THERMAL WAVE PROPAGATION IN HELIUM II

Thesis by

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ABSTRACT

Landau's equations for the two-fluid model of liquid helium II are used as the basis for an investigation of the properties of thermal wave propagation. A number of assumptions are made which reduce the four original equations to a system of two non-linear partial differential equations valid to first order in the relative velocity of the two components. These equations are analogous to Riemann's equations which describe pressure waves in a classical fluid.

This system of equations, when reduced to just one space dimension is shown to be hyperbolic and a set of characteristics and invariants is found. A particularly simple, one-dimensional problem is then formulated and an explicit solution is given. This solution is then studied in detail to show the distortion of a temperature pulse as it propagates and also to show effects such as non-linear breaking.

Subsequently, the restrictive assumptions are eliminated individually and the equations are then valid to second order in the relative velocity; the effects of including thermal expansion and using the relative velocity as a thermodynamic variable are given. Also, some effects due to the interaction of first and second sound are investigated. In all cases, the results are compared with other results based on equations differing from the Landau equations and with results found by using perturbation techniques.

Finally, equations based on the same Landau equations are derived and discussed which describe steady state shock (discontinuous) solutions.

Suggestions for further theoretical and experimental work are made.

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CHAPTER I

INTRODUCTION

A. A SURVEY OF LIQUID HELIUM

Liquid helium displays many very unusual and interesting properties that have no analogies in any other fluid. In fact, it has been likened to a "fourth" state of matter in addition to the usual three[1]. This uniqueness is due in part to the fact that in the low temperature regions of liquid helium the quantum nature of the liquid is important on a macroscopic scale.

Kamerlingh Onnes was the first to liquefy helium in 1908[2]. There is an additional transformation which separates liquid helium into two distinct phases; helium I, the higher temperature phase, is similar to other liquids in that it obeys the classical hydrodynamic equations. The lower temperature phase, helium II, is the "quantum liquid" which will be of interest here.

These two phases are separated by the λ-line which intersects the saturated vapor pressure curve at the λ-point ($T_\lambda = 2.172°K$)[3]. This is not the usual phase transition in that there is no latent heat associated with it nor is there a discontinuity in the density. Instead, it is a second order transition in which the second partial derivatives of the Gibbs potential are discontinuous at the λ-line[4]. This transition manifests itself by a logarithmic discontinuity in the specific heat and a discontinuity in the slope of the density curve[3]. The terms "λ-line" and "λ-point" are derived from the shape of the specific heat curve which resembles the Greek letter "λ".

Some other unusual properties of helium are readily discernible from the phase diagram in the P, T-plane[1],[3]. To begin with, He II remains a liquid even at zero temperature unless the pressure is raised above approximately 25 atmospheres; thus the liquid state must possess as much thermal order as the solid state at zero temperature. Also, there is no solid, liquid, gas triple point because the He II region separates the melting curve and the saturated vapor pressure line.

Among the many unusual properties of He II itself are

1. Superfluidity, the ability to flow with no apparent viscous drag. Measurements using fine capillaries have shown that the flow velocity depends very weakly on the pressure force but has a strong dependence upon the bore of the capillaries[5],[6]. In direct contrast to this are measurements of the damping of oscillating bodies which have shown that He II possess the usual drag characteristics associated with a viscous fluid[7]. Thus,it is clear that a description of He II using something other than the usual coefficient of viscosity is needed.

This introduction of superfluidity as flow without friction is complicated by the fact that there exists a critical velocity which limits the range of velocities of superfluid motion[8],[9]. Properties of He II which are dependent on the quantum state are still found to be present even when the critical velocity is exceeded so that the concept of a critical velocity is associated with a breakdown in the ability to flow without friction and not with the destruction of the basic quantum nature of He II. The concept of a critical velocity is still ill-defined and not very well understood at this time. The exact value of the critical velocity is dependent on the flow geometry in a very complicated manner[10]. Generally speaking one interpretation is that the critical velocity is exceeded whenever there is enough energy present in the flow field to create quantized vortex lines[11];

2. The thermomechanical effect or fountain effect[12], which demonstrates that, in He II, heat and mass can flow in opposite directions simultaneously. In a simplified experiment to demonstrate this effect two reservoirs filled with He II are connected by a capillary and both have a free surface at the same height. Heating the liquid in one of the reservoirs will cause a rise of the level of the liquid there. The name "fountain effect" comes from the fact that if the reservoir which is heated is simply a capillary then the liquid will shoot out of the top in a type of "fountain." Obviously the heated vessel is at a higher temperature than the other one yet fluid flows into the heated vessel; this is just opposite to what would happen in an experiment with an ordinary liquid. Relatively large convection rates can be produced by very small temperature differences between the two reservoirs.

A different demonstration of the same effect occurs when He II flows out of a vessel through a capillary under the force of gravity. The result is that the temperature of the fluid remaining in the vessel rises. This is often called the mechanocaloric effect[13],[14]. In this latter case a pressure difference and accompanying flow give rise to a temperature difference. This is just opposite to the cause and effect relationship occurring in the fountain effect. In either case, heat and mass transfer are definitely not related to each other in the classical manner for fluids and, once again, the conclusion is that a complete description of He II is not possible within the framework of ordinary fluid mechanics; and

3. The propagation of thermal waves[15],[16]. The ability of temperature waves to propagate virtually undamped with a finite speed shows that temperature must obey a hyperbolic wave equation rather than a parabolic diffusion equation as in other media. These thermal waves exist in addition to the usual pressure waves and travel with a velocity which is an order of magnitude smaller than the velocity of propagation of pressure waves[16]. Hence, this phenomenon is definitely distinct from ordinary sound waves and not just a different manifestation of them. There is no analogy to this type of wave propagation in classical fluid mechanics.

These phenomena, and others, all inescapably lead to the conclusion that an entirely new hydrodynamic description is required for He II, one that is based on new basic principles and not derived from equations within the frame of reference of classical hydrodynamics. The most comprehensive and successful theory to date is the two-fluid model as first proposed by Tisza[13],[15] and developed by Landau[17].

B. THE HYDRODYNAMICS OF HELIUM II

The two-fluid theory is an attempt to create a consistent macroscopic description of He II which can give a satisfactory theoretical explanation of the large number of unusual and sometimes contradictory experimental results. This continuum model can be interpreted on the basis of two distinct microscopic theories but a comprehensive and consistent derivation of it from these theories is not possible at this time. It is not necessary or desirable to wait for this link to be completed before investigating in detail the implications of the two-fluid model. Moreover, it should be emphasized that all of the consequences of the two-fluid theory can be considered as derived from a basic set of postuates wholly independent of any microscopic interpretation.

The justification for this theory, as for any theory, is how accurately it describes observed phenomena and how well predictions based on this theory are borne out by further experimentation. As far as these aspects go the two-fluid model has proven remarkably successful. Hence any further sophistication of present theories, or any entirely new approach, aimed at explaining the behavior of He II must contain the two-fluid theory as an accurate first approximation.

The following assumptions form the basic framework of the two-fluid model:

1. He II is composed of two mutually interpenetrating parts, called the "normal" and "superfluid" parts, each of which has its own density and velocity fields. These two parts are neither components nor phases in the usual sense. For one reason, the fluid can not be separated into the two parts in any way. Secondly, a given fluid element can not be said to contain either the normal or superfluid part no matter how small that element may be defined. In other words, from a continuum point of view two densities and two

velocities can be specified at each point in the fluid volume.

All quantities corresponding to the normal part are denoted by a subscript "n" while an "s" denotes those quantities associated with the superfluid part. The sum of the normal density, ρ_n, and the superfluid density, ρ_s, gives the total macroscopic mass density of He II. Figure 1 shows the temperature dependence of ρ_n and ρ_s as first measured by Andronikashvili[18].

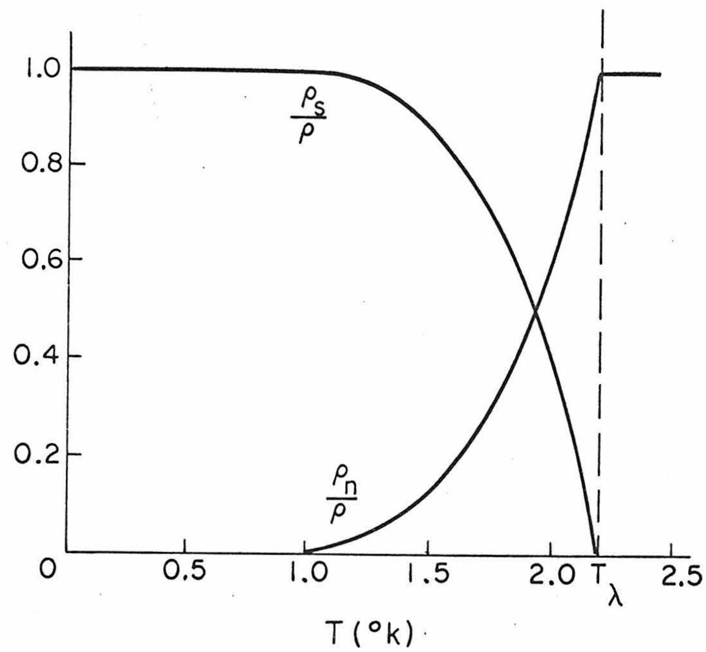


Figure 1 - Normal and Superfluid Density

The normal and superfluid parts may move in a manner that is completely different from, although not independent of each other; each part requires a separate differential equation to describe its motion.

2. The entire superfluid component is assumed to be a macroscopic manifestation of a single quantum state as opposed to a thermal

average over all existing states. Since there is no possibility of thermal collisions within the superfluid part it follows that the viscosity of the superfluid part is zero. This is not to say that the superfluid and normal parts do not interact. Indeed, there are $\rho_n \rightleftarrows \rho_s$ interactions, for instance due to temperature changes, but the superfluid part still remains a single quantum system. Another consequence of this assumption is that when $\rho_n \rightleftarrows \rho_s$ interactions do occur they can not change the local superfluid velocity since this would imply a change in the entire quantum state. Hence the quantum state and thus the superfluid state are determined by the macroscopic boundary conditions.

3. The entropy of the entire liquid is assumed to be contained in the normal part. Thus

$$\rho s = \rho_n s_n \quad . \tag{1.1}$$

Here s and s_n represent the specific entropy of the bulk liquid and normal part respectfully. This normal part is analogous to a classical viscous fluid. The normal velocity, $\overline{v}_n$, in contrast to $\overline{v}_s$, is a thermal average so the normal fluid has viscosity. The origin of the name "normal" is now clear. This third assumption is not absolutely essential to the two-fluid model and London[1] discusses some of the consequences of allowing the superfluid to have entropy.

Even with this cursory introduction to the two-fluid model one can gain an intuitive understanding of some of the properties of He II which were discussed in Section A. For instance, it is the normal part which causes the damping of immersed oscillating bodies and the superfluid part which flows freely through narrow capillaries.

A brief and heuristic discussion of the two main microscopic theories of liquid helium provides an interesting illustration of the origin of some of the basic assumptions in the two-fluid model. London[19],[20] treats liquid helium as an imperfect gas in which intermolecular forces can not be entirely neglected while Landau[21],[22] treats it as a solid in which the atoms are very weakly bound to the lattice sites. These approaches are in sharp contrast but they both assume a particular molecular picture and then attempt to explain liquid helium in terms of that picture.

London based his approach on the fact that a condensation and higher order phase transition occur in a Bose-Einstein gas which are qualitatively similar to the λ-transition. The validity of this theory is further strengthened by the fact that He^4 atoms are somewhat like hard spheres as in a perfect gas and must obey Bose-Einstein statistics since they are composed of an even number of fundamental particles. In addition, He^3, which does not exhibit a λ-transition, is composed of atoms that do not contain an even number of fundamental particles so they are not Bosons and, as such, do not obey Bose-Einstein statistics. This approach automatically provides an explanation for the λ-transition and a method for identifying the two parts of He II. The atoms which have condensed into the ground state are associated with the superfluid part while those atoms still in excited states correspond to the normal part.

From the other point of view, Landau proposed a scheme of quantized excitations for quasi-particles or thermal excitations.

These excitations, which he called "phonons" and "rotons", are assumed to have an effective mass and momentum. Feynman[23] has proposed a quantum mechanical wave function which gives this type of excitation spectrum. London[1] argues that phonons alone are capable of representing the liquid. The correspondence with the two-fluid model is made by associating the normal part with these excitations while the superfluid part is composed of the remaining mass.

Actually these two approaches are complementary rather than contradictory since the first supplies a reasonable mechanism to explain the λ-transition while the latter one is able to account for the existence of a critical velocity. Also, it has been explicitly shown that phonon excitations correspond to the lowest excited states in a hard-sphere Bose gas[24]. Both approaches have faults and deficiencies and neither one can be said to be the correct one with the exclusion of the other; both need a great deal of additional work before they represent the actual situation in liquid helium. In any case, the two-fluid model provides an excellent description of He II, at least from a phenomenological point of view, and it will be assumed valid in all further discussions of the hydrodynamics of He II.

Several different approaches have been taken in an attempt to formulate a set of hydrodynamic equations and concomitant boundary conditions that will adequately describe the flow of He II. Complete agreement on the final form of the equations has not been reached yet so a rigorous derivation of them can not be given. At the present time virtually all hydrodynamic theories of liquid helium are based on continuum mechanical arguments and their development is guided by intuitive relationships with microscopic theories. Direct analogies to classical hydrodynamics are also very important.

A complete derivation actually has two distinct steps that must be performed sequentially. First, a set of local macroscopic variables must be chosen that are adequate to describe the complete flow field. In ordinary hydrodynamics this is a very simple task to do. One variable is almost always the mass flux velocity and the remainder are two thermodynamic variables, usually pressure, P, and temperature, T, or density, ρ, and entropy, s. However, this step is a matter of some importance and difficulty for He II. We have already seen that more than three local variables are needed to describe He II. Some of the various possibilities for He II are $\overline{v}_n$, $\overline{v}_s$, ρ_n, ρ_s, ρ, s, P, T, or other combinations of these variables; it is possible that additional variables are necessary for a complete flow description.

The second step is the derivation of the equations of motion describing the time and spatial history of the chosen variables. The type of derivation is somewhat determined by which variables are chosen as basic flow quantities. As mentioned previously, one particular microscopic picture is usually chosen as a guide to the selection of the appropriate macroscopic variables then the derivation proceeds from there making use of ideas borrowed from the derivation of the equations of motion of classical hydrodynamics.

All of these derivations can be classified as being based on either variational principles or conservation laws; incidentally this is true for ordinary hydrodynamics as well as for He II. However the variational approach is beset by difficulties particular to He II as described by the two-fluid model. When variational principles are applied to a continuum the Lagrangian density must be integrated over all volume elements moving with the fluid. Yet there are two distinct velocities at each point in He II so it is manifestly impossible to integrate over a volume element moving with the fluid since this is a meaningless phrase in terms of the two-fluid model. This is not a minor difficulty overcome by a reformulation of the two-fluid model; in fact, London questions ". . . whether the two-fluid concept is actually compatible with the principles of classical particle mechanics"[1]. In addition, $\rho_n \rightleftarrows \rho_s$ interactions combined with the fact that a given fluid element can not be said to contain either normal or superfluid parts leads to the conclusion that Hamilton's principle can not be applied to He II. Zilsel[25] has recognized this inherent obstacle and uses an entirely different variational principle first stated by Eckart[26]. This principle uses an integration over volume elements fixed in space rather than moving mass elements and consequently avoids the objections raised above. Even this new variational principle restricts the class of solutions which can be used to satisfy the resulting equations of motion.

One very important point that must be taken into consideration by all types of derivations concerns any assumptions made on the vorticity of the superfluid part. Restrictive conditions on the vorticity must be explicitly stated before the derivation can proceed to completion. The derived equations of motion can not resolve this question because the derivation itself is not generally valid until some assumption is made on the vorticity. The question whether the superfluid part can rotate is a very complex one and has almost become a field of study unto itself[27].

One of the three following approaches to the problem is usually taken:

1. Most simple theories assume that the superfluid part has zero vorticity everywhere, i.e.

$$\nabla \times \bar{v}_s = 0 \quad . \tag{1.2}$$

Landau[21] proposed the first complete set of non-linear equations of motion. Khalatnikov[28] later derived this set of equations using the two-fluid model, conservation laws, the Galilean relativity principle, and assumption (1.2). Another implicit assumption is that there is no momentum exchange between the two parts except for $\rho_n \rightleftharpoons \rho_s$ interactions. The derivation starts by neglecting all irreversible processes and writing differential equations in a fixed frame of reference which express the conservation of mass, momentum, entropy, and energy plus an equation for $\bar{v}_s$ which insures that condition (1.2) is satisfied for all time. These equations are not all independent and contain unknown scalars and vector and tensor fluxes. Next, these quantities are transformed to a frame of reference where $\bar{v}_s = 0$. In this frame all quantities are assumed to behave like classical hydrodynamic variables. Finally, the interdependence of the conservation laws is used to obtain explicit expressions for the unknown quantities in the moving frame which can then be transformed back to the original, fixed reference frame. The resulting equations, called the Landau equations, are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \bar{v} = 0 \tag{1.3a}$$

$$\rho_n\left[\frac{\partial \bar{v}_n}{\partial t} + (\nabla \bar{v}_n)\bar{v}_n\right] = -\frac{\rho_n}{\rho}\nabla P - \rho_s s \nabla T - \frac{\rho_n \rho_s}{\rho}\nabla \frac{\bar{w}^2}{2} - \Gamma \bar{w} \tag{1.3b}$$

$$\rho_s\left[\frac{\partial \bar{v}_s}{\partial t} + (\nabla \bar{v}_s)\bar{v}_s\right] = -\frac{\rho_s}{\rho}\nabla P + \rho_s s \nabla T + \frac{\rho_n \rho_s}{\rho}\nabla \frac{\bar{w}^2}{2} \tag{1.3c}$$

$$\frac{\partial \rho s}{\partial t} + \nabla \cdot \rho s \bar{v}_n = 0 \tag{1.3d}$$

where

$$\bar{v} = \frac{\rho_n}{\rho}\bar{v}_n + \frac{\rho_s}{\rho}\bar{v}_s \tag{1.4}$$

- the mass flux velocity,

$$\bar{w} = \bar{v}_n - \bar{v}_s \tag{1.5}$$

- the relative velocity of the two parts,

$$\Gamma = \frac{\partial \rho_n}{\partial t} + \nabla \cdot \rho_n \bar{v}_n = -\left(\frac{\partial \rho_s}{\partial t} + \nabla \cdot \rho_s \bar{v}_s\right) \tag{1.6}$$

- a source term due to $\rho_n \rightleftarrows \rho_s$ interactions, and

s = specific entropy.

Equation (1.3a) is the usual conservation of mass equation, Eqs. (1.3b) and (1.3c) are conservation of momentum equations for the $\bar{v}_n$ and $\bar{v}_s$ velocities, and Eq. (1.3d) expresses the fact that entropy flows with the normal part only and is conserved. In addition, three equations of state are needed to make a complete formulation; for instance, these equations can take the form

$$\rho = \rho(P, T, \overline{w}) \tag{1.7a}$$

$$s = s(P, T, \overline{w}) \tag{1.7b}$$

$$\rho_n = \rho_n(P, T, \overline{w}) \quad . \tag{1.7c}$$

Notice that $\overline{w}$ serves as a thermodynamic variable in addition to being a mechanical velocity. This fact makes Eqs. (1.3) considerably more complicated than they first appear to be. The Γ term appears only in the equation for $\overline{v}_n$ due to the assumption stated earlier that $\overline{v}_s$ represents a single quantum state and not a thermal average.

It should be emphasized that these equations are not a unique result of the derivation as outlined unless $\nabla \times \overline{v}_s = 0$. This derivation has a number of faults and is criticized by Clark[29] who also shows that these same equations can be uniquely derived without assuming $\nabla \times \overline{v}_s = 0$ if a literal interpretation of the two-fluid model is assumed. He also gives an excellent review and critique of almost all other derivations of the equations of motion and shows some possible generalizations of some existing derivations.

Equations (1.3) are the same ones obtained by Zilsel[25] using a variational approach. However, his formulation also includes a restrictive equation on the curl of $\overline{v}_n$;

$$\nabla \times \overline{v}_n = \nabla s_n \times \nabla \beta$$

where s_n is defined by Eq. (1.1) and β is a Lagrangian multiplier. Another drawback to the variational approach is that it can not deal with irreversible flows. Nevertheless Landau's equations can be generalized to include viscous terms by following the same general procedure outlined above with the additional requirement that dissipation terms due to viscous effect be positive. The Landau equations have been well verified experimentally and usually serve as a starting point for further hydrodynamic analysis.

2. A different approach is to assign a special role to the vorticity of the superfluid part. One way of doing this is to assume that $\nabla \times \bar{v}_s = 0$ everywhere except on singular lines in the fluid[23]. These vortex lines can then be quantized using arguments based on the necessary symmetry of the wave function for the helium atoms. This approach has additional appeal because these vortex lines can be intuitively related to the rotons in Landau's microscopic theory. Hall and Vinen[30] have succeeded in developing a set of macroscopic hydrodynamic equations based on this approach.

Another line of reasoning is taken by Bekarevich and Khalatnikov[31] in which they assume that $\omega = |\nabla \times \bar{v}_s|$ is an additional thermodynamic variable and that the internal energy of a rotating superfluid depends on ω. The hydrodynamic equations resulting from this approach are the same as those derived by Hall and Vinen.

3. Finally, a completely different approach has been taken by Lin[32]. He rejects the two-fluid model altogether and attempts to write a hydrodynamic theory of He II without recourse to any microscopic theory which is a generalized one-fluid version of ordinary hydrodynamics. This line of reasoning has some drawbacks[34]

and, at best, offers no advantages over the other approaches based on the two-fluid model. Even so, the equations of motion which result from this approach are equivalent to the Landau equations.

Another important point to keep in mind is that the Landau equations represent the true flow situation only in the very simplest situation, that is, when all irreversible processes are negligible and when $\overline{v}_s$ is less than the relevant critical velocity. Whenever these conditions are not met Eqs. (1.3) must be modified by adding new terms to represent mutual friction between the two flows and irreversible processes. The nature and origin of these additional mutual friction terms are a very complex problem and their exact form has not been completely agreed upon and is an open question. The Hall-Vinen and Bekarevich-Khalatnikov equations mentioned above are the identical results of two different attempts to take these additional complications into account. Still another version of the equations of motion which include mutual friction is the semi-empirical Gorter-Mellink formulation[33]. Both of these sets of equations reduce to the Landau equations in the limit of reversible flow with negligible mutual friction. These principal theories have been concisely summarized by Hsieh[34].

The last important question concerning a complete hydrodynamic theory deals with boundary conditions. The appropriate boundary conditions that should be used depend on the particular set of governing equations of motion that are being used to describe the flow field. Nevertheless, most formulations have certain points in common. For instance, at a fixed, solid boundary there usually is a no-slip condition on $\overline{v}_n$ and requirements that the perpendicular component of the mass flux is zero and that the heat flux is continuous across the boundary; these conditions are all analogous to those in ordinary hydrodynamics. However, when heat flows from a solid into liquid helium it is accompanied by a temperature discontinuity known as the Kapitza boundary effect[35]; on the other hand, no discontinuity is found when the heat flux is directed from the fluid into the solid. The magnitude of the discontinuity is proportional to the heat flux per unit area. This effect has also been observed in He^3[36] where the phenomena associated with superfluidity do not appear so that it does not depend on the quantum nature of He II for its existence. The boundary conditions imposed on $\overline{v}_s$ depend strongly on the particular formulation which is chosen to represent the flow.

In summary, a continuum formulation exists which can be derived in a number of different ways and which has good experimental support. However this formulation is valid only under a limited set of conditions and more comprehensive theories are needed. The entire field of study, from a microscopic theory with its connection to a complete continuum model on to a satisfactory derivation of the thermohydrodynamic equations of motion and boundary conditions, is certainly open to further theoretical and experimental study.

CHAPTER II

SELECTED APPROACHES TO THERMAL WAVE PROPAGATION

A. SOME ELEMENTARY ASPECTS OF THERMAL WAVES

The propagation, rather than the diffusion of temperature variations is probably the most unusual characteristice of He II. This type of wave motion is usually called "second sound" to distinguish it from the familiar propagation of pressure waves which is called "first sound" in He II. The existence of these temperatures waves can be easily explained on the basis of the two-fluid theory. Since there are two distinct densities each with its own velocity field, two separate modes of energy transport exist as compared to only one velocity and one mode of energy transport by wave motion in ordinary one-fluid hydrodynamics. This additional mode in He II is heat transport by a purely mechanical process which is the second sound phenomenon.

