Design Issues in Communications Networks: Reliability and Traffic Analysis

Thesis by

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To my parents, my brothers Frank, Jim, Bo,
and especially to my dear wife Xuan (Jessica) Fan

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Abstract

This thesis aims to investigate two rather separate issues: network reliability and traffic analysis. The first concerns the reliability for unreliable systems, including communications networks with possible link failures, and more general fault-tolerant systems. The second concerns the traffic characteristics specifically in ATM networks with respect to the performance of statistical multiplexers.

One way in which we studied the reliability issue is via *mean time to failure* (MTTF) which considers systems that have component failures and repairs with exponential distributions. Such systems can be modeled by continuous-time discrete-state Markov chains. We investigated the MTTF from a more general framework of *fault-tolerant systems* (FTS), and developed two systematic approaches, the *all-path-weight approach* and the *signal-flow-graph approach*, to compute the MTTF. We also derived a simple asymptotic formula for estimating the MTTF, and obtained asymptotically the optimal networks in terms of the MTTF.

The other way in which we studied the reliability issue is via reliability polynomials for a system with component failures with certain fixed probability that is independent of time, but a function of the size of the system. No repair is allowed. We modeled such systems by *random graphs*, and analyzed reliability polynomials in a framework of random graph theory. We specifically focused on certain regular random graphs and analyzed the evolution of the regular random graphs, by showing a *transition phenomenon* when such a regular random graph evolves from edge probability zero to probability one because of the expansion of graph size, and identified its *threshold function*. Our work extends the study of the evolution of random graphs to regular random graphs which do not appear in the literature of random graphs, and our results are generalizations of some famous previously known results in random graph theory.

As for the second issue of traffic analysis in ATM networks, we first studied,

via the approach of generating functions, *Markov on-off traffic* and the performance behavior of a statistical multiplexer with such traffic. We developed a heuristic procedure which allowed us to compute the expected buffer occupancy of a statistical multiplexer with Markov on-off traffic, and obtained closed form formulas showing that the expected buffer occupancy under such traffic not only depends on the incoming traffic intensity, but also largely on the burstiness of incoming traffic. The expected buffer occupancy becomes unbounded with large enough traffic burstiness, even though the traffic intensity is small. These results showed that burstiness control of traffic was very critical in designing ATM networks.

We then introduced a class of burst-constrained traffic sources, the *periodic interchangeable* (PI) traffic, and applied generalized Ballot theorems to analyze the buffer occupancy in a statistical multiplexer with PI traffic. We derived closed form formulas for survivor functions, expected buffer occupancy, and simple asymptotic formula that can be used as a rule of thumb for dimensioning buffer size in designing a statistical multiplexer. The results obtained could shed light on the study of worst case performance of statistical multiplexers for burst-constrained traffic sources in ATM networks.

Contents

Dedication iii

Acknowledgements iv

Abstract vii

1 Introduction 1

1.1 Network Reliability . . . 1

1.2 Traffic Analysis in ATM Networks . . . 2

1.3 The Contributions of the Thesis . . . 4

1.4 Structure of the Thesis . . . 6

2 Network Reliability and MTTF 8

2.1 Introduction . . . 8

2.2 Overview of Network Reliability . . . 9

2.2.1 Network Models . . . 9

2.2.2 Performance Measures . . . 10

2.3 Fault-Tolerant Systems and MTTF . . . 10

2.3.1 Markov Process Models . . . 11

2.3.2 Fundamental Theorems on MTTF . . . 14

2.4 Techniques for Computing MTTF . . . 17

2.4.1 The All-Path-Weight Approach . . . 17

2.4.2 The Signal-Flow-Graph Approach . . . 19

2.4.3 Examples . . . 22

2.5 Asymptotic Analysis of MTTF . . . 30

2.6 Network Reliability and Optimal Networks . . . 33

2.7 Concluding Remarks . . . 39

3 Random Graphs and Reliability Polynomials **41**

3.1 Introduction 41

3.2 Reliability Polynomials 42

3.2.1 The Definition of Reliability Polynomials 42

3.2.2 Exact Algorithms 43

3.2.3 Computational Complexity 46

3.3 The Models of Random Graphs 47

3.3.1 The Models and Threshold Functions 47

3.3.2 Definitions and Auxiliary Theorems 50

3.4 The Reliability Polynomial of Regular Random Graphs 53

3.4.1 The Isolated Vertices of Regular Random Graphs 54

3.4.2 The Components of Regular Random Graphs 60

3.4.3 The Main Theorem 66

3.5 The Slotted Model and MTTF 69

3.6 Concluding Remarks 71

4 ATM Networks and Markov On-Off Traffic **73**

4.1 Introduction 73

4.2 Fundamentals of ATM 74

4.2.1 Asynchronous and Synchronous 75

4.2.2 Transfer Modes 76

4.2.3 ATM Protocol Reference Model 78

4.2.4 Traffic Parameters and Quality of Service 80

4.3 Traffic Sources and Statistical Multiplexers 81

4.4 Multiplexing Markov On-Off Traffic in ATM 83

4.4.1 The Model and Generating Functions 83

4.4.2 Homogeneous and Heterogeneous On-Off Traffic 85

4.4.3 Buffer and Expected Buffer Occupancy Analysis 90

4.4.4 The Expected Buffer Occupancy for Homogeneous Traffic . . . 96

4.4.5 The Expected Buffer Occupancy for Heterogeneous Traffic . . 98

4.5 Concluding Remarks 100

5 Periodic Interchangeable Traffic in ATM Networks **102**

5.1 Introduction 102

5.2 The Ballot Theorems and Interchangeable Random Variables 104

5.3 Periodic Interchangeable Traffic 106

5.4 Multiplexing Periodic Interchangeable Traffic 107

5.4.1 The Queueing Model and Buffer Occupancy 108

5.4.2 Uniform Periodic Interchangeable Traffic 111

5.4.3 Unit Uniform Periodic Interchangeable Traffic 113

5.5 Asymptotic Analysis 117

5.6 Concluding Remarks 119

6 Summary and Conclusions **122**

A The proof of Lemma 2 in Section 2.4.3 **125**

B The upper bound for $f_1(K, \rho)$ in Section 5.4.3 **127**

C The derivation of formula (5.29) **129**

D The upper bound for $f_2(K, \rho)$ in Section 5.4.3 **131**

Bibliography **133**

List of Figures

2.1 A fault-tolerant system with three warning states. 13

2.2 The ring topology and its parallel-type state transition diagram. . . . 23

2.3 The chain state transition diagram. 26

2.4 The state diagram for the proof of Theorem 2.9. 35

2.5 The state diagram for the complete graph with four nodes. 38

2.6 Two non-isomorphic Harary graphs $H(9,4)$. 39

3.1 The transition phenomenon of a property Q. 49

3.2 An illustrated figure for the proof of (3.22): τ nontrivial components induced by vertices $v_1, v_2, \ldots, v_s$. 60

4.1 The service types offered by B-ISDN 75

4.2 The ATM protocol reference model. 78

4.3 The ATM cell format. 80

4.4 The statistical multiplexer in ATM networks. 82

4.5 The statistical multiplexer with homogeneous on-off traffic. 85

4.6 The statistical multiplexer with heterogeneous on-off traffic. 87

4.7 The expected buffer occupancy with homogeneous on-off traffic. . . . 98

4.8 The expected buffer occupancy with heterogeneous on-off traffic. . . . 100

5.1 The survivor functions for UPI traffic (I). 118

5.2 The survivor functions for UPI traffic (II). 119

5.3 The survivor functions for UPI traffic (III). 120

5.4 The expected buffer occupancy for UUPI traffic. 121

List of Tables

3.1 The examples of threshold functions for certain properties 49

5.1 An example of cyclically interchangeable random variables 105

Chapter 1 Introduction

This thesis studies two topics: topic one concerns the reliability of general networks; topic two concerns traffic and performance behavior of statistical multiplexers in ATM networks.

1.1 Network Reliability

Reliability theory is the study of the overall performance of a system built from failure-prone components. That is, the components of the system are not perfect in operation, but their failure is assumed to be governed by certain probabilistic statistics. It is thus of interest to describe the statistical behavior of the system in terms of the characteristics of its components. Not only are the reliabilities of individual components important, but the manner in which they are assembled can have a significant influence on the overall performance of the system. For example, Moore and Shannon [34] over forty years ago showed that it is possible to obtain a reliable system by properly configuring unreliable components via the use of redundancy.

The problem of determining the reliability of a complex system, whose components are subject to failure, has received considerable attention in the statistical, engineering, and operations research literature. Reliability analysis can be applied to a variety of practical systems, ranging from large-scale telecommunication, transportation, and mechanical systems, to the microelectronic scale of integrated circuits.

We are mainly concerned here with network reliability, in which the underlying system arises from the interconnection of various components in the form of a *network*, or *graph*, such as is exemplified by telecommunication, distribution, and computer networks. For example, the *nodes* of a computer communication network might represent the physical locations of computers and its *edges* might represent existing communication links between computer sites. In realistic situations, the components of a network, its nodes and edges or both, are subject to failure. At any instant, each component is either working or failed, and as a result, the network itself is also either working or failed. In the computer communication example, working might mean that a computer is able to communicate over operational links of the network with another computer.

There are mainly two models considered in real systems for reliability study.

(1) Systems with component failures and repairs with exponential distributions. Such a system can be modeled by a continuous-time discrete-state Markov chain, in which there is a so-called *failure* state in the Markov chain representing that the system is failed (according to certain objective), due to the failures of components. The reliability measure in such systems is usually the *mean time to failure* (MTTF) [40] [10] [27] [25], which is the average time for the system to migrate from its perfect state (no component failure) to the failure state.

(2) Systems which allow component failures with a certain "fixed" probability p that is independent of time, but no repair. Such a system can be modeled by a *random graph* [11]. The reliability measure in such systems is usually the *reliability polynomial* in p [13] [39], which is the probability that a certain objective (e.g., there exists an operational path between any two distinct nodes) for the system is maintained.

1.2 Traffic Analysis in ATM Networks

ATM is a standard that is recognized throughout the world [35] [36], which provides for the first time a method for universal information exchange, independent of the end systems and the type of information (data, audio, video). ATM stands for *asynchronous transfer mode.* It is the most modern telecommunications switching technique. It is a highly efficient switching technique which is able to switch connections for a wide range of different information types at a wide range of different rates.

ATM networks provide a *Quality of Service* (QOS) guarantee to user connections. The QOS parameters include: cell error ratio, cell loss ratio, cell transfer delay, mean cell transfer delay, and cell delay variation, etc. The method used by ATM networks to provide QOS is by maintaining a contract between network user and network service provider. When a connection is required by a network user, there is a contract set-up between the network and the user. The user describes the connection in terms of its traffic parameters and QOS requirements. A traffic parameter is a specification of a particular traffic aspect. Three main traffic parameters are *Peak Cell Rate* (PCR), *Sustainable Cell Rate* (SCR), and *Maximum Burst Size* (MBS). After the user describes its traffic parameters and QOS requirements, the network uses a connection admission control scheme to determine if the connection can be admitted to the network, while providing that QOS required to the incoming connection and also to maintain the QOS to the other connections that have already set up. If the new connection violates the traffic parameters in the contract, the violated cells will be dropped by the network.

One characteristic of ATM traffic is its *burstiness.* The burstiness of a traffic is defined as the ratio of the peak traffic rate to the average traffic rate. A traffic source is said to be "bursty" when this ratio is much larger than one. Bursty traffic sources do not require fixed allocations of bandwidth at their peak rates from ATM networks. The ATM scheme makes efficient use of bandwidth by statistically multiplexing a large amount of bursty traffic sources. One of the most important components in ATM technology is the *statistical multiplexer*, which is a multiplexer that combines a number of traffic sources over a single output path such that the transmission bandwidth (capacity) of the output path is not permanently allocated to any given input channel, but instead transmits ("serves") the incoming cells on a first-come-first-served (FCFS) basis. Cells from incoming traffic sources are multiplexed into an output link. Because the aggregate cell arrival rate may temporarily exceed the bandwidth of the output link, a buffer is provided at the output port to hold cells during overflow periods.

Since ATM networks provide QOS guarantee to user connections, it is very important to study ATM traffic with certain "typical characteristics", and the performance behavior (e.g., the distributions of buffer occupancy and the expected buffer occupancy) of statistical multiplexers with such traffic. There are two types of ATM traffic sources we have investigated in this thesis: *Markov on-off* traffic sources which are not burst-constrained, and *periodic interchangeable* (PI) traffic sources which are burst-constrained.

A Markov on-off traffic source is described by a two-state (state 1 and state 0) continuous-time Markov chain such that, when the source in state 1 ("on" state), it sends one cell per time slot; while in state 0 ("off" state), it sends nothing. When many Markov on-off traffic sources are superimposed, we obtain homogeneous Markov on-off traffic (when all Markov chains are identical), or heterogeneous Markov on-off traffic (when all Markov chains are not identical). Markov on-off traffic is not *burst-constrained*, that is, it can generate an infinitely long sequence of consecutive cells. We will develop a heuristic procedure to compute the expected buffer occupancy for a statistical multiplexer with homogeneous and heterogeneous incoming traffic.

A periodic interchangeable (PI) traffic source is one such that all the cells are generated periodically with certain period, and within a period, the sequence of random variables are interchangeable, so that the summation of the random variables is not a random number, but a deterministic number. A sequence of random variables is called interchangeable if all the permutations of the sequence of random variables is equally likely. A PI source is burst-constrained, that is, it can never generate an infinitely long sequence of consecutive cells. We will show how to apply generalized Ballot theorems to analyze the buffer occupancy for a statistical multiplexer with PI traffic.

1.3 The Contributions of the Thesis

The contributions of the thesis can be briefly summarized as following:

- We studied reliability in terms of the MTTF for systems that allow component failures and repairs with exponential distributions, in the more general framework of *fault-tolerant systems.* We developed two systematic approaches, the *all-path-weight approach* and the *signal-flow-graph approach*, to compute the MTTF for fault-tolerant systems. With these approaches, we derived simple asymptotic formulas for estimating the MTTF, and identified asymptotically the optimal networks in terms of the MTTF.

- We studied reliability in terms of the reliability polynomials for systems that allow component failures with certain time-independent probabilities, but no repair. We modeled such systems by *random graphs*, and analyzed reliability polynomials in a framework of random graph theory. We specifically focused on certain regular random graphs and analyzed the evolution of them, by proving the *transition phenomenon* when such a regular random graph evolves from edge probability zero to probability one because of the expansion of graph size, and identified the associated *threshold function.* Our work extended the study of the evolution of random graphs to regular random graphs which do not appear in the literature of random graphs, and our results are generalizations of some famous previously known results in random graph theory.

- We studied Markov on-off traffic and the performance behavior of a statistical multiplexer with such traffic in ATM networks using the approach of generating functions. We developed a heuristic procedure which allowed us to compute the expected buffer occupancy of statistical multiplexers with Markov on-off traffic, and showed that the expected buffer occupancy under such traffic not only depends on the incoming traffic intensity, but also on the burstiness of the incoming traffic. These results showed that burstiness control of traffic is very critical in designing ATM networks.

- We introduced a class of burst-constrained traffic sources, the *periodic interchangeable* (PI) traffic, and applied generalized Ballot theorems to analyze the buffer occupancy in a statistical multiplexer with PI traffic, which resulted in closed form formulas for survivor functions, expected buffer occupancy, and a simple asymptotic formula that can be used as a rule of thumb for dimensioning buffer size in designing a statistical multiplexer. Our results could shed light on the study of worst case performance in statistical multiplexers with burst-constrained traffic in ATM networks.

1.4 Structure of the Thesis

In this introduction, we have provided a general introduction to the two general problems we address in this thesis: network reliability and traffic analysis in ATM networks. The remainder of the thesis is organized as follows.

In Chapter 2, we study reliability in terms of the MTTF for systems that allow components failed and repaired with exponential distributions, in the more general framework of *fault-tolerant systems.* Some efficient algorithms are developed and asymptotic analysis is performed.

In Chapter 3, we study reliability in terms of the reliability polynomials for systems that allow components failed with certain time-independent probability, but no repair. We model such systems by *random graphs*, and analyze their reliability polynomials in a frame work of random graph theory. We specifically focus on certain regular random graphs and analyze the evolution of the regular random graphs, by proving a transition phenomenon when such a regular random graph evolves, and identify the associated threshold functions.

In Chapter 4, we analyze Markov on-off traffic and the performance behavior of a statistical multiplexer with such traffic in ATM networks. We develop a heuristic procedure which allows us to compute the expected buffer occupancy of statistical multiplexers with Markov on-off traffic. The results show that burstiness control of traffic is very critical in ATM networks.

In Chapter 5, we introduce a class of burst-constrained traffic sources, the periodic interchangeable (PI) traffic, and show how to apply generalized Ballot theorems to analyze the buffer occupancy in a statistical multiplexer with PI traffic. These results shed light on the study of worst case performance of statistical multiplexers with burst-constrained traffic sources in ATM networks.

In Chapter 6, we summarize and conclude our results.

Chapter 2 Network Reliability and MTTF

2.1 Introduction

Many systems in the engineering world consist of independent components which are imperfect. Each component can be in one of two states: *working* or *failed*. The system is capable of performing a large and complex task, even if some of its components have failed. A communications network including imperfect processors and links is an example of such a system. A major issue for such a network is its *reliability*.

In this chapter, we first give an overview of network reliability. We introduce some network models and performance measures. We analyze network reliability in terms of a figure-of-merit, *mean time to failure* (MTTF), by considering a more general class of systems, namely, *fault-tolerant systems*. There has been extensive previous research on the MTTF for fault-tolerant systems [40] [10] [27] [25], but computationally efficient methods for computing, and asymptotic analysis for the MTTF are not readily available.

We then develop several techniques for computing and estimating the MTTF, which we have obtained by combining techniques from linear systems, Markov processes, directed weighted graphs, and signal-flow-graphs. We develop two techniques, namely the *all-path-weight approach* and the *signal-flow-graph approach*, for computing the MTTF exactly. In real systems, the repair rate is usually much larger than failure rate, and the *safety factor*, defined as the ratio of the repair rate to the failure rate, is therefore a large number. We obtain good approximations for the MTTF for systems with large safety factors. We finally apply the techniques developed to analyze communication networks whose edges are subject to failure but whose vertices are not, and for which the criterion for network failure is network disconnection. We derive a simple asymptotic formula for computing MTTF of the network, and show that the Harary graphs [20], which have the largest edge connectivity for a given number of edges and nodes, are asymptotically optimal networks, with respect to MTTF.

2.2 Overview of Network Reliability

Reliability theory is the study of the overall performance of a system comprising of failure-prone components, i.e., the components of the system are not perfect in operation, but their failure is assumed to be governed by certain probabilistic statistics. Reliability analysis can be applied to a variety of practical systems, ranging from large-scale telecommunication, transportation, and mechanical systems, to the microelectronic scale of integrated circuits.

We are mainly concerned here with network reliability, in which the underlying system arises from the interconnection of various components (e.g., computers) in the form of a *network*, or *graph*. For example, the *nodes* of a computer communication network might represent the locations of computers and its *edges* might represent existing communication links between computers. In realistic situations, the components of a network, its nodes and edges or both, are subject to failure. At any instant, each component is either working or failed, and as a result, the network itself is also either working or failed. In the computer communication example, working might mean that a computer is able to communicate over operational links of the network with another computer.

2.2.1 Network Models

We use a simple model for topology of computer networks, the *probabilistic graph*. A probabilistic graph $G = (V, E)$ is a set V of n nodes, together with a collection E of m edges. The graph incorporates information about the network's topology, but does not include information about component failure. A probabilistic graph has, in addition, a probability of operation associated with each component (node and/or edge). There are two major probabilistic laws of component failure considered in this paper. Firstly, we allow failed components to be *repaired.* Specifically, each component of the system is subject to failure, and repair, according to independent exponential distributions. This model is analyzed in this chapter through considering more general systems, called fault-tolerant systems. Secondly, each component is subject to failure with a certain probability, and no repair is allowed for failed components. This model is analyzed in next chapter via the celebrated *random graph* approach.

2.2.2 Performance Measures

Reliability is concerned with the ability of a network to carry out a desired operation. An important first step is to identify network objectives and the performance measures associated with the objectives. A common objective is communication from a source node s to a destination node t. For a probabilistic graph G and specified nodes s and t, we define a measure, the *two-terminal reliability* to be the probability that there exists at least one operational path from s to t. Another common objective in networks is communication from any node to any other node, or overall connectivity. We define another measure, the *all-terminal reliability*, or simply *reliability*, to be the probability that for any pair v_1 and v_2 of nodes there is an operational path from v_1 to v_2. In other words, this is the probability that the graph contains at least a spanning tree. For a network in which each edge is subject to failure, and repair, according to independent exponential distributions, and in which the objective is to maintain overall connectivity, a figure-of-merit, called the *mean time to failure*, i.e., the expected time it takes for the system to migrate from perfect state to disconnected state, is another subject of major interest.

2.3 Fault-Tolerant Systems and MTTF

We consider a large system composed of many interdependent components, in which each component can be in one of two states: working or failed. The system is capable of performing a large and complex task, even if some of its components have failed, i.e., it is a *fault-tolerant system (FTS)*. The fault-tolerant systems considered here allow the failed components to be repaired. Specifically, every component of the system is subject to failure, and repair, according to independent exponential distributions. When the configuration of failure components is such that the system can no longer operate properly, we say that the system is in the *failure state.*

A fault tolerant system can be modeled by a continuous-time, finite-state Markov chain. Assume the system starts in a state in which no component has failed. The time until the system first enters the failure state is called the *time to failure* (TTF), a random variable. An important measure of the reliability of a fault tolerant system is the *mean time to failure* (MTTF).

2.3.1 Markov Process Models

In a fault-tolerant system, we assume each system component is in either one of two states: working or failed. A component is subject to failure if it is in the working state and repair if it is in the failure state, according to independent exponential distributions. The states of all components are statistically independent. The "atomic" state of the system is defined to be the list of states of its components. It is frequently possible to find equivalences among the atomic states, thereby simplifying the system state diagram, and in many of our examples, we shall utilize the fact. In any case, the equivalence classes must be such that if a and a' are equivalent atomic states, written $a \equiv a'$, and if b and b' are atomic states corresponding to single component failures in a and a', then $b \equiv b'$. Such a system can be modeled by a Markov chain. The states of the Markov chain are of the three kinds :

(1) The *pristine* state 1, in which no components of the system are failed.

(2) The *failure* state F, in which the system is failed, because of component failures.

(3) The *warning* states, in which there are some component failures but the system is still operating.

A Markov chain corresponding to the state diagram can be specified by the transition rates between states. The transition rate from a state to another state which corresponds to a failure (repair) event is called the *uplink* (*downlink*) rate, respectively, in the state transition diagram. Note that if there exists a positive uplink rate between two states, there must also exist a positive downlink rate, and vice versa. Our problem is to evaluate how long a system can avoid the failure state F if it is initially in the pristine state; we allow no transitions from failure state F to any other state, which makes the failure state F an *absorbing* state. Furthermore, there can be no transition from a state to itself, i.e., the Markov chain for the system has no self-loops.

Consider a Markov chain for a fault-tolerant system with a state transition rate diagram with pristine state 1, $n-1$ warning states, numbered $2, \ldots, n$, and failure state F. Let us define the $n \times n$ matrix

$$A = \begin{pmatrix} -a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & -a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & -a_{nn} \end{pmatrix}, \tag{2.1}$$

where

$$a_{ij} = \begin{cases} \text{transition rate from state } i \text{ to state } j & \text{if } i \neq j \\ \text{total flow out of state } i & \text{if } i = j \end{cases} \tag{2.2}$$

for $1 \leq i, j \leq n$. Figure 2.1 is an example of Markov chain for a fault-tolerant system with three warning states. Note that the matrix A is the transition rate matrix for the Markov chain *excluding* the failure (absorbing) state F, which is called the *transition rate matrix* for the fault-tolerant system. Our first task is to show that A is nonsingular. We begin with a fact:

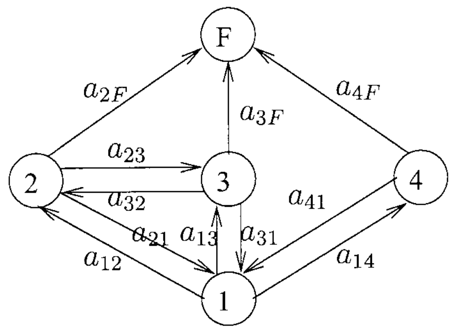

Figure 2.1: A fault-tolerant system with three warning states.

Fact *For the transition rate matrix* A *in an* FTS,

$$a_{ii} \geq \sum_{j \neq i} a_{ij} \tag{2.3}$$

for $1 \leq i \leq n$. *Furthermore, there exists at least one* i, $1 \leq i \leq n$, *such that* $a_{ii} > \sum_{j \neq i} a_{ij}$.

Proof. Since, by definition, a_{ii} is the total flow out of state i, we have

$$a_{ii} \begin{cases} = \sum_{j \neq i} a_{ij} & \text{if state } i \text{ does not connect to the failure state } F \\ > \sum_{j \neq i} a_{ij} & \text{if state } i \text{ connects to the failure state } F. \end{cases} \tag{2.4}$$

Since there is at least one state in the diagram which connects to the failure state, there is at least one inequality that does not hold for equality. ■

To prove the non-singularity of A, we need the following definitions [21].

Definition 1. Let $M_n = (m_{ij})$ for $1 \leq i, j \leq n$. The matrix M_n is said to be *diagonally dominant* if

$$|m_{ii}| \geq \sum_{j \neq i} |m_{ij}| \tag{2.5}$$

for all $1 \leq i \leq n$. It is said to be *strictly diagonally dominant* if no equality holds in (2.5).

Definition 2. The *directed graph* of the matrix M_n, denoted by $\Gamma(M_n)$, is the directed graph on n nodes $v_1, \ldots, v_n$ such that there is a directed edge in $\Gamma(M_n)$ from v_i to v_j if and only if $m_{ij} \neq 0$.

A directed graph Γ is *strongly connected* if between every pair of distinct nodes v_i, v_j in Γ there is a directed path of edges from v_i to v_j. Note that our assumptions about the system state diagram imply that it is strongly connected, since any non-failure state can be reached from the pristine state and vice versa. For diagonally dominant matrices, we have the following theorem [21]:

Theorem 2.1 (Geršgorin's Theorem) *Let $M_n = (m_{ij})$ for $1 \leq i, j \leq n$. If M_n is strictly diagonally dominant, then M_n is invertible; if M_n is diagonally dominant and $\Gamma(M_n)$ is strongly connected, then M_n is invertible.* ■

By (2.3), the matrix A for a system is diagonally dominant, though possibly not strictly diagonally dominant. The directed graph $\Gamma(A)$ is the transition rate diagram excluding the failure state F, which is strongly connected. Therefore, by Geršgorin's Theorem, we have proved the following theorem:

Theorem 2.2 *The transition rate matrix A for an* FTS *is nonsingular.* ■

2.3.2 Fundamental Theorems on MTTF

Consider a Markov chain for an FTS with $n-1$ warning states. State 1 is the pristine state, states $2, \ldots, n$ are warning states. Let us define

$$Q(t) = (Q_1(t), Q_2(t), \ldots, Q_n(t)), \tag{2.6}$$

where $Q_i(t)$ is the probability that the system is in state i at time t, and has not yet been in the failure state. Then the state equation describing the Markov chain (excluding the failure state F) is [40]

$$\frac{d}{dt}Q(t) = Q(t)A. \tag{2.7}$$

The MTTF is defined as the average length of time it takes a system to migrate from the pristine state 1 to the failure state F. Therefore, by a known formula [40],

$$\begin{aligned} \text{MTTF} &= \int_0^\infty (Q_1(t) + Q_2(t) + \cdots + Q_n(t))dt \\ &= \sum_{i=1}^{n} T_i, \end{aligned} \tag{2.8}$$

where

$$T_i = \int_0^\infty Q_i(t)dt \tag{2.9}$$

for $1 \le i \le n$. Let $T = (T_1, T_2, \ldots, T_n)$. Integrating equations (2.7) from $t = 0$ to $t = \infty$, we obtain

$$-T \cdot A = Q(0) = (\underbrace{1, 0, \ldots, 0}_{n}). \tag{2.10}$$

By Theorem 2.2, it follows the linear system (2.10) has a *unique* solution, simply, $Q(0) \cdot (-A)^{-1}$. Hence we have proved the following fundamental theorem:

Theorem 2.3 *For an* FTS *with an $n \times n$ transition rate matrix A, the* MTTF *for the system is given by the formula*

$$\text{MTTF} = \sum_{i=1}^{n} T_i, \tag{2.11}$$

where $T = (T_1, T_2, \ldots, T_n)$ is the unique solution for the linear system (2.11). ■

Theorem 2.3 says that the system MTTF is the sum of the entries in the first row of the matrix $(-A)^{-1}$. The formula in Theorem 2.3 was developed in [10], but the non-singularity of matrix A was not shown. Furthermore, by Cramer's rule [2], we have the following corollary:

Corollary 2.1 *For an* FTS *with an $n \times n$ transition rate matrix A, the* MTTF *for the system is given by the formula*

$$\text{MTTF} = \frac{1}{\det A} \sum_{i=1}^{n} (-1)^i \det A_i, \tag{2.12}$$

where A_i, $1 \leq i \leq n$, *denotes the matrix obtained by removing the* first column *and the* ith row *of* A. ■

Now we define matrix B as follows:

$$B = \begin{pmatrix} -1 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & -1 & \cdots & \frac{a_{2n}}{a_{22}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \cdots & -1 \end{pmatrix}. \tag{2.13}$$

Note that if the diagonal elements of B are all zeros, it is the transition probability matrix for the Markov chain excluding the failure state F. We call B the *pseudo-transition probability* matrix for the FTS. Consider the following linear system

$$-T' \cdot B = (\underbrace{1, 0, \ldots, 0}_{n}). \tag{2.14}$$

It can be easily shown that the solution $T' = (T'_1, \ldots, T'_n)$ for the linear system (2.14) and the solution $T = (T_1, \ldots, T_n)$ for the linear system (2.10) are related as follows:

$$T'_i = a_{ii} \cdot T_i \tag{2.15}$$

for $i = 1, \ldots, n$. Hence we have an alternate way to compute the MTTF, which we will exploit in the next section.

Corollary 2.2 *For an* FTS *with an* $n \times n$ *pseudo-transition probability matrix* B, *the* MTTF *for the system is given by the formula*

$$\text{MTTF} = \sum_{i=1}^{n} \frac{T'_i}{a_{ii}}. \tag{2.16}$$

■

2.4 Techniques for Computing MTTF

The theorems in preceding section require computing the inverse of a matrix (A^{-1} for Theorem 2.3 and Corollary 2.1, B^{-1} for Corollary 2.2). We want to develop techniques to calculate the MTTF without explicitly computing the inverse of a matrix. From Corollary 2.2, the major task in computing the MTTF is to solve the linear system (2.14). We present two approaches to get the solution T' for the linear system without explicitly computing B^{-1}. The new techniques developed also allow us to compute an asymptotic approximation to the MTTF for very general fault-tolerant systems under certain conditions, which will be shown in later section.

2.4.1 The All-Path-Weight Approach

Consider an edge weighted directed graph D with weight function $w(\cdot)$. Figure 2.1 is an example of an edge weighted directed graph where the weights are the transition rates. If $\gamma = e_1 e_2 \dots e_k$ is a path of length k, then the weight of the path is defined by

$$w(\gamma) = w(e_1)w(e_2)\cdots w(e_k). \tag{2.17}$$

For vertices v_i and v_j, denote Γ_{ij}^k the set of all paths connecting v_i to v_j with length k. Define

$$m_{ij}(k) = \sum_{\gamma \in \Gamma_{ij}^k} w(\gamma) \tag{2.18}$$

for $k > 0$. Then $m_{ij}(k)$ is the sum of path weights for all paths of length k from vertex v_i to vertex v_j. For $k = 0$, we define $m_{ij}(0) = \delta_{ij}$. For $k = 1$, Γ_{ij}^1 is simply the set of all edges which directly connect v_i to v_j.

