THEORY OF
ELECTRODYNAMICS OF MEDIA IN NON-INERTIAL FRAMES
AND APPLICATIONS

Thesis by

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ABSTRACT

In the first half of this thesis a local electrodynamics of media in given non-inertial frames*, within Maxwell-Einstein classical field theory, is constructed in terms of observable EM fields and co-moving local physical media parameters. Localization of tensors to observables is introduced and justified, and a relation is obtained connecting tensor transforms to instantaneous Lorentz transforms for observers in different frames. A constitutive tensor, explicitly expressed by the four-velocity and the local properties in co-moving frame of a linear medium, is found for the first time. Previous mistakes in confusing the tensors, in which forms the physical quantities combine with the non-flatness of frames to be used in covariant equations and thus make geometrical quantities, with observables are cleared. Also a Lagrangian formulation for both lossless and lossy media is constructed, and boundary conditions, local conservation laws, and energy momentum tensor are obtained.

The second half concerns application to motions in SRT, such as uniform linear (hyperbolic) acceleration and steady rotation. For these local Maxwell equations in co-moving frames are obtained, and approximate solutions are found for special cases. An EM wave propagating in the direction of acceleration is studied in the accelerating frame. The first order propagation shows a frequency shift and amplitude change which have very simple physical significances of instantaneous Doppler shift and photon density in media

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*The contribution of EM field to $g_{\mu\nu}$ is neglected.

and which agree with familiar results in the vacuum limit. A particle model for this wave shows that the "mass dressed" photon is dragged by the medium and does not follow a geodesic path. In the rotating medium case a plane wave scattered by a rotating sphere is solved by an integral iteration method in the laboratory frame. The scattered field purely associated to the rotation of the medium is separated from the Mie scattering. Its first order amplitudes are found and plotted for incidences perpendicular and parallel to the rotation axis. Particular symmetry and shapes of scattering amplitude in the results agree with intuition and resemble radiation patterns of appropriately induced traveling electric and magnetic dipole sheaths.

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I. INTRODUCTION

The theory and application of electrodynamics of media in inertial and non-inertial motions are subject to the recent interest of many authors (1-6,10,11,13). The purpose of this thesis is to construct a local electrodynamics, within Maxwell-Einstein's classical field theory (7,12,14,18,27,36) directly in terms of the observable EM fields and the rest-frame physical constitutive properties of a medium in a non-inertial frame. This may be produced either by a non-inertial motion in SRT, or by the presence of a tidal gravitation. Then application to simple problems in SRT is examined.

For a simple medium moving with uniform velocity, Lee and Papas (1-3) recently found the time harmonic Green's function and showed that dipole radiation in it has a forward-tilted far-zone Poynting vector. The time-dependent Green's functions are obtained by other workers (4,5). Many more studies (5,29,30) deal with different theoretical approaches and applications. The theory being used is Maxwell's theory and special relativity.

For media in non-inertial frames less work has been done (6,10, 11,13,25). Since macroscopic "photons" do not follow null paths nor geodesics in this case, in order to get any information of EM phenomena, we must start from Maxwell's equations. Then two problems arise which were not encountered in the previous inertial motions. The first concerns the physically observable EM fields to observers in a non-inertial frame and how it enters into the postulated covariant equations which govern the EM field space-time evolution. The second concerns a covariant formalism of the macroscopic media constitutive relations, which can only be determined locally in the medium co-moving frame and which should be constructed and built into the field equations.

In the first concern, Einstein's tetrad physics (14,20,33) and covariant Maxwell equations in general relativity are used to obtain a relation of the form

$$\underset{[1]}{e}{}^{\alpha}{}_{(\lambda)} \, \Lambda^{(\lambda)}{}_{(\overline{\delta})} \, \underset{[2]}{e}{}^{(\overline{\delta})}{}_{\overline{\beta}} \, \frac{\partial x^{\overline{\beta}}}{\partial x^{\gamma}} = \delta^{\alpha}{}_{\gamma} \tag{1}$$

where $A^{(\alpha)} = e^{(\alpha)}{}_{\beta} A^{\beta}$ localizes the tensor components to be physically observable in a general frame. Equation 1 reduces the tensor transform to instantaneous Lorentz transform for observables of observers in different frames. It explicitly states the form in which physical observables, whose measurements locally in GRT are identical with that of SRT, combine with the non-inertialness of the frame and/or space-time to make the physics laws in a covariant form. This localized transform is important and very useful, especially when one is interested in the local physics, e.g., electrodynamics in media, for which general local Maxwell equations are obtained later. These equations also show the extent of the approximation in using the usual 3-vector Maxwell equations for a neighborhood of non-inertial space-time.

In the second concern, based on a covariance assumption for medium EM equations, a constitutive tensor expressed by the 4-velocity $\underline{u}$ and the rest-frame local properties $\varepsilon^{(i)}{}_{(j)}, K^{(i)}{}_{(j)}$ of linear medium is constructed for the first time,

$$C^{\mu\nu\alpha\beta} = \frac{1}{2} K_{\gamma\delta}(*u)^{\mu\nu\gamma}\ (*u)^{\alpha\beta\delta} + \frac{1}{2}[u^{\alpha}(\varepsilon^{\nu\beta}u^{\mu} - \varepsilon^{\mu\beta}\ u^{\nu})$$

$$- u^{\beta}(\varepsilon^{\nu\alpha}u^{\mu} - \varepsilon^{\mu\alpha}u^{\nu})] \qquad (2)$$

such that $G^{\mu\nu} = C^{\mu\nu\alpha\beta}\ F_{\alpha\beta}$.

Previous errors and confusions (6,10) in considering an example and not distinguishing physical observables from their tensors, and thus leading to misinterpretations of $C^{\mu\nu\alpha\beta}$ as physical properties of media, are all cleared. With this covariant formalism to build the constitutive parameters of media into EM theory, we also find the Lagrangian formulations for the lossless and lossy media, the boundary conditions, local conservation laws, and energy momentum tensor.

In the latter half of this work applications to motions of media in SRT such as uniform linear (hyperbolic) acceleration and steady rotation are considered. For both cases exact local Maxwell equations in co-moving frames are found. In the rotational case the error in a previous work (11) is corrected. Then special problems are solved in detail.

In a uniform linear-accelerated simple medium the EM wave propagating along the direction of acceleration is studied by co-moving observers. The first order solution gives two terms that correspond to traveling against and traveling with the apparent gravitation in that frame. A frequency shift and amplitude decrease (or increase) result for this first order propagation and have the simple meanings of equivalent gravitational red (or blue) shift and instantaneous "photon" density. The coordinate phase velocity is time dependent. If we identify the instantaneous frequency and phase velocity of the wave as energy and velocity of the corresponding "mass-dressed" photon* (24), then the photon has a time-dependent mass and does not follow a geodesic. Physically it means photons are dragged by the non-inertial motion of the medium.

In the rotational case a plane wave scattered by a rotating simple sphere is studied by using integral iteration method in the laboratory frame. The scattered field purely due to the rotation of the medium is separated from the Mie scattering (35) . This is the only scattering, providing that the rotating medium is the same as its surrounding medium. The first order amplitude of this rotational scattered field is evaluated and plotted for incidences perpendicular and parallel to the axis of rotation. Particular symmetry and the shapes of scattering amplitude result; they agree with intuition and resemble the radiation patterns of appropriately induced traveling electric and magnetic dipole sheaths.

Part II is the general theory in which II.1 introduces localization and equation 1 for frame co-moving observers; II.2 constructs a formalism for the constitutive relations; II.3 derives local equations, least action formalism, and boundary conditions, and investigates local conservation laws and energy momentum tensor. Part III

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*In media, even in an inertial frame, this identification is arbitrary and a ± sign is included in the definition of the dressed mass. Also it is obvious that the path is not null.

gives the application: III.1 on linear accelerated medium and III.2 on steady rotating medium. Appendices contain remarks and some derivations. Fundamental knowledge of GRT and tensor calculus is assumed and some references are given (15-23). Also geometrized unit is used in the text for convenience and A-1 shows conversion to mks units.

II. THEORY

II.1 Local Physical Tensors of Frame Co-Moving Observers and their Transforms

From the principle of equivalence, local physical bases and their tensor components as corresponding physical observables are introduced (14,20,26,33). For co-moving observers in different frames physical bases reduce tensor transforms to instantaneous Lorentz transforms.

II.1.1 Reviews and coordinate bases vectors. Consider a 4-dimensional differentiable manifold S_4 labelled with permissible coordinate frame $\{x^\mu\}$ * which represents a space-time continuum. An affine connected geometry is constructed in the following usual way (21-23): Define parallel transport of vectors** by a set of affine numbers $\Gamma^\mu_{\alpha\beta}$, then define geodesic as a path generated by parallel transport dx^μ; define geometrical scalar distance $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ between neighboring points by symmetric metric tensor $g_{\mu\nu}$ and define path of extreme length by $\delta \int ds = 0$; then identify the path of extreme length to geodesic. This so-constructed geometry is identified with physics space-time by the postulate that free-falling neutral particles follow a geodesic***.

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*If $\{x^\mu\}$, $\{x^{\bar{\mu}}\}$ are two coordinates of S_4, then both $x^\mu = x^\mu(x^{\bar{\mu}})$ and $x^{\bar{\mu}} = x^{\bar{\mu}}(x^\mu)$ exist and are differentiable. Also, $\mu = 0$ for time, and $\mu = i = 1,2,3$ for space. Minkovskian signature $(+,-,-,-)$ and coordinate frame $\{x^\mu\} \equiv \{x^\mu\}$ are used.

**Vectors and tensors defined in their usual transform senses.

***This is possible because Etvos' experiment showed $\mathcal{M}_{grav} = \mathcal{M}_{inertial}$; present accuracy, 10^{-11}.

Now at a point $P(x^\mu)$ the coordinate contravariant and covariant basis vectors (App.2), $\{\underline{e}_\mu\}$ and $\{\underline{e}^\mu\}$ of $\{x^\mu\}$ are defined by

$$d\underline{x} \equiv dx^\mu \underline{e}_\mu \equiv dx_\mu \underline{e}^\mu, \qquad \mu = 0,1,2,3 \tag{1}$$

where $dx_\mu \equiv g_{\mu\nu} dx^\nu$ thus $\underline{e}^\mu \equiv g^{\mu\nu} \underline{e}_\nu$ and dx^μ are infinitesimal coordinate increments of x^μ at $P(x^\mu)$. The scalar length $ds^2 \equiv d\underline{x} \cdot d\underline{x} = g_{\mu\nu} dx^\mu dx^\nu$ then implies the scalar products of basis vectors satisfy:

$$\underline{e}_\mu \cdot \underline{e}_\nu = g_{\mu\nu}, \quad \underline{e}^\mu \cdot \underline{e}^\nu = g^{\mu\nu}, \quad \underline{e}^\mu \cdot \underline{e}_\nu = \delta^\mu{}_\nu \tag{2}$$

Thus $\underline{e}_\mu$ is a vector with length $|g_{\mu\mu}|^{\frac{1}{2}}$ pointing in the coordinate tangent direction of x^μ for $\{x^\mu\}$. Compare eq. 1 in $\{x^{\bar{\mu}}\}$ to eq. 1, then,

$$\underline{e}_\mu = \frac{\partial x^{\bar{\alpha}}}{\partial x^\mu} \underline{e}_{\bar{\alpha}} \quad , \quad \underline{e}^\mu = \frac{\partial x^\mu}{\partial x^{\bar{\alpha}}} \underline{e}^{\bar{\alpha}} \tag{3}$$

where $\{\underline{e}_{\bar{\alpha}}\}$, $\{\underline{e}^{\bar{\alpha}}\}$ are basis vectors to $\{x^{\bar{\mu}}\}$ at the same point P .*

Similar to equation 1, now any vector field $\underline{V}$ at P can be expressed as

$$\underline{V} \equiv V^\mu \underline{e}_\mu \equiv V_\mu \underline{e}^\mu \tag{4}$$

then equation 2 implies

$$V^\mu = \underline{V} \cdot \underline{e}^\mu \quad , \quad V_\mu = \underline{V} \cdot \underline{e}_\mu \tag{5}$$

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*Notice that the "μ" for basis vector labelling is not a contra- or a co-variant tensor components labelling; it only describes the coordinate direction to which $\underline{e}_\mu$ is tangent. The super and subscript positions are used to keep sum convention and distinguish the two sets.

and equation 3 implies

$$V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\bar{\alpha}}} V^{\bar{\alpha}} \quad , \quad V_{\mu} = \frac{\partial x^{\bar{\alpha}}}{\partial x^{\mu}} V_{\bar{\alpha}} \tag{6}$$

Thus contra- and co-variant vectors are actually components of a vector on the respective coordinate basis as in equation 4. The above equations 4 - 6 apply to tensors of higher ranks with more indices written, e.g.,

$$\underline{\underline{\underline{T}}} = T^{\mu\nu\alpha} \underline{e}_{\mu} \underline{e}_{\nu} \underline{e}_{\alpha} \quad , \quad T^{\mu\nu\alpha} = \underline{e}^{\alpha} \underline{e}^{\nu} \underline{e}^{\mu} \cdots T \tag{7}$$

II.1.2 Localization of tensor to observable on physical basis of frame co-moving observers. Consider a flat space-time (curvature tensor $R^{\mu}{}_{\nu\alpha\beta} = 0$, no tidal gravitation) in which an inertial $\bar{K}$ frame with Cartesian coordinates $\{X^{\bar{\mu}}\}$ such that $g_{\overline{\mu\nu}} \equiv \eta_{\mu\nu}$ exists*; we can also describe this space-time by a $\{x^{\mu}\}$ so its co-moving observers $\{0\}$ with world lines $\{\Gamma\} \equiv \{x^{i} = \text{const.}, x^{0} \text{ varies}\}$ are in a non-inertial frame K. Now consider an 0 observer passing an inertial $\bar{0}$ of $\bar{K}$ momentarily at P. The equivalence principle states that in a small neighborhood of P the physics of 0 is identical to that of an instantaneously inertial co-moving $0'$ with Minkovskian $\{X^{\mu'}\}$ and $\{\underline{e}_{\mu'}\}$ whose physics is related to 0 by a Lorentz transform. Thus a vector $\underline{dx}$ observed by $\bar{0}$ as $dx^{\bar{\mu}}$ will be observed by 0 as

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* $\eta_{\overline{\mu\nu}} \equiv \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ Notice that the local spatial orthogonalization to time must be done for geometrical model to coincide with physics; spatial orthogonality is just a convenient choice.

Also, spatial normalization means length measured with the same unit of rule; time normalization means constancy of light propagation.

$$dx^{(\mu)} \equiv dX^{\mu'} = \Lambda^{\mu'}{}_{\bar{\alpha}}\, dx^{\bar{\alpha}} = \Lambda^{\mu'}{}_{\bar{\alpha}} \frac{\partial X^{\bar{\alpha}}}{\partial x^{\lambda}} dx^{\lambda}, \quad \Lambda^{\mu'}{}_{\bar{\alpha}} \equiv \Lambda^{(\mu)}{}_{\bar{\alpha}} \qquad (8)$$

where $\Lambda^{\mu'}{}_{\bar{\alpha}}$ is a Lorentz transform using the relative instantaneous velocity and spatial axes orientation of $0'$ to $\bar{0}$. Thus equation 8 provides a relation between the observed quantity $dx^{(\mu)}$ and the contravariant components dx^{μ} of a vector $d\underline{x}$ in $\{x^{\mu}\}$ for its co-moving $\{0\}$. This applies to any vectors and tensors representing physical quantity; we call it localization and define $e^{(\mu)}{}_{\lambda}$ by

$$dx^{(\mu)} = e^{(\mu)}{}_{\lambda}\, dx^{\lambda}, \quad e^{(\mu)}{}_{\lambda} \equiv \Lambda^{(\mu)}{}_{\bar{\alpha}} \frac{\partial X^{\bar{\alpha}}}{\partial x^{\lambda}} \qquad (9)$$

From equations 1 and 2 we have $dx^{\mu}\, \underline{e}_{\mu} = d\underline{x} = dX^{\mu'}\, \underline{e}_{\mu'} \equiv dx^{(\mu)}\, \underline{e}_{(\mu)}$, then equation 9 implies

$$\underline{e}_{(\mu)} \cdot \underline{e}_{(\nu)} = \eta_{\mu\nu}, \quad \underline{e}_{(\mu)} \cdot \underline{e}^{(\nu)} = \delta_{\mu}{}^{\nu}, \quad \underline{e}^{(\mu)} \cdot \underline{e}^{(\nu)} = \eta^{\mu\nu}$$

$$e^{(\mu)}{}_{\lambda} = \underline{e}^{(\mu)} \cdot \underline{e}_{\lambda}, \quad e^{(\mu)\lambda} = \underline{e}^{(\mu)} \cdot \underline{e}^{\lambda}, \quad e_{(\mu)}{}^{\lambda} = \underline{e}_{(\mu)} \cdot \underline{e}^{\lambda}, \qquad (10)$$

$$e_{(\mu)\lambda} = \underline{e}_{(\mu)} \cdot \underline{e}_{\lambda}$$

and

$$dx^{\lambda} = e^{\lambda}{}_{(\alpha)}\, dx^{(\alpha)} \qquad (11)$$

where $\underline{e}_{(\mu)} \equiv \underline{e}_{\mu'}$ and $\underline{e}_{\mu'} \cdot \underline{e}_{\lambda'} \equiv \eta_{\mu'\lambda'}$ of $0'$ have been used. Thus we see that to physically observe or measure a vector or tensor quantity by some $\{0\}$ in a frame $\{x^{\mu}\}$ is to observe its local components as equation 8 on a local Minkovskian basis $\{\underline{e}_{(\mu)}\}$ of $\{0\}$. But this local result also applies to $\{\bar{0}\}$ in $\bar{K}$, thus we do not need $\overline{K}$ to be inertial, i.e., we do not need to be in a flat space-time. In that case equation 9 becomes

$$dx^{(\overline{\mu})} = \underset{[2]}{e}{}^{(\overline{\mu})}{}_{\overline{\lambda}}\, dx^{\overline{\lambda}} , \qquad dx^{(\mu)} = \underset{[1]}{e}{}^{(\mu)}{}_{\lambda}\, dx^{\lambda} \tag{12}$$

and equation 8 becomes

$$dx^{(\mu)} = \Lambda^{(\mu)}{}_{(\overline{\alpha})}\, \underset{[2]}{e}{}^{(\overline{\alpha})}{}_{\overline{\beta}}\, \frac{\partial x^{\overline{\beta}}}{\partial x^{\lambda}}\, dx^{\lambda} \tag{13}$$

or, with equation 11,

$$\frac{\partial x^{\lambda}}{\partial x^{\overline{\beta}}} = \underset{[1]}{e}{}^{\lambda}{}_{(\mu)}\, \Lambda^{(\mu)}{}_{(\overline{\alpha})}\, \underset{[2]}{e}{}^{(\overline{\alpha})}{}_{\overline{\beta}} \tag{14}$$

for arbitrary frames co-moving observers $\{0\}$, $\{\overline{0}\}$ with physical tetrad basis $\{ \underset{[1]}{\underline{e}}{}_{(\mu)}\}$, $\{ \underset{[2]}{\underline{e}}{}_{(\overline{\mu})}\}$ in $\{x^{\mu}\}$, $\{x^{\overline{\mu}}\}$ respectively, which explicitly states measurements in GRT are locally identical with SRT, and localizes tensor transforms to instantaneous Lorentz transforms for observables of any tensor. Equation 14 is an important and very useful relation, especially when local physics is emphasized, e.g., electrodynamics in media. Analogous relations of equation 11 or 12 hold for observations on any vector $\underline{V}$ or tensor $\underline{\underline{T}}$ by $\{0\}$ in $\{x^{\mu}\}$ with $\{\underline{e}_{(\mu)}\}$ local physical basis

$$V^{(\alpha)} = e^{(\alpha)}{}_{\beta}\, V^{\beta} \qquad , \qquad V^{\alpha} = e^{\alpha}{}_{(\beta)}\, V^{(\beta)}$$

$$T^{(\alpha)(\beta)} = e^{(\alpha)}{}_{\gamma}\, e^{(\beta)}{}_{\delta}\, T^{\gamma\delta} \quad , \quad T^{\alpha\beta} = e^{\alpha}{}_{(\gamma)}\, e^{\beta}{}_{(\delta)}\, T^{(\gamma)(\delta)} \quad ; \text{etc.} \tag{15}$$

Fig. 1 graphically sketches equation 14.

