 0$ at $x = 0$. In addition, we obtain an estimate for the divergence on the boundary.

Lemma 2.4.1 *Energy estimate for divergence. If $c^* \leq 0$, then*

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2 + \nu\|\nabla\delta\|^2 + \frac{|c^*|}{2}\int_0^1 \delta^2|_{x=0}\,dy = 0. \tag{2.4.4}$$

If $c^* < 0$, then

$$\int_0^1 \delta^2|_{x=0}\,dy \leq \frac{1}{|c^*|}\left(\|\delta\|^2 + \|\delta_t\|^2\right). \tag{2.4.5}$$

Proof.

$$(\delta, \delta_t) + \nu\|\nabla\delta\|^2 = \int_0^1 (\frac{c}{2}\delta^2 - \nu\delta\delta_x)|_{x=0}\,dy \tag{2.4.6}$$

We assume for simplicity that the initial velocity is divergence free; otherwise the divergent part can be subtracted off. Hence, we take $\delta := u_x + v_y \equiv 0$.

Estimates for any number of y and t derivatives are assumed throughout.

The energy estimate for the velocity is obtained in the same way. The integrand in the boundary integral is

$$\begin{aligned} G(y,t) &:= \left[\frac{c}{2}(u^2+v^2) + up - \nu(uu_x + vv_x)\right]|_{x=0} \\ &= -\left[(\gamma - \frac{c}{2})u^2 - \frac{c}{2}v^2\right]|_{x=0}. \end{aligned} \tag{2.4.7}$$

The contribution of the boundary term to the estimate has the right sign, provided

$$\gamma \geq \frac{c^*}{2}, \quad \text{and} \quad c^* \leq 0.$$

Furthermore, if either of these is a strict inequality then the boundary conditions are dissipative.

Lemma 2.4.2 *Energy estimate for u, v.*

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \nu\|\nabla\mathbf{w}\|^2 + \int_0^1 \left[(\gamma - \frac{c}{2})u^2 - \frac{c}{2}v^2\right]|_{x=0}\,dy = (\mathbf{w}, \mathbf{f}) \tag{2.4.8}$$

Lemma 2.4.3 *Energy estimates for u and v on the boundary.*

$$\int_0^1 u^2|_{x=0}\,dy \leq \frac{1}{\gamma - \frac{c^*}{2}}\left(\|\mathbf{w}\|^2 + \|\mathbf{w}_t\|^2 + \|\mathbf{f}\|^2\right) \tag{2.4.9}$$

$$\int_0^1 v^2|_{x=0}\, dy \leq \frac{2}{|c^*|}\left(\|\mathbf{w}\|^2 + \|\mathbf{w}_t\|^2 + \|\mathbf{f}\|^2\right) \tag{2.4.10}$$

Lemma 2.4.4 *Energy estimate for u_x.*

$$\|u_x\| = \|v_y\| \tag{2.4.11}$$

To obtain an estimate for v_x which does not depend on ν we need first to estimate the pressure. Estimates for the pressure are obtained from the pressure equation. However, at this point we do not have an estimate for p on the boundary. We do have a relation between p and u_x on the boundary, which gives the boundary estimate

$$\int_0^1 (p - \nu u_x)|_{x=0}\, dy = \gamma^2 \int_0^1 u^2|_{x=0}\, dy. \tag{2.4.12}$$

This estimate corresponds to the estimate for p on the boundary which held for the inflow boundary conditions considered above. Again, it provides a constraint preventing the limit $\gamma \to \infty$ from being taken. It also indicates that the choice $\gamma = 0$ is optimal for this estimate.

Lemma 2.4.5 *Energy estimate for p_x, p_y, and p on the boundary.*

$$\|\nabla p\|^2 + \frac{|c^*|}{2\nu}\int_0^1 p^2|_{x=0}\, dy \leq \text{const}\left(\int_0^1 \Phi^2|_{x=0}\, dy + \|\mathbf{w}\|^2 + \|\mathbf{f}\|^2\right) \tag{2.4.13}$$

where

$$\Phi = \frac{1}{|c^*|}\left[\gamma^2 u^2 + \nu^2(u^2 + v^2 + u_t^2 + u_y^2 + \nu^2 u_{yy}^2 + f_1^2)\right]|_{x=0}. \tag{2.4.14}$$

Proof. The same estimate (2.3.12) which was found for the inflow case applies here. It remains to estimate the boundary integral

$$I := -\int_0^1 \left[p\left(p_x + c_x u + c_y v - f\right)\right]|_{x=0}\, dy. \tag{2.4.15}$$

We claim that the $-p\,p_x$ term has the right sign and that it dominates the other terms.

To see this we first notice that on the boundary $x = 0$ the u momentum equation reduces to

$$p_x + cu_x = r_1 := -u_t - du_y + \nu u_{yy} + f_1,$$

since $u_{xx} = 0$, and also that

$$p - \nu u_x = -\gamma u.$$

Hence

$$p_x + \frac{c}{\nu} p = \frac{1}{\nu} r_2 := r_1 - \frac{\gamma c}{\nu} u,$$

so

$$-p\, p_x = \frac{c}{\nu}\, p^2 - \frac{1}{\nu}\, p\, r_2.$$

For $c < 0$ the term $\frac{c}{\nu}\, p^2$ has the right sign and it is also strong enough to dominate all the other terms. The integrand of the boundary integral can be written as

$$-p\,(p_x + c_x u + c_y v - f) = \frac{c}{\nu}\, p^2 - \frac{1}{\nu}\, p\, r_3$$

where

$$\begin{aligned} r_3 &:= r_2 + \nu\,(c_x u + c_y v - f) \\ &= -\gamma\, c\, u + \nu\,(r_1 + c_x u + c_y v - f). \end{aligned}$$

Thus, it can be estimated by

$$\frac{c}{\nu}\, p^2 - \frac{1}{\nu}\, p\, r_3 \leq \frac{c}{2\nu}\, p^2 + \frac{1}{2|c|}\, r_3^2.$$

This completes the proof of the lemma.

Observe that this is a strong estimate for p on the boundary. As in the inflow case we can now subtract off a function to make $p = 0$ on the boundary and then proceed to estimate p.

Lemma 2.4.6 *Estimate for p. Let $\alpha > 0$ be arbitrary. Then*

$$\|e^{-\alpha x}p\|^2 \le \text{const} \int_0^1 p^2|_{x=0}\,dy + \frac{\text{const}}{\alpha^2}\left(\|\nabla p\|^2 + (1+\frac{1}{\alpha})\|\mathbf{w}\|^2\right). \qquad (2.4.16)$$

Next we estimate v_x. The argument is similar to that which was used in the inflow case. The equation for v_x has the same form as (2.3.14). This time we use the fact that $c^* < 0$ to get the estimate. Taking the inner product of v_x with (2.3.14) gives

$$\begin{aligned} c^*\|v_x\|^2 &\ge (v_x, cv_x) = (v_x, H) + \nu(v_x, v_{xx}) \\ &\ge \frac{c^*}{2}\|v_x\|^2 + \frac{1}{2c^*}\|H\|^2 - \frac{\nu}{2}\int_0^1 v_x^2|_{x=0}\,dy, \end{aligned}$$

provided $c^* < 0$. Hence

$$|c^*|\,\|v_x\|^2 \le \frac{1}{|c^*|}\|H\|^2 + \nu\int_0^1 v_x^2|_{x=0}\,dy.$$

Now by the boundary condition $v_x = 0$ the estimate for v_x follows.

Lemma 2.4.7 *Estimate for v_x. If $c^* < 0$, then*

$$\|v_x\| \le \frac{1}{|c^*|}\|H\|. \qquad (2.4.17)$$

At this point we have estimates for u, v, p and u_x, v_x, p_x and all their y and t derivatives. Estimates for a couple more x derivatives are obtained by continuing this process as follows. R can be estimated now that v_x has been estimated. p_{xx} can be estimated in terms of R and p_{yy}. An estimate for δ_x is obtained in the same way as the estimate for v_x. Then u_{xx} is estimated.

This is as far as we can go. The process fails when we try to estimate v_{xx}. There is no boundary condition at $x = 0$ for v_{xx} and so the technique used in estimating v_x and δ_x fails. In contrast to the situation at the inflow boundary we cannot continue indefinitely making these estimates.

2.4.2 Type IV Boundary Conditions

Next we consider type IV boundary conditions (2.4.2) which are also suitable for an outflow boundary. There is a major difference between these boundary conditions and the others which we have considered so far. With these boundary conditions there is no constraint on the divergence on the boundary. Consequently, the estimates must be made in a different order, since we cannot begin with an estimate for the divergence.

The energy estimates for the velocity and the pressure are treated together. In the usual way we obtain for the velocity components by integration by parts

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \nu\|\nabla\mathbf{w}\|^2 = (u, p_x) + (v, p_y) + (\mathbf{w}, \mathbf{f}) + \int_0^1 G(y,t)\, dy \qquad (2.4.18)$$

where

$$\begin{aligned} G(y,t) &:= \left[\frac{c}{2}(u^2+v^2) - \nu(uu_x + vv_x)\right]\Big|_{x=0} \\ &= \frac{c}{2}(u^2+v^2)\Big|_{x=0} \end{aligned} \qquad (2.4.19)$$

Note that the terms involving the pressure are not integrated by parts. The contribution of the boundary term to the estimate has the right sign provided $c^* \leq 0$. If this is a strict inequality then the boundary terms are dissipative.

To obtain an estimate for the velocity components it is necessary to estimate the gradient of the pressure in terms of the velocity. Consider the pressure equation (2.1.4). Integration by parts leads to (2.3.12), in which the boundary terms vanish since $p = 0$ on the boundary. Hence

$$\|\nabla p\|^2 \leq \text{const}\left(\|\mathbf{w}\|^2 + \|\mathbf{f}\|^2\right).$$

This estimate is used to eliminate the pressure terms from (2.4.18). Thus we have proved the following.

Lemma 2.4.8 *Energy estimates for u, v, p_x and p_y.*

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 + \nu\|\nabla\mathbf{w}\|^2 + \int_0^1 \left[-\frac{c}{2}(u^2+v^2)\right]|_{x=0}\,dy \le \text{const}\left(\|\mathbf{w}\|^2 + \|\mathbf{f}\|^2\right)$$

$$\|\nabla p\|^2 \le \text{const}\left(\|\mathbf{w}\|^2 + \|\mathbf{f}\|^2\right)$$

Lemma 2.4.9 *Energy estimates for u and v on the boundary.*

$$\int_0^1 (u^2+v^2)|_{x=0}\,dy \le \frac{2}{|c^*|}\left(\|\mathbf{w}\|^2 + \|\mathbf{w}_t\|^2 + \|\mathbf{f}\|^2\right)$$

Now that we have this estimate for v on the boundary, we get immediately an estimate for v_y on the boundary. This means that we can bound the divergence on the boundary also.

Lemma 2.4.10 *Estimate for the divergence on the boundary.*

$$\int_0^1 \delta^2|_{x=0}\,dy = \int_0^1 v_y^2|_{x=0}\,dy \le \frac{2}{|c^*|}\left(\|\mathbf{w}_y\|^2 + \|\mathbf{w}_{yt}\|^2 + \|\mathbf{f}_y\|^2\right)$$

We use this to subtract off the part of the velocity field associated with the non-zero divergence on the boundary, and then we can estimate the divergence of the rest. The velocity which we subtract off is

$$\widetilde{v} = v(0,y,t)\phi(x)$$

where $\phi(x)$ is a C^∞ function satisfying

$$\phi(0) = 1, \qquad \phi_x(0) = 0, \qquad \phi(x) \equiv 0 \quad \text{for } x \ge 1.$$

This velocity satisfies the boundary conditions

$$\widetilde{v}_x(0,y,t) = 0, \qquad \widetilde{v}(0,y,t) = v(0,y,t),$$

and furthermore

$$\|\widetilde{v}_y\|^2 = \int_0^1 v_y^2|_{x=0}\,dy \int_0^\infty \phi(x)^2\,dx.$$

Let $v' := v - \widetilde{v}$. Then (u, v', p) satisfies the same equations as (u, v, p) with f_2 replaced by

$$f_2' = f_2 - (\widetilde{v}_t + c\widetilde{v}_x + d\widetilde{v}_y - \nu\Delta\widetilde{v})$$

and the boundary conditions at $x = 0$ are

$$p = 0, \qquad u_x = 0, \qquad v_x' = 0, \qquad v' = 0.$$

So, in particular,

$$\delta' := u_x + v_y' = 0$$

at $x = 0$. Hence, the usual estimate works for δ', and we have proved the following result.

Lemma 2.4.11 *Energy estimate for the divergence.*

$$\|\delta\| \leq \|\delta'\| + \|\widetilde{v}_y\|$$

is bounded.

The estimates for u_x, p, v_x, and p_{xx} now go through exactly as for the outflow boundary conditions considered above.

This time the process fails when we try to estimate δ_x. There is no boundary condition at $x = 0$ for δ_x and so the technique used in estimating v_x fails.

CHAPTER 3

Semi-discrete Linear Navier-Stokes Equations

3.1 Introduction

In this chapter we study the discretization in space of the linear Navier-Stokes equations. This is the method of lines approach for analysis of finite difference methods. As in the previous chapter the aim is to establish estimates for the solution in terms of the data.

The techniques which are used in this analysis are the discrete analogues of those used for the continuous equations. The most important technique for the energy method is integration by parts. Integration by parts is essential to the estimates for the velocity components and the estimate for the divergence in the pressure formulation. The discrete analogue of integration by parts is summation by parts.

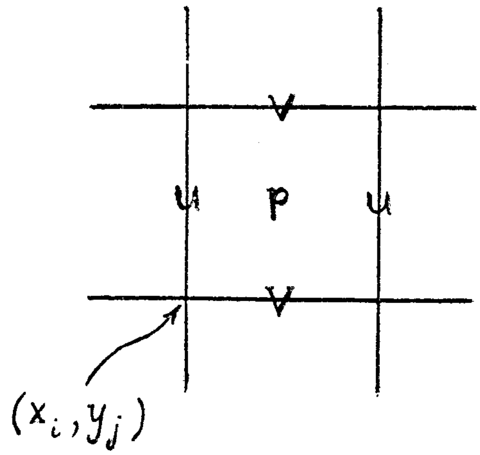
We discretize the linear Navier-Stokes equations (2.1.1) in space using a uniform staggered grid on the domain $x \geq 0$, $0 \leq y \leq 1$. The arrangement of the variables on the grid is shown in Figure 3.1. The discrete flow variables are

$$u_{i,j+1/2}(t) := u(x_i, y_{j+1/2}, t),$$

$$v_{i+1/2,j}(t) := v(x_{i+1/2}, y_j, t),$$

$$p_{i+1/2,j+1/2}(t) := p(x_{i+1/2}, y_{j+1/2}, t)$$

for $i = -1, 0, 1, \ldots$ and $j = 0, 1, \ldots, N-1$. The variables u, v, p here are not the same as the continuous u, v, p of the previous chapter; this duplicate notation should not be confusing.



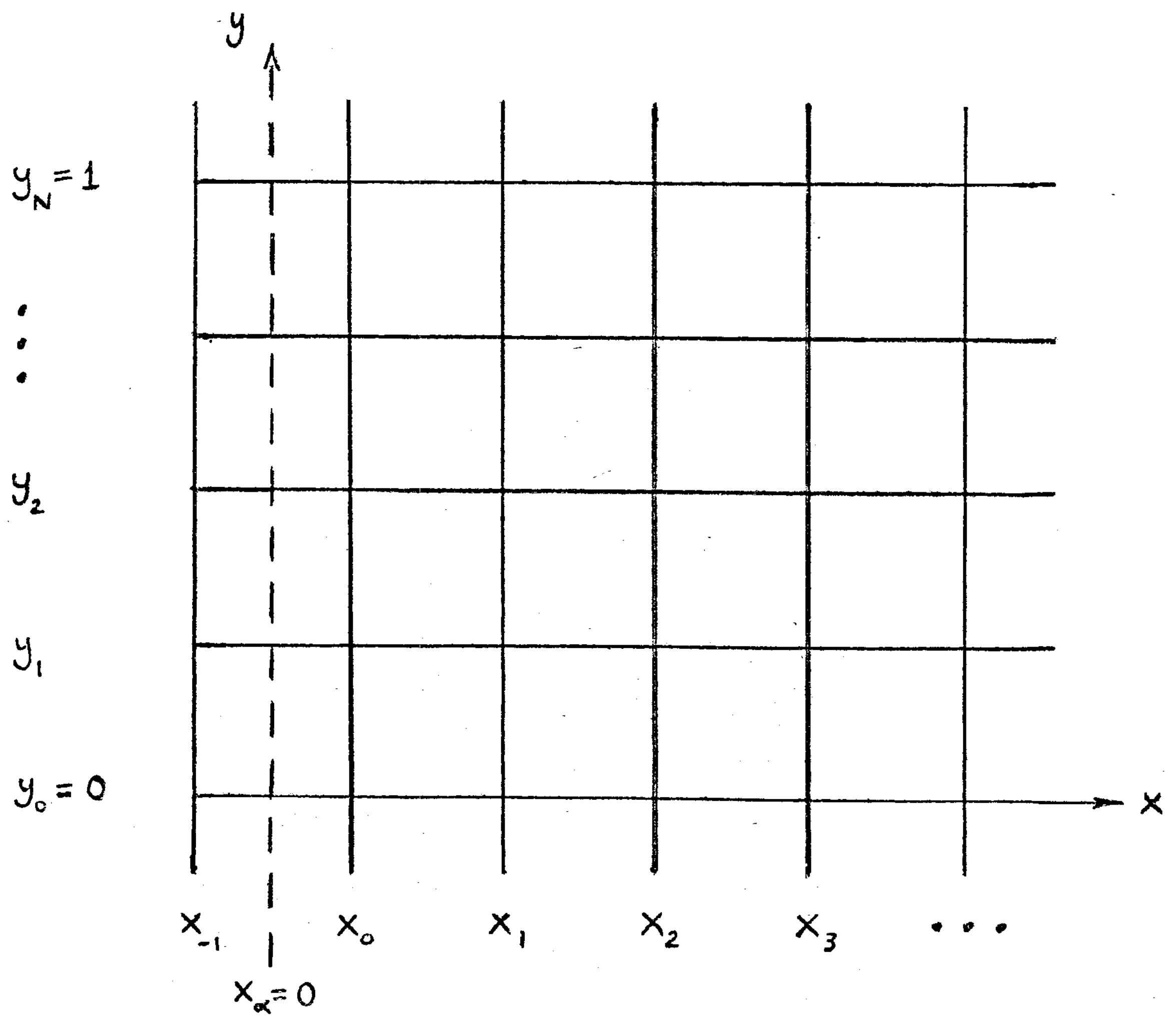


Figure 3.1. Staggered grid

The location of the boundary $x = 0$ on the grid can be chosen in more than one way. Two possible choices are for the boundary $x = 0$ to lie along the line $x = x_0$ where the u velocity is defined or to lie along the line $x = x_{-1/2}$ where v and p are defined. In general, we define the grid to be

$$\{(x_i, y_j) : \ i = -1, 0, 1, \ldots, \ j = 0, 1, \ldots, N-1, N\}$$

where

$$x_i = (i - \alpha)h, \qquad y_j = jh$$

and $Nh = 1$. The parameter α determines the location of the boundary line — the boundary line is $x_\alpha = 0$. The assumption that the x and y mesh spacings are the same plays no role and is made purely for the sake of simplicity. The half points $x_{i+1/2}$ and $y_{j+1/2}$ are the midpoints of the grid.

The space derivatives in the partial differential equations are replaced by finite differences. The divided difference operators are defined by

$$\begin{aligned} D_x w_{i,j} &:= \frac{w_{i+1/2,j} - w_{i-1/2,j}}{h}, \\ D_{0x} w_{i,j} &:= \frac{w_{i+1,j} - w_{i-1,j}}{2h}, \\ D_{+x} D_{-x} w_{i,j} &:= D_x D_x w = \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} \end{aligned}$$

and similarly for the y differences. The discrete Laplacian operator is defined by

$$\Delta_h w_{i,j} := \left(D_{+x} D_{-x} + D_{+y} D_{-y}\right) w_{i,j}.$$

Also we define the averaging operators S_x, S_y by

$$S_x w_{i,j} := \frac{w_{i+1/2,j} + w_{i-1/2,j}}{2}, \qquad S_y w_{i,j} := \frac{w_{i,j+1/2} + w_{i,j-1/2}}{2}.$$

The *semi-discrete linear Navier-Stokes equations* can be written as

$$u_t + c\,D_{0x} u + d\,D_{0y} u + D_x p = \nu \Delta_h u + f_1, \tag{3.1.1a}$$

$$v_t + c\,D_{0x} v + d\,D_{0y} v + D_y p = \nu \Delta_h v + f_2, \tag{3.1.1b}$$

$$\delta = \delta_{i+1/2,j+1/2} := D_x u + D_y v = 0, \tag{3.1.1c}$$

where the appropriate subscripts have been omitted from the variables. The u momentum equation is centred at the points $(x_i, y_{j+1/2})$, the v momentum equation is centred at $(x_{i+1/2}, y_j)$, and the divergence equation is centred at $(x_{i+1/2}, y_{j+1/2})$. The range of values of (i,j) for which the equations (3.1.1) apply is $i = 0, 1, 2, \ldots$ and $j = 0, 1, \ldots, N-1$. The coefficients c, d will mostly be taken to be constants since we have verified in the context of the continuous problem that the variable coefficients do not play an important role in the analysis.

The semi-discrete *pressure formulation* is obtained by the same steps as in the continuous case. Taking the discrete divergence of the momentum equations gives

$$\delta_t + Q + \Delta_h p = \nu \Delta_h \delta + \nabla_h . \mathbf{f}$$

where Q is the divergence of the convective terms

$$Q = Q_{i+1/2,j+1/2} = D_x \left(c\, D_{0x} u + d\, D_{0y} u \right) + D_y \left(c\, D_{0x} v + d\, D_{0y} v \right).$$

If c, d are constants, then Q has a simple form

$$Q = c\, D_{0x} \delta + d\, D_{0y} \delta.$$

If c, d are functions of x and y, then there are additional terms and we must expand further.

$$\begin{aligned}
Q &= S_x c\, D_{0x} D_x u + S_x d\, D_{0y} D_x u + S_y c\, D_{0x} D_y v + S_y d\, D_{0y} D_y v \\
&\quad + D_x c\, D_{0x} S_x u + D_x d\, D_{0y} S_x u + D_y c\, D_{0x} S_y v + D_y d\, D_{0y} S_y v \\
&= \frac{1}{2}(S_x + S_y) c\, D_{0x} \delta + \frac{1}{2}(S_x + S_y) d\, D_{0y} \delta \\
&\quad + \frac{1}{2}(S_x - S_y) c\, D_{0x} \left(D_x u - D_y v \right) + \frac{1}{2}(S_x - S_y) d\, D_{0y} \left(D_x u - D_y v \right) \\
&\quad + D_x c\, D_{0x} S_x u + D_x d\, D_{0y} S_x u + D_y c\, D_{0x} S_y v + D_y d\, D_{0y} S_y v.
\end{aligned}$$

Define the function R by

$$R := Q - \nabla_h . \mathbf{f} - \left(\tilde{c}\ D_{0x} \delta + \tilde{d}\ D_{0y} \delta \right),$$

where $\widetilde{c}$ and $\widetilde{d}$ stand for the four point averages $\frac{1}{2}(S_x + S_y)c$ and $\frac{1}{2}(S_x + S_y)d$, respectively. Then the pressure formulation consists of the two momentum equations together with the elliptic *pressure equation*

$$\Delta_h p + R = 0. \tag{3.1.2}$$

The equation for the divergence in the pressure formulation is

$$\delta_t + \widetilde{c}\, D_{0x}\delta + \widetilde{d}\, D_{0y}\delta = \nu\Delta_h\delta. \tag{3.1.3}$$

Furthermore, if the coefficient functions c, d are both sufficiently smooth and bounded, then each of the terms in R can be estimated in terms of the velocity and its gradient and the forcing.

Both formulations consist of a system of ordinary differential equations with algebraic constraints. We are interested in the evolution for $t \geq 0$ of this system. To close the system we must add initial and boundary conditions.