It is possible to deduce a number of the fundamental properties of second sound from a linear perturbation analysis of Eqs. (1.3). The procedure that will be used here is the same as employed by Atkins[10] among others[1],[28]. Consider a stationary bulk of He II at equilibrium; write all thermodynamic variables as

$$\left.\begin{aligned} P &= P_o + \epsilon P_1 + \ldots \\ T &= T_o + \epsilon T_1 + \ldots \\ \rho_n &= \rho_{no} + \epsilon \rho_{n1} + \ldots, \text{ etc.} \end{aligned}\right\} \qquad (2.1)$$

The subscript "o" indicates a constant, equilibrium value and a subscript "1" labels all variable perturbation quantities. Also, since both velocities have a zero equilibrium value,

$$\left.\begin{aligned} \bar{v}_n &= \epsilon \bar{v}_{n1} + \ldots \\ \bar{v}_s &= \epsilon \bar{v}_{s1} + \ldots \quad . \end{aligned}\right\} \tag{2.1}$$

Substituting Eqs. (2.1) into Eqs. (1.3) and keeping only order ϵ terms will give the desired equations for the perturbation quantities. By its definition, Γ is order ϵ so the first order equations are

$$\frac{\partial \rho_1}{\partial t} + \rho_{no} \nabla \cdot \bar{v}_{n1} + \rho_{so} \nabla \cdot \bar{v}_{s1} = 0 \tag{2.2a}$$

$$\rho_{no} \frac{\partial \bar{v}_{n1}}{\partial t} = - \frac{\rho_{no}}{\rho_o} \nabla P_1 - \rho_{so} s_o \nabla T_1 \tag{2.2b}$$

$$\rho_{so} \frac{\partial \bar{v}_{s1}}{\partial t} = - \frac{\rho_{so}}{\rho_o} \nabla P_1 + \rho_{so} s_o \nabla T_1 \tag{2.2c}$$

$$\rho_o \frac{\partial s_1}{\partial t} + s_o \frac{\partial \rho_1}{\partial t} + \rho_o s_o \nabla \cdot \bar{v}_{n1} = 0 \quad . \tag{2.2d}$$

Differentiating (2.2a) and (2.2d) with respect to time and using the other two equations to eliminate derivatives of $\bar{v}_n$ and $\bar{v}_s$ gives

$$\frac{\partial^2 \rho_1}{\partial t^2} = \nabla^2 P_1 \tag{2.3a}$$

$$\frac{\partial^2 s_1}{\partial t^2} = \frac{\rho_{so}}{\rho_{no}} s_o^2 \nabla^2 T_1 \quad . \tag{2.3b}$$

Equation (2.3a) is the same as the equation which describes pressure waves in classical hydrodynamics while Eq. (2.3b) is the new equation which governs second sound waves. Writing the equations of state in the form $P = \rho(\rho, s)$ and $T = T(\rho, s)$ implies

$$P_1 = \left(\frac{\partial P}{\partial \rho}\right)_s \rho_1 + \left(\frac{\partial P}{\partial s}\right)_\rho s_1$$

and

$$T_1 = \left(\frac{\partial T}{\partial \rho}\right)_s \rho_1 + \left(\frac{\partial T}{\partial s}\right)_\rho s_1 \quad .$$

Using these equations in (2.3) gives two coupled wave equations for ρ_1 and s_1;

$$\frac{\partial^2 \rho_1}{\partial t^2} = \left(\frac{\partial P}{\partial \rho}\right)_s \nabla^2 \rho_1 + \left(\frac{\partial P}{\partial s}\right)_\rho \nabla^2 s_1$$

$$\frac{\partial^2 s_1}{\partial t^2} = \frac{\rho_{so}}{\rho_{no}} s_o^2 \left(\frac{\partial T}{\partial s}\right)_\rho \nabla^2 s_1 + \frac{\rho_{so}}{\rho_{no}} s_o^2 \left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \rho_1 \quad .$$

Plane wave solutions to this set of equations having the form

$$s_1 = a e^{i\omega(t-x/u)}$$

and

$$\rho_1 = b e^{i\omega(t-x/u)}$$

are possible if

$$(i\omega)^2 b = \left(\frac{\partial P}{\partial \rho}\right)_s \left(-\frac{i\omega}{u}\right)^2 b + \left(\frac{\partial P}{\partial s}\right)_\rho \left(-\frac{i\omega}{u}\right)^2 a$$

$$(i\omega)^2 a = \frac{\rho_{so}}{\rho_{no}} s_o^2 \left(\frac{\partial T}{\partial s}\right)_\rho \left(-\frac{i\omega}{u}\right) a + \frac{\rho_{so}}{\rho_{no}} s_o^2 \left(\frac{\partial T}{\partial \rho}\right)_s \left(-\frac{i\omega}{u}\right)^2 b$$

which is equivalent to

$$\left.\begin{aligned} (u^2 - u_1^2) b - \left(\frac{\partial P}{\partial s}\right)_\rho a &= 0 \\ -\frac{\rho_{so}}{\rho_{no}} s_o^2 \left(\frac{\partial T}{\partial \rho}\right)_s b + (u^2 - u_2^2) a &= 0 \end{aligned}\right\} \qquad (2.4)$$

where

$$u_1^2 = \left(\frac{\partial P}{\partial \rho}\right)_s \tag{2.5}$$

$$u_2^2 = \frac{\rho_s}{\rho_n} s^2 \left(\frac{\partial T}{\partial s}\right)_\rho \quad . \tag{2.6}$$

Setting the determinant of Eqs. (2.4) equal to zero gives

$$\left[\left(\frac{u}{u_1}\right)^2 - 1\right]\left[\left(\frac{u}{u_2}\right)^2 - 1\right] = \left(\frac{\partial P}{\partial T}\right)_\rho \left(\frac{\partial T}{\partial P}\right)_s = \frac{C_P - C_V}{C_P} \tag{2.7}$$

where C_P and C_V are the specific heats at constant pressure and volume, respectively. If the right hand side of Eq. (2.7) is neglected then

$u = u_1$, the velocity of first sound

or

$u = u_2$, the velocity of second sound.

As London shows

$$C_P - C_V \propto \alpha_P^2$$

where α_P is the coefficient of thermal expansion. Hence, Eqs. (2.3) are de-coupled if the thermal expansion is negligible. At $T = 1.5°K$

$$\frac{C_P - C_V}{C_P} = 7 \times 10^{-4}$$

so the coupling is very small and can be neglected as a valid first approximation. Figure 2 shows the variation of u_2 with temperature[3]. For the purpose of comparison u_1 equals 235 m/sec at 1.5°K and 218 m/sec at the λ-point. Below .9°K u_2 rises rapidly and apparently approaches 190 m/sec as T goes to 0°K[37].

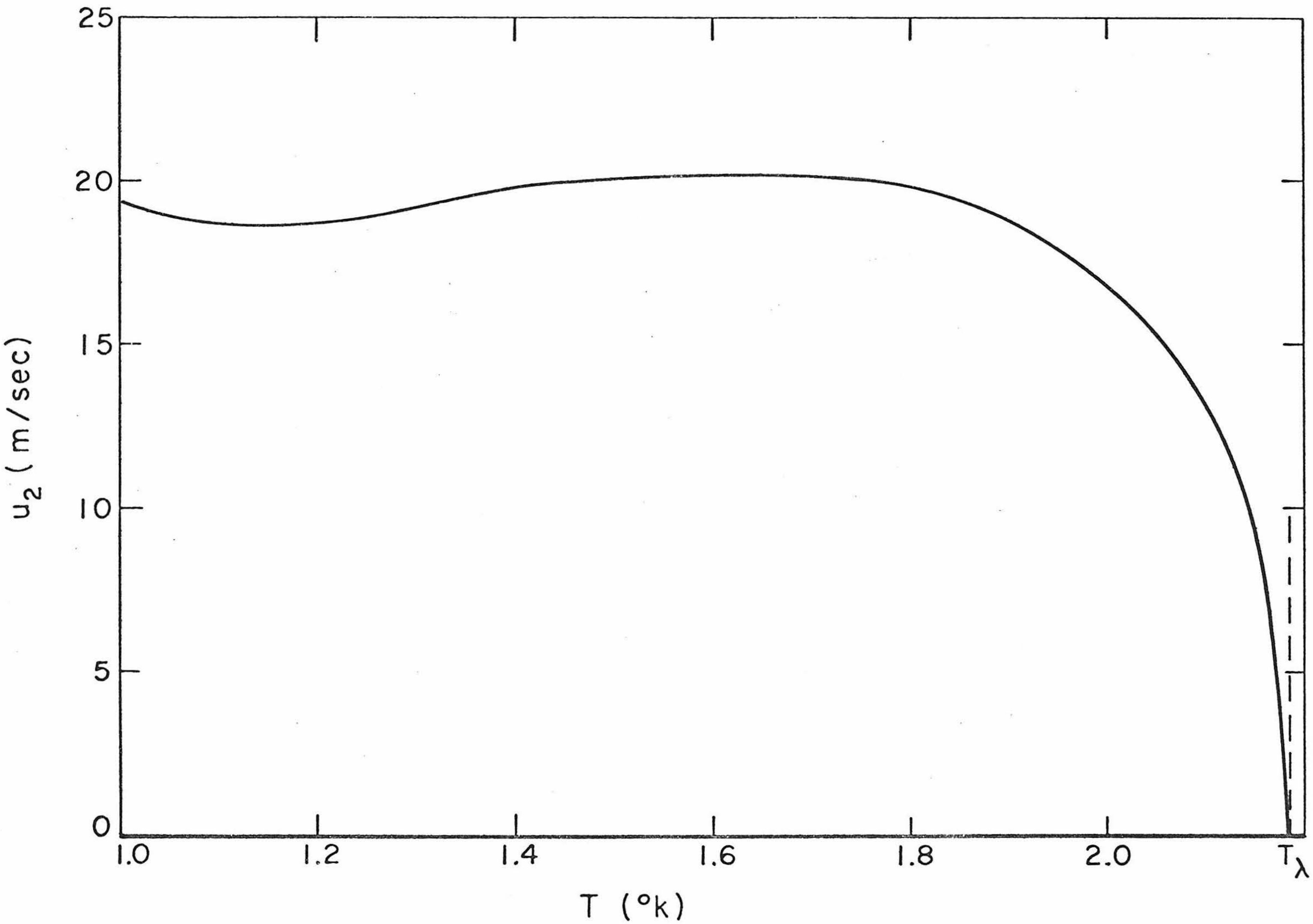


Figure 2 - Velocity of Second Sound

Additional properties of the two types of wave propagation can be found from further analysis of Eqs. (2.2); for the sake of definiteness neglect the linear coupling between first and second sound by assuming $\alpha_P = 0$. To begin, consider the properties of first sound by choosing the root $u = u_1$ of Eq. (2.7). Then, by Eq. (2.4), $a = 0$ so this root represents the familiar pressure and density oscillations at constant entropy usually found in ordinary hydrodynamics. With $T_1 = 0$ in Eqs. (2.2b) and (2.2c), these two equations imply, for first sound, that $\overline{v}_n = \overline{v}_s$ or

$$\overline{w} = 0 \quad . \tag{2.8}$$

This means that the normal and superfluid parts move together in phase during first sound propagation.

Similarly, we can discover some of the properties of second sound by taking the root $u = u_2$. Then $b = 0$ so the density and pressure are constant. Now, Eqs. (2.2b) and (2.2c) with $\rho_1 = 0$ combine to yield

$$\rho_{no}\overline{v}_{n1} + \rho_{so}\overline{v}_{s1} \equiv \overline{v}_1 = 0 \quad . \tag{2.9}$$

Thus, we have the result that the mass flux velocity is zero during second sound propagation or, in other words, second sound is a type of heat transfer without an accompanying mass transfer. Under these same conditions it is possible to show that heat flow is proportional to, and in phase with, the temperature[35]. Given the traveling wave solution for the entropy in the form

$$s_1 = ae^{i\omega(t-x/u_2)}$$

it follows that

$$T_1 = a\left(\frac{\partial T}{\partial s}\right)_\rho e^{i\omega(t-x/u_2)} \quad .$$

Set $P_1 = 0$ and use this expression for T_1 in Eq. (2.2b);

$$\rho_{no} \frac{\partial v_{n1,x}}{\partial t} = - \rho_{so} s_o a \left(\frac{\partial T}{\partial s}\right)_\rho \left(- \frac{i\omega}{u_2}\right) e^{i\omega(t-x/u_2)}$$

or

$$\frac{\partial v_{n1,x}}{\partial t} = \frac{u}{u_2} \frac{\rho_{so}}{\rho_{no}} s_o \left(\frac{\partial T}{\partial s}\right)_\rho [i\omega e^{i\omega(t-x/u_2)}] \quad .$$

Therefore

$$v_{n1,x} = \frac{s_1}{s_o} u_2 \quad . \qquad (2.10)$$

The entropy flux is $\rho s \overline{v}_n$ [see Eq. (1.4d)] so the heat flux is

$$\overline{Q} = \rho s T \overline{v}_n \quad .$$

In this scheme

$$Q_{1,x} = \rho_o s_o T_o v_{n1,x}$$

$$= \rho_o s_o T_o \frac{s_1}{s_o} u_2$$

$$Q_{1,x} = \rho_o T_o u_2 s_1 \propto T_1 \qquad (2.11)$$

This is the desired result; the striking difference between (2.11) and ordinary hydrodynamics, where heat flux is proportional to the gradient of temperature, arises because temperature satifies a hyperbolic equation here rather than a parabolic equation as it does in other fluids.

It should be emphasized that the above results regarding the basic properties of first and second sound are valid only as long as thermal expansion is negligible. In actual fact, there are small fluctuations in entropy and temperature carried along with a first sound wave and, conversely, small changes in the pressure and density accompany a second sound wave. Even so, a natural division of the variables into two distinct sets seems to be suggested here. One group contains P, $\overline{v}$, ρ and other variables which have a strong pressure dependence; it is these variables which are of primary importance in a first sound situation. The other group contains T, $\overline{w}$, s and other temperature dependent variables whose variation is of most concern in a second sound problem.

B. PREVIOUS SECOND SOUND INVESTIGATIONS AND A SUMMARY OF THIS WORK

Several investigators have already used a variety of approaches in addition to the straightforward linear analysis given above to study the nature of second sound. The best review of the most important properties of second sound and an extensive bibliography are contained in the book by Atkins[10]. The detailed temperature dependence of the amplitude-independent velocity of propagation, energy flow theorems, attentuation, and the effects of various types of boundary conditions are some of the points of interest that are discussed; an analogy between second sound propagation and an equivalent electrical circuit which was first proposed by Dingle[39] is also developed further. The discussions of the amplitude-dependent velocity of propagation and the distortion of a pulse type waveform, two of the properties which will be of particular interest in this paper, are based on the theories of Khalatnikov[38]. He derives expressions for the velocity of propagation of both first and second sound which are valid to first order in the relative velocity, $\overline{w}$. This derivation is open to some criticism and will be repeated and discussed in detail in Appendix B in order to clarify it. Khalatnikov uses a relatively simple linear perturbation analysis which is encumbered by a minimum number of restrictive assumptions so his results will provide a valuable check on the formulae derived herein by a non-linear analysis. Unfortunately, Khalatnikov includes very little detailed discussion of his results along with the analysis.

Temperley[40] has done the only non-linear analysis up to this time but his work is not based on the Landau equations; his work is also reviewed in Appendix B. A procedure similar to the one used by Temperley will be used here but the results are not comparable because different equations of motion are used. Temperley only derives the equations which specifically govern the motion of second sound but, like Khalatnikov, does not discuss any of the consequences of these equations in detail.

In both the theoretical investigations mentioned above the authors have been explicitly concerned with solutions which are continuous functions of time and spatial dimensions. Nevertheless, Khalatnikov[28] and Temperley, in the same paper, also consider discontinuous (shock) solutions. Both use a perturbation scheme although Khalatnikov has done a more general analysis; their general conclusions are virtually the same even though Temperley does not use the Landau equations. The most important results based on their analysis deal with the propagation speed of the discontinuity and with order of magnitude estimates which relate the size of the jump of the important variables across the discontinuity. These estimates are made for both pressure and temperature discontinuities.

A large amount of experimental data has been collected concerning all the various aspects of second sound but only two experiments are of direct interest as far as this paper is concerned. One of them entailed the use of a pulse technique by Osborne[41] to clearly demonstrate the phenomenon of non-linear "breaking" of a given pulse type temperature waveform. The method consisted of generating a heat pulse at one end of a tube filled with He II and photographing oscillascope traces representing the temperature waveform which pictorially show the deformation of a pulse as it travels down the tube. Qualitative measurements of the attenuation of the size of the pulse as a function of the distance traveled were taken for various temperatures and heat inputs. These experiments were the first direct experimental confirmation of some of the non-linear aspects of temperature wave propagation.

The other experimental investigation was specifically concerned with obtaining quantitative measurements of the temperature variation of the amplitude-dependent propagation speed; this series of experiments also used pulse techniques[42]. The theoretical predictions of Khalatnikov and Temperley were expressed in terms of a convenient dimensionless parameter which was indirectly measured experimentally for small heat pulses. For temperatures above 1.4°K the two theoretical results were found to be virtually identical and were verified by the experimental data. Since experimental verification was sought for predictions based on a linear theory, the magnitude of the total change of temperature in the heat pulse was kept very small so no non-linear effects were observed.

All but one of the analyses that have been done up to the present time are based on some perturbation procedure in which the relative velocity, $\overline{w}$, is the small parameter. And yet the Landau equations are definitely non-linear in character so, by analogy with ordinary hydrodynamics where linear and non-linear theories of pressure waves can be strikingly different, a non-linear approach will be taken here with one goal being the comparison of the results for small $\overline{w}$ with the previous theories. However, additional complications are present in He II which prevent even a non-linear analysis from being valid outside a rather limited range of conditions. One of these limitations is the existence of a critical velocity as mentioned earlier.

A second motivating purpose behind this work is the desire to apply the general theory of second sound to a specific problem and to discuss the flow in some detail. This type of additional analysis and detailed discussion of the results has not been done previously and will yield new insight into the nature of second sound. In particular, we will consider the behavior of a temperature pulse as it propagates through He II. As mentioned, this type of flow situation has been used to measure certain characteristics of second sound so it is of more than just academic interest. Moreover, this point of view provides a contract to analyses based on a sinusoidal temperature variation.

Keeping in mind the goal of attempting to describe in detail the nature of second sound, a number of simplifying assumptions will be made so that all considerations which appear to be second order effects are neglected and in order to theoretically isolate second sound from first sound. Also, all irreversible effects are implicitly neglected since the Landau equations, which do not include viscous effects or mutual friction, are used. Then the "Riemann theory of second sound" will be developed with the ultimate objective being a set of characteristics and invariants which describe the flow in a large class of problems. This approach is used because it works so well for the description of wave propagation in ordinary hydrodynamics and is especially suited to non-oscillatory driving functions.

This set of characteristics and invariants will then be used to obtain an explicit solution for the problem of the propagation of a temperature pulse in a semi-infinite, one-dimensional channel. This solution has some properties usually associated with pressure waves in classical hydrodynamics and other new features that have no analogy at all with wave motion in ordinary media. Subsequently, the initial assumptions will be dropped individually in order to study their relative importance and the order of magnitude of their effect on the first solution. This includes considering the interaction of first and second sound. Finally, to complete the theoretical investigation, the complete non-linear conservation laws describing the motion of discontinuous solutions are derived. These equations are extremely complex and are discussed from a general point of view since a perturbation analysis has already been done.

All these results are then summarized and some concluding remarks are made about the nature of second sound as well as about the advantages and disadvantages of the procedures used in the preceeding analysis. Several new areas of theoretical and experimental investigation are suggested by this work and some of these are discussed in detail. It is also shown that it might be worthwhile to reconsider some of the previous theoretical work that has been done so some suggestions for additional research are also made along this line.

CHAPTER III

THE RIEMANN THEORY OF SECOND SOUND

A. SIMPLIFICATION OF THE LANDAU EQUATIONS

The Landau equations serve as the starting point for this analysis of thermal waves. Within the other limitations mentioned earlier, this basic set of equations is valid from approximately 1°K up to the λ-point. Below this lower limit the normal density is very small -- ρ_n/ρ is less than .0086 for temperatures below 1°K -- so mean free path effects are becoming increasingly important. This leads to the conclusion that a two-fluid, continuum theory of He II is not valid at these very low temperatures and a different approach, possibly one similar to rarified gas dynamics, is needed.

The Landau equations as they are now written in Eqs. (1.3) are not in a convenient form for an analysis of first and second sound. They are much more amenable to a Riemann analysis when they are written entirely in terms of variables which can clearly be put into one of the two groups mentioned on page 25 in Chapter II. The mass flux velocity, $\bar{v}$, and the relative velocity, $\bar{w}$, are the mechanical velocity variables that will be used. In addition the total mass density, ρ, and the normalized relative density, δ, will be used as the independent thermodynamic variables; δ is defined by

$$\delta = \frac{\rho_n - \rho_s}{\rho} \quad . \tag{3.1}$$

The motivation for choosing this variable rather than ρ_n or ρ_s comes from the linear analyses of Khalatnikov and Temperley where it has been shown that the temperature where $\rho_n = \rho_s$, or where $\delta = 0$, is of particular importance. Thus it might be necessary to expand our results for small δ hence the choice of the normalized relative density and also its symbol "δ." The relative density varies between -1 and +1 as the temperature ranges from zero to the λ-point and is a very weak function of pressure.

The transformation equations relating ρ, δ, $\bar{v}$, and $\bar{w}$ to ρ_n, ρ_s, $\bar{v}_n$ and $\bar{v}_s$ are

$$\left.\begin{aligned} \rho &= \rho_n + \rho_s & \qquad \delta &= \frac{\rho_n - \rho_s}{\rho} \\ \bar{v} &= \frac{\rho_n}{\rho}\bar{v}_n + \frac{\rho_s}{\rho}\bar{v}_s & \qquad \bar{w} &= \bar{v}_n - \bar{v}_s \end{aligned}\right\} \tag{3.2a}$$

and

$$\left.\begin{aligned} \rho_n &= \frac{1+\delta}{2}\rho & \qquad \rho_s &= \frac{1-\delta}{2}\rho \\ \bar{v}_n &= \bar{v} + \frac{1-\delta}{2}\bar{w} & \qquad \bar{v}_s &= \bar{v} - \frac{1+\delta}{2}\bar{w} \end{aligned}\right\} . \tag{3.2b}$$

When the transformations (3.2b) are substituted into Eqs. (1.3) the resulting equations can be written in the following form

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\bar{v} = 0 \tag{3.3a}$$

$$\frac{\partial \rho\bar{v}}{\partial t} + \nabla\cdot\rho\bar{v}\bar{v} = -\nabla P - \nabla\cdot\frac{1-\delta^2}{4}\rho\bar{w}\bar{w} \tag{3.3b}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1+\delta}{2}\rho\bar{w}\right) + \nabla\cdot\frac{1+\delta}{2}\rho\bar{w}\bar{v} &= -\rho s\nabla T - \frac{1+\delta}{2}\rho\nabla\frac{\bar{w}^2}{2} \\ &- \frac{1+\delta}{2}\rho(\bar{w}\cdot\nabla)\left(\bar{v} - \frac{1+\delta}{2}\bar{w}\right) - \nabla\cdot\frac{1-\delta^2}{4}\rho\bar{w}\bar{w} \end{aligned} \tag{3.3c}$$

$$\frac{\partial \rho s}{\partial t} + \nabla\cdot\rho s\bar{v} = -\nabla\cdot\frac{1-\delta}{2}\rho s\bar{w} \quad . \tag{3.3d}$$

The left hand side of each of these equations is the material derivative moving with the mass flux velocity $\overline{v}$ of a scalar or vector quantity. The right hand side of the equations describing the motion of $\overline{v}$ and $\overline{w}$ contains forces due to the gradient of a thermodynamic variable plus additional terms involving the relative velocity, $\overline{w}$. These equations clearly show that a pressure gradient is the important thermodynamic force accelerating $\overline{v}$ while a temperature gradient is the most important thermodynamic force as far as $\overline{w}$ is concerned. This explains the natural pairing of P and $\overline{v}$, and T and $\overline{w}$ for the first and second sound respectfully. Nevertheless, there is a coupling between the two velocities and thermodynamic gradients by the other forcing terms involving $\overline{w}$.

Equations (3.3) constitute a set of eight equations for eight dependent variables which we will assume to be ρ, δ, $\overline{v}$, and $\overline{w}$. Three additional equations of state are necessary to make this formulation complete; assume these equations take the form

$$P = P(\rho, \delta, \overline{w}) \tag{3.4a}$$

$$T = T(\rho, \delta, \overline{w}) \tag{3.4b}$$

$$s = s(\rho, \delta, \overline{w}) \quad . \tag{3.4c}$$

If it is assumed that $\overline{w} = 0$ then Eq. (3.3c) implies that temperature is a constant, entropy is conserved by Eq. (3.3d) and the remaining equations including Eqs. (3.4) form the classical set of inviscid hydrodynamic equations.

It is now necessary to make the following three assumptions in order to reduce these equations as much as possible and still not destroy their ability to describe temperature waves.

1. Consider only those solutions for which $\overline{v} = 0$. This restriction makes it impossible for first sound to be generated so that the interaction of first and second sound is neglected and we will be dealing strictly with thermal waves.

2. Assume s_n, the specific entropy of the normal part, is constant. This is an empirical fact and not the consequence of some fundamental principle. This assumption is a very good first approximation for temperatures between about 1.4°K and the λ-point but is only a crude approximation for temperatures less than 1°K.

3. Assume it is possible to neglect the $\overline{w}$ dependence of the thermodynamic variables in Eqs. (3.4). The dependence of these variables on $\overline{w}$ should be weak but one of the objectives of subsequent analyses will be to calculate the effects of this dependence.

Equations (3.3) are considerably simplified by these assumptions. If the total mass density is a constant, say ρ_o, at some time everywhere in a given volume of the fluid then Eq. (3.3a) combined with the first assumption imply that $\rho = \rho_o$ for all time. The assumption that $\overline{v} = 0$, when used in conjunction with Eq. (3.3b) also yields a restrictive condition on the pressure:

$$\nabla P + \rho_o \nabla \cdot \frac{1-\delta^2}{4} \overline{w}\,\overline{w} = 0 \quad . \tag{3.5}$$

Under the first and third assumptions, Eqs. (3.4) have only a δ dependence

$$P = P(\delta) \quad , \quad T = T(\delta) \quad , \quad s = s(\delta) \quad . \tag{3.6}$$

Thus, Eq. (3.5) is considered to be a restrictive condition which limits the class of functions $P = P(\delta)$ that are allowable when the above assumptions are valid. Notice that a pressure gradient will accompany second sound even though it has been assumed that no first sound is present. Because its existence depends directly on a non-zero $\overline{w}$, this type of pressure variation will move with the relative velocity or, in other words, a pressure wave propagates with the velocity of thermal waves in a purely second sound flow.

The definition of s_n can be written

$$s = \frac{\rho_n}{\rho} s_n = \frac{1+\delta}{2} s_n \quad .$$

Consequently, under all the above assumptions, Eq. (3.4d) becomes an equation for δ;

$$\frac{\partial}{\partial t}\left(\frac{1+\delta}{2}\right) + \nabla\cdot \frac{1-\delta^2}{4} \overline{w} = 0 \quad .$$

Finally, Eq. (3.3c) is

$$\frac{1+\delta}{2}\frac{\partial \overline{w}}{\partial t} + \overline{w}\left[\frac{\partial}{\partial t}\left(\frac{1+\delta}{2}\right) + \nabla\cdot \frac{1-\delta^2}{4}\overline{w}\right] = - s\nabla T$$

$$- \frac{1+\delta}{2}\nabla\frac{\overline{w}^2}{2} + \frac{1+\delta}{2}(\overline{w}\cdot\nabla)\frac{1+\delta}{2}\overline{w} - \frac{1-\delta^2}{4}(\overline{w}\cdot\nabla)\overline{w} \quad . \quad (3.7)$$

The bracketed term vanishes by the previous equation.

In summary, under the three simplifying assumptions ∇P is a quadratic function of $\overline{w}$, and δ is the most convenient thermodynamic variable, and Landau's equations reduce to the following set of equations for δ and $\overline{w}$;

$$\frac{\partial \overline{w}}{\partial t} + \frac{2}{1+\delta} s\nabla T + \nabla \frac{\overline{w}^2}{2} - \delta(\overline{w}\cdot\nabla)\overline{w} - \frac{1}{2}\overline{w}\,\overline{w}\cdot\nabla\delta = 0 \tag{3.8a}$$

$$\frac{\partial \delta}{\partial t} + \frac{1-\delta^2}{2}\nabla\cdot\overline{w} - \delta\overline{w}\cdot\nabla\delta = 0 \quad . \tag{3.8b}$$

We can use the second of Eqs. (3.6) to write

$$\nabla T = \left(\frac{\partial T}{\partial s}\right)\left(\frac{\partial s}{\partial \delta}\right)\nabla\delta$$

$$= \frac{1}{1+\delta} s\left(\frac{\partial T}{\partial s}\right)\nabla\delta$$

and

$$\frac{2}{1+\delta} s\nabla T = \frac{2}{1-\delta^2} u_2^2 \nabla\delta \tag{3.9}$$

where u_2 is defined by Eq. (2.6); in this scheme $u_2 = u_2(\delta)$. Equations (3.8) can be further reduced by considering a one-dimensional problem. Then all variables depend only on one space dimension, for instance x, and time. Also

$$\overline{w} = w(x,t)\hat{e}_x \quad .$$

Thus, Eqs. (3.8) written in matrix form are

$$\begin{pmatrix} \frac{\partial w}{\partial t} \\ \\ \frac{\partial \delta}{\partial t} \end{pmatrix} + \begin{pmatrix} (1-\delta)w & \frac{2}{1-\delta^2}u_2^2 - \frac{1}{2}w^2 \\ \\ \frac{1-\delta^2}{2} & -\delta w \end{pmatrix} \begin{pmatrix} \frac{\partial w}{\partial x} \\ \\ \frac{\partial \delta}{\partial x} \end{pmatrix} = 0 \quad . \tag{3.10}$$

In addition, the one-dimensional, integrated form of Eq. (3.5) is

$$\frac{P - P_o}{\rho_o} = -\frac{1-\delta^2}{4} w^2 \quad . \tag{3.11}$$

B. DERIVATION AND APPLICATION OF THE CHARACTERISTICS AND INVARIANTS

Having derived the basic equations which describe thermal waves, Eqs. (3.10), we will now study them in detail to deduce some of the basic features of second sound. The following analysis is based on the theory of hyperbolic equations as discussed in Appendix A. The important details in this analysis and in the solution of the problem following it are explicitly shown in order to clearly illustrate the method which is being used. For subsequent analyses similar to this one we will only point out the essential differences between the two systems of governing equations and then quote the final results by analogy to this procedure.