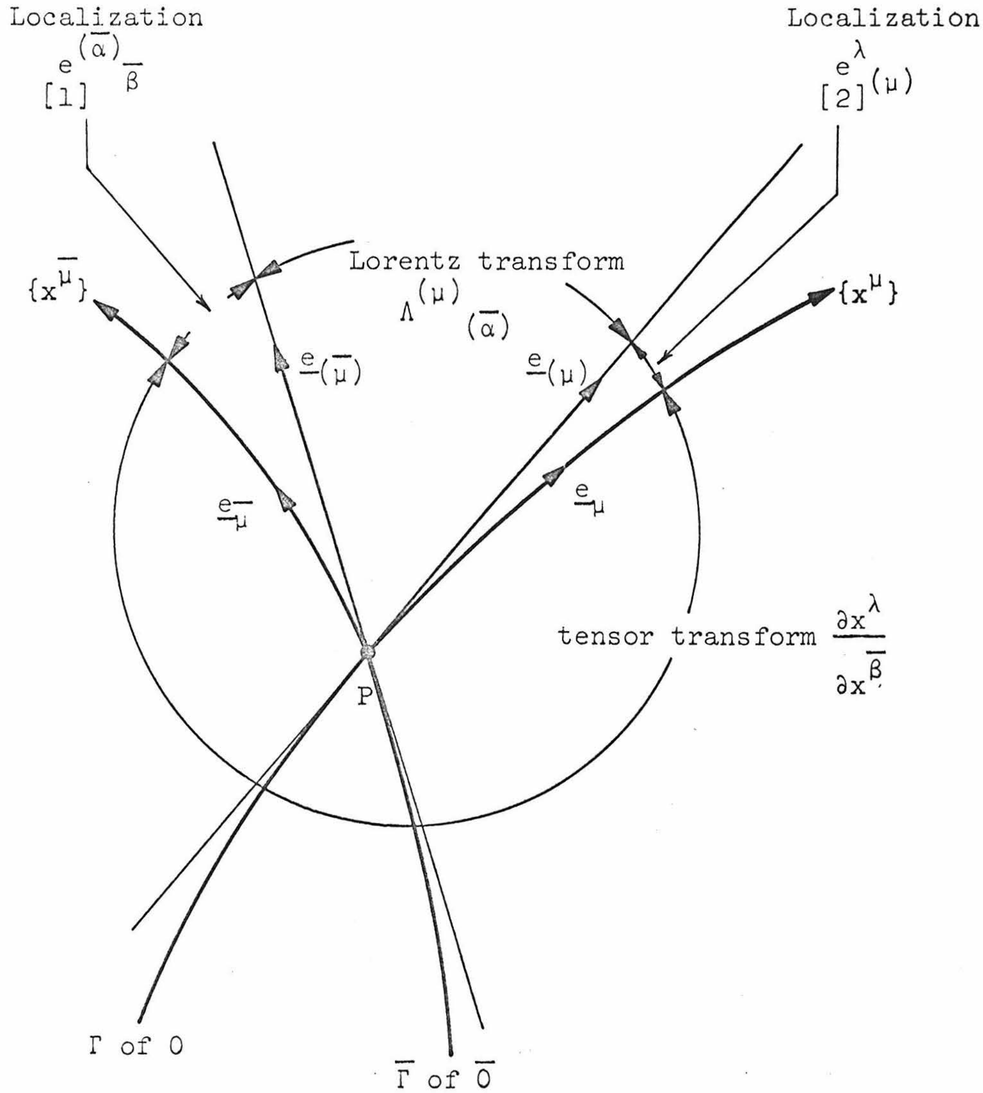

Fig.1. A sketch representing equation 14. $\Gamma, \bar{\Gamma}$ are world lines of $O, \bar{O}$ which are co-moving in $\{x^{\mu}\}, \{x^{\bar{\mu}}\}$ respectively.

To find $\{\underline{e}_{(\mu)}\}$ for $\{0\}$, first, for a co-moving observer in $\{x^\mu\}$, his proper time during a coordinate time lapse $dt \equiv dx^o$ is $d\tau \equiv ds = \sqrt{g_{oo}}\, dx^o$, thus

$$\underline{e}_{(o)} = \frac{1}{\sqrt{g_{oo}}}\, \underline{e}_o \equiv \underline{u} \tag{16}$$

Then local time direction for $\{0\}$ is the coordinate time direction there, only physically rescaled by $\frac{1}{\sqrt{g_{oo}}}$; here $\underline{u}$ is the 4-velocity of 0 . Since the locally pure spatial $\{\underline{e}_{(i)}\}$ are orthogonal to $\underline{e}_{(o)}$ thus to $\underline{e}_o$, and they are orthonormalized for convenience, thus within the restriction of equation 10 it follows that

$$\{\underline{e}_{(i)}\} = \text{O.N. } \{\underline{d}_i \equiv \underline{e}_i - \frac{g_{oi}}{g_{oo}}\, \underline{e}_o\} \qquad *(17)$$

which are defined within a spatial rotation, and $\underline{d}_i$ are just time orthogonal triads.

<u>II.1.3 Coordinate transport of co-moving basis</u>. Now equations 16 and 17 specify $\{\underline{e}_{(o)}\}$ but leave free the ways $\{0\}$ can carry $\{\underline{e}_{(i)}\}$ along their $\{\Gamma\}$. In order to have simple formalism for local physics, the medium-fixed observers should also keep their $\{\underline{e}_{(i)}\}$ as a whole not rotating with respect to the medium. Thus the simple Fermi transport** of $\{\underline{e}_{(i)}\}$ which preserves $\{\underline{e}_{(\mu)}\}$ but fixes $\{e_{(i)}\}$ directions with respect to distant stars is not convenient. Later studies on motions in SRT reveal that for frame $\{x^\mu\}$ with (Appendix II)

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*O.N. ≡ orthonormal combinations of.
**Appendix IV.

$$\frac{\gamma_{ij}^{\;2}}{\gamma_{ii}\,\gamma_{jj}} = \text{time independent}, \qquad \gamma_{ij} \equiv \underline{d}_i \cdot \underline{d}_j \tag{18}$$

Neighboring co-moving observers will not see each other's $\{\underline{e}_{(i)}\}$ rotating if $\{\underline{e}_{(i)}\}$ is carried along $\{\Gamma\}$ of $\{0\}$ by

$$\{\underline{e}_{(i)}\} = \text{O.N. } \{\underline{n}_i\} \quad \text{, with time independent coefficients and conditions (eq. 10)} \tag{19}$$

where $\underline{n}_i$ are just the Frenet-Serret normals to Γ *

along Γ

$$\begin{cases} \dfrac{D\underline{u}}{Ds} \equiv a_1\underline{n}_1 \\[2ex] \dfrac{D\underline{n}_1}{Ds} \equiv a_1\underline{u} + a_2\underline{n}_2 \\[2ex] \dfrac{D\underline{n}_2}{Ds} \equiv -a_2\underline{n}_1 + a_3\underline{n}_3 , \quad \underline{n}_i \cdot \underline{n}_i \equiv -1 \end{cases} \tag{20}$$

Thus the medium-co-moving-frame-attached observers should coordinate transport $\{\underline{e}_{(i)}\}$ as equation 19, such as to have a locally non-rotating spatial triad with respect to the frame. By equation 14, the coefficients in equation 19 can be chosen to make $\{\underline{e}_{(\mu)}\}$ as the instantaneous Lorentz transform of the $\{\underline{e}_{(\bar{\mu})}\}$ of some frame $\{x^{\bar{\mu}}\}$ which has particular simple geometry, although it may not co-move with the medium.

If in $\{x^\mu\}$ equation 18 does not hold, the frame of the medium to which $\{x^\mu\}$ co-moves is not "locally rigid", e.g.

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* a_i are curvatures of Γ in $\underline{n}_i$ directions. Also, equation 20 implies $\dfrac{D}{Ds}\,\underline{n}_3 = -a_3\underline{n}_2$.

$\frac{\bar{u}^i}{\bar{u}^o}$ = position dependent in the case of motion in SRT, where $\bar{u}^\mu$ are components in an inertial $\{\bar{X}^\mu\}$ of $\underline{u}$ of $\{0\}$ in $\{x^\mu\}$. Then it is impossible to have any orthogonal $\{\underline{e}_{(i)}\}$ for $\{0\}$ which also hold un-rotated to their neighbors in $\{x^\mu\}$. In this case we still coordinate transport $\{\underline{e}_{(i)}\}$ as equation 19 to keep it orthonormal and least local rotated.

II.2 Electromagnetic Descriptions of Media as Tensors

The formalism to enter the phenomenological media properties into EM theory is thoroughly investigated and a constitutive tensor expressed on the 4-velocity and rest-frame local physical parameters of a linear medium is constructed for the first time.

II.2.1 Physical constitutive relations. Splitting the interaction $-J^\mu A_\mu$ between EM field and microscopic charge-current of matter into one part for the macroscopic neutral medium and the other part for free net charges, and then averaging each spatially (8,28) gives the macroscopic medium EM field $\underset{\sim}{D}$, $\underset{\sim}{H}$ and explicit charge-current $(\rho, \underset{\sim}{J})$ as a result. This procedure averages the inertial frame vacuum Maxwell equations

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \rho \quad , \qquad \underset{\sim}{\nabla} \times \underset{\sim}{B} = \underset{\sim}{J} + \frac{\partial}{\partial T} \underset{\sim}{E} \tag{21}$$

into the medium equations

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{D} = \rho \quad , \qquad \underset{\sim}{\nabla} \times \underset{\sim}{H} = \underset{\sim}{J} + \frac{\partial}{\partial T} \underset{\sim}{D} \tag{22}$$

and leaves

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \quad , \quad \underset{\sim}{\nabla} \times \underset{\sim}{E} = - \frac{\partial}{\partial T} \underset{\sim}{B} \tag{23}$$

the same form for both cases, with the understanding that all quantities in the media equations are macroscopic ones. Now the dependence of $\underset{\sim}{D}$, $\underset{\sim}{H}$ on $\underset{\sim}{E}$, $\underset{\sim}{B}$ * for most media at rest and over particular ranges of fields E , B can be characterized by linear relations with constants $\varepsilon^{(i)}{}_{(j)}$, $K^{(i)}{}_{(j)}$ **

$$D^{(i)} = \varepsilon^{(i)}{}_{(j)} E^{(j)}$$

$$H^{(i)} = K^{(i)}{}_{(j)} B^{(j)} \tag{24}$$

which are the so-called constitutive relations that actually approximate the medium's response to the EM field to the first order. The $\varepsilon^{(i)}{}_{(j)}$, $K^{(i)}{}_{(j)}$ are the constitutive parameters which can only be obtained by experiments or detailed microscopic consideration of the medium. Since only when an observer has no motion with respect to the medium can he obtain the intrinsic properties of it, and in general the form equation 24 holds only for media at rest, so physically the medium co-moving frame is the natural frame to start its EM investigation and $\varepsilon^{(i)}{}_{(j)}$, $K^{(i)}{}_{(j)}$ in that frame is the only set of numbers that describes the EM properties of a linear medium.

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*These 3-vector symbols $\underset{\sim}{E},\underset{\sim}{B},\underset{\sim}{D},\underset{\sim}{H}$ only stand for local observable EM fields with components $(E^{(1)}, E^{(2)}, E^{(3)})$ throughout this work; similar remarks apply to other 3-vectors.

**This particular way of writing indices is just to keep up with the sum convention.

For a medium in a non-inertial frame, the observer $\{0\}$ attached to it can still locally perform his experiment or theoretical considerations in his neighborhood, and since local physics is not affected by the presence of an equivalent gravitation, $\{0\}$ can determine the constitutive parameters locally for the medium in equation 24 as if he were in an inertial frame. Now the EM field observed by $\{0\}$ is the physical field at his location (x^i), and it is these co-moving physical constitutive parameters he so obtained as equation 24 that we should use as a basis to formulate a tensor for media properties.

Notice that equation 24 can be rewritten in terms of electric and magnetic polarization $\underset{\sim}{P}$ and $\underset{\sim}{M}$ as

$$\underset{\sim}{D} = \underset{\sim}{E} + \underset{\sim}{P}$$

$$\underset{\sim}{H} = \underset{\sim}{B} - \underset{\sim}{M} \tag{25}$$

which is equivalent to equation 24. But equation 24 is much simpler in general formulation and provides a clearer physics, so we adopt it.

<u>II.2.2 Tensor representation of linear media.</u> With the principle of covariance of physics laws, the covariance of $\underset{\sim}{E}$, $\underset{\sim}{B}$ Maxwell equations implies a vacuum EM tensor $F^{\mu\nu}$. Now that we postulate the covariance of the $\underset{\sim}{D}$, $\underset{\sim}{H}$ Maxwell equations implies a media EM tensor* $G^{\mu\nu}$. But in a non-Minkovskian frame $\{x^\mu\}$, as explained in Chapter II.1, the physical observable EM fields to any observer $\{0\}$

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*See Chapter II.3

in $\{x^\mu\}$ are not $F^{\mu\nu}$, $G^{\mu\nu}$, but $F^{(\mu)(\nu)}$, $G^{(\mu)(\nu)}$.**

Now consider a medium co-moving in $\{x^\mu\}$, then equation 24 is

$$\begin{cases} G^{(i)(o)} = \varepsilon^{(i)}{}_{(j)} F^{(j)(o)} \\ (*G)^{(i)(o)} = K^{(i)}{}_{(j)} (*F)^{(j)(o)} \end{cases}$$

which, since $u_{(\lambda)} = (1,0,0,0)$ for $\{0\}$, can be written as

$$\begin{cases} G^{(i)(\nu)} u_{(\nu)} = \varepsilon^{(i)}{}_{(j)} F^{(j)(\nu)} u_{(\nu)} \\ (*G)^{(i)(\nu)} u_{(\nu)} = K^{(i)}{}_{(j)} F^{(j)(\nu)} u_{(\nu)} \end{cases} \qquad (26)$$

Now expand equation 26 to a 4-local tensor form by defining some $\varepsilon^{(\mu)}{}_{(\nu)}$ $K^{(\mu)}{}_{(\nu)}$ $\ni$

$$\begin{cases} G^{(\mu)(\nu)} u_{(\nu)} = \varepsilon^{(\mu)}{}_{(\lambda)} F^{(\lambda)(\nu)} u_{(\nu)} \\ (*G)^{(\mu)(\nu)} u_{(\nu)} = K^{(\mu)}{}_{(\lambda)} F^{(\lambda)(\nu)} u_{(\nu)} \end{cases} \qquad (27)$$

What is $\varepsilon^{(\mu)}{}_{(\nu)}$, $K^{(\mu)}{}_{(\nu)}$? Since equation 27 must contain equation 26 we have the pure spatial parts of $\varepsilon^{(\mu)}{}_{(\nu)}$ and $K^{(\mu)}{}_{(\nu)}$ identically

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** As a trivial example, the $\overline{F^{\mu\nu}}$ in a Cartesian coordinate transformed into a cylindrical coordinate $\{x^\mu\}$ by tensor transform yields $F^{\mu\nu}$, we see immediately that

$$F^{o2} = \frac{-E^{(\emptyset)}}{r} \neq - E^{(\emptyset)}$$

equal to the physical constitutive parameters $\varepsilon^{(i)}{}_{(j)}$ and $K^{(i)}{}_{(j)}$ defined in equation 24. Now $\mu = 0$ in equation 27 implies

$$\varepsilon^{(o)}{}_{(i)} \equiv 0$$

$$K^{(o)}{}_{(i)} \equiv 0 \qquad (28)$$

Also, since we know that all physics is contained in $\varepsilon^{(i)}{}_{(j)}$, $K^{(i)}{}_{(j)}$ and equation 26 implies $\varepsilon^{(i)}{}_{(o)}$, $K^{(i)}{}_{(o)}$ play no role at all in physics, we can assign arbitrary values to them; for simplicity

$$\varepsilon^{(i)}{}_{(o)} \equiv 0$$

$$K^{(i)}{}_{(o)} \equiv 0 \qquad (29)$$

Now whatever $\varepsilon^{(o)}{}_{(o)}$ is, physics is not altered. In order to have simple notations for local isotropic media $\ni \varepsilon^{(\mu)}{}_{(\nu)} = \varepsilon\delta^{\mu}{}_{\nu}$, $K^{(\mu)}{}_{(\nu)} = K\delta^{\mu}{}_{\nu}$ and treat all spatial directions with simple symmetric footing in their time participations, we put

$$\varepsilon^{(o)}{}_{(o)} \equiv \frac{1}{3}\left(\varepsilon^{(1)}{}_{(1)} + \varepsilon^{(2)}{}_{(2)} + \varepsilon^{(3)}{}_{(3)}\right)$$

$$K^{(o)}{}_{(o)} \equiv \frac{1}{3}\left(K^{(1)}{}_{(1)} + K^{(2)}{}_{(2)} + K^{(3)}{}_{(3)}\right) \qquad (30)$$

Thus equation 27 provides a local tensorial description of the medium in its co-moving frame, with $\varepsilon^{(\mu)}{}_{(\nu)}$, $K^{(\mu)}{}_{(\nu)}$ given by equations 24, 28, 29, and 30 in terms of physical, measurable, media parameters.

Since local tensors between observers of different frames are instantaneous Lorentz transform related, from equation 27 it follows that if in any $\{x^{\overline{\mu}}\}$ we define $\varepsilon^{(\overline{\mu})}{}_{(\overline{\nu})}$, $K^{(\overline{\mu})}{}_{(\overline{\nu})}$ to be the instantaneous Lorentz transform of $\varepsilon^{(\mu)}{}_{(\nu)}$, $K^{(\mu)}{}_{(\nu)}$ from $\{\underline{e}_{(\mu)}\}$ of $\{0\}$ to $\{\underline{e}_{(\overline{\mu})}\}$ of $\{\overline{0}\}$, then

$$\begin{cases} G^{(\overline{\mu})(\overline{\nu})}\, u_{(\overline{\nu})} = \varepsilon^{(\overline{\mu})}{}_{(\overline{\lambda})}\, F^{(\overline{\lambda})(\overline{\nu})}\, u_{(\overline{\nu})} \\ (*G)^{(\overline{\mu})(\overline{\nu})} u_{(\overline{\nu})} = K^{(\overline{\mu})}{}_{(\overline{\lambda})} (*F)^{(\overline{\lambda})(\overline{\nu})}\, u_{(\overline{\nu})} \end{cases} \tag{31}$$

are the physical constitutive relations in any $\{x^{\overline{\mu}}\}$, where $u_{(\overline{\nu})}$ is the 4-velocity $\underline{u}$ of $\{0\}$ observed by $\{\overline{0}\}$.