The initial conditions are

$$\begin{aligned} u_{i,j+1/2}(0) &= U^{(0)}(x_i, y_{j+1/2}), \qquad i \geq 1, \\ v_{i+1/2,j}(0) &= V^{(0)}(x_{i+1/2}, y_j), \qquad i \geq 0. \end{aligned} \tag{3.1.4}$$

For consistency between the initial conditions and the divergence equation it is required that the discrete divergence of the initial data is identically zero. This constraint is not required for the pressure formulation. The initial pressure field is determined by solving the elliptic pressure equation once the boundary conditions have been properly specified.

The number of boundary conditions needed is determined by a counting argument. The unknowns are

$$\left\{(u_{i,j+1/2},\, v_{i+1/2,j},\, p_{i+1/2,j+1/2}) :\; i = -1, 0, 1, \ldots,\; j = 0, 1, \ldots, N-1\right\}.$$

The differential equations and algebraic constraints (3.1.1) provide three equations for each grid point $i \geq 0$. Hence, three additional equations (i.e., boundary conditions) are needed. For the divergence formulation (3.1.1) one of these boundary conditions is the divergence equation on the line $x = x_{-1/2}$. The form of the boundary conditions will be discussed later.

We define the 1-D discrete inner product and norm for real functions u, $\widetilde{u}$ defined at the points x_i by

$$(u, \widetilde{u})_{hx} = (u_i, \widetilde{u}_i)_{hx} := (\frac{1}{2} - \alpha) h u_0 \widetilde{u}_0 + \sum_{i=1}^{\infty} h u_i \widetilde{u}_i, \qquad \|u\|_{hx}^2 := (u, u)_{hx},$$

and for functions v, $\widetilde{v}$ defined at the half points $x_{i+1/2}$ by

$$(v, \widetilde{v})_{hx} = (v_i, \widetilde{v}_i)_{hx} := (-\alpha) h v_{-1/2} \widetilde{v}_{-1/2} + \sum_{i=0}^{\infty} h v_{i+1/2} \widetilde{v}_{i+1/2}, \quad \|v\|_{hx}^2 := (v, v)_{hx}.$$

These definitions are for α in the range $-1/2 \leq \alpha \leq 0$. Similar definitions can be made for other choices of α, say α between 0 and 1/2. The dependence of the inner product on α has been suppressed here to simplify the notation. These inner products are both second order approximations to the continuous inner product from $x = x_\alpha = 0$ to $x = \infty$.

The two dimensional inner product is defined by

$$(u, \widetilde{u})_h = (u, \widetilde{u})_{h,\alpha} = (u_{i,j+1/2}, \widetilde{u}_{i,j+1/2})_h := \sum_{j=0}^{N-1} h (u_{i,j+1/2}, \widetilde{u}_{i,j+1/2})_{hx}$$

for functions defined at $(x_i, y_{j+1/2})$ and similarly for functions defined at the other staggered points $(x_{i+1/2}, y_j)$ and $(x_{i+1/2}, y_{j+1/2})$. As in the continuous case we define

$$\mathbf{w} = \mathbf{w}_{i,j} := \begin{pmatrix} u_{i,j+1/2} \\ v_{i+1/2,j} \end{pmatrix}, \qquad \mathbf{f} = \mathbf{f}_{i,j} := \begin{pmatrix} f_1(x_i, y_{j+1/2}) \\ f_2(x_{i+1/2}, y_j) \end{pmatrix}$$

with

$$\|\mathbf{w}\|_h^2 := \|u\|_{h,\alpha_1}^2 + \|v\|_{h,\alpha_2}^2,$$

$$\|\nabla_h \mathbf{w}\|_h^2 := \|D_x u\|_{h,\beta_1}^2 + \|D_y u\|_{h,\beta_1}^2 + \|D_x v\|_{h,\beta_2}^2 + \|D_y v\|_{h,\beta_2}^2.$$

3.2 Summation by Parts

We shall need some formulae for summation by parts. The boundary terms which result depend on the definition of the inner products; there are some cases of special interest and these are noted. It is only the summation by parts over x which produces boundary terms. The periodic boundary conditions in the y direction guarantee that the summation by parts over y is trivial. For notational convenience we ignore the y dependence throughout this section. All the appropriate summations over j are assumed.

Lemma 3.2.1 *u convection terms.*

$$\begin{aligned}(u, D_{0x}u)_{h,\alpha} &= -\frac{1}{2}u_0\left[\left(\frac{1}{2}+\alpha\right)u_1 + \left(\frac{1}{2}-\alpha\right)u_{-1}\right] \\ &= -\frac{1}{2}u_0\left[S_{0x}u_0 + 2\alpha h D_{0x}u_0\right]\end{aligned} \tag{3.2.1}$$

Special cases:

$$\alpha = 0: \qquad (u, D_{0x}u)_{h,0} = -\frac{1}{2}u_0 S_{0x}u_0,$$

$$\alpha = -1/2: \qquad (u, D_{0x}u)_{h,-1/2} = -\frac{1}{2}u_0 u_{-1}.$$

Proof.

$$\begin{aligned}(u, D_{0x}\widetilde{u})_{h,\alpha} &= \frac{1}{2}\left[\left(\frac{1}{2}-\alpha\right)u_0(\widetilde{u}_1 - \widetilde{u}_{-1}) + \sum_{i=1}^{\infty} u_i(\widetilde{u}_{i+1} - \widetilde{u}_{i-1})\right] \\ &= \frac{1}{2}\left[\left(\frac{1}{2}-\alpha\right)u_0(\widetilde{u}_1 - \widetilde{u}_{-1}) \right.\\ &\qquad \left. + \sum_{i=1}^{\infty}(u_{i-1}\widetilde{u}_i - u_{i+1}\widetilde{u}_i) - u_0\widetilde{u}_1 - u_1\widetilde{u}_0\right] \\ &= \frac{1}{2}\left[\left(\frac{1}{2}-\alpha\right)u_0(\widetilde{u}_1 - \widetilde{u}_{-1}) - u_0\widetilde{u}_1 - u_1\widetilde{u}_0 \right.\\ &\qquad \left. + \left(\frac{1}{2}-\alpha\right)(u_1 - u_{-1})\widetilde{u}_0\right] - (D_{0x}u, \widetilde{u})_{h,\alpha}\end{aligned}$$

Hence

$$\begin{aligned}(u, D_{0x}\widetilde{u})_{h,\alpha} + (D_{0x}u, \widetilde{u})_{h,\alpha} = \frac{1}{2}&\left[-\left(\frac{1}{2}+\alpha\right)(u_0\widetilde{u}_1 + u_1\widetilde{u}_0)\right. \\ &\left. -\left(\frac{1}{2}-\alpha\right)(u_0\widetilde{u}_{-1} + u_{-1}\widetilde{u}_0)\right].\end{aligned}$$

Lemma 3.2.2 *v convection terms.*

$$
\begin{aligned}
(v, D_{0x}v)_{h,\alpha} &= -\frac{1}{2}v_{-1/2}\left[(1+\alpha)\,v_{1/2} - \alpha v_{-3/2}\right] \\
&= -\frac{1}{2}v_{-1/2}\left[v_{1/2} + 2\alpha h D_{0x}v_{-1/2}\right]
\end{aligned}
\tag{3.2.2}
$$

Special case:

$$
\alpha = 0: \qquad (v, D_{0x}v)_{h,0} = -\frac{1}{2}v_{-1/2}v_{1/2}.
$$

Proof. Replace $\left(\frac{1}{2}-\alpha\right)$ by $(-\alpha)$ in the formula (3.2.1) above.

Lemma 3.2.3 *Pressure terms.*

$$
\begin{aligned}
(u, D_xp)_{h,\alpha} = &-2\left(\frac{1}{2}+\alpha\right)u_0S_xp_0 - 2(-\beta)S_xu_{-1/2}p_{-1/2} + 2(\alpha-\beta)u_0p_{-1/2} \\
&- (D_xu, p)_{h,\beta}
\end{aligned}
\tag{3.2.3}
$$

Special cases:

$$
\begin{aligned}
\alpha = 0,\ \beta = 0: &\qquad \textit{boundary terms} = -u_0S_xp_0, \\
\alpha = -1/2,\ \beta = -1/2: &\qquad \textit{boundary terms} = -S_xu_{-1/2}p_{-1/2}, \\
\alpha = -1/2,\ \beta = 0: &\qquad \textit{boundary terms} = -u_0p_{-1/2}.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
(u, D_xp)_{h,\alpha} &= \left(\frac{1}{2}-\alpha\right)u_0(p_{1/2}-p_{-1/2}) + \sum_{i=1}^{\infty}u_i(p_{i+1/2}-p_{i-1/2}) \\
&= \left(\frac{1}{2}-\alpha\right)u_0(p_{1/2}-p_{-1/2}) \\
&\quad + \sum_{i=0}^{\infty}(u_ip_{i+1/2} - u_{i+1}p_{i+1/2}) - u_0p_{1/2} \\
&= \left(\frac{1}{2}-\alpha\right)u_0(p_{1/2}-p_{-1/2}) - u_0p_{1/2} + (-\beta)(u_0-u_{-1})p_{-1/2} \\
&\quad - (D_xu, p)_{h,\beta} \\
&= -\left(\frac{1}{2}+\alpha\right)u_0p_{1/2} - \left(\frac{1}{2}-\alpha+\beta\right)u_0p_{-1/2} - (-\beta)u_{-1}p_{-1/2} \\
&\quad - (D_xu, p)_{h,\beta} \\
&= -\left(\frac{1}{2}+\alpha\right)u_0(p_{1/2}+p_{-1/2}) - (-\beta)(u_{-1}+u_0)p_{-1/2} \\
&\quad + 2(\alpha-\beta)u_0p_{-1/2} - (D_xu, p)_{h,\beta}
\end{aligned}
$$

Lemma 3.2.4 *u diffusion terms.*

$$\begin{aligned}(u, D_{+x}D_{-x}u)_{h,\alpha} = -2\left(\frac{1}{2}+\alpha\right)u_0 D_{0x}u_0 - 2(-\beta)S_x u_{-1/2}D_x u_{-1/2} \\ + 2(\alpha-\beta)u_0 D_x u_{-1/2} - \|D_x u\|^2_{h,\beta}\end{aligned} \tag{3.2.4}$$

Special cases:

$$\alpha = 0,\ \beta = 0: \qquad \textit{boundary terms} = -u_0 D_{0x}u_0,$$

$$\alpha = -1/2,\ \beta = -1/2: \qquad \textit{boundary terms} = -S_x u_{-1/2}D_x u_{-1/2},$$

$$\alpha = -1/2,\ \beta = 0: \qquad \textit{boundary terms} = -u_0 D_x u_{-1/2}.$$

Proof. Write

$$(u, D_{+x}D_{-x}u)_{h,\alpha} = (u, D_x w)_{h,\alpha}, \qquad \text{where } w := D_x u.$$

Then the result follows by using Lemma Lemma 3.2.3.

.

Lemma 3.2.5 *v diffusion terms.*

$$\begin{aligned}(v, D_{+x}D_{-x}v)_{h,\alpha} = -2(-\alpha)v_{-1/2}D_{0x}v_{-1/2} - 2\left(\frac{1}{2}+\beta\right)S_x v_0 D_x v_0 \\ - 2(\alpha-\beta)v_{-1/2}D_x v_0 - \|D_x v\|^2_{h,\beta}\end{aligned} \tag{3.2.5}$$

Special cases:

$$\alpha = 0,\ \beta = 0: \qquad \textit{boundary terms} = -S_x v_0 D_x v_0,$$

$$\alpha = 0,\ \beta = -1/2: \qquad \textit{boundary terms} = -v_{-1/2}D_x v_0.$$

Proof. Let $w := D_x v$. Then

$$\begin{aligned}(v, D_x w)_{h,\alpha} &= (-\alpha)v_{-1/2}(w_0 - w_{-1}) + \sum_{i=0}^{\infty} v_{i+1/2}(w_{i+1} - w_i) \\ &= (-\alpha)v_{-1/2}(w_0 - w_{-1}) + \sum_{i=1}^{\infty}(v_{i-1/2}w_i - v_{i+1/2}w_i) - v_{1/2}w_0 \\ &= (-\alpha)v_{-1/2}(w_0 - w_{-1}) - v_{1/2}w_0 \\ &\quad + \left(\frac{1}{2}-\beta\right)(v_{1/2} - v_{-1/2})w_0 - (D_x v, w)_{h,\beta} \\ &= -(-\alpha)v_{-1/2}(w_0 + w_{-1}) - \left(\frac{1}{2}+\beta\right)(v_{1/2} + v_{-1/2})w_0 \\ &\quad + (-2\alpha + 2\beta)v_{-1/2}w_0 - (D_x v, w)_{h,\beta} \\ &= -2(-\alpha)v_{-1/2}S_x w_{-1/2} - 2\left(\frac{1}{2}+\beta\right)S_x v_0 w_0 \\ &\quad - 2(\alpha-\beta)v_{-1/2}w_0 - (D_x v, w)_{h,\beta}.\end{aligned}$$

3.3 Estimates for the Divergence and the Velocity

The summation by parts formulae can be used to obtain energy estimates for the divergence and the velocity components. The estimate for the divergence in the pressure formulation is obtained by taking the inner product of δ with the equation (3.1.3) satisfied by the divergence. The inner product is taken from $\alpha = 0$ since δ is defined only for $x_{i+1/2}$, $i \geq -1$. Otherwise, if $\alpha < 0$ then the summation by parts brings in $\delta_{-3/2}$. The result is

$$\frac{1}{2}\frac{d}{dt}\|\delta\|_{h,0}^2 + \nu\|D_x\delta\|_{h,\beta}^2 + \nu\|D_y\delta\|_{h,0}^2 = \sum_{j=0}^{N-1} h\, H_{j+1/2} \tag{3.3.1}$$

where

$$\begin{aligned} H_{j+1/2} &:= \frac{c}{2}\delta_{-1/2}\delta_{1/2} - 2\left(\frac{1}{2}+\beta\right)\nu S_x\delta_0 D_x\delta_0 - 2(-\beta)\nu\delta_{-1/2}D_x\delta_0 \\ &= \frac{c}{2}\delta_{-1/2}\delta_{1/2} - \nu\left[\left(\frac{1}{2}+\beta\right)\delta_{1/2} + \left(\frac{1}{2}-\beta\right)\delta_{-1/2}\right]D_x\delta_0 \\ &= \frac{c}{2}\delta_{-1/2}\delta_{1/2} - \nu S_x\delta_0 D_x\delta_0 - \beta h\nu\left(D_x\delta_0\right)^2. \end{aligned} \tag{3.3.2}$$

Each of the δ terms in this expression also depends on $j + 1/2$. Two particular choices of the inner product for the gradient terms are of special interest.

$$\begin{aligned} \beta = 0: &\qquad H = \frac{c}{2}\delta_{-1/2}\delta_{1/2} - \nu S_x\delta_0 D_x\delta_0, \\ \beta = -1/2: &\qquad H = \frac{c}{2}\delta_{-1/2}\delta_{1/2} - \nu\delta_{-1/2}D_x\delta_0. \end{aligned}$$

The energy estimate for the velocity is obtained by taking $(u, (3.1.1a))_{h,\alpha} + (v, (3.1.1b))_{h,0}$.

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}&\left(\|u\|_{h,\alpha}^2 + \|v\|_{h,0}^2\right) + \nu\left(\|D_xu\|_{h,\beta_1}^2 + \|D_yu\|_{h,\alpha}^2\right. \\ &\left. + \|D_xv\|_{h,\beta_2}^2 + \|D_yv\|_{h,0}^2\right) = (\delta, p)_{h,0} + \sum_{j=0}^{N-1} hG_j \end{aligned} \tag{3.3.3}$$

where the boundary terms are

$$
\begin{aligned}
G_j := \frac{c}{2}&\left\{u_0\left[\left(\frac{1}{2}+\alpha\right)u_1+\left(\frac{1}{2}-\alpha\right)u_{-1}\right]+v_{-1/2}v_{1/2}\right\}\\
&+u_0\left[\left(\frac{1}{2}+\alpha\right)p_{1/2}+\left(\frac{1}{2}-\alpha\right)p_{-1/2}\right]\\
&-2\nu\left[\left(\frac{1}{2}+\alpha\right)u_0D_{0x}u_0-\beta_1S_xu_{-1/2}D_xu_{-1/2}-(\alpha-\beta_1)u_0D_xu_{-1/2}\right]\\
&-\nu\left[\left(\frac{1}{2}+\beta_2\right)v_{1/2}+\left(\frac{1}{2}-\beta_2\right)v_{-1/2}\right]D_xv_0.
\end{aligned}
\tag{3.3.4}
$$

Special cases: (i) $\alpha = 0$:

$$
\begin{aligned}
G_j = \frac{c}{2}&\left[u_0S_{0x}u_0+v_{-1/2}v_{1/2}\right]+u_0S_xp_0\\
&-\nu\left[u_0D_{0x}u_0+2\beta_1h\left(D_xu_{-1/2}\right)^2\right]\\
&-\nu\left[\left(\frac{1}{2}+\beta_2\right)v_{1/2}++\left(\frac{1}{2}-\beta_2\right)v_{-1/2}\right]D_xv_0.
\end{aligned}
$$

If we take $\beta_1 = \beta_2 = 0$ also, the formula becomes

$$
\begin{aligned}
G_j = \frac{c}{2}&\left[u_0S_{0x}u_0+v_{-1/2}v_{1/2}\right]+u_0S_xp_0\\
&-\nu\left[u_0D_{0x}u_0+S_xv_0D_xv_0\right].
\end{aligned}
$$

(ii) $\alpha = -1/2$:

$$
\begin{aligned}
G_j = \frac{c}{2}&\left[u_0u_{-1}+v_{-1/2}v_{1/2}\right]+u_0p_{-1/2}\\
&-\nu\left[(1+\beta_1)u_0-\beta_1u_{-1}\right]D_xu_{-1/2}\\
&-\nu\left[\left(\frac{1}{2}+\beta_2\right)v_{1/2}++\left(\frac{1}{2}-\beta_2\right)v_{-1/2}\right]D_xv_0.
\end{aligned}
$$

The terms depending on β_1 and β_2, respectively, in this formula reduce to simpler expressions in special cases of interest.

$$
\begin{aligned}
\beta_1 = -1/2: &\qquad -\nu\, S_xu_{-1/2}D_xu_{-1/2},\\
\beta_1 = 0: &\qquad -\nu\, u_0D_xu_{-1/2},\\
\beta_2 = -1/2: &\qquad -\nu\, v_{-1/2}D_xv_0,\\
\beta_2 = 0: &\qquad -\nu\, S_xv_0D_xv_0.
\end{aligned}
$$

3.4 Boundary Conditions

We discuss the types of boundary conditions which will guarantee that the boundary terms G and H arising in the estimates for the velocity components and the divergence have the right sign. The boundary conditions are classified into the same four types which we studied for the continuous equations. We consider two particular choices for the location of the boundary line $x = 0$.

First consider the case where the boundary line is $x_0 = 0$. The u velocity points lie on the boundary but the boundary lies midway between v and p points. This is the case $\alpha = 0$, and we take $\beta = \beta_1 = \beta_2 = 0$ also. The formulae for G and H are

$$\begin{aligned} G_j &= \frac{c}{2}\left[u_0 S_{0x} u_0 + v_{-1/2} v_{1/2}\right] + u_0 S_x p_0 \\ &\quad - \nu\left[u_0 D_{0x} u_0 + S_x v_0 D_x v_0\right], \\ H_j &= \frac{c}{2}\delta_{-1/2}\delta_{1/2} - \nu S_x \delta_0 D_x \delta_0. \end{aligned}$$

Type I: Velocity boundary conditions.

$$u_0 = 0$$

$$S_x v_0 = 0 \qquad \left\{\begin{array}{l} G = -\dfrac{c}{2} v_{1/2}^2 \\ H = -\dfrac{c}{2}\delta_{1/2}^2 \end{array}\right\}$$

$$D_{0x} u_0 = 0$$

$$(\ S_x \delta_0 = 0\ )$$

Type I b: Velocity boundary conditions for Outflow.

$$u_0 = 0$$

$$D_x v_0 = 0 \qquad \left\{\begin{array}{l} G = \dfrac{c}{2} v_{1/2}^2 \\ H = \dfrac{c}{2}\delta_{1/2}^2 \end{array}\right\}$$

$$D_{+x} D_{-x} u_0 = 0$$

$$(\ D_x \delta_0 = 0\ )$$

Type II: Inflow boundary conditions.

$$S_x p_0 + \frac{c}{2} S_{0x} u_0 = -\sigma u_0$$

$$S_x v_0 = 0 \qquad \left\{ \begin{array}{r} G = -\sigma u_0^2 - \frac{c}{2} v_{1/2}^2 \\ H = -\frac{c}{2} \delta_{1/2}^2 \end{array} \right\}$$

$$D_{0x} u_0 = 0$$

$$(\; S_x \delta_0 = 0 \;)$$

Type III: Outflow boundary conditions.

$$S_x p_0 + \gamma u_0 - \nu D_{0x} u_0 = 0$$

$$D_x v_0 = 0 \qquad \left\{ \begin{array}{r} G = -\left(\gamma - \frac{c}{2}\right) u_0^2 + \frac{c}{2} v_{1/2}^2 \\ H = \frac{c}{2} \delta_{1/2}^2 \end{array} \right\}$$

$$D_{+x} D_{-x} u_0 = 0$$

$$(\; D_x \delta_0 = 0 \;)$$

Type IV: Outflow boundary conditions.

$$S_x p_0 = 0$$

$$D_x v_0 = 0 \qquad \left\{ G = -\sigma u_0^2 + \frac{c}{2} v_{1/2}^2 \right\}$$

$$\frac{c}{2} S_{0x} u_0 - \nu D_{0x} u_0 = -\sigma u_0$$

Next consider the case where the boundary line is $x_{-1/2} = 0$. The v velocity points and the pressure points lie on the boundary; the boundary lies midway between the u velocity points. This is the case $\alpha = -1/2$ and we also take $\beta = \beta_1 = \beta_2 = -1/2$. Then

$$\begin{aligned} G_j = \frac{c}{2} &\left[u_0 u_{-1} + v_{-1/2} v_{1/2} \right] + u_0 p_{-1/2} \\ &- \nu \left[S_x u_{-1/2} D_x u_{-1/2} + v_{-1/2} D_x v_0 \right\} \end{aligned}$$

$$H_j = \frac{c}{2} \delta_{-1/2} \delta_{1/2} - \nu \delta_{-1/2} D_x \delta_0 .$$

Type I: Velocity boundary conditions.

$$S_x u_{-1/2} = 0$$

$$v_{-1/2} = 0 \qquad \left\{ \begin{array}{l} G = 0 \\ H = 0 \end{array} \right\}$$

$$D_x u_{-1/2} = 0$$

$$(\ \delta_{-1/2} = 0\ )$$

Type II: Inflow boundary conditions.

$$p_{-1/2} + \gamma S_x u_{-1/2} = 0$$

$$v_{-1/2} = 0 \qquad \left\{ \begin{array}{r} G = -\left(\gamma - \frac{c}{2}\right) u_0^2 \\ H = 0 \end{array} \right\}$$

$$D_x u_{-1/2} = 0$$

$$(\ \delta_{-1/2} = 0\ )$$

Type III: Outflow boundary conditions. There is trouble in this case to eliminate the $S_x u_{-1/2} D_x u_{-1/2}$ term. To fix this, change to $\beta_1 = 0$. Then the boundary conditions which yield an estimate are not centred on $x_{-1/2} = 0$. They are

$$p_{-1/2} + \frac{c}{2} u_{-1} - \nu D_x u_{-1/2} = -\sigma u_0$$

$$D_x v_0 = 0 \qquad \left\{ \begin{array}{r} G = -\sigma u_0^2 + \frac{c}{2} v_{1/2}^2 \\ H = \frac{c}{2} \delta_{1/2}^2 \end{array} \right\}$$

$$D_{+x} D_{-x} u_0 = 0$$

$$(\ D_x \delta_0 = 0\ )$$

Type IV: Outflow boundary conditions.

$$p_{-1/2} = 0$$

$$D_x v_0 = 0 \qquad \left\{ G = \frac{c}{2} \left(u_0^2 + v_{1/2}^2 \right) \right\}$$

$$D_x u_{-1/2} = 0$$

CHAPTER 4

Fully Discrete Linear Navier-Stokes Equations

4.1 Introduction

We now consider discretizing in time. We are interested in choosing a method which will enable us to make energy estimates of the solution as we have above in the continuous and the semi-discrete cases.