For an edge weighted directed graph D with n vertices, define an $n \times n$ matrix $M = (m_{ij})$, where the (i, j)th entry of M is given by

$$m_{ij} = m_{ij}(1) = \sum_{e} w(e), \tag{2.19}$$

where the sum is over all edges e from vertex i to vertex j. The matrix M is called the *adjacency matrix* for D. We have the following fundamental theorem [43]:

Theorem 2.4 *The (i,j)th entry of M^k is equal to $m_{ij}(k)$ for any non-negative integer k.* ■

Now back to our problem. For an FTS, consider the state transition probability diagram (excluding the failure state F), with the pseudo-transition probability matrix B defined in (2.13), as an edge weighted directed graph (the weights are the corresponding transition probabilities). Define

$$C = B + I, \tag{2.20}$$

where I is the identity matrix. Then

$$C = \begin{pmatrix} 0 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 & \cdots & \frac{a_{2n}}{a_{22}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \cdots & 0 \end{pmatrix}. \tag{2.21}$$

Note that C is simply the transition probability matrix for the Markov chain *excluding* the failure state F for the FTS. C is also the adjacency matrix M when the state transition probability diagram (excluding the failure state F) of the FTS is viewed as a weighted directed graph. From (2.14) and (2.20), we have

$$T' = \underbrace{(1, 0, \ldots, 0)}_{n} \cdot (I - C)^{-1}. \tag{2.22}$$

It follows that,

$$T' = \underbrace{(1, 0, \ldots, 0)}_{n} \cdot \sum_{k=0}^{\infty} C^k, \tag{2.23}$$

since $(I - C)^{-1} = \sum_{k=0}^{\infty} C^k$.

By Theorem 2.4, the ijth entry in the matrix C^k is the sum of the weights of all paths of length k from state i to state j in the state transition probability diagram. Hence the ijth entry in the matrix $\sum_{k=0}^{\infty} C^k$ is the sum of weights for all possible paths from state i to state j. Therefore, T'_i is the summation of weights for all possible paths from state 1 to state i. Applying Corollary 2.2, we obtain the following theorem:

Algorithm 2.1 *The* MTTF *for an* FTS *is the summation of all path weights from the pristine state to every state, except the failure state, in the state transition probability diagram for the system.* ■

2.4.2 The Signal-Flow-Graph Approach

Another way to compute T' for the linear system (2.14) is to view the transition probability diagram as a signal-flow-graph, which is just an edge weighted directed graph. Each node i in the graph is identified with a signal x_i. A *source* of a signal-flow-graph is a node having only outgoing edges; a *sink* is a node having only incoming edges. A *forward* path is a path from source to sink along which no node is encountered more than once. A *loop* is a closed path. For a signal-flow-graph, there is one source and at least one sink. The node signals are related by the following equations:

$$\sum_i w(ij)x_i = x_j \tag{2.24}$$

for all j. If we denote the source signal by x_0 and the sink signal by x_s, then Mason's formula [33] can be used to compute the signal transfer function $G = x_s/x_0$:

Theorem 2.5 (Mason's Formula) *Let G denote the transfer function of a signal-flow-graph. Then*

$$G = \frac{\sum_k F^{(k)}\Delta^{(k)}}{\Delta}, \tag{2.25}$$

where

(1) $F^{(k)}$ = weight of the kth forward path from source to sink,

(2) $\Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2} - \sum_m P_{m3} + \cdots$,

where P_{mr} is the weight product of the mth possible combination of r non-touching loops in the graph. Δ is called the determinant *of the signal-flow-graph, and*

(3) $\Delta^{(k)}$ = *the value of* Δ *for the part of the graph not touching the* k*th forward path (i.e., removing the* k*th forward path from the graph).* ■

In fact, Mason's formula can be used to compute the ratio of signals from *any* node (not necessarily the sink) to the source. For example, if we want to compute x_i/x_0 for some node i, we can attach an artificial node i', and make an edge from node i to node i' with edge weight 1. Then node i' becomes a sink in the signal flow graph, and $x_i = x_{i'}$. Note that this does not change any quantity in Mason's formula. Hence when we apply Mason's formula to compute $x_{i'}/x_0$, we actually obtain x_i/x_0. Therefore, every node can be considered as a sink in a signal-flow-graph, and the transfer function from this node to any other node can be computed by Mason's formula.

Let $X = (x_1, x_2, \ldots, x_n)$. By (2.24), it is easy to check that a signal-flow-graph *without* self-loops is equivalent to the following linear system:

$$X \cdot W = (\underbrace{0, 0, \ldots, 0}_{n}), \tag{2.26}$$

where $W = (w(ij))$ is the transition weight matrix of the graph in which $w(ii) = -1$ for all i. Note that the matrix W is exactly the pseudo transition probability matrix if the signal-flow-graph is viewed as a transition probability diagram. Therefore, Mason's formula can be used to compute the ratio x_i/x_j in a linear system (2.26).

For a system with $n-1$ warning states, let us add an *artificial* state 0 and make an edge, with edge weight 1, from this state 0 to the pristine state 1 in the state transition probability diagram for the system. The resulting diagram is called the *modified* state transition probability diagram for the system. If a modified transition probability diagram is viewed as a signal-flow-graph, with $x_0 = 1$, then the signal-flow-graph corresponds to the linear system:

$$X' \cdot W' = (1, \underbrace{0, 0, \ldots, 0}_{n}), \tag{2.27}$$

where $X' = (1, x_1, x_2, \ldots, x_n)$ and

$$W' = \begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix}. \tag{2.28}$$

The above linear system is equivalent to

$$-T' \cdot B = (1, \underbrace{0, 0, \ldots, 0}_{n}), \tag{2.29}$$

where $T' = X'$ and

$$B' = -W' = \begin{pmatrix} -1 & 0 \\ 0 & W \end{pmatrix}. \tag{2.30}$$

The matrix B' is exactly the pseudo transition probability matrix for the modified state transition probability diagram, i.e., the linear system (2.29) is the same as the linear system (2.14) that we want to solve. Hence, the transfer functions T_i' are the same T_i' as in Corollary 2.2 which we need to compute. Since $x_0 = 1$, if we apply Mason's formula to compute the transfer function $T_i' = \frac{x_i}{x_0}$, $1 \leq i \leq n$, for the modified state transition probability diagram, we obtain the value T_i'. Therefore, we have showed the following algorithm:

Algorithm 2.2 *The* MTTF *for an* FTS *with $n-1$ warning states can be computed by*

$$\text{MTTF} = \sum_{i=1}^{n} \frac{T_i'}{a_{ii}}, \tag{2.31}$$

where T_i', $1 \leq i \leq n$, is the signal transfer function (x_i/x_0) from the sink i (node i) to the source 0 (the artificial state) in the modified state transition probability diagram (considered as a signal-flow-graph), which furthermore can be computed by Mason's formula:

$$T_i' = \frac{\sum_k F_{0i}^{(k)} \Delta_{0i}^{(k)}}{\Delta}, \tag{2.32}$$

where the sum is over all forward paths from the artificial state (node) 0 to the state (node) i, $F_{0i}^{(k)}$ is the weight of the kth such forward path, Δ is the determinant of the *signal-flow-graph (node 0 as the source and node i as the sink), and* $\Delta_{0i}^{(k)}$ *is the value of* Δ *for the part of graph by removing the kth forward path (from node 0 to node i).*

■

Note that the determinant Δ can be computed by

$$\Delta = \sum_k F_{0F}^{(k)} \Delta_{0F}^{(k)}, \tag{2.33}$$

where the sum is over all forward paths from the artificial state 0 to the failure state F, $F_{0F}^{(k)}$ is the weight of the kth such forward path, and $\Delta_{0F}^{(k)}$ is the value of Δ of the signal-flow-graph (node 0 as the source and node F as the sink) for the part of graph by removing the kth forward path (from node 0 to node F).

2.4.3 Examples

We apply the theorems developed in the preceding section to compute the MTTF for two typical fault-tolerant systems.

- **Parallel-Type State Systems**

Consider a fault-tolerant system which has the ring topology of n nodes and n edges. Assume edge i in the ring has failure rate λ_i and repair rate μ_i, and if we define state i to be the state in which the ith edge, and no other, has failed and define network failure as disconnection, then the ring can be illustrated by a fault-tolerant system with a *parallel*-type state transition diagram with n warning states, as shown in Figure 2.2. The transition rate matrix for such a system is

$$A = \begin{pmatrix} -z_1 & \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \mu_1 & -z_2 & 0 & \cdots & 0 \\ \mu_2 & 0 & -z_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_n & 0 & 0 & \cdots & -z_{n+1} \end{pmatrix}, \tag{2.34}$$

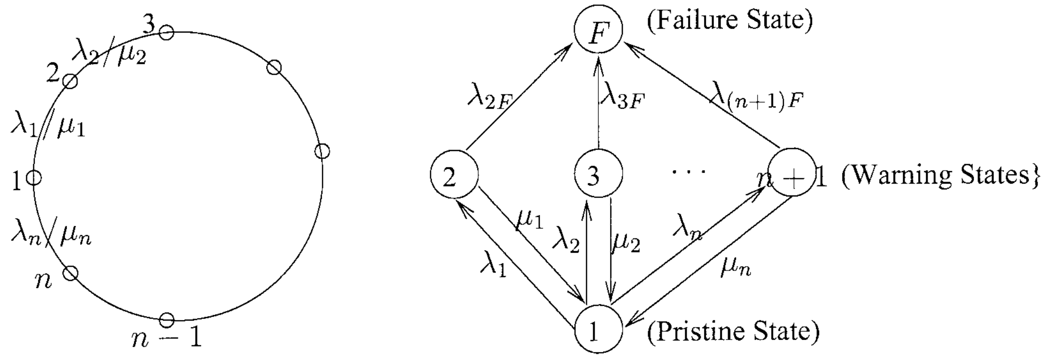

Figure 2.2: The ring topology and its parallel-type state transition diagram.

where

$$z_i = \begin{cases} \sum_{j=1}^{n} \lambda_j & i = 1 \\ \mu_{i-1} + \lambda_{iF} & 2 \leq i \leq n+1, \end{cases} \tag{2.35}$$

and

$$\lambda_{iF} = \sum_{j \neq i-1} \lambda_j \tag{2.36}$$

for $2 \leq i \leq n+1$. We have the following theorem for computing the MTTF:

Theorem 2.6 *For a parallel-type* FTS *with* n *warning states with the state rate transition diagram shown in Figure 2.2, the* MTTF *of the system is given by the formula*

$$\text{MTTF} = \frac{1 + \sum_{i=1}^{n} \frac{\lambda_i}{z_{i+1}}}{\sum_{i=1}^{n} \lambda_{i+1F} \frac{\lambda_i}{z_{i+1}}}, \tag{2.37}$$

where z_i *and* λ_{iF} *are defined in (2.35) and (2.36).*

Proof. We give two proofs for the theorem.

Proof 1: Apply Corollary 2.1.

To calculate the determinant of the matrix A, we multiply the jth column, $2 \leq j \leq n+1$, by $\frac{\mu_{j-1}}{z_j}$ and then add it to the first column to make the lower entries of the matrix zero. Then the determinant of the matrix A is the product of the diagonal entries, i.e.,

$$\det A = (-z_1 + \sum_{j=2}^{n+1} \lambda_{j-1}\frac{\mu_{j-1}}{z_j}) \prod_{j=2}^{n+1}(-z_j)$$
$$= (-1)^{n+1}(z_1 - \sum_{j=2}^{n+1} \lambda_{j-1}\frac{\mu_{j-1}}{z_j}) \prod_{j=2}^{n+1} z_j.$$

Define A_i, $1 \leq i \leq n+1$, the matrix obtained by removing the *first column* and the *ith row* of A. We have

$$\det A_1 = \prod_{i=2}^{n+1}(-z_i) = (-1)^n \prod_{i=2}^{n+1} z_i.$$

Note that the $(i-1)$th column in A_i, $2 \leq i \leq n+1$, is λ_{i-1} followed with all zeros. Expanding at $(i-1)$th column in A_i, we get

$$\det A_i = (-1)^{(i-1)+1}\lambda_{i-1} \cdot \prod_{j\neq 1, j\neq i} (-z_j)$$
$$= (-1)^{i+n-1}\lambda_{i-1} \prod_{j\neq 1, j\neq i} z_j$$

for $2 \leq i \leq n+1$. Hence, by Corollary 2.1, it follows that

$$\text{MTTF} = \frac{1}{\det A}\sum_{i=1}^{n+1}(-1)^i \det A_i$$
$$= \frac{-(-1)^n \prod_{i=2}^{n+1} z_i + \sum_{i=2}^{n+1}(-1)^i \cdot (-1)^{i+n-1}\lambda_{i-1}\prod_{j\neq i, j\neq 1} z_j}{(-1)^{n+1}(z_1 - \sum_{i=2}^{n+1}\lambda_{i-1}\frac{\mu_{i-1}}{z_i})\prod_{i=2}^{n+1} z_i}$$
$$= \frac{\prod_{i=2}^{n+1} z_i + \sum_{i=2}^{n+1}\lambda_{i-1}\prod_{j\neq i, j\neq 1} z_j}{(z_1 - \sum_{i=2}^{n+1}\lambda_{i-1}\frac{\mu_{i-1}}{z_i})\prod_{i=2}^{n+1} z_i}$$
$$= \frac{\prod_{i=2}^{n+1} z_i + \sum_{i=2}^{n+1}\lambda_{i-1}\prod_{j\neq i, j\neq 1} z_j}{(\sum_{i=1}^{n}(1 - \frac{\mu_i}{z_{i+1}})\lambda_i)\prod_{i=2}^{n+1} z_i}$$
$$= \frac{1 + \sum_{i=1}^{n}\frac{\lambda_i}{z_{i+1}}}{\sum_{i=1}^{n}\lambda_{i+1F}\frac{\lambda_i}{z_{i+1}}}.$$

Proof 2: Apply the signal-flow-graph technique (Algorithm 2.2).

There are n loops in the modified state transition probability diagram for the system, with the loop weights:

$$L_i = \frac{\lambda_i}{z_1} \cdot \frac{\mu_i}{z_{i+1}}$$

for $1 \leq i \leq n$. These loops all touch each other. Hence, the determinant of the signal flow graph is

$$\Delta = 1 - \sum_{i=1}^{n} \frac{\lambda_i}{z_1} \cdot \frac{\mu_i}{z_{i+1}}.$$

From node 0 to node i, there is only one forward path, and it has weight

$$F_{0i} = \begin{cases} 1 & i = 1 \\ \frac{\lambda_{i-1}}{z_1} & 2 \leq i \leq n+1, \end{cases}$$

and

$$\Delta_{0i} = 1$$

for $1 \leq i \leq n+1$. Hence, by (2.31) and (2.32),

$$\begin{aligned} \text{MTTF} &= \sum_{i=1}^{n+1} \frac{1}{z_i} \cdot \frac{F_{0i} \cdot \Delta_{0i}}{\Delta} \\ &= \frac{\frac{1}{z_1} + \sum_{i=2}^{n+1} \frac{1}{z_i} \cdot \frac{\lambda_{i-1}}{z_1}}{1 - \sum_{i=1}^{n} \frac{\lambda_i}{z_1} \cdot \frac{\mu_i}{z_{i+1}}} \\ &= \frac{1 + \sum_{i=2}^{n+1} \frac{\lambda_{i-1}}{z_i}}{z_1 - \sum_{i=1}^{n} \lambda_i \cdot \frac{\mu_i}{z_{i+1}}} \\ &= \frac{1 + \sum_{i=1}^{n} \frac{\lambda_i}{z_{i+1}}}{\sum_{i=1}^{n} \lambda_{i+1F} \frac{\lambda_i}{z_{i+1}}}. \end{aligned}$$

■

As a special case, consider a system with ring topology as shown in Figure 2.2 such that all edge failure and repair rate are equal, i.e., $\lambda_i = \lambda$ and $\mu_i = \mu$, then $z_i = \mu + (n-1)\lambda$. By Theorem 2.6, we obtain

$$\text{MTTF} = \frac{1}{\lambda}\left(\frac{1}{n} + \frac{1}{n-1} + \frac{\mu/\lambda}{n(n-1)}\right). \tag{2.38}$$

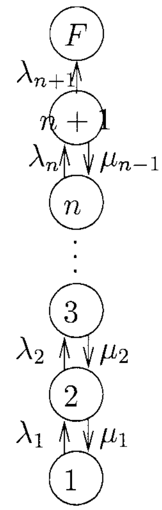

Figure 2.3: The chain state transition diagram.

- **Chain-Type State Systems**

Consider a fault-tolerant system which can be modeled by a transition state diagram with n warning states arranged in a chain, as shown in Figure 2.3. Such an FTS is called a *chain*-type FTS. The transition rate matrix for such a system can be written as

$$A = \begin{pmatrix} -z_1 & \lambda_1 & 0 & \cdots & 0 & 0 & 0 \\ \mu_1 & -z_2 & \lambda_2 & \cdots & 0 & 0 & 0 \\ 0 & \mu_2 & -z_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \mu_{n-1} & -z_n & \lambda_n \\ 0 & 0 & \cdots & \cdots & 0 & \mu_n & -z_{n+1} \end{pmatrix}, \tag{2.39}$$

where

$$z_i = \begin{cases} \lambda_1 & i = 1 \\ \mu_{i-1} + \lambda_i & 2 \le i \le n+1. \end{cases} \tag{2.40}$$

Note that the matrix A is a *tridiagonal matrix* with the property that $a_{ij} = 0$ if $|i - j| > 1$.

Applying Corollary 2.1, we can prove the following theorem:

Theorem 2.7 *For a chain-type* FTS *with* n *warning states with the state rate transition diagram shown in Figure 2.3, the* MTTF *of the system is given by the formula*

$$\text{MTTF} = \sum_{i=1}^{n+1} \frac{\omega_i}{\lambda_i}, \tag{2.41}$$

where the ω_i *satisfy the following recursive relation:*

$$\omega_i = \begin{cases} 1 & i = 1 \\ 1 + \rho_{i-1}\omega_{i-1} & 2 \leq i \leq n+1, \end{cases} \tag{2.42}$$

where $\rho_i = \frac{\mu_i}{\lambda_i}$ *for* $1 \leq i \leq n+1$.

Proof. We prove the theorem by applying Corollary 2.1. To compute the determinant of the matrix A in Corollary 2.1, we transfer the matrix A into an upper diagonal matrix by row and column operations. In particular, we add the 1st column of A to the 2nd column, then add the 2nd column to the 3rd column, ..., finally add the nth column to the $(n+1)$th column, which makes the upper diagonal entries of the matrix all zeros. Hence,

$$\det A = (-1)^{n+1} z_1 \lambda_2 \cdots \lambda_{n+1} = (-1)^{n+1} \prod_{i=1}^{n+1} \lambda_i.$$

Now we want to compute $\det A_i$ for $1 \leq i \leq n+1$. Recall that A_i is the matrix obtained from A by removing the first column and the ith row. Applying row operations to make the upper diagonal entries of the matrix A_i all zeros, we obtain

$$\det A_{n+1} = \lambda_1 \cdots \lambda_n = \prod_{i=1}^{n} \lambda_i$$

and

$$\det A_i = \Big(\prod_{0 \leq k \leq i-1} \lambda_k\Big)(-z'_{i+1})(-z'_{i+2}) \cdots (-z'_{n+1})$$

for $1 \le i \le n$, where $\lambda_0 = 1$ and z'_j satisfy the following recursive relation:

$$z'_j = \begin{cases} z_{i+1} & j = i+1 \\ z_j - \mu_{j-1}\frac{\lambda_{j-1}}{z'_{j-1}} & i+1 < j \le n+1. \end{cases}$$

Now let us define a function

$$Y(i,j) = \begin{cases} (\prod_{k \le i-1} \lambda_k)(-z'_{i+1})(-z'_{i+2}) \cdots (-z'_j) & 1 \le i < j \le n+1 \\ \lambda_1 \cdots \lambda_{j-1} & i = j \end{cases} \tag{2.43}$$

for $1 \le i \le j \le n+1$. Hence,

$$\det A_i = Y(i, n+1)$$

for $1 \le i \le n+1$. Now we need to have the following lemma:

Lemma 1 *The function $Y(i,j)$ defined in (2.43) can be expressed as*

$$Y(i,j) = (-1)^{j-i}(\prod_{k=1}^{j} \lambda_k)(\frac{1}{\lambda_i} + \frac{\rho_i}{\lambda_{i+1}} + \frac{\rho_i\rho_{i+1}}{\lambda_{i+2}} + \cdots + \frac{\rho_i\rho_{i+1}\cdots\rho_{j-1}}{\lambda_j}),$$

where $\rho_i = \frac{\mu_i}{\lambda_i}$ for $1 \le i \le n$ and $\rho_i = 0$ for $i \ge j$.

The lemma can be proved by mathematical induction. See appendix A for the proof.

From the lemma, it follows that

$$\begin{aligned} \det A_i &= Y(i, n+1) \\ &= (-1)^{n+1-i}(\prod_{k=1}^{n+1} \lambda_k)(\frac{1}{\lambda_i} + \frac{\rho_i}{\lambda_{i+1}} + \frac{\rho_i\rho_{i+1}}{\lambda_{i+2}} + \cdots + \frac{\rho_i\rho_{i+1}\cdots\rho_n}{\lambda_{n+1}}). \end{aligned}$$

Now applying Corollary 2.1, we obtain

$$\text{MTTF} = \frac{1}{\det A}\sum_{i=1}^{n+1}(-1)^i \det A_i$$

$$= \sum_{i=1}^{n+1}(\frac{1}{\lambda_i} + \frac{\rho_i}{\lambda_{i+1}} + \frac{\rho_i\rho_{i+1}}{\lambda_{i+2}} + \cdots + \frac{\rho_i\rho_{i+1}\cdots\rho_n}{\lambda_{n+1}})$$
$$= \sum_{i=1}^{n+1}\frac{\omega_i}{\lambda_i},$$

where

$$\omega_i = 1 + \rho_{i-1} + \rho_{i-1}\rho_{i-2} + \cdots + \rho_{i-1}\cdots\rho_1$$

for $2 \le i \le n+1$. Finally, it is easy to check that ω_i satisfy the following recursive relation:

$$\omega_i = \begin{cases} 1 & i = 1 \\ 1 + \rho_{i-1}\omega_{i-1} & 2 \le i \le n+1, \end{cases}$$

which completes the proof. ■

As a special case, let $\lambda_i = \lambda$, $\mu_i = \mu$, then $\rho_i = \rho = \frac{\mu}{\lambda}$. Hence, $\omega_i = \frac{1-\rho^i}{1-\rho}$. By Theorem 2.7, the result is

$$\text{MTTF} = \frac{1}{\lambda}(\frac{n+1}{1-\rho} - \frac{\rho(1-\rho^{n+1})}{(1-\rho)^2}). \tag{2.44}$$

Note that when $\rho \to \infty$, MTTF $\sim \rho^n/\lambda$.

All modern computers have memories built from VLSI RAM chips. A chip "failure" means any situation in which one or more of the bits written on the chip cannot be reliably read. These failures are traditionally classified as either *hard* failures, which means that the affected memory cells are permanently damaged, or *soft* failures, which means that the damage is only temporary. Almost all large computer memories are protected by error-correcting codes [7] and hence computer memories can be viewed as fault-tolerant systems. The result of Theorem 2.7 can be applied to evaluate the reliability of computer memories with soft-error scrubbing and protected with error-correcting codes.

2.5 Asymptotic Analysis of MTTF

Consider an FTS in which the repair rate is much larger than failure rate. This is the case for real situation. We want to evaluate certain asymptotic behavior of the MTTF for an FTS under this condition. Assume the failure rate from state i to state j is λ_{ij} and the repair rate from state j to state i is μ_{ji}, and furthermore, assume

$$\begin{cases} \lambda_{ij} & = & \alpha_{ij}\lambda \\ \mu_{ji} & = & \beta_{ji}\mu, \end{cases} \tag{2.45}$$

where α_{ij} and β_{ji} are constants, while λ and μ are variables. Define

$$\rho = \frac{\mu}{\lambda}, \tag{2.46}$$

which is called the *safety factor* for the FTS. For a real system, normally ρ is much larger than 1. We want to see how the MTTF behaves as $\rho \to \infty$. Let us consider the modified state transition probability diagram (as introduced in section 2.3.2) of the FTS. The edge weights (state transition probabilities) of the state diagram for the FTS are given by

$$w(ij) = \begin{cases} 1 & i = 0, j = 1 \\ \frac{\alpha_{ij}\lambda}{a_{ii}} & ij \text{ is a failure transition} \\ \frac{\beta_{ij}\mu}{a_{ii}} & ij \text{ is a repair transition,} \end{cases} \tag{2.47}$$

where a_{ii} is the sum of total rates out of state i. It is easy to check that

$$a_{ii} = \begin{cases} 1 & i = 0 \\ O(\lambda) & i = 1 \\ O(\lambda) + O(\mu) & \text{otherwise.} \end{cases} \tag{2.48}$$

The transition probability diagram can be viewed as a signal-flow-graph (see Section 2.4.2). Hence, for any loop L in the graph, if $|L|$ is the length of loop L, then we have

$$w(L) = \begin{cases} O(\frac{\lambda}{\mu}) & \text{if } |L| > 2, \text{ or } |L| = 2 \text{ and } L \text{ does not touch state } 1 \\ O(\lambda) & \text{if } |L| = 2 \text{ and } L \text{ touches state } 1. \end{cases} \tag{2.49}$$

Since any forward path in the graph includes state 1, any loop weight in computing Δ_{0i} (the determinant for the part of graph by removing a forward path from node 0 to node i) by Mason's formula, $i \neq 1$, will be $O(\frac{\lambda}{\mu})$. Hence,

$$\Delta_{0i} = 1 + O(\frac{\lambda}{\mu}) \tag{2.50}$$

for $i \neq 1$. Furthermore, it is easy to check that the weight of any forward path from node 0 to node i is as follows:

$$F_{0i} = \begin{cases} 1 & i = 1 \\ O(\lambda) & \text{state } i \text{ is adjacent to state } 1 \\ O(\frac{\lambda}{\mu}) & \text{otherwise.} \end{cases} \tag{2.51}$$

Denote $F_{0i}^{(k)}$ the kth forward path from node 0 to node i. Then, by (2.31) and (2.32) in Algorithm 2.2, and by (2.50), we can compute

$$\begin{aligned} \text{MTTF} &= \sum_{i=1}^{n} \frac{\sum_k F_{0i}^{(k)} \Delta_{0i}^{(k)}}{a_{ii}} \\ &= \frac{1}{\Delta} \cdot (\frac{1}{a_{11}} \sum_k F_{01}^{(k)} \Delta_{01}^{(k)} + \frac{1}{a_{22}} \sum_k F_{02}^{(k)} \Delta_{02}^{(k)} + \cdots + \frac{1}{a_{nn}} \sum_k F_{0n}^{(k)} \Delta_{0n}^{(k)}) \\ &= \frac{1 + O(\frac{\lambda}{\mu})}{a_{11} \cdot \Delta}. \end{aligned} \tag{2.52}$$

By (2.33) and (2.50),

$$\Delta = \sum_k F_{0F}^{(k)} \Delta_{0F}^{(k)} = \sum_k F_{0F}^{(k)} (1 + O(\frac{\lambda}{\mu})), \tag{2.53}$$

where the sum is over all forward paths from state 0 to state F, and $F_{0F}^{(k)}$ is the weight of the kth such forward path. Note that a_{11} is the total rate out of the pristine state 1, and $F_{0F}^{(k)} = F_{1F}^{(k)}$ because the weight from the artificial state 0 to the pristine state 1 is one. Therefore, we have proved the following fundamental theorem on asymptotic MTTF:

Theorem 2.8 *For an* FTS *with the state transition rate diagram such that the rates are defined in (2.45) and the safety factor ρ defined in (2.46), when $\rho \to \infty$,*

$$\mathrm{MTTF} = \frac{1}{R} \cdot \frac{1 + O(\frac{1}{\rho})}{\sum_k F_{1F}^{(k)}(1 + O(\frac{1}{\rho}))}, \tag{2.54}$$

where the sum is over all forward paths from the pristine state 1 to the failure state F in the corresponding state transition probability diagram of the FTS, *R is the total rate out of the pristine state 1, and $F_{1F}^{(k)}$ is the k-th forward path weight.* ■

It is easy to check that the weight of a long forward path is smaller than the weight of a short forward path at least by a scale of $O(1/\rho)$ in the state transition probability diagram. When ρ is sufficiently large, the shortest forward paths play the dominant role in computing MTTF in Theorem 2.8. The following corollary follows immediately:

Corollary 2.3 *For an* FTS *with the state transition rate diagram such that the rates are defined in (2.45) and the safety factor ρ defined in (2.46), when $\rho \to \infty$,*

$$\mathrm{MTTF} = \frac{1}{R} \cdot \frac{1 + O(\frac{1}{\rho})}{\sum_{k^*} F_{1F}^{(k^*)}(1 + O(\frac{1}{\rho}))}, \tag{2.55}$$

where the sum is over all shortest forward paths from the pristine state 1 to the failure state F in the corresponding state transition probability diagram of the FTS, *R is the total rate out of the pristine state 1, and $F_{1F}^{(k^*)}$ is the k^*th shortest forward path weight.*

■

2.6 Network Reliability and Optimal Networks

Now we study the reliability for connected communication networks which are subject to edge failure, and repair, according to exponential distributions. We apply the asymptotic formulas derived in the preceding section to analyze the MTTF for communication networks. The following theorem gives an asymptotic formula for the MTTF for any such network in terms of the *edge connectivity* and the number of *minimum cut-sets.* A cut-set in a graph G is a set of edges, which, when removed from the graph, disconnects the graph, and the edge connectivity of G is the smallest cardinality of any cut-set.

Theorem 2.9 *For a network which is represented by a graph with edge connectivity κ, and which has $C_{\min}$ cut-sets of size κ, if the failure rate for every edge is λ and repair rate μ, then as $\rho = \mu/\lambda \to \infty$,*

$$\text{MTTF} = \frac{1}{\lambda} \cdot \frac{\rho^{\kappa-1}}{\kappa C_{\min}} + O(\rho^{\kappa-2}). \tag{2.56}$$

Proof. Assume there are m edges in the graph G, among which ν edges are contained in at least one minimum cut set. These ν edges are called cut-set edges. We enumerate the edges of the graph in such a way that the ν cut-set edges are numbered as $e_0, e_1, \ldots, e_{\nu-1}$. Define the $C_{\min}$ minimum cut-sets of the network by $S_1, S_2, \ldots, S_{C_{\min}}$. Then

$$|S_i| = \kappa \tag{2.57}$$

for $1 \leq i \leq C_{\min}$. Assume the cut-set edge e_i, $0 \leq i \leq \nu - 1$, is contained in m_i minimum cut-sets $T_i^1, \ldots, T_i^{m_i}$. Let

$$T_i = \{T_i^1, \ldots, T_i^{m_i}\}. \tag{2.58}$$

Then

$$\sum_{i=0}^{\nu-1} m_i = \sum_{i=1}^{C_{\min}} |S_i| = \kappa C_{\min}. \tag{2.59}$$

There are 2^m atomic states for the system, but we can simplify the system state diagram by defining equivalence classes of atomic states. Let us define the system states and the state transition diagram of the network in the following manner:

(i) the pristine state 1 represents no edge failure in the network.

(ii) the failure state F represents all subgraphs which are disconnected.

(iii) state i, $2 \leq i \leq \kappa - 1$, represents $i - 1$ edge failures in the network, so that the failure rate from state i to state $i + 1$ (uplink) is

$$\lambda_{i,i+1} = (m - i + 1) \cdot \lambda \tag{2.60}$$

for $1 \leq i \leq \kappa - 2$, and the repair rate from state $i + 1$ to state i (downlink) is

$$\mu_{i+1,i} = i \cdot \mu \tag{2.61}$$

for $1 \leq i \leq \kappa - 1$.