Now that equation 31 in general is mixed coupled in $\overline{\underset{\sim}{D}},\overline{\underset{\sim}{H}}$ to $\overline{\underset{\sim}{E}},\overline{\underset{\sim}{B}}$, we want to construct a physical tensor formula $\ni$

$$G^{(\mu)(\nu)} = C^{(\mu)(\nu)(\alpha)(\beta)}\, F_{(\alpha)(\beta)} \tag{32}$$

which gives $\underset{\sim}{D},\underset{\sim}{H}$ directly in terms of $\underset{\sim}{E},\underset{\sim}{B}$ in $\{x^{\mu}\}$ and thus in all $\{x^{\overline{\mu}}\}$. In constructing $C^{(\mu)(\nu)(\alpha)(\beta)}$, first we know that $\varepsilon^{(\mu)}{}_{(\lambda)}$ $K^{(\mu)}{}_{(\lambda)}$ contain all physics for media at rest; for moving media the only additional physics is its velocity $\underline{u}$. Also $C^{(\mu)(\nu)(\alpha)(\beta)}$ for linear media should be independent of field intensities, thus it should be made of $\underline{u}$, $\underline{\underline{\varepsilon}}$, $\underline{\underline{K}}$.

Comparing equation 32 with equation 27 in the medium co-moving $\{x^{\mu}\}$ reveals

$$\begin{cases} C^{(o)(i)(o)(j)} = \frac{1}{2}\,\varepsilon^{(i)(j)} \\ C^{(k)(\ell)(m)(n)} = -\frac{1}{2}\,K_{(i)(j)}\,\eta^{oik\ell}\,\eta^{mnoj} \\ \text{all other components} \equiv 0 \end{cases} \tag{33}$$

where $\eta^{\mu\nu\alpha\beta} = 1$ if $\pi(\mu\nu\alpha\beta) =$ even $\pi(0123)$, -1 if $\pi(\mu\nu\alpha\beta) =$ odd $\pi(0123)$, and 0 if neither of previous cases; and the symmetric parts of $C^{(\mu)(\nu)(\alpha)(\beta)}$ in $(\mu \overset{\rightarrow}{\leftarrow} \nu)$, $(\alpha \overset{\rightarrow}{\leftarrow} \beta)$ which enter into no physics has been set at 0 . Thus we have

$$C^{(o)(i)(o)(j)} = \frac{1}{2}\,\varepsilon^{(i)(j)}\,u^{(o)}u^{(o)} \longrightarrow$$

$$(C^{(\mu)(\nu)(\alpha)(\beta)})_{\underline{\underline{\varepsilon}}\text{-part}} \sim \frac{1}{2}\,\varepsilon^{(\nu)(\beta)}u^{(\mu)}u^{(\alpha)}$$

$$\xrightarrow[(\mu \rightleftarrows \nu)\ ,\ (\alpha \rightleftarrows \beta)]{\text{antisymmetrizing w.r.t.}}$$

$$(C^{(\xi)(\nu)(\alpha)(\beta)})_{\underline{\underline{\varepsilon}}\text{ part}} = \frac{1}{2}\,[(\varepsilon^{(\nu)(\beta)}u^{(\mu)} - \varepsilon^{(\mu)(\beta)}u^{(\nu)})u^{(\alpha)} - (\alpha \rightleftarrows \beta)] \tag{34a}$$

Similarly,

$$C^{(k)(\ell)(m)(n)} \text{ in } (33) \longrightarrow (C^{(\mu)(\nu)(\alpha)(\beta)})_{\underline{\underline{K}}\text{ part}} \sim \frac{-1}{2}\,K_{(i)(j)}\,\eta^{oi\mu\nu}\,\eta^{\alpha\beta oj}$$

$$\xrightarrow{\text{locate index } 0}$$

$$(C^{(\mu)(\nu)(\alpha)(\beta)})_{\underline{\underline{K}}\text{ part}} = \frac{+1}{2}\,K_{(\gamma)(\delta)}\,({}^{*}u)^{(\mu)(\nu)(\gamma)}\,({}^{*}u)^{(\alpha)(\beta)(\delta)} \tag{34b}$$

Thus for $\{0\}$ we have constructed for equation 32

$$C^{(\mu)(\nu)(\alpha)(\beta)} = \frac{1}{2} K_{(\gamma)(\delta)} \, (*u)^{(\mu)(\nu)(\gamma)} (*u)^{(\alpha)(\beta)(\delta)}$$

$$+ \frac{1}{2}[(\varepsilon^{(\nu)(\beta)} u^{(\mu)} - \varepsilon^{(\mu)(\beta)} u^{(\nu)}) \, u^{(\alpha)} - (\alpha \rightleftarrows \beta)] \qquad (35)$$

which, being a local tensor expression, is valid for $\{\bar{0}\}$ in any $\{x^{\bar{\mu}}\}$ with all indices "¯" (barred) as the instantaneous Lorentz transform of equation 35. The proof of validity of equation 35 is straightforward by using equation 27.

Either equation 27 or equations 32 and 35 provide a complete local tensorial description for linear media.* The corresponding tensors (global tensors) are obtained simply by de-localization with equation 15, thus in $\{x^\mu\}$ we have

$$\begin{cases} G^{\mu\nu} \, u_\nu = \varepsilon^\mu{}_\lambda \, F^{\lambda\nu} \, u_\nu \\ (*G)^{\mu\nu} u_\nu = K^\mu{}_\lambda \, (*F)^{\lambda\nu} \, u_\nu \end{cases} \qquad (36)$$

or

$$\begin{cases} G^{\mu\nu} = C^{\mu\nu\alpha\beta} \, F_{\alpha\beta} \\ C^{\mu\nu\alpha\beta} = \frac{1}{2} K_{\gamma\delta} (*u)^{\mu\nu\gamma} (*u)^{\alpha\beta\delta} + \frac{1}{2}[u^\alpha(\varepsilon^{\nu\beta} u^\mu - \varepsilon^{\mu\beta} \, u^\nu) \\ \qquad - u^\beta(\varepsilon^{\nu\alpha} u^\mu - \varepsilon^{\mu\alpha} \, u^\nu)] \end{cases} \qquad (37)$$

where $\{e^\alpha{}_{(\mu)}\}$ of $\{0\}$ is used to de-localize. In any $\{x^{\bar{\mu}}\}$ the "indices barred" equivalent of equations 36 and 37 is obtained by tensor transform of equations 36 and 37 from $\{x^\mu\}$ to $\{x^{\bar{\mu}}\}$, or by de-localization of equation 31 and the local "barred" equations 32 and 35 in that frame $\{x^{\bar{\mu}}\}$ itself, i.e., using equation 14.

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* Appendix X .

As special cases, if we consider a co-moving-local-isotropic medium $\ni$ in the rest frame

$$K^{(\mu)}{}_{(\nu)} = K\delta^{\mu}{}_{\nu} \quad , \quad \varepsilon^{(\mu)}{}_{(\nu)} = \varepsilon\, \delta^{\mu}{}_{\nu} \tag{38}$$

then equations 36 and 37 take the respective simple forms

$$\begin{cases} G^{\mu\nu}\, u_{\nu} = \varepsilon\, F^{\mu\nu}\, u_{\nu} \\ (*G)^{\mu\nu} u_{\nu} = K(*F)^{\mu\nu}{}_{\mu\nu} \end{cases} \tag{39}$$

$$C^{\mu\nu}{}_{\alpha\beta} = \frac{1}{2\mu}\, \delta^{\mu\nu}{}_{\alpha\beta} + \frac{1}{2}\left(\frac{1}{\mu} - \varepsilon\right) [u^{\mu}(\delta^{\nu}{}_{\alpha} u_{\beta} - \delta^{\nu}{}_{\beta}\, u_{\alpha}) - u^{\nu}(\delta^{\mu}{}_{\alpha}\, u_{\beta} - \delta^{\mu}{}_{\beta}\, u_{\alpha})] \tag{*40}$$

in any frame; which for vacuum $\mu = 1 = \varepsilon$ yields $C^{\mu\nu}{}_{\alpha\beta} = \frac{1}{2}\, \delta^{\mu\nu}{}_{\alpha\beta}$, thus $G^{\mu\nu} = F^{\mu\nu}$ follows independent of the observers frame. For lossless media the $\varepsilon^{(\mu)(\nu)}$, $K^{(\mu)(\nu)}$, and thus $\varepsilon^{\mu\nu}$, $K^{\mu\nu}$ are symmetric (Appen. V), then $C^{\mu\nu\alpha\beta}$ is symmetric with respect to $(\mu\nu) \rightleftarrows (\alpha\beta)$. All losses in a lossy medium are due to the antisymmetric part of $\varepsilon^{\mu\nu}$, $K^{\mu\nu}$, or the $(\mu\nu) \rightleftarrows (\alpha\beta)$ antisymmetric part of $C^{\mu\nu\alpha\beta}$.

Thus we construct a tensor formalism to enter the local constitutive properties of linear media into EM theory. No pure theoretical parameters are involved. Equation 27 or 35 tells the constitutive physics directly. Equation 36 or 37 tells the form the physics combine with frame and/or space-time non-inertialness to enter global tensor

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* $\mu \equiv \frac{1}{K}$ here as permeability constant should not cause any confusion with the μ-index.

formalism in EMT; any direct interpretation to the elements of this $c^{\mu\nu\alpha\beta}$ as intrinsic physical properties of the medium is wrong. All the above result from the postulations of the covariance of macroscopic Maxwell equations in media.

II.3 General Formalism of Local Electrodynamics in Media

Maxwell-Einstein equations in their local forms are found in the medium co-moving frame and arbitrary frames. A least action principle is constructed for lossless and lossy media, and boundary conditions and local conservation laws are obtained.

II.3.1 Local Maxwell equations in medium co-moving frame.

If in an inertial frame $\{X^\mu\} \equiv \{T,X,Y,Z\}$, then equations 21 and 23 can be rewritten as

$$F^{\mu\nu}{}_{,\nu} = -J^\mu \tag{41a}$$

$$(*F)^{\mu\nu}{}_{,\nu} = 0 \quad \leftrightarrow \quad F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{41b}$$

where

$$(\ )_{,\nu} \equiv \frac{\partial}{\partial X^\nu}(\ ) \tag{42a}$$

$$J^\mu \equiv (\rho, J^X, J^Y, J^Z) \tag{42b}$$

$$(*F)^{\mu\nu} \equiv \frac{1}{2}\eta^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{42c}$$

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & -E^X & -E^Y & -E^Z \\ E^Z & 0 & -B^Z & +B^Y \\ E^Y & B^Z & 0 & -B^X \\ E^Z & -B^Y & B^X & 0 \end{pmatrix} \tag{42d}$$

The SRT covariance of physical laws postulates that Maxwell equations in vacuum in the form of equations 41 and 42 holds for all inertial frames and implies that A^μ, J^μ are Lorentz tensors. Now in a medium, we postulate the above for the macroscopic EM fields, then equations 22 and 23 still result in the same forms except that equation 41a is replaced by

$$G^{\mu\nu}{}_{,\nu} = -J^\mu \tag{43a}$$

$$G^{\mu\nu} \equiv \begin{pmatrix} 0 & -D^X & -D^Y & -D^Z \\ D^X & 0 & -H^Z & H^Y \\ D^Y & H^Z & 0 & -H^X \\ D^Z & -H^Y & H^X & 0 \end{pmatrix} \tag{43b}$$

Equations 41b - 43 are SRT Maxwell equations in media, with equation 36 or 37 for linear media.

Now in GRT, in a general coordinate $\{x^\mu\}$ the Einstein-Maxwell theory postulates in vacuum the equations 41 still hold but with "," partial derivatives replaced by ";" covariant derivatives and $\eta^{\mu\nu\alpha\beta}$, $\eta_{\alpha\beta}$ replaced by $\varepsilon^{\mu\nu\alpha\beta}$, $g_{\alpha\beta}$

$$\begin{cases} V^\mu{}_{;\lambda} \equiv V^\mu{}_{,\lambda} + \Gamma^\mu{}_{\lambda\alpha} V^\alpha \\ T^{\mu\nu}{}_{;\lambda} \equiv T^{\mu\nu}{}_{,\lambda} + \Gamma^\mu{}_{\lambda\alpha} T^{\alpha\nu} + \Gamma^\nu{}_{\lambda\alpha} T^{\mu\alpha} \qquad \text{, etc.} \\ \varepsilon^{\mu\nu\alpha\beta} \equiv (-g)^{-1/2}\, \eta^{\mu\nu\alpha\beta} \end{cases} \tag{44}$$

where the Christoffel symbols are defined by the metric $g_{\mu\nu}$ of $\{x^\mu\}$

$$\Gamma^{\mu}{}_{\alpha\beta} \equiv g^{\mu\lambda}\, \Gamma_{\lambda|\alpha\beta}$$

$$\Gamma_{\lambda|\alpha\beta} \equiv \frac{1}{2}\,(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}) \tag{45}$$

Then postulating the same for EM field in media gives

$$G^{\mu\nu}{}_{;\nu} = -\,J^{\mu} \tag{46a}$$

$$(*F)^{\mu\nu}{}_{;\nu} = 0 \leftrightarrow \exists A_{\mu} \ni F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu} \tag{46b}$$

as macroscopic Maxwell-Einstein tensor equations for any frame $\{x^{\mu}\}$. But now the observable EM field and current by observers $\{0\}$ co-moving in $\{x^{\mu}\}$ along $\{\Gamma\!: \; x^{i} = \text{fixed},\; x^{0} = \text{varying}\}$ are the locally measured $F^{(\mu)(\nu)}$, $G^{(\mu)(\nu)}$, $J^{(\mu)}$ on their $\{\underline{e}_{(\mu)}\}$, with equations 42b,d and 43b no longer valid, but

$$F^{(\mu)(\nu)} = \begin{pmatrix} 0 & -E^{(1)} & -E^{(2)} & -E^{(3)} \\ E^{(1)} & 0 & -B^{(3)} & B^{(2)} \\ E^{(2)} & B^{(3)} & 0 & -B^{(1)} \\ E^{(3)} & -B^{(2)} & B^{(1)} & 0 \end{pmatrix}$$

$$G^{(\mu)(\nu)} = \begin{pmatrix} 0 & -D^{(1)} & -D^{(2)} & -D^{(3)} \\ D^{(1)} & 0 & -H^{(3)} & H^{(2)} \\ D^{(2)} & H^{(3)} & 0 & -H^{(1)} \\ D^{(3)} & -H^{(2)} & H^{(1)} & 0 \end{pmatrix} \tag{47}$$

$$J^{(\mu)} = (\rho,\, J^{(1)},\, J^{(2)},\, J^{(3)})$$

which enter the postulated covariant simple-formed tensor equations 46 by using equation 15 such that the EM system is described by a set of differential equations in $\{x^\mu\}$ which, in principle, can be integrated to determine its global space-time evolution. The local constitutive physics for a linear medium is just equation 24 in its co-moving frame and equation 31 or 35 in any frame $\{\bar{x}^\mu\}$, and the tensor formalism of it is equation 36 or 37 where $\underline{u}$ is the 4-velocity of the part of the medium under consideration.

Now suppose a $\{x^\mu\}$ is the medium co-moving frame in order to find the EM equations for observables in equation 47 of $\{0\}$ in $\{x^\mu\}$; first, equations 16 and 17 or 19 imply for $\{0\}$ in $\{x^\mu\}$

$$\begin{cases} (\underline{e}_{(o)})^\mu \equiv e_{(o)}{}^\mu = \dfrac{1}{\sqrt{g_{oo}}}(1,0,0,0) \\ (\underline{e}_{(o)})_\mu \equiv e_{(o)\mu} = \dfrac{1}{\sqrt{g_{oo}}}\; g_{o\mu} \\ (\underline{e}_{(i)})_o \equiv e_{(i)o} = 0 \end{cases} \tag{48}$$

Then substituting equation 47 into 46 with equation 15 gives the local Maxwell equations for $\{0\}$ at rest with respect to the medium (Appen. III) as follows:

$$\begin{cases} [\sqrt{-g}\, e_{(i)}{}^j \dfrac{D^{(i)}}{\sqrt{g_{oo}}}]_{,j} + [\sqrt{-g}\, \eta^{oijk}\, e_{(i)}{}^o\, e_{(j)}{}^\ell\, H_{(k)}]_{,\ell} \\ \qquad\qquad = \sqrt{-g}\,(\dfrac{\rho}{\sqrt{g_{oo}}} + J^{(i)}\, e_{(i)}{}^o) \quad (49a) \\ [\sqrt{-g}\, \eta^{oik\ell}\, e_{(i)}{}^j\, e_{(k)}{}^\nu\, H_{(\ell)}]_{,\nu} = [\sqrt{-g}\, e_{(i)}{}^j \dfrac{D^{(i)}}{\sqrt{g_{oo}}}]_{,o} + \sqrt{-g}\, J^{(i)} e_{(i)}{}^j \quad (49b) \end{cases}$$

$$\left\{\begin{array}{l} [\sqrt{-g}\, e^{j}{}_{(i)} \dfrac{B^{(i)}}{\sqrt{g_{oo}}}]_{,j} - [\sqrt{-g}\, \eta^{oijk}\, e_{(i)}{}^{o}\, e_{(j)}{}^{\ell}\, E_{(k)}]_{,\ell} = 0 \qquad (49c) \\ \\ [\sqrt{-g}\, \eta^{oik\ell}\, e_{(i)}{}^{j}\, e_{(k)}{}^{\nu}\, E_{(\ell)}]_{,\nu} = -[\sqrt{-g}\, e_{(i)}{}^{j} \dfrac{B^{(i)}}{\sqrt{g_{oo}}}]_{,o} \qquad (49d) \end{array}\right.$$

$$D^{(i)} = \varepsilon^{(i)}{}_{(j)}\, E^{(j)} \quad , \quad H^{(i)} = K^{(i)}{}_{(j)}\, B^{(j)} \qquad (49e)$$

Physically in equation 46a the mix of local current density $\underset{\sim}{J}$ into charge density ρ and the presence of the curl-like term of $\underset{\sim}{H}$ compensate for the fact that the coordinate divergence of $\underset{\sim}{D}$ is not taken purely spatially; similar remarks apply to equations 49b,c,d. In fact, if we express coordinate differential operators by local differential operators through equation 11, the local flat equations

$$\left\{\begin{array}{ll} \dfrac{\partial G^{(\mu)(\alpha)}}{\partial X^{(\alpha)}} = -J^{(\mu)} & (50a) \\ \dfrac{\partial (*F)^{(\mu)(\alpha)}}{\partial X^{(\alpha)}} = 0 & (50b) \end{array}\right.$$

are valid in a small enough neighborhood only after the space-time dependence of $g_{\mu\nu}$ is neglected (Appen. III).