In this instance there are only two unknown dependent variables, δ and w, so $n = 2$. Also, the matrix A in Eq. (A2) is the unit matrix, I, so the condition determining the characteristics is

$$\begin{vmatrix} (1-\delta)w-c & \dfrac{2}{1-\delta^2}\, u_2^2 - \dfrac{1}{2}\, w^2 \\ \dfrac{1-\delta^2}{2} & -\delta w - c \end{vmatrix} = 0 \quad .$$

Expanding this determinant gives the following quadratic for c;

$$c^2 - (1-2\delta)wc - u_2^2 + \frac{1}{4}(3\delta^2-4\delta+1)w^2 = 0 \quad . \tag{3.12}$$

Therefore, the characteristic lines are

$$L_+ : \left(\frac{dx}{dt}\right)_+ = c^+ = \left(\frac{1}{2} - \delta\right) w + \left(u_2^2 + \frac{1}{4}\,\delta^2 w^2\right)^{\frac{1}{2}}$$

$$L_- : \left(\frac{dx}{dt}\right)_- = c^- = \left(\frac{1}{2} - \delta\right) w - \left(u_2^2 + \frac{1}{4}\,\delta^2 w^2\right)^{\frac{1}{2}} \quad .$$

Since the c's are always real and distinct the system of Eqs. (3.10) is always hyperbolic. This means that the temperature always propagates with a finite speed rather than obeying a diffusion equation.

In this case the equation for the left eigenvectors reduces to

$$\bar{\ell}^k B = c^k \bar{\ell}^k$$

or

$$\ell_i^{\pm} b_{ij} = c^{\pm} \ell_j^{\pm} \quad .$$

Using the second column of the B matrix these equations are

$$\left(\frac{2}{1-\delta^2} u_2^2 - \frac{1}{2} w^2 \right) \ell_1^{\pm} = \left[\frac{1}{2} w \pm \left(u_2^2 + \frac{1}{4} \delta^2 w^2 \right)^{\frac{1}{2}} \right] \ell_2^{\pm} \quad .$$

Pick

$$\ell_1^{\pm} = \frac{1-\delta^2}{2}$$

$$\ell_2^{\pm} = -\frac{1}{2} w \pm \left(u_2^2 + \frac{1}{4} \delta^2 w^2 \right)^{\frac{1}{2}} \quad .$$

Since $A = I$, the invariants are given by

$$\ell_1^{\pm} dw_{\pm} + \ell_2^{\pm} d\delta_{\pm} = 0$$

with $\ell_1^{\pm}$ and $\ell_2^{\pm}$ given above. In these equations $dw_{\pm}$ represents the differential of w taken along the $L_{\pm}$ characteristic, respectively. A similar interpretation holds for $d\delta_{\pm}$. These results are summaries in the following table.

$$L_+ : \text{Characteristic,} \quad \left(\frac{dx}{dt} \right)_+ = \left(\frac{1}{2} - \delta \right) w + \left(u_2^2 + \frac{1}{4} \delta^2 w^2 \right)^{\frac{1}{2}} \qquad (3.13)$$

$$\text{Invariant,} \quad \frac{1-\delta^2}{2} dw_+ - \left[\frac{1}{2} w - \left(u_2^2 + \frac{1}{4} \delta^2 w^2 \right)^{\frac{1}{2}} \right] d\delta_+ = 0 \qquad (3.14)$$

$$L: \text{Characteristic,} \quad \left(\frac{dx}{dt}\right)_- = \left(\frac{1}{2} - \delta\right) w - \left(u_2^2 + \frac{1}{4}\delta^2 w^2\right)^{\frac{1}{2}} \qquad (3.15)$$

$$\text{Invariant,} \quad \frac{1-\delta^2}{2}\, dw_- - \left[\frac{1}{2} w + \left(u_2^2 + \frac{1}{4}\delta^2 w^2\right)^{\frac{1}{2}}\right] d\delta_- = 0 \qquad (3.16)$$

This set of characteristics and invariants provides a useful tool for solving a large class of problems for which Eqs. (3.10) are adequate to describe the flow.

We will now apply these results to a simple problem to illustrate the procedure by which characteristics and invariants are used to obtain a solution and to discover some of the basic features of second sound. Suppose a semi-infinite, one-dimensional channel filled with He II is initially at a uniform temperature T_o and assume the fluid is at rest. At the one end, which we will take to be $x = 0$, the temperature is a given function of time. It will be implicitly assumed that this boundary condition is some general temperature pulse. The set of equations which describe this problem has been formulated and analyzed above using δ, rather than T, as the independent thermodynamic variable. However, δ is a monotonically increasing function of temperature so there is no difficulty in reformulating the initial and boundary conditions in terms of δ. It will also be a simple matter to write the final results in terms of either δ or T. Initially, the fluid in equilibrium is represented by $w = 0$ and $\delta = \delta_o = \delta(T_o)$. The boundary condition is now $\delta = \delta[T(t)] = \delta(t)$ for $t > 0$. A complete solution is known when the variables $w(x,t)$ and $\delta(x,t)$ are known for all $x > 0$, $t > 0$ in terms of δ_o and the given boundary condition $\delta(t)$.

A type of iteration scheme will be used to find this solution.

We begin by considering a point P near the x-axis as shown in Fig. 3.

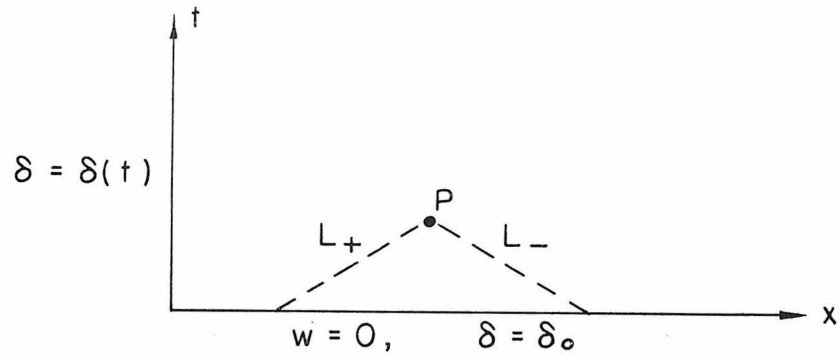


Figure 3

By evaluating the coefficients of the differentials in the invariants on the x-axis we can write,

$$\left.\begin{aligned} \text{Along } L_+ &: \frac{1-\delta_o^2}{2}\, dw_+ + u_2(\delta_o)d\delta_+ = 0 \\ \text{Along } L_- &: \frac{1-\delta_o^2}{2}\, dw_- - u_2(\delta_o)d\delta_- = 0 \quad . \end{aligned}\right\} \tag{3.17}$$

Since P has been chosen so that $L_\pm$ are short line segments, we can also write

$$\text{Along } L_+ : \delta(P) = \delta_o + d\delta_+ \quad , \quad w(P) = 0 + dw_+$$

$$\text{Along } L_- : \delta(P) = \delta_o + d\delta_- \quad , \quad w(P) = 0 + dw_- \quad .$$

The requirement that the solution be single-valued at P implies that

$$dw_+ = dw_- = dw$$

and

$$d\delta_+ = d\delta_- = d\delta \quad .$$

Equations (3.17) now give the solutions $dw = 0$ and $d\delta = 0$; hence

$$\delta(P) = \delta_o \tag{3.18a}$$

and

$$w(P) = 0 \quad . \tag{3.18b}$$

This result is valid for all points P whose domain of dependence is restricted to just the x-axis. This in turn is determined by the slope of the L_+ characteristic which passes through the origin of the x, t-plane; by Eqs. (3.18) this characteristic, called the "wavefront", is a straight line. We will call the slope of this line u_2°;

$$u_2^\circ = u_2(\delta_o) = u_2[\delta(T_o)] \quad . \tag{3.19}$$

Hence, the equation for the wavefront is

$$\frac{dx}{dt} = u_2^\circ$$

or

$$x - u_2^\circ \cdot t = 0 \quad . \tag{3.20}$$

Thus far we have found the expected result that there is no disturbance ahead of the wavefront whose slope in the x, t-plane is determined by the velocity of second sound at the initial temperature.

Next, consider some time τ on the t-axis close to $t = 0$ (see Fig. 4). Some characteristic L_- must exist which intersects the t-axis at τ and which goes into the equilibrium zone ahead of the

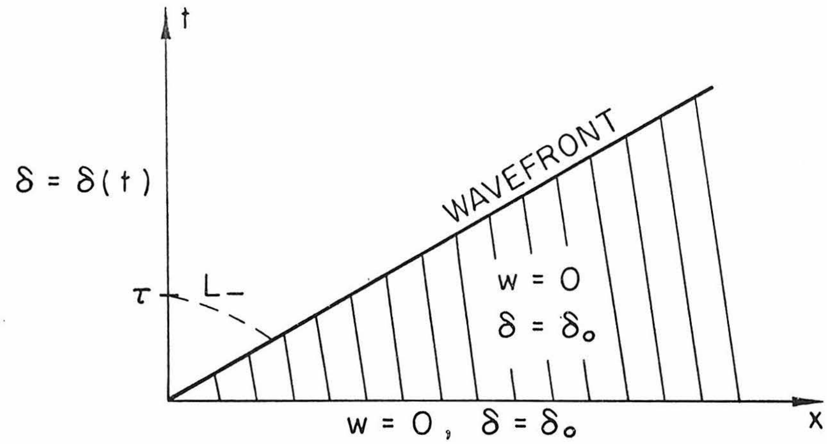


Figure 4

wavefront. Reasoning as before, we can write

$$\text{Along } L_- : \frac{1-\delta_o^2}{2}\, dw_- - u_2^\circ d\delta_- = 0 \quad .$$

Also

$$\delta(\tau) = \delta_o + d\delta_-$$

and

$$w(\tau) = 0 + dw_- \quad .$$

In this instance, however, $\delta(\tau)$ is known by the given boundary condition so the above equations combine to give an expression for $w(\tau)$ for small τ,

$$w(\tau) = 2\,\frac{\delta(\tau)-\delta_o}{1-\delta_o^2}\, u_2^\circ \quad . \tag{3.21}$$

The next step is to consider a point P which is close to both the t-axis and the wavefront as shown in Fig. 5. Using the same arguments as before

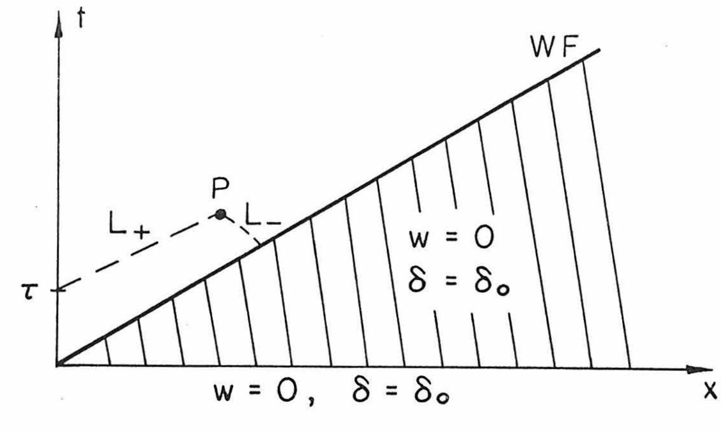


Figure 5

$$\text{Along } L_+ : \frac{1-\delta^2(\tau)}{2} dw_+ - \left\{ \frac{w(\tau)}{2} - \left[u_2^2(\tau) + \frac{1}{4}\delta^2(\tau)w^2(\tau)\right]^{\frac{1}{2}} \right\} d\delta_+ = 0$$

$$\text{Along } L_- : \frac{1-\delta_o^2}{2} dw_- - u_2^\circ d\delta_- = 0$$

and

$$\text{Along } L_+ : \delta(P) = \delta(\tau) + d\delta_+ \quad , \qquad w(P) = w(\tau) + dw_+$$

$$\text{Along } L_- : \delta(P) = \delta_o + d\delta_- \quad , \qquad w(P) = 0 + dw_- \quad .$$

Again the unknowns are $w(P)$ and $\delta(P)$ while $w(\tau)$ is now known from Eq. (3.21). Eliminating $dw_\pm$ and $d\delta_\pm$ from these equations gives two equations for $\delta(P)$ and $w(P)$ which have solutions

$$w(P) = w(\tau)$$

and

$$\delta(P) = \delta(\tau) \quad .$$

Consequently, the L_+ characteristic emanating from the t-axis at τ is a straight line and the values of w and δ on this characteristic are constant and equal to their values on the t-axis.

Finally, consider two neighboring L_+ characteristics which intersect the t-axis at τ_1 and τ_2 and assume that τ_1 and τ_2 are close to each other. Following the general scheme used in the preceeding calculations it is possible to show that all L_+ characteristics are straight with w and δ constant along them. In addition we find

$$w_2 - w_1 = 2\,\frac{\delta_2 - \delta_1}{1 - \delta_1^2}\left\{\frac{w_1}{2} + \left[u_2^2(\delta_1) + \frac{1}{4}\delta_1^2 w_1^2\right]^{\frac{1}{2}}\right\}$$

where the subscripts "1" and "2" indicate that the variables are evaluated at τ_1 and τ_2 respectfully on the t-axis. Now let $\tau_2 \to \tau_1$; from the above expression we get an ordinary differential equation relating w and δ at the boundary $x = 0$.

$$\frac{1-\delta^2}{2}\,\frac{dw}{d\delta} = \frac{w}{2} + \left(u_2^2 + \frac{1}{4}\delta^2 w^2\right)^{\frac{1}{2}} \quad . \tag{3.22}$$

Since u_2 is a known function of δ this equation can be solved in principle, to give $w = w(\delta) = w[\delta(t)] = w(t)$ on the t-axis, or equivalently, at the end of the tube. With w now known the slope of the L_+ characteristic intersecting the t-axis at τ is given by

$$\left(\frac{dx}{dt}\right)_+ = c(\tau) = \left[\frac{1}{2} - \delta(\tau)\right]w(\tau) + \left[u_2^2(\delta) + \frac{1}{4}\delta^2(\tau)w^2(\tau)\right]^{\frac{1}{2}} \quad . \tag{3.23}$$

This can be directly integrated to give an explicit equation for L_+;

$$x - c(\tau)(t-\tau) = 0 \quad . \tag{3.24}$$

We now have a complete solution. It can be summarized as follows:

$$w(x,t) = \begin{cases} 0 & \text{for} \quad x > u_2^\circ \cdot t \\ w(\tau) & \text{for} \quad x < u_2^\circ \cdot t \end{cases} \tag{3.25a}$$

and

$$\delta(x,t) = \begin{cases} \delta_o & \text{for} \quad x > u_2^\circ \cdot t \\ \delta(\tau) & \text{for} \quad x < u_2^\circ \cdot t \end{cases} \tag{3.25b}$$

where, for a given point (x,t), τ is implicitly given by Eq. (3.24) with $c(\tau)$ given by Eq. (3.23), $\delta(\tau)$ is the boundary condition at $x = 0$, $u_2 = u_2(\delta)$ is a given function, and $w(\tau)$ is determined from Eq. (3.22) [see Appendix D].

Only Eq. (3.22) needs to be modified when the temperature is used as the fundamental thermodynamic variable. We can relate w to T rather than δ by writing

$$\frac{dw}{d\delta} = \left(\frac{\partial T}{\partial \delta}\right)\frac{dw}{dT} .$$

Since s_n is a constant

$$\left(\frac{\partial \delta}{\partial T}\right) = \frac{2}{s_n}\left(\frac{\partial s}{\partial T}\right) = (1-\delta)\,\frac{s}{u_2^2} \tag{3.26}$$

so

$$\frac{dw}{dT} = \frac{2}{1+\delta}\,\frac{s}{u_2^2}\left[\frac{1}{2}\,w + \left(u_2^2 + \frac{1}{4}\,\delta^2 w^2\right)^{\frac{1}{2}}\right] . \tag{3.27}$$

The other parts of the solution remain unaltered except for the fact that δ and u_2 are now considered to be functions of T.

There are places in the derivation of this solution where significant difficulties could arise. For instance, it might be possible to obtain different solutions than the one found for Eq. (3.17) if the determinant of that system of equations vanishes. However, it is a simple matter to show that this can not occur and Eqs. (3.18) do indeed represent a unique set of solutions.

Another point of concern arises in connection with Fig. 4 where we have shown an L_- characteristic intersecting the t-axis. It is possible that the L_- lines may either intersect each other before crossing the t-axis or they may curve away from it completely and become vertical at some point. In either case Eq. (3.21) would no longer be valid. It is subsequently shown, however, that both w and δ are constant along all L_+ lines so that all members of the L_- family of characteristics are parallel at the point where they cross the line $x - c(\tau)(t-\tau) = 0$. Since this is true for all τ, the L_- characteristics never do intersect.

Equation (3.15) is the differential equation for the L_- characteristics. If an L_- line ever does become vertical in the x,t-plane then at that point

$$\left(\frac{dx}{dt}\right)_- = 0 \quad . \tag{3.28}$$

Thus

$$\left(\frac{1}{2} - \delta\right) w - \left(u_2^2 + \frac{1}{4}\,\delta^2 w^2\right)^{\frac{1}{2}} = 0$$

or

$$\frac{(1-3\delta)(1-\delta)}{4}\, w^2 = u_2^2 \quad .$$

A study of these equations shows that $w > 2/3\, u_2$ if Eq. (3.27) is true. But the original assumption that w is not important as a thermodynamic variable (assumption 3) is no longer valid if this inequality is true. Therefore this consideration is not a limiting factor as far as the validity of the derivation of the solution is concerned.

It is also possible that the L_+ characteristics will intersect each other with the result that the one parameter family of L_+ lines forms an envelope and discontinuities appear in the solution. This actually does occur and will be discussed in detail in the next Section.

One final point should be made here concerning this entire analysis and much of the work contained in the next Section. It has been mentioned previously that the Landau equations represent the actual flow only under a very restrictive set of circumstances. In particular, they are valid only to order w^2; in addition, terms of the same order of magnitude have been neglected as a result of the assumption that w is not an important thermodynamic variable. Consequently, all of the above calculations are valid to order w, at best. It is more correct, then, to rewrite the differential equation giving w at the boundary as

$$\frac{1-\delta^2}{2}\frac{dw}{d\delta} = u_2\left[1 + \frac{1}{2}\frac{w}{u_2} + O\left(\frac{w^2}{u_2^2}\right)\right] \qquad (3.29)$$

or

$$\frac{dw}{dT} = \frac{2}{1+\delta}\,\frac{s}{u_2}\left[1 + \frac{1}{2}\frac{w}{u_2} + O\left(\frac{w^2}{u_2^2}\right)\right] \qquad (3.30)$$

and the slope of the L_+ characteristics as

$$c(\tau) = u_2\left[1 + \left(\frac{1}{2} - \delta\right)\frac{w}{u_2} + O\left(\frac{w^2}{u_2^2}\right)\right] \quad . \qquad (3.31)$$

However, a numerical analysis has shown that neglecting the higher order terms has very little actual effect on the solution. The result of the numerical integration of Eq. (3.30) to give $w(T)$ for different values of T_o is shown in Fig. 6 for $|w| < u_2$. The values for u_2 and s were taken from Donnelly[3] and it was assumed that

$$\frac{\rho_n}{\rho} = \left(\frac{T}{T_\lambda}\right)^{5.6}$$

which London[1] has shown to be true in the same region where s_n is a constant. These results differ from those obtained by integrating Eq. (3.27) rather than Eq. (3.30) by less than 1 meter/sec for all initial temperatures and for w less than u_2.

Equation (3.31) is the only one that can be verified to $O(w)$ by Khalatnikov's linear theory. We can reduce Eq. (B7) so that it is consistent with this theory by

1. Neglecting $\left(\frac{\partial \rho_n}{\partial T}\right)$ so the expansion of the thermodynamic variables is not included,
2. Setting $v = 0$ and hence $u \rightarrow c$, and
3. Assuming s_n is a constant.

Under the third assumption

$$\frac{u_2^2}{\rho_s s}\left(\frac{\partial \rho_n}{\partial T}\right) = 1$$

and the quadratic for c is

$$c^2 - (1-2\delta)wc - u_2^2 = 0$$

which is the same as Eq. (3.12) if $O(w^2)$ terms are neglected.

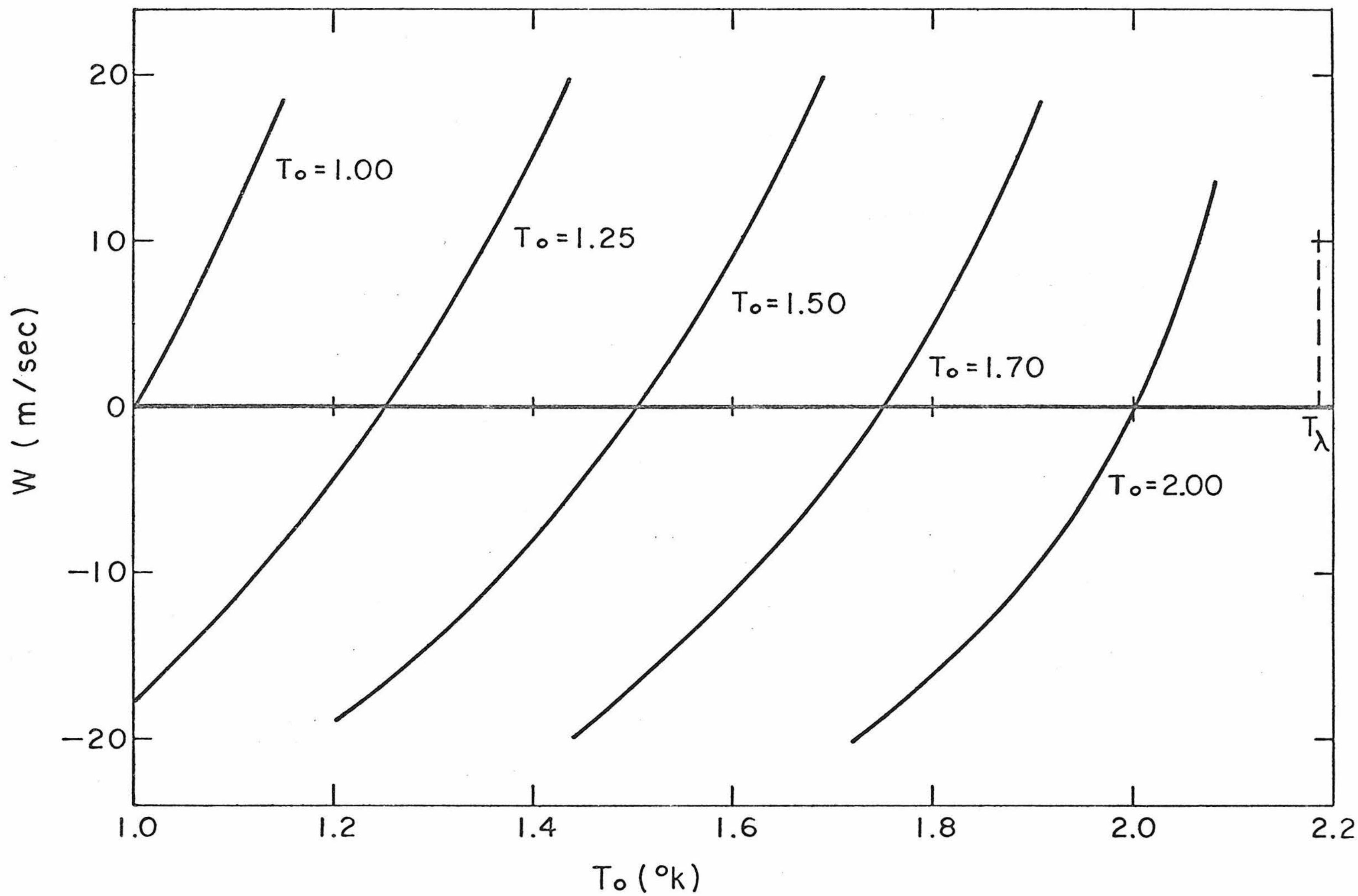


Figure 6 - Relative Velocity Versus Temperature

C. THE NATURE OF THE SOLUTION

Equations (3.25) represent the time and spatial dependence of the flow parameters δ and w for the propagation of a temperature fluctuation given at the end of a semi-infinite tube containing He II. Although this solution has a limited range of validity it displays many of the important characteristics of second sound and will serve as a basis of comparison with more general solutions derived later and with other theoretical results. It also has the distinct advantage of being an explicit solution and can therefore be studied in detail with the use of some relatively simple analytical tools.

By inspection of this solution it is clear that it represents the undamped propagation of δ and w with c, as defined by Eq. (3.23), being the propagation speed. In this approximation c is a function of temperature only. As Temperley[40] has pointed out, the terms involving w have a much greater influence on the temperature dependence of the velocity of propagation than the temperature variation of u_2 itself. This is in contrast to the case in ordinary gas dynamics where just the opposite situation occurs. It is not surprising that there is no attenuation since all irreversible processes and the interaction of first and second sound have been neglected.

Some non-linear aspects of this solution such as the deformation of a given temperature pulse as it propagates through He II are found by considering the temperature dependence of the velocity of propagation, c. Since the L_+ characteristics are straight lines and c measures their slope with respect to the t-axis, it is clear that two adjacent L_+ lines will converge towards each other and intersect if

$$c'(\tau) = \frac{dc}{d\tau} > 0 \quad . \tag{3.32}$$

If this condition is met then the L_+ lines will form an envelope. Equation (3.23) is the equation for these L_+ lines;

$$x - c(\tau)(t-\tau) = 0 \quad . \tag{3.23}$$

To find the equation of the envelope differentiate this equation with respect to the parameter τ,

$$c(\tau) - c'(\tau)(t-\tau) = 0$$

and combine these two equations to write the parametric equation for the envelope as

$$t = \tau + \frac{c(\tau)}{c'(\tau)} = t(\tau) \tag{3.33a}$$

$$x = \frac{c^2(\tau)}{c'(\tau)} = x(\tau) \quad . \tag{3.33b}$$

These equations can also be written

$$t - \tau = -c^{-1}\left[\frac{d}{d\tau}\,(c^{-1})\right]^{-1}$$

$$x = -\left[\frac{d}{d\tau}\,(c^{-1})\right]^{-1} \quad .$$

Therefore, if

$$\frac{d}{d\tau}\left\{\left[\frac{d}{d\tau}\,(c^{-1})\right]^{-1}\right\} = 0 \qquad \text{at} \quad \tau = \tilde{\tau}$$

then the envelope has a cusp at

$$t_c = \tilde{\tau} + \frac{c(\tilde{\tau})}{c'(\tilde{\tau})} \tag{3.34a}$$

$$x_c = \frac{c^2(\tilde{\tau})}{c'(\tilde{\tau})} \quad . \tag{3.34b}$$

The pair (x_c, t_c) is the point in the x, t-plane where a discontinuity first appears in the solution given by Eqs. (3.25) and this solution is no longer valid.