(iv) state $\kappa + i$, $0 \leq i \leq \nu - 1$, represents those subgraphs obtained by removing $(\kappa - 1)$ cut-set edges from each T_i^j, $1 \leq j \leq m_i$ and only edge e_i is remained in each T_i^j. Clearly, every state $\kappa + i$ actually represents m_i connected subgraphs which all contain the edge e_i, and further removing the edge e_i will result in the m_i subgraphs disconnected. Hence, the failure rate from state $\kappa + i$ to the failure state F is

$$\lambda_{\kappa+i,F} = m_i \cdot \lambda \tag{2.62}$$

for $0 \leq i \leq \nu - 1$. (v) states $\kappa + \nu, \kappa + \nu + 1, \ldots,$ N, for certain integer N, represent all those subgraphs obtained by removing $(\kappa - 1)$ edges from G such that removing *any* one more edge will *not disconnect* the graph. The state transition rate diagram is illustrated in Figure 2.4.

Clearly, among the states $1, \ldots, \kappa - 1, \kappa, \kappa + 1, \ldots, \kappa + \nu - 1, \kappa + \nu, \ldots$,N, only the ν states $\kappa, \kappa + 1, \ldots, \kappa + \nu - 1$ are directly connected to the failure state F. Now we need to compute the failure rate $\lambda_{\kappa-1,\kappa+i}$ and the repair rate $\mu_{\kappa+i,\kappa-1}$, $0 \leq i \leq \nu - 1$. Let

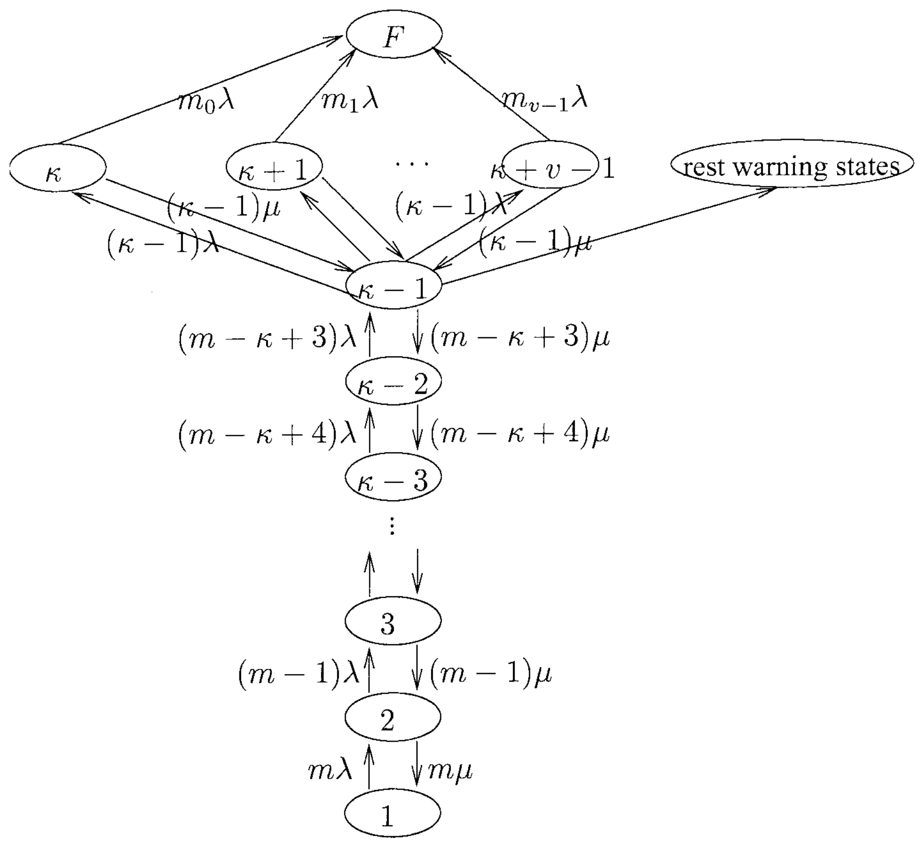

Figure 2.4: The state diagram for the proof of Theorem 2.9.

$G_{\kappa-1}$ denote the set of all subgraphs with $\kappa-2$ edges removed. Clearly $|G_{\kappa-1}| = \binom{m}{\kappa-2}$. Hence, the state $\kappa-1$ represents the set $G_{\kappa-1}$, and

$$\begin{aligned} \lambda_{\kappa-1,\kappa+i} &= \sum_{j_1 \in G_{\kappa-1}} \sum_{j_2 \in T_i} \Pr\{j_1\}\lambda_{j_1,j_2} \\ &= \frac{m_i \cdot (\kappa-1)\lambda}{\binom{m}{\kappa-2}}, \end{aligned} \tag{2.63}$$

and

$$\begin{aligned} \mu_{\kappa+i,\kappa-1} &= \sum_{j_1 \in T_i} \sum_{j_2 \in G_{\kappa-1}} \Pr\{j_1\}\mu_{j_1,j_2} \\ &= m_i \cdot (\kappa-1)\mu \end{aligned} \tag{2.64}$$

for $0 \le i \le \nu - 1$.

There exist ν shortest paths $P_{1F}^{(i)} = \{1, \ldots, (\kappa-1), (\kappa+i), F\}$, $0 \le i \le \nu - 1$, of length κ from the pristine state 1 to the failure state F. Note that

$$a_{ii} = \begin{cases} m \cdot \lambda & i = 1 \\ \mu_{i(i-1)} + o(1) & i \ge 2. \end{cases} \tag{2.65}$$

The path weight for the shortest path $P_{1F}^{(i)} = \{0, 1, \ldots, (\kappa-1), (\kappa+i), F\}$, $0 \le i \le \nu-1$, in the corresponding state transition probability diagram is

$$\begin{aligned} F_{0F}^{(i)} &= \left(\prod_{j=1}^{\kappa-2} \frac{\lambda_{j(j+1)}}{a_{jj}}\right) \cdot \frac{\lambda_{(\kappa-1)(\kappa+i)}}{a_{(\kappa-1)(\kappa-1)}} \cdot \frac{\lambda_{(\kappa+i)F}}{a_{(\kappa+i)(\kappa+i)}} \\ &= \frac{m\lambda}{m\lambda} \cdot \frac{(m-1)\lambda}{\mu + O(\frac{\lambda}{\mu})} \cdots \frac{(m-(\kappa-3))\lambda}{(\kappa-3)\mu + O(\frac{\lambda}{\mu})} \cdot \frac{\frac{m_i \cdot (\kappa-1)\lambda}{\binom{m}{\kappa-2}}}{(\kappa-2)\mu + O(\frac{\lambda}{\mu})} \cdot \frac{m_i \lambda}{m_i \cdot (\kappa-1)\mu + O(\frac{\lambda}{\mu})} \\ &= \frac{1}{m} \cdot \frac{m_i}{\rho^{\kappa-1} + O(\rho^{\kappa-2})}. \end{aligned} \tag{2.66}$$

Applying Corollary 2.3 and (2.59), we obtain

$$\begin{aligned} \text{MTTF} &= \frac{1}{a_{11}} \cdot \frac{1 + O(\frac{1}{\rho})}{\sum_{i=0}^{\nu-1} F_{0F}^{(i)} (1 + O(\frac{1}{\rho}))} \\ &= \frac{1}{m\lambda} \cdot \frac{1 + O(\frac{1}{\rho})}{\frac{1}{m} \cdot \sum_{i=0}^{\nu-1} \frac{m_i}{\rho^{\kappa-1} + O(\rho^{\kappa-2})}} \\ &= \frac{1}{\lambda} \cdot \frac{\rho^{\kappa-1} + O(\rho^{\kappa-2})}{\sum_{i=0}^{\nu-1} m_i} \\ &= \frac{1}{\lambda} \cdot \frac{\rho^{\kappa-1}}{\kappa C_{\min}} + O(\rho^{\kappa-2}), \end{aligned}$$

which completes the proof of the theorem. ■

As an example, let us consider a network which is represented by a complete graph with 4 vertices. Then $\kappa = 3$ and $C_{\min} = 4$. Assume every edge has failure rate λ and repair rate μ. There are $2^6 = 64$ atomic states for this system. If we let each warning state represent a set of isomorphic induced connected subgraphs (which is an equivalence class), and let the failure state F represent the set of all induced disconnected subgraphs, then we obtain a state diagram including the pristine state

(state 1) , five warning states (states 2, 3, 4, 5, and 6), and the failure state F, as shown in Figure 2.5. In the figure, the pristine state 1 indicates that all the six edges of the network are all working; state 2 represents a set of six isomorphic subgraphs, each of which is obtained by removing one edge from the network; state 3 represents a set of four isomorphic subgraphs, each of which is obtained by removing one edge from a subgraph represented by state 2; and so on. Applying Theorem 2.9, we immediately have the asymptotic formula

$$\text{MTTF} = \frac{1}{\lambda} \cdot \frac{\rho^{3-1}}{3 \cdot 4} + O(\rho^{3-2}) = \frac{1}{\lambda} \cdot \frac{\rho^2}{12} + O(\rho). \tag{2.67}$$

We can check this result by computing the exact MTTF by Corollary 2.1. The state transition rate matrix (excluding the failure state F) is given by

$$A = \begin{pmatrix} -6\lambda & 6\lambda & 0 & 0 & 0 & 0 \\ \mu & -(5\lambda+\mu) & \lambda & 4\lambda & 0 & 0 \\ 0 & 2\mu & -(4\lambda+2\mu) & 0 & 4\lambda & 0 \\ 0 & 2\mu & 0 & -(4\lambda+2\mu) & 2\lambda & \lambda \\ 0 & 0 & \mu & 2\mu & -(3\lambda+3\mu) & 0 \\ 0 & 0 & 0 & 3\mu & 0 & -(3\lambda+3\mu) \end{pmatrix}.$$

On the other hand, we can apply Corollary 2.1 to compute the MTTF, After considerable computation, we obtain

$$\text{MTTF} = \frac{1}{\lambda} \cdot \frac{636 + 1211\rho + 859\rho^2 + 299\rho^3 + 61\rho^4 + 6\rho^5}{720 + 1144\rho + 552\rho^2 + 72\rho^3} = \frac{1}{\lambda}\left(\frac{1}{12}\rho^2 + \frac{5}{24}\rho + \frac{133}{108} + O\left(\frac{1}{\rho}\right)\right), \tag{2.68}$$

which agrees to the result from (2.67) that was obtained much easier from Theorem 2.9.

From Theorem 2.9, we can see that the MTTF for a network is, asymptotically, determined by the edge connectivity κ and the number of minimum cut-sets $C_{\min}$ of the network with large safety factor ρ. The MTTF increases as κ increases and $C_{\min}$

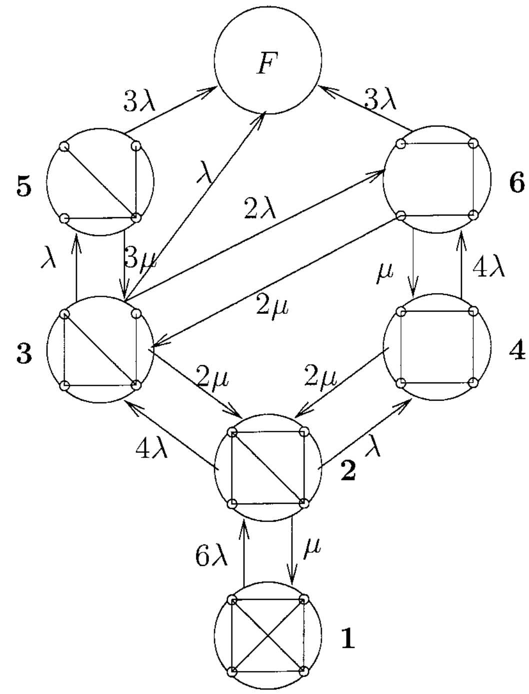

Figure 2.5: The state diagram for the complete graph with four nodes.

decreases, which agrees with intuition. Since the MTTF increases exponentially as κ increases, we see that the most important determiner of the reliability is the edge connectivity of the network.

A *regular* graph is one for which each node has the same degree, i.e., the same number of edges incident to the node. For a graph with a given number of nodes v and edges e, Harary [20] showed that the maximum edge connectivity is $\kappa_{max} = [\frac{2e}{v}]$ when $e \geq v-1$, and 0 when $e < v-1$. Harary gave a construction of regular graphs with degree $[\frac{2e}{v}]$, now called *Harary graphs*, which achieve the maximum edge connectivity. A Harary graph $H(n,r)$ is one with n vertices and degree r. Figure 2.6 is an example of a two non-isomorphic Harary graphs $H(9,4)$. To check the two graphs are non-isomorphic, one only need to check that any vertex in the left graph is contained in three triangles of the graph, while any vertex in the right graph is contained in two triangles of the graph.

By Theorem 2.9, the following corollary follows:

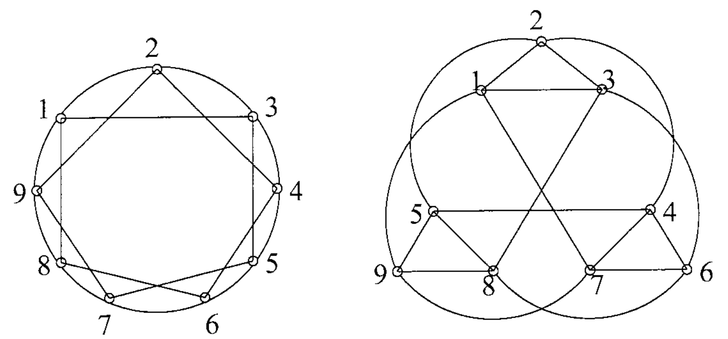

Figure 2.6: Two non-isomorphic Harary graphs $H(9,4)$.

Corollary 2.4 *The asymptotic* MTTF *for a network with v nodes and e edges, such that the failure rate for every edge is λ, repair rate μ, and the safety factor $\rho = \mu/\lambda$, is bounded by the following formula:*

$$\text{MTTF} \leq \frac{1}{\lambda} \cdot \frac{\rho^{\lceil \frac{2e}{v} \rceil - 1}}{\lceil \frac{2e}{v} \rceil v} + O(\rho^{\lceil \frac{2e}{v} \rceil - 2}), \tag{2.69}$$

where equality is achieved asymptotically if the network is a Harary graph. ■

2.7 Concluding Remarks

In this chapter, we studied network reliability in terms of the MTTF for fault-tolerant systems with exponential rates of component failure and repair. We have recast the problem as a linear system problem, and proved that the linear system has unique solution. We have developed two systematic approaches, the all-path-weight approach and the signal-flow-graph approach, to solve the linear system and compute the MTTF without explicitly computing the inverse of a matrix. The significance of the two approaches developed is not only to provide new techniques for computing the MTTF for fault tolerant systems, but also to provide insight. The techniques developed resulted in a simple asymptotic formula for estimating the MTTF when the safety factor is large, which is usually the case for real systems. Finally, we studied communication networks with link failure and repair, we derived a simple asymptotic formula for computing the MTTF for such networks, and showed the Harary graph networks, which have the largest edge connectivity for the given number of links and nodes, are asymptotically the optimal networks in terms of the MTTF.

Chapter 3 Random Graphs and Reliability Polynomials

3.1 Introduction

The theory of random graphs was founded by Erdös and Renyi [16] after Erdös had discovered that probabilistic methods were often useful in tackling extremal problems in graph theory. Since then, random graphs have been investigated extensively [11]. In the set of all possible graphs with a given number of vertices, there are two natural ways of estimating the proportion if graphs have a certain property. One may obtain exact formulas using combinatorial enumeration. This approach is deterministic. The other approach has little connection with enumeration. One is not interested in (or it is too difficult to find) exact formulas but rather in approximating a variety of exact values by appropriate probability distributions and using probabilistic ideas, whenever possible. As shown by Erdos and Renyi [16], this probabilistic approach is often more powerful than the deterministic one.

A communication network consists of nodes and edges. We assume edges, but not nodes, are subject to failure, and no repair is allowed. We are interested in the ability of the network to carry out a desired network objective, e.g., the *connectedness* of the network. Assume each edge operates independently with probability p, or fails with probability $1 - p$ ($p = p_n$ is usually a function of n, the number of nodes of the network). Then the probability that the network is connected can be expressed by a polynomial in p, which is called the *reliability polynomial* of the network. It has been shown [5] that the calculation of reliability polynomials is one of the hardest computational problems, in fact, it is $\#P$-complete as explained in Section 3.2.3. Hence, it is very unlikely there exists a good algorithm which computes general reliability polynomials effectively.

In this chapter, we study reliability polynomials for a special type of networks, *regular graph networks*, using the approach of random graph theory. We will investigate a special class of random graphs, i.e., *regular random graphs*, which is much less studied in the literature of random graph theory. We analyze the evolution (as the number of nodes n of a graph increases) of the class of regular random graphs and prove a series of theorems regarding to reliability polynomials of networks based on the regular random graphs. We will show that the expected value of the number of isolated vertices plays a crucial role in determining the connectedness of the regular random graphs. We will also show an interesting phenomenon, called *transition* phenomenon, when a regular random graph evolutes, i.e., the number of vertices increases, and when edge probability p increases from 0 to 1, the reliability polynomial of the graph will jump from 0 to 1 at certain moment, which is called *phase transition*. We identify the so-called *threshold function* for the special type of regular random graphs. Finally we will introduce a slotted model to easily compute the approximate MTTF for regular graph networks from the reliability polynomials we obtain.

3.2 Reliability Polynomials

3.2.1 The Definition of Reliability Polynomials

Consider a communication network consisting of nodes and edges. Assume edges, but not nodes, are subject to failure, and no repair is allowed. The reliability is concerned with the ability of the network to carry out a desired network objective. The model for this network is a probabilistic graph, $G = (V, E)$, where V is the set of n nodes together with a collection E of m edges. We define a *state* of G to be the subset $S \subseteq E$, which means that all edges in S are operational and all edges in $E - S$ are failed. The universe of possible states is then just the powerset 2^E of E. Hence, a network objective is specified by defining a set $OP(G) \subseteq 2^E$, where $OP(G)$ is the set of states considered to be operational. We call the members of $OP(G)$ *pathsets*.

Specifying the pathsets for G defines the network objective. An alternate approach is to employ *cutsets*, which is a set $S \subseteq E$ for which $E - S$ is a failed state. In other words, for a state S, either S is a pathset (operational state) or $E - S$ is a cutset (S is a failed state). Minimal pathsets are called *minpaths* and minimal cutsets are called *mincuts*. Therefore, reliability is the probability of obtaining a pathset.

The objective of the network considered here is that for every pair v_1, v_2 of nodes in G there is an operational path from v_1 to v_2. The *reliability* of a network is then defined as the probability that for every pair of nodes in the network there is an operational path connecting them. Under this objective, a pathset is hence a node-induced subgraph of G which contains a spanning tree, and the reliability is the probability that the graph G contains at least a spanning tree as a subgraph. Assume each edge operates independently with probability p (or failed with probability $1-p$). Then a pathset with i edges will occur with probability

$$p^i(1-p)^{m-i},$$

where $|E| = m$. Let N_i be the number of i-edge pathsets, C_i be the number of i-edge cutsets, and κ be the edge connectivity of the network. Then the reliability $\mathrm{Rel}(G_p)$ of the network G is given by

$$\mathrm{Rel}(G_p) = \sum_{i=0}^{m} N_i p^i (1-p)^{m-i} = 1 - \sum_{i=\kappa}^{m} C_i (1-p)^i p^{m-i}. \tag{3.1}$$

$\mathrm{Rel}(G_p)$ is a polynomial in p with degree at most m, called the *reliability polynomial.* Clearly, the problem of computing reliability is equivalent to computing the sequence of numbers $N_0, \ldots, N_m$ for the graph G.

3.2.2 Exact Algorithms

We briefly examine some exact algorithms for computing the network reliability (*State-Space Enumeration* algorithm, and *Transformations and Reductions* algorithm

[39], *Inclusion-Exclusion* algorithm [13]).

- **State-Space Enumeration**

This algorithm simply enumerates all states, that is, all possible subgraphs, and then determines which are pathsets, and sums the occurrence probabilities of each pathset. The enumeration of all possible states is just the enumeration of all subsets of E. To determine whether a state is a pathset is accomplished by checking if the pathset provides the objective for the network. For example, in our all-terminal reliability problem, a state (subgraph) is a pathset if it contains a spanning tree. Finally, the probability of obtaining a pathset is the product of the operational probabilities of edges in the pathset and the failure probabilities of edges not in the pathset.

Another way of viewing state-space enumeration comes from the binary nature of the states assumed by each edge. Rather than fully specifying the states of all m edges at the same time, we can instead choose a particular edge $e \in E$ and 'condition' on the status of e, either perfect ($p_e = 1$) or failed ($p_e = 0$). Then, we obtain two new systems, G/e in which edge e is perfect, and $G - e$ in which e is failed. This results in the *pivotal decomposition formula*:

$$\text{Rel}(G) = p_e\text{Rel}(G/e) + (1 - p_2)\text{Rel}(G - e).$$

This formula shows how reliability computation for a given system can be decomposed into that for two smaller systems. By properly selecting the edges for conditioning, sometimes, substantial computational savings can be achieved.

- **Transformations and Reductions**

(1) *Parallel* reduction: suppose that G contains b parallel edges $e_1, \ldots, e_b$ from node v_1 to v_2, with operation probabilities $p_1, \ldots, p_b$. Then it is easy to verify that we can transfer G to a graph which replaces the b edges by an edge e (between nodes v_1 and

v_2) with an operation probability

$$1 - \prod_{i=1}^{b} (1 - p_i).$$

This parallel reduction ensures that algorithms need only consider simple graphs.

(2) *Series* reduction: suppose that G contains b edges in series, with operation probabilities $p_1, \ldots, p_b$. Then it is easy to verify that we can transfer G to a graph which replaces the b edges (and the intermediate nodes) by an edge e (between the first and last nodes) with an operation probability

$$p_1 p_2 \cdots p_b.$$

- **Using Minpaths: Inclusion-Exclusion**

Recall that in the case of all-terminal reliability, minpaths are spanning trees. We describe an inclusion-exclusion algorithm for generating a spanning tree. For a given graph $G = (V, E)$, select a "root" node s and set $X = \{s\}$ initially. X is the set of nodes already in the spanning tree; T is the (initially empty) set of edges in the spanning tree so far; $C \subseteq E$ is the (initially $C = E$) set of edges which remain as candidates for inclusion in the spanning tree. The following is a procedure *stgen* to generate a spanning tree.

stgen(X, T, C):
(i) if $X = V$, *output* T *and exit.*
(ii) locate an edge $e = \{x, y\}$ *in* C *with* $x \in X$ *and* $y \in V - X$.
(iii) if e *is a cutedge in* $(V, T \cup C)$, *set* $X = X \cup \{y\}, T = T \cup \{e\}$, *and goto (i).*
(iv) call stgen$(X \cup \{y\}, T \cup \{e\}, C - \{e\})$.
(v) call stgen$(X, T, C - \{e\})$.

Note that the number of calls to *stgen* is linear in the number of spanning trees, and each call to *stgen* requires time which is linear in the size of G.

Now suppose that the minpaths $P_1, \ldots, P_s$ have been listed. Let E_i be the event that all edges in minpath P_i are operational. Then the reliability is just the probability that at least one of the events $\{E_i\}$ occurs. Note that $\{E_i\}$ are not disjoint events. Hence,

$$\mathrm{Rel}(G) = \sum_{j=1}^{s}(-1)^{j+1} \sum_{I\subseteq\{1,\ldots,s\},|I|=j} \Pr\{E_I\},$$

where E_I is the event that all paths P_i with $i \in I$ are operational. This is the standard inclusion-exclusion expansion [13].

3.2.3 Computational Complexity

The previous section describes several algorithms for calculating network reliability. It has been shown that all *known* algorithms have a running time that grows exponentially with problem size. Network reliability problems are intrinsically difficult, in fact among the most challenging of all computational problems, namely, *#P-complete* problems [19].

Ideally, one would like an algorithm whose running time grows polynomially with the "size" of the input (for our case the number of nodes and edges in the network). A *decision problem* is one that asks whether a certain property holds or not for the input instance: e.g., whether a given network contains a *Hamiltonian* cycle (which visits every node exactly once). If a decision problem can be decided by an algorithm with complexity function $f(z)$ bounded by a polynomial in z, then the problem belongs to the class P. A presumably broader class NP consists of those decision problems with the property that a proposed solution can be verified, though not necessarily found, in polynomial time. The *NP-complete* (or *NP-hard*) problems represent the most difficult problems in NP; any polynomial-time algorithm for solving one *NP-complete* problem would enable all problems in the class NP to be solved in polynomial time. However, no one has discovered a polynomial-time algorithm for any *NP-complete* problem, although the list of such problems is large and expanding.

Reliability problems are not normally cast as decision (yes/no) problems, but rather as computations of probabilities. A more appropriate problem class is that denoted by $\#P$, consisting of all counting problems associated with decision problems in NP. For example, counting the number of Hamiltonian cycles in a network falls into the class $\#P$. In fact, this specific problem is as hard as any problem in $\#P$. It belongs to the class called *#P-complete* (or *#P-hard*), characterized by the fact that a polynomial-time solution algorithm for such a problem could be efficiently transformed into a polynomial-time algorithm for all problems in $\#P$. Clearly *#P-complete* problems are at least as difficult as *NP-complete* problems, since knowing the number of solutions to a decision problem easily settles the existence question. It can be shown [5] that the calculation of the reliability is *#P-complete*, which indicates that it is very unlikely that there exists a good algorithm for computing general reliability polynomials.

Because of this, in the rest of the chapter, we will focus on a class of specific networks, namely *regular graph* networks, to analyze their reliability polynomials. We will study a special class of regular random graphs using the approach of random graph theory to compute reliability polynomials in an asymptotic sense.

3.3 The Models of Random Graphs

3.3.1 The Models and Threshold Functions

The two most frequently occurring models of random graphs are $G(n, M)$ and $G_{n,p}$ [1] [11]. The first model consists of all graphs with vertex set $V = \{1, 2, \ldots, n\}$ having M edges, in which the graphs all have the same probability. Let $N = \binom{n}{2}$. Then $G(n, M)$ has $\binom{N}{M}$ elements and each element occurs with probability $\binom{N}{M}^{-1}$. Almost always M is a function of $n : M = M_n$.

The second model $G_{n,p}$ consists of all graphs with vertex set $V = \{1, 2, \ldots, n\}$ in which the edges are chosen independently and with probability p. In other words, if

G_0 is a graph with vertex set V and it has m edges, then

$$\Pr(G_0) = p^m(1-p)^{N-m}.$$

Usually, p is a function of $n : p = p_n$. This is the probabilistic way to view $G_{n,p}$. Another equivalent dynamic way to view $G_{n,p}$ is to imagine $G_{n,p}$ as having no edges at time 0; at each time unit a randomly chosen edge is added to $G_{n,p}$. Then $G_{n,p}$ *evolves* from empty to full. It can be shown that, in most situation, the models $G(n, M)$ and $G_{n,p}$ are practically interchangeable (i.e., they are identical models), provided M is close to pN. In most investigations, it is much easier to work in $G_{n,p}$ than in $G(n, M)$. In the rest of the chapter, we will only work in the model $G_{n,p}$.

A special case of $G_{n,p}$ is $G_{H,p}$, where H is a fixed graph. In this case, we select the edges of H with probability p independently of each other, and the edges not belonging to H are not selected. In fact, the graph H is called the *initial graph* for building the random graph $G_{H,p}$. Clearly, $G_{n,p}$ is a random graph built on the initial graph K^n, the complete graph, i.e., $G_{K^n,p}$ is the same as $G_{n,p}$.

$G_{K^n,p}$ represents the set of all subgraphs from the complete graph K^n. We call a subset Q of $G_{n,p}$ a *property* of graphs of order n if $G \in Q$, $H \in G_{n,p}$ and $G \simeq H$ (G and H are isomorphic) imply that $H \in Q$. A property Q is called *monotone increasing* or simply *monotone* if whenever $G \in Q$ and $G \subset H$ then also $H \in Q$. Thus the properties, such as containing a certain subgraph, and the connectedness of all vertices, are monotone. Erdös and Renyi [16] [17] discovered the important fact that most monotone properties appear rather suddenly: for some $p = p_n$ almost no $G_{n,p}$ has property Q while for 'slightly' larger p_n almost every $G_{n,p}$ has Q. Given a monotone increasing property Q, a function p_n^* is said to be a *threshold function* for Q if

(i) $\lim_{n\to\infty} \frac{p_n}{p_n^*} = 0$ implies $\lim_{n\to\infty} \Pr(G_{n,p} \text{ has } Q) = 0$;

(ii) $\lim_{n\to\infty} \frac{p_n}{p_n^*} = \infty$ implies $\lim_{n\to\infty} \Pr(G_{n,p} \text{ has } Q) = 1$.

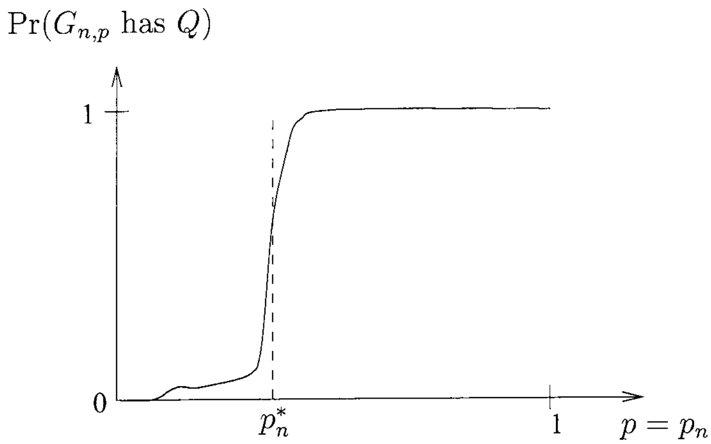

Figure 3.1: The transition phenomenon of a property Q.

Property Q	*Threshold Functions* p_n^*
Contains path of length k	$n^{-(k+1)/k}$
Is not planar	$1/n$
Contains a Hamiltonian path	$(\ln n)/n$
Is connected	$(\ln n)/n$
Contains a clique on k vertices	$n^{-2/(k-1)}$

Table 3.1: The examples of threshold functions for certain properties

Set $f(p) = f_Q(n) = \Pr(G_{n,p}$ has $Q)$. When p_n^* is a threshold function, the function $f(p)$ jumps from zero to one around $p = p_n^*$, for n large. Equivalently, in the dynamic view, the graph $G_{n,p}$ almost certainly does not have property Q when there are $<< p_n^* n^2$ edges, and almost certainly does have property Q when there are $>> p_n^* n^2$ edges, so $G_{n,p}$ attains property Q at some time $p_n^* n^2$. Figure 3.1 illustrates the transition phenomenon. Table 3.1 shows some examples of threshold functions for certain properties [42].

Random graphs based on the complete graph as the initial graph have been extensively investigated, while random graphs based on other initial graphs are relatively less studied. In the rest of the chapter, we will focus on studying random graphs based on a special class of regular graph as initial graphs, and prove a series of theorems regarding to the reliability polynomial for the special type of regular random graphs.

3.3.2 Definitions and Auxiliary Theorems

We first give some definitions related to (undirected) graphs.

Definition 3.1 *For a given graph G with vertex set V, an induced subgraph of $S \subset V$ is the subgraph which consists of all vertices of S and those edges of G, each of which has two end vertices that belong to S. The order of a subgraph is the number of its vertices.*

Definition 3.2 *A set of vertices S in a graph G is called independent if there are no edges of G connecting any two vertices in S; otherwise, they are called dependent.*

Definition 3.3 *A component of a graph G is a connected subgraph; a nontrivial component is one which has order at least two.*

For later use of showing some converging distributions, we introduce the following notations. We write $\mu(X)$ for the distribution of a random variable X. Given integer-valued random variables X and Y, the *total variation distance* of $\mu(X)$ and $\mu(Y)$ is defined as

$$d(\mu(X), \mu(Y)) = sup\{|P(X \in A) - P(Y \in A)| : A \subset \mathcal{Z}\}, \tag{3.2}$$

where $\mathcal{Z}$ is the domain of all integers. Let $X, X_1, X_2, \ldots$ be non-negative integer valued random variables. We say that the sequence of random variables X_n tends to X in *distribution*, denoted as

$$X_n \xrightarrow{d} X,$$

if $\lim_{n\to\infty} P(X_n = k) = P(X = k)$ for every k. Clearly, $X_n \xrightarrow{d} X$ if and only if $d(X_n, X) \to 0$. Define $(x)_r = x(x-1)\cdots(x-r+1)$. The *rth factorial moment* of a random variable X is defined as

$$E_r[X] = E[(X)_r] = E[X(X-1)\cdots(X-r+1)]. \tag{3.3}$$

Note that if X denotes the number of objects in a certain class, then $E_r[X]$ is the expected number of ordered r-tuples of objects of that class.