When the $\{x^{\mu}\}$ has synchronous metric $g_{oi} \equiv 0$ all $e_{(o)}{}^{i}$, $e_{(o)i}$, $e^{(o)i}$, $e^{(o)}{}_{i}$ vanish, we can rescale time by letting $g_{oo} \equiv 1$ and define 3-spatial operator $\underset{\sim}{\nabla}\text{x}$ and $\underset{\sim}{\nabla}\cdot$ according to equation 49 ∋

$$\left\{\begin{array}{l} \underset{\sim}{\nabla} \cdot \underset{\sim}{D} = \rho \\ \\ \underset{\sim}{\nabla} \cdot \underset{\sim}{H} = \underset{\sim}{J} + \dfrac{\partial \underset{\sim}{D}}{\partial t} + \underset{\approx}{\alpha} \cdot \underset{\sim}{D} \end{array}\right.$$

$$\begin{cases} \underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \\ \underset{\sim}{\nabla} \times \underset{\sim}{E} = - \dfrac{\partial \underset{\sim}{B}}{\partial t} - \underset{\approx}{\alpha} \cdot \underset{\sim}{B} \end{cases} \tag{51}$$

where

$$\alpha^{(j)}{}_{(i)} \equiv e^{(j)}{}_{k} \, (\sqrt{-g} \, e^{k}{}_{(i)})_{,o}$$

If this $\{x^{\mu}\}$ has even simpler $g_{\mu\nu} \neq$ functions of time, then equation 51 just reduces to the ordinary 3-vector equations 22, 23 in curvilinear coordinates.

II.3.2 Local Maxwell equations in frames not co-moving with media. For $\{\bar{0}\}$ in a $\{x^{\bar{\mu}}\}$ not medium-co-moving, all results obtained in the previous section still hold only with $\{\underline{e}_{(\mu)}\}$ replaced by $\{\underline{e}_{(\bar{\mu})}\}$ and equation 49e replaced by equation 31 or 32. The velocity for instantaneous Lorentz transform from $\{0\}$ to $\{\bar{0}\}$ is just $V^{(\bar{i})} \equiv u^{(\bar{i})}/u^{(\bar{o})}$. Although coordinate conditions (15, 20, 27) can be imposed to $\{x^{\bar{\mu}}\}$ to simplify the local equations 49a,b,c,d in it, in general such a frame is not medium co-moving and the mixed constitutive relations then make it difficult to decouple the equations. Also, the physics is obscured because the relative motion of the coordinate-conditioned frame with respect to the medium enters the equations. The wave equations of potentials $A_{\bar{\mu}}$ in any frame $\{x^{\bar{\mu}}\}$ expressed by the constructed $c^{\mu\nu\alpha\beta}$ in a linear medium, from equation 46, are

$$J^{\bar{\mu}} = 2(c^{\bar{\mu}\bar{\nu}\bar{\alpha}\bar{\beta}} \, A_{\bar{\alpha};\bar{\beta}})_{;\bar{\nu}} \tag{52}$$

in which a gauge condition can be used on the divergence of $\underline{A}$. For the special case of a medium as equation 38 and a frame in which

$\frac{Du^\mu}{Dx^\lambda} \equiv 0$, then equation 52 becomes

$$[-\frac{1}{\mu} A^{\mu;\nu} + (\frac{1}{\mu} - \varepsilon) u_\beta u^\nu A^{\mu;\beta}]_{;\nu} + [\frac{1}{\mu} R^{\lambda\mu} + (\frac{1}{\mu} - \varepsilon) R^{\beta\lambda\mu\nu} u_\beta u_\nu] A_\lambda$$

$$= - [J^\mu + (\frac{1}{\mu} - \varepsilon) u^\mu u_\beta J^\beta] \qquad (53)$$

where an invariant gauge condition of the form

$$\frac{1}{\mu} A^\nu{}_{;\nu} - (\frac{1}{\mu} - \varepsilon) u_\beta u^\nu A^\beta{}_{;\nu} = 0 \qquad (54)$$

was used. The $R^{\mu\nu\alpha\beta}$ and $R^{\mu\nu} \equiv R_\lambda{}^{\mu\lambda\nu}$ are the Riemann and Ricci curvature tensors respectively (26, p.43). Equations 53 and 54 in the Minkovski limit of inertial frames reduce to the equations 7-105,106 of Ref. 19.

<u>II.3.3 Lagrangian formulation of EM fields in media, boundary conditions, local conservation laws, and energy momentum tensor.</u> Whether lossy or not, Maxwell equations 46 can be obtained from an invariant integral as follows:

<u>A. Lossless media.</u> For a lossless linear medium $c^{\mu\nu\alpha\beta} = c^{\alpha\beta\mu\nu}$(Appen. V); then the Maxwell equations 46 can be obtained from

$$\delta I \equiv \int \mathcal{L} \sqrt{-g}\, d^4x = 0 \qquad (55)$$

by varying only the EM potentials A_μ ; in equation 55 the expressions are

$$\mathcal{L} \equiv \mathcal{L}_{em} + \mathcal{L}_{matter} + \mathcal{L}_{int} + \mathcal{L}_g$$

$$\mathcal{L}_{em} = -\frac{1}{4} G^{\mu\nu} F_{\mu\nu} \quad , \quad G^{\mu\nu} = c^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

$$\mathcal{L}_{int} = - J^{\mu} A_{\mu}$$

$$\mathcal{L}_{g} = R \tag{55'}$$

$\mathcal{L}_{matter}$: for that of matter, except above,

with $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ serving as a definition of potentials, since equation 46b is valid with or without media. Thus equation 55 gives

$$\delta \int (\mathcal{L}_{em} + \mathcal{L}_{int}) \sqrt{-g}\, d^4x = 0 \tag{56a}$$

or

$$\int \sqrt{-g}\, d^4x \left[\frac{\partial \mathcal{L}_{int}}{\partial A_{\mu}} \delta A_{\mu} + \frac{\partial \mathcal{L}_{em}}{\partial A_{\mu,\nu}} \delta A_{\mu,\nu}\right] = 0 \tag{56b}$$

Now we have

$$\frac{\partial \mathcal{L}_{int}}{\partial A_{\mu}} = - J^{\mu}$$

$$\int \sqrt{-g}\, d^4x \frac{\partial \mathcal{L}_{em}}{\partial A_{\mu,\nu}} \delta A_{\mu,\nu} \equiv \int d\Sigma^* \left[\left(\sqrt{-g} \frac{\partial \mathcal{L}_{em}}{\partial A_{\mu,\nu}} \delta A_{\mu}\right)_{,\nu} - \left(\sqrt{-g} \frac{\partial \mathcal{L}_{em}}{\partial A_{\mu,\nu}}\right)_{,\nu} \delta A_{\mu} \right]$$

$$= -\int d\Sigma^* \delta A_{\mu} \left(\sqrt{-g} \frac{\partial \mathcal{L}_{em}}{\partial A_{\mu,\nu}}\right)_{,\nu} + \oint d\Sigma^*_{\nu} \left(\sqrt{-g} \frac{\partial \mathcal{L}_{em}}{A_{\mu,\nu}}\right) \delta A_{\mu} \tag{57}$$

in which the last integral vanishes because $\delta A_{\mu} \equiv 0$ at the initial and final x^{o} = const. 3-hypersurfaces and $A_{\mu} \equiv 0$ on the space-time 3-volume evaluated at spatial infinity; also Gauss' theorem

$$\int_{V_4} T^{\mu\nu}{}_{,\nu}\, d\Sigma^* = \oint_{V_3} T^{\mu\nu}\, d\Sigma^*_{\nu} \tag{58}$$

$$d\Sigma^* \equiv - \eta_{\mu\nu\alpha\beta} \underset{[o]}{dx^{\mu}} \underset{[1]}{dx^{\nu}} \underset{[2]}{dx^{\alpha}} \underset{[3]}{dx^{\beta}} = d^4x$$ if we choose coordinates as differential legs

Thus, calculating

$$\frac{\partial \mathcal{L}_{em}}{\partial A_{\mu,\nu}} = -\frac{1}{4}\frac{\partial G^{\alpha\beta}F_{\alpha\beta}}{\partial A_{\mu,\nu}} = -\frac{1}{2}[G^{\nu\mu} + c^{\alpha\beta\mu\nu}F_{\beta\alpha}] = G^{\mu\nu}$$

equations 56 and 57 give

$$\int \sqrt{-g}\, d^4x\, [-J^\mu - \frac{1}{\sqrt{-g}}(\sqrt{-g}\, G^{\mu\nu})_{,\nu}] = 0 \tag{59}$$

which immediately implies media EM equation 46a.

B. Lossy media. In a lossy medium the above $\mathcal{L}_{em}$ only picks up the part of $G^{\mu\nu}$ corresponding to the symmetric part of $\varepsilon^{(\mu)(\nu)}$, $K^{(\mu)(\nu)}$; the "antisymmetric"part of $G^{\mu\nu}$ should be introduced by a lossy term in the Lagrangian $\ni$ to have the total $G^{\mu\nu}$ in the medium EM equations. Fortunately, we can decompose

$$\varepsilon^{(i)}{}_{(j)} \equiv \frac{1}{2}(\varepsilon^{(i)}{}_{(j)} + \varepsilon_{(j)}{}^{(i)}) + \frac{1}{2}(\varepsilon^{(i)}{}_{(j)} - \varepsilon_{(j)}{}^{(i)}) \equiv \underset{S}{\varepsilon}{}^{(i)}{}_{(j)} + \underset{A}{\varepsilon}{}^{(i)}{}_{(j)}$$

$$K^{(i)}{}_{(j)} \equiv \frac{1}{2}(K^{(i)}{}_{(j)} + K_{(j)}{}^{(i)}) + \frac{1}{2}(K^{(i)}{}_{(j)} - K_{(j)}{}^{(i)}) \equiv \underset{S}{K}{}^{(i)}{}_{(j)} + \underset{A}{K}{}^{(i)}{}_{(j)} \tag{60}$$

or similarly decompose $\varepsilon^{\mu\nu}$, $K^{\mu\nu}$ by using equation 15, or equivalently decompose

$$c^{\mu\nu\alpha\beta} \equiv \frac{1}{2}(c^{\mu\nu\alpha\beta} + c^{\alpha\beta\mu\nu}) + \frac{1}{2}(c^{\mu\nu\alpha\beta} - c^{\alpha\beta\mu\nu}) \equiv \underset{S}{c}{}^{\mu\nu\alpha\beta} + \underset{A}{c}{}^{\mu\nu\alpha\beta} \tag{60'}$$

and associate all losses to the"antisymmetric"part of $G^{\mu\nu}$ which has the form (Appen. V)*

$$\delta W_{loss} = \frac{1}{2} \underset{A}{G}^{\mu\nu} \delta F_{\mu\nu}$$

$$\underset{A}{G}^{\mu\nu} \equiv \underset{A}{c}^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{61}$$

Then the action principle in lossy media can be stated as

$$\delta I \equiv \delta \int d^4x \sqrt{-g} \, (\mathcal{L} - W_{loss}) = 0 \tag{62}$$

where $\mathcal{L}$ is given by equation 55'.

Performing as before, from equation 62 we have

$$0 = \int \sqrt{-g} \, d^4x \, (\delta \mathcal{L} - \delta W_{loss})$$

$$= \int \sqrt{-g} \, d^4x \left[\frac{\partial \mathcal{L}}{\partial A_\mu} - \frac{1}{\sqrt{-g}} \left(\sqrt{-g} \frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}}\right)_{,\nu} - \frac{1}{\sqrt{-g}} \left(\sqrt{-g} \, \underset{A}{G}^{\mu\nu}\right)_{,\nu}\right] \delta A_\mu \tag{63}$$

and the vanishing of the integrand in the above square bracket gives the required equations. From equation 55' we have

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = - J^\mu$$

$$- \frac{1}{\sqrt{-g}} \left(\sqrt{-g} \frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}}\right)_{,\nu} = - \frac{1}{\sqrt{-g}} \left[\sqrt{-g} \frac{\partial}{\partial A_{\mu,\nu}} \left(- \frac{1}{4} \underset{S}{G}^{\alpha\beta} F_{\alpha\beta}\right)\right]_{,\nu}$$

where $\underset{S}{G}^{\mu\nu} \equiv \underset{S}{c}^{\mu\nu\alpha\beta} F_{\alpha\beta}$

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* $\underset{A}{G}^{\mu\nu} \delta F_{\mu\nu}$ has meaning only through $\underset{A}{G}^{\mu\nu}(\delta F_{\mu\nu}/\delta t)\delta t$ as energy density loss at a fixed point in $\{x^\mu\}$ during δx^o, thus the covariant form of δW_{loss} does not mean energy loss is a scalar.

then equation 63 gives

$$-J^{\mu} - \frac{1}{\sqrt{-g}} (\sqrt{-g}\, \underset{A}{G}^{\mu\nu} + \sqrt{-g}\, \underset{S}{G}^{\mu\nu})_{,\nu} = 0 \tag{64}$$

which are the required equations 46a after rewriting them by using identity

$$(\sqrt{-g}\, T^{\mu\nu})_{,\nu} = \sqrt{-g}\, T^{\mu\nu}{}_{;\nu} + \sqrt{-g}\, \Gamma^{\mu}{}_{\mu\lambda} T^{\mu\lambda} \tag{65}$$

Now the conservation of physical charge to $\{0\}$ in any $\{x^{\mu}\}$ is just rewriting $J^{\mu}{}_{;\mu} = 0$, i.e.,

$$\left(\frac{\rho}{\sqrt{g_{oo}}} + g^{oj}\, e_{(i)j}\, J^{(i)}\right)_{;o} = -\left(e^{i}{}_{(j)}\, J^{(j)}\right)_{;i} \tag{66}$$

or in integral form

$$0 = \oint \sqrt{-g}\, d\Sigma^{*}{}_{\lambda}\, J^{\lambda} \quad , \qquad d\Sigma^{*}{}_{\lambda} \equiv - \eta_{\lambda\alpha\beta\gamma}\, \underset{[1]}{dx^{\alpha}}\ \underset{[2]}{dx^{\beta}}\ \underset{[3]}{dx^{\gamma}} \tag{67a}$$

which, if taken between two x^{o} = const. space-like 3-hypersurfaces, gives

$$\begin{aligned} \begin{matrix}\text{total}\\ \text{conserved}\\ \text{charge}\end{matrix} &= \int \sqrt{-g}\, d^3x\, J^{o} = \int \sqrt{-g}\, d^3x \left(\frac{\rho}{\sqrt{g_{oo}}} + g^{oj}\, e_{(i)j}\, J^{(i)}\right) \\ &= \int \sqrt{-\det(g_{ij})}\, \rho\, d^3x \quad \text{if } \{x^{\mu}\} \text{ time orthogonal} \end{aligned} \tag{67b}$$

The boundary conditions at the interface of two different media are obtained by integrating Maxwell equations over appropriate infinitesimal spatial and time-spatial 3-volumes and making use of Gauss' theorem

$$\frac{1}{2}\oint_{V_2} S^{\alpha\mu\nu}\, d\Sigma^{*}_{\mu\nu} = \int_{V_3} S^{\alpha\mu\nu}{}_{,\nu}\, d\Sigma^{*}_{\mu} \tag{68}$$

If we choose a time orthogonal frame, the immediate results can be expressed by local physical quantities for observers $\{0\}$ in that frame as (Appen.VI) their usual forms

$$\begin{cases} \underset{\sim}{n} \cdot (\underset{\sim}{D}_{II} - \underset{\sim}{D}_{I}) = \sigma \\ \underset{\sim}{n} \cdot (\underset{\sim}{B}_{II} - \underset{\sim}{B}_{I}) = 0 \end{cases}$$

$$\begin{cases} \underset{\sim}{n} \cdot (\underset{\sim}{E}_{II} - \underset{\sim}{E}_{I}) = 0 \\ \underset{\sim}{n} \cdot (\underset{\sim}{H}_{II} - \underset{\sim}{H}_{I}) = \underset{\sim}{k} \end{cases} \tag{69}$$

where $\underset{\sim}{n}$ is unit normal of the intersurface pointing from medium II to medium I and $\sigma, \underset{\sim}{k}$ are the physical surface charge and current densities on that surface to $\{0\}$. But equation 69 as a local result then holds for any observer. So the boundary conditions of the observable EM field for observers in any frame are given by equation 67, which is being anticipated since the geometry $g_{\mu\nu}$ is continuous without abrupt changes and thus contributes nothing to local limiting processes.

The energy momentum tensor of EM field in media (15) has meaning only in a limited sense, since the EM field includes averaged interaction with media and is not a closed system. Physically, extrapolation of static cases and intepretation of energy relation in SRT to

$$-\underset{\sim}{\nabla} \cdot (\underset{\sim}{E} \times \underset{\sim}{H}) = \underset{\sim}{E} \cdot \underset{\sim}{J} + (\underset{\sim}{E} \cdot \frac{\partial \underset{\sim}{D}}{\partial T} + \underset{\sim}{H} \cdot \frac{\partial \underset{\sim}{B}}{\partial T})$$

enable us to interpret $\frac{1}{2}(\underset{\sim}{E} \cdot \underset{\sim}{D} + \underset{\sim}{B} \cdot \underset{\sim}{H})$ as macroscopic EM energy density $T^{(o)(o)}$ in lossless time independent media (Appen. V). Rewriting the relation as*

$$\frac{\partial}{\partial T} T^{(o)(o)} = -\underset{\sim}{E} \cdot \underset{\sim}{J} - \underset{\sim}{\nabla} \cdot (\underset{\sim}{E} \times \underset{\sim}{H}) + \frac{1}{2} (+\underset{\sim}{B}\,\underset{\sim}{B} : \frac{\partial}{\partial T} \underset{\approx}{K} - \underset{\sim}{E}\,\underset{\sim}{E} : \frac{\partial}{\partial T} \underset{\approx}{\varepsilon}) \tag{70}$$

leads us to identify $\underset{\sim}{E} \times \underset{\sim}{H}$ as EM field momentum density**

$$T^{(o)(i)} = (\underset{\sim}{E} \times \underset{\sim}{H})^{(i)} \tag{71}$$

and the last term in equation 70 as power stored in media to maintain a fixed EM field during a change of media properties. Now an infinitesimal displacement of the medium will give the force acted on it by EM field, through

$$\delta \int d^3X \, \frac{1}{2} (\underset{\sim}{H} \cdot \underset{\sim}{B} + \underset{\sim}{E} \cdot \underset{\sim}{D}) = - \int d^3X \, \underset{\sim}{f} \cdot \delta \underset{\sim}{X} \tag{72}$$

which gives (Appen. VII)

$$\underset{\sim}{f} = \rho \underset{\sim}{E} + \underset{\sim}{J} \times \underset{\sim}{B} + \frac{1}{2} ((\nabla \underset{\approx}{K}) : \underset{\sim}{B}\,\underset{\sim}{B} - (\nabla \underset{\approx}{\varepsilon}) : \underset{\sim}{E}\,\underset{\sim}{E}) + \frac{\partial}{\partial T} (\underset{\sim}{D} \times \underset{\sim}{B}) \tag{73}$$

---

*All 3-vector notations are done in 3-vector conventions.