The simplest discretization which gives energy estimates is Crank-Nicholson. The estimates go through easily in this case. This method is fully implicit and hence for the nonlinear Navier-Stokes equations the solution of a nonlinear system is required at each time step.

Another possibility is to use a semi-implicit method which treats the nonlinear terms explicitly. We consider the leap frog–Crank-Nicholson method (LF-CN), which uses leap frog for the convective terms and the pressure terms and Crank-Nicholson over two time steps for the diffusive terms. The result is a scheme which is stable and second order accurate. The technique for obtaining energy estimates for the LF-CN method is well known for hyperbolic and parabolic problems. It involves defining an equivalent energy norm which can be estimated directly.

As in the continuous and semi-discrete cases boundary terms arise in the energy estimates. For the energy method to succeed the boundary conditions must be chosen such that these boundary terms have the right sign. We find that the LF-CN time discretization leads to a coupling between the different time levels in the boundary conditions.

The sequence of the estimates in each of the four cases is exactly the same as those which were made above in the continuous case and so we shall not go through all the details. We must look at how the time discretization affects these estimates and how the boundary conditions are modified to account for this.

The semi-discrete linear Navier-Stokes equations can be written in symbolic form as

$$\begin{aligned} \mathbf{w}_t &= A\mathbf{w} + B\mathbf{w} + Cp, \\ \Delta_h p + R(\mathbf{w}) &= 0 \end{aligned} \tag{4.1.1}$$

where $\mathbf{w}$ represents the velocity components, $A\mathbf{w}$ is the convective terms, $B\mathbf{w}$ is the diffusive terms, and Cp is the pressure terms. That is

$$\mathbf{w} = \mathbf{w}(x_i, y_j, n\Delta t) = \mathbf{w}^n_{i,j} = \begin{pmatrix} u^n_{i,j+1/2} \\ v^n_{i+1/2,j} \end{pmatrix},$$

$$p = p(x_{i+1/2}, y_{j+1/2}, n\Delta t) = p^n_{i+1/2,j+1/2},$$

$$A\mathbf{w} = -(cD_{0x}\mathbf{w} + dD_{0y}\mathbf{w}) = -\begin{pmatrix} cD_{0x}u + dD_{0y}u \\ cD_{0x}v + dD_{0y}v \end{pmatrix},$$

$$B\mathbf{w} = \nu\Delta_h\mathbf{w} = \begin{pmatrix} \nu\Delta_h u \\ \nu\Delta_h v \end{pmatrix}, \qquad Cp = -\nabla_h p = -\begin{pmatrix} D_x p \\ D_y p \end{pmatrix}.$$

The Crank-Nicholson scheme is

$$\begin{aligned} \frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t} &= A\widetilde{\mathbf{w}^n} + B\widetilde{\mathbf{w}^n} + C\widetilde{p^n}, \\ \Delta_h\widetilde{p^n} + R(\widetilde{\mathbf{w}^n}) &= 0 \end{aligned} \tag{4.1.2}$$

where the snook denotes the average of the time levels t_n and t_{n+1}

$$\widetilde{u^n} := \frac{u^{n+1} + u^n}{2}.$$

The leap frog–Crank-Nicholson scheme is

$$\frac{\mathbf{w}^{n+1} - \mathbf{w}^{n-1}}{2\Delta t} = A\mathbf{w}^n + B\frac{\mathbf{w}^{n+1} + \mathbf{w}^{n-1}}{2} + Cp^n, \tag{4.1.3a}$$

$$\Delta_h p^n + R(\mathbf{w}^n) = 0. \tag{4.1.3b}$$

4.2 Crank-Nicholson Method

The stability of the Crank-Nicholson method follows easily from the estimates for the semi-discrete problem. The boundary conditions which we found in Section 3.4 are applied at each time level. The estimates for the divergence and the velocity components and the role of the boundary conditions mimic closely the situation for the semi-discrete equations discussed above. We outline the details.

In the pressure formulation the divergence $\delta = D_x u + D_y v$ satisfies

$$\frac{\delta^{n+1} - \delta^n}{\Delta t} = A\widetilde{\delta^n} + B\widetilde{\delta^n}.$$

Taking the inner product of $\widetilde{\delta^n} = \left(\delta^n + \delta^{n+1}\right)/2$ with this equation gives

$$\begin{aligned}\frac{\|\delta^{n+1}\|_h^2 - \|\delta^n\|_h^2}{2\Delta t} &= (\widetilde{\delta^n}, A\widetilde{\delta^n})_h + (\widetilde{\delta^n}, B\widetilde{\delta^n})_h \\ &= -\nu\|\nabla_h\widetilde{\delta^n}\|_h^2 + \sum_{j=0}^{N-1} h\,\widetilde{H}_j^n.\end{aligned}$$

The right hand side of this equation can be summed by parts in the same way as in the semi-discrete case. The boundary terms which result are of the same form as H defined in (3.3.2) with δ replaced by the time average $\widetilde{\delta^n}$.

Similarly, the estimate for the velocity terms comes from the inner product of $\widetilde{\mathbf{w}^n}$ with (4.1.2). The result is

$$\begin{aligned}\frac{\|\mathbf{w}^{n+1}\|_h^2 - \|\mathbf{w}^n\|_h^2}{2\Delta t} &= (\widetilde{\mathbf{w}^n}, A\widetilde{\mathbf{w}^n})_h + (\widetilde{\mathbf{w}^n}, B\widetilde{\mathbf{w}^n})_h + (\widetilde{\mathbf{w}^n}, C\widetilde{p^n})_h \\ &= -\nu\|\nabla_h\widetilde{\mathbf{w}^n}\|_h^2 + (\widetilde{\delta^n}, \widetilde{p^n})_h + \sum_{j=0}^{N-1} h\,\widetilde{G}_j^n\end{aligned}$$

where $\widetilde{G}$ has the same form as G of the continuous case defined in (3.3.4) with u, v, p replaced by $\widetilde{u^n}$, $\widetilde{v^n}$, $\widetilde{p^n}$.

In Section 3.4 we discussed the choice of boundary conditions which guarantee that G and H are both non-positive at each point on the boundary (i.e., for any j) at any time t. The same boundary conditions can be used with Crank-Nicholson time discretization if they are applied at each time level t_n. If the boundary conditions are applied at each time level separately then they can also be averaged in time — for example, if $u(t_n) = u^n = 0$ for all n then $\widetilde{u^n} = 0$ also for all n. Hence it follows that $\widetilde{G} = \widetilde{G}_j^n$ and $\widetilde{H} = \widetilde{H}_j^n$ are both non-positive for all y_j and all t_n.

4.3 Leap Frog–Crank-Nicholson Method

We study the stability of the leap frog–Crank-Nicholson method for the linear Navier-Stokes equations. The energy method requires modification to be applied with leap frog time discretization of the convective terms. The usual estimates for the L^2 norms of the divergence and the velocity which we have used repeatedly above fail here because the terms appearing cannot be removed by summation by parts. The type of modification required is well known; it is discussed by Richtmyer and Morton [1967]. An equivalent norm is introduced involving the solution at two time levels. The estimates for the divergence and the velocity are made in the equivalent norm.

There is a stability limit on the time step Δt in terms of the grid spacing h. This is the usual CFL condition. It arises in the analysis in enforcing the requirement that the new norm be equivalent to the usual L^2 norm.

4.3.1 Boundary Conditions

Before going into the details of the equivalent norms we jump ahead to examine the boundary conditions. The boundary conditions are chosen in the (by now) familiar way to bound the boundary terms $\overline{G}$, $\overline{H}$ which result from the summation by parts. The boundary terms $\overline{G}$, $\overline{H}$ involve a coupling between the solution at the time level t_n and the average over t_{n-1}, t_{n+1}. For all the types of boundary conditions which involve the pressure (i.e., types II, III, IV) the boundary conditions couple the different time levels.

As we have found previously, the boundary terms depend on the location of the boundary line and the precise way the discrete inner products are defined. The general form will be given later. The special case which leads to the simplest boundary conditions is the case $x_{-1/2} = 0$ where the v velocity and the pressure are defined on the boundary line. In this case ($\alpha = -1/2$) the boundary terms have the form

$$\begin{aligned}\overline{G_j^n} = & \frac{c}{2}\left[\overline{u_0^n u_{-1}^n} + \overline{v_{1/2}^n v_{-1/2}^n}\right] + \overline{u_0^n p_{-1/2}^n} \\ & - \nu\left[(1+\beta_1)\overline{u_0^n} - \beta_1 \overline{u_{-1}^n}\right] D_x \overline{u_{-1/2}^n} \\ & - \nu\left[\left(\frac{1}{2}+\beta_2\right)\overline{v_{1/2}^n} + + \left(\frac{1}{2}-\beta_2\right)\overline{v_{-1/2}^n}\right] D_x \overline{v_0^n},\end{aligned}$$

$$\overline{H_j^n} := \frac{c}{2}\overline{\delta_{1/2}^n \delta_{-1/2}^n} - \nu\left[\left(\frac{1}{2}+\beta\right)\overline{\delta_{1/2}^n} + \left(\frac{1}{2}-\beta\right)\overline{\delta_{-1/2}^n}\right] D_x \overline{\delta_0^n}$$

where the bar denotes the average over t_{n-1} and t_{n+1}

$$\overline{u^n} := \frac{u^{n-1} + u^{n+1}}{2}.$$

The boundary conditions are classified into the same four types which we have studied in the previous chapters.

Type I: Velocity boundary conditions. Take $\beta = \beta_1 = \beta_2 = -\frac{1}{2}$. The boundary conditions are applied at each time level separately.

$$\begin{aligned} S_x u_{-1/2} &= 0 \\ v_{-1/2} &= 0 \qquad\qquad \left\{\begin{matrix} \overline{G} = 0 \\ \overline{H} = 0 \end{matrix}\right\} \\ D_x u_{-1/2} &= 0 \\ (\ \delta_{-1/2} &= 0\ ) \end{aligned}$$

Type II: Inflow boundary conditions. Take $\beta = \beta_1 = \beta_2 = -\frac{1}{2}$. The pressure boundary condition involves coupling between the time levels.

$$\begin{aligned} p_{-1/2} + \frac{c}{2} S_x u_{-1/2} &= -\sigma \overline{S_x u^n_{-1/2}} \\ v_{-1/2} &= 0 \\ D_x u_{-1/2} &= 0 \\ (\delta_{-1/2} &= 0) \end{aligned} \qquad \left\{ \begin{aligned} \overline{G} &= -\sigma(\overline{u^n_0})^2 \\ \overline{H} &= 0 \end{aligned} \right\}$$

Type III: Outflow boundary conditions. Take $\beta_1 = 0$, $\beta = \beta_2 = \frac{1}{2}$. All the boundary conditions involve coupling between the time levels.

$$\begin{aligned} p_{-1/2} + \frac{c}{2} u_{-1} - \nu \overline{D_x u^n_{-1/2}} &= -\sigma_1 \overline{u^n_0} \\ \frac{c}{2} v_{-1/2} - \nu D_x \overline{v^n_0} &= -\sigma_2 \overline{v^n_{1/2}} \\ \frac{c}{2} D_x u_{-1/2} - \nu D_{+x} D_{-x} \overline{u^n_0} &= -\sigma_2 \overline{D_x u^n_{1/2}} \\ (\frac{c}{2} \delta_{-1/2} - \nu D_x \overline{\delta^n_0} &= -\sigma_2 \overline{\delta^n_{1/2}}) \end{aligned} \qquad \left\{ \begin{aligned} \overline{G} &= -\sigma_1(\overline{u^n_0})^2 - \sigma_2(\overline{v^n_{1/2}})^2 \\ \overline{H} &= -\sigma_2(\overline{\delta^n_{1/2}})^2 \end{aligned} \right\}$$

Type IV: Outflow boundary conditions. Take $\beta_1 = 0$, $\beta_2 = \frac{1}{2}$. The Neumann boundary conditions on the velocity components involve coupling between the time levels.

$$\begin{aligned} p_{-1/2} &= 0 \\ \frac{c}{2} v_{-1/2} - \nu D_x \overline{v^n_0} &= -\sigma_2 \overline{v^n_{1/2}} \\ \frac{c}{2} u_{-1} - \nu D_x \overline{u^n_{-1/2}} &= -\sigma_1 \overline{u^n_0} \end{aligned} \qquad \left\{ \overline{G} = -\sigma_1(\overline{u^n_0})^2 - \sigma_2(\overline{v^n_{1/2}})^2 \right\}$$

4.3.2 Equivalent Norms and Estimates

Now we go through the calculations in detail. First we seek an estimate for the divergence. The divergence $\delta = D_x u + D_y v$ satisfies

$$\frac{\delta^{n+1} - \delta^{n-1}}{2\Delta t} = A\delta^n + B\overline{\delta^n}. \tag{4.3.1}$$

Taking the inner product of $\delta^{n+1} + \delta^{n-1}$ with this equation gives

$$\frac{\|\delta^{n+1}\|_h^2 - \|\delta^{n-1}\|_h^2}{2\Delta t} = (\delta^{n+1} + \delta^{n-1}, A\delta^n)_h + 2(\overline{\delta^n}, B\overline{\delta^n})_h. \tag{4.3.2}$$

The hitch with applying the usual energy argument can be seen in this equation. It is the terms

$$(\delta^{n+1} + \delta^{n-1}, A\delta^n)_h = -(\delta^{n+1} + \delta^{n-1}, c\,D_{0x}\delta^n + d\,D_{0y}\delta^n)_h \tag{4.3.3}$$

which cause the trouble. The summation by parts formula for the operator A is

$$(\delta, A\widehat{\delta})_{h,\alpha} = \text{boundary terms} - (A\delta, \widehat{\delta})_{h,\alpha}$$

for any two discrete functions δ, $\widehat{\delta}$ defined at the half points $x_{i+1/2}$. In all the cases treated previously the A terms arising in the equations corresponding to (4.3.2) have had the form $(\delta, A\delta)$, enabling the A terms to be replaced by boundary terms. An equivalent norm can be constructed in the present case to restore this property.

The boundary terms from the summation by parts have a symmetric form. They are

$$\begin{aligned}\text{boundary terms} &= \frac{c}{2}\Big[(1+\alpha)\left(\delta_{1/2}\widehat{\delta}_{-1/2} + \delta_{-1/2}\widehat{\delta}_{1/2}\right) \\ &\qquad -\alpha\left(\delta_{-1/2}\widehat{\delta}_{-3/2} + \delta_{-3/2}\widehat{\delta}_{-1/2}\right)\Big] \\ &= g(\delta, \widehat{\delta}) + g(\widehat{\delta}, \delta)\end{aligned} \tag{4.3.4}$$

where g is defined by

$$g(\delta, \widehat{\delta}) := \frac{c}{2}(1+\alpha)\,\delta_{1/2}\widehat{\delta}_{-1/2} - \alpha\delta_{-1/2}\widehat{\delta}_{-3/2}. \tag{4.3.5}$$

We concentrate on the special case $\alpha = 0$ since we have defined δ only for $x_{i+1/2}$, $i \geq -1$. In the special case $\alpha = 0$

$$g(\delta, \widehat{\delta}) := \frac{c}{2}\delta_{1/2}\widehat{\delta}_{-1/2}.$$

Define the equivalent norm $N = N(\delta)$ for the divergence by

$$N^n := \|\delta^n\|_h^2 + \|\delta^{n-1}\|_h^2 - 2\Delta t(\delta^n, A\delta^{n-1})_h + 2\Delta t \sum_{j=0}^{N-1} h\, g(\delta^n, \delta^{n-1}). \qquad (4.3.6)$$

Then

$$\begin{aligned}
\frac{N^{n+1} - N^n}{2\Delta t} &= (\delta^{n+1} + \delta^{n-1}, A\delta^n)_h + 2(\overline{\delta^n}, B\overline{\delta^n})_h \\
&\quad - (\delta^{n+1}, A\delta^n)_h + (\delta^n, A\delta^{n-1})_h + g(\delta^{n+1}, \delta^n) - g(\delta^n, \delta^{n-1}) \\
&= (\delta^{n-1}, A\delta^n)_h + (\delta^n, A\delta^{n-1})_h + g(\delta^{n+1}, \delta^n) - g(\delta^n, \delta^{n-1}) \\
&\quad + 2(\overline{\delta^n}, B\overline{\delta^n})_h \\
&= g(\delta^{n-1}, \delta^n) + g(\delta^n, \delta^{n-1}) + g(\delta^{n+1}, \delta^n) - g(\delta^n, \delta^{n-1}) \\
&\quad + 2(\overline{\delta^n}, B\overline{\delta^n})_h \\
&= -2\nu \|\nabla_h \overline{\delta^n}\|_h^2 + 2\sum_{j=0}^{N-1} h\overline{H}_j
\end{aligned}$$

where the sum over j in all the g terms is assumed. The boundary summand simplifies to

$$\overline{H}_j = g(\overline{\delta^n}, \delta^n) - 2\nu \left[\left(\frac{1}{2} - \beta\right) \overline{\delta^n_{-1/2}} + \left(\frac{1}{2} + \beta\right) \overline{\delta^n_{1/2}} \right] D_x \overline{\delta^n_0}.$$

The coefficients β arise from summation by parts of the diffusion terms since there is some freedom in the choice of the inner product.

The equivalence between N and the L^2 norm for δ is determined as follows.

$$\begin{aligned}
(\delta, D_{0x}\widehat{\delta})_{h,0} &= \sum_{i=0}^{\infty} h\delta_{i+1/2} D_{0x}\widehat{\delta}_{i+1/2} \\
&= \frac{1}{2}\sum_{i=0}^{\infty} \delta_{i+1/2}\left(\widehat{\delta}_{i+3/2} - \widehat{\delta}_{i-1/2}\right) \\
&= \frac{1}{2h}\left[(\delta_{i+1/2}, \widehat{\delta}_{i+3/2})_{h,0} - (\delta_{i+3/2}, \widehat{\delta}_{i+1/2})_{h,0}\right] - \frac{1}{2}\delta_{1/2}\widehat{\delta}_{-1/2},
\end{aligned}$$

so

$$\left|(\delta, D_{0x}\widehat{\delta})_{h,0} + \frac{1}{2}\delta_{1/2}\widehat{\delta}_{-1/2}\right| \leq \frac{1}{2h}\left(\|\delta\|_{h,0}^2 + \|\widehat{\delta}\|_{h,0}^2\right).$$

Likewise,

$$\left|(\delta, D_{0y}\widehat{\delta})_h\right| \leq \frac{1}{2h}\left(\|\delta\|_h^2 + \|\widehat{\delta}\|_h^2\right).$$

From these results it follows that

$$\left|(\delta^n, A\delta^{n-1})_h - g(\delta^n, \delta^{n-1})\right| \leq \frac{1}{2h}\left(|c|_\infty + |d|_\infty\right)\left[\|\delta^n\|_h^2 + \|\delta^{n-1}\|_h^2\right],$$

and so

$$\left[1 - \frac{K\Delta t}{h}\right]\left(\|\delta^n\|_h^2 + \|\delta^{n-1}\|_h^2\right) \leq N_n \leq \left[1 + \frac{K\Delta t}{h}\right]\left(\|\delta^n\|_h^2 + \|\delta^{n-1}\|_h^2\right)$$

for $K = |c|_\infty + |d|_\infty$. Hence, N_n is a norm and is equivalent to $\|\delta^n\|_h^2 + \|\delta^{n-1}\|_h^2$, provided

$$(|c|_\infty + |d|_\infty)\Delta t < h. \tag{4.3.7}$$

For the velocity components the analysis goes the same way. The evolution of the norm of the velocity is obtained by taking

$$(\mathbf{w}^{n+1} + \mathbf{w}^{n-1}, (4.1.3\text{a}))_h = 0,$$

where the discrete norm starts at $x = x_\alpha = 0$ for the first component and at $x = x_0$ for the second component. The resulting evolution equation for the norm of the velocity is

$$\frac{\|\mathbf{w}^{n+1}\|_h^2 - \|\mathbf{w}^{n-1}\|_h^2}{2\Delta t} = (\mathbf{w}^{n+1} + \mathbf{w}^{n-1}, A\mathbf{w}^n + Cp^n)_h + 2(\overline{\mathbf{w}^n}, B\overline{\mathbf{w}^n})_h.$$

Define the equivalent norm for the velocity components by

$$M^n(\mathbf{w}) := \|\mathbf{w}^n\|_h^2 + \|\mathbf{w}^{n-1}\|_h^2 - 2\Delta t(\mathbf{w}^n, A\mathbf{w}^{n-1}) + 2\Delta t\sum_{j=0}^{N-1} h\,\widetilde{g}(\mathbf{w}^n, \mathbf{w}^{n-1}),$$

where $\widetilde{g}$ is defined by

$$\widetilde{g}(\mathbf{w}, \widehat{\mathbf{w}}) = \frac{c}{2}\left[\left(\frac{1}{2} + \alpha_1\right)u_1\widehat{u}_0 + \left(\frac{1}{2} - \alpha_1\right)u_0\widehat{u}_{-1}\right.$$
$$\left. + (1 + \alpha_2)\,v_{1/2}\widehat{v}_{-1/2} - \alpha_2 v_{-1/2}\widehat{v}_{-3/2}\right].$$

The summation by parts formula to be used for the velocity is

$$(\mathbf{w}, A\widehat{\mathbf{w}})_h + (\widehat{\mathbf{w}}, A\mathbf{w})_h = (u, A\widehat{u})_{h,\alpha_1} + (\widehat{u}, Au)_{h,\alpha_1} + (v, A\widehat{v})_{h,\alpha_2} + (\widehat{v}, Av)_{h,\alpha_2}$$
$$= \widetilde{g}(\mathbf{w}, \widehat{\mathbf{w}}) + \widetilde{g}(\widehat{\mathbf{w}}, \mathbf{w}).$$

We concentrate once again on the special case $\alpha_2 = 0$ and we omit the subscript from α_1.

Then the resulting equation for the equivalent norm for the velocity is

$$\begin{aligned}\frac{M^{n+1} - M^n}{2\Delta t} &= (\mathbf{w}^{n-1}, A\mathbf{w}^n)_h + (\mathbf{w}^n, A\mathbf{w}^{n-1})_h \\ &\quad + 2(\overline{\mathbf{w}^n}, C\mathbf{w}^n)_h + 2(\overline{\mathbf{w}^n}, B\overline{\mathbf{w}^n})_h \\ &\quad + \sum_j \left[\widetilde{g}(\mathbf{w}^{n+1}, \mathbf{w}^n) - \widetilde{g}(\mathbf{w}^n, \mathbf{w}^{n-1})\right] \\ &= 2(\overline{\delta^n}, p^n)_h - 2\nu\|\nabla_h \overline{\mathbf{w}^n}\|_h^2 + 2\sum_{j=0}^{N-1} h\overline{G}_j\end{aligned}$$

where the boundary terms $\overline{G}_j$ are

$$\begin{aligned}\overline{G}_j = \quad & \widetilde{g}(\overline{\mathbf{w}^n}, \mathbf{w}^n) \\ & + \overline{u_0^n}\left[\left(\frac{1}{2} - \alpha\right) p_{-1/2}^n + \left(\frac{1}{2} + \alpha\right) p_{1/2}^n\right] \\ & - 2\nu\left[\left(\frac{1}{2} + \alpha\right)\overline{u_0^n} D_{0x}\overline{u_0^n} - \beta_1 S_x \overline{u_{-1/2}^n} D_x \overline{u_{-1/2}^n} - (\alpha - \beta_1)\,\overline{u_0^n} D_x \overline{u_{-1/2}^n}\right] \\ & - 2\nu\left[\left(\frac{1}{2} - \beta_2\right)\overline{v_{-1/2}^n} + \left(\frac{1}{2} + \beta_2\right)\overline{v_{1/2}^n}\right] D_x \overline{v_0^n}.\end{aligned}$$

The equivalence between M and the L^2 norm for $\mathbf{w}$ is determined by the same steps as for the divergence. The result is that M_n is a norm and is equivalent to $\|\mathbf{w}^n\|_h^2 + \|\mathbf{w}^{n-1}\|_h^2$, provided the time step satisfies the *CFL* stability condition (4.3.7).