A random variable X is *Poisson* distributed with rate γ, denoted by $\mu(X) = \mathcal{P}(\gamma)$, if $P(X = i) = \frac{\gamma^i}{i!}e^{-\gamma}$ for any i. To show a sequence of random variables $\{X_n\}$ converges to a random variable with Poisson distribution $\mathcal{P}(\gamma)$, we have the following theorem.

Theorem 3.1 **([11])** *Let $\gamma = \gamma_n$ be a non-negative bounded function on $\mathcal{N}$. Suppose the non-negative integer valued random variables $X_1,X_2, \ldots$ are such that*

$$\lim_{n\to\infty}\{E_r[X_n] - \gamma^r\} = 0, \quad r = 0, 1, \ldots \tag{3.4}$$

Then

$$d(\mu(X_n), \mathcal{P}(\gamma)) \to 0. \tag{3.5}$$

■

For a given graph G, the following theorem guarantees that a set of nodes in the graph has at least a certain lower bound for the number of vertices adjacent to the set.

Theorem 3.2 **([12])** *Let G be a graph of order n and suppose the maximum degree $\Delta(G) \leq \Delta$, $2e(G) = nd$ (i.e. d is the average degree of G), and $\Delta+1 \leq u \leq n-\Delta-1$. Let $\Gamma(U) = \{x \in G$: $xy \in e(G)$ for some $y \in U\}$. Then there exists a u-set U, $|U| = u$, of vertices such that*

$$|N(U)| = |U \cup \Gamma(U)| \geq n\frac{d}{\Delta}(1 - exp(-\frac{u(\Delta+1)}{n})). \tag{3.6}$$

■

Let V be the vertex set of a graph G. For any set S, $T \subset V$, let $E(S,T)$ denote the set of all edges connecting S and T, and $\mathcal{E}(S)$ stand for $\mathcal{E}(S,S)$. Then $|\mathcal{E}(S)|$ is the number of inter-connecting edges among vertices in set S, and $|\mathcal{E}(S, V \setminus S)|$ is the number of external-connecting edges from vertices in set S to vertices in set $V \setminus S$.

$\mathcal{E}(S, V \setminus S)$ is called the *edge boundary* of S, denoted by $\mathcal{B}(S)$. For a r-regular graph $G_{n,r}$ with n nodes and degree r, select a set of vertices S from the graph with $|S| = s$. It is easy to see that

$$2|\mathcal{E}(S)| + |\mathcal{B}(S)| = r \cdot s. \tag{3.7}$$

We define $b(s) = min(|\mathcal{B}(S)|)$. For a graph G with n vertices, let us randomly select a vertex set S of size s from the graph. Assume every vertex of G is selected equally likely, i.e., with probability $1/n$. Then the size of edge boundary of S is a random variable. The following theorem gives the expected value for the size of edge boundary for a set of vertices in a regular graph.

Theorem 3.3 *For a r-regular graph $G_{n,r}$, let S be a set of vertices with $|S| = s \geq 2$, then*

$$E[b(s)] = rs(1 - \frac{s}{n}). \tag{3.8}$$

In particular, when $2 \leq s \leq \frac{n}{2}$,

$$E[b(s)] \geq \frac{rs}{2}. \tag{3.9}$$

Proof. We color the n vertices of the graph with two colors, R(red) and B(blue). Color the s vertices of the set S the color R, the rest $n - s$ vertices the color B. Then the probability of a vertex being colored by R is

$$\Pr(R) = \frac{s}{n},$$

and the probability of a vertex being colored by B is

$$\Pr(B) = \frac{n-s}{n}.$$

Hence, the probability of an edge connecting two red vertices is

$$\Pr(RR) = (\frac{s}{n})^2,$$

and the probability of an edge connecting two blue vertices is

$$\Pr(BB) = (\frac{n-s}{n})^2.$$

Therefore, the probability of an edge connecting two vertices with different colors, i.e., an edge in the edge boundary $\mathcal{B}(S)$, is

$$\begin{aligned}\Pr(RB \cup BR) &= 1 - \Pr(RR) - \Pr(BB) \\ &= 1 - (\frac{s}{n})^2 - (\frac{n-s}{n})^2 \\ &= \frac{2s(n-s)}{n^2}.\end{aligned}$$

Since there are totally $rn/2$ edges in the graph, it follows that the expected value of the size of the edge boundary $\mathcal{B}(S)$ is

$$E[b(s)] = \frac{rn}{2} \cdot \frac{2s(n-s)}{n^2} = rs(1 - \frac{s}{n}).$$

Furthermore, when $2 \le s \le \frac{n}{2}$, we have $1 - \frac{s}{n} \ge \frac{1}{2}$, and hence, $E[b(s)] \ge \frac{rs}{2}$. ■

3.4 The Reliability Polynomial of Regular Random Graphs

A regular graph G_{n,r_n} is a graph with n vertices in which every vertex has degree r_n. Clearly G_{n,r_n} has $nr_n/2$ edges. Taking G_{n,r_n} as the initial graph, we can obtain a random graph G_{n,r_n,p_n} by choosing every edge of G_{n,r_n} with probability p_n. The resulting regular random graph G_{n,r_n,p_n} is our object to analyze. A regular random graph is also called a *regular graph network* when it represents a network such that G_{n,r_n} is its underlying topology graph, and $1 - p_n$ is the edge failure probability in the network. Our objective is to study the reliability polynomial for a regular graph network.

3.4.1 The Isolated Vertices of Regular Random Graphs

Consider a regular random graph G_{n,r_n,p_n}. Define a random variable

$$X_{n,i} = \begin{cases} 1 & \text{if vertex } i \text{ is isolated} \\ 0 & \text{if not.} \end{cases}$$

Let

$$X_n = \sum_{i=0}^{n-1} X_{n,i}, \tag{3.10}$$

the number of isolated vertices in G_{n,r_n,p_n}. We first prove the following theorem which computes the expected value and variance of X_n:

Theorem 3.4 *For a regular random graph G_{n,r_n,p_n} and random variables X_n as defined, the number of isolated vertices of the random graph satisfies*

$$E[X_n] = n(1-p_n)^{r_n}, \tag{3.11}$$

and

$$\mathrm{Var}[X_n] = n(1-p_n)^{r_n} + \frac{(2r_n+1)p_n - 1}{1-p_n} \cdot n(1-p_n)^{2r_n}. \tag{3.12}$$

Proof. Since each vertex i has degree r_n, the probability that vertex i is isolated is

$$\Pr\{X_{n,i} = 1\} = (1-p_n)^{r_n}.$$

Clearly, we have

$$E[X_{n,i}] = \Pr\{X_{n,i} = 1\} = (1-p_n)^{r_n},$$

and also

$$E[X_{n,i}^2] = (1-p_n)^{r_n}.$$

Hence,

$$\mathrm{Var}[X_{n,i}] = E[X_{n,i}^2] - E[X_{n,i}]^2 = (1-p_n)^{r_n} - (1-p_n)^{2r_n}.$$

Note that

(1) when vertex i and vertex j are not adjacent in the original graph, $X_{n,i}$ and $X_{n,j}$ are independent, which results in $E[X_{n,i}X_{n,j}] = E[X_{n,i}] \cdot E[X_{n,j}] = E[X_{n,i}]^2$.

(2) when vertex i and vertex j are adjacent in the original graph, $X_{n,i}$ and $X_{n,j}$ are dependent, which results in $E[X_{n,i}X_{n,j}] = (1-p_n)^{r_n+r_n-1} = (1-p_n)^{2r_n-1}$.

Hence,

$$\begin{aligned}
\text{Cov}(X_{n,i}, X_{n,j}) &= E[X_{n,i}X_{n,j}] - E[X_{n,i}] \cdot E[X_{n,j}] \\
&= \begin{cases} 0 & \text{if } i \text{ and } j \text{ are not adjacent} \\ (1-p_n)^{2r_n-1} - (1-p_n)^{2r_n} & \text{if } i \text{ and } j \text{ are adjacent} \end{cases}
\end{aligned}$$

Therefore,

$$E[X_n] = n \cdot E[X_{n,i}] = n(1-p_n)^{r_n},$$

and

$$\begin{aligned}
\text{Var}(X_n) &= \sum_i \text{Var}(X_{n,i}) + 2\sum_{i \neq j} \text{Cov}(X_{n,i}, X_{n,j}) \\
&= n \cdot [(1-p_n)^{r_n} - (1-p_n)^{2r_n}] + 2n \cdot r_n \cdot [(1-p_n)^{2r_n-1} - (1-p_n)^{2r_n}] \\
&= n(1-p_n)^{r_n} + \frac{(2r_n+1)p_n - 1}{1-p_n} \cdot n(1-p_n)^{2r_n}.
\end{aligned}$$

■

We next define

$$\gamma_n = n(1-p_n)^{r_n}. \tag{3.13}$$

Hence, $\gamma_n = E[X_n]$, which is the expected number of isolated vertices in the random graph. We can write

$$p_n = 1 - (\frac{\gamma_n}{n})^{1/r_n}. \tag{3.14}$$

Define

$$\text{Rel}(G_{n,r_n,p_n}) = \Pr\{G_{n,r_n,p_n} \text{ is connected}\}, \tag{3.15}$$

the usual reliability polynomial for the graph G_{n,r_n,p_n}. We now prove the following theorem, which is one part of our main theorem.

Theorem 3.5 *For a regular random graph G_{n,r_n,p_n} and γ_n as defined in (3.13), if the parameters r_n and p_n are such that $\gamma_n \to \infty$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \mathrm{Rel}(G_{n,r_n,p_n}) = 0. \tag{3.16}$$

Proof. By Chebyshev's inequality,

$$\Pr\{X_n = 0\} \le \Pr\{|X_n - E[X_n]| \ge E[X_n]\} \le \frac{\mathrm{Var}(X_n)}{E[X_n]^2}.$$

It follows from Theorem 3.4 that

$$\begin{aligned}\Pr\{X_n = 0\} &\le \frac{1}{n(1-p_n)^{r_n}} + \frac{1}{n}\cdot\frac{(2r_n+1)p_n - 1}{1-p_n} \\ &= \frac{1}{\gamma_n} - \frac{1}{n} + \frac{2p_n r_n}{n(1-p_n)} \\ &= \frac{2p_n r_n}{n(1-p_n)} + o(1).\end{aligned}$$

Now we consider the following two cases:

(i) $r_n = o(n)$.

From (3.13), it follows

$$n(1-p_n) = n\left(\frac{\gamma_n}{n}\right)^{1/r_n}.$$

Then,

$$\frac{2p_n r_n}{n(1-p_n)} = \frac{2p_n}{\gamma_n^{1/r_n}} \cdot \frac{r_n}{n^{1-1/r_n}} < \frac{2r_n}{n^{1-1/r_n}} = o(1),$$

and hence $\Pr\{X_n = 0\} = o(1)$.

(ii) $r_n = O(n)$.

Recall that we have the condition $\gamma_n \to \infty$. Combining with the condition $r_n = O(n)$, the equation (3.13) implies $p_n = o(1)$. Then,

$$\frac{2p_n r_n}{n(1-p_n)} < \frac{2p_n}{1-p_n} = o(1),$$

and hence again $\Pr\{X_n = 0\} = o(1)$. Therefore,

$$\begin{aligned} \mathrm{Rel}(G_{n,r_n,p_n}) &\leq 1 - \Pr\{G_{n,r_n,p_n} \text{ contains isolated vertices}\} \\ &= \Pr\{X_n = 0\} \\ &= o(1). \end{aligned}$$

■

We give an example for the parameters r_n and p_n satisfying the conditions in Theorem 3.5. Let $r_n = k \ln n$ for some constant k, and $p_n = 1 - e^{-\frac{1}{2k}}$. Then

$$\gamma_n = n(1-p_n)^{r_n} = n \cdot e^{-\frac{1}{2k} \cdot k \ln n} = n \cdot e^{-\ln \sqrt{n}} = \sqrt{n} \to \infty,$$

as required.

Theorem 3.5 says that when $p_n = 1 - (\frac{\gamma_n}{n})^{1/r_n}$ with $\gamma_n \to \infty$ as $n \to \infty$, G_{n,r_n,p_n} is disconnected with probability one. It in fact implies that if p_n or r_n is fairly small (i.e., the random graph grows not too fast), the graph is asymptotically disconnected, which is intuitively clear. The following theorem shows that the number of s isolated vertices in G_{n,r_n,p_n} asymptotically converges to Poisson distribution $\mathcal{P}(\gamma_n)$ under certain conditions.

Theorem 3.6 *For a regular random graph G_{n,r_n,p_n} with $r_n = o(n)$ and γ_n as defined in (3.13), the number of isolated vertices X_n in the graph is asymptotically Poisson distributed with the mean γ_n, i.e.,*

$$d(\mu(X_n), \mathcal{P}(\gamma_n)) \to 0. \tag{3.17}$$

Proof. Let us evaluate the s-th factorial moment of X_n. Recall that $E_s[X_n]$ is the expected number of ordered s-tuples of isolated vertices in the regular graph. Hence,

$$E_s[X_n] \geq (n)_s[(1-p_n)^{r_n}]^s = \frac{(n)_s}{n^s}(n(1-p_n)^{r_n})^s = \gamma_n^s + o(1). \tag{3.18}$$

Now we want to compute an upper bound for $E_s[X_n]$. Let $E_s^{(1)}[X_n]$ be the expected number of ordered s-tuples of isolated vertices $(v_1, v_2, \ldots, v_s)$, which consist of an independent set in the initial graph G_{n,r_n}. Let $E_s^{(2)}[X_n]$ be the expected number of ordered s-tuples of isolated vertices $(v_1, v_2, \ldots, v_s)$, which induce a subgraph of the random graph G_{n,r_n,p_n} containing at least one nontrivial component (i.e., the s vertices in G_{n,r_n,p_n} are dependent). It then follows that

$$E_s[X_n] = E_s^{(1)}[X_n] + E_s^{(2)}[X_n]. \tag{3.19}$$

We evaluate above two terms separately.

(1) Clearly,

$$E_s^{(1)}[X_n] \leq n(n-s)^{s-1}[(1-p_n)^{r_n}]^s \leq n^s[(1-p_n)^{r_n}]^s = \gamma_n^s. \tag{3.20}$$

(2) For s dependent vertices $(v_1, v_2, \ldots, v_s)$, assume the subgraph induced by $(v_1, v_2, \ldots, v_s)$ has τ, $1 \leq \tau \leq [\frac{s}{2}]$, nontrivial connected components of orders $t_1, t_2, \ldots, t_\tau$, where

$$t_1 + t_2 + \cdots + t_\tau = \omega \tag{3.21}$$

for some ω, $2\tau \leq \omega \leq s$. Figure 3.2 illustrates the situation. Now let us consider the jth component, $1 \leq j \leq \tau$. The probability that each vertex of the component is isolated is

$$t_j[(1-p_n)^{r_n}]^{t_j}(1-p_n)^{-\frac{t_j(t_j-1)}{2}}.$$

So the jth component can be chosen in at most

$$n \underbrace{r_n(r_n-1)\cdots(r_n-1)}_{(t_j-1)\ terms} \le n r_n^{t_j-1}$$

different ways. Therefore, the probability of having the jth component P_j is

$$\begin{aligned} P_j &\le t_j[(1-p_n)^{r_n}]^{t_j}(1-p_n)^{-\frac{t_j(t_j-1)}{2}} n r_n^{t_j-1} \\ &= n[(1-p_n)^{r_n}]^{t_j} r_n^{t_j-1} t_j (1-p_n)^{-\frac{t_j(t_j-1)}{2}}. \end{aligned}$$

There are $s-\omega$ independent vertices in the subgraph, which exists with probability $[(1-p_n)^{r_n}]^{s-\omega}$. Hence, the probability of having this subgraph P_{sg} is

$$\begin{aligned} P_{sg} &\le \prod_{j=1}^{\tau}\{n[(1-p_n)^{r_n}]^{t_j} r_n^{t_j-1} t_j (1-p_n)^{-\frac{t_j(t_j-1)}{2}}\} \cdot [(1-p_n)^{r_n}]^{s-\omega} \\ &= n^{\tau}[(1-p_n)^{r_n}]^{\sum_{i=1}^{\tau} t_i} r_n^{\sum_{i=1}^{\tau}(t_i-1)}(t_1\cdots t_\tau)(1-p_n)^{-\sum_{i=1}^{\tau}\frac{t_i(t_i-1)}{2}}[(1-p_n)^{r_n}]^{s-\omega} \\ &\le n^{\tau}[(1-p_n)^{r_n}]^{s} r_n^{\omega-\tau}(t_1\cdots t_\tau)(1-p_n)^{-\frac{\omega(\omega-1)}{2}} \\ &= \gamma_n^s \cdot \frac{1}{n^{s-\tau}} r_n^{\omega-\tau} \cdot (t_1\cdots t_\tau)(1-p_n)^{-\frac{\omega(\omega-1)}{2}}. \end{aligned}$$

Therefore, since $2\tau \le \omega \le s$, $t_i \le \omega \le s$ for $1 \le i \le \tau$, and since $r_n = o(n)$,

$$\begin{aligned} E_s^{(2)}[X] &\le \sum_{\tau=1}^{[\frac{s}{2}]}\sum_{\omega=2\tau}^{s} \gamma_n^s \cdot \frac{1}{n^{s-\tau}} r_n^{\omega-\tau}\cdot(t_1\cdots t_\tau)(1-p_n)^{-\frac{\omega(\omega-1)}{2}} \\ &\le \gamma_n^s \cdot \sum_{\tau=1}^{[\frac{s}{2}]}\sum_{\omega=2\tau}^{s}(t_1\cdots t_\tau)(1-p_n)^{-\frac{\omega(\omega-1)}{2}}\cdot(\frac{r_n}{n})^{s-\tau} \\ &= \gamma_n^s \cdot o(1). \end{aligned} \tag{3.22}$$

It then follows that

$$\begin{aligned} E_s[X_n] &= E_s^{(1)}[X_n] + E_s^{(2)}[X_n] \\ &\le \gamma_n^s(1+o(1)) + \gamma_n^s \cdot o(1) \\ &= \gamma_n^s(1+o(1)). \end{aligned} \tag{3.23}$$

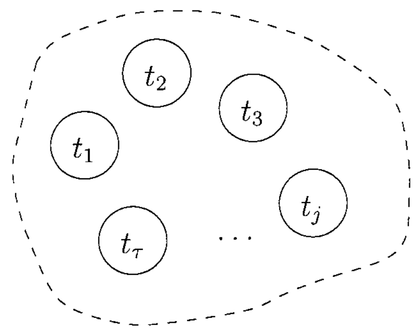

Figure 3.2: An illustrated figure for the proof of (3.22): τ nontrivial components induced by vertices $v_1, v_2, \ldots, v_s$.

Therefore, by (3.18) and (3.23), we obtain

$$\lim_{n\to\infty} \{E_s[X_n] - \gamma_n^s\} = 0, \; s = 0, 1, \ldots$$

and the proof is completed by Theorem 3.1. ■

Theorem 3.6 showed that the number of isolated vertices X_n in the special regular random graph G_{n,r_n,p_n} is Poisson distributed with mean γ_n. In the next section, we will show that it is (asymptotically) because of the isolated vertices that cause the random graph disconnected.

3.4.2 The Components of Regular Random Graphs

It is fascinating that the obstacle for some random graphs to be connected is asymptotically the existence of isolated vertices [17] [12]. The probability that those graphs contain components with size at least 2 and at most $[\frac{n}{2}]$ vertices tends to zero. We now prove the following theorem which shows that this is also the case for the regular random graphs we consider.

Theorem 3.7 *For a regular random graph G_{n,r_n,p_n}, assume $r_n = O(\log n)$ and $p_n = 1 - (\frac{\gamma_n}{n})^{\frac{1}{r_n}}$ for some finite number γ_n. Let $\mathcal{C}_s$ be the family of induced connected* *subgraphs with size s, and* $\mathcal{C} = \bigcup_{s=2}^{[\frac{n}{2}]} \mathcal{C}_s$. *Then*

$$\lim_{n\to\infty} \sum_{T\in\mathcal{C}} \Pr(T) = 0. \tag{3.24}$$

Proof. Let $T_s \in \mathcal{C}_s$, with edge boundary $\mathcal{B}(T_s)$ and $|\mathcal{B}(T_s)| = b(T_s)$. It is clear that

$$\Pr(T_s) \leq (1 - p_n)^{b(T_s)}.$$

Define

$$Z = \sum_{T\in\mathcal{C}} \Pr(T) = \sum_{s=2}^{[\frac{n}{2}]} \sum_{T_s\in\mathcal{C}_s} \Pr(T_s) \leq \sum_{s=2}^{[\frac{n}{2}]} \sum_{T_s\in\mathcal{C}_s} (1-p_n)^{b(T_s)}. \tag{3.25}$$

Then proving the theorem is equivalent to showing

$$Z = o(1).$$

We first give two upper bounds for $|\mathcal{C}_s|$. Clearly we have

$$|\mathcal{C}_s| \leq \binom{n}{s} \sim (\frac{en}{s})^s. \tag{3.26}$$

On the other hand, let us count the ways to choose each vertex for a connected subgraph T_s of size s. After relabeling the vertices of T_s in a certain order, we can see that there are at most n ways to choose the first vertex, at most r_n ways to choose the second vertex, at most $2r_n$ ways to choose the third vertex,..., at most $(i-1)r_n$ ways to choose the ith vertex,.... Hence,

$$|\mathcal{C}_s| \leq n \cdot s! \cdot r_n^{s-1} \sim n \cdot \sqrt{2\pi s} \cdot (\frac{s}{e})^s \cdot r_n^{s-1}. \tag{3.27}$$

Note that (3.26) is a better estimation for large s, while (3.27) is a better estimation for small s. Let $b(s) = \min(b(T_s))$, from (3.25), we then have

$$Z \leq \sum_{s=2}^{[\frac{n}{2}]} |\mathcal{C}_s| \cdot (1-p_n)^{b(s)} = \sum_{s=2}^{[\frac{n}{2}]} |\mathcal{C}_s| \cdot (\frac{\gamma_n}{n})^{\frac{b(s)}{r_n}}. \tag{3.28}$$

We now evaluate the number Z in the following four ranges for s.

Case 1: $2 \leq s \leq r_n/2$

Define

$$Z_1 = \sum_{s=2}^{r_n/2} |\mathcal{C}_s| \cdot (\frac{\gamma_n}{n})^{b(T_s)}. \tag{3.29}$$

For a vertex set S of size s in a regular graph G_{n,r_n,p_n}, there are at most $\binom{s}{2}$ edges among the vertices of S. Hence,

$$b(s) \geq r_n s - 2\binom{s}{2} = s(r_n - s + 1).$$

Therefore,

$$Z_1 \leq \sum_{s=2}^{r_n/2} n \cdot s! \cdot r_n^{s-1} \cdot (\frac{\gamma_n}{n})^{\frac{s}{r_n}(r_n - s + 1)}.$$

Let $\alpha_s = n \cdot s! \cdot r_n^{s-1} \cdot (\frac{\gamma_n}{n})^{\frac{s}{r_n}(r_n-s+1)}$, then

$$\begin{aligned}
\log(\alpha_s \cdot \frac{r_n s^s}{s!}) &= \log(\frac{r_n s^s}{s!} \cdot n \cdot s! \cdot r_n^{s-1} \cdot (\frac{\gamma_n}{n})^{\frac{s}{r_n}(r_n-s+1)}) \\
&= \log(r_n^s \cdot s^s \cdot n \cdot (\frac{\gamma_n}{n})^{\frac{s}{r_n}(r_n-s+1)}) \\
&= s\log r_n + s\log s + \log n + \frac{s}{r_n}(r_n - s + 1)(\log\gamma_n - \log n) \\
&\leq 2s\log r_n + \log n + s(1 - \frac{s-1}{r_n})\log\gamma_n - s(1 - \frac{s-1}{r_n})\log n \\
&= 2s\log r_n + s(1 - \frac{s-1}{r_n})\log\gamma_n + [1 - s(1 - \frac{s-1}{r_n})]\log n \\
&= 2s\log r_n + s(1 - \frac{s-1}{r_n})\log\gamma_n - (s-1)\frac{(r_n - s)}{r_n}\log n \\
&\leq 2s\log r_n + s(1 - \frac{s-1}{r_n})\log\gamma_n - \frac{s-1}{2}\log n \\
&\leq 0,
\end{aligned}$$

for n sufficiently large. Hence $\alpha_s \cdot \frac{r_n s^s}{s!} \leq 1$, or $\alpha_s \leq \frac{s!}{s^s} \cdot \frac{1}{r_n}$, and

$$\sum_{s=2}^{r_n/2} \frac{s!}{s^s} \cdot \frac{1}{r_n} = o(1),$$

since $r_n \to \infty$. Therefore,

$$Z_1 = o(1). \tag{3.30}$$

Case 2: $r_n/2 < s \leq O(\frac{n^{1/3}}{\log n})$

Define

$$Z_2 = \sum_{s=r_n/2}^{O(\frac{n^{1/3}}{\log n})} \sum_{T_s \in \mathcal{C}_s} (1-p_n)^{b(T_s)}. \tag{3.31}$$

Since $|\mathcal{C}_s| \leq n \cdot s! \cdot r_n^{s-1} \sim n \cdot \sqrt{2\pi s} \cdot (\frac{s}{e})^s \cdot r_n^{s-1}$, and by Theorem 3.3, $E[b(T_s)] \geq \frac{r_n s}{2}$, it follows that

$$\begin{aligned}
|\mathcal{C}_s| \cdot (1-p_n)^{E[b(T_s)]} &\leq n \cdot \sqrt{2\pi s} \cdot (\frac{s}{e})^s \cdot r_n^{s-1} \cdot (\frac{\gamma_n}{n})^{\frac{s}{2}} \\
&= \sqrt{2\pi} \cdot (n^{\frac{2}{s}} \cdot s^{\frac{1}{s}} \cdot \frac{s^2}{e^2} \cdot r_n^{\frac{2(s-1)}{s}} \cdot \frac{\gamma_n}{n})^{\frac{s}{2}} \\
&= \sqrt{2\pi} \cdot (\frac{\gamma_n \cdot s^{2+\frac{1}{s}} \cdot r_n^{\frac{2(s-1)}{s}}}{e^2 \cdot n^{1-\frac{2}{s}}})^{\frac{s}{2}} \\
&= o(1),
\end{aligned}$$

since $r_n = O(\log n)$ and $s \leq O(\frac{n^{1/3}}{\log n})$. Therefore, it follows that

$$Z_2 = o(1). \tag{3.32}$$

Case 3: $O(\frac{n^{1/3}}{\log n}) < s \leq O(\sqrt{n})$

Define

$$Z_3 = \sum_{s=O(\frac{n^{1/3}}{\log n})}^{O(\sqrt{n})} \sum_{T_s \in \mathcal{C}_s} (1-p_n)^{b(T_s)}. \tag{3.33}$$

We consider the following two subcases:

Subcase 1: Assume $\mathcal{C}_s^-$ is the family of T_s satisfying $b(T_s) > r_n s(1 - \frac{\log s}{\log n} + \frac{\log r_n}{\log n})$.

Then

$$
\begin{aligned}
Z_3^{(1)} &= \sum_{s=2}^{[\frac{n}{2}]} \sum_{T_s \in \mathcal{C}_s^-} (1-p_n)^{b(T_s)} \\
&\le \sum_{s=2}^{[\frac{n}{2}]} \binom{n}{s} \cdot (\frac{\gamma_n}{n})^{s(1-\frac{\log \frac{s}{r_n}}{\log n})} \\
&\le \sum_{s=2}^{[\frac{n}{2}]} (\frac{en}{s})^s \cdot (\frac{\gamma_n^{1-\frac{\log \frac{s}{r_n}}{\log n}}}{n^{1-\frac{\log \frac{s}{r_n}}{\log n}}})^s \\
&= \sum_{s=2}^{[\frac{n}{2}]} (\frac{e\gamma_n^{1-\frac{\log \frac{s}{r_n}}{\log n}}}{r_n})^s \\
&\sim \sum_{s=2}^{[\frac{n}{2}]} (\frac{1}{r_n})^s = o(1). \qquad (3.34)
\end{aligned}
$$

Subcase 2: Assume $\mathcal{C}_s^+$ is the family of T_s satisfying $b(T_s) \le r_n s(1 - \frac{\log s}{\log n} + \frac{\log r_n}{\log n})$. Then,

$$b(T_s) + s[r_n(\frac{\log s}{\log n} - \frac{\log r_n}{\log n})] \le r_n s.$$

Let d be the average degree of the component T_s, since $b(T_s) + s \cdot d = r_n s$, it follows

$$d \ge r_n(\frac{\log s}{\log n} - \frac{\log r_n}{\log n}).$$

Let $u = 3s/r_n$. By Theorem 3.2, in component T_s, there exists a subset U, $|U| = u$, such that

$$
\begin{aligned}
|N(U)| &\ge n(\frac{\log s}{\log n} - \frac{\log r_n}{\log n})(1 - e^{-\frac{u(r_n+1)}{s}}) \\
&= n(\frac{\log s}{\log n} - \frac{\log r_n}{\log n})(1 - e^{-3\frac{r_n+1}{r_n}}) \\
&\ge s(\frac{\log \frac{\sqrt{n}}{(\log n)^2}}{\log n} - \frac{\log r_n}{\log n})(1 - e^{-3\frac{r_n+1}{r_n}}) \\
&\ge \frac{s}{3}.
\end{aligned}
$$

This shows that the set T_s in $\mathcal{C}_s^+$ can be selected as follows: first we select u vertices, which determines $s/3$ vertices that are connected with each other, then we select the other $2s/3$ vertices. Hence,

$$|\mathcal{C}_s^+| \leq \binom{n}{u} \cdot \binom{n}{\frac{2s}{3}} \sim (\frac{en}{u})^u \cdot (\frac{3en}{2s})^{\frac{2s}{3}} = (\frac{enr_n}{3s})^u \cdot (\frac{3en}{2s})^{\frac{2s}{3}},$$

and

$$\begin{aligned} |\mathcal{C}_s^+| \cdot (1-p_n)^{E[b(T_s)]} &\leq (\frac{enr_n}{3s})^u \cdot (\frac{3en}{2s})^{\frac{2s}{3}} \cdot (\frac{\gamma_n}{n})^{\frac{s}{2}} \\ &\leq (\frac{enr_n(\log n)^2}{3\sqrt{n}})^{\frac{3s}{r_n}} \cdot (\frac{3enr_n(\log n)^2}{2\sqrt{n}})^{\frac{2s}{3}} \cdot (\frac{\gamma_n}{n})^{\frac{s}{2}} \\ &= (\frac{er_n(\log n)^2}{3})^{\frac{3s}{r_n}} \cdot (\frac{3er_n(\log n)^2}{2})^{\frac{2s}{3}} \cdot (\gamma_n)^{\frac{s}{2}} \cdot n^{\frac{3s}{2r_n} - \frac{s}{6}} \\ &= o(1), \end{aligned}$$

since $r_n = O(\log n)$, γ_n is finite, and $s > O(\frac{n^{1/3}}{\log n})$. It follows that

$$Z_3^{(2)} = \sum_{s=O(\frac{n^{1/3}}{\log n})}^{O(\sqrt{n})} \sum_{T_s \in \mathcal{C}_s^+} (1-p_n)^{b(T_s)} = o(1). \tag{3.35}$$

Therefore, from (3.34) and (3.35), we obtain

$$Z_3 = Z_3^{(1)} + Z_3^{(2)} = o(1). \tag{3.36}$$

Case 4: $O(\sqrt{n}) < s \leq n/2$

Define

$$Z_4 = \sum_{s=O(\sqrt{n})}^{\frac{n}{2}} \sum_{T_s \in \mathcal{C}_s} (1-p_n)^{b(T_s)}. \tag{3.37}$$

Recall $|\mathcal{C}_s| \leq \binom{n}{s} \sim (\frac{en}{s})^s$, and $E[b(T_s)] \geq \frac{r_n s}{2}$. Then,

$$|\mathcal{C}_s| \cdot (1-p_n)^{E[b(T_s)]} \leq (\frac{en}{s})^s \cdot (\frac{\gamma_n}{n})^{\frac{s}{2}} = (\frac{e\sqrt{\gamma_n} \cdot \sqrt{n}}{s})^s = o(1),$$

since $s > O(\sqrt{n})$ implies $\frac{e\sqrt{\gamma_n}\cdot\sqrt{n}}{s} < 1$. Hence,

$$Z_4 = o(1). \tag{3.38}$$

Therefore, by (3.30), (3.32), (3.36), and (3.38), we have shown that

$$Z \le Z_1 + Z_2 + Z_3 + Z_4 = o(1),$$

which completes the proof of the theorem. ■

Theorem 3.7 says that, for a regular random graph G_{n,r_n,p_n} as defined, the probability that it contains connected subgraphs (components) with size between 2 and $[\frac{n}{2}]$ is asymptotically zero. Therefore, the random graph decomposes with probability one into a "giant" connected subgraph and a set of isolated vertices. Hence, the reliability of the random graph is actually the probability of no isolated vertices, as we state in the following corollary:

Corollary 3.1 *For a regular random graph G_{n,r_n,p_n} such that $r_n = O(\log n)$ and $p_n = 1 - (\frac{\gamma_n}{n})^{\frac{1}{r_n}}$ for some finite number γ_n,*

$$\mathrm{Rel}(G_{n,r_n,p_n}) \to \Pr\{G_{n,r_n,p_n} \textit{ contains no isolated vertices}\}, \tag{3.39}$$

for n sufficiently large. ■

3.4.3 The Main Theorem

Now we present the following theorem which is another part of our main theorem.