**By definition, a momentum density must associate with an energy flux (34).

Now the form of $T^{(o)(o)}$, $T^{(o)(i)}$ can be written as the $(o)(\nu)$ components of a $T^{(\mu)(\nu)}$

$$T^{(\mu)(\nu)} \equiv -F^{(\mu)(\lambda)}\, G^{(\nu)}{}_{(\lambda)} + \frac{1}{4}\, \eta^{(\mu)(\nu)}\, F^{(\alpha)(\beta)}\, G_{(\alpha)(\beta)} \qquad (74)$$

Furthermore, if we compute $T^{(i)(j)}{}_{,(j)}$ of equation 74, we will obtain exactly the $\underset{\sim}{f}$ of equation 73. Thus $T^{(\mu)(\nu)}$ of form 74 meets the requirements of an energy momentum tensor for the EM field in media in its meaning on $T^{(o)(\mu)}$, $T^{(o)(\mu)}{}_{,(\mu)}$, $T^{(i)(j)}{}_{,(j)}$ only. Its non-symmetry just results from our forcing an energy momentum concept to a non-closed system in which an arbitrary averaged interaction is included in the EM field through $\varepsilon^{(i)(j)}$, $K^{(i)(j)}$. In GRT, a use of equation 15 immediately gives

$$T^{\mu\nu} = -F^{\mu\lambda}\, G^{\nu}{}_{\lambda} + \frac{1}{4}\, g^{\mu\nu}\, F^{\alpha\beta}\, G_{\alpha\beta} \qquad (75)$$

as a tensor. Other formal ways to obtain $T^{\mu\nu}$ are either by rearranging terms in the explicit Lorentz force density or directly varying the Lagrangian $\mathcal{L}_{em} + \mathcal{L}_{int}$ with respect to metric $g_{\mu\nu}$ (Appen.VIII) which lead either to equation 75 or its symmetrization; but the physical significance is only in $T^{(o)(\mu)}$, $T^{(o)(\mu)}{}_{,(\mu)}$, $T^{(i)(j)}{}_{,(j)}$ of equation 74.

If we assume equation 75 for lossy media, since the anti-symmetric part of $\underline{\underline{\varepsilon}}, \underline{\underline{K}}$ is cancelled out in equation 70, $T^{o\mu}$ still have the same interpretations as in the lossless case. But now we have

$$T^{\mu\nu}{}_{;\nu} = F^{\mu\lambda} J_\lambda + \frac{1}{4} g^{\mu\nu} \underset{S}{C}{}^{\alpha\beta\gamma\delta}{}_{;\nu} F_{\alpha\beta} F_{\gamma\delta} - \frac{1}{2} g^{\mu\nu} \underset{A}{C}{}^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta;\nu} \tag{76}$$

which means, in addition to an explicit Lorentz force and an averaged conservative force to the medium, the EM field supplies another power-force to the medium losses.

II.3.4 Discussions of EM theory in media in non-inertial frames. For media in inertial frames the assumption of covariance of $\underset{\sim}{D}$, $\underset{\sim}{H}$ equations and their $G^{\mu\nu}$ tensor and thus the whole formalism actually does not assume any new law of macroscopic EM physics, even if observed in an accelerated frame. The formalism is just a convenient way to present the well-accepted rest-frame physics in equations 22, 23, 24 to other frames by using $G^{\mu\nu}$ as an intermediate concept. But for a medium in a non-inertial frame, equation 46a is an additional assumption on the behavior of the averaged medium-EM interactions even in the medium co-moving frame. One reason for us to make this assumption is that the so-formulated theory surely holds for non-inertial observers moving with respect to a medium at an inertial frame; the principle of relativity (12) then suggests this assumption.

The path of a charged particle free falling under EM and gravitational forces, if we neglect collision in the medium, is still

$$m \frac{Du^\mu}{Ds} = q F^{\mu\nu} u_\nu \tag{77}$$

which also can be derived from a variational integral (Appen. IX) .

III. APPLICATIONS TO MEDIA MOTIONS IN SRT

III.1 Medium in Linear Uniform Acceleration

In this section local EM field equations are first formulated for general media in hyperbolic motion in a SRT space-time; then propagation along the direction of acceleration in a simple medium is studied in detail.

III.1.1 Formulations. Let $\{\bar{X}^{\mu}\}$* be an inertial Minkovskian frame. A uniform accelerational motion is described by

$$X = \frac{1}{a}(\cosh a\tau - 1)$$

$$T = \frac{1}{a}(\sinh a\tau) \tag{1}$$

where a is the constant acceleration, τ is the proper time of the accelerated point. If a medium is in uniform linear acceleration, its co-moving frame can be described by an $\{x^{\mu}\}$ related to $\{X^{\mu}\}$ through

$$T = \frac{1}{a}\sinh at$$

$$X = \frac{1}{a}(\cosh at - 1) + x$$

$$Y = y$$

$$Z = z \tag{2}$$

---

* $\{\bar{X}^{\mu}\} \equiv \{T,X,Y,Z\}$, $\{x^{\mu}\} \equiv \{t,x,y,z\}$ in this section.

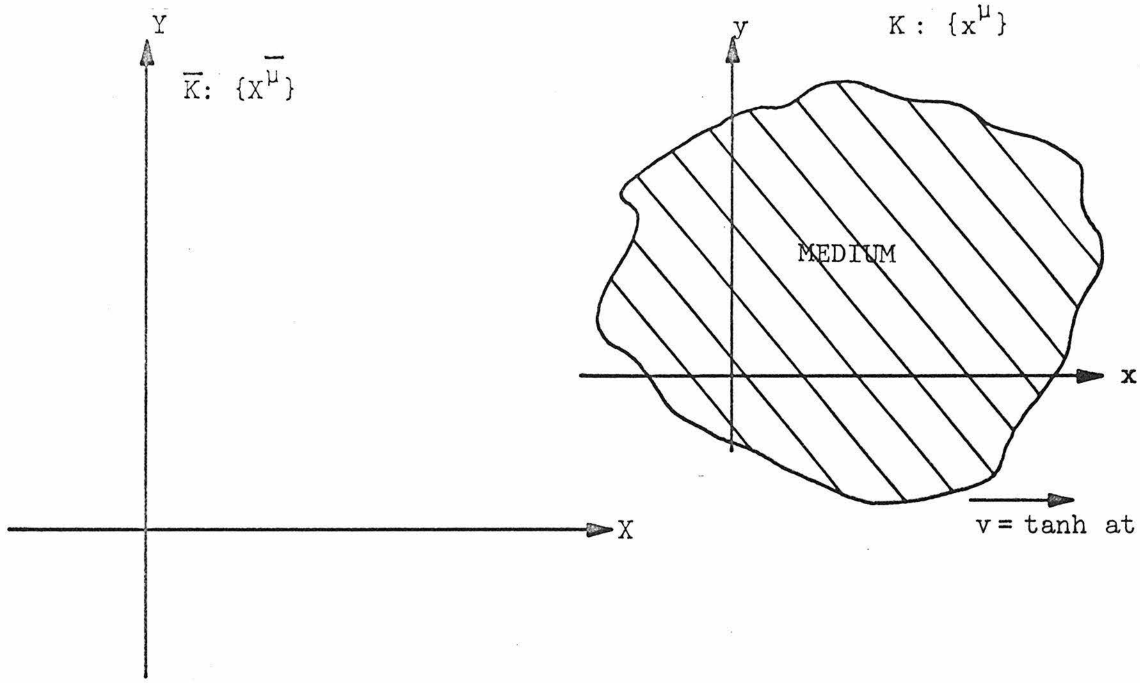
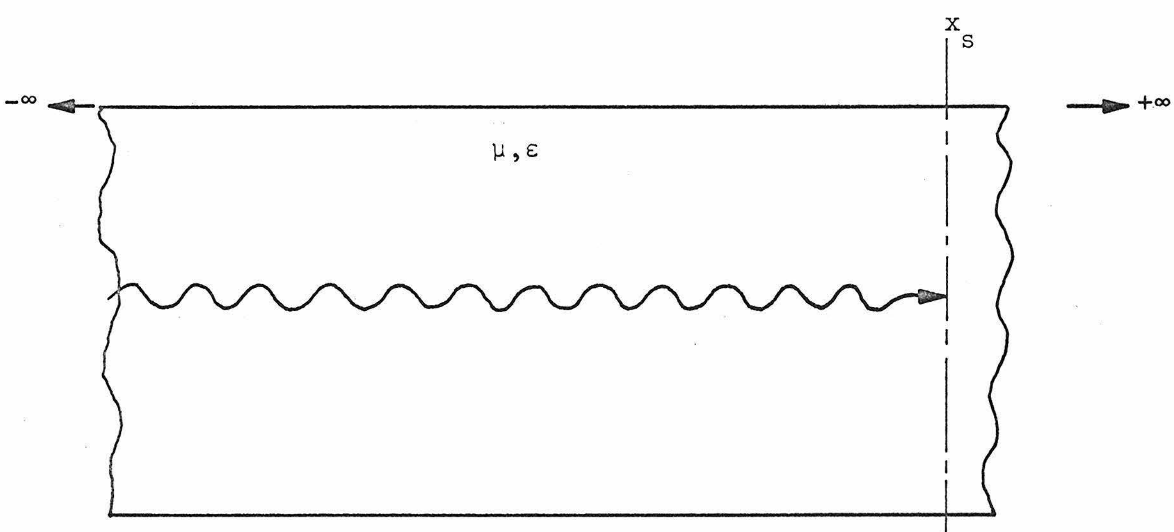


Fig. 2a. Linear accelerated medium



at t = 0 it begins to accelerate, whence the propagation obeys a new law

Fig. 2b. EM-wave propagation in an accelerated simple medium where origins have been adjusted such that at $t = 0 = T$ they have zero relative velocity and $\{x^\mu\}$ is co-moving in the sense that each point of fixed (x^i) has X-uniform-accelerational motion

$$v = \tanh at \tag{3}$$

with respect to $\{X^{\bar{\mu}}\}$. The $\{X^{\bar{\mu}}\}$ metric $\eta_{\bar{\mu}\bar{\nu}}$ implies the metric $g_{\mu\nu}$, $g^{\mu\nu}$ and the time orthogonal spatial metric γ_{ij} of $\{x^\mu\}$ as:

$$g_{\mu\nu} = \begin{pmatrix} 1 & -\sinh at & 0 & 0 \\ -\sinh at & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma_{ij} = \begin{pmatrix} -\cosh^2 at & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{\cosh^2 at}\begin{pmatrix} 1 & -\sinh at \\ -\sinh at & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \end{pmatrix} \tag{4}$$

This $\{x^\mu\}$ is just a convenient frame describing the medium motion and it is neither synchronous nor static. However, t is the proper time of a medium co-moving observer $\{0\}$. The observable value of a tensor to $\{0\}$ in $\{x^\mu\}$ on a conveniently coordinate transported $\{\underline{e}_{(\mu)}\}$ is obtained either by choosing $\{\underline{e}_{(\mu)}\} \ni$ it is the instantaneous Lorentz transform of the $\{\underline{e}_{(\bar{\mu})}\}$ of $\bar{K}$

$$\begin{aligned} \underline{e}_{(o)} &= \underline{e}_o \\ \underline{e}_{(1)} &= \frac{1}{\cosh at}(\underline{e}_1 + \sinh at\ \underline{e}_o) \\ \underline{e}_{(2)} &= \underline{e}_2 \\ \underline{e}_{(3)} &= \underline{e}_3 \end{aligned} \tag{5}$$

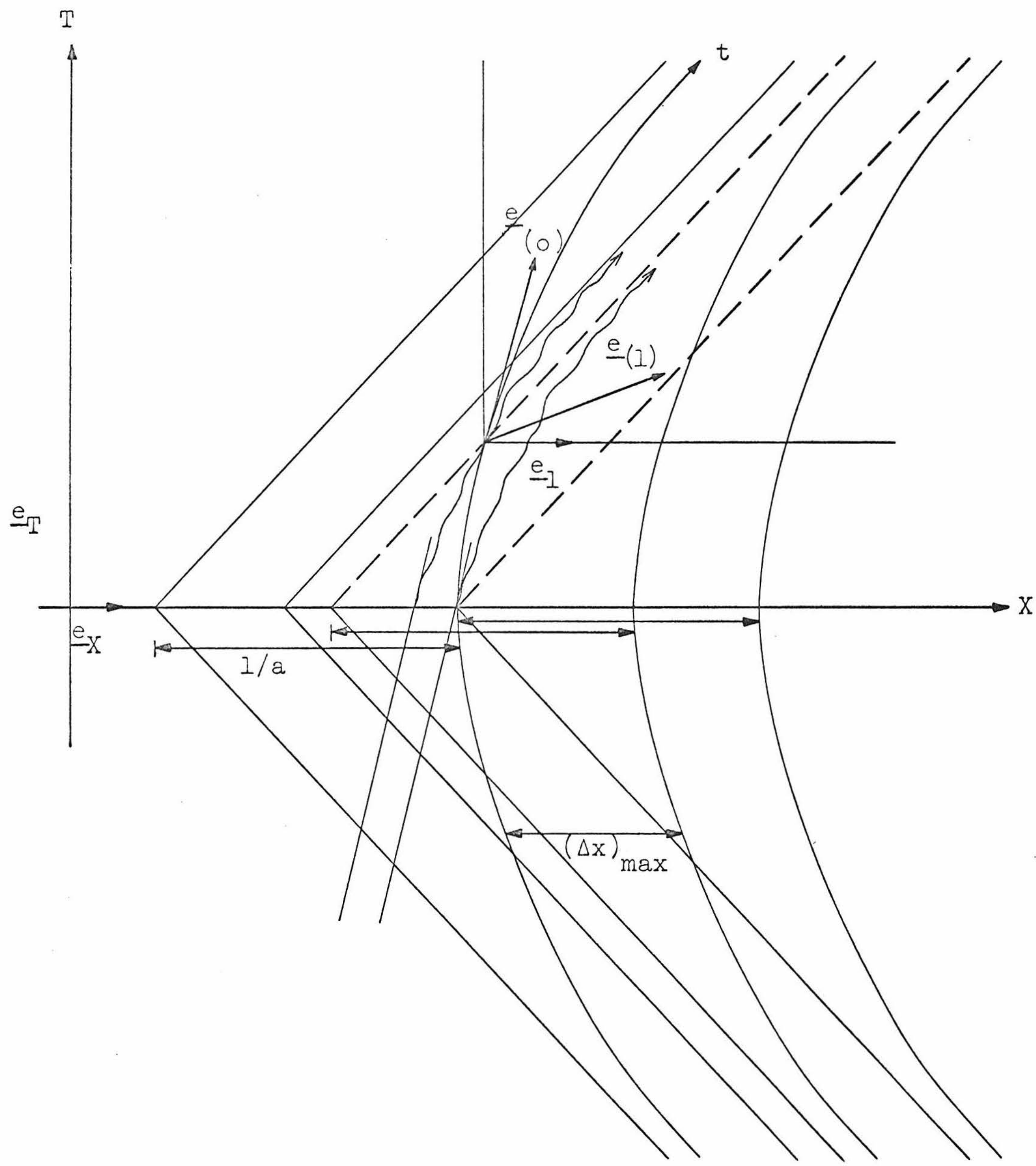


Fig. 3. Space-time diagram of accelerated medium

or by the tensor transform $\partial X^{\bar{\alpha}}/\partial x^{\beta}$ and instantaneous Lorentz transform $\Lambda^{\bar{\alpha}}{}_{(\lambda)}$ which are

$$\frac{\partial X^{\bar{\alpha}}}{\partial x^{\beta}} = \begin{pmatrix} \cosh at & 0 & 0 & 0 \\ \sinh at & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \frac{\partial x^{\beta}}{\partial x^{\bar{\alpha}}} = \begin{pmatrix} \frac{1}{\cosh at} & 0 & 0 & 0 \\ -\tanh at & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

$$\Lambda^{\bar{\alpha}}{}_{(\lambda)} = \begin{pmatrix} \cosh at & \sinh at & 0 & 0 \\ \sinh at & \cosh at & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

because of the relation I-14. With these formulas the current J^{μ} and field tensors $F^{\mu\nu}, G^{\mu\nu}$ in the medium-co-moving $\{x^{\mu}\}$ are connected to their respective local physical quantities as

$$J^{\mu} = \frac{\partial x^{\mu}}{\partial X^{\bar{\alpha}}} \Lambda^{\bar{\alpha}}{}_{(\lambda)} J^{(\lambda)} \qquad \text{or} \qquad J^{\mu} = e^{\mu}{}_{(\lambda)} J^{(\lambda)}$$

$$\Longrightarrow J^{\mu} = \left(\rho + \tanh at\, J^{(x)}, \frac{J^{(x)}}{\cosh at}, J^{(y)}, J^{(z)}\right) \tag{8}$$

$$F^{\mu\nu} = \frac{\partial x^{\mu}}{\partial X^{\bar{\alpha}}} \frac{\partial x^{\nu}}{\partial X^{\bar{\beta}}} \Lambda^{\bar{\alpha}}{}_{(\lambda)} \Lambda^{\bar{\beta}}{}_{(\tau)} F^{(\lambda)(\tau)} \quad \text{or} \quad F^{\mu\nu} = e^{\mu}{}_{(\alpha)} e^{\nu}{}_{(\beta)} F^{(\alpha)(\beta)}$$

$$\Longrightarrow F^{\mu\nu} = \begin{pmatrix} 0 & \frac{-E^{(x)}}{\cosh at} & -(E^{(y)} + \tanh at\, B^{(z)}) & -E^{(z)} + \tanh at\, B^{(y)} \\ & 0 & \frac{-B^{(z)}}{\cosh at} & \frac{B^{(z)}}{\cosh at} \\ & & 0 & -B^{(x)} \\ - & & & 0 \end{pmatrix} \tag{9}^{*}$$

---

* - in the matrix means negative values of the upper-right elements of the matrix.

Similarly, we have

$$G^{\mu\nu} = \begin{pmatrix} 0 & \frac{-D^{(x)}}{\cosh at} & -(D^{(y)} + \tanh at\ H^{(z)}) & -D^{(z)} + \tanh at\ H^{(y)} \\ & 0 & \frac{-H^{(z)}}{\cosh at} & \frac{H^{(y)}}{\cosh at} \\ \text{—} & & 0 & -H^{(x)} \\ & & & 0 \end{pmatrix} \quad (10)$$

The physical explanation of equations 8, 9, 10 can easily be seen from the space-time diagram, Fig. 3.