References

G. K. Batchelor, *An Introduction to Fluid Mechanics*, Cambridge University Press, Cambridge, 1967.

G. Browning and H.-O. Kreiss, *Problems with different time scales for nonlinear partial differential equations*, SIAM J. Appl. Math., **42**, 704–718, 1982.

A. J. Chorin, *A numerical method for solving incompressible viscous flow problems*, J. Comp. Phys., **2**, 12–26, 1967.

A. J. Chorin, *Numerical solution of the Navier-Stokes equations*, Math. Comp., **22**, 745–762, 1968.

A. J. Chorin, *On the convergence of discrete approximations to the Navier-Stokes equations*, Math. Comp., **23**, 341–353, 1969.

J. Guerra and B. Gustafsson, *A semi-implicit method for hyperbolic problems with different time scales*, preprint.

J. Guerra and B. Gustafsson, *A numerical method for incompressible and compressible flow problems with smooth solutions*, preprint.

B. Gustafsson, H.-O. Kreiss and A. Sundström, *Stability theory of difference approximations for mixed initial boundary value problems. II*, Math. Comp., **26**, 649–686, 1972.

B. Gustafsson and A. Sundström, *Incompletely parabolic problems in fluid mechanics*, SIAM J. Appl. Math., **35**, 343–357, 1978.

B. Gustafsson and J. Oliger, *Stable boundary approximations for implicit time discretizations for gas mechanics*, SIAM J. Sci. Stat. Comp., **3**, 408–421, 1982.

B. Gustafsson and H.-O. Kreiss, *Difference approximations of hyperbolic problems with different time scales. I: the reduced problem*, SIAM J. Numer. Anal., **20**, 46–58, 1983.

F. Harlow and J. E. Welch, *Numerical calculation of time dependent viscous incompressible flow of fluid with free surface*, Phys. Fluids, **8**, 2182–2189, 1965.

S. Klainerman and A. Majda, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids*, Comm. Pure Appl. Math., **34**, 481–524, 1981.

H.-O. Kreiss, *Initial boundary value problems for hyperbolic systems*, Comm. Pure Appl. Math., **23**, 277–298, 1970.

H.-O. Kreiss, *Problems with different time scales for partial differential equations*, Comm. Pure Appl. Math., **33**, 399–440, 1980.

H.-O. Kreiss, *Problems with different time scales*, preprint.

O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, 1969.

A. Majda and S. Osher, *Initial boundary value problems for hyperbolic equations with uniformly characteristic boundaries*, Comm. Pure Appl. Math., **28**, 607–675, 1975.

D. Michelson, *Stability theory of difference approximations for multidimensional initial-boundary value problems*, Math. Comp., **40**, 1–45, 1983.

J. Oliger and A. Sundström, *Theoretical and practical aspects of some initial boundary value problems in fluid mechanics*, SIAM J. Appl. Math., **35**, 419–446, 1978.

R. Peyret and T. D. Taylor, *Computational Methods for Fluid Flow*, Springer-Verlag, New York, 1983.

T. A. Porsching, *Error estimates for MAC-like approximations to the linear Navier-Stokes equations*, Numer. Math., **29**, 291–306, 1978.

R. D. Richtmyer and K. W. Morton, *Difference Methods for Initial Value Problems*, Interscience, New York, 1967.

P. J. Roache, *Computational Fluid Dynamics*, Hermosa Publications, Albuquerque, 1972.

J. C. Strikwerda, *Initial boundary value problems for incompletely parabolic problems*, Comm. Pure Appl. Math., **30**, 797–822, 1977.

J. C. Strikwerda, *Initial boundary value problems for the method of lines*, J. Comp. Phys., **34**, 94–107, 1980.

J. C. Strikwerda, *Finite difference methods for the incompressible Navier-Stokes equations*, preprint.

Part II: Boundary Conditions for the Stream Function Vorticity Equations

CHAPTER 1

Introduction

One of the difficulties associated with the stream function vorticity equations for viscous incompressible flows is the numerical implementation of the no-slip boundary conditions $\psi = 0$ and $\partial\psi/\partial n = 0$ at a rigid wall. These boundary conditions place constraints on the stream function and its normal derivative at the boundary. There is, however, no explicit boundary condition for the vorticity. This lack of a vorticity boundary condition seems to be at the root of the problems which appear.

In this study we isolate an appropriate model problem and analyse the process of discretization in space. We investigate the error between the solution of the continuous model problem and the solution of the semi-discrete model problem.

The stream function vorticity equations describe the flow of a viscous incompressible fluid in two space dimensions. They are

$$\varsigma_t + u\,\varsigma_x + v\,\varsigma_y = \nu\,\Delta\varsigma + f,$$

$$\varsigma = \Delta\psi$$

where

$$\varsigma := v_x - u_y, \qquad u = -\psi_y, \qquad v = \psi_x.$$

The stream function vorticity equations are an alternative formulation of the incompressible Navier-Stokes equations which are derived from the primitive variable formulation. The vorticity ς is defined to be the curl of the velocity. For 2-D flow

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The work in this part was performed together with William Henshaw.

the vorticity vector has only one non-zero component, which points in the z direction, and hence it can be treated as a scalar quantity. The vorticity equation is obtained by taking the curl of the momentum equations. In this way the pressure is eliminated from the equations. The stream function ψ is defined in such a way that $u_x + v_y = 0$ is satisfied identically.

To close the system initial and boundary conditions are also needed. For flow in a bounded region Ω with solid walls the *no-slip* boundary conditions

$$u = v = 0 \qquad \text{on } \partial\Omega$$

become conditions on the stream function and its normal derivative

$$\psi = \partial\psi/\partial n = 0 \qquad \text{on } \partial\Omega.$$

Initial conditions are

$$u(x,y,0) = u^{(0)}(x,y), \qquad v(x,y,0) = v^{(0)}(x,y)$$

satisfying $u_x^{(0)} + v_y^{(0)} = 0$ for (x,y) in Ω, which become

$$\psi(x,y,0) = \psi^{(0)}(x,y).$$

The stream function vorticity formulation is very popular for use in 2-D numerical calculations. Its primary advantage over the primitive variable formulation is that it involves two dependent variables (ψ, ς) instead of three (u, v, p) which is an advantage for storage and possibly also in computational speed, depending on the numerical method used. Also, with the stream function vorticity formulation, the divergence is guaranteed to remain zero, which is not the case for other formulations. The stream function vorticity formulation also presents the same advantages for the calculation of axisymmetric flows.

On the other hand, the stream function vorticity formulation has its limitations in comparison to the primitive variable formulation. Since the pressure has been eliminated from the equations, the only boundary conditions which can be used are those which involve the velocity components (or possibly the vorticity). This is a disadvantage for computation of flows with open boundaries where boundary conditions involving the pressure have desirable characteristics as discussed elsewhere in this thesis. Also, the stream function vorticity formulation can only be applied to incompressible flows. For 3-D flows the primary advantage of the stream function vorticity formulation disappears — the vorticity vector has three non-zero components and so the number of dependent variables is the same as in the primitive variable formulation.

Some reviews of numerical methods for the stream function vorticity equations can be found in Peyret and Taylor [1983], Orszag and Israeli [1974] and Roache [1972].

The two boundary conditions $\psi = 0$ and $\partial\psi/\partial n = 0$ both must be applied on the boundary. The first condition specifies that there is no flow of fluid through the boundary, the *zero flux* condition, and the second condition specifies that there is no flow of fluid along the boundary, the *no-slip* condition. Usually, there is no trouble satisfying the zero flux boundary condition if the grid is chosen such that the boundary lies on a curve of grid points; the boundary condition is applied at each of the grid points on the boundary. There are (at least) two ways to look at the numerical approximation of the no-slip boundary condition. The first and more common approach is to use the normal derivative condition to determine an expression for the vorticity on the wall in terms of interior and boundary values of the stream function. This requires that a special one-sided formula be used for the vorticity on the boundary, and that the no-slip boundary condition be incorporated in this formula. The alternative approach, which we prefer, is to think not of approximating the vorticity at the wall but rather of approximating $\partial\psi/\partial n$. The vorticity on the wall is defined as it is in the interior. The discrete approximation to $\partial\psi/\partial n$ will determine the values of ψ_{ij} needed to apply the formula for ς on the boundary.

To achieve accurate answers, many investigators advocate the use of higher order approximations to the boundary conditions. When an implicit time marching method is used, or when a steady state solution is required, the stream function vorticity equations may have to be solved by iteration. When such iterations are required, some investigators have encountered difficulties with higher order boundary conditions and have abandoned them in preference to lower order schemes. The trouble is that the iterations to solve the nonlinear equations at each time step may converge slowly or not at all. We have found in implicit time stepping calculations that higher order methods are stable provided an appropriate iteration scheme is used, such as the one developed by Israeli [1970].

There has been some work performed at trying to obtain accurate and stable boundary conditions and to try to understand the difficulties present in this problem including the work of Briley [1971], Bontoux, Gilly and Roux [1980], Israeli [1970] and Orszag and Israeli [1974]. It appears that most results are heuristic or only qualitative in nature, although Orszag and Israeli study a model problem similar to the one we look at here.

Numerical experience shows that the difficulties seem to be related to boundary layers in the vorticity. We were led to this problem through observations of the errors in numerical calculations whilst testing a code for the stream function vorticity equations on a bounded domain. The exact solution of the equations was known — the solution was prescribed and the corresponding forcing was calculated. The equations were solved numerically and the error could be calculated exactly. The error in the stream function was observed to be uniformly second order in the local mesh spacing. The error in the vorticity was also $O(h^2)$ except for a boundary layer which was $O(h)$. This $O(h)$ boundary layer in the vorticity error was generated immediately at $t = 0$ and then it diffused and decayed away in time.

1.1 Reduction to a Model Problem

In the neighbourhood of a no-slip boundary the convective terms are small compared with the other terms. This suggests that it is reasonable to neglect the convective terms and to consider the stream function vorticity formulation of the Stokes equations. The initial boundary value problem is

$$\left.\begin{aligned} \varsigma_t &= \nu \Delta \varsigma + f \\ \varsigma &= \Delta \psi \end{aligned}\right\} \qquad \text{for } (x,y) \in \Omega,\ t > 0,$$

with

$$\psi = \partial \psi / \partial n = 0 \qquad \text{for } (x,y) \in \partial\Omega,\ t > 0,$$

$$\psi(x,y,0) = \psi^{(0)}(x,y) \qquad \text{for } (x,y) \in \Omega.$$

Near the boundary the flow is often one dimensional in character, varying in the normal direction to the boundary. Tangential derivatives of the stream function and vorticity are small. A reasonable model problem to study thus seems to be the following initial boundary value problem in one space dimension

$$\begin{aligned} &\varsigma_t = \nu \varsigma_{xx} + f(x,t), \\ &\varsigma = \psi_{xx}, \\ &\psi = \psi_x = 0 \qquad \text{at } x = 0, 1\,, \\ &\psi(x,0) = \psi^{(0)}(x). \end{aligned} \tag{1.1.1}$$

We shall call this the *continuous model problem.* The terms which have been neglected are assumed to be small or to vary smoothly in which case one can argue that they can be absorbed into the forcing term f.

In studying this model problem we are interested in the influence of the boundary conditions. We hope that its structure is rich enough to contain the salient features of the full problem and at the same time simple enough to be analysed and explained. In fact, we do find that the continuous model and its space discretization have interesting properties which seem to explain the source of the difficulty and its proper resolution.

In Chapter 2 we study the continuous model problem in detail. We show how ψ and any number of its x and t derivatives can be estimated in terms of the forcing, the initial data and the viscosity coefficient. These estimates also involve the time derivatives of ψ and ψ_x at $t = 0$. We prove the following result.

Theorem 1.1.1 *For each integer $l \geq 0$ there exist constants K, α such that*

$$\|\frac{\partial^l}{\partial x^l}\psi(x,t)\|^2 \leq Ke^{\alpha t}\sum_{j=0}^{J(l)}\|\frac{\partial^{j+1}}{\partial x \partial t^j}\psi(x,0)\|^2 + F(x,t) \tag{1.1.2}$$

where

$$J(l) = \begin{cases} 0 & l = 0, 1, \\ l/2 & l \text{ even}, l \geq 2, \\ (l+1)/2 & l \text{ odd}, l \geq 3, \end{cases} \tag{1.1.3}$$

and $F(x,t)$ depends on the forcing function and its derivatives.

This theorem can be rephrased. If ψ_x has J time derivatives bounded at $t = 0$, then ψ has $2J$ space derivatives bounded for all time in the L^2 norm.

The question of *compatibility* between the initial data and the boundary conditions arises. We show that there are constraints which the initial data must satisfy for the time derivatives of ψ and ψ_x to be bounded at $t = 0$. A sequence of *compatibility conditions* is obtained by requiring that $\partial^i\psi/\partial t^i$ and $\partial^{i+1}\psi/\partial x\partial t^i$ be continuous up to the boundary at $t = 0$ for $i = 0, 1, 2, \ldots$. We make the following definition.

Definition 1.1.2 *The initial boundary value problem (1.1.1) is compatible to order m if $\partial^i\psi/\partial t^i$ and $\partial^{i+1}\psi/\partial x\partial t^i$ are both continuous up to the boundary at $t = 0$ for $i = 0, 1, \ldots, m$.*

The zeroth order compatibility conditions are just the boundary conditions of the problem. These are of local type. The higher order compatibility conditions are *global* constraints which in two dimensions involve an integral of the initial data around the boundary.

If some of the compatibility conditions are not satisfied, then the process of estimating derivatives of ψ breaks down at some point. To any given order the solution for ψ can be decomposed into a compatible part and an incompatible part. This is achieved by decomposing the initial data as

$$\psi^{(0)}(x) = \widetilde{\psi}^{(0)}(x) + \overline{\psi}^{(0)}(x)$$

where the $\widetilde{\psi}$ part is compatible and the $\overline{\psi}$ part contains the incompatibility. A precise description of this decomposition is given in Chapter 2. The breakdown of the estimates can be traced to singularities in derivatives of the incompatible part of the solution at the boundary at $t = 0$. We show that the singularity in the solution due to incompatibility diffuses and decays away in time; for large time the incompatible part of the solution decays away exponentially to zero.

It is already clear from the energy estimates above that the solution retains some smoothness even if the compatibility conditions are violated at some order. By examining in detail the nature of the solutions in such incompatible cases it is in fact possible to estimate one more x derivative than is given by (1.1.2).

Theorem 1.1.3 *Consider the continuous model problem (1.1.1) with zero forcing and special initial data*

$$\psi^{(0)}(x) = x^n(1-x)^n\,[a + bx] \tag{1.1.4}$$

where a and b are constants. The solution $\psi(x,t)$ satisfies

$$\|\frac{\partial^l}{\partial x^l}\psi(x,t)\|^2 \leq Ke^{-\nu(2\pi)^2 t}\left(|a| + |b|\right) \tag{1.1.5}$$

for $l = 0, 1, \ldots, 2n-1$ *for some constant* K *depending on* n.

For these initial data we shall see that ψ has exactly $J = n-1$ time derivatives bounded at $t = 0$; the number of x derivatives of $\psi(x,t)$ which can be bounded is $2n - 1 = 2J + 1$.

1.2 Space Discrete Model Problem

The process of discretization in space consists of laying down a grid on the domain, replacing the dependent variables by corresponding grid functions and replacing the space derivatives by finite differences. A uniform grid on the interval $[0,1]$ is shown in Figure 1.1; it is given by

$$x_\mu = \mu h, \qquad \mu = -1, 0, 1, \ldots, N, N+1$$

with $Nh = 1$. The grid is located such that grid points lie at both the end points $x = 0$ and $x = 1$; the grid extends by one grid point beyond each end point.

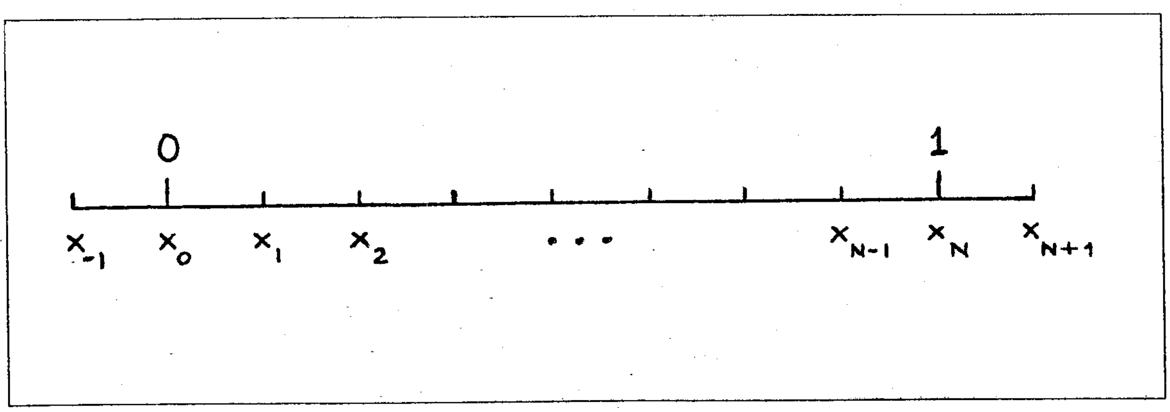

Figure 1.1

The discrete solution consists of the discrete stream function $\phi_\mu(t)$ and the discrete vorticity $z_\mu(t)$. $\phi_\mu(t)$, which is defined for $\mu = -1, 0, 1, \ldots, N-1, N, N+1$, $t \geq 0$ is an approximation to $\psi_\mu(t) := \psi(x_\mu, t)$, $\mu = 0, 1, \ldots, N$. The points outside the boundary, ϕ_{-1} and ϕ_{N+1}, are fictitious points which enable z_μ to be calculated on the boundary by the same centred second order formula as in the interior. $z_\mu(t)$ is defined for $\mu = 0, 1, \ldots, N-1, N$, and is an approximation to $\varsigma_\mu(t) := \varsigma(x_\mu, t)$. Second order centred finite differences are used to approximate d^2/dx^2

$$D_+D_-u_\mu := \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

The derivative boundary conditions are differenced with approximations of order q. $D_{l,q}$ will be the approximation to d/dx at the left boundary and $D_{r,q}$ the approximation at the right boundary; we shall define these operators below.

The initial conditions specify ϕ_μ at the interior grid points $\mu = 1, \ldots, N-1$. The boundary conditions are applied all the way to $t = 0$ and so they are used to calculate $\phi_\mu(0)$ on the boundary and at the fictitious points. The discrete initial data is taken to be a second order accurate approximation to the continuous initial data. By this we mean that

$$\phi_\mu^{(0)} = \psi^{(0)}(x_\mu) + O(h^2).$$

Often it is convenient to think of the discrete initial data $\phi_\mu^{(0)}$ as the point values of some continuous function $\phi_h^{(0)}(x)$ where

$$\phi_h^{(0)}(x) = \psi^{(0)}(x) + O(h^2).$$

Likewise, the forcing is a second order accurate discretization of the continuous forcing; we shall usually take f^h to be the point values of the continuous forcing; i.e., $f_\mu^h(t) = f(x_\mu, t)$.

The resulting problem is the *space discrete model problem*

$$\begin{aligned}
\frac{\partial z_\mu}{\partial t} &= \nu D_+D_-z_\mu + f_\mu^h, \qquad && \mu = 1, 2, \ldots, N-1, \\
z_\mu &= D_+D_-\phi_\mu, && \mu = 0, 1, \ldots, N, \\
\phi_0 &= \phi_N = 0, && \qquad\qquad\qquad (1.2.1) \\
D_{l,q}\phi_0 &= D_{r,q}\phi_N = 0, \\
\phi_\mu(0) &= \phi_\mu^{(0)}, && \mu = 1, 2, \ldots, N-1.
\end{aligned}$$

We consider a particular class of numerical approximations to the no-slip boundary condition. The approximations which we consider use the fictitious points outside the boundary and minimize the number of points used inside the boundary for a given order of accuracy. The general form of the no-slip boundary conditions of order q is

$$D_{l,q}\phi_0 := \frac{1}{h}\sum_{i=-1}^{q-1} a_i\phi_i = 0, \qquad D_{r,q}\phi_N := \frac{1}{h}\sum_{i=-1}^{q-1}(-a_i)\phi_{N-i} = 0,$$

where the coefficients a_i must be determined. The *second order* form of the discrete boundary conditions is

$$D_{l,2}\phi_0 := D_0\phi_0 = 0, \qquad D_{r,2}\phi_N := D_0\phi_N = 0, \tag{1.2.2}$$

where D_0 is the usual centred second order approximation for d/dx

$$D_0 u_\mu := \frac{u_{\mu+1} - u_{\mu-1}}{2h}.$$

The fictitious points can be eliminated from the scheme by using special formulae for the vorticity on the boundary. This corresponds to an approximation to the vorticity on the left hand boundary of

$$z_0 = \frac{2\phi_1}{h^2}. \tag{1.2.3}$$

This approximation is alternatively referred to as the *conventional approximation* (Gupta and Manohar [1979]) or Thom's formula. The truncation error in formula (1.2.3) is formally only $O(h)$. We shall also be interested in using higher order approximations for the no-slip condition. The second order approximation was centred but the higher order schemes are one sided; there are more values of the stream function used from within the interval but still only a single point outside the boundary (i.e., at x_{-1} or x_{N+1}). The *third order* approximations can be derived from Taylor expansions of ψ and are

$$D_{l,3}\phi_0 := \frac{-2\phi_{-1} - 3\phi_0 + 6\phi_1 - \phi_2}{6h} = 0,$$
$$D_{r,3}\phi_N := \frac{\phi_{N-2} - 6\phi_{N-1} + 3\phi_N + 2\phi_{N+1}}{6h} = 0.$$

The *fourth order* approximations are

$$D_{l,4}\phi_0 := \frac{-3\phi_{-1} - 10\phi_0 + 18\phi_1 - 6\phi_2 + \phi_3}{12h} = 0,$$
$$D_{r,4}\phi_N := \frac{-\phi_{N-3} + 6\phi_{N-2} - 18\phi_{N-1} + 10\phi_N + 3\phi_{N+1}}{12h} = 0.$$

In Chapter 3 we study the semi-discrete model problem in detail. We show how ϕ and any number of its divided differences and t derivatives can be estimated in terms of the data. As in the continuous case the estimates involve the time derivatives of $D\phi$ at $t = 0$, where D is the usual compact two point divided difference operator which approximates d/dx

$$Du_{\mu+1/2} := \frac{u_{\mu+1} - u_\mu}{h}.$$

This difference approximation is thought of as being centred at the midpoints of the grid, where it is second order accurate. If *second order* boundary conditions are used, all the estimates go through exactly as in the continuous case. If *higher order* boundary conditions are used, the estimates require modification. Let $D^{(l)}$ denote the compact $l+1$ point divided difference approximation for d^l/dx^l; i.e.,

$$D^{(l)}u_\mu := (D)^l u_\mu \qquad l \text{ even},$$
$$D^{(l)}u_{\mu+1/2} := (D)^l u_{\mu+1/2} \qquad l \text{ odd}.$$

We prove the following result.

Theorem 1.2.1 *If the no-slip boundary conditions are approximated by the second order formulae (1.2.2), then for each integer $l \geq 0$ there exist constants K, α such that*

$$\|D^{(l)}\phi(t)\|_h^2 \leq Ke^{\alpha t}\sum_{j=0}^{J(l)}\|\frac{\partial^j}{\partial t^j}D\phi(0)\|_h^2 + F^h(t). \tag{1.2.4}$$

where $J(l)$ is given by (1.1.3) and $F^h(t)$ depends on the forcing function and its divided differences and derivatives.