In order to write Eq. (3.32) as condition on the temperature, define the function F by

$$F(T) = \frac{\partial c}{\partial T} \tag{3.35}$$

so $c'(\tau) > 0$ if

$$F(T)\,\frac{dT}{d\tau} = F(T)T' > 0 \quad . \tag{3.36}$$

Using Eq. (3.27) to eliminate $\frac{dw}{dT}$ we can write F as

$$F(T) = \frac{s}{u_2}\left(1 + \frac{1}{4}\,\delta^2\,\frac{w^2}{u_2^2}\right)^{-\frac{1}{2}}\left\{h + \frac{1-2\delta}{1+\delta} - \frac{(1-\delta)(1+3\delta)}{2(1+\delta)}\,\frac{w}{u_2}\left[\left(1 + \frac{1}{4}\,\delta^2\,\frac{w^2}{u_2^2}\right)^{\frac{1}{2}} - \frac{1}{2}\,\delta\,\frac{w}{u_2}\right]\right\} \tag{3.37}$$

$$= \frac{s}{u_2}\left[h + \frac{1-2\delta}{1+\delta} - \frac{(1-\delta)(1+3\delta)}{2(1+\delta)}\,\frac{w}{u_2} + O\left(\frac{w^2}{u_2^2}\right)\right] \tag{3.38}$$

where

$$h = \frac{u_2}{s}\left(\frac{\partial u_2}{\partial T}\right) \tag{3.39}$$

This function h is dimensionless and provides a measure of the importance of the temperature dependence of u_2. The function h(T) is shown in Fig. 7. This function must vanish at the λ-point since $u_z(T_\lambda) = 0$ and all other factors are finite. Also, it appears that $h \to -\infty$ as $T \to 0$. This is to be expected since $s \to 0$ by the third law of thermodynamics but the detailed T dependence of u_z must be known before this can be stated as a certainty.

Solving Eq. (3.27) for the function $w = w(T)$ involves using the condition $w = 0$ at $T = T_o$ to eliminate the integration constant. Hence, the influence of T_o on condition (3.32) comes only through the dependence of w on T_o in $F(T)$. Notice that this condition depends on T, T_o, and the sign of T' but not on the magnitude of T'. However, if the L_+ characteristics do converge both x_c and $t_c - \tilde{\tau}$ are inversely proportional to the magnitude of T'.

We can now state that a discontinuity will appear in the solution if

$$F(T)\begin{cases} > 0 & \text{for} \quad T' > 0 \\ < 0 & \text{for} \quad T' < 0 \end{cases} \quad . \tag{3.40}$$

This leads one to define a "critical" temperature T_c by

$$F(T_c) = 0 \tag{3.41}$$

with its corresponding critical $\delta = \delta_c = \delta(T_c)$. This critical temperature is important because, for a given initial temperature, it divides the temperature range of interest into separate regions in which the nature of the solution is quite different. Using this concept of the critical temperature we can write Eq. (3.40) as a condition directly on the temperature as follows:

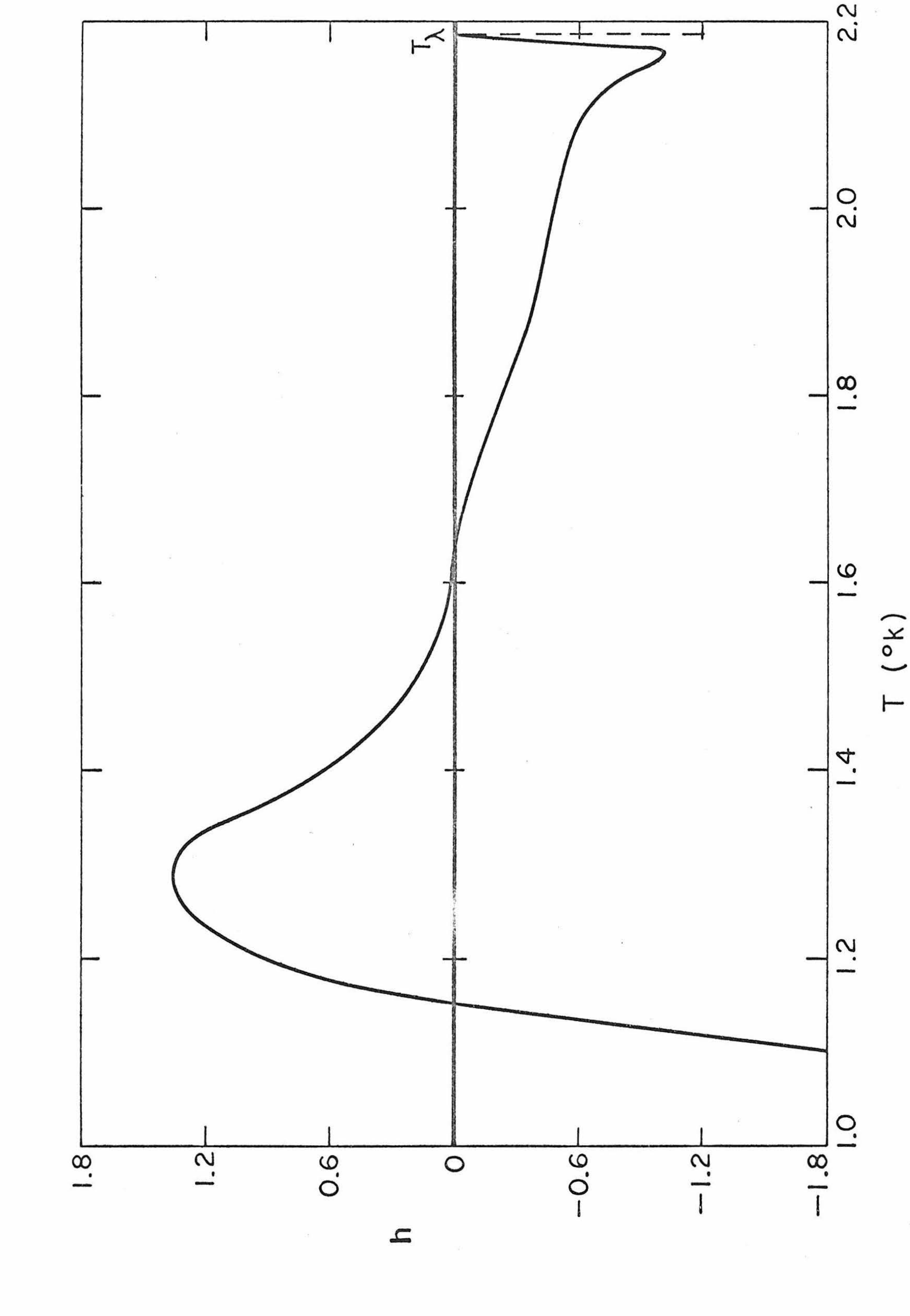


Figure 7 - h Versus Temperature

$$T \begin{cases} \gtrless T_c & \text{when} \quad T' \gtrless 0 \quad \text{for} \quad F(T) \gtrless T_c) \gtrless 0 \\ \gtrless T_c & \text{when} \quad T' \lesseqgtr 0 \quad \text{for} \quad F(T \gtrless T_c) \lesseqgtr 0 . \end{cases} \tag{3.42}$$

Equation (3.41) is considerably simplified at the initial conditions. Defining $f(T)$ as $F(T)$ when $w = 0$, Eq. (3.41) reduces to

$$f(T_c) = \frac{s(T_c)}{u_2(T_c)} \left[h(T_c) + \frac{1 - 2\delta_c}{1 + \delta_c} \right] = 0 \quad . \tag{3.43}$$

This $f(T)$ is strictly a function of T and does not depend on the detailed solution given by Eq. (3.25). Although $c = u_2$ when $w = 0$, this equation shows that

$$\frac{\partial c}{\partial T} \neq \frac{\partial u_2}{\partial T}$$

since

$$\frac{dw}{dT} \neq 0 \quad \text{at} \quad T = T_o \quad .$$

If the inequalities stated by Eq. (3.42) with $T = T_o$ and F replaced by f are valid initially, then the discontinuity will first appear at the wavefront. If we could neglect the temperature dependence of u_2, h would be identially zero and

$$\delta_c = 1/2 \quad \text{or} \quad T_c = 2.11°K \quad . \tag{3.44}$$

This first estimate of the critical temperature could have been made directly from Eq. (3.31) with u_2 a constant.

The function $f(T)$ is shown in Fig. 8. Again there is only one critical temperature; it has a value of

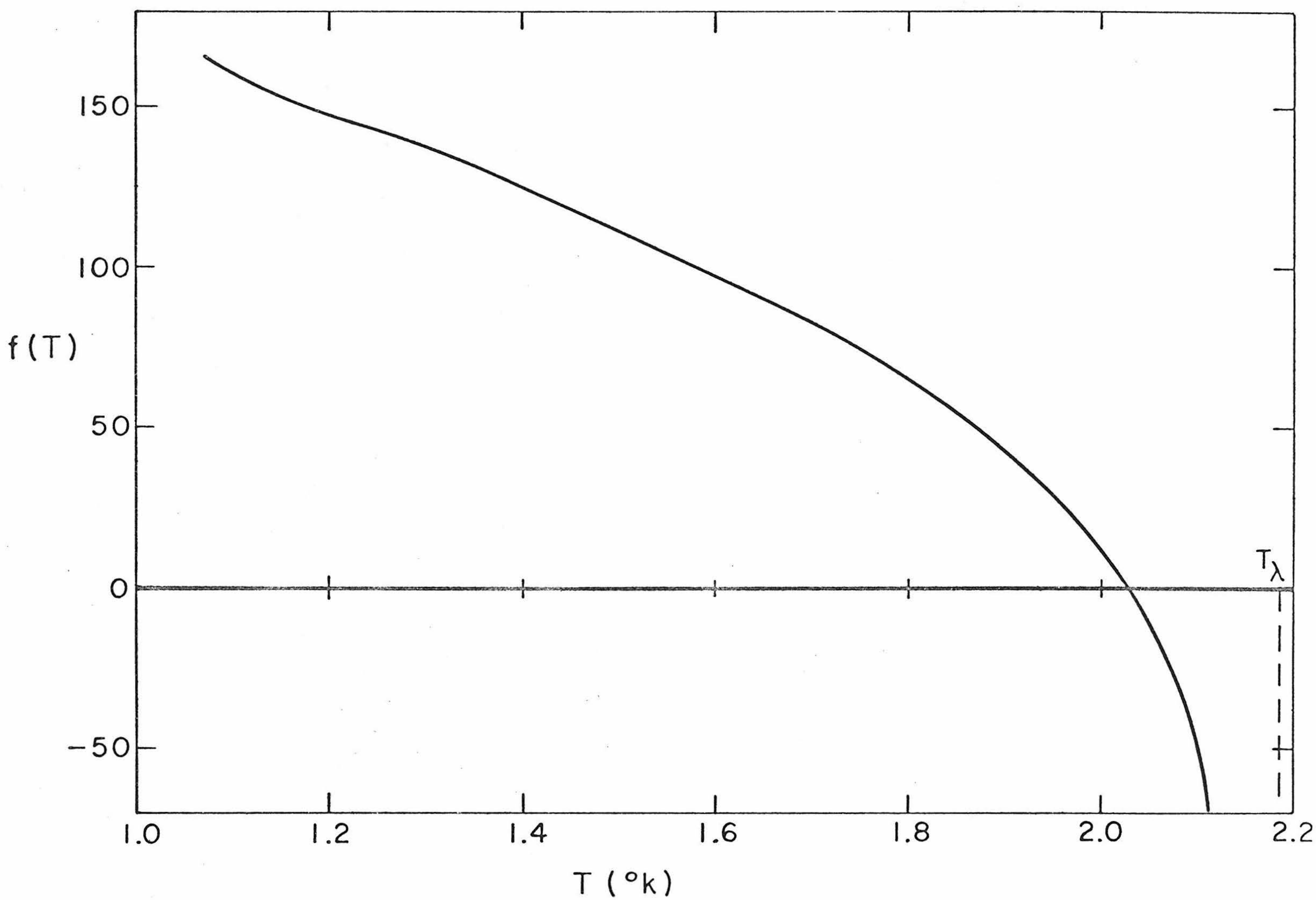


Figure 8 - f Versus Temperature

$$T_c = 2.02°K \quad . \tag{3.45}$$

Since $f(T \gtrless T_c) \lesseqgtr 0$, Eq. (3.42) is simply

$$T_o \gtrless T_c \qquad \text{when} \qquad T_o' \gtrless 0 \quad . \tag{3.46}$$

For example, if $T_o < 2.00°K$ then a discontinuity will appear at the wavefront if T_o', the initial time rate of change of T at the boundary, is positive.

The complete function $F(T)$ does depend on the specific solution and this is shown in Fig. 9 for various initial temperatures. Again $F(T \gtrless T_c) \gtrless 0$ so Eq. (3.46) is still the condition that determines whether a discontinuity will appear in the solution. As in Fig. 6 only the results for the temperature range where $|w| < u_2$ are shown. When the initial temperature is less than 1.75°K there is no critical temperature and, by Eq. (3.36) a discontinuity will appear in the solution only if $T' > 0$. For $T_o > 1.80°K$ there is a critical temperature which is a function of T_o as shown in Fig. 10. However, the low temperature portion of each $F(T)$ curve is also pointed toward the abscissa so, in a better theory, we might expect the existence of two critical temperatures for each T_o if this trend is continued. At $T = 2.01°K$, the initial and critical temperatures have the same value. The temperature at which $T_o = T_c$, which will be defined as T^*, is a very important one because it, like the critical temperature, separates regions of different flow characteristics. In fact, the prediction of this temperature is one of the major objectives of this theory. We will discuss this point in greater detail in the next

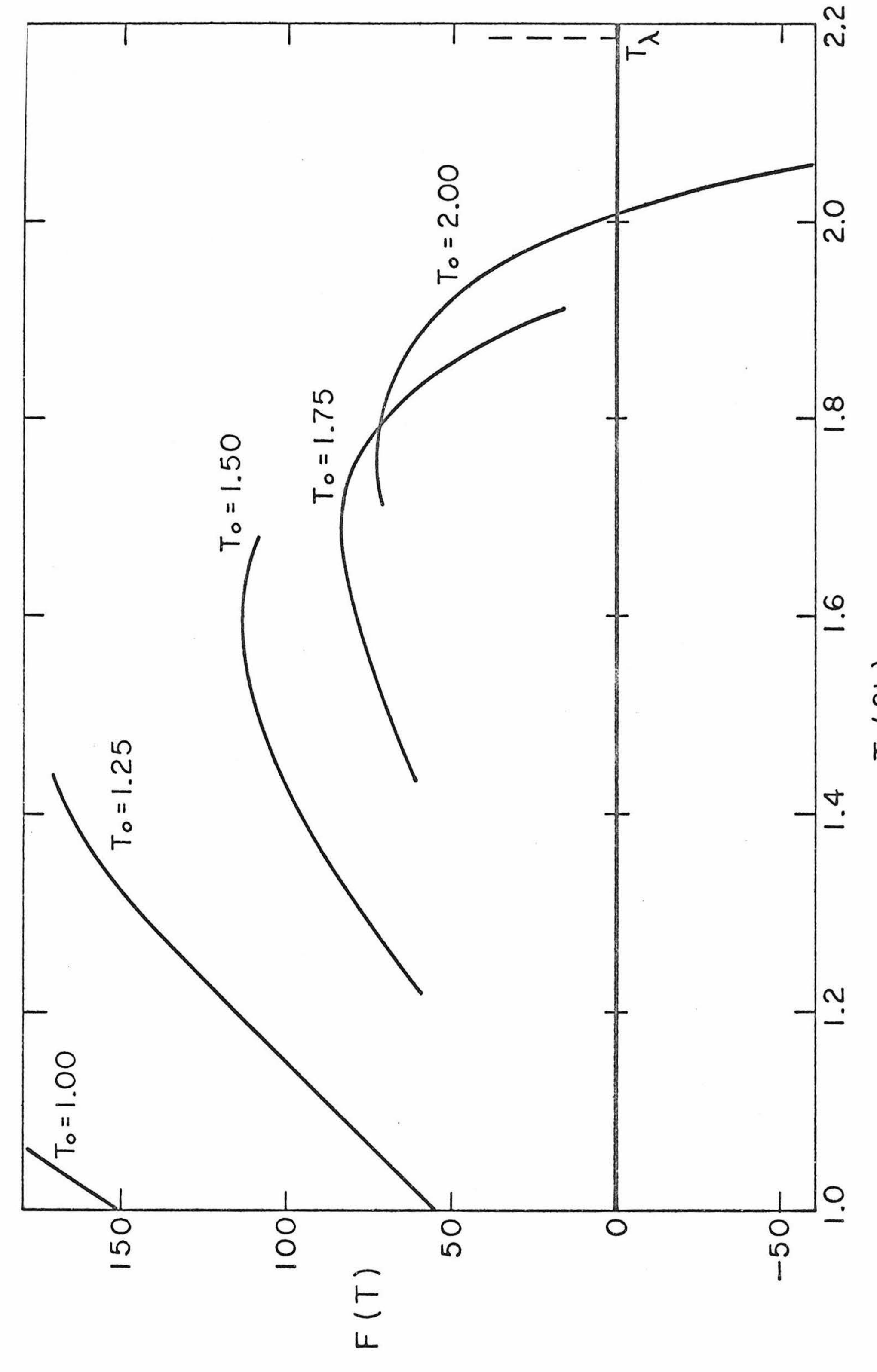


Figure 9 - F Versus Temperature

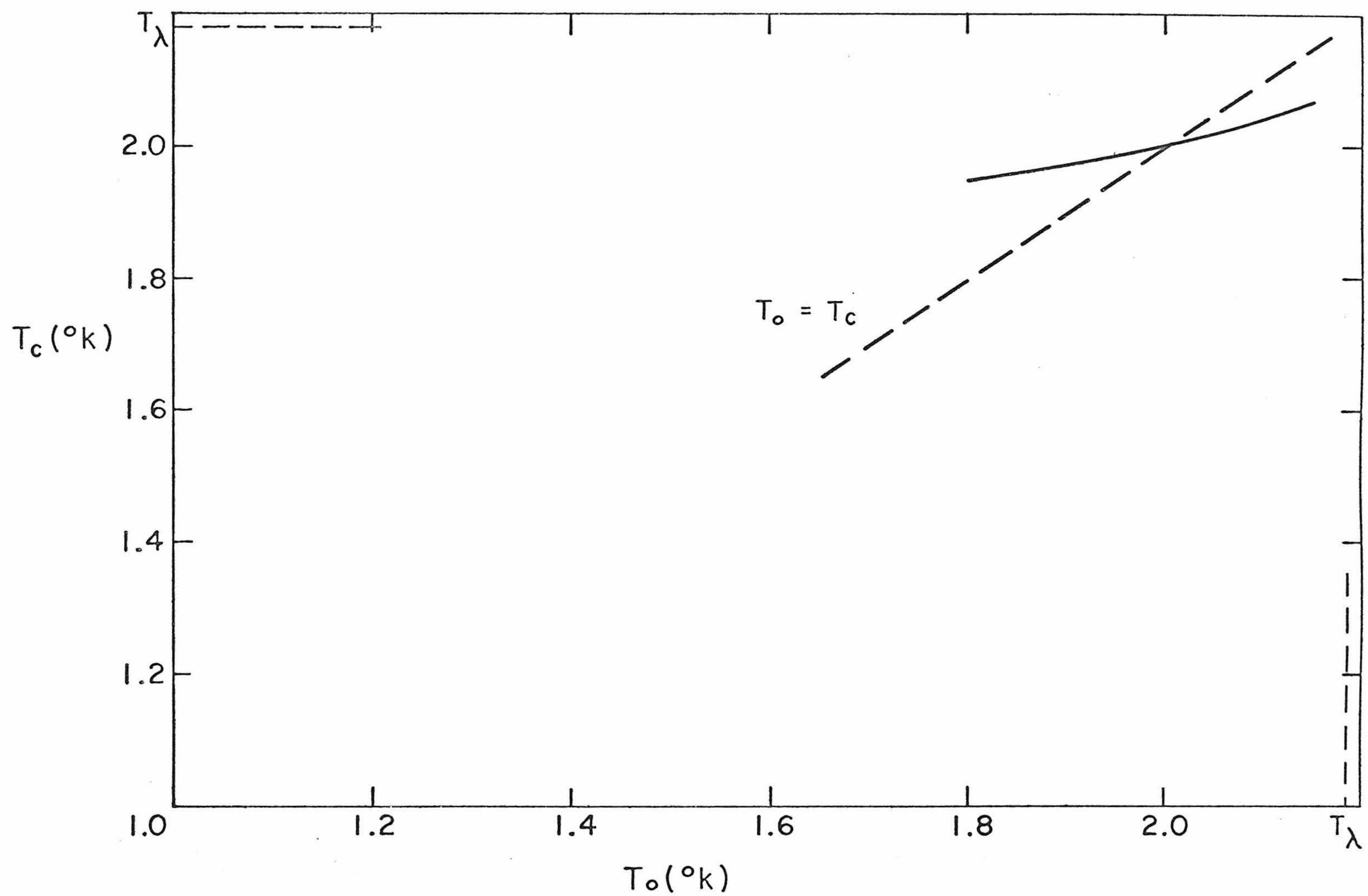


Figure 10 - Critical Temperature Versus Initial Temperature

section.

D. MORE GENERAL ANALYSIS

We have now developed the Riemann theory of second sound in some detail and have discussed most of the important features of the propagation of these thermal waves. However, it is desirable to develop the same type of theory but one that has a greater range of validity. This can be done by eliminating the restrictive assumptions stated on page 34 . We will thereby also gain a quantitative measure of the relative importance of these assumptions; the qualitative features of second sound as presented above will still remain the same.

It is relatively easy to drop the assumption that s_n is a constant and not specify any direct relationship between δ and s as implied by Eq. (1.1). One of the consequences of this is that Eq. (3.3d) does not reduce to an equation for δ. Retaining the other two assumptions that $\overline{v} = 0$ and that the thermodynamic importance of $\overline{w}$ is negligible means that the density is still a constant and Eq. (3.5) is still valid. Therefore, we must again start by simplifying Eqs. (3.3c) and (3.3d). In one dimension these equations are

$$\frac{1+\delta}{2}\frac{\partial w}{\partial t} + \frac{1}{2} w \frac{\partial \delta}{\partial t} + s \frac{\partial T}{\partial x} + 3\,\frac{1-\delta^2}{4}\, w \frac{\partial w}{\partial x} - \frac{1+3\delta}{4} w^2 \frac{\partial \delta}{\partial x} = 0 \quad (3.47a)$$

$$\frac{\partial s}{\partial t} + \frac{1-\delta}{2} w \frac{\partial s}{\partial x} + \frac{1-\delta}{2} s \frac{\partial w}{\partial x} - \frac{1}{2} s w \frac{\partial \delta}{\partial x} = 0 \quad . \qquad (3.47b)$$

It is more convenient to use T as the independent thermodynamic variable so we will assume $\delta = \delta(T)$, $s = s(T)$ and write

$$\left(\frac{\partial s}{\partial T}\right) = \frac{1-\delta}{1+\delta} \frac{s^2}{u_2^2} \quad . \tag{3.48}$$

Equations (3.47) can now be written in the following form:

$$\begin{pmatrix} \frac{1+\delta}{2} & \frac{1}{2}\left(\frac{\partial \delta}{\partial T}\right) w \\ 0 & \frac{1-\delta}{1+\delta} \frac{s}{u_2^2} \end{pmatrix} \begin{pmatrix} \frac{\partial w}{\partial t} \\ \frac{\partial T}{\partial t} \end{pmatrix} +$$

$$\begin{pmatrix} 3\,\frac{1-\delta^2}{4}\, w & s - \frac{1+3\delta}{4} \frac{\partial \delta}{\partial T} w^2 \\ \frac{1-\delta}{2} & \frac{1}{2}\left[\frac{(1-\delta)^2}{1+\delta} \frac{s}{u_2^2} - \left(\frac{\partial \delta}{\partial T}\right)\right] w \end{pmatrix} \begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial T}{\partial x} \end{pmatrix} = 0 \quad . \tag{3.49}$$

The analysis to be done on these equations is based on the theory presented in Appendix A and is quite similar to the analysis of Eq. (3.10). Therefore, we will eliminate the details and simply quote the results based on the previous work. The quadratic equation for the characteristic speeds is

$$c^2 + \left[\frac{1}{1-\delta} \frac{u_2^2}{s} \left(\frac{\partial \delta}{\partial T}\right) - 2(1-\delta)\right] wc - u_2^2 + O(w^2) = 0 \quad .$$

Under the appropriate assumptions this is identical to Eq. (B7), Khalatnikov's result. The roots of this equation are

$$c^{\pm} = \pm u_2 + \left[(1-\delta) - \frac{1}{2} \frac{1}{1-\delta} \frac{u_2^2}{s} \left(\frac{\partial \delta}{\partial T}\right)\right] w + O(w^2) \quad . \tag{3.50}$$

As before, there are two distinct roots so the system of Eqs. (3.49) is hyperbolic and the solution as given by Eqs. (3.25) with $T(\tau)$, the given boundary condition, replacing $\delta(\tau)$ in Eq. (3.25b) is still valid.

However, the expression for the temperature dependence of w, for instance, must be modified; from Eq. (3.50) it is clear that the velocity of propagation is given by

$$c = u_2 + \left[(1-\delta) - \frac{1}{2}\frac{1}{1-\delta}\frac{u_2^2}{s}\left(\frac{\partial\delta}{\partial T}\right)\right] w + O(w^2) \quad . \qquad (3.51)$$

The differential equation for $w(T)$ at the boundary is

$$\frac{dw}{dT} = \frac{2}{1+\delta}\frac{s}{u_2}\left\{1 + \frac{1}{2}\left[\frac{\delta}{1-\delta}\frac{u_2^2}{s}\left(\frac{\partial\delta}{\partial T}\right) + (1-\delta)\right]\frac{w}{u_2} + O\left(\frac{w^2}{u_2^2}\right)\right\} \quad . \qquad (3.52)$$

The result of the numerical integration of this equation is shown in Fig. 11 for $|w| < u_2$. This temperature dependence of the relative velocity is qualitatively similar to that shown in Fig. 6 for the case where s_n is a constant. And, as expected, the differences are more pronounced at lower temperatures although there is some change over the entire temperature region of interest. The most important difference is that w does not vary as rapidly with T and, consequently, the region over which this analysis is valid is greater than before.