Now in $\{x^{\mu}\}$ the only non-zero Christoffel symbols are

$$\Gamma^{0}_{00} = a \tanh at\ , \quad \Gamma^{1}_{00} = a \operatorname{sech} at, \quad \Gamma_{1|00} = -a \cosh at;$$

thus $G^{\mu\nu}{}_{;\nu} = -J^{\mu}$ with equations 8 and 10 gives the local Maxwell equations in the accelerated frame:

$$\left\{\begin{aligned} &\frac{1}{\cosh at}\frac{\partial}{\partial x} D^{(x)} + \frac{\partial}{\partial y} D^{(y)} + \frac{\partial}{\partial z} D^{(z)} + \tanh at\left[\frac{\partial}{\partial y} H^{(z)} - \frac{\partial}{\partial z} H^{(y)}\right] \\ &\qquad = \rho + J^{(x)} \tanh at \qquad (11a) \\ &\frac{\partial}{\partial y} H^{(z)} - \frac{\partial}{\partial z} H^{(y)} = J^{(x)} + \frac{\partial}{\partial t} D^{(x)} \qquad (11b) \\ &\frac{\partial}{\partial z} H^{(x)} - \frac{1}{\cosh at}\frac{\partial}{\partial x} H^{(z)} - aH^{(z)} - \tanh at \cdot \frac{\partial}{\partial t} H^{(z)} \\ &\qquad = \frac{\partial}{\partial t} D^{(y)} + J^{(y)} + a \tanh at \cdot D^{(y)} \qquad (11c) \end{aligned}\right.$$

$$\frac{1}{\cosh at}\frac{\partial}{\partial x}H^{(y)}-\frac{\partial}{\partial y}H^{(x)}+aH^{(y)}+\tanh at\cdot\frac{\partial}{\partial t}H^{(y)}$$

$$=\frac{\partial}{\partial t}D^{(z)}+J^{(z)}+a\tanh at\cdot D^{(z)} \qquad \text{(11d)}$$

With a particular convention that $\underset{\sim}{\nabla}\equiv\left(\frac{1}{\cosh at}\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$ and $\frac{\partial}{\partial t}$ do not act on $\{\underline{e}_{(i)}\}$, then equation 11 can be written in the 3-vector form*

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{D}+\underset{\sim}{V}\cdot(\underset{\sim}{\nabla}\times\underset{\sim}{H})=\rho+\underset{\sim}{V}\cdot\underset{\sim}{J} \qquad \text{(12a)}$$

$$\underset{\sim}{\nabla}\times\underset{\sim}{H}+\underset{\sim}{a}\times\underset{\sim}{H}+\underset{\sim}{V}\times\frac{\partial}{\partial t}\underset{\sim}{H}=\underset{\sim}{J}+\frac{\partial}{\partial t}\underset{\sim}{D}-\underset{\sim}{V}\times(\underset{\sim}{a}\times\underset{\sim}{D}) \qquad \text{(12b)}$$

where 3-vectors are componented on $\{\underline{e}_{(i)}\}$ and $\underset{\sim}{V}\equiv\tanh at\ \underline{e}_{(x)}$, $\underset{\sim}{a}\equiv a\ \underline{e}_{(x)}$. Similar equations for $(*F)^{\mu\nu}{}_{;\nu}=0$ are obtained just by replacing $(\underset{\sim}{D},\underset{\sim}{H})$ by $(\underset{\sim}{B},-\underset{\sim}{E})$ and putting $J=0$ in equation 12, i.e.,

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{B}=\underset{\sim}{V}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{E} \qquad \text{(13a)}$$

$$\underset{\sim}{\nabla}\times\underset{\sim}{E}+\underset{\sim}{a}\times\underset{\sim}{E}+\underset{\sim}{V}\times\frac{\partial}{\partial t}\underset{\sim}{E}=-\frac{\partial}{\partial t}\underset{\sim}{B}+\underset{\sim}{V}\times(\underset{\sim}{a}\times\underset{\sim}{B}) \qquad \text{(13b)}$$

The local form of the continuity equation $J^{\mu}{}_{;\mu}=0$ implied by equation 12 now reads

$$-\underset{\sim}{\nabla}\cdot\underset{\sim}{J}-\underset{\sim}{a}\cdot\underset{\sim}{J}=\frac{\partial}{\partial t}\rho+\underset{\sim}{V}\cdot\frac{\partial}{\partial t}\underset{\sim}{J}+\rho\,\underset{\sim}{a}\cdot\underset{\sim}{V} \qquad \text{(14)}$$

which as seen in Appen. III and Fig. 3 again shows the non-orthogonality of the coordinate $\{x^i\}$ and the non-constancy of its metric. Also

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*Here all 3-notations are the 3-vector analysis ones, including $\times$ and $\cdot$.

quation 13b implies 13a.

In addition to equations 12 and 13, since $\{x^\mu\}$ is the medium o-moving frame, the constitutive relations are simply described by quation II.24. The values of $\varepsilon^{(i)(j)}$, $K^{(i)(j)}$ are determined locally or the medium by co-moving $\{0\}$ just as if there is no acceleration, hich, if neglecting the effect due to mechanical accelerational strain n the medium, are simply equal to their values when the medium is in an nertial frame. Equations 12, 13 and II.24 describe the local EM fields n the co-moving $\{x^\mu\}$.

Now the electric and magnetic constitutive tensors in $\{x^\mu\}$ are

$$\varepsilon^{\mu\nu} = e^{\mu}{}_{(\alpha)} e^{\nu}{}_{(\beta)} \varepsilon^{(\alpha)(\beta)} = \begin{pmatrix} \varepsilon^{(o)(1)} + \varepsilon^{(1)(1)}\tanh^2 at & \text{sech } at \cdot \tanh at \cdot \varepsilon^{(1)(1)} & \tanh at \cdot \varepsilon^{(1)(2)} & \tanh at \cdot \varepsilon^{(1)(3)} \\ \text{sech } at \cdot \tanh at \cdot \varepsilon^{(1)(1)} & \text{sech}^2 at \cdot \varepsilon^{(1)(1)} & \text{sech } at \cdot \varepsilon^{(1)(2)} & \text{sech } at \cdot \varepsilon^{(1)(3)} \\ \tanh at \cdot \varepsilon^{(2)(1)} & \text{sech } at \cdot \varepsilon^{(2)(1)} & \varepsilon^{(2)(2)} & \varepsilon^{(2)(3)} \\ \tanh at \cdot \varepsilon^{(3)(1)} & \text{sech } at \cdot \varepsilon^{(3)(1)} & \varepsilon^{(3)(2)} & \varepsilon^{(3)(3)} \end{pmatrix} \tag{15}$$

$$K^{\mu\nu} = (\text{same as equation 15 with } \varepsilon^{(\alpha)(\beta)} \text{ replaced by } K^{(\alpha)(\beta)}) \tag{16}$$

Then the $\overline{K^{\mu\nu}}$, $\overline{\varepsilon^{\mu\nu}}$ in any $\{\overline{x^\mu}\}$ is simply obtained by tensor transforming equations 15 and 16 into $\{\overline{x^\mu}\}$ and thus $\overline{c^{\mu\nu\alpha\beta}}$ is constructed according to equation II.37 with $\underline{u}$ as the 4-velocity of the medium. In particular, for the inertial laboratory frame $\{X^{\overline{\alpha}}\}$, we have

$$u^{\bar{\alpha}} = \frac{\partial X^{\bar{\alpha}}}{\partial x^{\beta}} u^{\beta} = \frac{1}{\sqrt{g_{oo}}} \frac{\partial X^{\bar{\alpha}}}{\partial x^{o}} = (\cosh at, \sinh at, 0, 0) \tag{17}$$

Then substituting equation 17 and $K^{\overline{\alpha\beta}}$, $\varepsilon^{\overline{\alpha\beta}}$ into equation II.36 or II.37 yields the constitutive relations in the laboratory frame which, since $g_{\overline{\mu\nu}} = \eta_{\mu\nu}$, directly gives*

$$(D^{\bar{i}}) = \begin{pmatrix} \varepsilon^{(1)}{}_{(1)} & \cosh at \cdot \varepsilon^{(1)}{}_{(2)} & \cosh at \;{}^{(1)}{}_{(3)} \\ \cosh at \cdot \varepsilon^{(2)}{}_{(1)} & (\cosh^2 at \cdot \varepsilon^{(2)}{}_{(2)} - \sinh^2 at \cdot K^{(3)}{}_{(3)}) & (\cosh^2 at \;{}^{(2)}{}_{(3)} + \sinh^2 at\, K^{(3)}{}_{(2)}) \\ \cosh at \cdot \varepsilon^{(3)}{}_{(1)} & (\cosh^2 at \cdot \varepsilon^{(3)}{}_{(2)} + \sinh^2 at \cdot K^{(2)}{}_{(3)}) & (\cosh^2 at \;{}^{(3)}{}_{(3)} - \sinh^2 at\, K^{(2)}{}_{(2)}) \end{pmatrix} \cdot (E^{\bar{i}}) + \sinh at \cdot$$

$$\begin{pmatrix} 0 & \varepsilon^{(1)}{}_{(3)} & -\varepsilon^{(1)}{}_{(2)} \\ K^{(3)}{}_{(1)} & \cosh at & \begin{pmatrix} K^{(3)}{}_{(2)} + \varepsilon^{(2)}{}_{(3)}, & K^{(3)}{}_{(3)} - \varepsilon^{(2)}{}_{(2)} \\ -K^{(2)}{}_{(2)} + \varepsilon^{(3)}{}_{(3)}, & -K^{(2)}{}_{(3)} - \varepsilon^{(3)}{}_{(2)} \end{pmatrix} \\ -K^{(2)}{}_{(1)} & & \end{pmatrix} \cdot (B^{\bar{j}}) \tag{18}$$

and

$$(H^{\bar{i}}) = (\varepsilon^{(i)}{}_{(j)} \to K^{(i)}{}_{(j)},\ K^{(i)}{}_{(j)} \to \varepsilon^{(i)}{}_{(j)},\ \underset{\sim}{\bar{E}} \to \underset{\sim}{\bar{B}},\ \underset{\sim}{\bar{B}} \to -\underset{\sim}{\bar{E}} \text{ in eq. } 18) \tag{19}$$

---

*Eqs. 18 and 19 can also be obtained by substituting instantaneous Lorentz transform into equation II.37 and solving $D^{\bar{i}}, H^{\bar{i}}$ in the laboratory, because of equation II.14.

Equations 18, 19, and the ordinary Maxwell equations II.22,23 describe the EM field in the inertial laboratory $\{\bar{X}^{\mu}\}$ with respect to which the medium is X-direction uniformly accelerated as equation 3.

The boundary conditions equation II.69 directly apply to observable EM fields in the accelerated frame. If there are no local surface charge and current, then the quantities

$$\underset{\sim}{n} \cdot \underset{\sim}{D}\,, \quad \underset{\sim}{n} \cdot \underset{\sim}{B}\,, \quad \underset{\sim}{n} \times \underset{\sim}{E}\,, \quad \underset{\sim}{n} \times \underset{\sim}{H} \tag{20}$$

are continuous across the boundary of different media. In particular, if the boundary is perpendicular to the x-direction $(\underset{\sim}{n} \parallel \underline{e}_{(x)})$, then $D^{(x)}$, $B^{(x)}$, $E^{(y)}$, $E^{(z)}$, $H^{(y)}$, $H^{(z)}$ are continuous; if the boundary has its normal in the y-direction $(\underset{\sim}{n} \parallel \underline{e}_{(y)})$, then $D^{(y)}$, $B^{(y)}$, $E^{(x)}$, $E^{(z)}$, $H^{(x)}$, $H^{(z)}$ are continuous.

<u>III.1.2 Wave propagation in the co-moving frame along direction of acceleration for simple medium.</u> For a simple accelerated medium, in its co-moving frame the constitutive relation is equation II.38. Then in the sourceless region equation 12 becomes

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} + \frac{1}{\mu\varepsilon} \underset{\sim}{V} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{B}) = 0 \tag{21a}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} + \underset{\sim}{a} \times \underset{\sim}{B} + \underset{\sim}{V} \times \frac{\partial}{\partial t} \underset{\sim}{B} = \mu\varepsilon\left(\frac{\partial}{\partial t} \underset{\sim}{E} - \underset{\sim}{V} \times (\underset{\sim}{a} \times \underset{\sim}{E})\right) \tag{21b}$$

Consider now a plane-like wave propagating along the x-direction; the medium homogeneity leads to the assumption that nothing is y,z dependent. Then equations 13a and 21a become

$$\frac{\partial}{\partial x} B^{(x)} = 0 \quad , \quad \frac{\partial}{\partial x} E^{(x)} = 0 \tag{22}$$

so $E^{(x)}, B^{(x)}$ are spatial constants; but no source can produce a time varying field which is spatially uniform, thus they are space-time constants and we can put it to be 0. Then the other Maxwell equations of this transverse TEM wave are*

$$(\text{sech at} \frac{\partial}{\partial x} + a + \tanh \text{at} \frac{\partial}{\partial t}) \underset{\sim}{e}_{(x)} \times \underset{\sim}{E} = -(\frac{\partial}{\partial t} + a \tanh \text{at}) \underset{\sim}{B} \tag{23a}$$

$$(\text{sech at} \frac{\partial}{\partial x} + a + \tanh \text{at} \frac{\partial}{\partial t}) \underset{\sim}{e}_{(x)} \times \underset{\sim}{B} = \mu\varepsilon(\frac{\partial}{\partial t} + a \tanh \text{at}) \underset{\sim}{E} \tag{23b}$$

Now for abbreviation define operators

$$\underset{\sim}{\Phi} \equiv \Phi \underset{\sim}{e}_{(x)} \equiv \underset{\sim}{e}_{(x)}(\text{sech at} \frac{\partial}{\partial x} + a + \tanh \text{at} \frac{\partial}{\partial t})$$

$$\psi \equiv (\frac{\partial}{\partial t} + a \tanh \text{at}) \tag{24}$$

$$[\Phi, \psi] = a\, \text{sech}^2 \text{at} \,(\sinh \text{at} \frac{\partial}{\partial x} - \cosh \text{at} \frac{\partial}{\partial t} \frac{1}{\cosh \text{at}})$$

Rewrite equation 23

$$\underset{\sim}{\Phi} \times \underset{\sim}{E} = -\psi \underset{\sim}{B}$$

$$\underset{\sim}{\Phi} \times \underset{\sim}{B} = +\mu\varepsilon\psi \underset{\sim}{E} \tag{25}$$

We see that $\underset{\sim}{\Phi}, \psi$ are the natural corresponding operators of $\underset{\sim}{\nabla}, \frac{\partial}{\partial T}$ in an inertial frame. We thus solve the problem approximately with respect to these operators. From equations 24 and 25 we have

$$(\Phi^2 - \mu\varepsilon\psi^2) \underset{\sim}{E} = [\Phi, \psi] \underset{\sim}{e}_x \times \underset{\sim}{B}$$

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*See footnotes on p. 44 and convention in equations 12, $\underset{\sim}{e}_{(i)}$ same as $\underline{e}_{(i)}$ except $\partial/\partial x^\mu$ does not act on it and 3-vector convention is used.

$$(\Phi^2 - \mu\varepsilon\psi^2)\ \underset{\sim}{B} = -\mu\varepsilon[\Phi,\psi]\ \underset{\sim}{e}_x \times \underset{\sim}{E} \qquad (26)$$

from which $(\Phi + \sqrt{\mu\varepsilon}\psi)$ will give the +x traveling wave and $(\Phi - \sqrt{\mu\varepsilon}\psi)$ will give the -x traveling wave. We solve equation 26 using the following approximation scheme: First neglect the mixed effect corresponding to interaction of opposite traveling waves $[\Phi,\psi]$ on the right side of equation 26 which is of order a and approaches to zero for large (at) as $\text{sech}^2 at$; obtaining this E,B as the first approximation, then put it back into the right side of equation 2 and proceed to solve for the next approximation, and so on. Now assume that when the medium is at rest with respect to the laboratory at $t = 0$, a plane wave is propagating in the +x direction, i.e., $\underset{\sim}{E},\underset{\sim}{B} \sim e^{i(kx-\omega t)}$ at $t \sim 0$. Rewrite equation 26 as

$$(\Phi - \sqrt{\mu\varepsilon}\ \psi)(\Phi + \sqrt{\mu\varepsilon}\psi)\ \underset{\sim}{E} = [\Phi,\psi]\ (-\sqrt{\mu\varepsilon}\underset{\sim}{E} + \underset{\sim}{e}_x \times \underset{\sim}{B}) \qquad (27a)$$

$$(\Phi - \sqrt{\mu\varepsilon}\ \psi)(\Phi + \sqrt{\mu\varepsilon}\psi)\ B = -\sqrt{\mu\varepsilon}\ [\Phi,\psi]\ (\underset{\sim}{B} + \sqrt{\mu\varepsilon}\ \underset{\sim}{e}_x \times \underset{\sim}{E}) \qquad (27b)$$

Now neglect $[\Phi,\psi]$, solve $\underset{\sim}{E}$ or $E^{(y)}$, $E^{(z)}$ by substituting the Fourier transform of $\underset{\sim}{E}$ into equation 27a and breaking it into two first order d.e.'s, thus

$$(\Phi - \sqrt{\mu\varepsilon}\ \psi)\ \underset{\sim}{F}(k,t) = 0 \qquad (28a)$$

$$(\Phi + \sqrt{\mu\varepsilon}\ \psi)\ \underset{\sim}{E}(k,t) = \underset{\sim}{F}(k,t) \qquad (28b)$$

where $\partial/\partial x$ is replaced by ik in Φ. Then their solutions are

$$\underset{\sim}{F}(\underset{\sim}{k},t) = \underset{\sim}{\alpha}\, e^{-\int_0^t \frac{a(\cosh at - \sqrt{\mu\varepsilon}\sinh at) + ik}{\sinh at - \sqrt{\mu\varepsilon}\cosh at}\, dt} \tag{29a}$$

$$\underset{\sim}{E}(\underset{\sim}{k},t) = e^{-\int_0^t \frac{a(\cosh at + \sqrt{\mu\varepsilon}\sinh at) + ik}{\sinh at + \sqrt{\mu\varepsilon}\cosh at}\, dt} \cdot \left[\underset{\sim}{\beta} + \int_0^t \frac{\underset{\sim}{F}(k,\eta)}{\sqrt{\mu\varepsilon} + \tanh a\eta}\, e^{\int_0^{\eta} d\xi\, \frac{ik + a(\cosh a\xi + \sqrt{\mu\varepsilon}\sinh a\xi)}{\sinh a\xi + \sqrt{\mu\varepsilon}\cosh a\xi}}\, d\eta \right] \tag{29b}$$

Examining the behavior near $t \sim 0$ reveals that the $\underset{\sim}{F}$ term of $\underset{\sim}{E}$ in equation 29b represents a $-x$ traveling wave with increasing amplitude as $t \gtrsim 0$. Since initially we have only $+x$ traveling wave and the boundary conditions, equation 20, preclude reflected waves for propagation along the direction of acceleration in a simple medium, then $\underset{\sim}{\alpha} = 0$ and we can orient yz axis to have the solution

$$E^{(x)} = 0 \quad , \quad E^{(z)} = 0$$

$$E^{(y)} = \int \frac{dk}{2\pi}\, e^{ikx}\, \beta(k)\, e^{-\int_0^t \frac{a(\cosh at + \sqrt{\mu\varepsilon}\sinh at) + ik}{(\sinh at + \sqrt{\mu\varepsilon}\cosh at)}\, dt} \tag{30a}$$

as the first approximation ; the corresponding $\underset{\sim}{B}$ is obtained by equation 27 with an initial $B^{(z)} = \sqrt{\mu\varepsilon}\, e^{ik_o x}$ and no y,x components:

$$B^{(x)} = 0 \;, \quad B^{(y)} = 0 \;, \quad B^{(z)} = \sqrt{\mu\varepsilon}\, E^{(y)} \tag{30b}$$

Higher N^{th} order solution is obtained by substituting equation 30 into the right side of equation 27 and solving the two first order d.e.'s similar to equation 28.