Theorem 1.2.2 *If no-slip boundary conditions of order q are used, then*

$$\phi = \tilde{\phi} + h^2\tilde{\tilde{\phi}},$$

where $\tilde{\phi}$ satisfies the estimates (1.2.4) and $\tilde{\tilde{\phi}}$ satisfies

$$\begin{aligned} &\int_0^t e^{-2\nu\sigma_0 t}|\tilde{\tilde{\phi}}_\mu(t)|^2\,dt \leq \text{const}\int_0^t e^{-2\nu\sigma_0 t}|D^{(3)}\tilde{\phi}(t)|_\infty^2\,dt, \\ &\int_0^t e^{-2\nu\sigma_0 t}|\tilde{\tilde{z}}_\mu(t)|^2\,dt \leq \frac{\text{const}}{h^2}\int_0^t e^{-2\nu\sigma_0 t}|D^{(3)}\tilde{\phi}(t)|_\infty^2\,dt \end{aligned} \tag{1.2.5}$$

for some constant $\sigma_0 > 0$.

The estimates (1.2.5) are quite crude, but they demonstrate that the space discrete approximation is *stable*. These estimates can be improved.

We also prove estimates for incompatible problems.

Theorem 1.2.3 *Consider the space discrete model problem with inhomogeneous boundary data*

$$\begin{aligned} &\frac{\partial}{\partial t}D_+D_-\psi_\mu = \nu(D_+D_-)^2\psi_\mu, \qquad \mu = 1,2,\ldots,N-1, \\ &\phi_0 = \phi_N = 0, \\ &D_{l,q}\phi_0 = g_0(t), \qquad D_{r,q}\phi_N = g_1(t), \\ &\phi_\mu(0) \equiv 0. \end{aligned} \tag{1.2.6}$$

The solution ϕ and $z = D_+D_-\phi$ satisfy the estimates

$$\begin{aligned} \int_0^t e^{-2\nu\sigma_0 t}|\phi_\mu(t)|^2\,dt &\le \text{const}\int_0^t e^{-2\nu\sigma_0 t}\left[|g_0(t)|^2+|g_1(t)|^2\right]\,dt \\ \int_0^t e^{-2\nu\sigma_0 t}|z_\mu(t)|^2\,dt &\le \frac{\text{const}}{h^2}\int_0^t e^{-2\nu\sigma_0 t}\left[|g_0(t)|^2+|g_1(t)|^2\right]\,dt \end{aligned} \tag{1.2.7}$$

for some constant $\sigma_0 > 0$.

Theorem 1.2.4 *Consider the space discrete model problem (1.2.1) with zero forcing and special initial data*

$$\phi^{(0)}(x) = x^n(1-x)^n\left[a+bx\right]. \tag{1.2.8}$$

where a and b are constants. The solution $\phi_\mu(t)$ satisfies

$$\int_0^t e^{-2\nu\sigma_0 t}|D^{(l)}\phi_\mu(t)|^2\,dt \le \text{const}\,\frac{1-e^{-2\nu\sigma_0 t}}{2\nu\sigma_0}\left(|a|^2+|b|^2\right) \tag{1.2.9}$$

for $l = 0, 1, \ldots, 2n-1$, and

$$\int_0^t e^{-2\nu\sigma_0 t}|D^{(2n)}\phi_\mu(t)|^2\,dt \le \frac{\text{const}}{h^2}\,\frac{1-e^{-2\nu\sigma_0 t}}{2\nu\sigma_0}\left(|a|^2+|b|^2\right) \tag{1.2.10}$$

for some constant $\sigma_0 > 0$.

It is convenient to introduce a shorthand notation for the model problems. Define the continuous and semi-discrete operators L and L_h by

$$\begin{aligned} L\psi(x) &:= \psi_{xxt} - \nu\psi_{xxxx}, \\ L_h\phi_\mu &:= \frac{\partial}{\partial t}D_+D_-\phi_\mu - \nu(D_+D_-)^2\phi_\mu \end{aligned}$$

and the boundary operators B and B_h by

$$B\psi := \begin{bmatrix} \psi(0) \\ \psi(1) \\ \psi_x(0) \\ \psi_x(1) \end{bmatrix}, \qquad B_h\phi = B_{h,q}\phi := \begin{bmatrix} \phi_0 \\ \phi_N \\ D_{l,q}\phi_0 \\ D_{r,q}\phi_N \end{bmatrix}.$$

With this notation the continuous and semi-discrete model problems are written as

$$L\psi = f, \qquad B\psi = 0, \qquad \psi(x,0) = \psi^{(0)}(x) \tag{1.2.11}$$

and

$$L_h\phi = f^h, \qquad B_h\phi = 0, \qquad \phi_\mu(0) = \phi_\mu^{(0)} = \phi^{(0)}(x_\mu), \tag{1.2.12}$$

respectively. Proceeding one step further we will write (1.2.11) and (1.2.12) as

$$\mathbf{L}\psi = \mathbf{F},$$

$$\mathbf{L}_h\phi = \mathbf{F}^h.$$

1.3 Error Analysis

We now investigate the properties of the semi-discrete solution $(\phi_\mu(t), z_\mu(t))$ of (1.2.1) as an approximation of the continuous solution $(\psi(x,t), \varsigma(x,t))$ of (1.1.1). The analysis of the error makes use of the estimates for the solutions of the continuous and semi-discrete model problems. We are interested in the following questions:

1. What is the error in the approximation to the stream function ?
2. What is the error in the approximation to the vorticity ?

The distinction must be made between *compatible* and *incompatible* problems. Our discussion will focus mostly on compatible problems. First we analyse the errors in the approximation of compatible problems, and then we make some remarks about incompatible problems.

Compatible Problems

We need to define what we mean by a compatible problem. The error analysis involves an expansion of the semi-discrete stream function $\phi_\mu(t)$ in terms of the continuous stream function $\psi(x,t)$. This asymptotic expansion technique relies on the smoothness of $\psi(x,t)$. The precise number of bounded derivatives needed can be found by an inspection of the asymptotic expansion below.

Definition 1.3.1 *The continuous model problem (1.2.11) is called compatible if $\psi(x,t)$ has a sufficient number of bounded derivatives.*

From a practical viewpoint this notion of a compatible problem represents an *a priori* knowledge or assumption that the fluid flow which is being calculated is smooth. By this assumption we rule out the situation of an impulsive start. In an impulsive start problem the error estimates which follow will not be good.

Error Equations

To analyse the error in the stream function we form the equations for the error. The normalized error in the stream function is defined by

$$\psi(x_\mu, t) = \psi_\mu(t) = \phi_\mu(t) + h^2 e_\mu(t) \tag{1.3.1}$$

and satisfies the *semi-discrete error equations*

$$L_h e = f^{(1)}, \qquad B_h e = g^{(1)}, \qquad e_\mu(0) = e_\mu^{(0)}, \tag{1.3.2}$$

where $f^{(1)}$, $g^{(1)}$ are the truncation errors of the difference equations divided by h^2 and $e^{(0)}$ is the normalized error in the initial conditions $e_\mu^{(0)} := \left(\psi_\mu^{(0)} - \phi_\mu^{(0)}\right)/h^2$. The error equations can also be written as $\mathbf{L}_h e = \mathbf{F}^{(1)}$.

Estimates for the forcing terms in the error equations are provided by the estimates for $\psi(x,t)$ and its x, t derivatives. These estimates depend on the data $f(x,t)$, $\psi^{(0)}(x)$ and they also involve the time derivatives of ψ_x at $t = 0$.

In the compatible case the truncation errors in approximation (1.2.1) are all $O(h^2)$ or higher. The discrete stream function computed from (1.2.1) will thus be second order accurate provided that the normalized error satisfies $||e|| = ||\mathbf{L}_h^{-1}\mathbf{F}^{(1)}|| = O(1)$. The discrete vorticity $z_\mu = D_+D_-\phi_\mu$ will also be second order provided that the error e_μ is sufficiently smooth. This can been seen as follows.

The computed stream function $\phi_\mu(t)$ is related to the true solution $\psi(x_\mu, t)$ by $\phi_\mu(t) = \psi(x_\mu, t) - h^2 e_\mu$. The discrete vorticity is thus given by

$$z_\mu := D_+D_-\phi_\mu = D_+D_-\psi_\mu - h^2 D_+D_-e_\mu$$

$$= \psi_{xx}(x_\mu) + O(h^2) - h^2 D_+D_-e_\mu.$$

If the error e_μ is sufficiently smooth, then $D_+D_-e_\mu$ looks like a second derivative and so $\|D_+D_-e_\mu\|$ will be $O(1)$. The approximation to the vorticity will then be *second order*. It seems useful to consider the problem of obtaining an accurate answer to the vorticity in this way; that is, to think of obtaining an approximation to the stream function with a smooth error. Divided difference approximations to higher derivatives of the stream function will then have the same accuracy as the stream function itself. This is one reason why we consider approximating $\partial\psi/\partial n = 0$ rather than the vorticity on the wall.

The forcing function in the error equation $f^{(1)}$ is given explicitly by

$$f_\mu^{(1)}(t) := f^{(1)}(x_\mu, t),$$

$$f^{(1)}(x,t) := \frac{1}{h^2}(L_h - L)\psi(x,t)$$

$$= \left[\left(D_+D_- - \frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial t} - \nu\left((D_+D_-)^2 - \frac{\partial^4}{\partial x^4}\right)\right]\psi(x,t)$$

$$= \frac{1}{12}\frac{\partial^5\psi}{\partial x^4 \partial t} + \frac{2h^2}{6!}\frac{\partial^7\psi}{\partial x^6\partial t} + O(h^4) - \nu\left(\frac{1}{6}\frac{\partial^6\psi}{\partial x^6} + O(h^2)\right).$$

The truncation errors in the boundary terms are of the form $g^{(1)} = (0, 0, g_0^{(1)}, g_1^{(1)})'$

$$g_0^{(1)} := \frac{1}{h^2}\left(D_{l,q} - \frac{\partial}{\partial x}\right)\psi(0,t) = C_{0q}h^{q-2}\frac{\partial^{q+1}}{\partial x^{q+1}}\psi(0,t) + O(h^{q-1}),$$

$$g_1^{(1)} := \frac{1}{h^2}\left(D_{r,q} - \frac{\partial}{\partial x}\right)\psi(1,t) = C_{1q}h^{q-2}\frac{\partial^{q+1}}{\partial x^{q+1}}\psi(1,t) + O(h^{q-1}).$$

The approximations are accurate to order q. In particular, for the conventional approximation $D_{l,q} = D_0$ and $D_{r,q} = D_0$ we have

$$g_0^{(1)} := \frac{1}{h^2}\left(D_0 - \frac{\partial}{\partial x}\right)\psi(0,t) = \frac{1}{6}\psi_{xxx}(0,t) + \frac{h^2}{5!}\psi_{xxxxx}(0,t) + O(h^4),$$

$$g_1^{(1)} := \frac{1}{h^2}\left(D_0 - \frac{\partial}{\partial x}\right)\psi(1,t) = \frac{1}{6}\psi_{xxx}(1,t) + \frac{h^2}{5!}\psi_{xxxxx}(1,t) + O(h^4).$$

Asymptotic Expansion of the Error

The error can be expanded in an asymptotic series if the incompatible part is subtracted off at each order. The error $e_\mu(t)$ can be decomposed into a smooth part, an incompatible part and a higher order part

$$\begin{aligned} e_\mu(t) &= \widetilde{e}_\mu(t) + \bar{e}_\mu(t) \\ &= \widetilde{\widetilde{e}}(x_\mu, t) + \bar{e}_\mu(t) + h^2 e_\mu^{(2)}(t). \end{aligned}$$

The decomposition is made in two steps. The first step is to split the error equation (1.3.2) into

$$L_h \widetilde{e} = f^{(1)}, \qquad B_h \widetilde{e} = 0, \qquad \widetilde{e}_\mu(0) = \widetilde{e}_\mu^{(0)} = \widetilde{e}^{(0)}(x_\mu), \tag{1.3.3a}$$

$$L_h \bar{e} = 0, \qquad B_h \bar{e} = g^{(1)}, \qquad \bar{e}_\mu(0) = \bar{e}_\mu^{(0)} = e_\mu^{(0)} - \widetilde{e}_\mu^{(0)}. \tag{1.3.3b}$$

The initial data $\widetilde{e}^{(0)}(x)$ is chosen such that the *continuous* model problem corresponding to (1.3.3a) is *compatible* — that is, $\{f^{(1)}(x,t), \widetilde{e}^{(0)}(x)\}$ is a compatible data set. Then $\widetilde{e}$ can be approximated by the smooth solution $\widetilde{\widetilde{e}}$ of the continuous problem plus a remainder term $h^2 e^{(2)}$ of higher order

$$L\widetilde{\widetilde{e}} = f^{(1)}, \qquad B\widetilde{\widetilde{e}} = 0, \qquad \widetilde{\widetilde{e}}(x,0) = \widetilde{e}^{(0)}(x), \tag{1.3.4a}$$

$$L_h e^{(2)} = f^{(2)}, \qquad B_h e^{(2)} = g^{(2)}, \qquad e_\mu^{(2)}(0) = 0. \tag{1.3.4b}$$

The forcing terms $f^{(2)}$ and $g^{(2)}$ in the equation for $e^{(2)}$ are the truncation errors in the approximating $\widetilde{e}$ by $\widetilde{\widetilde{e}}$. Hence they have the same form as $f^{(1)}$ and $g^{(1)}$ except that they involve derivatives of $\widetilde{\widetilde{e}}$ instead of ψ. The assumption of compatibility guarantees that $f^{(2)}(x,t)$ and $g^{(2)}(t)$ are as smooth as necessary. It is clear that (1.3.4b) has the same form as (1.3.2) and so these steps can be repeated to generate more terms in the expansion.

Now each of the equations for $\tilde{\tilde{e}}$, $\bar{e}$, $e^{(2)}$ is in the right form to be estimated using the theorems stated above. Loosely speaking the result is that

$$\begin{aligned} \tilde{\tilde{e}} &= O(1), \qquad & D_+D_-\tilde{\tilde{e}} &= O(1), \\ \bar{e} &= O(1), \qquad & D_+D_-\bar{e} &= O(\frac{1}{h}g^{(1)}, \frac{1}{h^3}\phi_{hx}^{(0)}|_{x=0,1}, \frac{1}{h^4}\phi_h^{(0)}|_{x=0,1}), \\ e^{(2)} &= O(1), \qquad & D_+D_-e^{(2)} &= O(\frac{1}{h}g^{(2)}). \end{aligned}$$

The conclusion is that the error is second order except for the terms in $D_+D_-\bar{e}$. One contribution comes from the initial data for the semi-discrete approximation which is assumed to be at least $O(h^2)$. If this initial data $\phi_h^{(0)}(x)$ satisfies the no-slip boundary conditions then this contribution is not present.

The other contribution comes from $g^{(1)}$, the normalized truncation error in the approximation of the no-slip boundary condition. The magnitude of $g^{(1)}$ depends on the order of the numerical boundary conditions. If qth order boundary conditions are used, then $g^{(1)}$ is $O(h^{q-2})$, causing an error in the vorticity of order h^{q-1}. The conclusion is that if *second order* boundary conditions are used then the vorticity is in general accurate to $O(h)$ on the boundary, whilst if *higher order* boundary conditions are used then the vorticity is $O(h^2)$.

A more precise statement of the error bounds is the following.

Theorem 1.3.2 *Suppose $\psi(x,t)$ satisfies the continuous model problem (1.2.11) and $\phi_\mu(t)$ satisfies the semi-discrete model problem (1.2.12) with*

$$\psi^{(0)}(x) = \phi^{(0)}(x) + h^2 e^{(0)}(x),$$

$$f_\mu^h(t) = f(x_\mu, t).$$

Assume that the continuous problem for ψ is compatible. Then the normalized error $e_\mu(t)$ can be decomposed into

$$e_\mu(t) = \tilde{\tilde{e}}(x_\mu, t) + \bar{e}_\mu(t) + h^2 e_\mu^{(2)}(t),$$

where

- $\tilde{\tilde{e}}$ *satisfies estimates of the form (1.1.2) with* $\{f, \psi^{(0)}\}$ *replaced by* $\{f^{(1)}, \tilde{e}^{(0)}\}$,
- $\bar{e}$ *is the sum of terms which satisfy estimates of the forms*

  (1.2.7) with g *replaced by* $g^{(1)}$, *and*

  (1.2.9) with $\phi^{(0)}$ *replaced by* $\bar{e}^{(0)}$,
- $e^{(2)}$ *is the sum of terms which satisfy estimates of the forms*

  (1.2.4) with $\{f, \phi^{(0)}\}$ *replaced by* $\{f^{(2)}, \tilde{e}^{(2)(0)}\}$,

  (1.2.7) with g *replaced by* $g^{(2)}$, *and*

  (1.2.9) with $\phi^{(0)}$ *replaced by* $\tilde{e}^{(2)(0)}$.

The initial conditions $\tilde{e}^{(2)(0)}$ *are chosen to ensure that* $\{f^{(2)}, \tilde{e}^{(2)(0)}\}$ *satisfies the compatibility conditions at least to order 2.*

Incompatible Problems

If the initial data and the boundary conditions are incompatible, then the forcing functions and the inhomogeneous boundary data in the error equations are not bounded. It is not clear that the error in either the stream function or the vorticity can be estimated well.

However, for large time the incompatible part of the continuous solution dies off exponentially. Likewise, we expect that the incompatible part of the discrete solution also dies off exponentially although the stability results we have proved are only sufficient to show a positive exponential bound on the growth rate. Assuming that this expectation is correct, it follows that asymptotically for large time second order accuracy is obtained.

CHAPTER 2

Continuous Model Problem

We consider the model problem for $\psi(x,t)$

$$\begin{aligned} \psi_{xxt} &= \nu\psi_{xxxx} + f \qquad 0 \le x \le 1,\ t \ge 0, \\ \psi &= \psi_x = 0 \qquad x = 0,1,\ t \ge 0, \\ \psi(x,0) &= \psi^{(0)}(x) \qquad 0 < x < 1, \end{aligned} \tag{2.0.1}$$

where $f = f(x,t)$ is the external forcing and $\psi^{(0)}$ is the initial data. Our first goal is to investigate the properties of this system. We show that ψ and its derivatives can be estimated in terms of the data. The estimates relate the norm of the solution at any time to the forcing and the initial data. Estimates for the derivatives of ψ also involve the time derivatives of ψ_x at $t = 0$. We shall see that boundedness of the initial time derivatives of ψ places constraints on the initial data which we call *compatibility conditions.* We investigate the form of these compatibility conditions and their consequences.

The norms which we use here are the L^2 norm defined in the usual way in terms of the L^2 inner product on $[0,1]$

$$(u,v) := \int_0^1 u(x)\, v(x)\, dx, \qquad \|u\|^2 := (u,u),$$

and the maximum norm

$$|u|_\infty := \max_{0\le x\le 1} |u(x)|.$$

We assume throughout that the forcing and the initial data are smooth functions of (x,t) and x, respectively, in the interior and as x approaches the boundary.

We allow the possibility that the limit of the initial data from the interior may not match the boundary conditions. For the forcing function this means that $\|f(\cdot,t)\|$ is finite for all t and the norms of all the x and t derivatives of f are finite.

2.1 Estimates for the Compatible Problem

First we present the estimates under the assumption that the initial data and the boundary data are compatible to any order. That is, we assume that all the time derivatives of ψ at $t=0$ are as smooth as needed. Hence, once any estimate is made for ψ, then estimates for any number of t derivatives follow immediately.

Lemma 2.1.1 *(Sobolev's Inequality) For any function $u \in C^1[0,1]$ and any $\epsilon > 0$ there exists a constant $C_1(\epsilon)$ such that*

$$|u|_\infty^2 \le C_1(\epsilon)\|u\|^2 + \epsilon\|u_x\|^2. \tag{2.1.1}$$

Proof. Let x_{min} and x_{max} denote the x-values where

$$|u(x_{min})| = \min_{0\le x\le 1}|u(x)|, \qquad |u(x_{max})| = \max_{0\le x\le 1}|u(x)|.$$

Then

$$|u(x_{min})|^2 \le \int_0^1 |u|^2 dx$$

and

$$|u(x_{max})|^2 - |u(x_{min})|^2 = \int_{x_{min}}^{x_{max}} 2uu_x dx$$

imply

$$\begin{aligned}|u(x_{max})|^2 &\le \|u\|^2 + 2\|u\|\|u_x\|^2\\ &\le \|u\|^2 + \epsilon\|u_x\|^2 + \frac{1}{\epsilon}\|u\|^2.\end{aligned}$$

This proves the lemma with $C_1(\epsilon) = 1 + 1/\epsilon$.

Lemma 2.1.2 *(Poincaré's Inequality) Let $u \in C^1[0,1]$. Then*

$$\|u\|^2 \le \left(\int_0^1 u(x)dx\right)^2 + \frac{1}{2}\|u_x\|^2. \tag{2.1.2}$$

Proof. Let x and y be between 0 and 1. Then

$$u(y) - u(x) = \int_x^y u_x dx.$$

So

$$u^2(x) + u^2(y) - 2u(x)u(y) \le |\int_x^y u_x^2 dx| \le \int_0^1 u_x^2 dx.$$

Integrate this inequality from 0 to 1 with respect to x and then with respect to y. The result of the lemma follows.

Lemma 2.1.3 *Let $u \in C^1[0,1]$ and $u(0) = u(1) = 0$. Then*

$$\pi^2 \|u\|^2 \le \|u_x\|^2. \tag{2.1.3}$$

In particular

$$\pi^2 \|\psi\|^2 \le \|\psi_x\|^2,$$

$$\pi^2 \|\psi_x\|^2 \le \|\psi_{xx}\|^2.$$

This result follows from Parseval's relation applied to the Fourier sine series of u.

We now make estimates for ψ and its x derivatives, beginning with ψ_x.

Lemma 2.1.4 *Estimate for $\|\psi_x\|$. For any $\alpha > 0$ there exists a constant $C_3(\alpha, \nu)$ such that*

$$\frac{1}{2}\frac{d}{dt}\|\psi_x\|^2 + \nu\|\psi_{xx}\|^2 \le \frac{\alpha}{2}\|\psi_x\|^2 + C_3\|f\|^2 \tag{2.1.4}$$

and

$$\begin{aligned}\|\psi_x\|^2 &\le e^{(\alpha - 2\nu\pi^2)t}\|\psi_x^{(0)}\|^2 + C_3\int_0^t e^{\alpha(t-t')}\|f(\cdot,t')\|^2 dt' \\ &\le e^{(\alpha - 2\nu\pi^2)t}\|\psi_x^{(0)}\|^2 + C_3\theta(\alpha,t)\max_{0\le t'\le t}\|f(\cdot,t')\|\end{aligned} \tag{2.1.5}$$

where $\theta(\alpha,t) := (e^{\alpha t} - 1)/\alpha$.

Proof. Take the inner product of ψ with the equation (2.0.1), integrate by parts and use the boundary conditions to eliminate the boundary terms which arise. The resulting equation is

$$\frac{1}{2}\frac{d}{dt}\|\psi_x\|^2 + \nu\|\psi_{xx}\|^2 = -(\psi, f) \le \frac{\delta}{2}\|\psi\|^2 + \frac{1}{2\delta}\|f\|^2 \tag{2.1.6}$$

for any $\delta > 0$ using the Cauchy-Schwarz inequality. Now use the result of Lemma Lemma 2.1.3 to estimate $\|\psi\|$ in terms of $\|\psi_x\|$. The inequality (2.1.5) follows from integrating (2.1.4) with respect to t and using Lemma Lemma 2.1.3 as a lower bound for $\|\psi_{xx}\|$ in terms of $\|\psi_x\|$.

The same arguments used to estimate ψ_x also give estimates for ψ_{xt} and higher t derivatives. The only difference for ψ_{xt} is that the estimates are in terms of f_t and $\psi_{xt}(x,0)$. The estimates for time derivatives will be used to obtain estimates for higher x derivatives. We state the results for ψ_{xt}.