Finally, the expression for F is

$$\begin{aligned} F = \frac{u_2}{s}\Bigg\{ & h + 2\,\frac{1+\delta}{1-\delta} - \frac{1}{1-\delta^2}\frac{u_2^2}{s}\left(\frac{\partial\delta}{\partial T}\right) \\ & + \Bigg[\frac{3}{2}\frac{(1-\delta)^2}{1+\delta} - \frac{2}{1+\delta}\frac{u_2^2}{s}\left(\frac{\partial\delta}{\partial T}\right) - \frac{\delta}{(1-\delta)^2(1+\delta)}\frac{u_2^4}{s^2}\left(\frac{\partial\delta}{\partial T}\right)^2 \\ & - \frac{1}{1-\delta}\,h\,\frac{u_2^2}{s}\left(\frac{\partial\delta}{\partial T}\right) - \frac{1}{2}\frac{1}{1-\delta^2}\frac{u_2^4}{s^2}\left(\frac{\partial^2\delta}{\partial T^2}\right)\Bigg]\frac{w}{u_2} + O\left(\frac{w^2}{u_2^2}\right)\Bigg\} \quad . \end{aligned}$$

where h is defined by Eq. (3.39). This function is depicted in Fig. 12 and the dependence of the critical temperature on the initial temperature on the initial temperature is shown in Fig. 13; these

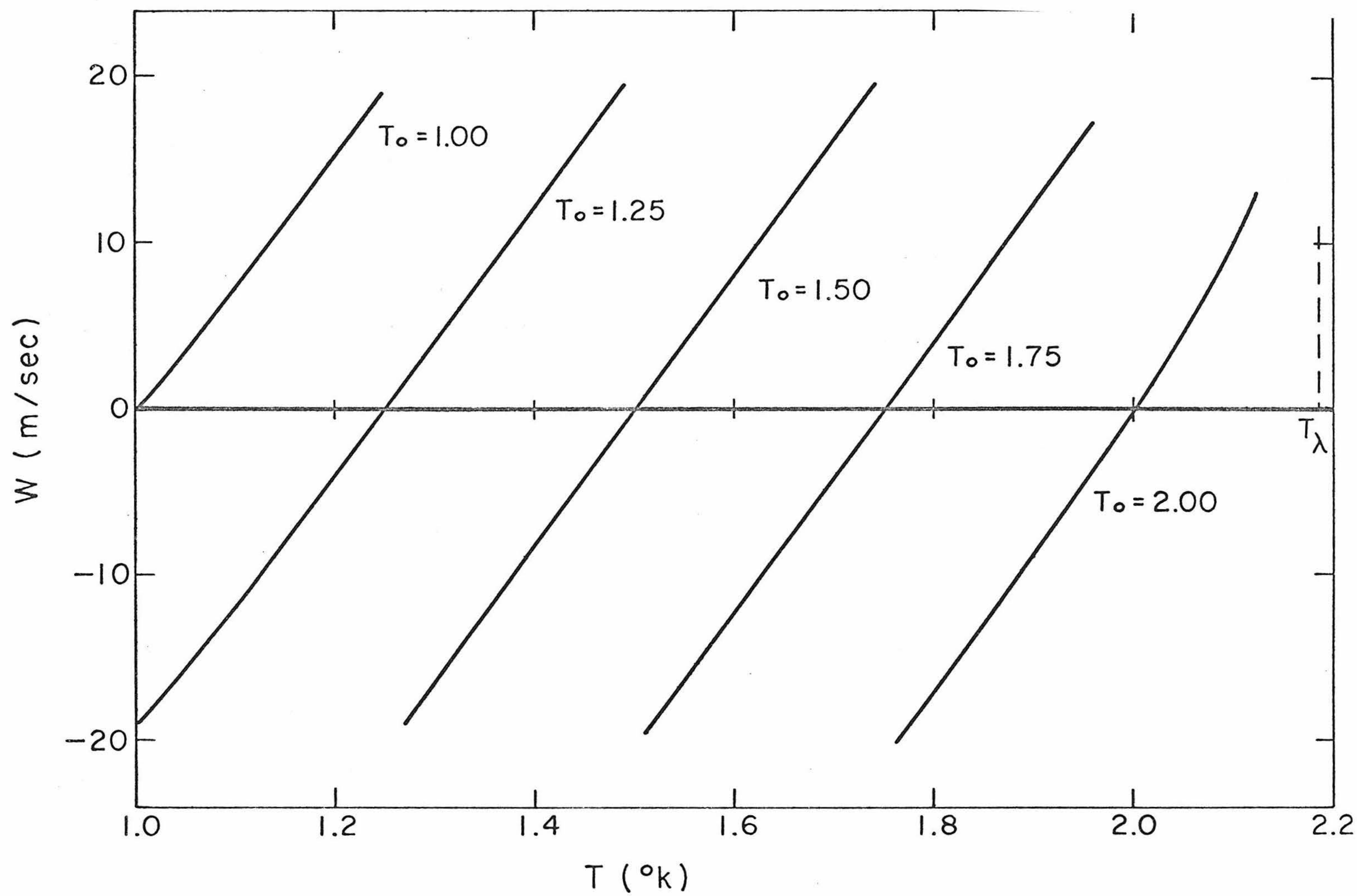


Figure 11 - Relative Velocity Versus Temperature

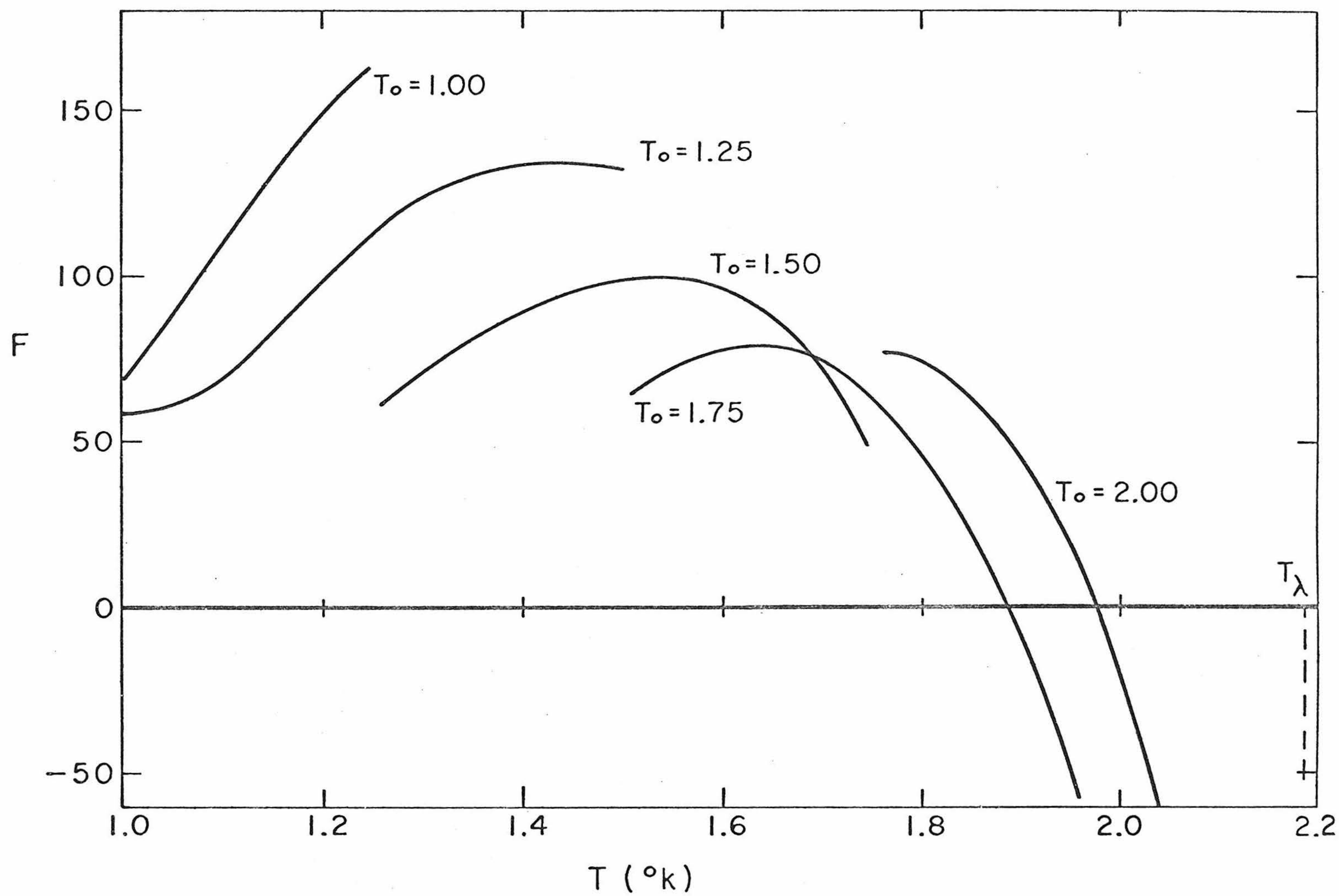


Figure 12 - F Versus Temperature

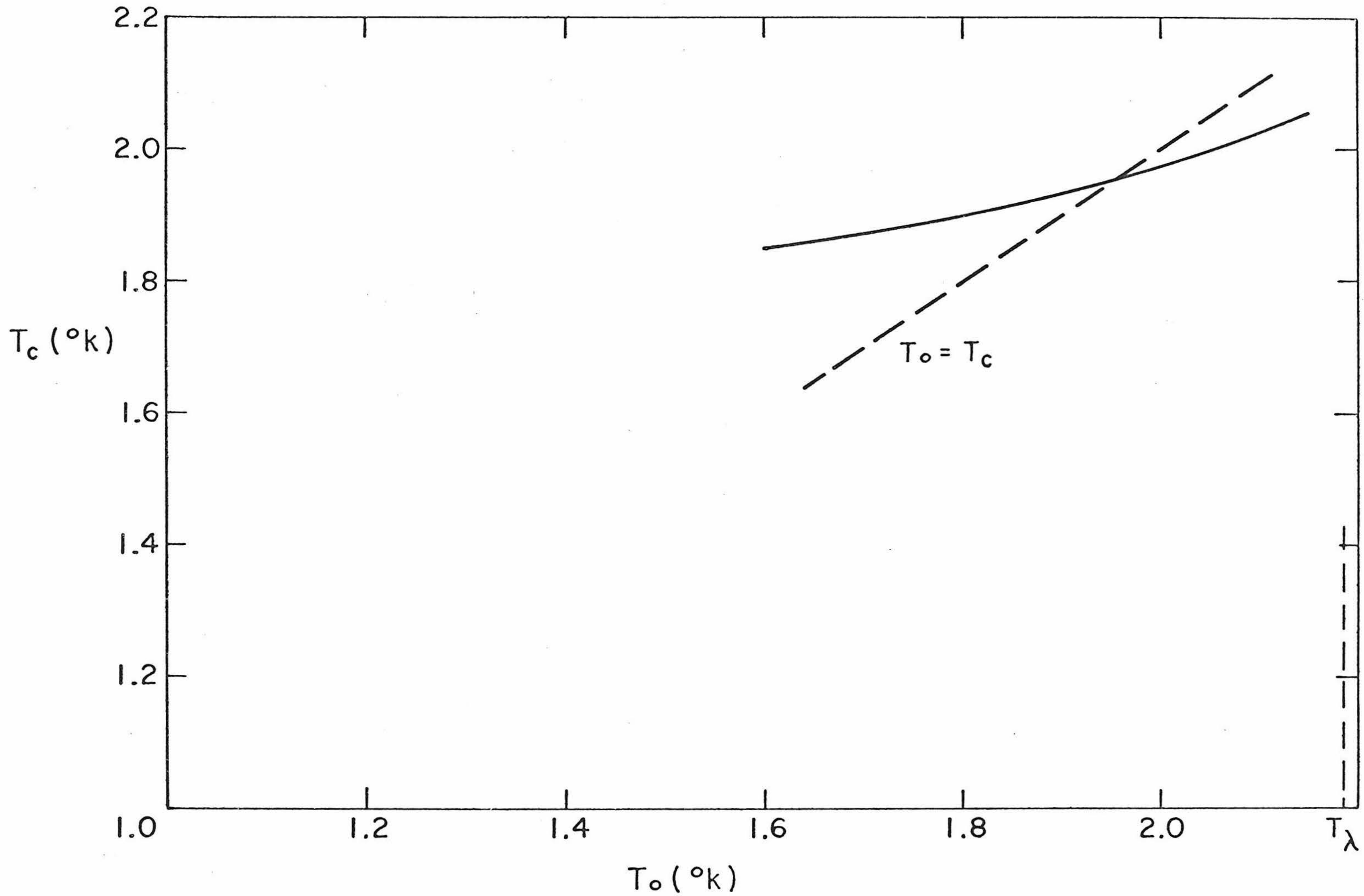


Figure 13 - Critical Temperature Versus Initial Temperature

correspond to Fig. 9 and Fig. 10, respectively. There is a significant change in F at all temperatures as compared to the previous theory. The temperature where $T_o = T_c$ is 1.95°K, which coincidentally, is also the temperature where $\delta = 0$. In addition, a critical temperature exists for initial temperatures as low as 1.60°K. These results are unexpected since the assumption that s_n is a constant is considered to be a very good approximation for $T > 1.4°K$ [1], [42]. Since $s_n = \frac{\rho s}{\rho_n}$, we can conclude that F and, consequently, the critical temperature are very sensitive to the total entropy and/or the normal density which, in turn, means that they are very sensitive to the thermodynamic functions $s = s(T)$ and $\delta = \delta(T)$. This has been previously observed as far as the critical temperature is concerned[42].

In summary, we have seen that it is possible to generalize the original equations and yet not need an entirely new analysis to study the new solution and its basic characteristics. With this in mind, we will now generalize the first system of equations, Eqs. (3.8), in a number of different ways and still retain the same fundamental solution; nevertheless, some important changes will be observed. It will be possible to isolate and compare the relative importance of the effects of the different generalizations in the final results.

In particular, we want to study the importance of $\overline{w}$ as a thermodynamic variable. To do this, choose P and T as the independent thermodynamic variables and rewrite Eqs. (3.4) as

$$\rho = \rho(P, T, \overline{w}) \qquad (3.53a)$$

$$s = s(P, T, \overline{w}) \qquad (3.53b)$$

$$\delta = \delta(P, T, \overline{w}) \quad . \qquad (3.53c)$$

The lowest order terms in the expansion of ρ and s in terms of $\overline{w}$ are given by Eqs. (B2);

$$s = s_o + \left(\frac{\partial \alpha_o}{\partial T}\right) \frac{\overline{w}^2}{2} + O(\overline{w}^4) \qquad (3.54a)$$

$$\frac{1}{\rho} = \frac{1}{\rho_o} - \left(\frac{\partial \alpha_o}{\partial P}\right) \frac{\overline{w}^2}{2} + O(\overline{w}^4) \quad . \qquad (3.54b)$$

A subscript "o" is used to denote quantities depending only on P and T. Since $\overline{w}$ is a thermodynamic as well as a mechanical variable even a non-linear analysis is valid, at best, for $|\overline{w}| < u_2$ when the expansions above are used to represent the role of $\overline{w}$ as a thermodynamic variable. Using these expansions, it should be possible to extend the preceeding analyses so that they are valid to $O(\overline{w}^2)$. And yet, it can be seen from Eq. (B7) that the $O(\overline{w}^2)$ terms in Eqs. (3.54) affect the velocity of second sound to first order so these terms are more than just a second order correction.

To avoid repetition of the previous work, it will again be assumed that s_n is a constant;

$$\delta = 2 \frac{s}{s_n} - 1$$

and hence

$$\delta = \delta_o + \frac{1}{s_n}\left(\frac{\partial \alpha_o}{\partial T}\right) \overline{w}^2 + O(\overline{w}^4) \qquad (3.54c)$$

where

$$\delta_o = 2\,\frac{s_o}{s_n} - 1 \quad .$$

In this notation Eq. (3.26) is written

$$\left(\frac{\partial \delta_o}{\partial T}\right) = (1-\delta_o)\,\frac{s_o}{u_2^2}$$

and

$$u_2^2 = \frac{1-\delta_o}{1+\delta_o}\, s_o^{\,2} \left(\frac{\partial s_o}{\partial T}\right)^{-1} \quad .$$

Similarly

$$\left(\frac{\partial \delta_o}{\partial P}\right) = -\,\frac{1+\delta_o}{s_o}\left(\frac{\partial}{\partial T}\,\frac{1}{\rho_o}\right) = \frac{1+\delta_o}{\rho_o}\,\frac{g}{u_2^2}$$

where

$$g = \frac{u_2^2}{\rho_o s_o}\left(\frac{\partial \rho}{\partial T}\right) = \frac{u_2^2}{s_o}\,\alpha_P \qquad (3.55)$$

in which α_P is the coefficient of thermal expansion. The function g provides a measure of the importance of thermal expansion and is shown as a function of temperature in Fig. 14. Equation (3.54) can now be written

$$s = s_o\left[1 + \frac{1-\delta_o}{4}\,\frac{\overline{w}^2}{u_2^2} + O(\overline{w}^4)\right] \qquad (3.56a)$$

$$\rho = \rho_o\left[1 + \frac{1+\delta_o}{4}\, g\,\frac{\overline{w}^2}{u_2^2} + O(\overline{w}^4)\right] \qquad (3.56b)$$

$$\delta = \delta_o + \frac{1-\delta_o^{\,2}}{4}\,\frac{\overline{w}^2}{u_2^2} + O(\overline{w}^4) \quad . \qquad (3.56c)$$

Once again assuming that $\overline{v} = 0$ implies that $\rho(P, T, \overline{w})$ is a constant, Eq. (3.11) is still valid, and Eqs. (3.8) serve as the basic equations for this analysis. Consider the term $\frac{\partial \delta}{\partial t}$; by using

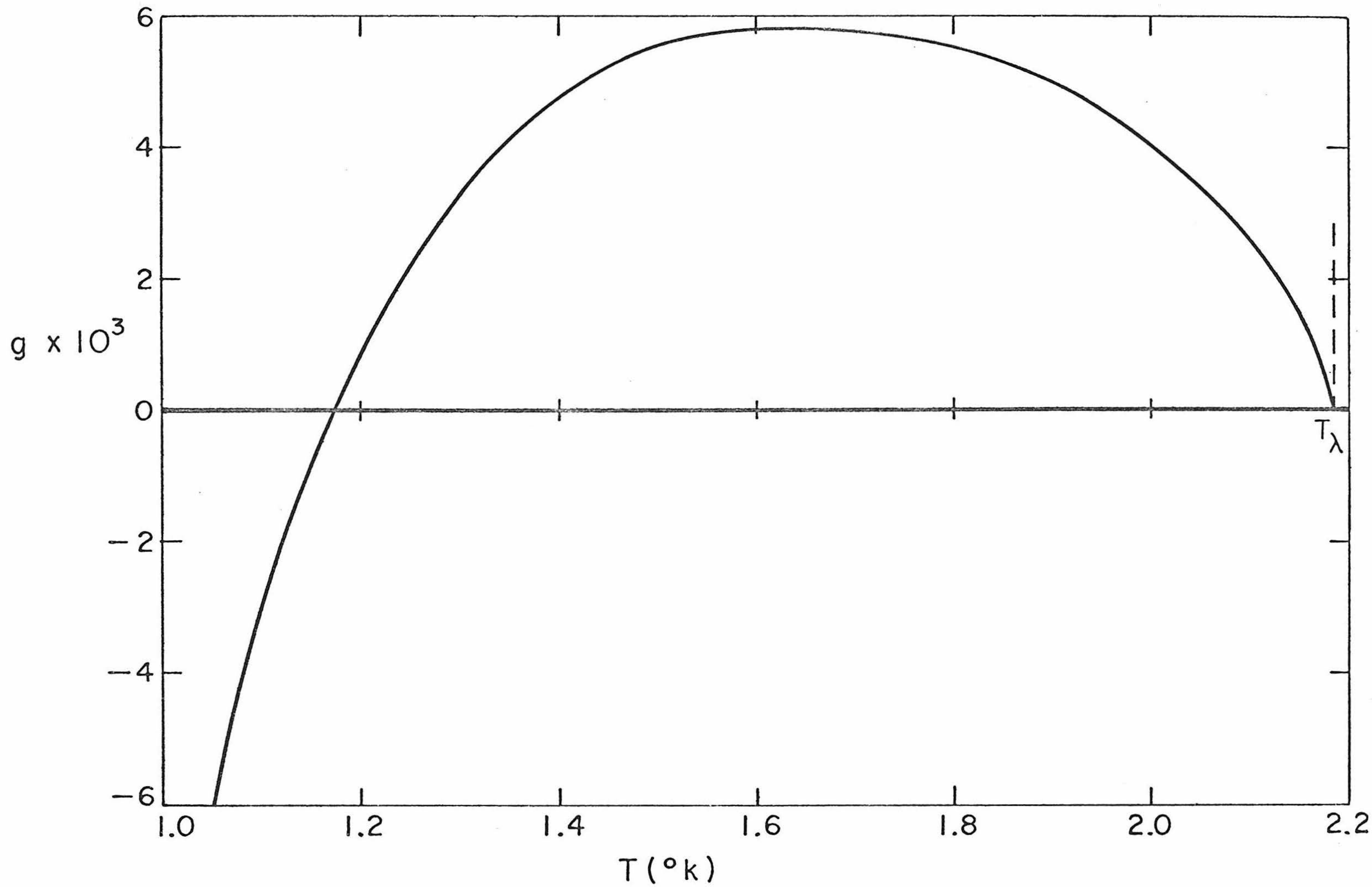


Figure 14 - g Versus Temperature

Eq. (3.56c) it can be written

$$\frac{\partial \delta}{\partial t} = \left(\frac{\partial \delta_o}{\partial T}\right)\frac{\partial T}{\partial t} + \left(\frac{\partial \delta_o}{\partial P}\right)\frac{\partial P}{\partial t} + \cdots \; .$$

Therefore, the pressure variations must be considered in this analysis. By Eq. (3.11) we can write

$$\frac{\partial P}{\partial t} = -\frac{1-\delta_o^2}{2}\rho_o w\frac{\partial w}{\partial t} + \frac{1-\delta_o}{2}\delta_o\rho_o s_o\frac{\overline{w}^2}{u_2^2}\frac{\partial T}{\partial t} + O\left(\frac{w^3}{u_2^3}\right) \; . \qquad (3.57)$$

Consequently, Eqs. (3.8) can be written only in terms of w and T derivatives. In one space dimension, these equations are

$$\begin{pmatrix} \frac{1+\delta}{2}\left[1-(1+\delta)g\right] w & s\left\{1-\frac{1}{2}\left[\delta-(1+\delta)\delta g+(1+\delta)h\right]\frac{w^2}{u_2^2}\right\} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial w}{\partial t} \\ \frac{\partial T}{\partial t} \end{pmatrix} +$$

$$\begin{pmatrix} \frac{1+\delta}{2}\left\{1-\delta\left[\frac{3}{2}+(1+\delta)g\right]\frac{w^2}{u_2^2}\right\} & -\delta s w \\ (1-\delta)w & \frac{2}{1+\delta}s\left(1-\frac{1-\delta^2}{4}\frac{w^2}{u_2^2}\right) \end{pmatrix}\begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial T}{\partial x} \end{pmatrix} = 0 \; . \qquad (3.58)$$

All $O(w^2)$ terms have been retained in these equations and, for convenience, the subscript "o" has been dropped.

As before, there are two distinct characteristic speeds for this set of equations and Eqs. (3.25) with $T(\tau)$ replacing $\delta(\tau)$, represent the solution. Consequently, the introduction of $\overline{w}$ as a thermodynamic variable does not give rise to any attenuation of second sound. The velocity of propagation is

$$c = u_2\left\{1 - \left[\delta - \frac{1+\delta}{2}\, g\, \frac{w}{u_2} - \frac{1+\delta}{4}\left[\frac{1-\delta}{2} + \delta g - h - \frac{1+\delta}{2}\, g^2\right]\frac{w^2}{u_2^2} + O\left(\frac{w^3}{u_2^3}\right)\right]\right\} .$$

(3.59)

By comparison with Eq. (3.31) it is clear that the inclusion of thermal expansion and $\overline{w}$ in Eqs. (1.7) affect c to $O(w)$. Once again, this result is consistent with Eq. (B9) when $g = 0$.

The differential equation for $w(T)$ is obtained by the same procedure as before; it is

$$\frac{dw}{dT} = \frac{2}{1+\delta}\,\frac{s}{u_2}\left\{1 + \left(1 - \frac{1+\delta}{2}\, g\right)\frac{w}{u_2} + \frac{1}{4}\left[\frac{1+4\delta^2}{2} - (1+\delta)(4-\delta)g \right.\right.$$
$$\left.\left. - (1+\delta)h + \frac{(1+\delta)^2}{2}\, g^2\right]\frac{w^2}{u_2^2} + O\left(\frac{w^3}{u_2^3}\right)\right\}. \qquad (3.60)$$

There is also a first order change in this expression as compared to Eq. (3.30) even for $g = 0$. The temperature dependence of w is shown in Fig. 15. In this approximation w varies more rapidly with T as compared to $w(T)$ as given by the first theory. This is just opposite to the effect noticed when the assumption that s_n equals a constant was eliminated; the order of magnitude of the change is the same and is also more pronounced at lower temperatures. The effect of thermal expansion on $w(T)$ is very small, less than $\pm.1$ meters/sec at all temperatures, which is also the same as the order of magnitude of the error.