If near $t \sim 0$ the propagation is single wave-lengthed $2\pi/k_o$, then we have $\beta(k) = 2\pi\delta(k-k_o)$ and $E^{(y)}$ is

$$E^{(y)} = e^{ik_o x}\, e^{-ik_o \int_0^t \frac{dt}{\sinh at + \sqrt{\mu\varepsilon}\cosh at}}\, e^{-a\int_0^t \frac{1+\sqrt{\mu\varepsilon}\tanh at}{\sqrt{\mu\varepsilon} + \tanh at}\, dt} \qquad t > 0 \tag{31}$$

This is the first order (neglecting $[\Phi,\psi]$) steady state wave propagation in an accelerated simple medium with its phase and amplitude chosen with respect to an arbitrary origin of coordinate time t, namely $t = 0$ at which time the physical wavelength to all $\{0\}$ is $2\pi/k_o$. Physically we can interpret that for $t < 0$ a $\omega_o \equiv k_o/\sqrt{\mu\varepsilon}$ plane wave has already been propagating in the inertial simple medium, then at $t = 0$ the medium is a-accelerated and the wave begins to obey the new law, equation 26. Since no reflection exists whether there is acceleration or not, it propagates according to equation 31 in first order; also for region $x > x_s$ in which the wave has not reached $t = 0$, a step function $S(x - x_s - w(t))$ is multiplied to equation 31. Here

$$w(t) \equiv \frac{1}{\sqrt{\mu\varepsilon}\cosh at + \sinh at} \tag{32}$$

is the new coordinate phase velocity, but its physical value to $\{0\}$ is, by using a relation as equation 8 for $d\underline{x}$,

$$w^{(x)} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{33}$$

which shows to local observers acceleration does not affect the phase velocity. Since if the wave is once $e^{ik_o x}$ x-dependent, it is always so, the acceleration can begin at any time $t < 0$ and equation 31 still holds with normalization fixed with respect to that $t = 0$ arbitrarily. The instantaneous red shifted frequency to $\{0\}$ is

$$\omega = \frac{k_o}{\sqrt{\mu\varepsilon}\cosh at + \sinh at} \xrightarrow{at \to \text{large}} \frac{2k_o}{1 + \sqrt{\mu\varepsilon}} e^{-at} \tag{34}$$

which results as propagating in the simple medium against an equivalent gravity. The constant phase wave front can propagate a maximum coordinate distance

$$\Delta x)_{max} = \int_0^\infty w(t)\, dt = \left.\begin{array}{ll} \dfrac{2}{a} \dfrac{\tan^{-1}\dfrac{\sqrt{\mu\varepsilon-1}}{\sqrt{\mu\varepsilon}+1}}{\sqrt{\mu\varepsilon-1}} & \text{if } \mu\varepsilon > 1 \\ \\ \dfrac{2}{a} \dfrac{\tanh^{-1}\dfrac{\sqrt{1-\mu\varepsilon}}{1+\sqrt{\mu\varepsilon}}}{\sqrt{1-\mu\varepsilon}} & \text{if } \mu\varepsilon < 1 \end{array}\right\} \xrightarrow{\mu\varepsilon \to 1} \frac{1}{a} \tag{35}$$

which shows in the equivalent gravitation an EM wave can propagate arbitrarily far into a medium iff $\mu\varepsilon \to 0$ which is case of infinite phase velocity in the rest frame of the medium.

The third term in equation 31

$$e^{-a\int_0^t \frac{1+\sqrt{\mu\varepsilon}\tanh at}{\sqrt{\mu\varepsilon}+\tanh at}dt} = \frac{1}{\cosh at + \frac{1}{\sqrt{\mu\varepsilon}}\sinh at} \qquad (36)$$

shows amplitude decreasing, which can be interpreted as the slowing down of the coordinate phase velocity which reduces the number of waves in unit time to $\{0\}$, or as the decrease of the "dressed photons" density with respect to the initial one, using instantaneous Lorentz transform (equation 7) to velocity equation 33 (Appen. XI). The special case for vacuum limit $\mu\varepsilon \to 1$ of equation 31 is clearly nothing but an instantaneous Lorentz transform of the vacuum plane wave $e^{i(k_o X - k_o T)}$ to the hyperbolically accelerated observers $\{0\}$. Also, equation 31 reveals that polarization of this propagation is not affected by acceleration.

Finally, if we want to make a particle-like photon model for this wave, then with

$$\omega = \text{proper energy of photon} = \underline{P}\cdot\underline{u} = P_o$$

$$\frac{P^1}{P^o} = \frac{w^1}{w^o} = \frac{dx^1}{dx^o} = w(t)$$

where $\underline{w}$ is the 4-velocity, we get a dressed mass

$$m = \frac{k_o\sqrt{\mu\varepsilon-1}}{\sqrt{\mu\varepsilon}(\sqrt{\mu\varepsilon}\cosh at + \sinh at)} \qquad (37a)$$

and w_1 or P_1

$$w_1 = \frac{-(\cosh at + \sqrt{\mu\varepsilon}\sinh at)}{\sqrt{\mu\varepsilon - 1}} \quad \text{or} \quad P_1 = \frac{-k_o(\cosh at + \sqrt{\mu\varepsilon}\sinh at)}{\sqrt{\mu\varepsilon}\,(\sqrt{\mu\varepsilon}\cosh at + \sinh at)} \qquad (37b)$$

which, since $g_{\mu\nu}$ independent of x^1 implies w_1 = constant for a massy particle and P_1 = constant for a massless particle along their geodesics (26), shows that this "photon in accelerated media" does not propagate along a geodesic, nor is it massless and path null. This just demonstrates again that a wave in a non-inertial moving medium is dragged by it.

All these results are caused partly by the particular behavior of the accelerated coordinate and partly by the presence of the medium. If we instantaneously Lorentz transform equation 31 to $\{\bar{X}^\mu\}$ (Appen.XII) then

$$E^Y = \frac{\sqrt{\mu\varepsilon}(1 + \sqrt{\mu\varepsilon}\tanh at)}{\sqrt{\mu\varepsilon} + \tanh at}\, e^{ik_o(X - \frac{1}{a}\sqrt{1+a^2T^2})}\, e^{-ik_oA(T)} \qquad (38)$$

where $A(T)$ is given in Appendix XII and approaches $\frac{T}{\sqrt{\mu\varepsilon}}$ for small at. The instantaneous phase velocity in $\{\bar{X}^\mu\}$ then is (see Fig. 3)

$$\left.\frac{dX}{dT}\right)_{\text{constant phase}} = \frac{1}{\sqrt{1+a^2T^2}}\left\{aT + 2(\sqrt{1+a^2T^2} - aT)\right.$$

$$\left.\times \frac{1}{(1+\sqrt{\mu\varepsilon}) + (\sqrt{\mu\varepsilon} - 1)(\sqrt{1+a^2T^2} - aT)^2}\right\} \qquad (38')$$

which approaches 1 in vacuum limit and is dragged along by the medium increasing (or decreasing) from $1/\sqrt{\mu\varepsilon}$ to 1 from $T = 0$ to $T = \infty$ as a result of the fact that when the medium velocity approaches +1 as

$T \to \infty$ then any velocity in its rest frame approaches 1 in $\bar{K}$. The instantaneous frequency $\bar{\omega}$ in $\bar{K}$ is just equation 38' multiplied by k_o which shows it changes from $\omega_o (\equiv k_o/\sqrt{\mu\varepsilon})$ to $\omega_o\sqrt{\mu\varepsilon}$ as the medium accelerates infinitely fast. The amplitude of E^Y changes from the initial 1 to $\sqrt{\mu\varepsilon}$ as $T \to \infty$. Compare (Appen. XII) the above effects at small $V = \tanh at \sim aT$

$$\frac{\bar{\omega}}{\omega_o} = 1 + V\sqrt{\mu\varepsilon} - \frac{V}{\sqrt{\mu\varepsilon}} + O(V^2)$$

$$\text{as } V \to 0 \qquad (39a)$$

$$|E^Y| = 1 + V\sqrt{\mu\varepsilon} - \frac{V}{\sqrt{\mu\varepsilon}} + O(V^2)$$

to the corresponding results of observing in $\bar{K}$ a steady x-propagating EM plane wave of frequency ω_o, amplitude 1 in a simple medium co-moving in K which has a small constant velocity β in +x direction relative to $\bar{K}$

$$\frac{\bar{\omega}}{\omega_o} = 1 + \beta\sqrt{\mu\varepsilon} + O(\beta^2)$$

$$|E^Y| = 1 + \beta\sqrt{\mu\varepsilon} + O(\beta^2) \qquad (39b)$$

We see that only when the medium is "dense" ($\sqrt{\mu\varepsilon} >> 1$) can it drag the wave along with it more. But however fast the medium can accelerate, there is a limit to this dragging effect.

III.2 Medium in Rotation

General local electrodynamics formalism is constructed for a steady rotating medium in this section, then rotational scattering of a plane wave by simple dielectric sphere in rigid rotation is solved.

III.2.1 Formulations. Consider an inertial frame $\overline{K}$ with spatial cylindrical coordinate frame $\{\overline{X^\mu}\} \equiv \{T,R,\Phi,Z\}$, then the transform

$$\begin{cases} T = t \\ R = r \\ \Phi = \phi + \Omega(r)t \\ Z = z \end{cases} \tag{40}$$

carries $\{\overline{X^\mu}\}$ to a steady rotating frame $\{x^\mu\} \equiv \{t,r,\phi,z\}$ in the sense that a fixed (x^i) point in $\{x^\mu\}$ rotates with a time-independent angular velocity

$$\Omega \equiv \left.\frac{d\Phi}{dT}\right)_{\text{fixed } x^i} = \Omega(R) \tag{41}$$

about the z-axis in $\overline{K}$. Because it is impossible to have Ω = const. rigid rotation for large media, $\Omega(r)$ is used in equation 40 $\ni$ $r\Omega < 1$ for all r . For a most possible "rigid-like" continuous rotation $\Omega(r)$ should be

$$\begin{cases} \Omega(r) = \Omega_o + 0\,(\Omega_o r) & \text{if} \quad \Omega_o r \ll 1,\ \Omega_o = \text{const.} \\ \Omega(r) \longrightarrow 1/r & \text{if} \quad r \to \infty \\ \Omega(r_1) > \Omega(r_2) & \text{if} \quad r_1 < r_2 \end{cases} \tag{42}$$

Now as a simplest analogue to the non-relativistic rigid rotation with centrifugal acceleration $a_c = r\Omega^2$, if we require the proper centrifugal acceleration of a Ω-rotating observer to be proportional to r with proportionality constant Ω_o^2 ; then

$$\Omega(r) = \frac{\Omega_o}{\sqrt{1 + \Omega_o^2 r^2}} \tag{43}$$

satisfies equation 42 and can be thought of as a simplest rigid-like rotation for a medium.

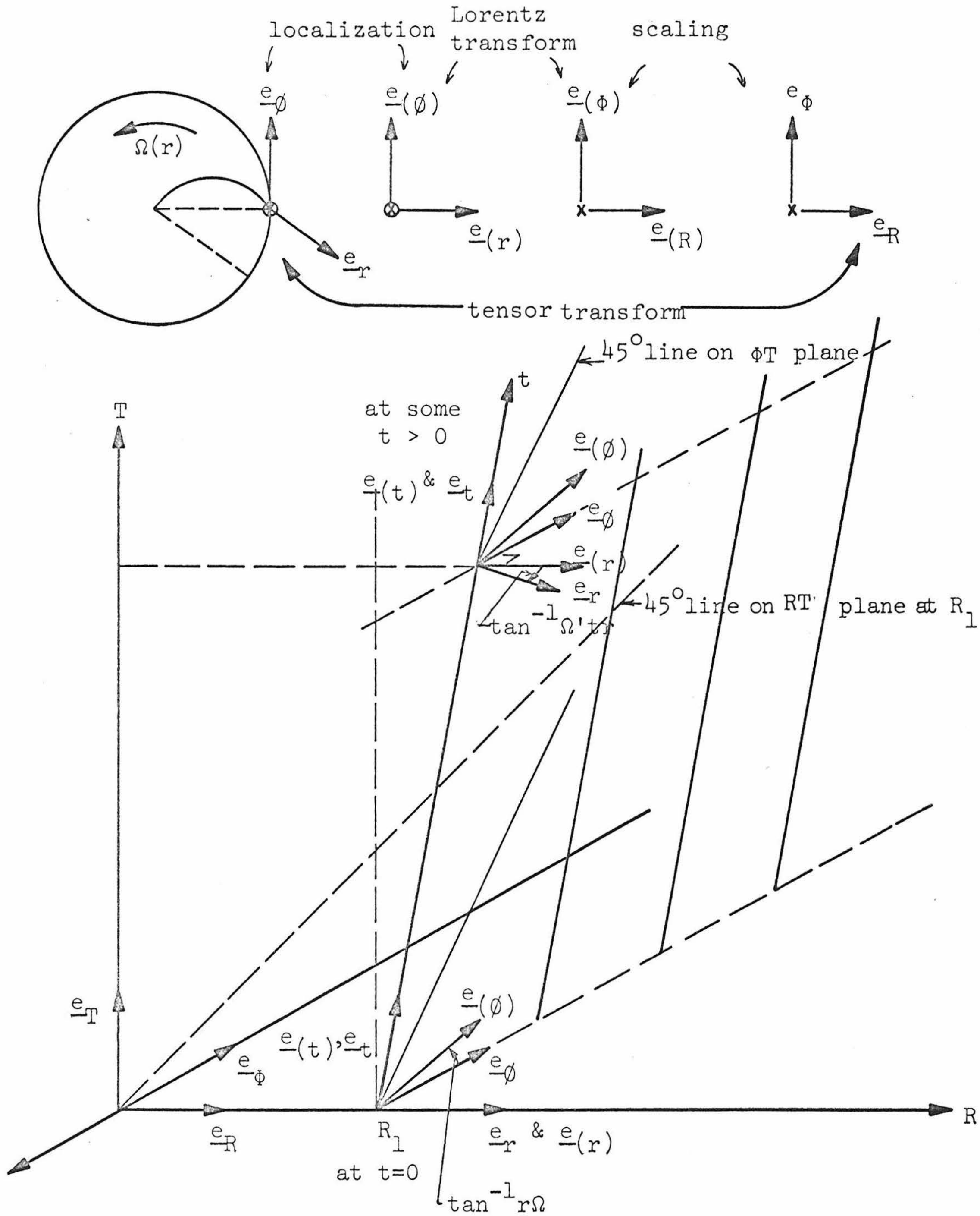
Now in the rotating frame $\{x^\mu\}$ of equation 40 we have metric

$$g_{\mu\nu} = \begin{pmatrix} 1 - r^2\Omega^2 & -r^2 t\ \Omega\Omega' & -r^2\Omega & 0 \\ -r^2 t\ \Omega\Omega' & -(1 + \Omega'^2 r^2 t^2) & -r^2 t\ \Omega' & 0 \\ -r^2\Omega & -r^2 t\ \Omega' & -r^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{44}$$

and tetrad for co-moving observers $\{0\}$ in a similar way as we did in Section III.1.1 , see Fig. 4,

$$\left\{ \begin{aligned} \underline{e}_{(o)} &= \frac{1}{\sqrt{1 - r^2\Omega^2}}\, \underline{e}_t \\ \underline{e}_{(1)} &= \underline{e}_r - \Omega' t\, \underline{e}_\phi \\ \underline{e}_{(2)} &= \frac{\sqrt{1 - r^2\Omega^2}}{r} \left(\underline{e}_\phi + \frac{r^2\Omega}{1 - r^2\Omega^2}\, \underline{e}_t\right) \\ \underline{e}_{(3)} &= \underline{e}_z \end{aligned} \right. \tag{45}$$

Here equation II-12 is used and $\underline{e}_{(1)}$, $\underline{e}_{(2)}$ are so combined out of the



$$\begin{cases} \underline{e}_t = \underline{e}_T + \Omega \underline{e}_\Phi \\ \underline{e}_\phi = \underline{e}_\Phi = r\ \underline{e}_{(\Phi)} \end{cases} \qquad \underline{e}_r = \underline{e}_R + \Omega' t\ \underline{e}_\Phi \qquad \underline{e}_z = \underline{e}_Z$$

Fig. 4. Space-time diagram and physical picture showing relation among $\{\underline{e}_{(\mu)}\}$, $\{\underline{e}_\mu\}$ and $\{\underline{e}_{\underline{\mu}}\}$.

time orthogonal $\underline{d}_i$ of equation II.17 that they are parallel to the instantaneous coincident Lorentz transform of $\underline{e}_R, \underline{e}_{(\Phi)}$ respectively, which is possible because of equation II.14.

Now consider a rotating medium described by equation 40 $\ni \{x^\mu\}$ is its co-moving frame and equation 45 is the local tetrad for co-moving observers $\{0\}$. Then similar to the way of obtaining equation III. 8,9, 10, we get

$$J = (\ell(\rho + r\Omega J^{(\emptyset)}),\ J^{(r)},\ \frac{J^{(\emptyset)}}{\ell r} - \Omega' t\ J^{(r)},\ J^{(z)}) \qquad (46)$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & \ell(-E^{(r)} + r\Omega B^{(z)}) & -E^{(\emptyset)}/r + \ell\Omega' t(E^{(r)} - r\Omega B^{(z)}) & -\ell(E^{(z)} + r\Omega B^{(r)}) \\ -\ell(-E^{(r)} + r\Omega B^{(z)}) & 0 & \frac{-B^{(z)}}{r} & B^{(\emptyset)} \\ E^{(\emptyset)}/r - \ell\Omega' t(E^{(r)} - r\Omega B^{(z)}) & \frac{B^{(z)}}{\ell r} & 0 & -B^{(r)}/\ell r - \Omega' t B^{(\emptyset)} \\ \ell(E^{(z)} + r\Omega B^{(r)}) & -B^{(\emptyset)} & B^{(r)}/\ell r + \Omega' t\ B^{(\emptyset)} & 0 \end{pmatrix} \qquad (47)$$

$$G^{\mu\nu} = (\text{same form as equation 47 with } (\underset{\sim}{E}, \underset{\sim}{B}) \longrightarrow (\underset{\sim}{D}, \underset{\sim}{H})) \qquad (48)$$

expressing the tensor current and EM field in terms of their local physical values to $\{0\}$, where $\ell \equiv \frac{1}{\sqrt{1-r^2\Omega^2}}$. Now the only non-vanishing $\Gamma^\nu_{\ \nu\lambda}$ in $\{x^\mu\}$ is

$$\Gamma^1{}_{11} = -rt^2\Omega'^2 \qquad \Gamma^2{}_{22} = r\Omega' t \qquad \Gamma^1{}_{10} = -r\Omega\Omega' t$$

$$\Gamma^1{}_{12} = -r\Omega' t \qquad \Gamma^2{}_{20} = r\Omega\Omega' t \qquad \Gamma^2{}_{21} = \frac{1}{r} + r\Omega'^2 t^2 \tag{49}$$