Lemma 2.1.5 *Estimate for $\|\psi_{xt}\|$. For any $\alpha > 0$ there exists a constant $C_4(\alpha,\nu)$ such that*

$$\frac{1}{2}\frac{d}{dt}\|\psi_{xt}\|^2 + \nu\|\psi_{xxt}\|^2 \leq \frac{\alpha}{2}\|\psi_{xt}\|^2 + C_4\|f_t\|^2 \tag{2.1.7}$$

and

$$\begin{aligned}\|\psi_{xt}\|^2 &\leq e^{(\alpha-2\nu\pi^2)t}\|\psi_{xt}(\cdot,0)\|^2 + C_4\int_0^t e^{\alpha(t-t')}\|f_t(\cdot,t')\|^2 dt' \\ &\leq e^{(\alpha-2\nu\pi^2)t}\|\psi_{xt}(\cdot,0)\|^2 + C_4\theta(\alpha,t)\max_{0\leq t'\leq t}\|f_t(\cdot,t')\|\end{aligned} \tag{2.1.8}$$

where $\theta(\alpha,t) := (e^{\alpha t}-1)/\alpha$.

An estimate for $\|\psi_{xx}\|$ is obtained in terms of ψ_x and ψ_{xt}.

Lemma 2.1.6 *Estimate for $\|\psi_{xx}\|$.*

$$\|\psi_{xx}\|^2 \leq C_4(\nu)\left(\|\psi_x\|^2 + \|\psi_{xt}\|^2 + \|f\|^2\right). \tag{2.1.9}$$

Proof. From (2.1.4)

$$\begin{aligned}\nu\|\psi_{xx}\|^2 &= -(\psi,f) - (\psi_x,\psi_{xt}) \\ &\leq \frac{1}{2}\left(\|u\|^2 + \|\psi_x\|^2 + \|\psi_x\|^2 + \|f\|^2\right).\end{aligned}$$

Estimates for $|\psi|_\infty$, $|\psi_x|_\infty$ now follow from Sobolev's inequality. Furthermore, we infer from the estimate for $\|\psi_{xx}\|$ a similar estimate for $\|\psi_{xxt}\|$, at the expense of involving ψ_{xtt}.

Lemma 2.1.7 *Estimate for* $\|\psi_{xxt}\|$.

$$\|\psi_{xxt}\|^2 \leq C_5(\nu)\left(\|\psi_{xt}\|^2 + \|\psi_{xtt}\|^2 + \|f_t\|^2\right). \tag{2.1.10}$$

The next estimate follows from the vorticity equation (2.0.1).

Lemma 2.1.8 *Estimates for* $\|\psi_{xxxx}\|$, $\|\psi_{xxx}\|$.

$$\nu\|\psi_{xxxx}\| \leq \|\psi_{xxt}\| + \|f\|, \tag{2.1.11}$$

$$\|\psi_{xxx}\|^2 \leq C_7(\epsilon)\|\psi_{xx}\|^2 + \epsilon\|\psi_{xxxx}\|^2. \tag{2.1.12}$$

Proof. The estimate for ψ_{xxxx} comes immediately from (2.0.1). The estimate for ψ_{xxx} is a Sobolev type inequality. By Poincaré's inequality $\|\psi_{xxx}\|$ can be estimated by ψ_{xx} on the boundary and $\|\psi_{xxxx}\|$. Then use Sobolev's inequality to estimate $|\psi_{xx}|_\infty$ in terms of $\|\psi_{xx}\|$ and $\|\psi_{xxx}\|$. By choosing the coefficients suitably the ψ_{xxx} terms can be moved to the left hand side and the results follow.

Lemma 2.1.9 *Estimate for* $|\psi_{xx}|_\infty$.

$$|\psi_{xx}|_\infty^2 \leq C_6(\epsilon)\|\psi_{xxx}\|^2 + \epsilon\|\psi_{xxxx}\|^2. \tag{2.1.13}$$

It is possible to continue in the same way making estimates for higher derivatives of ψ.

2.2 Compatibility

The sequence of energy estimates as they stand does not tell the whole story. It is necessary also to examine the question of *compatibility* between the initial data $\psi(x,0) = \psi^{(0)}(x)$ and the boundary conditions $\psi = \psi_x = 0$ at $x = 0, 1$. The estimates involve the time derivatives of ψ_x at $t = 0$ and these involve the initial data, the forcing function and the boundary conditions at $t = 0$.

We consider the boundary conditions as applying at $x = 0$ and $x = 1$ for $t \geq 0$ and the initial conditions as applying for $0 < x < 1$ at $t = 0$. The idea of compatibility is that the limits as $x \to 0$ and $x \to 1$ of the initial data should agree with the boundary data. Alternatively, we could take the initial conditions up to the boundary and define the boundary conditions for $t > 0$; in this case compatibility applies to the agreement between the limit as $t \to 0$ of the boundary data and the initial data.

It is well known from the study of the heat equation that an incompatible problem can still have a solution, but that this solution will be singular at the boundary at $t = 0$. For $t > 0$ the singularity spreads and decays by diffusion; in the presence of convective terms the singularity also moves in space.

The one dimensional initial boundary value problem which we study here resembles closely the heat equation. It is a heat equation for the vorticity $\varsigma = \psi_{xx}$. The twist is in the boundary conditions which are on ψ and ψ_x instead of ς.

A sequence of compatibility conditions is obtained by requiring that a number of time derivatives of ψ be bounded at $t = 0$. In the introduction we have made the definition (Definition 1.1.2) for a continuous model to be *compatible to order m*. It is straightforward to relate compatibility to conditions on the initial data.

The *zeroth order* compatibility conditions are

$$\psi^{(0)}(0) = \psi^{(0)}(1) = 0, \qquad \psi_x^{(0)}(0) = \psi_x^{(0)}(1) = 0; \tag{2.2.1}$$

i.e., $\psi(x,0)$, $\psi_x(x,0)$ are continuous up to the boundary. These conditions state simply that the initial data satisfies the no-slip boundary conditions.

The *first order* compatibility conditions are that $\psi_t(x,0)$ and $\psi_{xt}(x,0)$ are continuous up to the boundary. The conditions on the boundary are obtained by differentiating the boundary conditions in time:

$$\psi_t = \psi_{xt} = 0 \qquad \text{at } x = 0, 1 \text{ for } t \geq 0.$$

The time derivatives of ψ and ψ_x in the interior are obtained by integration of the vorticity equation in space

$$\begin{aligned} \psi_{xt}\big|_{x_1}^{x_2} &= \nu\, \psi_{xxx}\big|_{x_1}^{x_2} + \int_{x_1}^{x_2} f(x,t)\, dx, \\ \psi_t\big|_{x_1}^{x_2} - (x_2 - x_1)\psi_{xt}\big|_{x_1} &= \nu\left[\psi_{xx}\big|_{x_1}^{x_2} - (x_2 - x_1)\psi_{xxx}\big|_{x_1}\right] \\ &\quad + \int_{x_1}^{x_2}\int_{x_1}^{x} f(x',t)\, dx'\, dx \end{aligned}$$

for $0 \le x_1 < x_2 \le 1$. Taking these equations over the whole interval $x_1 = 0$, $x_2 = 1$ and evaluating at $t = 0$ gives

$$\begin{aligned} \psi_{xt}(x,0)\big|_{0^+}^{1^-} &= \nu\, \psi_{xxx}^{(0)}\big|_{0^+}^{1^-} + \int_0^1 f(x,0)\, dx, \\ \psi_t(x,0)\big|_{0^+}^{1^-} - \psi_{xt}(0^+,0) &= \nu\left[\psi_{xx}^{(0)}\big|_{0^+}^{1^-} - \psi_{xxx}^{(0)}(0^+)\right] + \int_0^1\int_0^x f(x',0)\, dx'\, dx. \end{aligned}$$

If the initial data is compatible to first order, then the left hand sides of these two equations vanish by continuity. This pair of equations becomes a pair of compatibility conditions on the initial data involving $\psi_{xx}^{(0)}$ and $\psi_{xxx}^{(0)}$. That is,

$$\begin{aligned} \nu\, \psi_{xxx}^{(0)}\big|_{0^+}^{1^-} + \int_0^1 f(x,0)\, dx &= 0, \\ \nu\left[\psi_{xx}^{(0)}\big|_{0^+}^{1^-} - \psi_{xxx}^{(0)}(0^+)\right] + \int_0^1\int_0^x f(x',0)\, dx'\, dx &= 0. \end{aligned} \qquad (2.2.2)$$

Similar constraints can be derived for higher order compatibility conditions to be satisfied. The next in the sequence are the conditions for $\psi_{tt}(x,0)$ and $\psi_{xtt}(x,0)$ to be continuous up to the boundary. They can be determined by differentiating the vorticity equation in time and then integrating in space. Differentiating in time gives an equation for ψ_{xxtt}

$$\psi_{xxtt} = \nu\psi_{xxxxt} + f_t = \nu^2\psi_{xxxxxx} + \nu f_{xx} + f_t.$$

The resulting second order compatibility conditions are

$$\begin{aligned}
0 &= \psi_{xtt}(x,0)\big|_{0^+}^{1^-} \\
&= \nu^2\,\psi^{(0)}_{xxxxx}\big|_{0^+}^{1^-} + \nu\, f_x(x,0)\big|_0^1 + \int_0^1 f_t(x,0)\,dx, \\
0 &= \psi_{tt}(x,0)\big|_{0^+}^{1^-} - \psi_{xtt}(0^+,0) \\
&= \nu^2\left[\psi^{(0)}_{xxxx}\big|_{0^+}^{1^-} - \psi^{(0)}_{xxxxx}(0^+)\right] \\
&\qquad + \nu\left[f(x,0)\big|_0^1 - f_x(0,0)\right] + \int_0^1\int_0^x f_t(x',0)\,dx'\,dx.
\end{aligned}$$

At this point it is clear that the assumption of compatibility is quite a stringent restriction on the initial data. The zeroth order compatibility is a local condition on $\psi^{(0)}$ at each point on the boundary. The other compatibility conditions are *global* conditions; in two space dimensions they involve an integral of the initial data around the boundary and an integral of the initial forcing over the whole domain.

The zeroth order compatibility conditions are also distinguished in that they consist of four conditions, whereas the other compatibility conditions each have two constraints. The mth order compatibility conditions give two linear constraints on the four quantities $d^{2m}\psi^{(0)}/dx^{2m}\big|_{x=0,1}$, $d^{2m+1}\psi^{(0)}/dx^{2m+1}\big|_{x=0,1}$. The general form is

$$\frac{d^{2m+1}\psi^{(0)}}{dx^{2m+1}}\Big|_{x=0}^{1} = H_{m,1}, \qquad \frac{d^{2m}\psi^{(0)}}{dx^{2m}}\Big|_{x=0}^{1} - \frac{d^{2m+1}}{dx^{2m+1}}\psi^{(0)}(0) = H_{m,0}. \tag{2.2.3}$$

2.3 Incompatible Problems

It is already clear from the energy estimates in Section 2.1 that the solution retains some smoothness even if the compatibility conditions are violated at some order. But we can show more than this. We now go on to show that the effect of incompatibility is to create singularities, that the singularities are present in the

$(2m+2)$th space derivative if the compatibility conditions are violated at order m, and that the singularities die off exponentially for large time.

The effects of incompatibility can be isolated and considered separately.

Lemma 2.3.1 *Consider the continuous model problem (2.0.1) with forcing $f(x,t)$ and initial data $\psi^{(0)}(x)$. For any given m the problem can be decomposed into*

$$\psi(x,t) = \widetilde{\psi}(x,t) + \overline{\psi}(x,t),$$

where

$$L\widetilde{\psi} = f, \qquad B\widetilde{\psi} = 0, \qquad \widetilde{\psi}(x,0) = \widetilde{\psi}^{(0)}(x) = \psi^{(0)}(x) - \overline{\psi}(x), \tag{2.3.1a}$$

$$L\overline{\psi} = 0, \qquad B\overline{\psi} = 0, \qquad \overline{\psi}(x,0) = \overline{\psi}^{(0)}(x). \tag{2.3.1b}$$

For any given m, the initial data $\overline{\psi}^{(0)}$ can be constructed in such a way that (2.3.1a) is compatible to order m. Furthermore, $\overline{\psi}^{(0)}(x)$ and its derivatives are bounded in terms of the data $\{f, \psi^{(0)}\}$.

Proof. The proof consists in constructing the appropriate function $\overline{\psi}^{(0)}$ successively to annihilate the compatibility conditions at each order. To make the construction we define two families of polynomials $p_n(x)$, $q_n(x)$ by

$$\begin{aligned} p_n(x) &:= x^n(1-x)^n, \\ q_n(x) &:= x^n(1-x)^n(\frac{1}{2}-x) \end{aligned} \tag{2.3.2}$$

for $n = 0, 1, 2, \ldots$.

We claim that p_n and q_n both satisfy the compatibility conditions up to order $n-2$ and that they violate the homogeneous compatibility conditions at order $n-1$. Furthermore, we claim that for any combination of $H_{n-1,0}$, $H_{n-1,1}$ there exist constants a_{n-1}, b_{n-1} such that

$$\psi^{(0)}_{n-1}(x) = a_{n-1}\, p_n(x) + b_{n-1}\, q_n(x) \tag{2.3.3}$$

satisfies the $(n-1)$th order compatibility conditions

$$\frac{d^{2n-1}\psi^{(0)}}{dx^{2n-1}}\Big|_{x=0}^{1} = H_{n-1,1}, \qquad \frac{d^{2n-2}\psi^{(0)}}{dx^{2n-2}}\Big|_{x=0}^{1} - \frac{d^{2n-1}}{dx^{2n-1}}\psi^{(0)}(0) = H_{n-1,0}.$$

These claims are easily verified by differentiation of p_n and q_n, keeping in mind that $p_n(x)$ is even about $x = 1/2$ whilst q_n is odd about $x = 1/2$.

It is now clear how the construction goes. The induction step is as follows. Assume that $\psi^{(0)}(x)$ satisfies the compatibility conditions up to order $m-1$. Then choose a_m, b_m such that $\psi_m^{(0)}(x)$ defined by (2.3.3) satisfies the mth order compatibility conditions. Replace the initial conditions $\psi^{(0)}(x)$ by $\widetilde{\psi}^{(0)}(x)$ defined by

$$\widetilde{\psi}^{(0)} := \psi^{(0)} - \psi_m^{(0)}.$$

Then $\widetilde{\psi}(x,t)$ is compatible to order m. This completes the proof.

As a corollary to this lemma it suffices to examine unforced problems of the form (2.3.1b) with special initial conditions in order to understand the effect of incompatible initial conditions in general. The special initial conditions are obviously the families $p_n(x)$ and $q_n(x)$.

We consider the problem

$$L\psi = 0, \qquad B\psi = 0, \qquad \psi(x,0) = \psi^{(0)}(x) \tag{2.3.4}$$

with $\psi^{(0)} = a_n\, p_n + b_n\, q_n$. The solution of this problem can be obtained either by Laplace transforms or by eigenfunction expansions. There are advantages to each approach — the Laplace transform method gives a better view of the behaviour for small time, whilst the eigenfunction approach gives the energy estimates cleanly.

The solution by eigenfunction expansions is made by expanding the initial data in a series

$$\psi^{(0)}(x) = \sum_m \widehat{\psi}_m^{(0)} \psi^{(m)}(x)$$

where $\psi^{(m)}(x)$ are the eigenfunctions of the eigenvalue problem

$$\psi_{xxxx} - s\psi_{xx} = 0, \qquad 0 \le x \le 1,$$
$$\psi = \psi_x = 0 \qquad \text{at } x = 0, 1 \tag{2.3.5}$$

with corresponding eigenvalues s_m. This is a self adjoint problem, but it differs from the form of eigenvalue problems which commonly arise in that the eigenvalue multiplies ψ_{xx} instead of ψ. That is, eigenvalue problems are usually written as $Lu = \lambda u$, whereas (2.3.5) is in the form $Lu = sMu$. We study the interesting properties of this eigenvalue problem in an appendix.

The solution of the initial boundary value problem is expressed as

$$\psi(x,t) = \sum_m \widehat{\psi}_m^{(0)} e^{s_m t} \psi^{(m)}(x).$$

The eigenvalues are all real and negative; they split into an even sequence and an odd sequence. The first even eigenvalue is $s_2 = -(2\pi)^2$. The first odd eigenvalue is $s_3 = -\omega_3^2$, where $\omega_3/2 = \tan(\omega_3/2)$, $2\pi < \omega_3 < 3\pi$.

In the appendix we prove the following results.

Lemma 2.3.2 *(a) The coefficients $\widehat{p}_{n,m}$, $\widehat{q}_{n,m}$ of the expansions of $p_n(x)$, $q_n(x)$ satisfy*

$$\widehat{p}_{n,m} = O(\frac{1}{m^{2n}}), \qquad \widehat{q}_{n,m} = O(\frac{1}{m^{2n}}) \tag{2.3.6}$$

as $m \to \infty$.

(b) Let $p_n(x,t)$, $q_n(x,t)$ denote the solutions with initial data $p_n(x)$, $q_n(x)$. Then for each $k = 0, 1, \ldots, 2n-1$ there exist constants C, $\widetilde{C}$ depending on n, k such that

$$\|\frac{\partial^k}{\partial x^k} p_n(x,t)\| \le C e^{-(2\pi)^2 \nu t},$$
$$\|\frac{\partial^k}{\partial x^k} q_n(x,t)\| \le \widetilde{C} e^{-\omega_3^2 \nu t}. \tag{2.3.7}$$

The rate of decay is higher for the higher modes.

The results of the lemma indicate that there is a singularity in the solution at $t = 0$. They show that the singularity decays exponentially in time. They also indicate in which derivative the singularity occurs. For $n = 1, 2, \ldots$ the indication is that there is a singularity in the $2n$th derivative. The case $n = 0$ is special since the lemma gives no L^2 estimates for the solution.

The details of the singularity at $t = 0$ are most apparent in the Laplace transform solution. The singularities in the solutions occur on the boundaries. Their forms are familiar from the study of the heat equation. We give the details for the initial data p_1 and q_1 which satisfy the Dirichlet boundary conditions $\psi = 0$ but not the derivative conditions $\psi_x = 0$.

We denote the Laplace transform of $\psi(x,t)$ by $\widehat{\psi}(x,s)$ and we scale time to make $\nu = 1$ for simplicity. Then $\widehat{\psi}$ satisfies

$$\widehat{\psi}_{xxxx} - s\widehat{\psi}_{xx} = -\psi^{(0)}_{xx}, \qquad \widehat{\psi} = \widehat{\psi}_x = 0 \text{ at } x = 0, 1.$$

In each case $\widehat{\psi}$ can be written down. The properties of $\psi(x,t)$ can be deduced from the transform by standard asymptotic arguments.

First consider the case $\psi^{(0)} = p_1 = x(1-x)$. The solution for the transform is

$$\widehat{\psi}(x,s) = \frac{1}{s}\left[x(1-x) - \widehat{\psi}^{(1)}\right],$$

where

$$\widehat{\psi}^{(1)}(x,s) = \frac{1}{\sqrt{s}\sinh\frac{\sqrt{s}}{2}}\left\{\cosh\frac{\sqrt{s}}{2} - \cosh\sqrt{s}(x - \frac{1}{2})\right\}. \tag{2.3.8}$$

The behaviour of ψ for small time is obtained by expanding $\widehat{\psi}$ for large s. The $\widehat{\psi}^{(1)}$ term is

$$\widehat{\psi}^{(1)}(x,s) \sim \frac{1}{\sqrt{s}}\left[1 - e^{\sqrt{s}\left(|x-\frac{1}{2}|-\frac{1}{2}\right)}\right]$$

$$\sim \begin{cases} \frac{1}{\sqrt{s}}\left[1 - e^{-\sqrt{s}x}\right] & 0 \le x \le 1/2. \\ \frac{1}{\sqrt{s}}\left[1 - e^{-\sqrt{s}(1-x)}\right] & 1/2 < x \le 1. \end{cases}$$

Hence

$$\widehat{\psi}(x,s) \sim \frac{x(1-x)}{s} - \frac{1}{s^{3/2}}\left[1 - e^{\sqrt{s}\left(|x-\frac{1}{2}|-\frac{1}{2}\right)}\right].$$

To invert this transform we make use of the transform pair

$$z(x,t) = \frac{1}{\sqrt{\pi t}} e^{-x^2/4t} \qquad \leftrightarrow \qquad \hat{z}(x,s) = \frac{1}{\sqrt{s}} e^{-\sqrt{s}x}.$$

At $t = 0$ the form of z is a delta function at $x = 0$; for $t > 0$ the delta function diffuses as a Gaussian with peak $1/\sqrt{\pi t}$.

ψ itself is continuous up to the boundary since it involves a convolution of z with g_0. The singularity in ψ_x at $t = 0$ is a step function at $x = 0$. The singularity in the second derivative is a delta function at the boundary at $x = 0$. That is, the limit as $t \to 0$ of the Laplace solution is

$$\begin{aligned} \psi(x,t) &\sim \psi^{(0)}(x) + O(t^{1/2}), \\ \psi_x(x,t) &\to \psi^{(0)}(x) + g_0 \left\{ H(x) - H(1-x) \right\}, \\ \psi_{xx}(x,t) &\to \psi^{(0)}_{xx}(x) + g_0 \left\{ z(x,t) + z(1-x,t) \right\}. \end{aligned}$$

Now we consider the case $\psi^{(0)} = q_1 = x(1-x)(\frac{1}{2} - x)$. The solution for the transform is

$$\hat{\psi}(x,s) = \frac{1}{s} q_1(x) - \frac{1}{2s} \hat{\psi}^{(2)}(x,s),$$

where

$$\hat{\psi}^{(2)}(x,s) = \frac{1}{\sqrt{s}\cosh\frac{\sqrt{s}}{2} - 2\sinh\frac{\sqrt{s}}{2}} \left\{ \sinh\sqrt{s}(x - \frac{1}{2}) - 2(x - \frac{1}{2}) \sinh\frac{\sqrt{s}}{2} \right\}. \tag{2.3.9}$$

The long time behaviour is found in the same way as the previous example. Once again the singularity is a pair of delta functions in the second derivative at $x = 0$ and $x = 1$. This time they have opposite signs instead of the same.

It is not difficult to analyse the next pair $p_2(x) = x^2(1-x)^2$ and $q_2(x) = x^2(1-x)^2(\frac{1}{2} - x)$ in the same way. In these cases the delta function appears in the fourth derivative. The general case, either p_n or q_n, is not amenable to analysis in this way since the algebra accumulates factorially in n.

CHAPTER 3

Space Discrete Model Problem

In this section we consider the semi-discrete problem (1.2.1) obtained from (1.1.1) by discretizing in space using centred second order differences in the interior and a class of difference approximations for the boundary conditions. The discrete approximation for $\psi(x,t)$ is $\phi_\mu(t)$ defined for $t \geq 0$ at the grid points $x_\mu = \mu h$, $\mu = -1, 0, 1, \ldots, N, N+1$ where $Nh = 1$. The equations of the model are

$$\begin{aligned} &D_+D_-\phi_t = \nu(D_+D_-)^2\phi + f^{(h)}, \qquad 1 \leq \mu \leq N, t \geq 0, \\ &\phi_0 = \phi_N = 0, \\ &D_{l,q}\phi_0 = D_{r,q}\phi_N = 0, \\ &\phi(0) = \phi^{(0)}, \qquad 0 \leq \mu \leq N \end{aligned} \tag{3.0.1}$$

where $f_\mu^{(h)}(t)$ is a discrete function related to f.

We define the discrete inner product of two functions $u_\mu, \tilde{u}_\mu$ defined at the grid points x_μ, $\mu = 0, 1, \ldots, N$ by

$$(u, \tilde{u})_h := \sum_{\mu=0}^{N}{}' h u_\mu \tilde{u}_\mu := \frac{h}{2} u_0 \tilde{u}_0 + \sum_{\mu=1}^{N-1} h u_\mu \tilde{u}_\mu + \frac{h}{2} u_N \tilde{u}_N,$$

and for functions $v_{\mu+1/2}, \tilde{v}_{\mu+1/2}$ defined at the midpoints of the grid $x_{\mu+1/2} := (\mu + 1/2)h$, $\mu = 0, 1, \ldots, N-1$ by

$$(v, \tilde{v})_h := \sum_{\mu=0}^{N-1} h v_{\mu+1/2} \tilde{v}_{\mu+1/2}.$$

These discrete formulae are second order approximations to the integrals of the continuous inner product. We use the notation D for the two point difference operator which approximates d/dx.