Theorem 3.8 *For a regular random graph G_{n,r_n,p_n} with $r_n = O(\log n)$,*

$$\lim_{n\to\infty} \mathrm{Rel}(G_{n,r_n,p_n}) = \begin{cases} e^{-\gamma} & \text{if } p_n = 1 - (\frac{\gamma_n}{n})^{\frac{1}{r_n}} \text{ for finite } \gamma_n = \gamma > 0 \\ 1 & \text{if } p_n = 1 - (\frac{\gamma_n}{n})^{\frac{1}{r_n}} \text{ for } \gamma_n \to 0. \end{cases} \tag{3.40}$$

Proof.

(1) $p_n = 1 - (\frac{\gamma_n}{n})^{\frac{1}{r_n}}$ for finite $\gamma_n = \gamma > 0$. By Corollary 3.1 and Theorem 3.6,

$$\mathrm{Rel}(G_{n,r_n,p_n}) \to \Pr\{G_{n,r_n,p_n} \text{ has no isolated vertices}\} \to e^{-\gamma}.$$

(2) $p_n = 1 - (\frac{\gamma_n}{n})^{\frac{1}{r_n}}$ for $\gamma_n \to 0$. Since

$$\Pr\{G_{n,r_n,p_n} \text{ contains isolated vertices}\} = \Pr\{X_n > 0\} \le E[X_n] = \gamma_n = o(1),$$

it follows, by Corollary 3.1, that

$$\begin{aligned} \mathrm{Rel}(G_{n,r_n,p_n}) &\to \Pr\{G_{n,r_n,p_n} \text{ contains no isolated vertices}\} \\ &= 1 - \Pr\{G_{n,r_n,p_n} \text{ contains isolated vertices}\} \\ &\to 1. \end{aligned}$$

■

Finally, combining Theorem 3.5 and Theorem 3.8, we obtain the main theorem of the chapter:

Theorem 3.9 *For a regular random graph G_{n,r_n,p_n}, if $p_n = 1 - (\frac{\gamma_n}{n})^{\frac{1}{r_n}}$, then γ_n is the expected number of isolated vertices in the random graph, and*

$$\lim_{n\to\infty} \mathrm{Rel}(G_{n,r_n,p_n}) = \begin{cases} 0 & \text{if } \gamma_n \to \infty \\ e^{-\gamma} & \text{if } \gamma_n = \gamma > 0 \text{ is finite and } r_n = O(\log n) \\ 1 & \text{if } \gamma_n \to 0 \text{ and } r_n = O(\log n). \end{cases} \tag{3.41}$$

■

Theorem 3.9 in fact exhibits the transition phenomenon when the regular random graph evolutes from $p_n = 0$ to $p_n = 1$, when n increases. The reliability polynomial $\mathrm{Rel}(G_{n,r_n,p_n})$ jumps from 0 to 1 around $p_n^* = 1 - (\frac{\omega_n}{n})^{\frac{1}{r_n}}$ for some function ω_n: when $\omega_n \to \infty$, the graph is disconnected; when $\omega_n \to 0$, the graph is connected; when

$\omega_n = \omega > 0$ is finite constant, the graph is connected with probability $e^{-\omega}$.

Corollary 3.2 *For a regular random graph G_{n,r_n,p_n} with $r_n = O(\log n)$, the threshold function is*

$$p_n^* = 1 - (\frac{\omega_n}{n})^{\frac{1}{r_n}} \tag{3.42}$$

for some function ω_n, and the regular random graph is asymptotically surely connected when $\omega_n \to 0$; it is asymptotically surely disconnected when $\omega_n \to \infty$; and it is asymptotically connected with probability $e^{-\omega}$ when $\omega_n = \omega > 0$ is a constant. ■

Theorem 3.9 actually shows that, no matter what p_n is, the probability that a regular random graph G_{n,r_n,p_n} with $r_n = O(\log n)$ is connected asymptotically approaches $e^{-\gamma_n} = e^{-E[X_n]}$, where X_n is the number of isolated vertices in the graph. Therefore, the expected value of the number of isolated vertices in such a regular random graph G_{n,r_n,p_n}, $n(1-p_n)^{r_n}$, is the most crucial quantity in determining the connectivity of the random graph.

Corollary 3.3 *For a regular random graph G_{n,r_n,p_n} with $r_n = O(\log n)$,*

$$\lim_{n\to\infty} \mathrm{Rel}(G_{n,r_n,p_n}) = \lim_{n\to\infty} e^{-n(1-p_n)^{r_n}}. \tag{3.43}$$

■

Let us consider an example. Assume we want to design a network with $n = 9$ nodes and $m = 18$ lines. To achieve the maximum connectivity, we design the network using the Harary graph $H(9,4)$, as shown in Figure 2.6. Assume each line of the network fails with probability $q = 1 - p = 0.1$. Then

$$\begin{aligned}
\mathrm{Rel}(H(9,4)) &= 1 - \sum_{i=\delta}^{m} C_i (1-p)^i p^{m-i} \\
&\simeq 1 - n(1-p)^{\delta} p^{m-\delta} \\
&= 1 - 9(0.1)^4(0.9)^{14} \\
&= 0.9992181,
\end{aligned}$$

whereas,

$$e^{-\gamma} = e^{-n(1-p)^r} = e^{-9(0.1)^4} = 0.9991004.$$

Hence, Theorem 3.9 indeed gives a quite accurate estimation for the network reliability.

Theorem 3.9 also implies a famous known result for a special class of regular random graphs, the *n-cubes*. An n-cube C^n is a regular graph with 2^n nodes such that each node has degree $n-1$. Clearly, for the C^n, we have $\gamma_n = E[X_n] = 2^n(1-p_n)^n = (2(1-p_n))^n$. From Theorem 3.9, it follows that $\Pr\{C^n \text{ is connected}\} \to e^{-(2(1-p_n))^n}$, and hence, $\Pr\{C^n \text{ is connected}\} \to 0$ when $p_n < 0.5$; e^{-1} when $p_n = 0.5$; 1 when $p_n > 0.5$, which is the main result in [17] [12].

3.5 The Slotted Model and MTTF

In Chapter 2, we introduced MTTF, the figure-of-merit which is commonly used to measure the reliability of a network. MTTF can also be defined as [40]

$$\text{MTTF} = \int_0^\infty \text{R(t)dt}, \tag{3.44}$$

where

$$R(t) = \Pr\{\text{a network has not yet failed at time t}\}.$$

We developed several algorithms and asymptotic formulas for computing MTTF in Chapter 2, but MTTF is generally difficult to compute. We now propose a model, which can be used to compute the MTTF for a regular graph network $G_{n,r}$ with much less difficulty, by applying reliability polynomials. The model is called the *slotted model*, and we define it as follows:

(1) All edges fail according to an exponential distribution, with the failure rate $\lambda = 1/\tau_f$. (Note that τ_f is actually the MTTF for a single edge).

(2) The time scale is divided into "slots", i.e., time intervals, with length τ; all repairs

occur spontaneously at slot boundaries. Hence, the repair rate is $\mu = 1/\tau$.

Typically, $\tau_f >> \tau$, i.e., $\rho = \mu/\lambda = \tau_f/\tau >> 1$. Hence, in each time slot, every edge fails with probability

$$q = 1 - p \approx 1 - e^{-\tau/\tau_f} \approx \frac{\tau}{\tau_f}. \tag{3.45}$$

Let

$$\pi = \Pr\{\text{a network failure occurs in a given slot}\}. \tag{3.46}$$

Then

$$\pi = 1 - \mathrm{Rel}(G_{n,r,p}).$$

Since the sequence of system failures during time slots is a binomial sequence, and since $1/\pi$ is the waiting time before the first failure occurs in a time slot, the MTTF for the slotted model is given by

$$\mathrm{MTTF}_{slotted} = \tau \cdot \frac{1}{\pi} = \frac{\tau}{1 - \mathrm{Rel}(G_{n,r,p})}. \tag{3.47}$$

For a network $G_{n,r,p}$, from (3.13), the expected number of the isolated nodes is given by

$$\gamma = n(1-p)^r = n(\tau/\tau_f)^r.$$

From Theorem 3.9,

$$\mathrm{Rel}(G_{n,r,p}) \approx e^{-\gamma} = e^{-n(\tau/\tau_f)^r}. \tag{3.48}$$

Hence, we have

$$\mathrm{MTTF}_{slotted} \approx \frac{\tau}{1 - e^{-n(\tau/\tau_f)^r}}. \tag{3.49}$$

When τ/τ_f is small (or ρ is large), we have $1 - e^{-n(\tau/\tau_f)^r} \approx n(\tau/\tau_f)^r = \frac{n}{\rho^r}$. Hence, by (3.49), we obtain $\mathrm{MTTF}_{slotted} = \frac{1}{\lambda} \cdot \frac{\rho^{r-1}}{n}$. Therefore, we have proved the following theorem:

Theorem 3.10 *For a network with a r-regular graph topology under the slotted model as defined above, if $\rho = \frac{\tau_f}{\tau} >> 1$, and the size of the network n is large enough, then*

$$\text{MTTF}_{slotted} \approx \frac{1}{\lambda} \cdot \frac{\rho^{r-1}}{n}. \tag{3.50}$$

■

Denote κ to be the connectivity of a network with a r-regular graph topology, then $\kappa = r$. From (3.50), we then have

$$\text{MTTF}_{slotted} \approx \frac{1}{\lambda} \cdot \frac{\rho^{\kappa-1}}{n} \approx \frac{1}{\lambda} \cdot \frac{\rho^{\kappa-1}}{\kappa C_{\min}} \cdot (\frac{\kappa}{n} C_{\min}). \tag{3.51}$$

Comparing (3.51) with (2.56) in Chapter 2, we can see that the MTTF for the slotted model is almost the same as that for general case, except for a factor $\frac{\kappa}{n}C_{\min}$, the number of minimum cut-sets. Hence, the slotted model introduced improves the MTTF for a system by a factor of $\frac{\kappa}{n}C_{\min}$.

As an example, let us design a network based on the Harary graph $H(9,4)$ as shown in Figure 2.6. Assume $\tau_f = 1$ month, and $\tau = 1$ day. By (3.45), the probability of line failure in the network is $q = 1 - p \approx \frac{1}{30}$. Therefore, by Theorem 3.9, the probability that the network is connected is $\approx e^{-n(1-p)^r} = e^{-\frac{9}{30^4}} = 0.9999888$, and by Theorem 3.10, the MTTF of the network $\approx \frac{30^4}{9} = 90000$ days ≈ 246 years, which indicates the network is extremely reliable.

3.6 Concluding Remarks

In this chapter, we studied reliability polynomials for communication networks. We briefly introduced some well-known algorithms for computing reliability polynomials exactly, and pointed out that the reliability polynomial problem is an $\#P$-complete problem, among the most challenging of all computational problems. We then took a totally different approach to analyzing reliability polynomials from a framework of random graph theory. We focused on a special class of random graphs, namely the regular random graphs, and analyzed the evolution of regular random graphs in terms of the expansion of network size. We proved that the number of isolated vertices in a regular random graph is asymptotically Poisson distributed. We also showed that the probability that a regular random graph contains an non-trivial component is asymptotically zero, and therefore, the expected value of the number of isolated vertices plays the most crucial role in determining the connectedness of a regular random graph. We showed a transition phenomenon when the regular random graph evolutes from edge probability zero to probability one, and identified the associated threshold functions, which completely characterized the evolution of the special class of regular random graphs. Finally, we introduced a "slotted model" for networks that have regular graph topology, by which the MTTF of such networks can be easily, asymptotically, computed by the reliability polynomial formula we developed.

Chapter 4 ATM Networks and Markov On-Off Traffic

4.1 Introduction

ATM is a standard which is recognized throughout the world [36]. It provides for the first time a method for universal information exchange, independent of the end systems and the type of information (data, audio, video). The architecture of ATM (53-byte cells) supports the design of massive parallel communication architectures and enables the implementation of networks with transfer rates in the gigabit range. With these high speed networks, it is possible to send huge amounts of data generated by the latest applications (video-mail, interactive TV, telemedicine etc.) at low cost in a real-time framework. Furthermore, ATM is suitable for local area networks as well as wide area networks. Thus the historical separation of local and wide area data transport, which has resulted in complex networks with numerous Internet components, such as routers, gateways, etc., will disappear in the future. The ability of ATM to emulate traditional LAN and WAN architectures will ensure a smooth transition from today's computer network infrastructure to ATM-based high speed technology.

The problem of analyzing buffer performance for a single link statistical multiplexer with Markov on-off traffic in ATM networks has been extensively studied [4] [8] [30]. The problem is challenging because of the bursty characteristics of the traffic sources. The reason for analyzing Markov on-off traffic is because it is widely believed, although without rigorous mathematical proof but vast numerical support, as the traffic which would result in worst case performance for ATM networks. A method using a concept of "effective bandwidth" has been designed to analyze buffer behavior [29]

[23] [48]. Using this method, Markov on-off traffic was proved [23] to be the worst case traffic in terms of overflow probability, for large number of incoming traffic sources.

In this chapter, we will first give a brief introduction to ATM and B-ISDN. We explain the fundamentals of ATM, and the advantages of ATM compared to other existing network technologies. We will focus on *statistical multiplexers*, one of the most important issues in ATM technology. We focus on studying *homogeneous* and *heterogeneous Markov on-off* traffic. These two types of Markov on-off traffic consist of the superposition of many single Markov on-off traffic sources, each of which is described by a Markov two-state Markov chain. We will use a generating function approach to study the performance of the buffer occupancy for a statistical multiplexer. We will derive closed form formulas for certain conditional generating functions of cells generated by homogeneous and heterogeneous Markov on-off traffic, and develop a heuristic procedure, which allows us to compute the expected buffer occupancy for homogeneous and heterogeneous Markov on-off traffic. A simple closed form formula can be developed for the expected buffer occupancy in the case of homogeneous Markov on-off traffic. The analysis and numerical results show that the expected buffer occupancy not only depends on the incoming traffic intensity (utilization), but also on the burstiness of the incoming traffic. In particular, the expected buffer occupancy becomes unbounded when the burstiness is large enough, even though the traffic intensity is not close to one. This shows that "burst control" is indispensable in ATM networks.

4.2 Fundamentals of ATM

ATM stands for *asynchronous transfer mode.* It is the most modern telecommunications switching technique. It is a highly efficient switching technique which is able to switch connections for a wide range of different information types at a wide range of different rates. It allows a network to be used simultaneously for the transfer of different signal types, such as telephone, data, video, etc. It is the integrated switching technique which will form the basis of the *Broadband Integrated Services Digital*

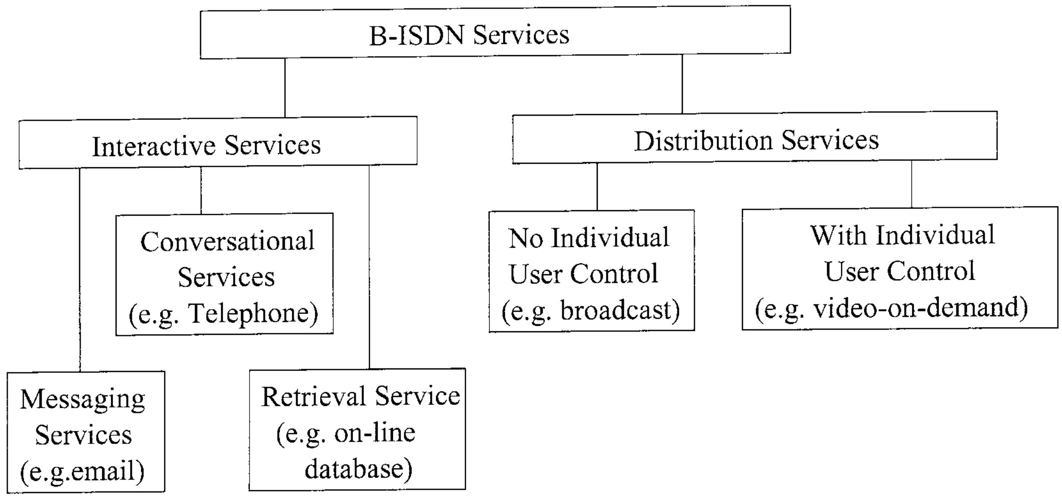

Figure 4.1: The service types offered by B-ISDN

Network (B-ISDN) [36].

B-ISDN is the most modern type of telecommunications network which offers simultaneous switching of different information types, for the carriage of *multimedia* applications. The services offered by B-ISDN can be classified into two categories: *interactive services* and *distribution services*, as shown in Figure 4.1.

B-ISDN is a complete network and management control architecture, and ATM is the switching technique at the heart of B-ISDN. The terminology *transfer mode* means that ATM is a telecommunications transport technique, i.e., a method by which information may be transferred (switched and transported) from one side of a network to the other. In ATM, user information is transmitted between communicating entities using fixed-size packets, referred to as the ATM cells; every cell is 53 bytes, consisting of 48-byte information field and 5-byte header.

4.2.1 Asynchronous and Synchronous

The term *asynchronous* distinguishes the technique from *synchronous* transfer techniques. Synchronous transfer mode (STM) is the method used in high speed transmission systems, such as SONET, *synchronous optical network*. In STM, the line capacity (bit rate) is structured in a strictly regular, and repeating pattern. Thus a 155 Mbit/s line transmission system, for example, is actually composed of a *frame* of 2430 bytes repeated 8000 times per second. There are no gaps between the frames, so the same part of the frame can be expected in the same place every 125 microseconds, i.e., the system is synchronous.

In ATM, frames (called cells) of information are only sent when necessary. Thus, for example, cells are only sent across the network to represent the alphabetic characters which one is typing and only when one types something. In between, nothing is sent. By comparison, STM would convey frames all the time, empty frames at times. Therefore, ATM is potentially the more efficient telecommunications transport technique.

4.2.2 Transfer Modes

The transfer mode defines how information supplied by network users is eventually mapped onto the physical network. We present various transfer modes [6] used in the current networks and discuss the motivation behind the concepts introduced with ATM.

(1) *Circuit Switching*:

Circuit switching is mainly used for telephone networks. A circuit is established between the two entities to exchange information for the complete duration of the connection. Each channel has a fixed bandwidth, for example, 64 Kbps for telephony. A number of channels can be multiplexed onto a link and switching is performed by translating the incoming channel to the outgoing channel number. Before data starts to transmit, a collection of consecutive channels are reserved from sender to receiver by signaling. Once the circuit is established, the traffic flows continuously during the duration of the connection. This transfer mode is called *circuit switching.* Circuit switching minimizes the end-to-end delay of connections.

Circuit switching is not suitable to support all B-ISDN applications, where different applications have significantly varying bandwidth requirements, ranging from 1 Kbps to 140Mbps. Also, it would waste network resources for applications with variable bit rates.

(2) *Message Switching*:

For data applications such as electronic mail, file transfer, and transaction processing, it is more efficient to treat each information unit as a logical entity (referred to as a message) that is transmitted in the network independently of other units. This can be done via adding a header to each message that defines the destination. Then when an intermediate node receives a message, it processes the message (including looking at the header and determining the next node to send towards its destination) and transmits the message. This transfer mode is called *message switching*. Note that no circuit establishment is required in message switching. The major disadvantage of this transfer mode is that it is not suitable for real time or delay-sensitive applications, such as voice, since the delay in such a network is quite unpredictable.

(3) *Packet Switching*:

Packet switching is an attempt to combine the advantages of both circuit switching and message switching. It is essentially the same as message switching except that the size of the information unit transmitted in the network is limited to some maximum value (a few Kbytes), called a *packet*. Each user message may be segmented into packets before they are transmitted. This transfer mode is called *packet switching*. Packet switching reduces the end-to-end delay of user message. The main disadvantage of packet switching is that it requires more overhead (packet headers) to transmit a message, thereby reducing the effective resource utilization in the network.

There are two approaches to handle packet streams in packet switching networks. The first method is called *datagram*, by which, each packet is treated independently and may follow different paths to its destination. The main disadvantage of this approach

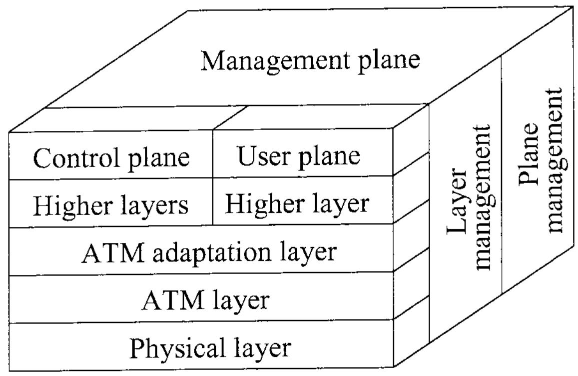

Figure 4.2: The ATM protocol reference model.

is that packets may arrive at the destination out of order and ordering packets to form information units is a processing intensive operation. The second method is called *virtual circuit.* An end-to-end logical connection is established before the transmission starts and all packets of a message follow the same path in the network. This guarantees the sequential delivery of packets to the destination, but requires a call set-up phase. One major problem introduced in packet switching with variable length packets is the complexity of the switching fabrics and the buffer management schemes as the transmission rates increase.

ATM is an attempt to combine all the advantages of existing transfer modes in a unique way. It is a virtual-circuit packet switching scheme in which all packets (called cells) have the same size.

4.2.3 ATM Protocol Reference Model

The ATM protocol model describes how two end systems communicate via ATM switches. As shown in Figure 4.2, the key layers are the ATM adaptation layer, the ATM layer, and the physical layer [35].

- *Physical layer*: This defines the physical transmission types which are suitable for ATM. Its specifications define the electrical, optical and transmission characteristics which should be used, as well as the interface required by the ATM layer.

- *ATM layer*: This is the next higher functionality added to the physical transmission. Its specifications define the 53-byte cell format, shown in Figure 4.3. It multiplexes and demultiplexes cells of different connections. Multiplexing refers to the process of taking several different data streams and consolidating them into a fast-flow data stream. At the other end of the communication path, demultiplexing reverses the process and directs the data back to its appropriate data stream and towards its final destination. It is also responsible for providing the appropriate routing information for cells in the form of VPI/VCI values, which are part of the control information found in a cell's 5-byte header. It translates VCI and/or VPI values at the switches. The VPI/VCI values (local to a specific switch) ensure that the cell will exit the correct switch output port. Finally, the ATM layer implements a flow control mechanism at the universal network interface (UNI) by using the general flow control (GFC) bits in the header.

- *ATM adaptation layer*: This provides services to the higher layers that support classes of service for transported data. Its specifications define how cells may be used to create connections suitable for a wide range of end-users (e.g., *constant bit rate* (CBS) connections, voiceband signal transport, data transport, etc.). It also performs the segmentation and reassemble of data. It takes this data and splits it up to multiple 48-byte cells.

The part of the layered architecture used for end-to-end data transfer is known as the *user plane (U plane)*. The control plane defines higher-level protocols used to support ATM signaling, and the *management plane (M plane)* provides control of an ATM node and consists of two parts: plane management and layer management. The plane management function manages all other planes and the layer management function is responsible for managing each of the ATM layers.

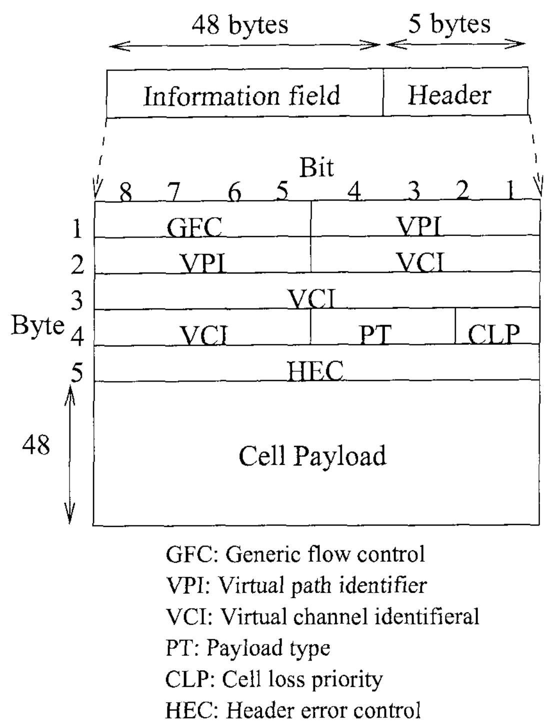

Figure 4.3: The ATM cell format.

4.2.4 Traffic Parameters and Quality of Service

A traffic parameter is a specification of a particular traffic aspect. Three main traffic parameters are *Peak Cell Rate* (PCR), *Sustainable Cell Rate* (SCR), and *Maximum Burst Size* (MBS).

- *Peak cell rate*: This is defined as the inverse of the minimum interarrival time between two consecutive cells. PCR provides an upper bound of the cell rate of a connection.

- *Sustainable cell rate*: This is defined as the long term average cell rate of a connection.

- *Maximum burst size*: This is defined as the maximum length of cell burst transmitted in a connection.

ATM networks provide a Quality of Service (QOS) guarantee to user connections. The parameters defined for QOS include: cell error ratio, cell loss ratio, cell transfer delay, mean cell transfer delay and cell delay variation, etc. The method used by ATM networks to provide QOS is by maintaining a contract between network user and network service provider. When a connection is required by a network user, there is a contract set-up between the network and the user. The user describes the connection in terms of its traffic parameters and QOS requirements. Then the network uses a connection admission control scheme to determine if the connection is to be admitted to the network, while providing the QOS required by the incoming connection and also to maintain the QOS to the other connections that have already set up. Once the connection is granted, it is important for the network to ensure that the connection is abiding by the contract. This is done by policing the connection to monitor the traffic parameters in the contract. One scheme of doing this for bursty connections is by means of the leaky bucket or generic cell rate algorithm [24] [41]. This algorithm allows cells to pass at the mean rate with burstiness constrained. If the new connection violates the traffic parameters in the contract, the violated cells will be dropped by the network.

4.3 Traffic Sources and Statistical Multiplexers

In order to study the performance of a network, the first step is to characterize the various traffic types that the network must support.

(1) **CBR sources**: traffic sources that produce information at fixed rates. Pulse code modulated (PCM) voice, digitized modem traffic, uncompressed video are examples of constant bit rate (CBR) traffic sources.

(2) **VBR sources**: traffic sources that produce information at varying rates. Image and video codec traffic, high definition TV (HDTV) sources, and high-speed data file transfer traffic are example of variable bit rate (VBR) traffic sources.

One characteristic of VBR traffic is its "burstiness," which represents the variation of traffic streams. The burstiness of traffic is defined as the ratio of the peak traffic

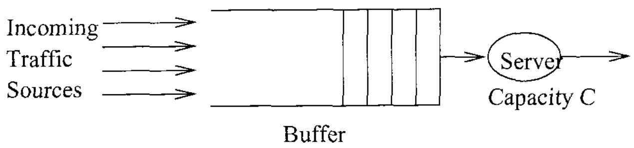

Figure 4.4: The statistical multiplexer in ATM networks.

rate to the average traffic rate. A traffic source is said to be "bursty" when this ratio is much larger than one. Most VBR traffic sources are also correlated, i.e., there exists dependence among the time-varying traffic rates or between the interarrivals of cells. The autocorrelations between traffic interarrival cells have a dramatic impact on network performance [32].

Bursty sources and VBR traffic do not require fixed allocations of bandwidth at their peak rates. The ATM scheme can make efficient use of bandwidth by statistically multiplexing a large amount of bursty and VBR traffic. A *statistical multiplexer* is a multiplexer that combines a number of virtual channels (traffic sources) from separate virtual paths over a single output virtual path such that the transmission bandwidth (capacity) of the output path is not permanently allocated to any given input channel, instead transmitting ("serving") the incoming cells on a first-come-first-serve (FCFS) basis, as shown in Figure 4.4. The information flows from individual sources may vary unpredictably, but the multiplexed traffic may exhibit a more regular, hence more predictable behavior. The saving in transmission bandwidth achieved is called the *statistical multiplexer gain.*

Cells from incoming traffic sources are multiplexed into an output link. Because the aggregate cell arrival rate may temporarily exceed the bandwidth of the output link, a buffer is provided at the output port to hold cells during overflow periods. For a statistical multiplexer to be stable, the average aggregate rate for incoming traffic should be less than output link bandwidth. The number of cells in the buffer fluctuates over time, but it should be empty often enough to achieve a satisfactory performance. A significant issue in the design of ATM networks is the analysis of the buffer occupancy behavior in a statistical multiplexer.

4.4 Multiplexing Markov On-Off Traffic in ATM

The problem of analyzing buffer performance for a single link statistical multiplexer with Markov on-off traffic in ATM networks has been extensively studied. The problem is challenging because of bursty characteristics of the traffic sources. Markov on-off traffic is widely believed to be the traffic, without burstiness constrained, which would cause worst case performance to ATM networks.

The generating function approach is one of the widely used methods to study statistical systems, especially queuing systems and communication networks [31] [30] [9] [44] [53]. This method can be used to derive generating functions for variables of interest for any statistical system modeled by a Markov chain. The method has been applied successfully for many classical queuing systems, especially for Poisson process systems. Unfortunately, for general input process systems, the generating function approach often leads to an infinite number of linear equations with infinite number of unknown functions to be solved, which makes it difficult to obtain closed form formulas on variables of interests, such as the distribution of buffer occupancy of a statistical multiplexer in ATM networks. However, in many practical applications, we are satisfied if we can obtain an explicit formula for the expected values of interests. In the next sections, we will analyze, using the generating function approach, the buffer occupancy for a statistical multiplexer with Markov on-off traffic.

4.4.1 The Model and Generating Functions

Let us consider a discrete time system where the time axis is divided into fixed-length slots. A statistical multiplexer, as shown in Figure 4.4, with infinite size buffer with N incoming traffics and one output channel, which has the fixed-length transmission rate (or service time) which is one slot of time, is considered. It is assumed that the service of a traffic cell can start only at a slot boundary, i.e., the transmission of cells in the output channel is synchronized to the occurrence of slot boundaries. Assume each of N incoming traffic sources for a statistic multiplexer is on-off traffic, described by a two-state continuous Markov chain. Two types of on-off traffic, homogeneous and heterogeneous traffic, will be specifically considered in the next section.