The other quantity of interest, F, is no longer defined by Eq. (3.35) because the effect of the variation in the pressure must be considered. In this theory

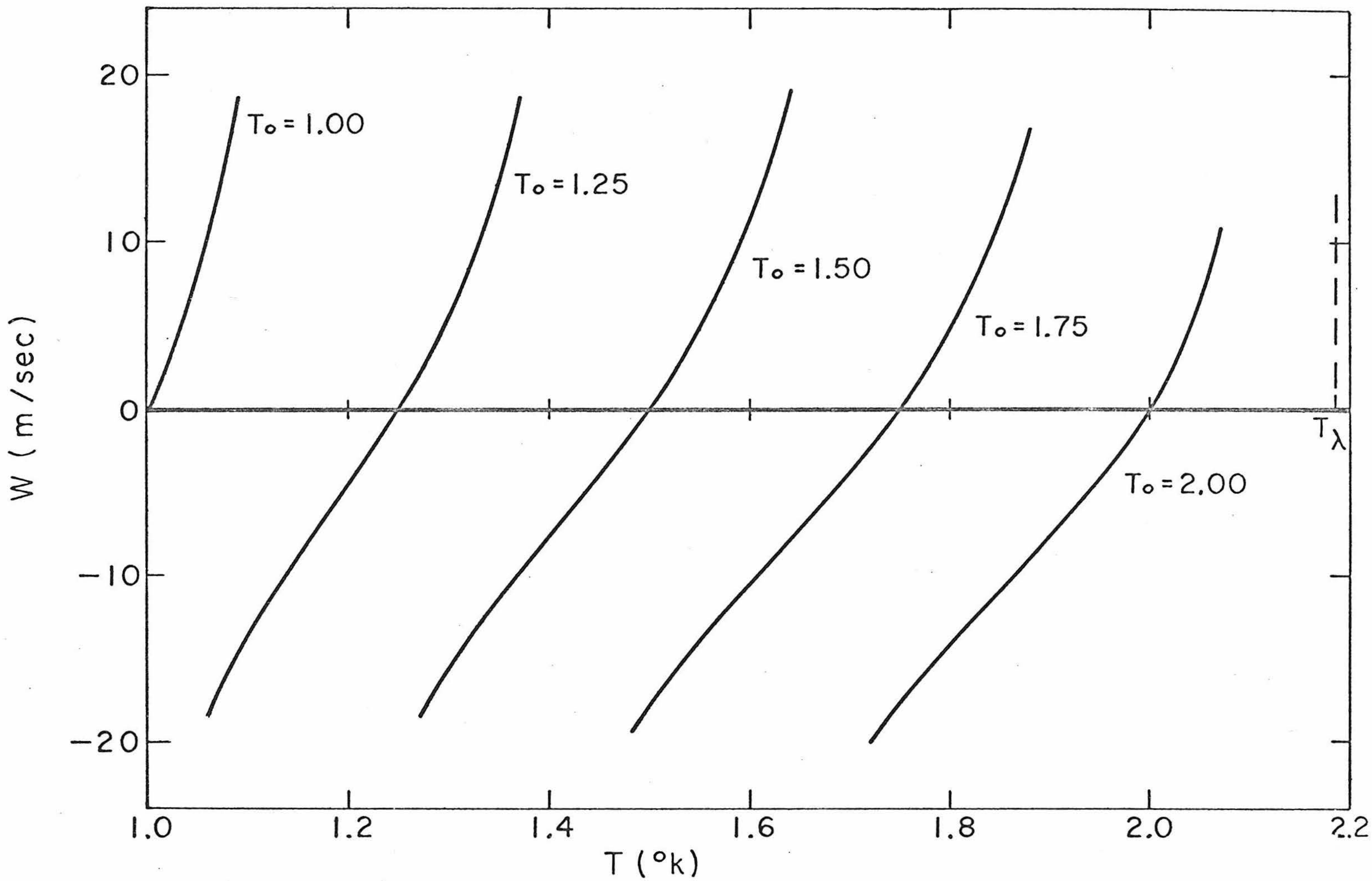


Figure 15 - Relative Velocity Versus Temperature

$$c = c(\tau) = c[P(\tau),\ T(\tau)]$$

so

$$\frac{dc}{d\tau} = \left(\frac{\partial c}{\partial T}\right)\frac{dT}{d\tau} + \left(\frac{\partial c}{\partial P}\right)\frac{dP}{d\tau} \quad .$$

By combining Eqs. (3.57) and (3.60) we can write

$$\frac{dP}{d\tau} = -\frac{1-\delta}{2}\,\rho s\,\frac{w}{u_2}\left\{2 + [2-\delta-(1+\delta)g]\,\frac{w}{u_2} + O\left(\frac{w^2}{u_2^2}\right)\right\}\frac{dT}{d\tau} \quad .$$

Therefore F is defined by

$$F = \left(\frac{\partial c}{\partial T}\right) - \frac{1-\delta}{2}\,\rho s\,\frac{w}{u_2}\left\{2 + \left[2-\delta-(1+\delta)g\right]\frac{w}{u_2}\right\}\left(\frac{\partial c}{\partial P}\right) \quad . \tag{3.61}$$

If we assume g and h are functions of T only, this becomes

$$F = \frac{s}{u_2}\left\{-2\,\frac{\delta}{1+\delta} + g + h_T + \left[\frac{3+4\delta-3\delta^2}{2(1+\delta)} + \frac{3-\delta}{2}\,g\right.\right.$$

$$\left.\left. + \frac{1+\delta}{2}\,\frac{u_2^2}{s}\,\frac{dg}{dT} + h_T - (1-\delta)h_P\right]\frac{w}{u_2} + O\left(\frac{w^2}{u_2^2}\right)\right\} \tag{3.62}$$

where

$$h_T = h = \frac{u_2^2}{s}\,\frac{\partial u_2}{\partial T}$$

$$h_P = \rho u_2\,\frac{\partial u_2}{\partial P} \quad .$$

The dimensionless function h_P measures the importance of the pressure dependence of u_2; this variation of u_2 is shown by Donnelly[3] and Atkins[10]. In the temperature region of interest h_P is the same order of magnitude as g so it represents a small effect. This F is shown in Fig. 16 and the critical temperatures in Fig. 17. As before, the temperature dependence of F is quite sensitive to the functions $\delta(T)$, $s(T)$, and $u_2(T)$. There is an appreciable

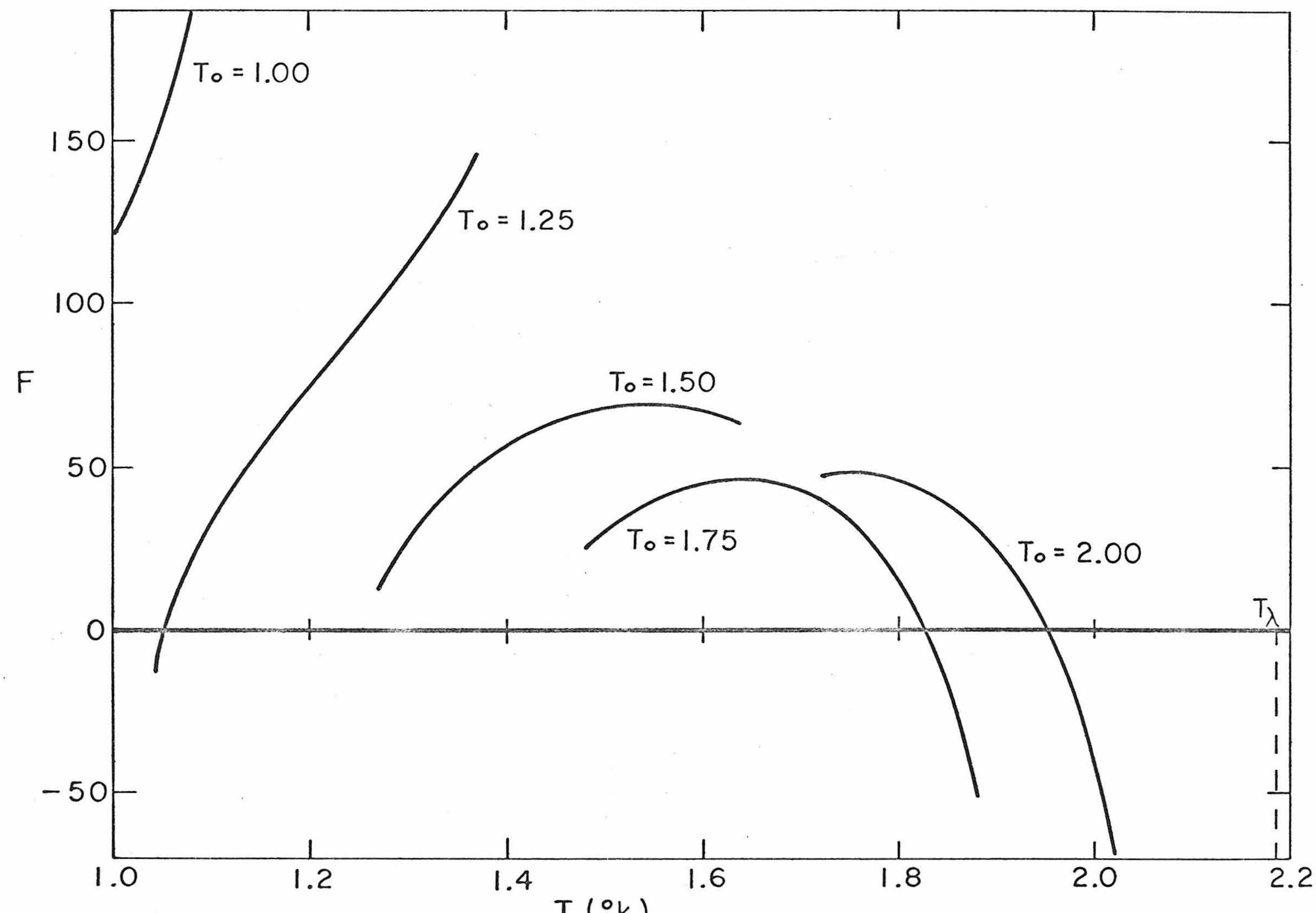


Figure 16 - F Versus Temperature

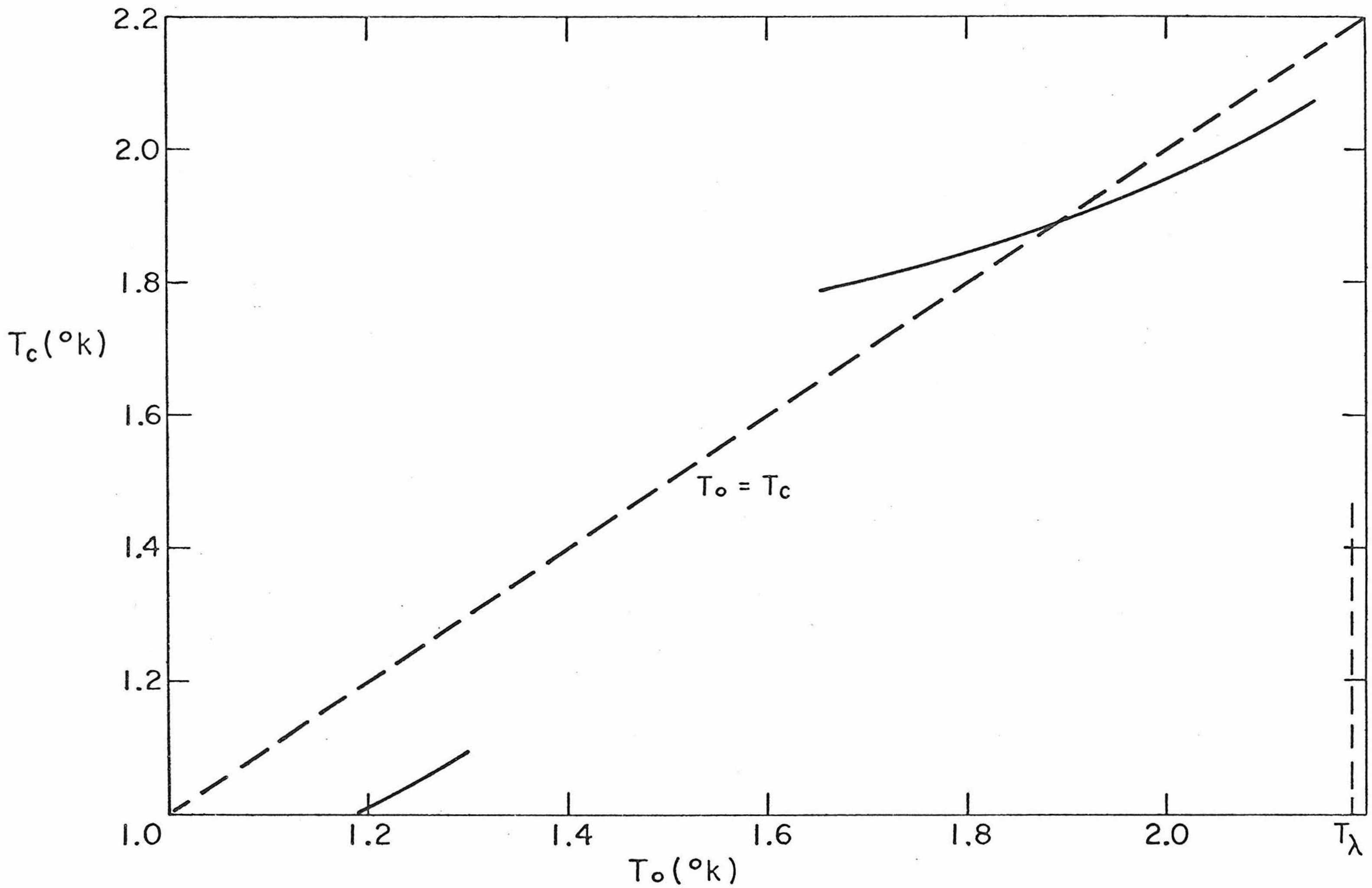


Figure 17 - Critical Temperature Versus Initial Temperature

difference between these curves and the ones in Fig. 9, particularly above approximately 1.25°K. For temperatures less than this, dropping the assumption that s_n is a constant has a greater effect on F than the inclusion of $\overline{w}$ as a thermodynamic variable.

One of the important differences is the appearance of a second set of critical temperatures at low temperatures as was anticipated earlier. This is a different branch of the function $T_c = T_c(T_o)$ but there is no additional temperature where $T_c = T_o$. However, this may not be the case in a more refined theory. On the upper branch $T^* = 1.89$°K.

The most significant change from the previous theories is in the zeroth order term of F. Compare Eq. (3.62) with $g = 0$ and Eq. (3.28), the expression for F in the first analysis:

$$F = \frac{s}{u_2}\left[h + \frac{1-2\delta}{1+\delta} + O\left(\frac{w}{u_2}\right)\right] \tag{3.38}$$

$$F = \frac{s}{u_2}\left[h - \frac{2\delta}{1+\delta} + O\left(\frac{w}{u_2}\right)\right]. \tag{3.62}$$

Hence, it is clear that including the second order terms in Eqs. (3.56) changes the zeroth order term in F. This difference is not a result of taking the pressure variations into account because, by Eq. (3.61) this can only change the first and higher order terms. Rather, it is because the velocity of propagation is changed to first order and F is defined as a linear sum of partial derivatives of c. In a similar manner, the $O(\overline{w}^4)$ terms in Eqs. (3.56) will affect F to $O(\overline{w}^2)$ so Eq. (3.62) is valid only to first order in w as shown.

There have been no previous direct experimental investigations of either $w(T)$ or $F(T)$. Motivated by the other linear theories Dressler and Fairbank[42] used a pulse technique to indirectly measure the dimensionless quantity γ where

$$\gamma = \frac{c - u_2}{v_n} \quad .$$

The relative velocity, defined by $w = \dot{H}/\rho_s s T$ where H is the heat current density, was kept small enough so that w was on the order of 1% of u_2 so that a linear theory is sufficient to describe the results. The only quantity of interest here that has been directly measured by this experiment is T^* which corresponds to the point where $\gamma = 0$. It was found that $\gamma = 0$ at $T = 1.873 \pm .005°K$ and $.946 \pm .01°K$. The higher value is very close to the value predicted here while it is felt that the lower temperature is out of the range of validity of this theory. The critical temperatures as shown in Fig. 17 are accurate to about $\pm 0.1°K$ due mostly to the uncertainty in the thermodynamic data.

We can now describe in detail the propagation of a temperature pulse and illustrate the change in the nature of the flow at the critical temperatures. Assuming a second, lower temperature exists where $T^* = T_o$ there are three distinct temperature regions to be considered.

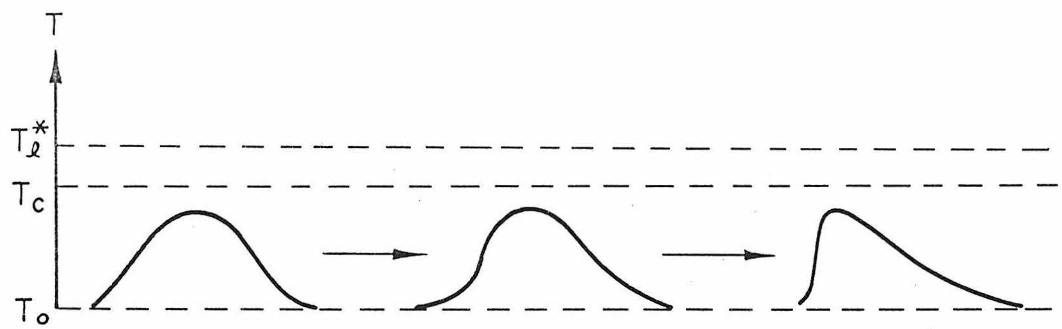
$$F \begin{cases} < 0 & \text{for} \quad T < T^*_\ell \\ > 0 & \text{for} \quad T^*_\ell < T < T^*_u \\ < 0 & \text{for} \quad T^*_u < T \end{cases} \tag{3.63}$$

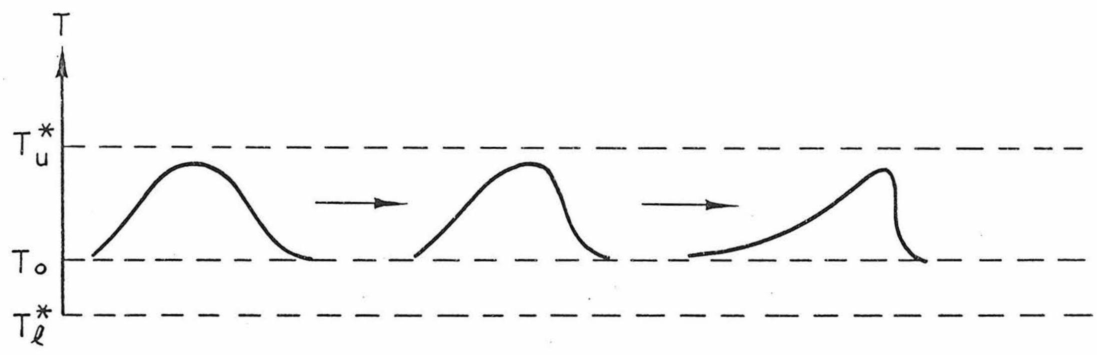
where

$$T^*_\ell \approx .95°K \text{ (experimentally)}$$

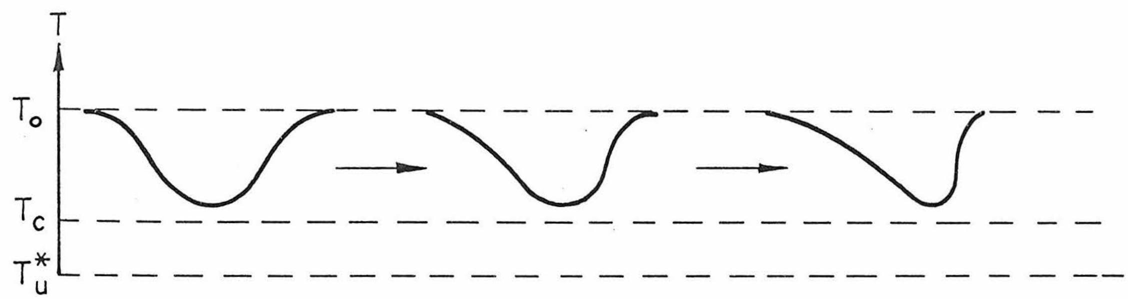
$$T^*_u = 1.89°K \text{ (theoretically)} \quad .$$

Since the sign of F determines whether the L_+ characteristics are convergent or divergent for a given boundary condition T(t), the solution as given by Eqs. (3.25) can now be explicitly shown. The sequence of events shown below illustrates the nature of the solution for an arbitrary pulse shape. The first curve shows the temperature pulse which is given at the end of a tube. The subsequent curves show the spatial distribution of the pulse at later times as it propagates down the tube.

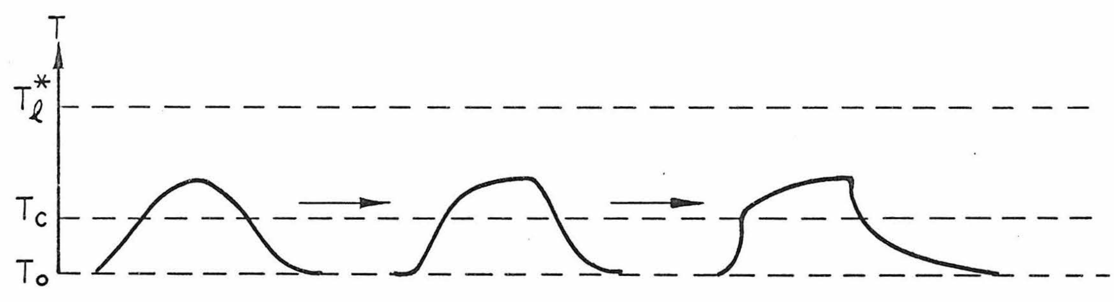


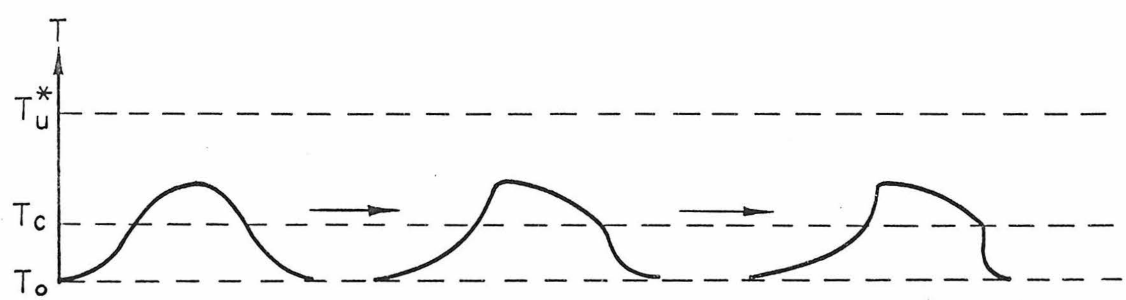


If $T_o > T_c(T_o) > T_u^*$ then the rear edge of the pulse steepens as shown above for $T_o < T_c(T_o) < T_\ell^*$. In all temperature regions a negative temperature pulse will also steepen into a shock but the opposite edge of the pulse as compared to a temperature rise is the one where the discontinuity appears. For instance, the following sequence will occur for $T_o > T_c(T_o) > T_u^*$.



The next two sets illustrate what happens when the critical temperature for a given initial temperature is exceeded during a positive pulse.





Summarizing these solutions, we can conclude that a temperature pulse, either positive or negative, always degenerates into a discontinuity which can appear at either the front or rear of the pulse. This discontinuity first occurs at the inflection point of the boundary condition $c(t)$ [see Appendix C]. Furthermore, if a critical temperature is exceeded during the pulse two separate shocks, one positive and the other negative, appear in the solution.

These conclusions have been qualitatively verified by the experiments performed by Osborne[41]. However, no observations were attempted which would have verified the prediction that a pulse may deform into two shocks since this was not known then. Using parameters for typical pulses used in these experiments in Eq. (3.34b) for the distance traveled by a pulse before it becomes discontinuous, we find that $x_c = 9.1$ cm. for a positive pulse at $T_o = 1.2°K$ and $x_c = 8.55$ cm. for a negative pulse at the same initial temperature; at $T_o = 2.1°K$, $x_c = 3.68$ cm. These distances are the same order of magnitude as indicated by the experimental results. A possible explanation for the anomalous short range behavior of positive pulses at $T_o = 2.12°K$ has been made by Dressler and Fairbank[42].

E. THE INTERACTION OF FIRST AND SECOND SOUND

It has previously been assumed that $\overline{v}$ was identically zero in order to rule out the excitation of first sound and to concentrate on the basic properties of second sound. This procedure also served to sufficiently simplify the equations of motion so that an explicit solution representing the propagation of a temperature pulse could be found and studied in detail. We will now eliminate this assumption and investigate the interaction of first and second sound. Hence, the solution that, up to now, has served as a basis for the study of pure second sound is no longer valid and a new one must be derived from the Landau equations.

A system of equations valid to second order in $\overline{w}$ that describes the propagation of both pressure and temperature waves can be derived from Eqs. (3.3) by making the following simplifications:

1. With the goal in mind of applying the results to the same physical problem as before, assume that all variables depend only on x and t.

2. Assume s_n is a constant in order to simplify the calculations as much as possible.

3. Include $\overline{w}$ as a thermodynamic variable but neglect the effects of thermal expansion so the equations of state are given by Eqs. (3.56) with $g = 0$.

These assumptions can be removed in more general analyses but provide here the simplest set of equations for the study of the interaction of both types of wave propagation in He II. This set of equations is

$$\frac{\partial P}{\partial t} + v \frac{\partial P}{\partial x} + \rho u_1^2 \frac{\partial v}{\partial x} = 0 \qquad (3.64a)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho}\left(1 + \frac{1-\delta^2}{4} \frac{w^2}{u_1^2}\right)\frac{\partial P}{\partial x} - \frac{1-\delta}{2} \delta s \frac{w^2}{u_2^2} \frac{\partial T}{\partial x} + \frac{1-\delta^2}{2} w \frac{\partial w}{\partial x} = 0$$

$$(3.64b)$$

$$\frac{1+\delta}{2}\left(1 + 3\,\frac{1-\delta}{4}\,\frac{w^2}{u_2^2}\right)\frac{\partial w}{\partial t} + \frac{1-\delta}{2}\,s\,\frac{w}{u_2^2}\,\frac{\partial T}{\partial t} + \frac{1+\delta}{2}\,w\,\frac{\partial v}{\partial x}$$

$$+ \frac{1-\delta^2}{4}\,\frac{1}{\rho}\,\frac{w^2}{u_1^2}\,\frac{\partial P}{\partial x} + \frac{1+\delta}{2}\left(v + 3\,\frac{1-\delta}{2}\,w + 3\,\frac{1-\delta}{4}\,v\,\frac{w^2}{u_2^2}\right)\frac{\partial w}{\partial x}$$

$$+ s\left(1 + \frac{1-\delta}{2}\,\frac{vw}{u_2^2} - 3\delta\,\frac{1-\delta}{4}\,\frac{w^2}{u_2^2}\right)\frac{\partial T}{\partial x} = 0 \qquad (3.64c)$$

$$\frac{2}{1+\delta}\,s\left\{1 - \frac{1}{2}[\delta + (1+\delta)h]\,\frac{w^2}{u_2^2}\right\}\frac{\partial T}{\partial t} + w\,\frac{\partial w}{\partial t} + \frac{1}{\rho}\,\frac{u_2^2}{u_1^2}\,w\,\frac{\partial P}{\partial x}$$

$$+ u_2^2\left(1 + \frac{vw}{u_2^2} - \frac{3}{2}\,\delta\,\frac{w^2}{u_2^2}\right)\frac{\partial w}{\partial x} + \frac{2}{1+\delta}\,s\left\{v - \delta w\right.$$

$$\left. - \frac{1}{2}\,v[\delta + (1+\delta)h]\,\frac{w^2}{u_2^2}\right\}\frac{\partial T}{\partial x} = 0 \qquad (3.64d)$$

where

$$u_1^2 = \left(\frac{\partial P}{\partial \rho}\right)_s$$

$$h = \frac{u_2}{s}\left(\frac{\partial u_2}{\partial T}\right)$$

and all thermodynamic variables do not depend on $\overline{w}$ (in other words, the subscript "o" in Eqs. (3.56) has been dropped). The partial derivative of u_2 with respect to T appears in these equations because the lowest order terms in the expansions of ρ and s contain u_2. The temperature derivative of u_1 would appear only if the higher order terms in these expansions were also included.

The Riemann analysis of these equations is also based on the theory presented in Appendix A and differs from the previous work in that there are four equations for the four unknowns P, v, T and w instead of just two equations as before. The quartic equation for the characteristic speeds is

$$U^4 + 2\delta w U^3 - \left(u_1^2 + u_2^2 + \frac{1-3\delta^2}{2} w^2\right) U^2 - 2\delta u_1^2 w U + u_1^2 u_2^2 - \frac{1-\delta^2}{4} u_2^2 w^2$$

$$+ \frac{1+\delta}{2} h u_1^2 w^2 - \frac{1+3\delta^2}{4} u_1^2 w^2 = 0 \qquad (3.65)$$

where

$$U = c - v \quad . \qquad (3.66)$$

The differences between this equation and Eq. (B11), Hsieh's result, are a result of including $\overline{w}$ as a thermodynamic variable and keeping all $O(w^2)$ terms.