3.1 Estimates for the Compatible Problem

We make estimates of ϕ and its divided differences in terms of the initial data and the forcing. The sequence of estimates is closely related to the estimates for the continuous case in the previous section. The major difference arises in obtaining the estimate for $D\phi$. The technique used in obtaining this estimate is integration by parts in the continuous case, summation by parts in the discrete case, and it relies on using the boundary conditions to eliminate the boundary terms which arise. In the case where second order boundary conditions are used for the derivative of ψ, the boundary terms do vanish. However, if higher order approximations are used, then they do not vanish and it is necessary to use Laplace transform techniques to obtain estimates.

Throughout this section we assume that $\|f^{(h)}(\cdot,t)\|_h$ is finite for all t and also that the norms of all the divided differences in x and derivatives in time of $f^{(h)}$ are finite.

We begin by stating the discrete analogues of the first three lemmas of the previous section. We omit the proofs which mimic those presented above.

Lemma 3.1.1 *(Sobolev's Inequality) For any discrete function u_μ and any $\epsilon \geq 0$ there exists a constant $C_1(\epsilon)$ such that*

$$|u|_\infty^2 \leq C_1(\epsilon)\|u\|_h^2 + \epsilon\|Du\|_h^2. \tag{3.1.1}$$

Lemma 3.1.2 *(Poincaré's Inequality)*

$$\|u\|_h^2 \leq \left(\sum_0^{N}{}' hu_\mu\right)^2 + \frac{1}{2}\|Du\|_h^2 \tag{3.1.2}$$

Lemma 3.1.3 *Let u be a discrete function on $[0,1]$ and $u_0 = u_N = 0$. Then there exists a constant C_2 such that*

$$\|u\|_h^2 \leq C_2\|Du\|_h^2. \tag{3.1.3}$$

Lemma 3.1.4 *Summation by parts formulae.*

$$\sum_{\mu=1}^{N-1} h\phi_\mu D_+D_-\tilde{\phi}_\mu = -\phi_0 D\tilde{\phi}_{1/2} + \phi_N D\tilde{\phi}_{N-1/2} - (D\phi, D\tilde{\phi})_h, \tag{3.1.4}$$

$$\begin{aligned}\sum_{\mu=1}^{N-1} h\phi_\mu (D_+D_-)^2\tilde{\phi}_\mu = &-\phi_0 D_+D_-(D\tilde{\phi}_{1/2}) + \phi_N D_+D_-(D\tilde{\phi}_{N-1/2}) \\ &+ D_0\phi_0 D_+D_-\tilde{\phi}_0 - D_0\phi_N D_+D_-\tilde{\phi}_N \\ &+ (D_+D_-\phi, D_+D_-\tilde{\phi})_h.\end{aligned} \tag{3.1.5}$$

Proof.

$$\begin{aligned}\sum_{\mu=1}^{N-1} h\phi_\mu D_+D_-\tilde{\phi}_\mu &= \sum_{\mu=1}^{N-1} h\phi_\mu D\tilde{\phi}_{\mu+1/2} - \sum_{\mu=0}^{N-2} h\phi_{\mu+1} D\tilde{\phi}_{\mu+1/2} \\ &= \phi_{N-1} D\tilde{\phi}_{N-1/2} - \sum_{\mu=1}^{N-2} hD\phi_{\mu+1/2} D\tilde{\phi}_{\mu+1/2} - \phi_1 D\tilde{\phi}_{1/2} \\ &= \phi_N D\tilde{\phi}_{N-1/2} - \sum_{\mu=0}^{N-1} hD\phi_{\mu+1/2} D\tilde{\phi}_{\mu+1/2} - \phi_0 D\tilde{\phi}_{1/2},\end{aligned}$$

$$\sum_{\mu=1}^{N-1} h\phi_\mu (D_+D_-)^2\tilde{\phi}_\mu = -\phi_0 D_+D_-D\tilde{\phi}_{1/2} + \phi_N D_+D_-D\tilde{\phi}_{N-1/2} - (D\phi, D_+D_-D\tilde{\phi})_h$$

and

$$\begin{aligned}(D\phi, D_+D_-D\tilde{\phi})_h &= \sum_{\mu=0}^{N-1} hD\phi_{\mu+1/2} D_+D_-D\tilde{\phi}_{\mu+1/2} \\ &= \sum_{\mu=0}^{N-1} D\phi_{\mu+1/2}\left(D_+D_-\tilde{\phi}_{\mu+1} - D_+D_-\tilde{\phi}_\mu\right) \\ &= \sum_{\mu=1}^{N} D\phi_{\mu-1/2} D_+D_-\tilde{\phi}_\mu - \sum_{\mu=0}^{N-1} D\phi_{\mu+1/2} D_+D_-\tilde{\phi}_\mu \\ &= D\phi_{N-1/2} D_+D_-\tilde{\phi}_N - \sum_{\mu=1}^{N-1} hD_+D_-\phi_\mu D_+D_-\tilde{\phi}_\mu \\ &\quad - D\phi_{1/2} D_+D_-\tilde{\phi}_0 \\ &= D\phi_{N-1/2} D_+D_-\tilde{\phi}_N + \frac{h}{2} D_+D_-\phi_N D_+D_-\tilde{\phi}_N \\ &\quad - (D_+D_-\phi, D_+D_-\tilde{\phi})_h\end{aligned}$$

$$- D\phi_{1/2} D_+ D_- \tilde{\phi}_0 + \frac{h}{2} D_+ D_- \phi_0 D_+ D_- \tilde{\phi}_0$$
$$= D_0 \phi_N D_+ D_- \tilde{\phi}_N - D_0 \phi_0 D_+ D_- \tilde{\phi}_0 - (D_+ D_- \phi, D_+ D_- \tilde{\phi})_h .$$

Lemma 3.1.5 *Estimate for $\|D\phi\|_h$. For any $\alpha > 0$ there exists a constant $C_3(\alpha, \nu)$ such that*

$$\frac{d}{dt} \|D\phi\|_h^2 + \nu \|D_+ D_- \phi\|_h^2 \leq |G| + \alpha \|D\phi\|_h^2 + C_3 \|f\|_h^2 \qquad (3.1.6)$$

where G denotes the boundary terms

$$G = \nu \left(D_0 \phi_N D_+ D_- \phi_N - D_0 \phi_0 D_+ D_- \phi_0 \right).$$

If second order boundary conditions are used (i.e., $q = 2$ in (3.0.1) so that $D_0 \phi_0 = D_0 \phi_N = 0$), then the boundary terms vanish (G=0) and

$$\|D\phi(t)\|_h^2 \leq e^{\alpha t} \|D\phi(0)\|_h^2 + C_3 \int_0^t e^{\alpha(t-t')} \|f^{(h)}(t')\|_h^2 dt'. \qquad (3.1.7)$$

Proof. Take the sum from $\mu = 1$ to $N - 1$ of $h\phi_\mu$ with the equation (3.0.1). Use the summation by parts formulae to obtain

$$\frac{1}{2} \frac{d}{dt} \|D\phi\|_h^2 + \nu \|D_+ D_- \phi\|_h^2 = G - \sum_{\mu=1}^{N-1} h \phi_\mu f_\mu^{(h)}$$

where

$$G = -\phi_0 D\phi_{1/2} + \phi_N D\phi_{n-1/2}$$
$$+ \nu \left(\phi_0 D_+ D_- D\phi_{1/2} - \phi_N D_+ D_- D\phi_{N-1/2} \right.$$
$$\left. - D_0 \phi_0 D_+ D_- \phi_0 + D_0 \phi_N D_+ D_- \phi_N \right).$$

The boundary conditions $\phi_0 = \phi_N = 0$ eliminate four of the six terms. The rest of the proof follows the continuous case.

For *second order* boundary conditions it is possible to continue as in the continuous case making estimates for higher divided differences of ϕ.

3.1.1 Higher Order Boundary Conditions

In the case of *higher order* boundary conditions the boundary terms G do not vanish and so we do not obtain an estimate of the form (3.1.6) for $\|D\phi\|_h$. One approach which we pursued without success was to try to eliminate the boundary terms by choosing the inner product in another way. We investigated the effect of changing the weighting of the terms near the boundary, but this did not appear to work.

Instead we take another approach. The idea is first to subtract out the solution with *second order* boundary conditions. Then the remainder satisfies the semi-discrete equations (3.0.1) with zero forcing, zero initial data and inhomogeneous boundary data for the derivative boundary conditions. This problem can be estimated using Laplace transforms in the same manner as the *incompatible problems* were treated in the continuous case.

Let ϕ be the solution of (3.0.1) with boundary conditions of order q, $B_{h,q}\phi = 0$, and let $\tilde{\phi}$ be the solution with second order boundary conditions, $B_{h,2}\tilde{\phi} = 0$. Then the normalized difference of these two solutions is

$$\tilde{\tilde{\phi}} := (\phi - \tilde{\phi})/h^2 \tag{3.1.8}$$

which satisfies

$$L_{t,h}\tilde{\tilde{\phi}} = 0 \qquad B_{h,q}\tilde{\tilde{\phi}} = g \qquad \tilde{\tilde{\phi}}(0) = 0, \tag{3.1.9}$$

where

$$h^2 g = B_{h,q}(\phi - \tilde{\phi}) = -B_{h,q}\tilde{\phi} = (B_{h,2} - B_{h,q})\,\tilde{\phi}.$$

So

$$g = \begin{bmatrix} 0 \\ 0 \\ g_0 \\ g_1 \end{bmatrix} = \frac{1}{h^2}\begin{bmatrix} 0 \\ 0 \\ (D_0 - D_{l,q})\tilde{\phi}_0 \\ (D_0 - D_{r,q})\tilde{\phi}_N \end{bmatrix}.$$

Both g_0 and g_1 can be written as divided differences of $\tilde{\phi}$. For example, for *third order* boundary conditions

$$g_0 = \frac{1}{6}\,\frac{\tilde{\phi}_2 - 3\tilde{\phi}_1 + 3\tilde{\phi}_0 - \tilde{\phi}_{-1}}{h^3} = \frac{1}{6}D_+D_-D\tilde{\phi}_{1/2}, \qquad g_1 = \frac{1}{6}D_+D_-D\tilde{\phi}_{N-1/2}.$$

Hence, g can be estimated in terms of $f^{(h)}$ via the estimates for $\tilde{\phi}$ and its divided differences.

It remains to estimate $\tilde{\tilde{\phi}}$ in terms of g and for this we use Laplace transforms.

3.2 Space Discrete Model with Inhomogeneous Boundary Data

We consider the space discrete model problem with inhomogeneous boundary data

$$\begin{aligned} &\frac{\partial}{\partial t}D_+D_-\phi_\mu = \nu(D_+D_-)^2\phi_\mu, \qquad 1 \le \mu \le N-1,\\ &\phi_0 = \phi_N = 0,\\ &D_{l,q}\phi_0 = g_0(t), \qquad D_{r,q}\phi_N = g_1(t),\\ &\phi_\mu(0) = 0 \qquad 0 \le \mu \le N \end{aligned} \tag{3.2.1}$$

for $0 \le t < \infty$. We seek estimates for the discrete stream function ϕ and the discrete vorticity $z = D_+D_-\phi$. These estimates are obtained using the method of Laplace transforms. They are contained in the following two lemmas.

Lemma 3.2.1 *Estimate for ϕ. There exists a constant $\sigma_0 > 0$ such that*

$$\int_0^t e^{-2\nu\sigma_0 t}|\phi_\mu(t)|^2 dt \le \text{const} \int_0^t e^{-2\nu\sigma_0 t}\left(|g_0(t)|^2 + |g_1(t)|^2\right) dt. \tag{3.2.2}$$

Lemma 3.2.2 *Estimate for z. There exists a constant $\sigma_0 > 0$ such that*

$$\int_0^t e^{-2\nu\sigma_0 t}|z_\mu(t)|^2 dt \le \text{const}\ \frac{4}{h^2}\int_0^t e^{-2\nu\sigma_0 t}\left(|g_0(t)|^2 + |g_1(t)|^2\right) dt. \tag{3.2.3}$$

Loosely speaking, the estimate for z says that the discrete vorticity behaves like $1/h$. This is in contrast to the estimate for ϕ and all the other estimates above which are independent of h. We shall examine below the asymptotic form of the solution for z for small time. In fact z is not everywhere large but the $1/h$ behaviour is contained in a boundary layer generated by the inhomogeneous boundary conditions. This boundary layer is set up at $t = 0$ and then it diffuses into the interior of the region and decays in time.

We proceed to prove these two estimates, and then to look at the asymptotic behaviour. The solution to (3.2.1) is found by the Laplace transform method. We denote the Laplace transform of ϕ by $\hat{\phi}$

$$\hat{\phi}_\mu(s) := \int_0^\infty e^{-st}\phi_\mu(t)dt.$$

Then $\hat{\phi}(s)$ satisfies

$$\begin{aligned} &sD_+D_-\hat{\phi}_\mu = \nu(D_+D_-)^2\hat{\phi}_\mu, \qquad 1 \le \mu \le N, \\ &\hat{\phi}_0 = \hat{\phi}_N = 0, \\ &D_{l,q}\hat{\phi}_0 = \hat{g}_0(s), \qquad D_{r,q}\hat{\phi}_N = \hat{g}_1(s). \end{aligned} \tag{3.2.4}$$

The estimates for ϕ and z will be made by estimating the Laplace transforms $\hat{\phi}$ and $\hat{z}$ in terms of the boundary data $\hat{g}_0$, $\hat{g}_1$ and then inverting the Laplace transforms.

First we show how the bounds for $\hat{\phi}$ and $\hat{z}$ are used to get the bounds on ψ and z. The argument is as follows. Take the solution for $\hat{\phi}(s)$ and invert the Laplace transform, using the inversion contour $s = \nu\sigma_0 + i\eta$, $-\infty < \eta < \infty$ for some constant σ_0

$$\phi_\mu(t) = \frac{1}{2\pi}\, e^{\nu\sigma_0 t}\int_{-\infty}^\infty e^{i\eta t}\hat{\phi}_\mu(s)d\eta.$$

By Parseval's relation

$$\begin{aligned} \int_0^\infty e^{-2\nu\sigma_0 t}|\phi_\mu(t)|^2dt &= \frac{1}{2\pi}\int_{-\infty}^\infty |\hat{\phi}_\mu(s)|^2 d\eta \\ &\le \frac{\text{const}}{2\pi}\int_{-\infty}^\infty \left(|\hat{g}_0(\nu\sigma_0 + i\eta)|^2 + |\hat{g}_1(\nu\sigma_0 + i\eta)|^2\right) d\eta \\ &= \text{const}\int_0^\infty e^{-2\nu\sigma_0 t}\left(|g_0(t)|^2 + |g_1(t)|^2\right) dt. \end{aligned}$$

The integrals from 0 to ∞ over time can be replaced by integrals from 0 to t by the observation that $\phi(t)$ depends on $g_0(\tau)$ and $g_1(\tau)$ only for $0 \leq \tau \leq t$ and so we can replace $g_0(\tau)$ and $g_1(\tau)$ by 0 for $\tau > t$. This gives the estimate for ϕ in the right form. The estimate for z uses the same steps.

Now we turn to the task of estimating $\hat{\phi}$ and $\hat{z}$. Consider the solution of (3.2.4) for $\hat{\phi}$. The homogeneous difference equation for $\hat{\phi}$ has the general solution

$$\hat{\phi}_\nu = A\kappa^{-\mu} + B\kappa^{\mu-N} + C + Dx_\mu \tag{3.2.5}$$

where κ is the larger root (in absolute value) of

$$(\kappa - 1)^2 - \sigma h^2 \kappa = 0, \qquad \sigma = s/\nu. \tag{3.2.6}$$

(Note that the two roots of (3.2.6) satisfy $\kappa_1\kappa_2 = 1$.) The larger root is

$$\kappa = 1 + \frac{\sigma h^2}{2} + \sqrt{\frac{\sigma h^2}{2}\left(2 + \frac{\sigma h^2}{2}\right)}. \tag{3.2.7}$$

There are two interesting limits, $|\sigma h^2| \ll 1$ and $|\sigma h^2| \gg 1$. In these limiting cases

$$\kappa \sim \begin{cases} 1 + \sigma^{1/2} h & |\sigma h^2| \ll 1 \\ \sigma h^2 & |\sigma h^2| \gg 1 \end{cases}.$$

The coefficients A, B, C, D are functions of σ and h. They are determined from the boundary conditions. Substituting into the boundary conditions gives

$$\begin{aligned} A + B\kappa^{-N} + C &= 0 \\ A\kappa^{-N} + B + C + D &= 0 \\ A\left[D_{l,q}\kappa^{-\mu}\right]_{\mu=0} + B\left[D_{l,q}\kappa^{\mu-N}\right]_{\mu=0} + D &= \hat{g}_0 \\ A\left[D_{r,q}\kappa^{-\mu}\right]_{\mu=N} + B\left[D_{r,q}\kappa^{\mu-N}\right]_{\mu=N} + D &= \hat{g}_1 \end{aligned} \tag{3.2.8}$$

We want to estimate A, B, C, D in terms of $\hat{g}_0$, $\hat{g}_1$. To solve this system of equations we first eliminate C and D

$$C = -\left(A + B\kappa^{-N}\right), \qquad D = (A - B)(1 - \kappa^{-N}).$$

Define

$$\theta_1 = \theta_1(\kappa, h) := \left[D_{r,q}\kappa^{\mu}\right]_{\mu=0} = -\left[D_{l,q}\kappa^{-\mu}\right]_{\mu=0}, \qquad (3.2.9)$$

$$\theta_2 = \theta_2(\kappa, h) := \left[D_{l,q}\kappa^{\mu}\right]_{\mu=0} = -\left[D_{r,q}\kappa^{-\mu}\right]_{\mu=0}.$$

Then A, B are given by

$$\begin{pmatrix} a & -b \\ b & -a \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \hat{g}_0 \\ \hat{g}_1 \end{pmatrix}, \qquad (3.2.10)$$

where

$$a = -\theta_1 + (1 - \kappa^{-N}), \qquad b = -\theta_2\kappa^{-N} + (1 - \kappa^{-N}). \qquad (3.2.11)$$

Hence

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{a^2 - b^2}\begin{pmatrix} a & -b \\ b & -a \end{pmatrix}\begin{pmatrix} \hat{g}_0 \\ \hat{g}_1 \end{pmatrix}. \qquad (3.2.12)$$

For it second order boundary conditions θ_1 and θ_2 are equal

$$\theta_1 = \theta_2 = \theta,$$

$$\theta = \theta(\kappa, h) := \frac{\kappa - \kappa^{-1}}{2h} = \sigma^{1/2}\left(1 + \frac{\sigma h^2}{4}\right)^{1/2}. \qquad (3.2.13)$$

In the two limits of interest

$$\theta \sim \begin{cases} \sigma^{1/2} & |\sigma h^2| \ll 1 \\ \sigma h/2 & |\sigma h^2| \gg 1 \end{cases}.$$

For higher order boundary conditions θ_1 and θ_2 are not equal, but they can be expressed as perturbations (not necessarily small) of θ. For third order boundary conditions

$$\begin{aligned} \theta_1 &= \frac{2\kappa + 3 - 6\kappa^{-1} + \kappa^{-2}}{6h} = \frac{\kappa - \kappa^{-1}}{2h} - \frac{h^2}{6}\,\frac{\kappa - 3 + 3\kappa^{-1} - \kappa^{-2}}{h^3} \\ &= \frac{\kappa - \kappa^{-1}}{2h}\left[1 - \frac{h^2}{3}\,\frac{(\kappa - 1)^2}{h^2\kappa(\kappa + 1)}\right] \\ &= \theta\left[1 - \frac{\sigma h^2}{3}\,\frac{1}{\kappa + 1}\right], \end{aligned} \qquad (3.2.14)$$

$$\begin{aligned} \theta_2 &= \frac{-\kappa^2 + 6\kappa - 3 - 2\kappa^{-1}}{6h} = \frac{\kappa - \kappa^{-1}}{2h} - \frac{h^2}{6}\,\frac{\kappa^2 - 3\kappa + 3 - \kappa^{-1}}{h^3} \\ &= \frac{\kappa - \kappa^{-1}}{2h}\left[1 - \frac{h^2}{3}\,\frac{(\kappa - 1)^2}{h^2(\kappa + 1)}\right] \\ &= \theta\left[1 - \frac{\sigma h^2}{3}\,\frac{\kappa}{\kappa + 1}\right] \end{aligned} \qquad (3.2.15)$$

using the definitions of θ and σ.

The estimates for A, B in terms of $\hat{g}_0$, $\hat{g}_1$ are obtained from the following bounds for a, b and $a^2 - b^2$. These bounds show that the θ_1 term in a dominates the other terms in the equations.

Lemma 3.2.3 *Suppose that $\sigma_0 > 0$ is sufficiently large. Then there exist positive constants M_a, M_b, M such that*

$$|a| \leq M_a|\theta|, \qquad |b| \leq M_b|\theta|, \qquad |a^2 - b^2| \geq M|\theta|^2 \tag{3.2.16}$$

for all σ with $\mathrm{Re}\ \sigma \geq \sigma_0$ and all $h < 1/2$, where θ is given by (3.2.13).

This lemma will provide the estimate for $\hat{\phi}$ in terms of $\hat{g}_0$ and $\hat{g}_1$ on the contour $s = \nu\sigma_0 + i\eta$, $-\infty < \eta < \infty$. The estimate is

$$\begin{aligned} |\hat{\phi}_\mu(s)|^2 &\leq \text{const}\left(|A|^2 + |B|^2 + |C|^2 + |D|^2\right) \\ &\leq \text{const}\ \frac{1}{|\theta|}\left(|\hat{g}_0(s)|^2 + |\hat{g}_1(s)|^2\right), \end{aligned}$$

where $1/|\theta|$ is bounded for $Re\ \sigma \geq \sigma_0$.

Likewise we can obtain the estimate for $z = D_+D_-\phi$. The solution for $\hat{z}$ is

$$\hat{z}_\mu(s) = D_+D_-\hat{\phi}_\mu(s) = \sigma\left(A\kappa^{-\mu} + B\kappa^{\mu-N}\right), \qquad \mu = 0, 1, \ldots, N,$$

where σA and σB can be estimated by

$$\left.\begin{matrix} |\sigma A| \\ |\sigma B| \end{matrix}\right\} \leq \text{const}\ |\sigma/\theta|\left(|\hat{g}_0(s)|^2 + |\hat{g}_1(s)|^2\right)^{1/2}$$

and

$$|\sigma/\theta|^2 = \left|\frac{\sigma}{1 + \sigma h^2/4}\right| \leq 4/h^2.$$

Proof of lemma Lemma 3.2.3. We first give the proof for second order boundary conditions and then consider higher order boundary conditions. Let

$$K_0 := \sup_{\mathrm{Re}\ \sigma \geq \sigma_0} |\kappa^{-N}|, \qquad K_1 := \inf_{\substack{\mathrm{Re}\ \sigma \geq \sigma_0 \\ h \geq 0}} |\theta|.$$

Clearly, $K_0 \leq 1$ since $|\kappa| \geq 1$, and $K_1 = \sigma_0^{1/2}$. Then

$$|a| \leq |\theta| + (1 + K_0) \leq M_a|\theta|,$$

$$|b| \leq K_0|\theta| + (1 + K_0) \leq M_b|\theta|,$$

where

$$M_a = 1 + \frac{1 + K_0}{K_1}, \quad M_b = K_0 + \frac{1 + K_0}{K_1}.$$

M_a and M_b are both positive numbers. Furthermore

$$\begin{aligned} a^2 - b^2 &= (a - b)(a + b) \\ &= \left(\theta_1 - \theta_2\kappa^{-N}\right)\left[\theta_1 + \theta_2\kappa^{-N} - 2(1 - \kappa^{-N})\right] \\ &= \left(1 - \kappa^{-N}\right)\left[\left(1 + \kappa^{-N}\right) - \frac{2}{\theta}\left(1 - \kappa^{-N}\right)\right]\theta^2, \end{aligned}$$

so

$$|a^2 - b^2| \geq (1 - K_0)\left[(1 - K_0) - \frac{2}{K_1}(1 + K_0)\right]|\theta|^2 = M|\theta|^2.$$

Hence M is positive provided

$$K_1 > 2\left(\frac{1 + K_0}{1 - K_0}\right) \text{ and } K_0 < 1. \tag{3.2.17}$$

We must check that these inequalities are satisfied for some σ_0. Since $K_1 = \sigma_0^{1/2}$ increases without bound as a function of σ_0 and K_0 is a decreasing function of σ_0, all we need show is that $K_0 < 1$ independently of h for some σ_0. Then (3.2.17) will be satisfied for σ_0 sufficiently large.