Let A_k denote the number of cells entering the buffer of the multiplexer during the k-th time slot. The r.v.'s $\{A_k\}$ form a discrete Markov chain with the transition probabilities

$$p_{i,j} = \Pr[A_{k+1} = j | A_k = i], \tag{4.1}$$

independent of time k. Let $P_i(z)$ denote the *conditional generating function* of the number of cell arrivals in a time slot which is proceeded by a time slot with i arrivals:

$$P_i(z) = \sum_{j=0}^{\infty} p_{i,j} z^j. \tag{4.2}$$

The stationary probabilities of the Markov chain are denoted by

$$p(j) = \Pr[j \text{ arrivals during a slot in the steady state}]. \tag{4.3}$$

Then, we have the following equilibrium equations:

$$p(j) = \sum_{i=0}^{\infty} p(i) p_{i,j}, \quad (j \geq 0).$$

Define the *generating function* for the stationary probabilities $p(j)$

$$P(z) = \sum_{j=0}^{\infty} p(j) z^j. \tag{4.4}$$

Then,

$$P(z) = \sum_{i=0}^{\infty} p(i) P_i(z).$$

Two classes of traffic sources, namely, homogeneous and heterogeneous sources, will be considered. They will be used as incoming traffic sources to a statistical multiplexer in an ATM network. We will study the buffer occupancy of the statistical multiplexer under such situation.

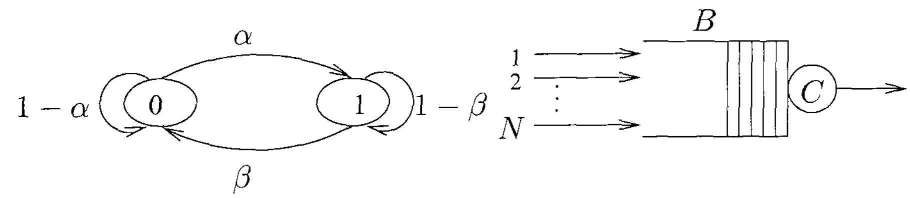

Figure 4.5: The statistical multiplexer with homogeneous on-off traffic.

4.4.2 Homogeneous and Heterogeneous On-Off Traffic

(1) *Homogeneous On-Off Traffic*

Assume there are N incoming traffic sources to a statistical multiplexer, each of which is identical to a two-state Markov on-off source, i.e., each traffic source can be described by a two-state continuous-time Markov chain, as shown in Figure 4.5. When the source is in state 1 ("on" state), it sends a cell per time slot; while in state 0 ("off" state), it sends nothing. The state transition probabilities are α from state 0 to state 1 ($1-\alpha$ from state 0 to itself), and β from state 1 to state 0 ($1-\beta$ from state 1 to itself). These N traffic sources are called *homogeneous* Markov on-off sources.

Assume that there are $A_k = i$ sources in the "on" state in slot k, i.e., there are i cell arrivals in slot k. A_k is a discrete time Markov chain, taking values 0, 1, 2, ..., N. The utilization of each source is $\frac{\alpha}{\alpha+\beta}$, and the total utilization is

$$\rho = \frac{N\alpha}{\alpha+\beta}. \tag{4.5}$$

In order to have a stable system, we need to have $\rho < 1$. As defined in (4.1), $p_{i,j}$ is the probability of having j cell arrivals, or j sources in the "on" state during time slot $k+1$ such that there are i cell arrivals during time slot k. We can compute

$$p_{i,j} = \begin{cases} \sum_{j_1=0}^{i} \binom{i}{j_1}(1-\beta)^{j_1}\beta^{i-j_1}\binom{N-i}{j-j_1}\alpha^{j-j_1}(1-\alpha)^{(N-i)-(j-j_1)} & 0 \le i \le N, 0 \le j \le N \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

(note that, by definition, $\binom{i}{j} = 0$ for any negative i,j, or $j > i$). The conditional generating function for the number of cell arrivals in a time slot which is proceeded by a time slot with i, $0 \le i \le N$, cell arrivals can be computed as

$$\begin{aligned}
P_i(z) &= \sum_{j=0}^{N}\sum_{j_1=0}^{i}\binom{i}{j_1}(1-\beta)^{j_1}\beta^{i-j_1}\binom{N-i}{j-j_1}\alpha^{j-j_1}(1-\alpha)^{(N-i)-(j-j_1)}z^j \\
&= \sum_{j=0}^{N}\sum_{j_1=0}^{i}\binom{i}{j_1}((1-\beta)z)^{j_1}\beta^{i-j_1}\binom{N-i}{j-j_1}(\alpha z)^{j-j_1}(1-\alpha)^{(N-i)-(j-j_1)} \\
&= \sum_{j_1=0}^{i}\binom{i}{j_1}((1-\beta)z)^{j_1}\beta^{i-j_1}\sum_{j=j_1}^{N}\binom{N-i}{j-j_1}(\alpha z)^{j-j_1}(1-\alpha)^{(N-i)-(j-j_1)} \\
&= \sum_{j_1=0}^{i}\binom{i}{j_1}((1-\beta)z)^{j_1}\beta^{i-j_1}[\sum_{j=j_1}^{N-i+j_1}\binom{N-i}{j-j_1}(\alpha z)^{j-j_1}(1-\alpha)^{(N-i)-(j-j_1)} \\
&+ \sum_{j=N-i+j_1+1}^{N}\binom{N-i}{j-j_1}(\alpha z)^{j-j_1}(1-\alpha)^{(N-i)-(j-j_1)}] \\
&= \sum_{j_1=0}^{i}\binom{i}{j_1}((1-\beta)z)^{j_1}\beta^{i-j_1}[((1-\alpha)+\alpha z)^{N-i} \\
&+ \sum_{j=N-i+j_1+1}^{N}\binom{N-i}{j-j_1}(\alpha z)^{j-j_1}(1-\alpha)^{(N-i)-(j-j_1)}].
\end{aligned}$$

Since $N-i \ge j-j_1$ implies $j \le N-i+j_1$. it follows that the second sum in the above square bracket is zero, Therefore,

$$\begin{aligned}
P_i(z) &= \sum_{j_1=0}^{i}\binom{i}{j_1}((1-\beta)z)^{j_1}\beta^{i-j_1}((1-\alpha)+\alpha z)^{N-i} \\
&= (\beta+(1-\beta)z)^i((1-\alpha)+\alpha z)^{N-i}.
\end{aligned}$$

Thus, we have proved the following theorem:

Theorem 4.1 *The conditional generating function of the number of cell arrivals in a time slot which is proceeded by a time slot with i, $0 \le i \le N$, arrivals for N homogeneous on-off sources is given by*

$$P_i(z) = (\beta+(1-\beta)z)^i((1-\alpha)+\alpha z)^{N-i}. \tag{4.7}$$

■

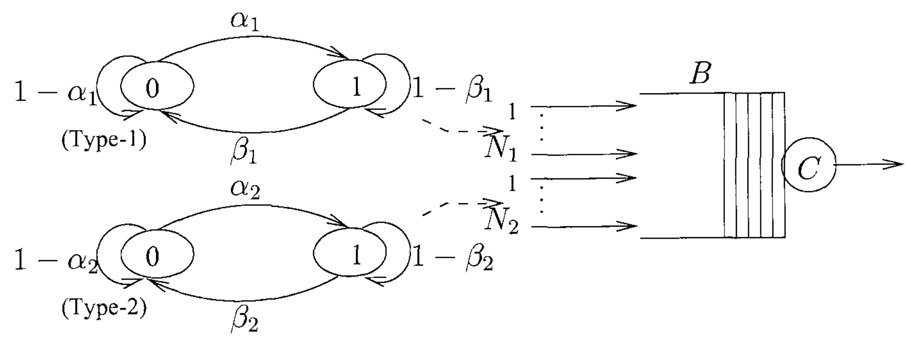

Figure 4.6: The statistical multiplexer with heterogeneous on-off traffic.

(2) *Heterogeneous On-Off Traffic*

Assume there are $N_1 + N_2$ incoming traffic sources to a statistical multiplexer. The traffic consists of two types of homogeneous on-off sources: N_1 homogeneous Markov on-off sources with transition probabilities α_1 and β_1, called type-1 sources, and N_2 homogeneous Markov on-off sources with transition probabilities α_2 and β_2, called type-2 sources, as shown in Figure 4.6. These $N_1 + N_2$ traffic sources are called (two-type) *heterogeneous* Markov on-off sources.

Since A_k denotes the number of cells generated by the heterogeneous sources in time slot k, or the total number of sources in the "on" state from two-type sources, A_k is a discrete time Markov chain, taking values 0, 1, 2, ..., $N_1 + N_2$. The utilizations are $\frac{N_1\alpha_1}{\alpha_1+\beta_1}$ for type-1 sources, $\frac{N_2\alpha_2}{\alpha_2+\beta_2}$ for type-2 sources, and in total

$$\rho = \frac{N_1\alpha_1}{\alpha_1+\beta_1} + \frac{N_2\alpha_2}{\alpha_2+\beta_2}. \tag{4.8}$$

Again we need $\rho < 1$ to have a stable system. Let $p_{i,j}^{(1)}$ and $p_{i,j}^{(2)}$ be the transition probabilities for type-1 sources and type-2 sources, respectively, given by (4.6) for their corresponding parameters; let $P_i^{(1)} = \sum_{j=0}^{\infty} p_{i,j}^{(1)} z^j$ and $P_i^{(2)} = \sum_{j=0}^{\infty} p_{i,j}^{(2)} z^j$ be conditional generating functions for type-1 sources and type-2 sources, respectively. Note that the probability of having i_1 type-1 sources on and i_2 type-2 sources on, given $i = i_1 + i_2$ sources on in total, is

$$\frac{\binom{N_1}{i_1}\binom{N_2}{i_2}}{\binom{N_1+N_2}{i}}.$$

Hence, the transition probabilities for the heterogeneous traffic sources can be expressed by

$$p_{i,j} = \sum_{i_1+i_2=i, j_1+j_2=j} \frac{\binom{N_1}{i_1}\binom{N_2}{i_2}}{\binom{N_1+N_2}{i}} p^{(1)}_{i_1,j_1} p^{(2)}_{i_2,j_2} = \sum_{i_1=0}^{i} \sum_{j_1=0}^{j} \frac{\binom{N_1}{i_1}\binom{N_2}{i_2}}{\binom{N_1+N_2}{i}} p^{(1)}_{i_1,j_1} p^{(2)}_{i-i_1,j-j_1} \tag{4.9}$$

for $0 \le j_1 \le N_1$, $0 \le j_2 \le N_2$. The conditional generating function of the number of cell arrivals in a time slot which is proceeded by a time slot of i cell arrivals can be computed as

$$\begin{aligned} P_i(z) &= \sum_{j=0}^{\infty} \sum_{i_1=0}^{i} \sum_{j_1=0}^{j} \frac{\binom{N_1}{i_1}\binom{N_2}{i_2}}{\binom{N_1+N_2}{i}} p^{(1)}_{i_1,j_1} p^{(2)}_{i-i_1,j-j_1} z^j \\ &= \sum_{i_1=0}^{i} \sum_{j=0}^{\infty} (\sum_{j_1=0}^{j} \frac{\binom{N_1}{i_1}\binom{N_2}{i_2}}{\binom{N_1+N_2}{i}} p^{(1)}_{i_1,j_1} z^{j_1} p^{(2)}_{i-i_1,j-j_1} z^{j-j_1}). \end{aligned}$$

By changing the order of summation, $\sum_{j=0}^{\infty} \sum_{j_1=0}^{j} = \sum_{j_1=0}^{\infty} \sum_{j=j_1}^{\infty}$, we obtain

$$\begin{aligned} P_i(z) &= \sum_{i_1=0}^{i} (\sum_{j_1=0}^{\infty} \sum_{j=j_1}^{\infty} \frac{\binom{N_1}{i_1}\binom{N_2}{i_2}}{\binom{N_1+N_2}{i}} p^{(1)}_{i_1,j_1} z^{j_1} p^{(2)}_{i-i_1,j-j_1} z^{j-j_1}) \\ &= \sum_{i_1=0}^{i} (\sum_{j_1=0}^{\infty} \frac{\binom{N_1}{i_1}\binom{N_2}{i_2}}{\binom{N_1+N_2}{i}} p^{(1)}_{i_1,j_1} z^{j_1} \sum_{r=0}^{\infty} p^{(2)}_{i-i_1,r} z^r) \\ &= \sum_{i_1=0}^{i} (\sum_{j_1=0}^{\infty} \frac{\binom{N_1}{i_1}\binom{N_2}{i_2}}{\binom{N_1+N_2}{i}} p^{(1)}_{i_1,j_1} z^{j_1} P^{(2)}_{i-i_1}(z)) \\ &= \sum_{i_1=0}^{i} \frac{\binom{N_1}{i_1}\binom{N_2}{i_2}}{\binom{N_1+N_2}{i}} P^{(1)}_{i_1}(z) P^{(2)}_{i-i_1}(z). \end{aligned}$$

Therefore, we have proved the following theorem:

Theorem 4.2 *The conditional generating function of the number of cell arrivals in* *a time slot which is proceeded by a time slot with i arrivals for heterogeneous on-off traffic sources, consisting of N_1 type-1 homogeneous on-off sources and N_2 type-2 homogeneous on-off sources, is given by*

$$P_i(z) = \frac{1}{\binom{N_1+N_2}{i}} \sum_{i_1=0}^{i} \binom{N_1}{i_1} \binom{N_2}{i-i_1} P_{i_1}^{(1)}(z) P_{i-i_1}^{(2)}(z), \tag{4.10}$$

where $P_{i_1}^{(1)}(z)$ and $P_{i-i_1}^{(2)}(z)$ are conditional generating functions for the two types homogeneous sources respectively, i.e.,

$$\begin{cases} P_{i_1}^{(1)}(z) &= (\beta_1 + (1-\beta_1)z)^{i_1}((1-\alpha_1)+\alpha_1 z)^{N_1-i_1} \\ P_{i-i_1}^{(2)}(z) &= (\beta_2 + (1-\beta_2)z)^{i-i_1}((1-\alpha_2)+\alpha_2 z)^{N_2-(i-i_1)}. \end{cases} \tag{4.11}$$

■

Theorem 4.2 is a generalization of Theorem 4.1. As a check, let $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, and $N_1 + N_2 = N$, then from Theorem 4.2,

$$P_i(z) = \frac{1}{\binom{N_1+N_2}{i}} \sum_{i_1=0}^{i} \binom{N_1}{i_1} \binom{N_2}{i_2} (\beta + (1-\beta)z)^i ((1-\alpha)+\alpha z)^{N-i}.$$

Applying the well-known Vandermonde convolution identity [8]:

$$\sum_{i_1=0}^{i} \binom{N_1}{i_1} \binom{N_2}{i-i_1} = \binom{N_1+N_2}{i},$$

we then obtain Theorem 4.1.

For heterogeneous on-off sources which consist of more than two types of homogeneous on-off sources, the formulas of conditional generating functions can be naturally generalized from the formula (4.10) in Theorem 4.2, provided changing binomial coefficients to multinomial coefficients and adding more dimensions of summation. We will see that the conditional generating functions are crucial in buffer occupancy analysis in the next section.

4.4.3 Buffer and Expected Buffer Occupancy Analysis

Define X_k as the buffer occupancy just after slot k. The evolution of the buffer occupancy is described by the following equation

$$X_{k+1} = A_{k+1} + (X_k - 1)^+ \tag{4.12}$$

for all k, where x^+ denotes max(0,x). Note that X_k and A_{k+1} are not statistically independent, since both of them depend on A_k. Hence, $\{X_k\}$ is not a Markov chain. However, if we describe the state of the system just after time slot k by (X_k, A_k), we obtain a 2-dimensional Markov chain. The Markovian nature of the arrival process implies that the knowledge of A_k suffices to characterize the probability distribution of A_{k+1}, while A_{k+1} and X_k together determine the probability distribution of X_{k+1}, as clearly shown from (4.12).

(1) **Buffer Occupancy of a Statistical Multiplexer**

The analysis of buffer occupancy using the generating function approach has been extensively studied. The following is a standard analysis summarized from [9].

Define the transition probabilities of the 2-dimensional Markov chain (X_k, A_k) as

$$q_{n,l|i,j} = \Pr[X_{k+1} = n, A_{k+1} = l | X_k = i, A_k = j]. \tag{4.13}$$

Since the number of cells in the buffer just after a time slot can never be less than the number of arrivals during the slot, we have $X_k \geq A_k$ for all k. Thus, the state space of the Markov chain is

$$S = \{(i, j) | i \geq j \geq 0\}.$$

Hence,

$$q_{n,l|i,j} = \Pr[A_{k+1} = l | X_k = i, A_k = j] \cdot \Pr[X_{k+1} = n | A_{k+1} = l, X_k = i, A_k = j]$$

$$= \quad p_{j,l} \cdot \delta(n-l-(i-1)^+),$$

where $\delta(\cdot)$ denotes the Kronecker delta function. Define the steady state probabilities of the 2-dimensional Markov chain (X_k, A_k) as

$$q_{n,l} = \lim_{k\to\infty} \Pr[X_k = n, A_k = l]. \tag{4.14}$$

Then,

$$q_{n,l} = \sum_{(i,j)\in S} q_{n,l|i,j} \cdot q_{i,j} = \sum_{(i,j)\in S} p_{j,l} \cdot \delta(n-l-(i-1)^+) \cdot q_{i,j}$$

for all $(n,l) \in S$. This shows

$$\begin{cases} q_{n,l} &= \sum_{j=0}^{n-l+1} p_{j,l} \cdot q_{n-l+1,j} \quad (n > l) \\ q_{l,l} &= \sum_{i=0}^{1} \sum_{j=0}^{i} p_{j,l} \cdot q_{i,j}. \end{cases} \tag{4.15}$$

Define the partial generating function for the steady state probabilities $q_{n,l}$ as

$$Q_l(z) = \sum_{n=l}^{\infty} q_{n,l} z^n \tag{4.16}$$

for all $l \geq 0$. Applying (4.15), and rearranging terms, we obtain

$$Q_l(z) = z^{l-1}[(1 - P'(1))p_{0,l}(z-1) + \sum_{j=0}^{\infty} p_{j,l} Q_j(z)] \tag{4.17}$$

for all $l \geq 0$. It can be shown [9] that $P'(1)$ is the utilization factor of the statistical multiplexer. Finally, let us define $X(Z)$ to be the generating function for the steady state probability of buffer occupancy, i.e.,

$$X(z) = \sum_{n=0}^{\infty} \Pr[X = n] z^n. \tag{4.18}$$

Then it is easy to check that

$$X(z) = \sum_{l=0}^{\infty} Q_l(z). \tag{4.19}$$

In general, an explicit solution for $Q_l(z)$ is difficult to obtain; it generally involves solving an infinite number of linear equations in (4.17). Hence, a closed form formula for the generating function of buffer occupancy $X(z)$ in (4.19), and hence the buffer occupancy distribution for the statistical multiplexer, is in general difficult to obtain.

(2) **Expected Buffer Occupancy of a Statistical Multiplexer**

The explicit solution for the buffer occupancy distribution is generally difficult to derive. However, we can compute the expected buffer occupancy, which is also an important parameter of great interest in practical network design.

Denote $\bar{Q}$ as the expected buffer occupancy. Then, by (4.19),

$$\bar{Q} = \frac{dX(z)}{dz}|_{z=1} = \sum_{l=0}^{\infty} Q_l'(1). \tag{4.20}$$

From (4.17), we compute

$$\begin{aligned} Q_l'(1) &= (1-P'(1))p_{0,l} + \sum_{j=0}^{\infty} p_{j,l} Q_j'(1) + (l-1)\sum_{j=0}^{\infty} p_{j,l} Q_j(1) \\ &= (1-P'(1))p_{0,l} + \sum_{j=0}^{\infty} p_{j,l} Q_j'(1) + (l-1)p(l) \end{aligned} \tag{4.21}$$

and

$$Q_l''(1) = \sum_{j=0}^{\infty} p_{j,l} Q_j''(1) + 2(l-1)Q_l'(1) - l(l-1)p(l). \tag{4.22}$$

Summing (4.22) over all $l \geq 0$, we obtain

$$X''(1) = X''(1) + 2\sum_{l=0}^{\infty} Q_l'(1)l - 2\bar{Q} - P''(1),$$

or,

$$\bar{Q} = r_1 - \frac{1}{2}P''(1),$$

where r_k is defined as the k-th moments of $Q_l'(1)$, i.e., for $k \geq 0$,

$$r_k = \sum_{l=0}^{\infty} Q_l'(1) l^k. \tag{4.23}$$

Now define the k-th moments of $p_{j,l}$ (defined in (4.1)) and $p(l)$ (defined in (4.3)) as

$$M_k(j) = \sum_{l=0}^{\infty} p_{j,l} l^k \tag{4.24}$$

and

$$M_k = \sum_{l=0}^{\infty} p(l) l^k. \tag{4.25}$$

Note that $M_0 = 1$. It is easy to check that

$$M_k = \sum_{j=0}^{\infty} p(j) M_k(j). \tag{4.26}$$

Combining (4.23) and (4.21), we obtain

$$r_k = (1 - P'(1)) M_k(0) + \sum_{j=0}^{\infty} Q_j'(1) M_k(j) + (M_{k+1} - M_k). \tag{4.27}$$

From (4.4) and (4.24), we obtain $P''(1) = M_2 - M_1$, and thus,

$$\bar{Q} = r_1 - \frac{1}{2}(M_2 - M_1). \tag{4.28}$$

From (4.28), we need to compute M_1, M_2, and r_1 to obtain $\bar{Q}$. By (4.26), we need to know $M_k(j)$ to obtain M_k, particularly to obtain M_1 and M_2. But note that from definitions (4.2) and (4.24), we can compute $M_k(j)$ from the conditional generating function $P_i(z)$ (which are known). In fact, by (4.2), we have

$$P_j^{(k)}(1) = \sum_{l=0}^{\infty} p_{j,l}\, l(l-1)\cdots(l-k+1).$$

Combining with (4.24), all $M_k(j)$ can be iteratively derived. Here are the first several $M_k(j)$:

$$\begin{cases} M_1(j) &= P_j'(1) \\ M_2(j) &= P_j''(1) + P_j'(1) \\ M_3(j) &= P_j'''(1) + 3P_j''(1) + P_j'(1) \\ \cdots\cdots \end{cases} \tag{4.29}$$

Computing r_1 in (4.28) is a harder problem. We first need to make some assumptions on $M_k(j)$. With the following assumptions, we can design a heuristic algorithm to compute r_1, and hence $\bar{Q}$. In particular, we need

For the functions $M_k(j)$ defined in (4.24), there exists an integer $\Omega > 0$ such that

Assumption 1 *$M_k(j)$ is a polynomial in the variable j with degree at most Ω,*

$$M_k(j) = \sum_{n=0}^{\Omega} m_k(n) j^n, \tag{4.30}$$

for $1 \leq k \leq \Omega$, and

Assumption 2 *$M_{\Omega+1}(j)$ is a polynomial in the variable j with degree at most $\Omega + 1$,*

$$M_{\Omega+1}(j) = \sum_{n=0}^{\Omega+1} m_{\Omega+1}(n) j^n, \tag{4.31}$$

where all the coefficients $m_k(n)$, $1 \leq k \leq \Omega + 1$, $0 \leq n \leq \Omega + 1$, are known.

With the above assumptions, applying (4.26) (4.30) (4.31), we obtain

$$\begin{cases} M_k &= \sum_{n=0}^{\Omega} m_k(n) M_n \quad \text{for } 1 \leq k \leq \Omega \\ M_{\Omega+1} &= \sum_{n=0}^{\Omega+1} m_{\Omega+1}(n) M_n. \end{cases} \tag{4.32}$$

From the $\Omega + 1$ linear equations in (4.32), we can solve all M_k, $1 \leq k \leq \Omega + 1$. Therefore, under the assumptions, all M_k become known.

From (4.27), applying (4.30) with the facts that $P'(1) = M_1$, $r_0 = \bar{Q}$, and $M_k(0) =$

$m_k(0)$, the unknown quantities r_k can be computed as

$$r_k = m_k(0)(1 - M_1 + \bar{Q}) + (M_{k+1} - M_k) + \sum_{n=1}^{\Omega} m_k(n) r_n \qquad (4.33)$$

for $1 \le k \le \Omega$. From the Ω equations in (4.33), r_k can be expressed in terms of $\bar{Q}$, as well as the quantities M_k and $m_k(n)$ (which are all known). This is particularly true for r_1. Thus we can derive r_1, and plug it into (4.28), $\bar{Q}$ is then finally determined.

The above computation can be presented in the following heuristic procedure:

(i) Start with $P_j(z)$, as given in (4.7) or (4.10).
(ii) Compute $M_k(j)$ from (4.29); check if assumptions (4.30) and (4.31) are satisfied; obtain Ω and $m_k(n)$.
(iii) Compute M_k, $1 \le k \le \Omega + 1$, from (4.32).
(iv) Compute r_1 from (4.33).
(v) Compute $\bar{Q}$ from the result of (iv) and (4.28).

Theoretically, this heuristic approach is applicable if the number of cell arrivals per slot is limited to a finite number of different values, which results in a finite integer Ω that satisfies the assumptions. Practically, the complexity of computation (or the "feasibility") for applying this approach depends on Ω. In general, the approach could be applied to those traffic sources which result in relatively small Ω.

Example: Now let us consider a special case: suppose that a traffic has $\Omega = 1$, i.e.,

$$\begin{cases} M_1(j) = P_j'(1) = m_1(0) + m_1(1)j \\ M_2(j) = P_j''(1) + P_j'(1) = m_2(0) + m_2(1)j + m_2(2)j^2. \end{cases} \qquad (4.34)$$

Since

$$\begin{cases} M_1 = m_1(0) + m_1(1)M_1 \\ M_2 = m_2(0) + m_2(1)M_1 + m_2(2)M_2, \end{cases}$$

it follows

$$\begin{cases} M_1 &= \frac{m_1(0)}{1-m_1(1)} \\ M_2 &= \frac{m_1(0)m_2(1)+(1-m_1(1))m_2(0)}{(1-m_1(1))(1-m_2(2))}. \end{cases} \tag{4.35}$$

From

$$\begin{cases} r_1 &= m_1(0)(1-M_1+\bar{Q})+(M_2-M_1)+m_1(1)r_1 \\ \bar{Q} &= r_1-\frac{1}{2}(M_2-M_1), \end{cases}$$

we obtain

$$\bar{Q}=\frac{m_1(0)(1-M_1)+\frac{1}{2}(1+m_1(1))(M_2-M_1)}{1-m_1(0)-m_1(1)}. \tag{4.36}$$

4.4.4 The Expected Buffer Occupancy for Homogeneous Traffic

We now consider a statistical multiplexer with N homogeneous on-off traffic sources, as introduced in Section 4.4.2. We can compute the expected buffer occupancy for the statistical multiplexer, by applying the approach developed in the previous section.

First, by Theorem 4.1, we can compute

$$\begin{cases} P_j'(1) &= N\alpha+(1-\alpha-\beta)j \\ P_j''(1) &= N(N-1)\alpha^2+(2N\alpha-(1+\alpha-\beta))(1-\alpha-\beta)j \\ & \quad +(1-\alpha-\beta)^2j^2. \end{cases} \tag{4.37}$$

Hence, by (4.34), we obtain

$$\begin{cases} M_1(j) &= N\alpha+(1-\alpha-\beta)j \\ M_2(j) &= N\alpha(1+(N-1)\alpha)+(2N\alpha-(\alpha-\beta))(1-\alpha-\beta)j+(1-\alpha-\beta)^2j^2. \end{cases}$$

Clearly, the assumptions (4.30) and (4.31) in the previous section are satisfied with $\Omega=1$, and

$$\begin{cases} m_1(0) &= N\alpha \\ m_1(1) &= 1-\alpha-\beta, \end{cases}$$

$$\begin{cases} m_2(0) &= N\alpha(1+(N-1)\alpha) \\ m_2(1) &= (2N\alpha-(\alpha-\beta))(1-\alpha-\beta) \\ m_2(2) &= (1-\alpha-\beta)^2. \end{cases}$$

Applying (4.35), we obtain

$$\begin{cases} M_1 &= \frac{N\alpha}{\alpha+\beta} \\ M_2 &= \frac{N\alpha(1+(N-1)\alpha)(\alpha+\beta)+N\alpha((2N-1)\alpha+\beta)(1-\alpha-\beta)}{(\alpha+\beta)(1-(1-\alpha-\beta)^2)}. \end{cases}$$

Finally, by (4.36), we get the expected buffer occupancy

$$\bar{Q} = \frac{N\alpha[2\beta^2+(2(N-1)-(3N-5)\beta)\alpha-3(N-1)\alpha^2]}{2(\alpha+\beta)^2(\beta-(N-1)\alpha)}.$$

Therefore, we have shown the following theorem:

Theorem 4.3 *The expected buffer occupancy in a statistical multiplexer for N homogeneous on-off traffic sources is given by*

$$\bar{Q} = \frac{N\alpha[2\beta^2+(2(N-1)-(3N-5)\beta)\alpha-3(N-1)\alpha^2]}{2(\alpha+\beta)^2(\beta-(N-1)\alpha)}, \tag{4.38}$$

where α and β are transition probabilities for the on-off traffic. ■

Since the expected number of cell arrivals per time slot, or the utilization in this unit-slot service rate case, for the homogeneous on-off sources is $\rho = \frac{N\alpha}{\alpha+\beta}$, we can rewrite

$$\bar{Q} = \frac{\rho}{1-\rho}(1-\frac{3N-1}{2N}\rho+\frac{(N-1)(N-\rho)}{N^2}\rho\cdot\frac{1}{\beta}). \tag{4.39}$$

Clearly, $\bar{Q}$ depends not only on ρ, but also on β (or α). In fact, when ρ is fixed, $\bar{Q}$ increases as β decreases, and even $\bar{Q} \to \infty$ as $\beta \to 0$. Note that $1/\beta$ is the expected burst-length of a single two-state on-off source. This result thus shows that $\bar{Q}$ increases as the expected burst-length increases, as expected. Figure 4.7 shows an example of the expected buffer occupancy for homogeneous traffic with $N = 10$ on-off sources. $\bar{Q}$ increases as ρ increases, and goes to infinity when the traffic utilization ρ approaches to one. The plot also shows that when β increases, i.e., the burstiness of each on-off source decreases, $\bar{Q}$ also decreases. Therefore, the control of traffic burstiness is very significant in the design of ATM networks.

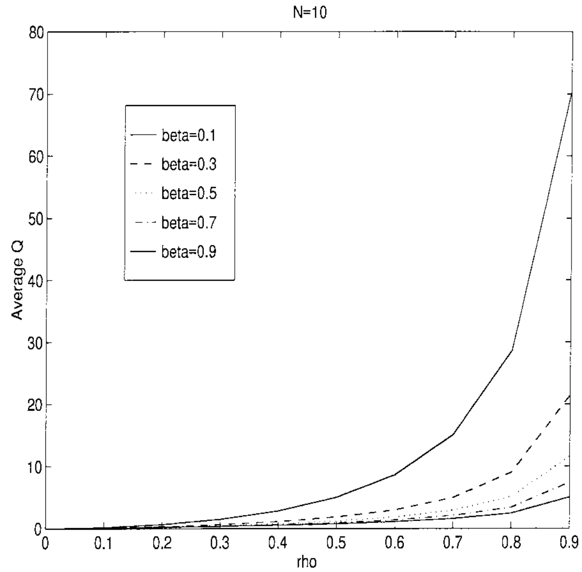

Figure 4.7: The expected buffer occupancy with homogeneous on-off traffic.

4.4.5 The Expected Buffer Occupancy for Heterogeneous Traffic

The expected buffer occupancy for heterogeneous on-off traffic which consists of two-type homogeneous on-off traffic source, as introduced in Section 4.4.2, can be computed in exactly the same way for homogeneous on-off traffic, as in the previous section.