The differential equations for the characteristics are found by solving Eq. (3.65) for c

$$\left(\frac{dx}{dt}\right)_{1,2} = c^{1,2} = v \pm u_1 \pm \frac{1-\delta^2}{8} \frac{3u_1^2+u_2^2}{u_1^2-u_2^2} \frac{w^2}{u_1} + O(w^3) \qquad (3.66)$$

$$\left(\frac{dx}{dt}\right)_{3,4} = c^{3,4} = v \pm u_2 - \delta w \mp \frac{1+\delta}{4} \left(\frac{1-\delta}{2} \frac{u_1^2+3u_2^2}{u_1^2-u_2^2} - h\right) \frac{w^2}{u_2} + O(w^3) . \qquad (3.67)$$

The corresponding invariants can be written in the form

$$\beta_1^{1,2} \frac{dP_{1,2}}{\rho} + \beta_2^{1,2} u_1 dv_{1,2} + \beta_3^{1,2} s dT_{1,2} + \beta_4^{1,2} u_1 dw_{1,2} = 0 \qquad (3.68)$$

$$\beta_1^{3,4} \frac{dP_{3,4}}{\rho} + \beta_2^{3,4} u_2 dv_{3,4} + \beta_3^{3,4} s dT_{3,4} + \beta_3^{3,4} u_2 dw_{3,4} = 0 \qquad (3.69)$$

where

$$\beta_1^{1,2} = 1 \pm \frac{w}{u_1} - \frac{1-\delta^2}{8}\frac{w^2}{u_1^2}$$

$$\beta_2^{1,2} = \pm 1 + \frac{w}{u_1}$$

$$\beta_3^{1,2} = \pm(1-\delta)\frac{w}{u_1^2-u_2^2}\left[u_1 \mp \frac{1}{2}\frac{\delta u_1^4-(2-3\delta)u_1^2u_2^2+2u_2^4}{u_1^2-u_2^2}\frac{w}{u_2^2}\right]$$

$$\beta_4^{1,2} = \frac{1-\delta^2}{2}\frac{u_1 w}{u_1^2-u_2^2}\left[1 \pm \frac{1}{2}\frac{(4-3\delta)u_1^2-(4+\delta)u_2^2}{u_1^2-u_2^2}\frac{w}{u_1}\right]$$

$$\beta_1^{3,4} = \mp\frac{1+\delta}{2}\frac{u_2^2}{u_1^2}\frac{u_1^2+u_2^2}{u_1^2-u_2^2}\frac{w}{u_2}\left[1 \pm \frac{1}{2}\frac{(1+\delta)u_1^4-6\delta u_1^2u_2^2-(1+\delta)u_2^4}{u_1^4-u_2^4}\frac{w}{u_2}\right]$$

$$\beta_2^{3,4} = -(1+\delta)\frac{u_2^2}{u_1^2-u_2^2}\frac{w}{u_2}\left[1 \pm \frac{1}{2}\frac{(1-2\delta)u_1^2-(1+2\delta)u_2^2}{u_1^2-u_2^2}\frac{w}{u_2}\right]$$

$$\beta_3^{3,4} = 1 \pm \frac{1-\delta}{2}\frac{w}{u_2} + \frac{1}{4}\left[\frac{(1-\delta)^2u_1^2+(1+\delta)(3-5\delta)u_2^2}{u_1^2-u_2^2} - (1+\delta)h\right]\frac{w^2}{u_2^2}$$

$$\beta_4^{3,4} = \pm\frac{1+\delta}{2}\left(1 \pm \frac{3-\delta}{2}\frac{w}{u_2} + 3\,\frac{1-\delta}{4}\frac{w^2}{u_2^2}\right) .$$

We can state by analogy to the previous application of a similar set of characteristics and invariants that the propagation speeds of first and second sound are given by Eqs. (3.66) and (3.67) using the upper signs. These results agree with those of Khalatnikov to first order and show that these propagation speeds are affected to $O(w^2)$ by the interaction of pressure and temperature waves. They also illustrate that both types of waves move relative to the mass flux velocity, v.

This set of characteristics and invariants can be applied to the same physical problem as before. However, the initial and boundary conditions must be stated more completely. At the end of the tube the temperature is still a given function of time but we also require the mass flux velocity to be zero at $x = 0$. Initially, the equilibrium stated is defined by $T = T_o$ and $P = P_o$ plus the requirement that both v and w vanish.

The procedure for solving this problem is very much like the one employed previously until the stage shown below is reached.

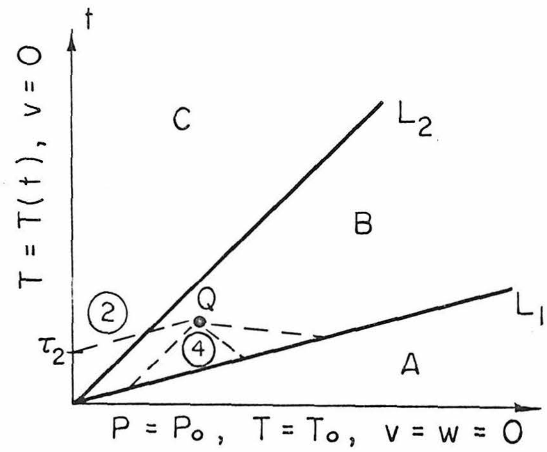


Figure 18a

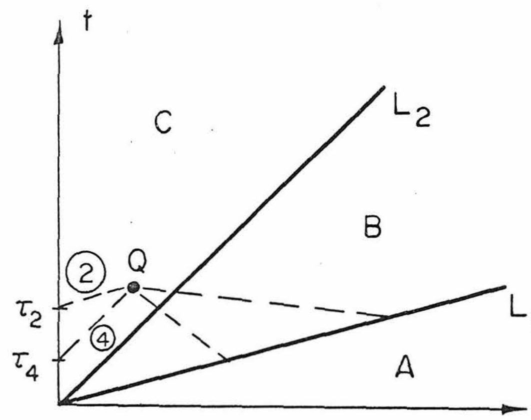


Figure 18b

The lines $L_{1,2}$ are defined by $x = u_{1,2} \cdot t$. Up to this point it has been shown that $P = P_o$, $T = T_o$ and $v = w = 0$ in region A. Also, it is known that

$$P(\tau) = P_o \qquad (3.70)$$

$$w(\tau) = \frac{2}{1+\delta_o} \frac{s_o}{u_2^o} [\,T(\tau)-T_o] \qquad (3.71)$$

where τ represents some time on the t-axis close to the origin. We now want to consider a point Q close to both the t-axis and the L_1

line. This can be done in either one of the two ways shown in Fig. 18 above. The distinguishing feature is whether the 4 characteristic first intersects the t-axis or the L_1 line. This, in turn, is determined by whether Q is above or below the L_2 line separating region B and C.

The values of the variables at Q are markedly different in these two different cases; the calculations in both cases proceed similarly to the ones used previously. It can be shown that throughout region B

$$\left.\begin{aligned} T(Q) &= T_o \qquad w(Q) = 0 \\ P(Q) - P_o &= \rho_o u_1^\circ v(Q) \\ v(Q) &= \frac{\beta_3^2(\tau_2)s_2(T_2 - T_o) + \beta_4^2(\tau_2)u_1(\tau_2)w^2}{u_1^\circ \beta_1^2(\tau_2) + u_1(\tau_2)\beta_2^2(\tau_2)} \end{aligned}\right\} \qquad (3.72)$$

where a subscript "2" denotes quantities evaluated at τ_2 and w_2 is given by Eq. (3.71) with $\tau = \tau_2$. However, since the β's contain powers of w_2, this expression for v(Q) must be expanded so that it includes only $O(w^2)$ terms. The result is very complicated and can be summarized as

$$\left.\begin{aligned} v(Q) &= 0 + O[(\Delta T_2)^2] \\ P(Q) &= P_o + O[(\Delta T_2)^2] \end{aligned}\right\} \qquad (3.73)$$

where

$$\Delta T_2 = T_2 - T_o \quad .$$

Therefore, we can conclude that no second sound has been excited in region B and that the magnitude of first sound in this region is

$O(w_2^2)$.

In region C both first and second sound waves are excited. At the point Q in Fig. 18b, it can be shown that

$$w(Q) = \frac{2}{1+\delta_o} \frac{s_o}{u_2^\circ} [T(Q) - T_o]$$

or, equivalently

$$w(Q) - w_4 = \frac{2}{1+\delta_o} \frac{s_o}{u_2^\circ} [T(Q) - T_4]$$

and, as in region B

$$P(Q) - P_o = \rho_o u_1^\circ v(Q)$$

$$v(Q) = o + O[(\Delta T_2)^2] \quad .$$

Therefore, the magnitude of first sound is still $O[(\Delta T)^2]$ but w is $O(\Delta T)$ or, in other words, w is an order of magnitude larger than v in the region where both first and second sound exist. Another important difference between regions B and C is the fact that the 2 characteristic is a straight line in region B but not in region C. This means that the velocity of first sound in region B depends only on the given boundary condition $T(t)$ but in region C both propagation speeds depend not only on the boundary condition but also on how far down the tube the pulse has traveled.

The same iterative procedure as illustrated in Fig. 18 can, in principle, be used to find the complete solution in region C. However, there are two main difficulties that have already been mentioned which prevent the details of the calculations from being completely carried out. First of all, any expression which contains w or ΔT in the denominator, such as the one for $v(Q)$ in Eqs. (3.72), must be expanded so that only $O(w^2)$ terms are retained in the solution. This creates unmanageably long expressions and the algebra of the problem itself becomes prohibitively complex.

The other difficulty is of a more serious and fundamental nature. When the equations describing pure second sound were studied, we were able to find a family of straight line characteristics with the dependent variables being constant along these lines; this is the only reason why we were able to obtain an explicit solution and discuss it in detail. But when the interaction of first and second sound is considered, there are four different families of characteristics none of which is a set of straight lines. This fact prevents us from extending the iterative scheme beyond one single step at a time. Consequently, it is impossible to find a useable, analytic representation for the solution or to obtain any thing more than a qualitative measure of the magnitude of both first and second sound. Nevertheless, the set of characteristics and invariants provides a very convenient tool which can be used to numerically study the problem and to obtain quantitative results for a given set of initial and boundary conditions. Finally, it should be pointed out that if a solution is sought which is valid to only $O(w)$ then the entire problem reduces to one that has already been solved because the magnitude of first sound is $O(w^2)$ so the distinction between regions A and B disappears and a non-trivial solution exists only in region C. This is exactly the type of solution depicted in Fig. 4.

A review of the qualitative results obtained above shows that there are two distinct wavefronts separating three distinct regions in the x, t-plane each of which has a different type of solution. There is no disturbance ahead of the first sound wavefront L_1 in region A. Only pressure waves have propagated into region B ahead of the second sound wavefront L_2 while in region C both first and second sound exist, the pressure waves being an order of magnitude smaller than the temperature waves.

F. DISCONTINUOUS SOLUTIONS

Throughout the preceeding analyses of second sound it has always been implicitly assumed that the given boundary condition T(t) and the accompanying solution were both continuously differentiable and single-valued. However, we have shown that there is some point in the x, t-plane where the solution possesses neither one of these properties for a pulse type boundary condition. Physically, this means that a shock is formed.

The hydrodynamic theory of shocks in He II has been discussed by Khalatnikov[28] and Temperley[40]. The shock is idealized by assuming that it has zero thickness and is manifested by a jump in the quantities across it. One method of deriving the equations which describe the discontinuity and its motion is illustrated below. The conservation of mass equation, Eq. (3.3a), written in one-dimensional, integrated form is

$$\frac{d}{dt}\int_a^b \rho\, dx + \rho v \Big|_a^b = 0 \quad . \tag{3.74}$$

Suppose the discontinuity is at $x = X(t)$ and is moving with speed $U = \dot{X}(t)$. Let $a \to X$ from the left and $b \to X$ from the right in Eq. (3.74). Then

$$- U[\rho] + [\rho v] = 0 \tag{3.75}$$

where $[Q] = Q_2 - Q_1$, the jump in Q across the discontinuity. The subscript "2" denotes quantities in front of the shock and a "1" those quantities behind it. Be defining

$$q_i = v_i - U \quad , \quad i = 1,2 \tag{3.76}$$

Eq. (3.75) can be written as

$$\rho_2 q_2 = \rho_1 q_1 \quad . \tag{3.77}$$

We can analyze all of the Landau equations, written in a form somewhat modified from Eqs. (3.3), in a similar manner. The conservation of mass and momentum as expressed by Eqs. (3.3a) and (3.3b), respectfully, will be retained as they are. It is possible to write the conservation of energy law based on the Landau equations[1] as

$$\frac{\partial}{\partial t}\left(\rho_n \frac{\overline{v}_n^2}{2} + \rho_s \frac{\overline{v}_s^2}{2} + \rho e\right) + \nabla\cdot\left[\rho_n \frac{\overline{v}_n^2}{2}\,\overline{v}_n + \rho_s \frac{\overline{v}_s^2}{2}\,\overline{v}_s + (P+\rho e)\overline{v}\right.$$
$$\left. + \rho s T(\overline{v}_n - \overline{v}) + \frac{1}{2}\,\frac{\rho_n \rho_s}{\rho}\,\overline{w}^3\right] = 0$$

where e is the specific internal energy. Finally, the equation of motion for $\overline{v}_s$ is

$$\frac{\partial \overline{v}_s}{\partial t} + \nabla\left(\mu + \frac{\overline{v}_s^2}{2}\right) = 0 \tag{3.78}$$

where μ is the specific chemical potential. This relation may be called a "conservation of zero vorticity" equation. It can be shown that[29]

$$\mu = e + P/\rho - sT - \frac{1}{2}\frac{\rho_n}{\rho}\overline{w}^2 \quad .$$

Writing these four conservation laws in one dimension and analyzing them as illustrated above gives the following equations describing the motion of a discontinuity in He II:

$$\rho_2 q_2 = \rho_1 q_1 = \text{const.} \qquad (3.79a)$$

$$P_2 + \rho_2 q_2^2 + \frac{\rho_{n2}\rho_{s2}}{\rho_2} w_2^2 = \text{const.} \qquad (3.79b)$$

$$e_2 + \frac{\rho_2}{\rho_2} - s_2 T_2 - \frac{1}{2} q_2^2 - \frac{\rho_{n2}}{\rho_2} q_2 w_2 - \frac{1}{2}\frac{\rho_{n2}\rho_{s2}}{\rho_2^2} w_2^2 = \text{const.} \qquad (3.79c)$$

$$(\rho_2 q_2 + \rho_{s2} w_2) s_2 T_2 + \rho_{n2} q_2^2 w_2 + 2\frac{\rho_{n2}\rho_{s2}}{\rho_2} q_2 w_2^2 + \frac{\rho_{n2}\rho_{s2}}{\rho_2^2} w_2^3 = \text{const.}$$

$$(3.79d)$$

where ρ, e etc. depend on w. Given the conditions ahead of the shock, these are four equations for both the state behind it and the shock speed U. By the definition of q_i these equations are invariant under a Gallilean transformation since w itself is invariant. Putting $w_1 = w_2 = 0$ reduces the first three equations to the classical ones while the fourth one implies that entropy is conserved.

It can be shown that these equations are the same as those derived by Khalatnikov when $w_1 = 0$. Temperley finds a different set not only because equations other than the Landau equations are used but also because he has imposed the conservation of entropy law across the discontinuity rather thab Eq. (3.79). Entropy is not conserved across a shock in an ordinary media and it is expected that the same result is true in He II; consequently, there are fundamental doubts about his shock front theory.

The ideal procedure to follow from this point would be to derive a function analogous to the Hugoniot[43] relation describing the shock transition in ordinary gas dynamics and to deduce the basic properties of shocks in He II from this function. This is exceedingly difficult to do because w appears in Eqs. (3.79) both as a thermodynamic variable and as a mechanical velocity. Thus, unlike the case in ordinary hydrodynamics, it is impossible to eliminate both v and w from any single equation and obtain a relation only on the thermodynamic variables. This increased complexity is due to the fact that Eqs. (3.79) describe both "first and second shocks" whereas in the classical case a small entropy rise is merely "carried along" with a pressure shock.

It is possible to obtain some results by assuming that the jump of the variables across the discontinuity is small and following a perturbation procedure. This is the method used by Khalatnikov; the most important results will be quoted here. In the limit of a weak pressure or temperature discontinuity the shock propagates with a velocity equal to the small-amplitude sound wave velocity. Considering only temperature discontinuities (liquid helium cavitates before a pressure discontinuity can be established) it can be shown that the jump in pressure is a second order quantity relative to w;

$$\Delta P = -\left[\frac{\rho_n \rho_s}{\rho} - \frac{1}{2}\rho^2 u_2^2 \frac{\partial \alpha}{\partial P}\right] w^2 \tag{3.80}$$

while there is a linear relation between ΔT and w,

$$\Delta T = \frac{\rho_n}{\rho} \frac{u_2}{s} w \quad . \tag{3.81}$$

These results are for a second shock moving into a state where $w_1 = 0$ and $w = w_2$, the relative velocity behind the shock. The order of magnitude relationships between ΔT, ΔP, and w in Eqs. (3.80) and (3.81) are the same as those for continuous solutions. The velocity of the discontinuity is

$$c_2 = u_2 + \frac{1}{2}\gamma(v_{n_1} + v_{s_1})$$

where γ is defined by Eq. (B8) and it has been assumed $v_1 = v_2 = 0$. Therefore the first order change in the velocity of propagation is one-half of its value for continuous solutions. These results include the lowest order expansions of ρ and s for small w, Eqs. (3.56).

CHAPTER IV

CONCLUDING REMARKS

A. A REVIEW OF THE RESULTS

The preceding work has clarified and extended the linear theories of second sound and exhibited the deficiencies in the previous Riemann analysis. In addition, the detailed behavior of the propagation of a temperature pulse has been given. Some of the important results and conclusions will be discussed further here and additional improvements in the theory will be suggested.

In general, the two-fluid model has provided a very useful and accurate continuum theory to describe the hydrodynamics of He II and, in particular, the propagation of thermal waves. The Riemann theory of second sound as presented in Chapter III can be better improved by refining the analysis within the two-fluid concept rather than by seeking a better continuum description of He II. Within this model, the Landau equations are the basic set of equations of motion upon which other, more general ones are based and, hence, are the natural starting point for an analysis of second sound. As with the two-fluid model, a better description of thermal waves can be found more readily by refining the Landau equations rather than by using a completely new set of equations of motion.

By making a number of simplifying assumptions we derived Eqs. (3.10), the simplest set of non-linear equations which describe the propagation of only thermal waves in He II; these equations were then used to study a particular physical problem. Under these assumptions a temperature pulse is propagated without attenuation down a one-dimensional channel and all quantities connected with this flow are functions of temperature only. In particular, the relative velocity is related to the temperature by a first order ordinary differential equation. The numerical integration of this equation shows that the rate of change of w with respect to T is always positive and w is nearly a linear function of temperature. Even though the density is a constant and no first sound exists in this approximation, a pressure variation which is second order in w travels with the velocity of thermal waves along with the temperature pulse.

Due to the non-linear nature of the Landau equations any given temperature pulse deforms as it travels down a channel and eventually degenerates into a temperature discontinuity, or a shock wave. The non-linear breaking occurs at the front or rear edge of a given temperature pulse depending on whether the initial temperature is greater or less than 1.89°K. This value agrees with the one determined experimentally. Since this breaking always occurs, care must be exercised in the use of pulses to measure the amplitude-dependent velocity of propagation of second sound to insure that a shock is not formed.

By subsequently eliminating the assumptions mentioned above we were able to study their qualitative and quantitative effects on the theory of second sound. Excluding the interaction of first and second sound, it was found that the basic description of the flow is not changed by making these assumptions but there are a number of differences which lead to some important conclusions. Any theoretical description of thermal waves must include the relative velocity as a thermodynamic variable to be valid to order $\bar{w}$. This conclusion is the principle source of errors in Temperley's analysis.

Furthermore, it was shown that the lowest order term of F, the function which determines the critical temperature, is changed by the inclusion of w as a thermodynamic variable. This naturally leads to the question of whether other important parameters might be similarly affected. For instance, the expression for the shearing stress on a surface contains the derivative of v_n or, equivalently, the derivative of w and thus this quantity might be strongly affected by this consideration. In any case, a complete hydrodynamic theory must include the thermodynamic effects of $\bar{w}$ and, consequently, it is necessary to review other theories in light of this conclusion.

It has been stated that the function F, and as a consequence T_c and T^*, are very sensitive to the exact temperature dependence of δ, s, and u_2. It can also be seen by comparison of Fig. 9 and Fig. 16 that the inclusion of the lowest order terms in the expansion of the thermodynamic variables in w also changes F significantly. Therefore it might be necessary to use some higher order terms in these expansions to get a good theoretical prediction of the critical temperatures depending on the magnitude of these additional terms. In other words, detailed knowledge of both the equations of state as given by Eqs. (3.53) and the function $u_2 = u_2\ (P, T)$ is necessary for accurate theoretical results.

Except for the discussion of the interaction of first and second sound, the same type of solution as given by Eqs. (3.25) was valid for all the analysis and, thus, no attentuation is present.

Therefore, we can conclude that effects such as thermal expansion do not cause any attenuation if first sound is negligible. However, these same effects may contribute to attenuation when both pressure and temperature waves are considered. Other effects which have not been considered here such as thermal conduction contribute to the attenuation of temperature waves whether both types of waves are considered or not.

When the interaction of first and second sound is considered the entire analytical description of wave propagation in He II is changed and a completely new one must be developed. At the level of analysis completed here it is only possible to write explicit expressions for the velocity of propagation of both pressure and temperature waves and to deduce the order of magnitude of the amplitude of the two types of waves. Clearly, any theoretical description of wave motion in He II must include both modes of propagation if it is to be valid to order $\bar{w}^2$.

The amplitude-dependent velocity of propagation of second sound is altered in two different ways by the consideration of first sound. First of all, it is now measured with respect to the mass flux velocity v; this is a small correction in the type of physical problem considered here because it is mainly temperature waves that are excited. Secondly, the coefficient of the order w^2 term is different. This is also a small correction because the change is $O(\frac{u_2^2}{u_1^2})$ and u_2 is an order of magnitude smaller than u_1 throughout the temperature range of interest. There is also an order w^2 change in the velocity of first sound. These changes are completely different from the corrections of the amplitude-independent propagation speeds u_1 and u_2 due to thermal expansion as given by London[1]. In his linear theory, first and second sound are decoupled if thermal expansion is negligible.

Further results cannot be stated until the complete solution is known and yet, due to the complexity of the calculations, it appears to be impossible to give any analytic expression for the solution. This type of approach to the interaction of first and second sound gives the complete description of the flow and, as such, cannot be handled as in the other, simpler cases. Therefore, within this analysis, some additional approximations or a numerical solution is called for.

A similar situation exists in the study of discontinuous solutions. The pressure jump across a temperature discontinuity cannot be completely neglected but the complexity in anything but a perturbation analysis makes the problem almost intractable. In this case, an extension of the procedures used in ordinary gas dynamics or a completely new approach must be used.

B. PROPOSALS FOR FURTHER RESEARCH

Since this work is the first and, to some degree, complete nonlinear theory of second sound only the essential features of the theory and its application to a physical problem have been considered in detail. Many other aspects of first and second sound need to be investigated using both the analytical approach taken here and also entirely different procedures before any hydrodynamic theory of wave propagation in He II can be considered to be complete.

There are four distinct places in this work where additional analysis can be done within the framework of the theory presented here. To begin with, it has been shown that the dependence of the thermodynamic variables on the relative velocity is very important and that the lowest order terms in the expansions of these variables in w have a significant effect on the solution. Hence, it would be worthwhile to include the next, higher order terms and investigate the importance of their effect on the results. These higher order terms are not known at the present time but, once known, they can be treated by the Riemann theory just as the lowest order terms were handled in Chapter III. The dependence of the thermodynamic variables on the relative velocity has never been experimentally investigated; some work needs to be done along this line due to the evident importance of this dependence.

Secondly, the analysis of pure second sound should be done without assuming s_n is a constant and including both the effects of thermal expansion and the dependence of the thermodynamic variables on the relative velocity simultaneously. This would not exhibit any new phenomena not already found in the other analyses but would be the most accurate theory for the prediction of quantities such as the critical temperature when first sound can be neglected. This extension of the current theory presents no inherent difficulties; this is not true, however, of the last two areas of investigation, the interaction of first and second sound and discontinuous solutions, that should be studied further.

As mentioned previously, the set of characteristics and invariants is difficult to use for detailed analytic work but it does provide a very useful tool for a numerical analysis of both pressure and temperature waves. However, discontinuities will still appear in the solution for almost any type of temperature pulse which means that derivatives in the x, t-plane become unbounded. This is not an insurmountable difficulty as far as a numerical solution is concerned but it is sufficient reason for using caution in setting up an iterative scheme and for carefully considering the possible errors in the solution. The possibility of further analytic work should not be completely ruled out.

Just the opposite situation exists in the case of discontinuous solutions where additional analytic results beyond those which are derived from the existing perturbation analyses are necessary before any numerical calculations are needed to complete the study of shocks. Rather than using the two-dimensional P, $1/\rho$ - space to study the shock transition as in ordinary gas dynamics, it will probably be necessary to generalize this to the three-dimensional s, T, w -space in order to study the properties of a temperature shock in He II. Questions about quantities such as the determinacy of the transition, the shock speed, and attenuation of a temperature shock should be considered. It has been shown that under certain conditions it is possible for a temperature pulse to degenerate into two distinct shocks. It would be interesting to study the relative motion of these two shocks to see if they converge or diverge and to question what happens if one of the shocks does overtake the other.

Another wide range of new problems that can also be treated by a similar Riemann analysis concerns various generalizations of the Landau equations. For instance, the effects of the viscosity of the normal part and of thermal conduction may be important in various temperature ranges and would cause attenuation of a temperature pulse even if first sound were neglected. The previous calculations and conclusions may be altered considerably in the temperature ranges where the attenuation becomes large. We can also speculate about an effect of viscosity other than attenuation that arises due to the strong temperature dependence of the normal viscosity, η_n. (This dependence is shown in London,[1] Fig. 28 and Atkins,[10] Fig. 39; the function $\eta_n(T)$ is qualitatively the same but these authors disagree on the temperature where the derivative of η is very large by about .5°K.) In any case, a small fluctuation of the temperature in a second sound wave will cause a considerable change in the viscosity at that temperature where the viscosity varies very rapidly. Hence, the normal part will experience less drag at the higher temperature phases of the second sound wave. Thus, the normal and superfluid parts will be driven out of phase which will manifest itself by the appearance of first sound or, as Hsieh[1] has suggested by the harmonic generation of thermal waves. This phenomenon has not been discussed theoretically nor observed experimentally. A detailed study of viscosity and thermal conduction effects constitutes one of the most important supplements to this current Riemann analysis.

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[1] Private Communication

Another way in which the Landau equations can be generalized is to include the mutual friction terms. This would not only add additional attenuation to the system of equations but also another mode of interaction of first and second sound. An analysis of the generalized Landau equations would also provide an additional method for contrasting the different formulations of the mutual friction terms.

Still another area of interest deals with the boundary conditions used to supplement the equations of motion. One important point concerning the specific physical problem studied here is the existence of the Kapitza boundary effect, the temperature jump that occurs at a solid-liquid interface in He II when the heat flow is directed from the solid into the liquid. This effect was completely neglected here because it is a small correction when the heat current is small. However, this effect cannot be neglected for a large positive heat pulse but, since no temperature discontinuity exists when the heat flows from the liquid into the solid, the conclusions of Chapter III are unchanged by the inclusion of this effect for negative temperature pulses. This is a very complicated problem because the magnitude of the temperature jump depends on the heat current, the temperature, the composition of the solid, and the smoothness of the interface. A better way to write the boundary condition is to require that the heat flux be continuous at the interface rather than specifying some condition on the temperature itself.