First consider σ real. In this case $\kappa = \kappa(\sigma h^2)$ is real also and satisfies

$$\kappa \geq 1 + \sigma^{1/2}h \geq 1 + \sigma_0^{1/2}h.$$

Hence

$$\kappa^N = \kappa^{1/h} \geq 1 + \sigma_0^{1/2}, \qquad |\kappa^{-N}| \leq \left(1 + \sigma_0^{1/2}\right)^{-1}.$$

Next, consider σ complex, $\sigma = \sigma_r + i\sigma_i$. We claim that the real part of κ is an increasing function of $|\text{Im } \sigma|$. That is,

$$\kappa = \kappa_r + i\kappa_i, \qquad \kappa_r(\sigma h^2) \geq \kappa_r(\sigma_r h^2) = \kappa(\sigma_r h^2).$$

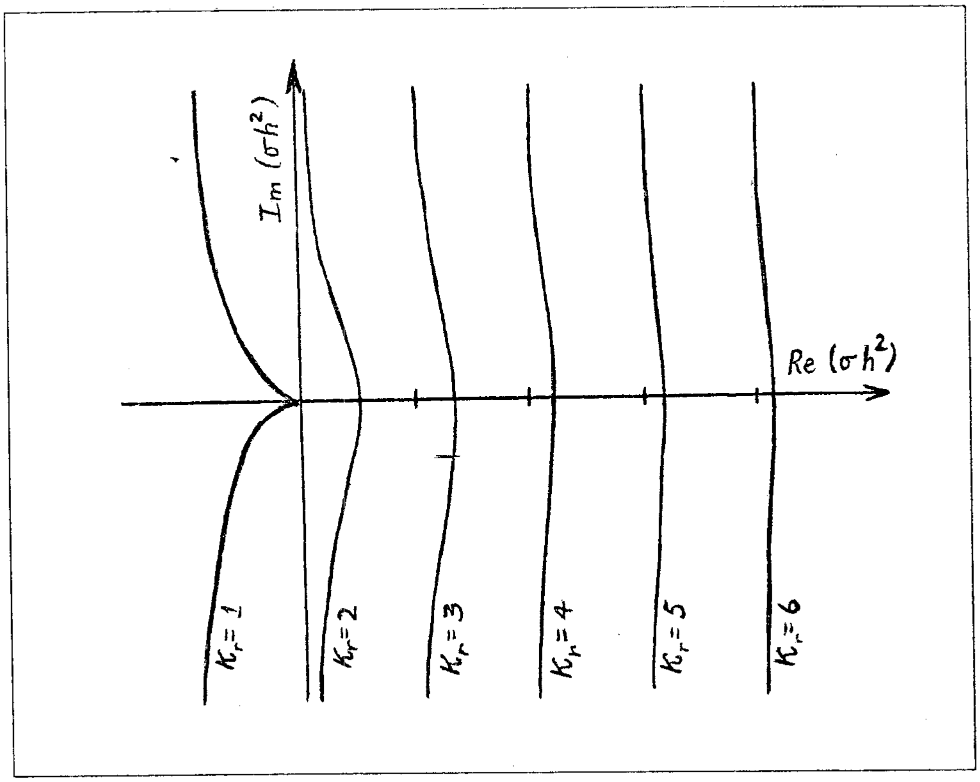

Figure 3.1

Hence

$$|\left(\kappa(\sigma h^2)\right)^{-N}| \leq \left(\kappa_r(\sigma h^2)\right)^{-N} \leq \left(\kappa(\sigma_r h^2)\right)^{-N} \leq \left(1+\sigma_0^{1/2}\right)^{-1}.$$

So, in order to complete the proof of the lemma for second order boundary conditions, we need only to prove this claim. Instead of considering κ as a function of σh^2 it is convenient to consider σh^2 as a function of κ given by

$$\sigma h^2 = \frac{(\kappa-1)^2}{\kappa}.$$

Consider the curves in the σh^2 plane which map onto the lines Re κ = const (≥ 1) in the κ plane. They are given by

$$\begin{aligned} \mathrm{Re}(\sigma h^2) &= \kappa_r - 2 + \frac{\kappa_r}{{\kappa_r}^2 + {\kappa_i}^2}, \\ \mathrm{Im}(\sigma h^2) &= \kappa_i\left(1 - \frac{1}{{\kappa_r}^2 + {\kappa_i}^2}\right). \end{aligned} \qquad (3.2.18)$$

These curves are sketched in Figure 3.1.

Clearly, for $\kappa_r > 0$ $\mathrm{Re}(\sigma h^2)$ is maximal when $\kappa_i = 0$. Now consider the curves in the κ plane with $|\kappa| \geq 1$ which map into the lines $\mathrm{Re}(\sigma h^2) = \mathrm{const} > 0$ in the σh^2 plane. It is clear from an inspection of the figure that Re κ increases as $|Im(\sigma h^2)|$ increases away from the real axis. This completes the proof of the lemma for second order boundary conditions.

The proof of the lemma for higher order boundary conditions follows the same steps. We give the details for third order boundary conditions.

First we show that

$$\frac{2}{3}|\theta| \leq |\theta_1| \leq \frac{4}{3}|\theta|, \qquad |\theta_2 \kappa^{-N}| \leq \frac{4}{3}\left(1 + \frac{1}{2}\sigma_0^{1/2}\right)^{-1} |\theta|. \tag{3.2.19}$$

To establish these inequalities observe that

$$\sigma_r h^2 + 1 \leq \kappa_r \leq \sigma_r h^2 + 2, \qquad |\sigma_i h^2| \leq |\kappa_i|$$

for $\sigma_r \geq 0$ from (3.2.18) and so

$$|\kappa|^2 \geq 1 + |\sigma h^2|^2, \qquad |\kappa| \geq \max\left\{1, |\sigma h^2|\right\}.$$

Hence

$$\left|\frac{\sigma h^2}{3}\frac{1}{\kappa+1}\right| \leq \min\left\{\frac{\sigma h^2}{6}, \frac{1}{3}\right\}.$$

The inequalities for θ_1 in (3.2.19) are obtained by taking the second of these two upper bounds and substituting it into the expression for θ_1

$$\theta_1 = \theta\left(1 - \frac{\sigma h^2}{3}\frac{1}{\kappa+1}\right).$$

Also

$$|\theta_2 \kappa^{-N}| = \left|\theta\left(1 - \frac{\sigma h^2}{3}\frac{\kappa}{\kappa+1}\right)\kappa^{-N}\right| \leq \left(1 + \frac{|\kappa|}{3}\right).|\kappa^{-N}|.|\theta| \leq \frac{4}{3}\,|\kappa^{1-N}|.|\theta|$$

and

$$|\kappa^{N-1}| \geq \left(1 + \sigma_0^{1/2} h\right)^{N-1} \geq 1 + \frac{N-1}{N}\sigma_0^{1/2} \geq 1 + \frac{1}{2}\sigma_0^{1/2}$$

for $N \geq 2$ (i.e., for $h \leq 1/2$). This proves the inequality for $\theta_2\kappa^{-N}$ in (3.2.19).

The inequalities for a, b and $a^2 - b^2$ follow.

$$|a| \leq \left(\frac{4}{3} + \frac{1+K_0}{K_1}\right)|\theta| = M_a|\theta|,$$

$$|b| \leq \left[\frac{4}{3}\left(1 + \frac{1}{2}\sigma_0^{1/2}\right)^{-1} + \frac{1+K_0}{K_1}\right]|\theta| = M_b|\theta|,$$

$$|a^2 - b^2| \leq \left\{\left[\frac{2}{3} - \frac{1+K_0}{K_1}\right]^2 - M_b^2\right\}|\theta|^2 = M|\theta|^2.$$

M_a, M_b and M are all positive for Re $\sigma \geq \sigma_0$ provided σ_0 is sufficiently large.

This completes the proof of Lemma 3 for third order boundary conditions.

3.2.1 Asymptotic Behaviour for Small Time

Through the Laplace transform there is a correspondence between t small and s large. For s large (i.e., $|\sigma h^2| \gg 1$)

$$|b| \ll |a| \quad \text{and} \quad a \sim -\theta_1 \sim -\frac{\kappa^{-1}}{qh} \sim -\frac{\sigma h}{q},$$

so

$$A \sim \frac{\hat{g}_0}{a} \sim -\frac{q}{h}.\frac{\hat{g}_0}{\sigma}, \qquad B \sim \frac{q}{h}.\frac{\hat{g}_1}{\sigma}.$$

Hence

$$\hat{z}_\mu(s) \sim -\frac{q}{h}\left[\frac{\hat{g}_0}{\sigma}(\sigma h^2)^{-\mu} - \frac{\hat{g}_1}{\sigma}(\sigma h^2)^{\mu-N}\right]$$

for s large. Thus

$$z_\mu(t) \sim -\frac{q}{h}\left[\frac{g_0(0)}{\mu!}(t/h^2)^\mu - \frac{g_1(0)}{(N-\mu)!}(t/h^2)^{N-\mu}\right]$$

for $t \ll h^2$.

The boundary layer form of this asymptotic solution is clear. The strength of the boundary layer is $O(1/h)$ times the initial value of the inhomogeneous boundary forcing. The boundary layer is set up instantaneously at $t = 0$ and it diffuses rapidly into the interior of the domain. Over time, however, this boundary layer decays as well as diffuses as can be seen by examining the asymptotic solution for large time.

APPENDIX 1

Eigenvalue Problem

We consider the eigenvalue problem for $\psi(x)$

$$\begin{aligned} \psi_{xxxx} - s\psi_{xx} &= 0 \qquad 0 \le x \le 1, \\ \psi = \psi_x &= 0 \qquad x = 0, 1. \end{aligned} \tag{A1.0.1}$$

In operator form this problem can be written as

$$L\psi = sM\psi, \qquad B\psi = 0,$$

where

$$L := \frac{d^4}{dx^4} = M^2, \qquad M := \frac{d^2}{dx^2}, \qquad B\psi := \begin{pmatrix} \psi(0) \\ \psi(1) \\ \psi_x(0) \\ \psi_x(1) \end{pmatrix}.$$

Both L and M are *self adjoint* operators with the usual L^2 inner product on $[0,1]$. By this we mean that for any two functions ψ, $\widetilde{\psi}$ satisfying the boundary conditions the equations $(\psi, L\widetilde{\psi}) = (L\psi, \widetilde{\psi})$ and $(\psi, M\widetilde{\psi}) = (M\psi, \widetilde{\psi})$ are satisfied. It is a simple consequence of this self-adjointness that all the eigenvalues are real and that the orthogonality properties

$$(\psi^{(l)}, M\psi^{(m)}) = \begin{cases} \|\psi_x^{(l)}\|^2 & \text{for } l = m \\ 0 & \text{for } l \ne m \end{cases},$$
$$(M\psi^{(l)}, M\psi^{(m)}) = 0 \qquad \text{for } l \ne m$$

hold. To verify these relations, consider $(\overline{s_l} - s_m)$ $(\psi^{(l)}, M\psi^{(m)})$ and $(\overline{s_l} - s_m)$ $(M\psi^{(l)}, M\psi^{(m)})$, respectively. In general,

$$(\psi^{(l)}, \psi^{(m)}) \ne 0 \qquad \text{for } l \ne m;$$

i.e., the usual orthogonality of the eigenfunctions does not hold.

The eigenvalues and eigenfunctions consist of an even family and an odd family. They are

$$\psi^{(2k)}(x) = 1 - \cos 2k\pi x, \qquad s_{2k} = -(2k\pi)^2, \tag{A1.0.2}$$

and

$$\begin{aligned} \psi^{(2k+1)}(x) &= \sin\Big[\omega_k(x-\frac{1}{2})\Big] - 2(x-\frac{1}{2})\sin\frac{\omega_k}{2}, \\ s_{2k+1} &= -\omega_k^2, \quad \frac{\omega_k}{2} = \tan\frac{\omega_k}{2}, \quad 2k\pi < \omega_k < (2k+1)\pi \end{aligned} \tag{A1.0.3}$$

for $k = 1, 2, \ldots$. The asymptotic form of the odd eigenvalues for large k is $s_{2k+1} = -\omega_k^2$, $\omega_k \sim (2k+1)\pi$.

We are interested in the expansion properties of these eigenfunctions. That is, we would like to know which functions have a convergent expansion in terms of the eigenfunctions. The answer is the following.

Theorem A1.0.1 *Let $f(x)$ be any continuous function on $[0,1]$ satisfying $f(0) = f(1) = 0$. Then f can be expanded as*

$$f(x) = \sum_{m=2}^{\infty} \widehat{f}_m \psi^{(m)}(x).$$

The theorem is proved by constructing the expansions of a dense subset of functions. The basis for this dense subset is the sequence of polynomials $p_n(x)$, $q_n(x)$ defined by

$$\begin{aligned} p_n(x) &:= x^n(1-x)^n, \\ q_n(x) &:= x^n(1-x)^n(\frac{1}{2} - x) \end{aligned} \tag{A1.0.4}$$

for $n = 1, 2, \ldots$. We show that convergent eigenfunction expansions can be constructed for each of the p_n, q_n for $n \geq 1$. The construction breaks down for the $n = 0$ polynomials $p_0(x) = 1$, $q_0(x) = \frac{1}{2} - x$, leading to the restriction $f(0) = f(1) = 0$ in the theorem.

Lemma A1.0.2 (a) *Each of the polynomials $p_n(x)$, $q_n(x)$ for $n \geq 1$ has a convergent expansion in terms of the eigenfunctions of (A1.0.1). The coefficients $\widehat{p}_{n,m}$, $\widehat{q}_{n,m}$ of the expansions of $p_n(x)$, $q_n(x)$ satisfy*

$$\widehat{p}_{n,m} = O(\frac{1}{m^{2n}}), \qquad \widehat{q}_{n,m} = O(\frac{1}{m^{2n}}) \tag{A1.0.5}$$

as $m \to \infty$.

(b) Estimates for the polynomials p_n, q_n for $n \geq 1$ from their eigenfunction expansions. For each $k = 0, 1, \ldots, 2n-1$ there exist constants C, $\widetilde{C}$ depending on n, k such that

$$\begin{aligned} \|\frac{\partial^k}{\partial x^k} p_n(x)\| &\leq C, \\ \|\frac{\partial^k}{\partial x^k} q_n(x)\| &\leq \widetilde{C}. \end{aligned} \tag{A1.0.6}$$

Remark: The norm estimates for the polynomials in this lemma are of no interest *per se*. Their usefulness is in their application to the separation of variables solution for the time dependent model problem studied above.

The case $n = 0$ is special since the eigenfunction expansion gives no L^2 estimates for the solution.

First we construct the expansions of the p_n. The expansions are in terms of the even eigenfunctions

$$p_n(x) = \sum_{m=1}^{\infty} \widehat{p}_{n,m} \psi^{(m)}(x) = \sum_{k=1}^{\infty} \widehat{p}_{n,2k} \psi^{(2k)}(x). \tag{A1.0.7}$$

The coefficients of $\widehat{p}$ are found in terms of the cosine series of p_n,

$$x^n(1-x)^n = \sum_{k=0}^{\infty} \widetilde{p}_{n,k} \cos 2k\pi x = \widetilde{p}_{n,0} + \sum_{k=1}^{\infty} \widetilde{p}_{n,k} \cos 2k\pi x.$$

For $n \geq 1$ the Fourier series converges pointwise since the periodic extension of p_n is continuous. Hence

$$0 = \widetilde{p}_{n,0} + \sum_{k=1}^{\infty} \widetilde{p}_{n,k}. \tag{A1.0.8}$$

Thus the coefficients $\widehat{p}$ are given by

$$\widehat{p}_{n,2k} = -\widetilde{p}_{n,k}.$$

The properties of Fourier series are thoroughly known. In particular, it is known that if f is a C^∞ function in the interior of the interval and the periodic extension has a jump in the nth derivative, then the coefficients $\widetilde{f}_m$ of the Fourier series of f are $O(1/m^{n+1})$ for m large. In particular

$$\widetilde{p}_{n,k} = O(1/k^{2n})$$

for k large since the periodic extension of p_n has a jump in the $(2n-1)$st derivative at $x = 0$.

The case $n = 0$ is special. In this case the eigenfunction expansion breaks down in a manner resembling the non-convergent Fourier series expansion of the delta function. The Fourier series of $p_0(x) \equiv 1$ has coefficients $\widetilde{p}_{0,0} = 1$, $\widetilde{p}_{0,k} = 0$, $k = 1, 2, \ldots$, so (A1.0.8) does not hold in this case. It is not clear how to calculate the expansion coefficients $\widehat{p}_{0,2k}$.

Next we construct the expansions of the q_n. The expansions are in terms of the odd eigenfunctions

$$\begin{aligned} q_n(x) &= \sum_{m=1}^{\infty} \widehat{q}_{n,m} \psi^{(m)}(x) \\ &= \sum_{k=1}^{\infty} \widehat{q}_{n,2k+1} \psi^{(2k+1)}(x) \\ &= \sum_{k=1}^{\infty} \widehat{q}_{n,2k+1} \left[\sin\left[\omega_k (x - \frac{1}{2})\right] - 2(x - \frac{1}{2}) \sin\frac{\omega_k}{2} \right]. \end{aligned} \tag{A1.0.9}$$

The direct approach to get the coefficients $\widehat{q}$ is to take the inner product of (A1.0.9) with each of the eigenfunctions $\psi^{(2k+1)}$. This approach runs into difficulty here because the eigenfunctions are not orthogonal. Instead, the orthogonality properties of the derivatives can be used by writing

$$\begin{aligned}\frac{d^2}{dx^2}q_n(x) &= \sum_{k=1}^{\infty}\widehat{q}_{n,2k+1}M\psi^{(2k+1)}(x)\\ &= \sum_{k=1}^{\infty}\widehat{q}_{n,2k+1}\left[-\omega_k^2\sin\left[\omega_k(x-\frac{1}{2})\right]\right].\end{aligned}$$

The coefficients $\widehat{q}$ are found in terms of the sine series of q_n. The argument requires calculating the expansion of $\sin 2l\pi x$ in terms of the functions $\sin\left[\omega_k(x-\frac{1}{2})\right]$. The sine series of q_n is

$$x^n(1-x)^n(x-\frac{1}{2}) = \sum_{l=1}^{\infty}\widetilde{q}_{n,l}\sin 2l\pi x.$$

We state some useful results in a lemma.

Lemma A1.0.3 *(a) The Fourier coefficients satisfy* $\widetilde{q}_{n,l} = O(1/l^{2n+1})$ *as* $l\to\infty$.

(b) Expansion coefficients of $\sin 2l\pi x$ *in terms of* $\sin\left[\omega_k(x-\frac{1}{2})\right]$.

$$\begin{aligned}\left(\sin\left[\omega_j(x-\frac{1}{2})\right],\sin 2l\pi x\right) &= (-1)^l\sin\left(\frac{\omega_j}{2}\right)\frac{-4l\pi}{\omega_j^2-4l^2\pi^2}\\ &\sim (-1)^{l+j+1}\frac{l/\pi}{(j+\frac{1}{2})^2-l^2}\end{aligned}$$

for j *large.*

(c) Orthogonality of the functions $\sin\left[\omega_k(x-\frac{1}{2})\right]$.

$$\begin{aligned}\left(\sin\left[\omega_j(x-\frac{1}{2})\right],\sin\left[\omega_k(x-\frac{1}{2})\right]\right) &= 0 \qquad \text{for } j\neq k,\\ \left(\sin\left[\omega_k(x-\frac{1}{2})\right],\sin\left[\omega_k(x-\frac{1}{2})\right]\right) &= \frac{1}{2}\left(1-\frac{\sin\omega_k}{\omega_k}\right).\end{aligned}$$

(4) Non-orthogonality of the odd eigenfunctions.

$$\left(\psi^{(2j+1)},\psi^{(2k+1)}\right) = \frac{1}{3}\sin\frac{\omega_j}{2}\sin\frac{\omega_k}{2}\sim\frac{1}{3}(-1)^{j+k}$$

for j, k *large.*

From these results the expansion of $\sin 2l\pi x$ follows as

$$\sin 2l\pi x = \sum_{k=1}^{\infty} r_{l,k}\sin\left[\omega_k(x-\frac{1}{2})\right].$$

Asymptotically, the behaviour of the coefficients $r_{l,k}$ is

$$r_{l,k} \sim (-1)^{k+l+1} \frac{2l/\pi}{(k+\frac{1}{2})^2 - l^2}$$
$$= \begin{cases} O(1/k^2) & \text{for } k \to \infty,\ l \text{ fixed} \\ O(1/l) & \text{for } l \to \infty,\ k \text{ fixed and large} \end{cases}$$

Now the calculation of the coefficients $\widehat{q}$ is

$$\begin{aligned} \frac{d^2}{dx^2} q_n(x) &= \sum_{k=1}^{\infty} \left(-\omega_k^2 \widehat{q}_{n,2k+1}\right) \sin\left[\omega_k(x - \frac{1}{2})\right] \\ &= \sum_{l=1}^{\infty} (-4l^2\pi^2)\widetilde{q}_{n,l} \sin 2l\pi x \\ &= \sum_{l=1}^{\infty} (-4l^2\pi^2)\widetilde{q}_{n,l} \sum_{k=1}^{\infty} r_{l,k} \sin\left[\omega_k(x - \frac{1}{2})\right] \\ &= \sum_{k=1}^{\infty} \left[\sum_{l=1}^{\infty} (-4l^2\pi^2)\widetilde{q}_{n,l} r_{l,k}\right] \sin\left[\omega_k(x - \frac{1}{2})\right]. \end{aligned}$$

The interchange of the order of summation is valid provided the series is absolutely convergent, and this depends on the decay rate of the coefficients. We claim that the series is absolutely convergent provided $n \geq 2$. Then the coefficients are given by

$$\widehat{q}_{n,2k+1} = \frac{1}{\omega_k^2} \sum_{l=1}^{\infty} 4l^2\pi^2 \widetilde{q}_{n,l} r_{l,k}.$$

From this it follows that $\widehat{q}_{n,2k+1} = O(1/k^{2n})$ for $n \geq 2$. For $n = 1$ it is easy to calculate the coefficients directly and to verify that the $\widehat{q}$ are $O(1/k^{2n})$.

The case $n = 0$ is $q_0(x) = \frac{1}{2} - x$. As with the $p_0(x) = 1$, this case is special. The eigenfunction expansion breaks down once again.

References

P. Bontoux, B. Gilly and B. Roux, *Analysis of the Effect of Boundary Conditions on the Numerical Stability of Navier Stokes Equations*, J. Comp. Phys., **15**, pp. 417-427, 1980.

W.R. Briley, *A Numerical Study of Laminar Separation Bubbles using the Navier Stokes Equations*, J. Fluid Mech., **47**, part 4, pp. 713-736, 1971.

M.M. Gupta and R.P. Manohar, *Boundary Approximations and Accuracy in Viscous Flow Computations*, J. Comp. Phys, **31**, pp. 265-288, 1979.

M. Israeli, *A Fast Implicit Numerical Method for Time Dependent Viscous Flows*, Studies in Applied Math, vol. XLIX, **4** , December 1970.

S.A. Orszag and M. Israeli, *Numerical Simulation of Viscous Incompressible Flows*, Annual Rev. Fluid Mech., **6**, pp. 281-318, 1974.

R. Peyret and T.D. Taylor, *Computational Methods for Fluid Flow*, Springer-Verlag, New York, 1983.

P.J. Roache, *Computational Fluid Dynamics*, Hermosa Publishers, Albuquerque, 1972.