By Theorem 4.2, we can compute

$$\begin{cases} P_j'(1) &= \frac{1}{\binom{N_1+N_2}{j}} \sum_{j_1=0}^{j} \binom{N_1}{j_1}\binom{N_2}{j-j_1}[P_{j_1}'^{(1)}(1) + P_{j-j_1}'^{(2)}(1)] \\ P_j''(1) &= \frac{1}{\binom{N_1+N_2}{j}} \sum_{j_1=0}^{j} \binom{N_1}{j_1}\binom{N_2}{j-j_1}[P_{j_1}''^{(1)}(1) + 2P_{j_1}'^{(1)}(1)P_{j-j_1}'^{(2)}(1) + P_{j-j_1}''^{(2)}(1)], \end{cases} \tag{4.40}$$

where $P_j^{(1)}(z)$ and $P_j^{(2)}(z)$ are conditional generating functions for homogeneous on-off traffic with parameters (α_1, β_1) and (α_2, β_2), as given in (4.11). Hence, we can apply (4.37) to compute (4.40). After some algebraic manipulation, we can show this is also the case that satisfies the assumptions (4.30) and (4.31) with $\Omega = 1$. Following the procedure as what we did for homogeneous traffic, the expected buffer occupancy can be computed. However, the computation involves much elementary algebra manipulation. Unlike the simple formula (4.38) we obtained for homogeneous traffic, the general formula of $\bar{Q}$ for heterogeneous traffic is too complicated to write. Nevertheless, following the procedure, we can always compute $\bar{Q}$ for any numerical example.

To illustrate this, let us consider a simple example: a two-type heterogeneous traffic with $N_1 = 1$ type-1 on-off source with parameters (α_1, β_1), and $N_2 = 2$ type-2 homogeneous on-off sources with parameters (α_2, β_2). Note that the utilization factor for this example is

$$\rho = \frac{\alpha_1}{\alpha_1 + \beta_1} + \frac{2\alpha_2}{\alpha_2 + \beta_2}.$$

Following the procedure described in previous section, we can compute

$$\begin{cases} m_1(0) &= \alpha_1 + 2\alpha_2 \\ m_1(1) &= \frac{1}{3}((1 - \alpha_1 - \beta_1) + 2(1 - \alpha_2 - \beta_2)) \end{cases}$$

$$\begin{cases} m_2(0) &= \alpha_1 + 2\alpha_2 + 4\alpha_1\alpha_2 + 2\alpha_2^2 \\ m_2(1) &= \frac{1}{3}[(6\alpha_2 + 2\beta_2 - 1)(1 - \alpha_1 - \beta_1) + (4\alpha_1 + 5\alpha_2 + \beta_2 + 1)(1 - \alpha_2 - \beta_2)] \\ m_2(2) &= \frac{1}{3}[2(1 - \alpha_1 - \beta_1)(1 - \alpha_2 - \beta_2) + (1 - \alpha_2 - \beta_2)^2]. \end{cases}$$

Hence, M_1 and M_2 can be computed by (4.35), and then $\bar{Q}$ can be computed by (4.36). The final formula for $\bar{Q}$ is too long to write. We plot some curves for the example in Figure 4.8. In the example, we assume $\beta_2 = 2\beta_1$, and $\alpha_1 = 2\alpha_2$. The figure shows $\bar{Q}$ versus ρ for different values of β_2. The curves are similar to those for homogeneous case. Again, it shows that $\bar{Q}$ increases as ρ increases, and $\bar{Q}$ can be unbounded even when ρ is not close to one, as long as β_2 is small enough. Hence, the values of β_2 and

β_1 play a very important role in the performance of the buffer. Burstiness control is indispensable in ATM networks. In the next chapter, we will introduce a special class of burstiness controlled traffic, called *Periodic Interchangeable* Traffic, and study the performance of a statistical multiplexer with such traffic.

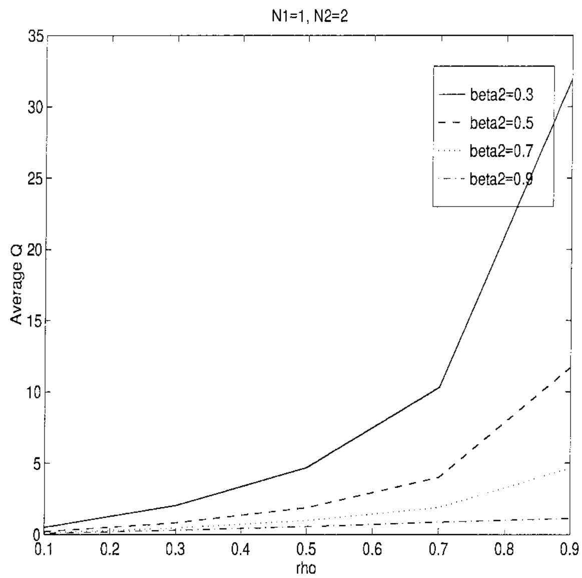

Figure 4.8: The expected buffer occupancy with heterogeneous on-off traffic.

4.5 Concluding Remarks

In this chapter, we first gave a brief introduction to ATM and B-ISDN. We focused on the performance analysis of buffer occupancy in a statistical multiplexer, which is one of the most important issues in ATM technology. We then especially focus on the homogeneous and heterogeneous Markov on-off traffic, and derived closed form formulas for certain conditional generating functions for cells generated by homogeneous and heterogeneous on-off traffic. We applied a generating function approach to study the performance of the buffer occupancy for a statistical multiplexer, and developed a heuristic procedure for computing the expected buffer occupancy, which allowed us to compute the expected buffer occupancy for homogeneous and heterogeneous on-off traffic. Simple closed form formulas for expected buffer occupancy are derived for the case of homogeneous on-off traffic. The analysis and some numerical results showed that the expected buffer occupancy not only depends on the incoming traffic intensity (utilization), but also largely on the burstiness of cells for incoming traffic. The expected buffer occupancy becomes unbounded with large enough traffic burstiness, even with small traffic intensity, which showed that burstiness control of traffic was very critical in ATM networks. Furthermore, since an important class of traffic in ATM networks can be modeled by Markov on-off traffic, and since numerical and asymptotic analysis suggests that on-off traffic is the worst case traffic, the simple result obtained here for on-off traffic is significant for dimensioning buffer size for a statistical multiplexer, as well as for the issues such as call admission control and bandwidth allocation, in designing ATM networks under the scenario of worst case performance.

Chapter 5 Periodic Interchangeable Traffic in ATM Networks

5.1 Introduction

An important concept in ATM networks is the efficient sharing of link capacities through statistical multiplexing of variable-rate traffic sources. Buffering is provided at a statistical multiplexer to manage traffic fluctuations when the instantaneous rate of the aggregate incoming traffic exceeds the capacity of the outgoing link. To provide quality-of-service (QOS) guarantees to customers in ATM networks, it is essential to estimate the buffer performance, which is especially important for designing higher level network management issues such as call admission, buffer allocation, and traffic control.

As we showed in the previous chapter, expected queue occupancy for a statistical multiplexer with Markov on-off sources, which are not burst-constrained, could be unbounded, even when the utilization factor (traffic intensity) is strictly less than one. Hence, traffic sources must be regulated entering an ATM network, and so performance analysis for a statistical multiplexer for burst-constrained traffic is very important.

A widely applied scheme for obtaining burst-constrained traffic is the leaky bucket, or (σ, ρ) regulator [14] [41]. In this chapter, we introduce a special type of burst-constrained traffic, namely, *periodic interchangeable* (PI) traffic, in which each traffic source is periodic with the same period. We will introduce a well-known combinatorial theorem, the classic *Ballot Theorem*, and show how to apply generalized Ballot theorems to analyze the buffer occupancy in a statistical multiplexer for such traffic sources. We then specifically consider the uniform PI (UPI) traffic, in which each traffic source generates cells which are uniformly distributed within a period. This traffic model is a generalization of the periodic traffic discussed in [15] [22] [28] [38], in which each source generates a single cell within a period. This type of special UPI traffic is called unit UPI (UUPI) traffic here. The periodic traffic model is neither stochastic, since all the cells are generated periodically, nor deterministic, since cells are generated in a random phase within a period. So we consider the traffic model *semi-deterministic*. We hope that our study can shed light on the worst case performance of a statistical multiplexer with burst-constrained traffic in ATM networks.

The analysis in this chapter is based on a study of generalized ballot theorems in [22], which originated from the classic *Ballot Theorems*. The rest of the chapter is organized as follows. In Section 5.2, we introduce the classic ballot theorem and some generalizations from Takàcs [46] [47]. We also introduce a special type of random variables, (cyclically) interchangeable random variables. In Section 5.3, we introduce a traffic model for a special class traffic sources, namely, the periodic interchangeable (PI) traffic. In Section 5.4, we show how to apply the generalized Ballot theorems to compute the distribution of buffer occupancy and expected buffer occupancy for a statistical multiplexer with PI traffic. We also consider a special kind of PI traffic, uniform PI (UPI) traffic, and compute the survivor functions and the expected buffer occupancy. Specifically, we will derive simple asymptotic formulas for the survivor function and expected queue occupancy for unit UPI (UUPI) traffic. In Section 5.5, we will show that, asymptotically, the UUPI traffic is the worst among all UPI traffic when the number of sources is sufficiently large. Numerical results are given, implying that UUPI traffic might be the worst traffic among all UPI traffic, even when the number of traffic sources is small. Finally, in Section 5.6, we make our conclusions.

5.2 The Ballot Theorems and Interchangeable Random Variables

The following theorem is originated in 1887, usually called *Bertrand's classic Ballot Theorem* [46] [47]:

Ballot Theorem. *If in a ballot candidate A scores a votes and candidate B scores b votes and if $a \geq b$, then the probability that throughout the counting the number of votes registered for A is always greater than the number of votes registered for B is given by*

$$P(a, b) = \frac{a-b}{a+b}, \tag{5.1}$$

provided that all the possible voting records are equally likely. ■

The Ballot Theorem can be proved either by mathematical induction [46] or by a clever method called the *reflection principle* [18]. The theorem is remarkable in that, under certain condition ("equally likely voting records"), the final outcome ("numbers a and b") of a certain process can completely determine the likelihood of an event which is a collection of sample paths that run through the whole process. The following interesting theorem is equivalent to the classic Ballot Theorem [46]:

Theorem 5.1 *Let a_i, $1 \leq i \leq M$, be nonnegative integers with $\sum_{i=1}^{M} a_i = N < M$. Then among the M cyclic permutations of $(a_1, a_2, \cdots, a_M)$, there are exactly $M - N$ for which the sum of the first r elements never reaches r for all $r \in [1, M]$.* ■

There is a class of important discrete time random processes, in which the random variables at different time moments within a time period are not independent, but have joint distribution invariant to permutations of the random variables. More precisely, we have the following definition:

Definition. Random variables $(X_1, X_2, \cdots, X_M)$ are *cyclically interchangeable* if $(X_1, X_2, \cdots, X_M)$ take the value $(x_1, x_2, \cdots, x_M)$, then they have the same probability for each of *cyclic* permutations of $(x_1, x_2, \cdots, x_M)$. Similarly, they are called

	A_1	A_2	A_3	A_4	A_5
1	1	1	0	1	0
2	1	0	1	0	1
3	0	1	0	1	1
4	1	0	1	1	0
5	0	1	1	0	1

Table 5.1: An example of cyclically interchangeable random variables

interchangeable if they have the same probability for each of *all* permutations.

Takàcs [46] made a further interesting generalization of the classic Ballot Theorem, which applies to cyclically interchangeable random variables.

Theorem 5.2 *If* $(A_1, A_2, \cdots, A_M)$ *are cyclically interchangeable random variables such that* $\sum_{i=1}^{M} A_i = N$*, then*

$$\Pr\{\sum_{j=1}^{i} A_j < i, \forall i \in [1, M]\} = \begin{cases} 1 - \frac{N}{M} & \textit{if } N < M \\ 0 & \textit{otherwise.} \end{cases} \tag{5.2}$$

■

Example: Consider cyclically interchangeable random variables $(A_1, A_2, A_3, A_4, A_5)$, $N = 3$ and $M = 5$, such that the five sequences in Table 5.1 occur with the same probability 1/5. It is easy to check that only sequences 3 and 5 satisfy the condition $\sum_{j=1}^{i} A_j < i, \forall i \in [1, 5]$. Hence, $\Pr\{\sum_{j=1}^{i} A_j < i, \forall i \in [1, 5]\} = 2/5$, which is exactly the number computed from Theorem 5.2.

With the generalized theorem, Takàcs [47] was able to compute the distribution of busy period in a queue and the distribution of the number of incident vertices for a given subset of vertices in a random graph. The following theorem follows from Takàcs' generalized theorem [22]:

Theorem 5.3 *If $u \geq 0$ is an integer and $(A_1, A_2, \cdots, A_M)$ are interchangeable random variables taking on nonnegative values such that $\sum_{i=1}^{M} A_i = N < M + u$, then*

$$\Pr\{\sum_{j=1}^{i} A_j < i + u, \forall i \in [1, M]\} = 1 - \sum_{r=1}^{N-u} \frac{M - N + u}{M - r} \Pr\{\sum_{j=1}^{r} A_j = r + u\}. \quad (5.3)$$

When $u = 0$, we then simply have

$$\Pr\{\sum_{j=1}^{i} A_j < i, \forall i \in [1, M]\} = 1 - \frac{N}{M}. \quad (5.4)$$

■

Theorem 5.3 provides a method for computing the survivor functions in a queue system when the arrival process to the queue is interchangeable, as defined in the next section. In fact, the theorem can be used to specify the partial-sum fluctuation in the context of random walks. We will show how to apply the theorem to analyze the buffer occupancy of a statistical multiplexer with certain traffic sources, namely, the *periodic interchangeable* (PI) traffic sources.

5.3 Periodic Interchangeable Traffic

An ATM traffic source can be considered as a discrete-time random process. Time is slotted, and during each time slot, a cell is either generated or not. Hence, during a period of time, a traffic source can be represented by a discrete-time random process which consists of a sequence of random variables taking values zero or one only. If the random variables are independently identically distributed (iid), then the traffic is a sequence of Bernoulli random variables, and the buffer occupancy of a statistical multiplexer with such a traffic source can be easily analyzed [26]. It is a more challenging problem when the random variables are not independent, such as the Markov on-off traffic studied in Chapter 4.

We consider a special class of traffic sources, namely, the *Periodic Interchangeable* (PI) traffic sources. A PI source is one such that all the cells are generated periodically with certain period, and within a period, the sequence of random variables are interchangeable, and the sum of the random variables is not random, but deterministic. When the period is M and the sum is N, the source is called a PI source with parameters (M, N). It is easy to show that the superposition of any number of PI traffic sources is also a PI source.

The reason we study PI traffic is due to its burst constraint properties. The burstiness of a PI source with parameters (M, N) is no more than N during a period of M time slots. Even though the traffic model may look artificial at first, it does capture the fundamental issue, that is, the cell burst within a period of time is always constrained to that deterministic number. The buffer behavior of a statistical multiplexer with such PI traffic may therefore shed light on the performance of statistical multiplexers for more general burst-constrained traffic. The property of interchangeability actually implies the "random phase" for each traffic source presented to a statistical multiplexer. Furthermore, the simplicity of the traffic model allows us to apply the combinatorial theorems introduced in the previous section to do the exact analysis. In fact, we will derive closed form formulas for the survivor functions and expected buffer occupancy in this case, and also some simple asymptotic formulas, which can serve as rules of thumb for the design of statistical multiplexers in ATM networks.

5.4 Multiplexing Periodic Interchangeable Traffic

In this section, we consider a statistical multiplexer with PI traffic sources. We will show how to apply the generalized Ballot Theorem to analyze the buffer occupancy of a statistical multiplexer with such traffic sources. We then consider some special type of PI traffic, and derive simpler closed form formulas which characterize the performance of statistical multiplexers.

5.4.1 The Queueing Model and Buffer Occupancy

Consider a statistical multiplexer with K input traffic sources which generate cells independently. Time is slotted, and it takes one slot for the transmission line (the server) of the statistical multiplexer to transmit a cell, i.e., the capacity of the system server is one. The cells arriving to the multiplexer are served in a first-come-first-served (FCFS) manner.

We assume each of the K sources is a PI traffic source such that source i, $1 \leq i \leq K$, generates n_i cells according to certain distribution among a period of M slots, and produces cells periodically with the period M. That is, source i is a PI source with parameters (M, n_i). Define

$$N = \sum_{i=1}^{K} n_i, \tag{5.5}$$

the number of total cells arriving to the multiplexer during a period of M slots. Note that the number N is fixed, not a random variable. Clearly, if $N > M$, the total input traffic exceeds the multiplexer capacity and congestion will occur. For stability of the queueing system, we need $N \leq M$, so if we define the utilization factor (or traffic intensity) as

$$\rho = \frac{N}{M}, \tag{5.6}$$

then, $\rho \leq 1$. In our analysis of the queueing system, we assume that the departures take place at the beginning of slots. Let us define

$$\begin{cases} Q_k &= \text{buffer occupancy at the end of the } k\text{th slot} \\ A_k &= \text{number of arriving cells in the } k\text{th slot.} \end{cases}$$

Then, by the PI assumption for each traffic source, the accumulated traffic is PI traffic with parameters (M, N), and the random variables $A_1, A_2, \cdots, A_M$ are interchangeable. Clearly, we have

$$\sum_{i=1}^{M} A_i = \sum_{i=1}^{K} n_i = N. \tag{5.7}$$

Now we want to compute the buffer occupancy for the statistical multiplexer under the situation. The evolution of the buffer occupancy is described by

$$Q_k = \max(Q_{k-1} - 1, 0) + A_k \tag{5.8}$$

for $k > 0$. Since the arriving pattern is periodic, hence $A_k = A_{k \bmod M}$. Let us assume $Q_0 = 0$, then by iterating on k in (5.8), we obtain

$$\begin{aligned} Q_k &= \max(A_k, A_k + Q_{k-1} - 1) \\ &= \max(A_k, A_k + A_{k-1} - 1, A_k + A_{k-1} + Q_{k-2} - 2) \\ &= \cdots \\ &= \max_{0 \le i < k}\left(\sum_{j=k-i}^{k} A_j - i\right) \end{aligned} \tag{5.9}$$

for $k \ge 0$. Since the arriving pattern is periodic and $K < M$, for $k \ge M$ a maximum must occur for $i < M$, hence,

$$Q_k = \max_{0 \le i < M}\left(\sum_{j=k-i}^{k} A_j - i\right)$$

for $k \ge M$. Therefore, Q_k is also periodic with period M. Without loss of generality, we can focus on the buffer occupancy at time M, denoted simply by Q, which can be expressed as

$$Q = \max_{0 \le i < M}\left(\sum_{j=M-i}^{M} A_j - i\right). \tag{5.10}$$

Hence, the survivor function can be computed as

$$\begin{aligned} \Pr\{Q > q\} &= 1 - \Pr\{Q \le q\} \\ &= 1 - \Pr\{\max_{0 \le i < M}\left(\sum_{j=M-i}^{M} A_j - i\right) \le q\} \\ &= 1 - \Pr\{\max_{1 \le i \le M}\left(\sum_{j=M-(i-1)}^{M} A_j - i + 1\right) \le q\} \end{aligned}$$

$$= \quad 1 - \Pr\{\max_{1\le i\le M}(\sum_{j=1}^{i} A_{M-i+j} - i) < q\}.$$

Recall that

$$\sum_{i=1}^{M} A_i = N < M + q$$

for $q \geq 0$. Applying Theorem 5.3, we obtain the following theorem:

Theorem 5.4 *The survivor function for the buffer occupancy in a statistical multiplexer, with K incoming PI traffic sources with parameters (M, N), is given by*

$$\Pr\{Q > q\} = \sum_{r=1}^{N-q} \frac{M - N + q}{M - r} \Pr\{\sum_{j=1}^{r} A_j = q + r\}. \tag{5.11}$$

Specifically, when $q = 0$, $\Pr\{Q > 0\} = \frac{N}{M} = \rho$. ■

Assume $K \geq 1$. Since $A_j \leq K$, $q + r \leq Kr$, hence $r \geq \lceil \frac{q}{K-1} \rceil$. To obtain nonzero survivor function, since $N - q \geq r$, it follows $q \leq \lfloor \frac{K-1}{K} N \rfloor$. Therefore, the maximum buffer occupancy is

$$q_{\max} = \lfloor \frac{K-1}{K} N \rfloor \leq N. \tag{5.12}$$

Define $\bar{Q}$ to be the expected buffer occupancy of the statistical multiplexer (excluding the cell in transmission). Note that the utilization factor ρ is the expected number of cells in transmission (i.e., the expected number of "customers" in service). Then

$$\bar{Q} = \sum_{q=0}^{q_{\max}} \Pr\{Q_M > q\} - \rho = \sum_{q=1}^{q_{\max}} \Pr\{Q_M > q\}.$$

Hence, we have the following theorem for the expected buffer occupancy:

Theorem 5.5 *The expected buffer occupancy in a statistical multiplexer, with K incoming PI traffic sources with parameters (M, N), is given by*

$$\bar{Q} = \sum_{q=1}^{q_{\max}} \sum_{r=1}^{N-q} \frac{M - N + q}{M - r} \Pr\{\sum_{j=1}^{r} A_j = q + r\}. \tag{5.13}$$

■

Hence, from Theorem 5.4 and Theorem 5.5, we can compute exactly the survivor functions and expected buffer occupancy of a statistical multiplexer with PI traffic, as long as we know $\Pr\{\sum_{j=1}^{r} A_j = q+r\}$ for all r. In the next section we will consider a special type of PI traffic, namely, uniform PI (UPI) traffic, and investigate the buffer occupancy for a statistical multiplexer with such traffic sources.

5.4.2 Uniform Periodic Interchangeable Traffic

Consider a special type of K PI traffic sources: each source i, $1 \leq i \leq K$, generates n_i cells *uniformly* among a period of M slots. The class of traffic is called *uniform* PI (UPI) traffic.

Let us consider a binary vector of length M. Suppose the weight of the vector is n where the n 1's are uniformly distributed among the vector. Define $a^{(r)}$ as the sum of any consecutive r bits of the vector, then the random variable $a^{(r)}$ satisfies the hypergeometric distribution [18], namely,

$$\Pr\{a^{(r)} = m\} = \frac{\binom{r}{m}\binom{M-r}{n-m}}{\binom{M}{n}}. \tag{5.14}$$

Since the K sources are independent, we have

$$\begin{aligned}
\Pr\{\sum_{j=1}^{r} A_j = q+r\} &= \Pr\{\sum_{i=1}^{K} a_i^{(r)} = q+r\} \\
&= \sum_{\substack{m_1+\cdots+m_K = q+r \\ 0 \leq m_i \leq n_i,\, 1 \leq i \leq K}} \prod_{i=1}^{K} \frac{\binom{r}{m_i}\binom{M-r}{n_i-m_i}}{\binom{M}{n_i}}.
\end{aligned} \tag{5.15}$$

Hence, by Theorem 5.4, we obtain the following theorem.

Theorem 5.6 *The survivor function for the buffer occupancy in a statistical multiplexer, with K incoming UPI traffic sources with parameters (M, N), is given by*

$$\Pr\{Q > q\} = \sum_{r=1}^{N-q} \frac{M-N+q}{M-r} \sum_{\substack{m_1+\cdots+m_K=q+r \\ 0\le m_i \le n_i, 1\le i\le K}} \prod_{i=1}^{K} \frac{\binom{r}{m_i}\binom{M-r}{n_i-m_i}}{\binom{M}{n_i}}. \qquad (5.16)$$

■

Denote $n_L = \max_{1\le i\le K} n_i$. If $q > N - n_L$, then

$$\begin{aligned} m_L + N - n_L &= m_L + \sum_{i\neq L} n_i \\ &\geq m_1 + \cdots + m_K \\ &= q + r \\ &> N - n_L + r, \end{aligned}$$

which implies $m_L > r$, hence $\Pr[Q > q] = 0$. Therefore, the maximum queue occupancy is

$$q_{\max} = N - \max_{1\le i\le K} n_i. \qquad (5.17)$$

Therefore, by Theorem 5.5, we obtain the following theorem:

Theorem 5.7 *The expected buffer occupancy for a statistical multiplexer with K UPI traffic sources with parameters (M, N), is given by*

$$\bar{Q} = \sum_{q=1}^{q_{\max}} \sum_{r=1}^{N-q} \frac{M-N+q}{M-r} \sum_{\substack{m_1+\cdots+m_K=q+r \\ 0\le m_i \le n_i, 1\le i\le K}} \prod_{i=1}^{K} \frac{\binom{r}{m_i}\binom{M-r}{n_i-m_i}}{\binom{M}{n_i}}. \qquad (5.18)$$

■

5.4.3 Unit Uniform Periodic Interchangeable Traffic

Now we further restrict the UPI traffic such that $n_i = 1$, $1 \le i \le K$, for each source. Hence, there is one and only one cell generated during a period of time by each source. This special class of UPI traffic is called *unit* UPI (UUPI) traffic. Clearly, for UUPI traffic, $N = K$, and $q_{\max} = K - 1$. Furthermore,

$$\begin{aligned}
\Pr\{\sum_{j=1}^{r} A_j = q + r\} &= \sum_{\substack{m_1 + \cdots + m_K = q + r \\ 0 \le m_i \le n_i, 1 \le i \le K}} \prod_{i=1}^{K} \frac{\binom{r}{m_i}\binom{M-r}{n_i - m_i}}{\binom{M}{n_i}} \\
&= \binom{K}{q+r} \frac{\binom{r}{1}^{q+r}\binom{M-r}{1}^{K-(q+r)}}{\binom{M}{1}^{K}} \\
&= \binom{K}{q+r} (\frac{r}{M})^{q+r} (1 - \frac{r}{M})^{K-(q+r)}.
\end{aligned}$$

Hence, by Theorem 5.7, we obtain

$$\Pr\{Q > q\} = \sum_{r=1}^{K-q} \frac{M - K + q}{M - r} \binom{K}{q+r} (\frac{r}{M})^{q+r} (1 - \frac{r}{M})^{K-(q+r)}. \tag{5.19}$$

The formula (5.19) has been derived in [22] and [38]. Since $\Pr\{Q > 0\} = \frac{N}{M} = \frac{K}{M} = \rho$, we have the identity

$$\sum_{r=1}^{K} \frac{M - K}{M - r} \binom{K}{r} (\frac{r}{M})^{r} (1 - \frac{r}{M})^{K-r} = \frac{K}{M}. \tag{5.20}$$

Hence, we have proved the following theorem:

Theorem 5.8 *The expected buffer occupancy for a statistical multiplexer, with K UUPI traffic sources with period M, is given by*

$$\bar{Q} = \sum_{q=1}^{K-1} \sum_{r=1}^{K-q} \frac{M - K + q}{M - r} \binom{K}{q+r} (\frac{r}{M})^{q+r} (1 - \frac{r}{M})^{K-(q+r)}. \tag{5.21}$$

■

The formula (5.21) can be simplified, especially when the number of traffic sources K is sufficiently large. Since $M = \frac{K}{\rho}$, we can rewrite formula (5.21) as

$$\begin{aligned}
\bar{Q} &= \sum_{r=1}^{K-1}\sum_{q=1}^{K-r} \frac{M-K+q}{M-r}\binom{K}{q+r}(\frac{r}{M})^{q+r}(1-\frac{r}{M})^{K-(q+r)} \qquad (5.22)\\
&= \sum_{r=1}^{K-1}\sum_{s=r+1}^{K} \frac{M-K+s-r}{M-r}\binom{K}{s}(\frac{r}{M})^{s}(1-\frac{r}{M})^{K-s}\\
&= \sum_{r=1}^{K-1}\sum_{s=r+1}^{K} [(1-\rho)+\rho\cdot\frac{s-\rho r}{K-\rho r}]\binom{K}{s}(\frac{\rho r}{K})^{s}(1-\frac{\rho r}{K})^{K-s}\\
&= (1-\rho)\cdot f_1(K,\rho)+\rho\cdot f_2(K,\rho), \qquad (5.23)
\end{aligned}$$

where

$$f_1(K,\rho) = \sum_{r=1}^{K-1}\sum_{s=r+1}^{K}\binom{K}{s}(\frac{\rho r}{K})^{s}(1-\frac{\rho r}{K})^{K-s} \qquad (5.24)$$

and

$$f_2(K,\rho) = \sum_{r=1}^{K-1}\sum_{s=r+1}^{K}\frac{s-\rho r}{K-\rho r}\binom{K}{s}(\frac{\rho r}{K})^{s}(1-\frac{\rho r}{K})^{K-s}. \qquad (5.25)$$

When $\rho < 1$, we can show that

$$f_1(K,\rho) \le \frac{\rho\alpha(1-\alpha^{K-1})}{1-\alpha}, \qquad (5.26)$$

where

$$\alpha = \rho e^{1-\rho}. \qquad (5.27)$$

(See appendix B for the proof). Note that $\alpha < 1$ when $\rho < 1$. Define the function

$$b(r,n;p) = \binom{n}{r}p^{r}(1-p)^{n-r}. \qquad (5.28)$$

Then we can show that

$$f_2(K,\rho) = \sum_{r=1}^{K-1}\frac{\rho r}{K}b(r,K-1;\frac{\rho r}{K}). \qquad (5.29)$$

(See appendix C for the proof). We now consider two cases.

(1) $\rho = 1$

When $\rho = 1$, from (5.23) and (5.29), it follows

$$\bar{Q} = f_2(K, \rho) = \sum_{r=1}^{K-1} \frac{r}{K} b(r, K-1; \frac{r}{K}).$$

Note that $b(r, K; p) = b(K-r, K; p)$. Assume K is even. Then

$$\begin{aligned}
\sum_{r=1}^{K-1} \frac{r}{K} b(r, K-1; \frac{r}{K}) &= \sum_{r=1}^{K-1} \frac{r}{K} b(r, K; \frac{r}{K}) \\
&= \sum_{r=1}^{K/2-1} \frac{r}{K} b(r, K; \frac{r}{K}) + \sum_{r=K/2+1}^{K-1} \frac{r}{K} b(r, K; \frac{r}{K}) + \frac{1}{2} b(\frac{K}{2}, K; \frac{1}{2}) \\
&= \sum_{r=1}^{K/2-1} b(r, K; \frac{r}{K}) + \frac{1}{2} b(\frac{K}{2}, K; \frac{1}{2}) \\
&= \frac{1}{2}[\sum_{r=1}^{K/2-1} b(r, K; \frac{r}{K}) + \sum_{r=K/2+1}^{K-1} b(r, K; \frac{r}{K})] + \frac{1}{2} b(\frac{K}{2}, K; \frac{1}{2}) \\
&= \frac{1}{2} \sum_{r=1}^{K-1} b(r, K; \frac{r}{K}). \qquad (5.30)
\end{aligned}$$

From [45], we have the following formula

$$\sum_{i=1}^{K} \binom{K}{i} (\frac{i}{K})^i (1 - \frac{i}{K})^{K-i} = \sqrt{\frac{\pi K}{2}} - \frac{1}{3} + o(1). \qquad (5.31)$$

Therefore, we have shown

$$\bar{Q} = \frac{1}{2}(\sqrt{\frac{\pi K}{2}} - \frac{1}{3}) + o(1). \qquad (5.32)$$

Note that, from above formula, the expected queue occupancy is proportional to the square root of the number of traffic sources. Thus the expected queue occupancy approaches infinity when the number of sources K approaches infinity.