There are numerous different problems, and boundary conditions, that can be studied with the use of a set of characteristics and invariants in addition to the one considered here. For instance, the deflection of thermal waves off of a solid boundary can be studied by considering a tube of finite length rather than the semi-infinite one as in Chapter III. This certainly corresponds more closely to an actual experimental situation. This second boundary could be held at constant temperature or the temperature of the boundary could follow that of a given second sound wave impinging upon it as in the case of an extremely thin piece of copper foil opposite the boundary where the temperature is given. A completely different method of exciting second sound is to place a plug which is only porous to the superfluid part in a channel filled with He II. Pressure waves on one side of such a plug will give rise to both pressure and temperature waves on the other side because $v_n = 0$ and $v_s \neq 0$ at the solid-liquid interface so neither v or w is zero there. In this case, and in contrast to the one we have studied, the two velocities are known at the boundary and the temperature is derived from them.

A different set of boundary conditions is used when a second sound wave impinges on a liquid-gas interface. This situation is realized experimentally in a container filled with He II and having a free surface. When thermal waves are generated at the bottom of the container they create a fluctuation of temperature at the surface causing the vapor pressure to change. Therefore, pressure waves are generated in the helium vapor. This experiment has been successfully performed[44] and Khalatnikov[28] has derived expressions for the transmission and reflection coefficients at the liquid surface for a sinusoidal second sound wave.

The boundary conditions may enter in yet another way if the channel is narrow. In this case the boundary conditions along the walls of the channel become important; usually a no-slip condition is imposed on the normal velocity, i.e., $v_n = 0$ on the walls of the channel. In the limit of very narrow channels v_n is zero throughout the liquid. The wave motion when the normal component is completely clamped by the walls has been briefly discussed by Atkins[45] and he calls the resulting mode of wave propagation "fourth sound." The general situation must be analyzed in two space dimensions. This problem is important not only because of the new type of boundary conditions but also because there is some knowledge of the critical velocity in narrow channels. (Notice that there is no known relationship between the critical velocity as discussed in the first chapter and the critical temperature defined later in this work; the names are coincidental.) Thus, narrow channel flow provides an opportunity to study the relationship between second sound and the critical velocity.

There is at least one other type of flow problem that is of interest in connection with second sound; this is the propagation of thermal waves through rotating He II. The equations best suited to an analysis of this type of problem are the generalized Landau equations using the HVBK[34] formulation for the mutual friction. This is a very complex problem because the problem of the rotation of He II is, by itself, very complicated. It is known that second sound experiences additional attenuation when propagated through a rotating fluid as compared to a fixed mass of He II.[46] Because quantized vortex lines exist and are important in all considerations of rotating He II, a classical continuum approach may not be able to describe this problem in detail and the quantum nature of the flow must be explicitely taken into consideration.

There are two other areas of interest to the propagation of waves in He II that cannot be adequately described by the two-fluid model and the Landau equations. One stated limitation of the entire analysis in Chapter III is that the continuum approach is not valid at very low temperatures. As mentioned earlier, a new set of equations of motion must be developed to describe the hydrodynamics of He II and, in particular, the propagation of second sound for temperatures between 0° K and approximately 1° K. Very little work has been done in this area. Again, the quantum nature of the elementary excitations at these low temperatures is quite important and an adequate set of continuum equations which describe the hydrodynamics of He II are not known.

Finally, a whole new field of investigation is introduced when He^3 - He^4 mixtures are considered. A new variable, the concentration of He^3, is introduced and this considerably enriches the class of solutions to the hydrodynamic equations. It is known that concentrations of He^3 less than 4.5% have a very considerable effect on the amplitude-independent velocity of propagation.[47] Only simple periodic solutions to a linearized set of thermo-hydrodynamical equations have been found so this is virtually an open field of investigation.

APPENDIX A

THEORY OF HYPERBOLIC EQUATIONS FOR FUNCTIONS OF TWO VARIABLES

We will present a brief discussion of characteristics and invariants. A detailed discussion of the general theory of hyperbolic equations and some examples of applications to fluid dynamic problems is contained in several different books[1]. Specifically, we are interested in a quasi-linear first order system of n partial differential equations with dependent variables u_i, $i = 1, 2, \ldots n$. The two independent variables are x and t. Write this system of equations as

$$a_{ij} \frac{\partial u_j}{\partial t} + b_{ij} \frac{\partial u_j}{\partial x} = 0 \qquad \text{(A1)}$$

or

$$A\bar{u}_t + B\bar{u}_x = 0 \qquad \text{(A2)}$$

where $\bar{u}$ is the column vector of the unknowns u_i,

$$\bar{u} = \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix},$$

and A and B are $n \times n$ matrices which can depend on $x, t, \bar{u}$.

[1] For example, R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves (Interscience Publishers, Inc., New York, 1948), Chapter 2 and 3.

By forming a linear combination of Eqs. (A1) it is possible to rewrite this system so that the total derivatives of all the unknowns are taken in the same direction in the x, t-plane. This linear combination, called the "normal form" of Eq. (A2), is

$$\ell_i \left(a_{ij} \frac{\partial u_j}{\partial t} + b_{ij} \frac{\partial u_j}{\partial x} \right) = 0$$

and we want this to be in the form

$$\ell_i \left(\alpha \frac{\partial u_j}{\partial t} + \beta \frac{\partial u_j}{\partial x} \right) = 0 \quad .$$

This is possible if

$$\ell_i a_{ij} = \alpha \ell_j \qquad \text{and} \qquad \ell_i b_{ij} = \beta \ell_j \quad .$$

By combining these two equations into a single condition we can write

$$\beta \ell_i a_{ij} = \alpha \ell_i b_{ij}$$

or

$$\ell_i (b_{ij} - c a_{ij}) = 0 \tag{A3}$$

where

$$c = \frac{\beta}{\alpha} \quad .$$

Non-trivial eigenvectors $\overline{\ell} = (\ell_1, \ell_2, \ldots \ell_n)$ exist if

$$\mathrm{DET}\{B - cA\} = 0 \quad . \tag{A4}$$

This equation determines the eigenvalues, c^k, $k = 1, 2, \ldots n$, and Eq. (A3) determines the corresponding left eigenvectors, $\overline{\ell}^k$.

If n real, distinct eigenvalues and eigenvectors exist then the system of Eqs. (A2) is said to be "hyperbolic."

Now the normal form can be written

$$\ell_i a_{uj}\left(\frac{\partial u_j}{\partial t} + c\,\frac{\partial u_j}{\partial x}\right) = 0 \quad . \tag{A5}$$

This is formally similar to a linear sum of the derivatives of u_i along a "characteristic direction" defined by

$$\left(\frac{dx}{dt}\right)_k = c^k \quad . \tag{A6}$$

The "invariants" along the characteristics are given by Eq. (A5);

$$\ell_i a_{ij} du_j = 0$$

or

$$\overline{\ell} A d\overline{u} = 0 \quad . \tag{A7}$$

Equations (A4), (A6), and (A7) are the explicit formulae for the characteristics and invariants of system (A2).

APPENDIX B

A REVIEW OF OTHER THEORIES

The perturbation analysis done by Khalatnikov[38] will be reviewed in detail since it serves as a basis of comparison with the results derived throughout the main body of this work. Only the assumptions and analysis will be discussed in this Appendix in order to clarify the essential features of the theory. The final results are considered in detail when they are compared with those derived in Chapter III.

The equations of motion used by Khalatnikov are the same as Eqs. (1.3) under the assumption that

$$\nabla \times \overline{v}_s = 0 \quad . \tag{B1}$$

His theory also includes the second order expansion of ρ and s;

$$\left.\begin{aligned} s &= s_o + \left(\underline{\frac{\partial\alpha}{\partial T}}\right)_o \frac{\overline{w}^2}{2} + O(\overline{w}^4) \\ \frac{1}{\rho} &= \frac{1}{\rho_o} - \left(\underline{\frac{\partial\alpha}{\partial P}}\right)_o \frac{\overline{w}^2}{2} + O(\overline{w}^4) \end{aligned}\right\} \tag{B2}$$

where

$$\alpha = \frac{\rho_n}{\rho}$$

and the subscript "o" denotes quantities which are only functions of the independent thermodynamic variables P and T and not dependent on $\overline{w}$. The partial derivatives $\underline{\frac{\partial\alpha}{\partial T}}$ and $\underline{\frac{\partial\alpha}{\partial P}}$ have been underlined to indicate that they originate from the expansion of ρ and s.

Following Khalatnikov, assume all quantities are functions of $(x-ct)$ and write $\overline{v}$ and $\overline{w}$ in the following form:

$$\overline{v} = v_x(x-ct)\hat{e}_x + v_y(x-ct)\hat{e}_y$$

$$\overline{w} = w_x(x-ct)\hat{e}_x + w_y(x-ct)\hat{e}_y$$

where $\hat{e}_x$ and $\hat{e}_y$ are unit vectors in a cartesian coordinate system.

By substituting these forms for the velocities into Eqs. (1.3) and neglecting all order $\overline{w}^2$ terms we obtain the following equations for the primed variables:

$$- U\left(\frac{\partial\rho}{\partial P}\right)_o P' - U\left(\frac{\partial\rho}{\partial T}\right)_o T' - \rho_o^2\left(\frac{\partial\alpha}{\partial T}\right) U w_x w_x'$$

$$- \rho_o^2\left(\frac{\partial\alpha}{\partial P}\right)_o U w_y w_y' + \rho_o v_x' = 0$$

$$- \rho_o U v_x' + P' + 2\,\frac{\rho_{no}\rho_{so}}{\rho_o}\, w_x w_x' = 0$$

$$- \rho_o U v_y' + \frac{\rho_{no}\rho_{so}}{\rho_o}\, w_y w_x' + \frac{\rho_{no}\rho_{so}}{\rho_o}\, w_x w_y' = 0$$

$$\left[- \rho_o U\left(\frac{\partial s}{\partial P}\right)_o + \left(\frac{\partial \rho_s s}{\partial P}\right)_o w_x\right] P' + \left[- \rho_o U\left(\frac{\partial s}{\partial T}\right)_o\right.$$

$$\left. + \left(\frac{\partial \rho_s s}{\partial T}\right)_o w_x\right] T' + \left[\rho_{so} s_o - \rho_o U\left(\frac{\partial\alpha}{\partial T}\right)_o w_x\right] w_x' - \rho_o U\left(\frac{\partial\alpha}{\partial T}\right)_o w_y w_y' = 0$$

$$- \rho_{no}\left(U - 3\,\frac{\rho_{so}}{\rho_o}\, w_x\right) w_x' + \rho_o\left[s_o - \left(\frac{\partial\alpha}{\partial T}\right)_o w_x U\right] T'$$

$$- \rho_o\left(\frac{\partial\alpha}{\partial P}\right)_o w_x U P' + \rho_{no} w_x v_x' + \rho_{no} w_y w_y' = 0$$

$$- \rho_{no}\left(U + \frac{\rho_{no} - \rho_{so}}{\rho_o}\, w_x\right) w_y' - \rho_o\left(\frac{\partial\alpha}{\partial t}\right)_o w_y U T'$$

$$- \rho_o\left(\frac{\partial\alpha}{\partial P}\right)_o w_y U P' + \rho_{no} w_x v_y' + \frac{\rho_{no}\rho_{so}}{\rho_o}\, w_y w_x' = 0$$

where

$$U = c - v_x$$

and a prime denotes differentiation with respect to the argument (x-ct). It should be re-emphasized that the plain and underlined α derivatives are actually identical but the underlined ones arise from the expansion of s or ρ while the others arise from the differentiation of α in the equations of motion. All but the last of these equations correspond with, but are not identical to, Eqs. (1) - (5). Equation (6) comes from one component of Eq. (B1) while the $\hat{e}_y$ - component of the equation for $\overline{w}$ is not used. Equation (B1) is a restriction on the class of solutions to Landau's equations and not an equation of motion so including it in the set of Eqs. (1) - (5) has no a priori justification. There are other significant differences between the two sets of equations.

The determinant of this set of equations must vanish for non-trivial solutions to exist; partially expanding this determinant and dropping the subscript "o" gives the following condition on U:

$$U\left(U + \frac{\rho_n - \rho_s}{\rho} w_x\right) \mathrm{Det} = 0 \qquad \text{(B3)}$$

where

$$\mathrm{Det}=\begin{vmatrix} \left(\frac{\partial\rho}{\partial P}\right)U & \left(\frac{\partial\rho}{\partial T}\right)U & \rho^2\left(\underline{\frac{\partial\alpha}{\partial P}}\right)w_xU & \rho \\ 1 & 0 & 2\,\frac{\rho_n\rho_s}{\rho}\,w_x & \rho U \\ \rho\left(\frac{\partial\alpha}{\partial P}\right)w_xU & \rho\left[\left(\frac{\partial\alpha}{\partial T}\right)w_xU-s\right] & \rho_n\left(U-3\,\frac{\rho_s}{\rho}\,w_x\right) & \rho_n w_x \\ \rho\left(\frac{\partial s}{\partial P}\right)U-\left(\frac{\partial\rho_s s}{\partial P}\right)w_x & \rho\left(\frac{\partial s}{\partial T}\right)U-\left(\frac{\partial\rho_s s}{\partial T}\right)w_x & \rho\left(\underline{\frac{\partial\alpha}{\partial T}}\right)w_xU-\rho_s s & 0 \end{vmatrix}$$

However, since we are neglecting $O(\overline{w}^2)$ terms throughout this calculation, Eq. (B3) can also be written

$$\left(U-\frac{\rho_s}{\rho}\,w_x\right)\left(U+\frac{\rho_n}{\rho}\,w_x\right)\mathrm{Det}=0 \tag{B4}$$

Thus, the fact that all $O(\overline{w}^2)$ terms are being neglected does not allow two of the roots for U to be uniquely determined to $O(w_x)$ and Eqs. (8) and (10) are not necessarily valid. Also note that there are no terms in Det which contain w_y.

Equation (7) can be derived by setting $\mathrm{Det}=0$ with $P_o = P_o(\rho_o)$;

$$\left[U^2\left(\frac{\partial P}{\partial\rho}\right)-1\right]\left\{\frac{\rho_n}{\rho}\left(\frac{\partial s}{\partial T}\right)U^2-\left[4\,\frac{\rho_n\rho_s}{\rho^2}\left(\frac{\partial s}{\partial T}\right)-s\left(\frac{\partial\alpha}{\partial T}\right)-s\left(\underline{\frac{\partial\alpha}{\partial T}}\right)\right]w_xU-\frac{\rho_s}{\rho}\,s^2\right\}=0 \quad . \tag{B5}$$

Notice that both $\left(\frac{\partial\alpha}{\partial P}\right)$ and $\left(\underline{\frac{\partial\alpha}{\partial P}}\right)$ no longer appear in the equation for U. Two of the roots of Eq. (B4) are

$$U^2 = u_1^2 + O(\overline{w}^2) \quad . \tag{B6}$$

and we can write the remaining factor as

$$U^2 - u_z^2 - w_x U \left\{ 4 \frac{\rho_s}{\rho} - \frac{u_z^2}{\rho_s s}\left[\left(\frac{\partial \rho_n}{\partial T}\right) + \left(\frac{\partial \rho_n}{\underline{\partial T}}\right)\right] \right\} = 0 \quad . \tag{B7}$$

Therefore, $\left(\frac{\partial \rho_n}{\partial T}\right)$ and $\left(\frac{\partial \rho_n}{\underline{\partial T}}\right)$ are equally important as far as this root is concerned. By assuming $\overline{v} = 0$ and neglecting the distinction between $\frac{\partial \rho_n}{\partial T}$ and $\frac{\partial \rho_n}{\underline{\partial T}}$, Khalatnikov solves this equation to first order in w_x to find

$$c = u_z + \gamma v_{nx} \tag{B8}$$

where

$$\gamma = \frac{\rho_n}{\rho_s}\left[2 \frac{\rho_s}{\rho} - \frac{s}{\rho_n}\left(\frac{\partial \rho_n}{\partial T}\right)\left(\frac{\partial T}{\partial s}\right)\right] \tag{B9}$$

and

$$v_{nx} = \frac{\rho_s}{\rho} w_x \quad .$$

Finally, Khalatnikov shows the temperature dependence of γ for $0 < T < T_\lambda$. However, this dependence should only extend to approximately 1°K since this is the lower temperature limit for the validity of any continuum approach. In conclusion, although there are a number of questionable points in his analysis, Khalatnikov does have the correct results and conclusions.

Hsieh[34] has also derived an equation similar to Eq. (B5) from the Landau equations but with an entirely different procedure. He first assumes $P = P(\rho)$, $T = T(s)$, and that s_n is a constant; therefore he has neglected thermal expansion and the dependence of P and T on $\overline{w}$. Then, by seeking non-trivial solutions for the jump of the independent variables across a characteristic surface, it is found that the characteristic speed c is either v_n, v_s, or must satisfy

$$c^4 + (3v_n+v_s)c^3 + [(2+\alpha)v_n^2 - (1-\alpha)v_s^2$$
$$+7v_nv_s - u_1^2 - u_2^2]c^2 + \{\alpha v_n^3 + (4-\alpha)v_n^2 v_s$$
$$+ (1-\alpha)v_n v_s^2 - (1-\alpha)v_s^3 - [(3-2\alpha)v_n - (1-2\alpha)v_s]u_1^2$$
$$- 2[\alpha v_n + (1-\alpha)v_s]\, u_2^2\}\, c + \alpha v_n^3 v_s + 2(1-\alpha)v_n^2 v_s^2$$
$$- (1-\alpha)v_n v_s^3 - [2(1-\alpha)v_n - (1-2\alpha)v_s]v_n u_1^2$$
$$- [\alpha v_n^2 + (1-\alpha)v_s^2]u_2^2 + u_1^2 u_2^2 = 0 \qquad \text{(B10)}$$

where

$$\alpha = \frac{\rho_n}{\rho} \quad .$$

The roots v_n and v_s confirm that Eq. (B4) rather than Eq. (B3) is the correct one and that Khalatnikov is indeed correct.

Defining

$$U = c - v$$

and using the transformations defined by Eqs. (3.2a), Eq. (B10) can be written

$$U^4 + (1-2\delta)wU^3 - \left[u_1^2 + u_2^2 + \frac{(1-\delta)(1+3\delta)}{2} w^2\right]U^2$$
$$- (1-2\delta)w\left(u_1^2 + \frac{1-\delta^2}{4} w^2\right) U + u_1^2 u_2^2$$
$$- \frac{(1-\delta)(1-3\delta)}{4} w^2 u_1^2 - \frac{1-\delta^2}{4} w^2 u_2^2 + \frac{(1-\delta^2)^2}{16} w^4 = 0$$

or

$$U^4 + (1-2\delta)wU^3 - (u_1^2 + u_2^2)U^2 - (1-2\delta)wu_1^2 U + u_1^2 u_2^2 = 0 \qquad \text{(B11)}$$

where we have correctly neglected all $O(w^2)$ terms in Eq. (B11). Comparison of Eqs. (B10) and (B11) clearly shows the distinct advantage of using v and w rather than v_n and v_s. Thus result is the same as Eq. (B5), Khalatnikov's result, when s_n is a constant and $\left(\frac{\partial \alpha}{\partial T}\right)$ is neglected.

Temperley[40] has done a brief introduction to a Riemann theory of second sound based on equations of motion which are derived in an Appendix to his main theory. In addition to the conservation of mass and entropy equations as given by Eqs. (1.3a) and (1.3d) he uses the following two equations to describe the motion of v_n and v_s:

$$\frac{\partial v_n}{\partial t} + v_n \frac{\partial v_n}{\partial x} - \frac{w}{2s_n}\left(\frac{\partial s_n}{\partial t} + v_n \frac{\partial s_n}{\partial x}\right) + \frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\rho_s}{\rho_n} s \frac{\partial T}{\partial x} = 0$$

$$\frac{\partial v_s}{\partial t} + v_s \frac{\partial v_s}{\partial x} - \frac{\rho_n}{\rho_s}\frac{w}{2s_n}\left(\frac{\partial s_n}{\partial t} + v_n \frac{\partial s_n}{\partial x}\right) + \frac{1}{\rho}\frac{\partial \rho}{\partial x} - s\frac{\partial T}{\partial x} = 0 \quad .$$

These are Eqs. (A8) and (A9) in his paper. By comparison with Eqs. (1.3b) and (1.3c) it can be seen that the term $\frac{\rho_n \rho_s}{\rho} \nabla \frac{\overline{w}^2}{2}$ has been left out. Thus, Temperley's equations can not reduce to the Landau equations under any simplifying assumptions and there is no ground for comparison between his theory and the one presented in this paper. By making the same simplifying assumptions as discussed on page 34 [which means that he has neglected $O(w^2)$ not $O(w^3)$ terms] Temperley derives the following two equations which describe the propagation of pure second sound:

$$u_2 \frac{\partial \sigma}{\partial x} + \frac{w(\rho_s - \rho_n)}{\rho}\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} = 0$$

$$u_2 \frac{\partial w}{\partial x} + \frac{w(\rho_s - \rho_n)}{\rho}\frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial t} = 0$$

where

$$d\sigma = \frac{\rho u_2}{\rho_n \rho_s} \, d\rho_n \quad .$$

The characteristic velocities for these equations are

$$c = u_2 \pm \frac{\rho_n - \rho_s}{\rho} \, w + O(w^2) \quad .$$

Therefore Temperley and Khalatnikov find the same result for the amplitude-dependent velocity of propagation to first order in w if s_n is a constant. But this is purely coincidental since neither the equations of motion nor the simplifying assumptions are the same in the two analyses.

APPENDIX C

A BRIEF DISCUSSION OF NON-LINEAR BREAKING

It will be shown in this Appendix that the discontinuity in the solution first appears at the inflection point of the velocity of propagation, $c(\tau)$.

When the temperature is given as a function of time at the boundary $x = 0$ then it has been shown that the pressure and relative velocity are also known and, consequently, all quantities are known as a function of time on the boundary. In particular, the propagation speed is known and we can speak about $c(t)$ being given at the boundary.

The one-parameter family of L_+ characteristics is given by

$$x - c(\tau)(t-\tau) = 0 \qquad (3.23)$$

and the solution by Eqs. (3.25). Calculate $\frac{\partial \delta}{\partial t}$ by

$$\frac{\partial \delta}{\partial t} = \frac{d\delta}{d\tau}\frac{\partial \tau}{\partial t} \quad .$$

By using Eq. (3.23) and denoting derivatives by a subscript, we can write this as

$$\delta_t = \frac{\partial \delta}{\partial t} = \frac{c}{c-(t-\tau)c_\tau}\,\delta\tau \quad .$$

Therefore the solution has a discontinuity $(\delta_t \rightarrow \infty)$ if

$$c - (t-\tau)c_\tau = 0$$

which, using Eq. (3.23), is

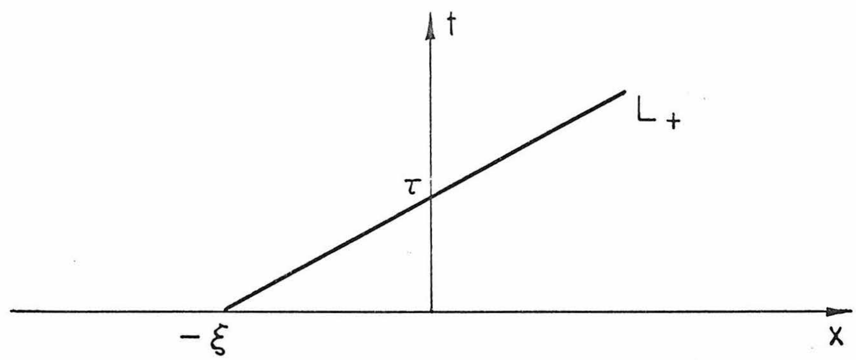
$$t = \tau + c/c_\tau \qquad (3.33a)$$

and

$$x = c^2/c_\tau \quad . \qquad (3.33b)$$

Thus we have illustrated a different method for finding the envelope of the L_+ lines, Eqs. (3.33).

Henceforth, it will be easier to use ξ rather than τ as the parameter in Eq. (3.23)



By definition

$$-\xi + c(\tau)\tau = 0$$

or

$$\xi = \tau c(\tau) \quad . \qquad (C1)$$

Thus

$$c_\tau = c_\xi \xi_\tau = c_\xi (c + \tau c_\tau)$$

and

$$c_\tau = \frac{c c_\xi}{1 - \tau c_\xi} \quad . \qquad (C2)$$

We will also need the second derivative

$$c_{\tau\tau} = \frac{d}{d\xi}\left(\frac{cc_\xi}{1-\tau c_\xi}\right)(c+\tau c_\tau)$$

$$= \frac{1}{c}\left(\frac{d\xi}{d\tau}\right)^3 c_{\xi\xi} \quad . \tag{C3}$$

The parametric equations for the envelope in terms of ξ are

$$t = \frac{1}{c_\xi}$$

$$x = \frac{c-\xi c_\xi}{c_\xi} \quad .$$

Therefore, the cusp of the envelope - - the point in the x, t-plane where a discontinuity first appears - - which occurs when t has a minimum, is at the point where c_ξ has a maximum;

$$\frac{d}{d\xi}(c_\xi) = 0$$

or

$$\frac{d^2c}{d\xi^2} = 0 \quad .$$

By Eq. (C3) this is $c_{\tau\tau} = 0$, which is the inflection point on the boundary condition $c(\tau)$.

APPENDIX D

PROOF THAT $v = 0$

Equations (3.25) represent the solution to Eqs. (3.10) for the chosen physical problem and its boundary conditions. It will be shown here that any solution to Eqs. (3.10) with $v = 0$ at a given point implies that v is identically zero.

Consider Eqs. (3.3c) and (3.3d), two of Landau's equations. If we assume s_n is a constant then these equations, in one dimension, can be written

$$\frac{\partial w}{\partial t} + [v + (1-\delta)w] \frac{\partial w}{\partial x} + \left[\frac{2}{1-\delta^2} u_2^2 - \frac{1}{2} w^2\right] \frac{\partial \delta}{\partial x} + w \frac{\partial v}{\partial x} = 0 \qquad \text{(D1)}$$

$$\frac{\partial \delta}{\partial t} + (v - \delta w) \frac{\partial \delta}{\partial x} + \frac{1-\delta^2}{2} \frac{\partial w}{\partial x} + \frac{1-\delta^2}{2} w \frac{1}{\rho} \frac{\partial \rho}{\partial x} = 0 \quad . \qquad \text{(D2)}$$

Now assume we know any solutions $w = w(x,t)$ and $\delta = \delta(x,t)$ which satisfy Eqs. (3.10). Then these equations can be used to simplify Eqs. (D1) and (D2); the result is

$$\begin{aligned} v \frac{\partial w}{\partial x} + w \frac{\partial v}{\partial x} &= 0 \\ v \frac{\partial \delta}{\partial x} + \frac{1-\delta^2}{2} w \frac{1}{\rho} \frac{\partial \rho}{\partial x} &= 0 \quad . \end{aligned} \qquad \text{(D3)}$$

Equation (D3) implies that

$$vw = \text{const.}$$

Thus, for a non-zero w if v is zero anywhere, for instance at a boundary, then it vanishes identically.

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