(2) $\rho < 1$

When $\rho < 1$, we can show the following upper bound

$$f_2(K,\rho) \leq \frac{\rho\alpha(1-\alpha^{K-1})}{2(1-\alpha)}\left(\sqrt{\frac{\pi}{2K}} - \frac{1}{3K}\right)(1+o(1)). \tag{5.33}$$

(See appendix D for the proof). By (5.23), (5.26), and (5.33), we obtain the upper bound for the expected queue occupancy:

$$\bar{Q} \leq \left(1 - \rho + \frac{1}{2}\sqrt{\frac{\pi}{2K}}\rho - \frac{1}{6K}\rho\right) \cdot \frac{\rho\alpha(1-\alpha^{K-1})}{1-\alpha}. \tag{5.34}$$

Note that when $K \to \infty$,

$$\bar{Q} \leq \frac{\rho\alpha(1-\rho)}{1-\alpha}, \tag{5.35}$$

i.e., the expected queue occupancy is always upper-bounded by a constant, independent of K. In summary, we have proved the following theorem:

Theorem 5.9 *The expected buffer occupancy for a statistical multiplexer, with K UUPI traffic sources with period M, is given by*

$$\bar{Q}\begin{cases} = \frac{1}{2}\left(\sqrt{\frac{\pi K}{2}} - \frac{1}{3}\right) + o(1) & \rho = 1 \\ \leq \left(1 - \rho + \frac{1}{2}\sqrt{\frac{\pi}{2K}}\rho - \frac{1}{6K}\rho\right) \cdot \frac{\rho\alpha(1-\alpha^{K-1})}{1-\alpha} = \frac{\rho\alpha(1-\rho)}{1-\alpha} & \rho < 1, \end{cases} \tag{5.36}$$

when K is sufficiently large. ■

From Theorem 5.9, the expected buffer occupancy for a statistical multiplexer with UUPI traffic is $O(\sqrt{K})$. This serves as a rule of thumb in dimensioning buffer size of a statistical multiplexer with traffic sources which are not "worse" than UUPI traffic. We will show the UUPI traffic is asymptotically, when the number of traffic sources is sufficiently large, the worst among all UPI traffic in next section. In fact, our numerical results in Figures 5.1, 5.2, 5.3, and 5.4 show that UUPI seems always the worst among general UPI traffic, even when K is small.

5.5 Asymptotic Analysis

Theorem 5.7 provides an exact formula for computing the expected buffer occupancy for general UPI traffic sources. Unfortunately the formula is rather complicated. We want to see what happens when the number of sources is large. Note that the hypergeometric distribution can be approximated by Poisson distribution under certain conditions [18]. Let

$$\lambda = r \cdot \frac{n}{M}, \tag{5.37}$$

then we have

$$\frac{\binom{r}{m}\binom{M-r}{n-m}}{\binom{M}{n}} \sim \frac{e^{-\frac{rn}{M}}(\frac{rn}{M})^m}{m!}, \tag{5.38}$$

when M is large enough. Hence, by Theorem 5.6,

$$\begin{aligned}
\Pr\{Q > q\} &\sim \sum_{r=1}^{N-q} \frac{M-N+q}{M-r} \sum_{\substack{\sum_{i=1}^{K} m_i = q+r \\ m_i \le \min(r, n_i)}} \frac{e^{-\frac{r}{M}\sum_{i=1}^{K} n_i}(\frac{r}{M})^{\sum_{i=1}^{K} m_i} n_1^{m_1}\cdots n_K^{m_K}}{m_1!\cdots m_K!} \\
&= \sum_{r=1}^{N-q} \frac{M-N+q}{M-r} \cdot e^{-\frac{rN}{M}}(\frac{r}{M})^{q+r} \sum_{\substack{\sum_{i=1}^{K} m_i = q+r \\ m_i \le \min(r, n_i)}} \frac{n_1^{m_1}\cdots n_K^{m_K}}{m_1!\cdots m_K!}. \quad (5.39)
\end{aligned}$$

Now if all the sources have the same rate, i.e., $n_i = n$, $1 \le i \le K$, then we have $N = Kn$, and

$$\begin{aligned}
\Pr\{Q > q\} &\sim \sum_{r=1}^{N-q} \frac{M-N+q}{M-r} \cdot e^{-\frac{rN}{M}}(\frac{r}{M})^{q+r} \sum_{\substack{\sum_{i=1}^{K} m_i = q+r \\ m_i \le \min(r, n_i)}} \frac{n^{m_1+\cdots+m_K}}{m_1!\cdots m_K!} \\
&= \sum_{r=1}^{N-q} \frac{M-N+q}{M-r} \cdot e^{-\frac{rN}{M}}(\frac{r}{M})^{q+r} \cdot n^{q+r} \sum_{\substack{\sum_{i=1}^{K} m_i = q+r \\ m_i \le \min(r, n_i)}} \frac{1}{m_1!\cdots m_K!}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{r=1}^{N-q} \frac{M-N+q}{M-r} \cdot e^{-\frac{rN}{M}} (\frac{rn}{M})^{q+r} \cdot \frac{K^{q+r}}{(q+r)!} \\
&= \sum_{r=1}^{N-q} \frac{M-N+q}{M-r} \cdot \frac{e^{-\frac{rN}{M}} (\frac{rN}{M})^{q+r}}{(q+r)!} \\
&\sim \sum_{r=1}^{N-q} \frac{M-N+q}{M-r} \cdot \binom{N}{q+r} (\frac{r}{M})^{q+r} (1-\frac{r}{M})^{N-(q+r)}. \qquad (5.40)
\end{aligned}
$$

The last expression is exactly the formula (5.19) for the survivor function for the UUPI traffic when the number of traffic sources equals N. Therefore, when ρ is fixed, and when M is sufficiently large and hence K is sufficiently large, the survivor functions and expected buffer occupancy for any UPI traffic will asymptotically approach those for UUPI traffic. So the result in Theorem 5.9 in fact gives the worst case buffer performance among all UPI traffic sources in asymptotic sense. Indeed, the numerical results in Figures 5.1, 5.2, 5.3, and 5.4 show this is the case, even when K is small.

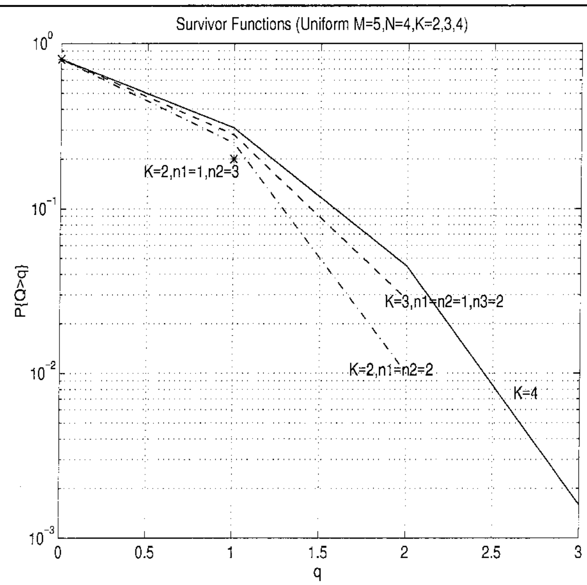

Figure 5.1: The survivor functions for UPI traffic (I).

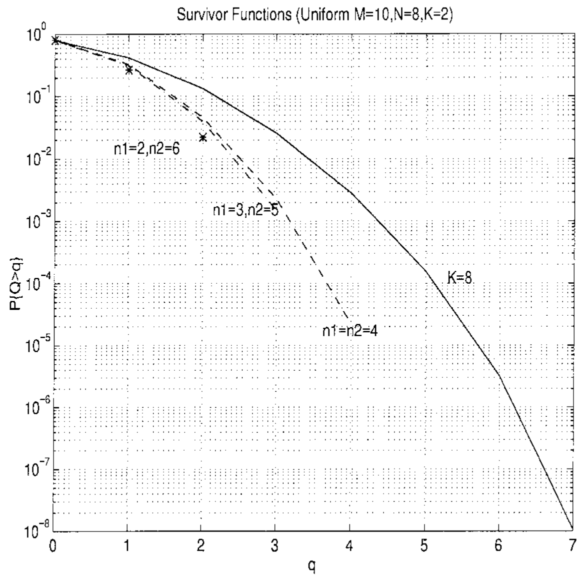

Figure 5.2: The survivor functions for UPI traffic (II).

5.6 Concluding Remarks

In this chapter we first introduced a special type of random variables, (cyclically) interchangeable random variables, and a class of burstiness constraint traffic sources, periodic interchangeable (PI) traffic. We introduced a celebrated combinatorial theorem, the classic Ballot Theorem, and then showed how to apply generalized Ballot theorems to analyze the buffer occupancy in an environment of statistical multiplexers in ATM networks for PI traffic sources. Explicit formulas for the distribution of buffer occupancy (survivor functions) and the expected buffer occupancy were derived for uniform PI (UPI) traffic, which is a special form of PI traffic sources. In particular, we obtained simple asymptotic formulas for survivor functions and expected buffer occupancy in a statistic multiplexer for unit UPI (UUPI) traffic sources, a special type of UPI traffic sources. Furthermore, we showed that UUPI sources create asymptotically the largest buffer occupancy for a statistical multiplexer among all UPI traffic with the same utilization factor (traffic intensity), when the number of traffic sources is sufficiently large. Numerical results implied that UUPI sources might be the worst traffic sources among all UPI sources, even when the number of

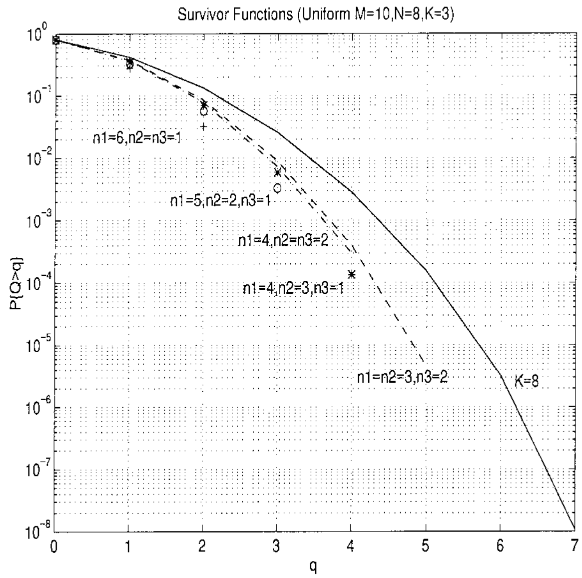

Figure 5.3: The survivor functions for UPI traffic (III).

traffic sources is small. The result of this chapter sheds light to the study of buffer occupancy for worst case traffic sources with burst-constrained and buffer management in a statistical multiplexer in ATM networks.

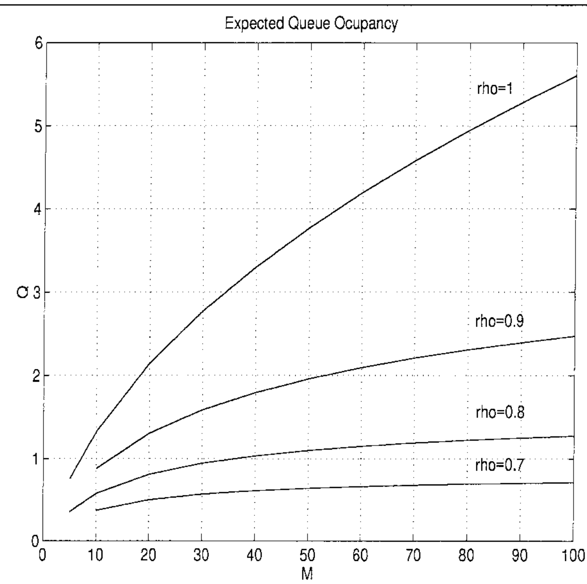

Figure 5.4: The expected buffer occupancy for UUPI traffic.

Chapter 6 Summary and Conclusions

This thesis aims to investigate two rather separate issues: network reliability and traffic analysis. The first concerns the reliability for unreliable systems, including communications networks with possible link failures, and more general fault-tolerant systems. The second concerns the traffic characteristics specifically in ATM networks regarding to the performance of statistical multiplexers.

(1) Reliability in terms of the MTTF

One way in which we studied the reliability issue is via the MTTF, which considers systems that have components failures and repairs with exponential distributions. Such systems can be modeled by continuous-time discrete-state Markov chains. In Chapter 2, we studied reliability in terms of general fault-tolerant systems with exponential rates of component failure and repair. We developed two systematic approaches, the all-path-weight approach and the signal-flow-graph approach, to compute the MTTF for fault-tolerant systems. The significance of these two approaches is not only to provide new techniques for computing the MTTF, but also to provide insight. The techniques developed allowed us to obtain a simple asymptotic formula for estimating the MTTF when a so-called safety factor is large, which is usually the case for real systems. The methods developed for general fault-tolerant systems were applied to study communications networks with link failure and repair. We derived a simple asymptotic formula for computing the MTTF for such networks. We were able to show that the networks based on Harary graphs, which have the largest edge connectivity for the given number of lines and nodes, are asymptotically the optimal networks in terms of the MTTF.

(2) Reliability in terms of reliability polynomials

The other way we studied the reliability issue is via reliability polynomials for networks, with component failures with certain fixed probabilities that are independent of time, but a function of the size of the network. No repair is allowed in such systems. In Chapter 3, we modeled such networks by random graphs, and analyzed reliability polynomials for such networks from a framework of random graph theory. We specifically focused on regular random graphs and analyzed the evolution of regular random graphs in terms of the expansion of network size. We proved that the number of isolated vertices in a regular random graph is asymptotically Poisson distributed. We also showed that the probability that a regular random graph contains a nontrivial component is asymptotically zero, and therefore, the expected value of the number of isolated vertices plays the most crucial role in determining the connectedness of a regular random graph. We showed the transition phenomenon when the regular random graph evolves from edge probability zero to probability one because of the expansion of network size, and identified the associated threshold functions. Our work accomplished the study of the evolution of regular random graphs which do not appear in the literature of random graphs, and our results are generalizations of some famous previously known results in random graph theory. Finally, we introduced a realistic slotted model for fault-tolerant systems that have a regular graph topology, by which the MTTF of such fault-tolerant system can be easily, asymptotically, computed using the reliability polynomial formula we developed.

(3) Markov on-off traffic

One traffic model we analyzed in ATM networks is homogeneous and heterogeneous Markov on-off traffic. In Chapter 4, we studied the buffer occupancy in a statistical multiplexer with such Markov on-off traffic. We applied a generating function approach and derived closed form formulas for certain conditional generating functions of cells generated by such traffic. We developed a heuristic procedure which allowed us to compute the expected buffer occupancy for homogeneous and heterogeneous Markov on-off traffic. This analysis, and some numerical calculations, showed that the expected buffer occupancy under such traffic was not only dependent on the incoming traffic intensity, but also on the burstiness of the incoming traffic. The expected buffer occupancy becomes unbounded with large enough traffic burstiness, even though the traffic intensity is small. These results showed that burstiness control of traffic is very critical in ATM networks. Furthermore, since a great deal of traffic in ATM networks can be modeled by Markov on-off traffic, and since much numerical and asymptotic analysis suggested that Markov on-off traffic is the worst case traffic, the result obtained here would be significant for dimensioning buffer size for a statistical multiplexer, as well as for the issues such as call admission control and bandwidth allocation, in designing ATM networks under the scenario of worst case performance.

(4) Periodic interchangeable traffic

Since burstiness control of traffic is critical to quality of service guarantees in ATM networks, the analysis of burst-constrained traffic, and hence the performance of a statistical multiplexer with burst-constrained traffic is important. In Chapter 5, we introduced a class of burst-constrained traffic sources, namely, the periodic interchangeable (PI) traffic sources, and applied generalized Ballot theorems to analyze the buffer occupancy in a statistical multiplexer with PI traffic. We derived explicit formulas for the distribution of buffer occupancy (survivor functions) and the expected buffer occupancy for such traffic sources. We further considered special classes of PI traffic, UPI and UUPI traffic, and obtained simple asymptotic formulas for survivor functions and expected buffer occupancy. We showed that UUPI sources create asymptotically the largest buffer occupancy for a statistical multiplexer among all UPI traffic with the same utilization factor (traffic intensity), when the number of traffic sources is sufficiently large. Numerically results implied that UUPI sources might be the worst traffic sources among all UPI sources, even when the number of traffic sources is small. Our results shed light on the study of worst case performance of statistical multiplexers with burst-constrained traffic in ATM networks.

Appendix A The proof of Lemma 2 in Section 2.4.3

Lemma 2:

$$Y(i,j) = (-1)^{j-i}(\prod_{k=1}^{j} \lambda_k)(\frac{1}{\lambda_i} + \frac{\rho_i}{\lambda_{i+1}} + \frac{\rho_i \rho_{i+1}}{\lambda_{i+2}} + \cdots + \frac{\rho_i \rho_{i+1} \cdots \rho_{j-1}}{\lambda_j}),$$

where $\rho_i = \frac{\mu_i}{\lambda_i}$ for $1 \leq i \leq n$ and $\rho_i = 0$ for $i \geq j$.

Proof: We prove the lemma by mathematical induction.

(i) The claim is true for $i = j$, since

$$Y(i,i) = \prod_{k=1}^{i} \lambda_k (\frac{1}{\lambda_i} + 0) = \prod_{k=1}^{i-1} \lambda_k.$$

(ii) The claim is true for $j = i+1$, since

$$\begin{aligned} Y(i,i+1) &= (\prod_{k \leq i-1} \lambda_k)(-z_{i+1}) \\ &= -(\prod_{k \leq i+1} \lambda_k)\frac{z_{i+1}}{\lambda_i \lambda_{i+1}} \\ &= -(\prod_{k \leq i+1} \lambda_k)\frac{\mu_i + \lambda_{i+1}}{\lambda_i \lambda_{i+1}} \\ &= (-1)^{i+1-i}(\prod_{k \leq i+1} \lambda_k)(\frac{1}{\lambda_i} + \frac{\rho_i}{\lambda_{i+1}}). \end{aligned}$$

(iii) Assume the claim is true for $Y(i, j-1)$ and $Y(i, j-2)$ for $j \geq i+2$, to prove it for $Y(i,j)$, we proceed as follows.

$$\begin{aligned} Y(i,j) &= (\prod_{k \leq i-1} \lambda_k)(-z'_{i+1})(-z'_{i+2}) \cdots (-z'_j) \\ &= (\prod_{k \leq i-1} \lambda_k)(-z'_{i+1})(-z'_{i+2}) \cdots (-z'_{j-1})(-(z_j - \mu_{j-1}\frac{\lambda_{j-1}}{z'_{j-1}})) \end{aligned}$$

$$
\begin{aligned}
&= Y(i,j-1)\cdot(-z_j) - Y(i,j-2)\cdot(\mu_{j-1}\lambda_{j-1}) \\
&= (-1)^{j-1-i}(\prod_{k=1}^{j-1}\lambda_k)(\frac{1}{\lambda_i}+\frac{\rho_i}{\lambda_{i+1}}+\cdots+\frac{\rho_i\cdots\rho_{j-2}}{\lambda_{j-1}})\cdot(-z_j) \\
&\quad -(-1)^{j-2-i}(\prod_{k=1}^{j-2}\lambda_k)(\frac{1}{\lambda_i}+\frac{\rho_i}{\lambda_{i+1}}+\cdots+\frac{\rho_i\cdots\rho_{j-3}}{\lambda_{j-2}})\cdot(\mu_{j-1}\lambda_{j-1}) \\
&= (-1)^{j-i}(\prod_{k=1}^{j}\lambda_k)(\frac{1}{\lambda_i}+\frac{\rho_i}{\lambda_{i+1}}+\cdots+\frac{\rho_i\cdots\rho_{j-3}}{\lambda_{j-2}}+\frac{\rho_i\cdots\rho_{j-2}}{\lambda_{j-1}})\cdot(\frac{z_j}{\lambda_j}) \\
&\quad -(-1)^{j-i}(\prod_{k=1}^{j}\lambda_k)(\frac{1}{\lambda_i}+\frac{\rho_i}{\lambda_{i+1}}+\cdots+\frac{\rho_i\cdots\rho_{j-3}}{\lambda_{j-2}})\cdot(\frac{\mu_{j-1}}{\lambda_j}) \\
&= (-1)^{j-i}(\prod_{k=1}^{j}\lambda_k)((\frac{1}{\lambda_i}+\frac{\rho_i}{\lambda_{i+1}}+\cdots+\frac{\rho_i\cdots\rho_{j-3}}{\lambda_{j-2}})\cdot(\frac{z_j-\mu_{j-1}}{\lambda_j}) \\
&\quad +\frac{\rho_i\cdots\rho_{j-2}}{\lambda_{j-1}}\cdot\frac{z_j}{\lambda_j}) \\
&= (-1)^{j-i}(\prod_{k=1}^{j}\lambda_k)(\frac{1}{\lambda_i}+\frac{\rho_i}{\lambda_{i+1}}+\cdots+\frac{\rho_i\cdots\rho_{j-3}}{\lambda_{j-2}}+\frac{\rho_i\cdots\rho_{j-2}}{\lambda_{j-1}}\cdot\frac{\mu_{j-1}+\lambda_j}{\lambda_j}) \\
&= (-1)^{j-i}(\prod_{k=1}^{j}\lambda_k)(\frac{1}{\lambda_i}+\frac{\rho_i}{\lambda_{i+1}}+\cdots+\frac{\rho_i\cdots\rho_{j-3}}{\lambda_{j-2}}+\frac{\rho_i\cdots\rho_{j-2}}{\lambda_{j-1}}+\frac{\rho_i\cdots\rho_{j-1}}{\lambda_j}),
\end{aligned}
$$

so that the claim is true for $Y(i,j)$.

Therefore, from (i), (ii) and (iii), we have proved the claim is always true for $Y(i,i)$ and for $Y(i,i+1)$ and hence for $Y(i,i+2)$, which completes the proof of the claim.

■

Appendix B The upper bound for $f_1(K,\rho)$ in Section 5.4.3

When $\rho < 1$, we want to obtain an upper bound for the following function:

$$f_1(K,\rho) = \sum_{r=1}^{K-1} \sum_{s=r+1}^{K} \binom{K}{s} (\frac{\rho r}{K})^s (1 - \frac{\rho r}{K})^{K-s}.$$

Clearly, for any $\mu \geq 0$, we have

$$\begin{aligned} & \sum_{s=r+1}^{K} \binom{K}{s} (\frac{\rho r}{K})^s (1 - \frac{\rho r}{K})^{K-s} \\ \leq & \sum_{s=r+1}^{K} \binom{K}{s} (\frac{\rho r}{K})^s (1 - \frac{\rho r}{K})^{K-s} \cdot e^{-\mu(r+1-s)} \\ \leq & \sum_{s=0}^{K} \binom{K}{s} (\frac{\rho r}{K} e^{\mu})^s (1 - \frac{\rho r}{K})^{K-s} \cdot e^{-\mu(r+1)} \\ = & (\frac{\rho r}{K} e^{\mu} + 1 - \frac{\rho r}{K})^K \cdot e^{-\mu(r+1)}. \end{aligned}$$

Define a function

$$g(K,\rho,\mu) = (\frac{\rho r}{K} e^{\mu} + 1 - \frac{\rho r}{K})^K \cdot e^{-\mu(r+1)}.$$

Taking the derivative with respect to μ,

$$g'(K,\rho,\mu) = K(\frac{\rho r}{K} e^{\mu} + 1 - \frac{\rho r}{K})^{K-1} \cdot \frac{\rho r}{K} e^{\mu} \cdot e^{-\mu(r+1)} - (\frac{\rho r}{K} e^{\mu} + 1 - \frac{\rho r}{K})^K \cdot (r+1) e^{-\mu(r+1)}.$$

Let $g'(K,\rho,\mu) = 0$, when $r < K-1$, we have

$$e^{\mu} = \frac{(r+1)(K - \rho r)}{\rho r (K - r - 1)}.$$

Hence,

$$\begin{aligned}
g(K,\rho,\mu) &\leq (\frac{\rho r}{K}\cdot\frac{(r+1)(K-\rho r)}{\rho r(K-r-1)}+1-\frac{\rho r}{K})^{K}\cdot(\frac{(r+1)(K-\rho r)}{\rho r(K-r-1)})^{-(r+1)} \\
&= (1+\frac{(1-\rho)r+1}{K-r-1})^{K-r-1}\cdot(1-\frac{1}{r+1})^{r+1}\cdot\rho^{r+1} \\
&\leq e^{(1-\rho)r+1}\cdot e^{-1}\cdot\rho^{r+1} \\
&= \rho\cdot(\rho e^{1-\rho})^{r} \\
&= \rho\cdot\alpha^{r},
\end{aligned}$$

where $\alpha=\rho e^{1-\rho}$. Therefore,

$$\begin{aligned}
f_1(K,\rho) &\leq \sum_{r=1}^{K-2}\rho\cdot\alpha^{r}+\binom{K}{K}(\frac{\rho(K-1)}{K})^{K}(1-\frac{\rho(K-1)}{K})^{K-K} \\
&\leq \sum_{r=1}^{K-2}\rho\cdot\alpha^{r}+\rho^{K}\cdot e^{-1} \\
&\leq \sum_{r=1}^{K-1}\rho\cdot\alpha^{r} \\
&= \rho\cdot\frac{\alpha(1-\alpha^{K-1})}{1-\alpha}.
\end{aligned}$$

■

Appendix C The derivation of formula (5.29)

We want to simplify the function

$$f_2(K,\rho) = \sum_{r=1}^{K-1} \sum_{s=r+1}^{K} \frac{s-\rho r}{K-\rho r} \binom{K}{s} (\frac{\rho r}{K})^s (1-\frac{\rho r}{K})^{K-s}.$$

Note that

$$\sum_{s=0}^{K} s \cdot \binom{K}{s} p^s q^{K-s} = \frac{\partial}{\partial p}(p+q)^K = Kp(p+q)^{K-1}.$$

Hence,

$$\sum_{s=0}^{K} (s-\rho r) \binom{K}{s} (\frac{\rho r}{K})^s (1-\frac{\rho r}{K})^{K-s} = K \cdot \frac{\rho r}{K} - \rho r = 0,$$

which results in

$$f_2(K,\rho) = \sum_{r=1}^{K-1} \frac{1}{K-\rho r} h(K,\rho,r),$$

where

$$h(K,\rho,r) = \sum_{s=0}^{r} (\rho r - s) \binom{K}{s} (\frac{\rho r}{K})^s (1-\frac{\rho r}{K})^{K-s}.$$

Since

$$\begin{aligned}
h(K,\rho,r) &= \rho r \sum_{s=0}^{r} \binom{K}{s} (\frac{\rho r}{K})^s (1-\frac{\rho r}{K})^{K-s} - \sum_{s=0}^{r} s \binom{K}{s} (\frac{\rho r}{K})^s (1-\frac{\rho r}{K})^{K-s} \\
&= \rho r [\sum_{s=0}^{r} \binom{K}{s} (\frac{\rho r}{K})^s (1-\frac{\rho r}{K})^{K-s} - \sum_{s=1}^{r} \binom{K-1}{s-1} (\frac{\rho r}{K})^{s-1} (1-\frac{\rho r}{K})^{K-s}] \\
&= \rho r [\sum_{s=1}^{r} (b(s,K;\frac{\rho r}{K}) - b(s-1,K-1;\frac{\rho r}{K})) + b(0,K;\frac{\rho r}{K})]
\end{aligned}$$

and

$$b(s,K;\frac{\rho r}{K}) = [\binom{K-1}{s} + \binom{K-1}{s-1}] (\frac{\rho r}{K})^s (1-\frac{\rho r}{K})^{K-s}$$

$$= (1-\frac{\rho r}{K})b(s,K-1;\frac{\rho r}{K}) + \frac{\rho r}{K}b(s-1,K-1;\frac{\rho r}{K}),$$

it follows,

$$\begin{aligned} h(K,\rho,r) &= \rho r[(1-\frac{\rho r}{K})\sum_{s=1}^{r}(b(s,K-1;\frac{\rho r}{K}) - b(s-1,K-1;\frac{\rho r}{K})) + b(0,K;\frac{\rho r}{K})] \\ &= \rho r[(1-\frac{\rho r}{K})(b(r,K-1;\frac{\rho r}{K}) - b(0,K-1;\frac{\rho r}{K})) + b(0,K;\frac{\rho r}{K})] \\ &= \rho r(1-\frac{\rho r}{K})b(r,K-1;\frac{\rho r}{K}). \end{aligned}$$

Therefore,

$$\begin{aligned} f_2(K,\rho) &= \sum_{r=1}^{K-1}\frac{1}{K-\rho r}\cdot \rho r(1-\frac{\rho r}{K})b(r,K-1;\frac{\rho r}{K}) \\ &= \sum_{r=1}^{K-1}\frac{\rho r}{K}b(r,K-1;\frac{\rho r}{K}). \end{aligned}$$

■

Appendix D The upper bound for $f_2(K,\rho)$ in Section 5.4.3

We have

$$f_2(K,\rho) = \sum_{r=1}^{K-1} \frac{\rho r}{K} b(r, K-1; \frac{\rho r}{K}).$$

When $\rho < 1$, we want to derive an upper bound for the function. Rewrite

$$\begin{aligned} f_2(K,\rho) &= \sum_{r=1}^{K-1} \frac{\rho r}{K} \binom{K-1}{r} (\frac{\rho r}{K})^r (1-\frac{\rho r}{K})^{K-1-r} \\ &= \sum_{r=1}^{K-1} \frac{\rho^{r+1} r}{K} \binom{K-1}{r} (\frac{r}{K})^r (1-\frac{r}{K})^{K-1-r} \cdot (\frac{1-\rho r/K}{1-r/K})^{K-1-r}. \end{aligned}$$

Since

$$\begin{aligned} (\frac{1-\rho r/K}{1-r/K})^{K-1-r} &= [(1+\frac{(1-\rho)r}{K-r})^{\frac{K-r}{(1-\rho)r}}]^{(1-\frac{1}{K-r})(1-\rho)r} \\ &\le e^{(1-\frac{1}{K-r})(1-\rho)r} \\ &< e^{(1-\rho)r}, \end{aligned}$$

it follows,

$$\begin{aligned} f_2(K,\rho) &< \sum_{r=1}^{K-1} \frac{\rho^{r+1} r}{K} \binom{K-1}{r} (\frac{r}{K})^r (1-\frac{r}{K})^{K-1-r} \cdot e^{(1-\rho)r} \\ &= \rho \sum_{r=1}^{K-1} \alpha^r \cdot \frac{r}{K} \binom{K-1}{r} (\frac{r}{K})^r (1-\frac{r}{K})^{K-1-r} \\ &= \rho \sum_{r=1}^{K-1} \alpha^r \cdot \frac{r}{K} \binom{K}{r} (\frac{r}{K})^r (1-\frac{r}{K})^{K-r}. \end{aligned}$$

Clearly, $a_r^{(1)} = \alpha^r$ is a decreasing function of r. It is easy to verify that to show $a_r^{(2)} = \frac{r}{K}\binom{K}{r}(\frac{r}{K})^r(1-\frac{r}{K})^{K-r}$ is an increasing function of r for $1 \le r \le K-1$ is equivalent to show $(1+1/x)^{1+x}(1-1/(K-x))^{K-x-1} \ge 1$ for $0 < x < K-1$. The latter inequality is clearly true when one realizes that $(1 + 1/x)^{1+x}$ is a decreasing function with the limit e (the lower bound), and $(1 - 1/(K - x))^{K-x-1}$ is also a decreasing function with the limit $\frac{1}{e}$ (the lower bound).

Now recall the following Chebyshev's inequality: if $0 \le a_1^{(1)} \le a_2^{(1)} \le \cdots \le a_n^{(1)}$ and $a_1^{(2)} \ge a_2^{(2)} \ge \cdots \ge a_n^{(2)} \ge 0$, then

$$\sum_{i=1}^{n} a_i^{(1)} a_i^{(2)} \le \frac{1}{n} (\sum_{i=1}^{n} a_i^{(1)}) \cdot (\sum_{i=1}^{n} a_i^{(2)}).$$

Therefore,

$$\begin{aligned} f_2(K,\rho) &< \rho \frac{1}{K-1} (\sum_{r=1}^{K-1} \alpha^r) \cdot (\sum_{r=1}^{K-1} \frac{r}{K} \binom{K}{r} (\frac{r}{K})^r (1 - \frac{r}{K})^{K-r}) \\ &= \frac{\rho\alpha(1-\alpha^{K-1})}{1-\alpha} \cdot \frac{1}{K-1} \sum_{r=1}^{K-1} \frac{r}{K} b(r, K; \frac{r}{K}). \end{aligned}$$

Applying (5.30), we obtain

$$\begin{aligned} f_2(K,\rho) &< \frac{\rho\alpha(1-\alpha^{K-1})}{1-\alpha} \cdot \frac{1}{K-1} \cdot \frac{1}{2} (\sqrt{\frac{\pi K}{2}} - \frac{1}{3} + o(1)) \\ &= \frac{\rho\alpha(1-\alpha^{K-1})}{2(1-\alpha)} (\sqrt{\frac{\pi}{2K}} - \frac{1}{3K})(1 + o(1)). \end{aligned}$$

■

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