Thus the local Maxwell equations to {0} follows as

$$\left\{\begin{aligned}
&[\frac{1}{\ell r}\frac{\partial}{\partial r}(\ell r D^{(r)}) + \frac{1}{\ell r}\frac{\partial}{\partial \phi} D^{(\phi)} + \frac{\partial}{\partial z} D^{(z)}] + \Omega' t \frac{\partial}{\partial \phi}(r\Omega\, H^{(z)} - D^{(r)}) \\
&\qquad = \rho + r\Omega J^{(\phi)} - r\Omega \frac{\partial}{\partial z} H^{(r)} + \frac{1}{\ell r}\frac{\partial}{\partial r}(\ell r^2 \Omega\, H^{(z)}) \\
&[\frac{1}{\ell r}\frac{\partial}{\partial \phi} H^{(z)} - \frac{\partial}{\partial z} H^{(\phi)}] + \ell r\Omega \frac{\partial}{\partial t} H^{(z)} = J^{(r)} + \ell \frac{\partial}{\partial t} D^{(r)} \\
&[\frac{\partial}{\partial z} H^{(r)} - \frac{\partial}{\partial r} H^{(z)}] + \ell^2 r\Omega^2 H^{(z)} + \Omega' t \frac{\partial}{\partial \phi} H^{(z)} = J^{(\phi)} + \ell \frac{\partial}{\partial t} D^{(\phi)} - \ell^2 r\Omega' D^{(r)} \\
&[\frac{1}{r}\frac{\partial}{\partial r}(rH^{(\phi)}) - \frac{1}{\ell r}\frac{\partial}{\partial \phi} H^{(r)}] - \ell r\Omega \frac{\partial}{\partial t} H^{(r)} - \Omega' t \frac{\partial H^{(\phi)}}{\partial \phi} = J^{(z)} + \ell \frac{\partial}{\partial t} D^{(z)}
\end{aligned}\right. \tag{50}$$

$$\left\{\begin{aligned}
&[\frac{1}{\ell r}\frac{\partial}{\partial r}(\ell r B^{(r)}) + \frac{1}{\ell r}\frac{\partial}{\partial \phi} B^{(\phi)} + \frac{\partial}{\partial z} B^{(z)}] - \Omega' t \frac{\partial}{\partial \phi}(r\Omega E^{(z)} + B^{(r)}) \\
&\qquad = r\Omega \frac{\partial}{\partial z} E^{(r)} - \frac{1}{\ell r}\frac{\partial}{\partial r}(\ell r^2 \Omega\, E^{(z)}) \\
&[\frac{1}{\ell r}\frac{\partial}{\partial \phi} E^{(z)} - \frac{\partial}{\partial z} E^{(\phi)}] + \ell r\Omega \frac{\partial}{\partial t} E^{(z)} = -\ell \frac{\partial}{\partial t} B^{(r)} \\
&[\frac{\partial}{\partial z} E^{(r)} - \frac{\partial}{\partial r} E^{(z)}] + \ell^2 r\Omega^2 E^{(z)} + \Omega' t \frac{\partial}{\partial \phi} E^{(z)} = -\frac{\partial}{\partial t} B^{(\phi)} + \ell^2 r\Omega'\, B^{(r)} \\
&[\frac{1}{r}\frac{\partial}{\partial r}(rE^{(\phi)}) - \frac{1}{\ell r}\frac{\partial}{\partial \phi} E^{(r)}] = -\ell \frac{\partial}{\partial t} B^{(z)} + \ell r\Omega \frac{\partial}{\partial t} E^{(r)} + \Omega' t \frac{\partial}{\partial \phi} E^{(\phi)}
\end{aligned}\right. \tag{51}$$

The behavior of the co-moving local $\{\underline{e}_{(\mu)}\}$ and coordinate $\{\underline{e}_{\mu}\}$ with respect to the inertial $\overline{K}$ directions can be easily visualized from Fig. 4 which explains the mixing and scaling of the tensors to make their respective physical values to $\{0\}$ as in equations 46, 47 and 48. Since $\{0\}$ is medium-corotating, the physical constitutive relations II.24 hold, together with which equations 50 and 51 complete the local EM equations for the rotating linear medium with respect to its co-moving $\{0\}$. The local continuity equation in the rotating frame from II.66 is

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r\, J^{(r)} + \frac{1}{\ell r}\frac{\partial}{\partial \phi} J^{(\phi)} + \frac{\partial}{\partial z} J^{(z)}\right] - \Omega' t \frac{\partial}{\partial \phi} J^{(r)} = -\,\ell \frac{\partial}{\partial t}\left(\rho + r\Omega J^{(\phi)}\right) \tag{52}$$

and the B.C. at the interface of media I,II is II.69 for their local EM fields.

Now the electric and magnetic tensor parameters $\underline{\underline{\varepsilon}}, \underline{\underline{K}}$ for the rotating linear medium have their values in the corotating $\{x^{\mu}\}$ obtained similar to equations 15 and 16

$$\varepsilon^{\mu\nu} = \left(\begin{array}{ll} \ell^2(\varepsilon^{(0)(0)} + r^2\Omega^2\varepsilon^{(2)(2)}) & \ell r\Omega\, \varepsilon^{(2)(1)} \\ \ell r\Omega\, \varepsilon^{(1)(2)} & \varepsilon^{(1)(1)} \\ -\ell r\Omega\Omega' t\cdot\varepsilon^{(1)(2)} + \Omega\varepsilon^{(2)(2)} & -\Omega' t\varepsilon^{(1)(1)} + \frac{1}{\ell r}\,\varepsilon^{(2)(1)} \\ \ell r\Omega\varepsilon^{(3)(2)} & \varepsilon^{(3)(1)} \end{array}\right.$$

$$
\left.\begin{matrix}
-\ell r\Omega\Omega' t\cdot\varepsilon^{(2)(1)} + \Omega\varepsilon^{(2)(2)} & \ell r\Omega\varepsilon^{(2)(3)} \\[2ex]
-\Omega' t\varepsilon^{(1)(1)} + \frac{1}{\ell r}\,\varepsilon^{(1)(2)} & \varepsilon^{(1)(3)} \\[2ex]
\Omega'^2 t^2\varepsilon^{(1)(1)} - \frac{\Omega' t}{\ell r}(\varepsilon^{(1)(2)} + \varepsilon^{(2)(1)}) + \frac{1}{\ell^2 r^2}\,\varepsilon^{(2)(2)} & -\Omega' t\varepsilon^{(1)(3)} + \frac{1}{\ell r}\,\varepsilon^{(2)(3)} \\[2ex]
-\Omega' t\,\varepsilon^{(3)(1)} + \frac{1}{\ell r}\,\varepsilon^{(3)(2)} & \varepsilon^{(3)(3)}
\end{matrix}\right) \tag{53}
$$

$$K^{\mu\nu} = \text{(same form as equation 53, with } \varepsilon^{(\mu)(\nu)} \longrightarrow K^{(\mu)(\nu)} \tag{54}$$

With these expressions the total constitutive tensor $c^{\mu\nu\alpha\beta}$ can be directly constructed by equation II.37 in terms of the local intrinsic properties of the medium with respect to its co-moving $\{0\}$. Equation II.46 and this $c^{\mu\nu\alpha\beta}$ present an observer-independent covariant formalism.

Consider the special case of a rotating medium which has $r\Omega < 1$ for all its parts and rotates rigidly. Thus for the co-moving $\{0\}$ $\Omega' = 0$ and the simplified equations 50, 51, 52 can be written in 3-vector forms respectively*

$$
\begin{cases}
\underset{\sim}{\nabla} \times \underset{\sim}{H} + \ell \frac{\partial}{\partial t}[(\underset{\sim}{\Omega} \times \underset{\sim}{r}) \times \underset{\sim}{H}] + \ell^2\, \underset{\sim}{\Omega} \times [\underset{\sim}{\Omega} \times (\underset{\sim}{r} \times \underset{\sim}{H})] = \underset{\sim}{J} + \ell\frac{\partial}{\partial t}\underset{\sim}{D} \\
\underset{\sim}{\nabla} \cdot \underset{\sim}{D} - \ell^2 \underset{\sim}{D} \cdot \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{r}) = \rho + (1+\ell^2)\underset{\sim}{\Omega} \cdot \underset{\sim}{H} + \underset{\sim}{\Omega} \times \underset{\sim}{r} \cdot (\underset{\sim}{J} - \underset{\sim}{\nabla} \times \underset{\sim}{H})
\end{cases} \tag{55}
$$

---

*Here $\underset{\sim}{r}$ stands for cylindrical radial vector and $\partial/\partial x^{\mu}$ do not act on basis vectors, Ω is in $\underset{\sim}{e}_{(z)}$ direction.

$$\begin{cases} \nabla \times E + \ell \frac{\partial}{\partial t}[(\Omega \times r) \times E] + \ell^2 \Omega \times [\Omega \times (r \times E)] = -\ell \frac{\partial}{\partial t} B \\ \nabla \cdot B - \ell^2 B \cdot \Omega \times (\Omega \times r) = -(1 + \ell^2)\Omega \cdot E + \Omega \times r \cdot \nabla \times E \end{cases} \quad (56)$$

$$\nabla \cdot J = -\ell \frac{\partial}{\partial t} (\rho + J \cdot \Omega \times r) \quad (57)$$

in which the familiar 3-vector analysis symbols in cylindrical coordinates have been identically adopted to achieve the above simple form; except a particular convention that $\frac{1}{\ell}\frac{\partial}{\partial \phi}$ replaces $\frac{\partial}{\partial \phi}$ in the symbol of usual notations. If the co-moving local constitutive properties of II.24 are known, equations 55 and 56 are directly subject to mathematical analysis for the physical EM field observed by corotating observers.

III.2.2 Plane wave scattering by a simple rotating dielectric sphere. For a simple dielectric sphere in rigid $\Omega' = 0$ rotation, if we neglect its deformation due to centrifugal and coriolis forces, then $D = \epsilon E$, $B = \mu H$ macroscopically holds with respect to corotating observers. Now the natural co-moving frame to fit its boundary and thus to simplify the mathematics is a spherical corotating frame $\{t, r, \theta, \phi\}$ * defined with respect to $\{T, R, \Theta, \Phi\}$ of the inertial laboratory $\overline{K}$ by

$$T = t, \quad R = r, \quad \Theta = \theta, \quad \Phi = \phi + \Omega t, \quad \Omega = \text{const.} \quad (58)$$

The co-moving observers $\{0\}$ for this spherical rotating frame are the same as those for the previous cylindrical rotating frame, but their

---

*The spherical r used in this section should not be confused with the previous cylindrical r which is denoted by r_c from here on.

spatial physical bases are locally rotated in respective cases just the way the familiar cylindrical and spherical coordinate unit vectors were in $\overline{K}$. Thus the local Maxwell equations in $\{t,r,\theta,\emptyset\}$ of equation 58 are just equations 55 and 56 reinterpreted in 3-vector analysis for a spherical coordinate, with the adopted convention of $\frac{1}{\ell}\frac{\partial}{\partial\emptyset}$ replacing $\frac{\partial}{\partial\emptyset}$ (Appen.XIII).

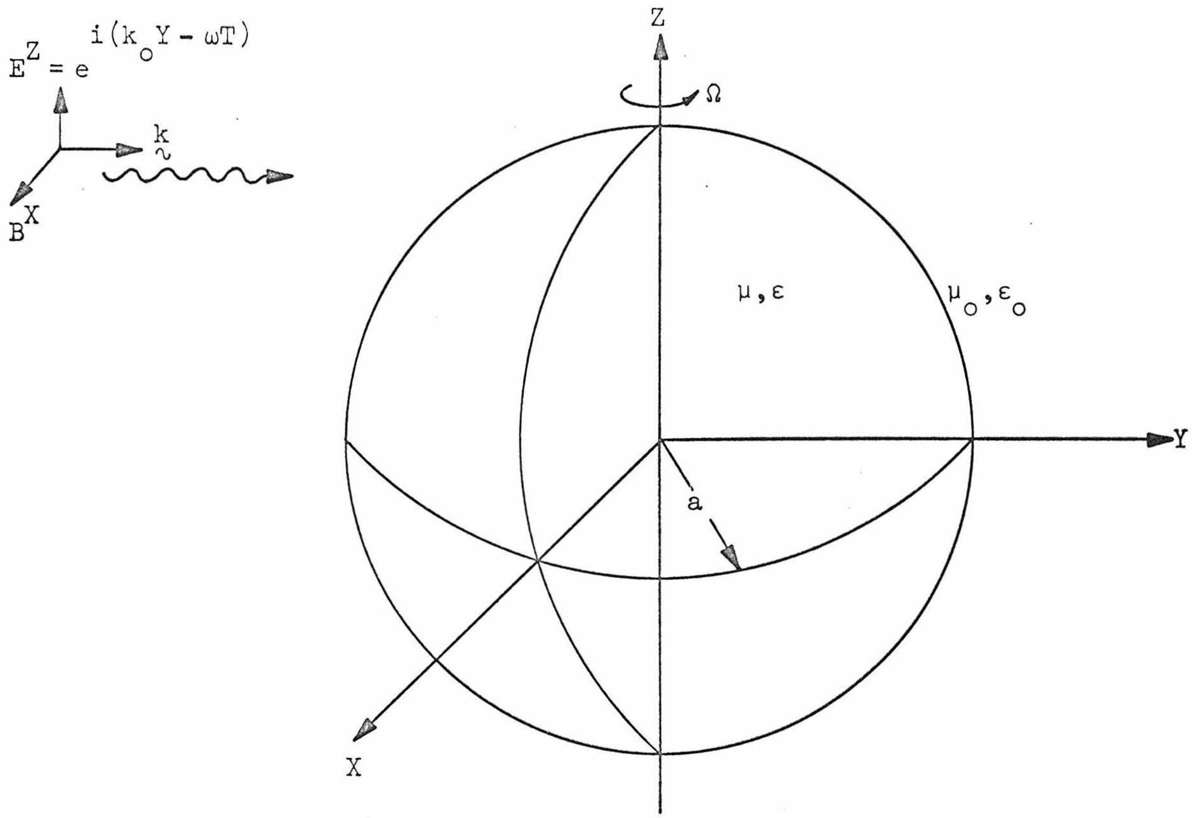
Now consider that the simple rotating sphere of radius a surrounded by a μ_o,ε_o medium at rest in $\overline{K}$ scatters a plane wave $\overline{E}^{inc}$ as shown in Fig. 5 for two kinds of incidences. The scattered field due to pure rotational effect in addition to the non-rotating Mie scattering part is sought.

Substituting $D = \varepsilon E$, $B = \mu H$, $J = 0$ in equation 55 gives (Appen. XIII)

$$\begin{cases} \nabla \times B + \ell\frac{\partial}{\partial t}[(\Omega\times r_c)\times B] + \ell^2\Omega\times[\Omega\times(r_c\times B)] = \ell\mu\varepsilon\frac{\partial}{\partial t}E \\ \nabla\cdot E - \ell^2 E\cdot\Omega\times(\Omega\times r_c) = \frac{1}{\mu\varepsilon}(1+\ell^2)\Omega\cdot B - \frac{1}{\mu\varepsilon}\Omega\times r_c\cdot\nabla\times B \end{cases} \qquad (59)$$

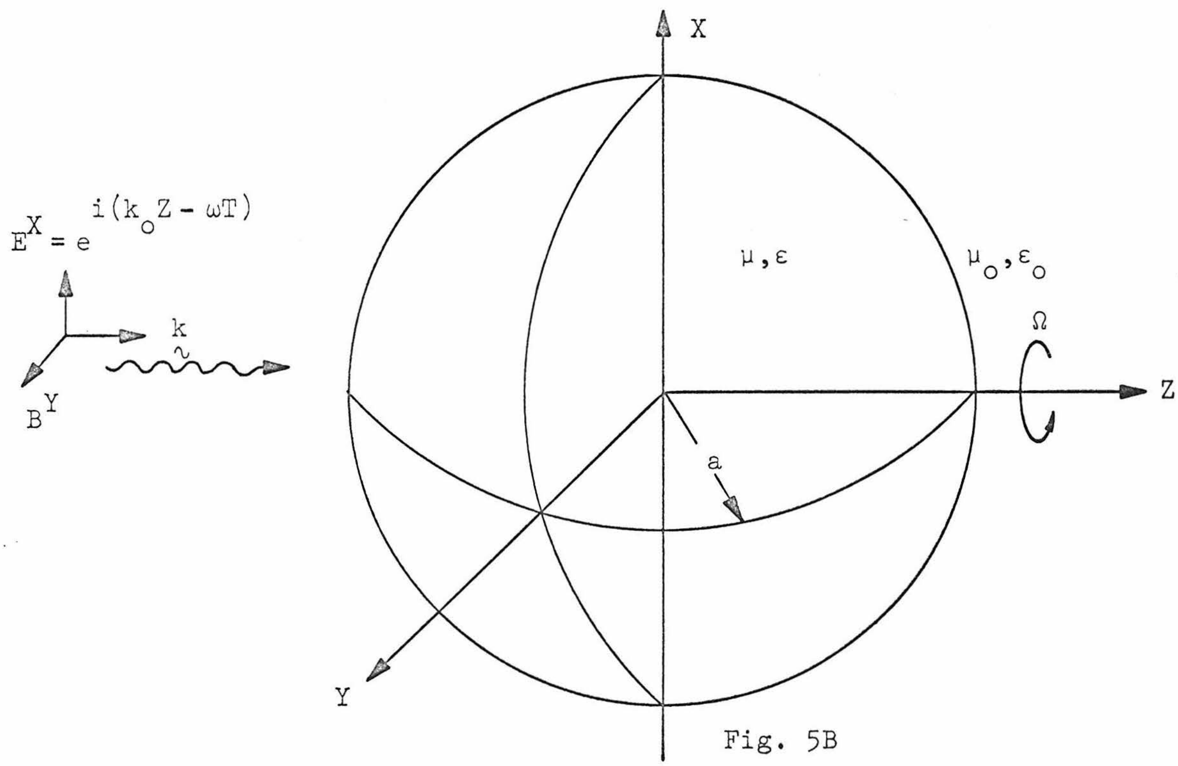
Thus equations 56 and 59 describe the wave propagation to rotating observers at $r < a$. But instead of solving EM field for rotating observers, we are interested in the scattered field in $\overline{K}$. Thus we must find the Maxwell equations in $\overline{K}$ obtained from equations 56 and 59 by substituting tensor and Lorentz transforms:

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial X^{\overline{\lambda}}}{\partial x^\alpha}\frac{\partial}{\partial X^{\overline{\lambda}}}, \qquad \begin{aligned} E &= \overline{E}_{||} + \ell(\overline{E}_{\perp} + V\times\overline{B}) \\ B &= \overline{B}_{||} + \ell(\overline{B}_{\perp} - V\times\overline{E}) \end{aligned} \qquad (60)$$



$E^Z = e^{i(k_o Y - \omega T)}$
k
B^X
Z
Ω
μ, ε
μ_o, ε_o
Y
a
X

Fig. 5A



X
$E^X = e^{i(k_o Z - \omega T)}$
k
B^Y
μ, ε
μ_o, ε_o
Ω
Z
a
Y

Fig. 5B

where

$$\frac{\partial \overline{X}^{\lambda}}{\partial x^{\alpha}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\overline{\underset{\sim}{E}}_{||}$, $\overline{\underset{\sim}{E}}$ are the parallel and perpendicular parts of $\overline{\underset{\sim}{E}}$ relative to $\underset{\sim}{V} \equiv r\Omega\, \underset{\sim}{e}_{(\Phi)}$. Then equation 59 gives* in $\overline{K}$ spherical coordinates

R < a :

$$\left\{ \begin{array}{l} \overline{\underset{\sim}{\nabla}} \times \overline{\underset{\sim}{B}} = -\left(\Omega \frac{\partial}{\partial \Phi} \overline{\underset{\sim}{E}} - \underset{\sim}{V}\, \overline{\underset{\sim}{\nabla}} \cdot \overline{E}\right) + \frac{\partial}{\partial T} + \Omega \frac{\partial}{\partial \Phi}) \quad [\mu\varepsilon \overline{\underset{\sim}{E}} + (\ell^2 - 1)(\mu\varepsilon - 1)\, \overline{\underset{\sim}{E}}_{\perp} \\ \qquad\qquad + \ell^2 (\mu\varepsilon - 1)\, \underset{\sim}{V} \times \overline{\underset{\sim}{B}}] \qquad (61a) \\ \\ \varepsilon \overline{\underset{\sim}{\nabla}} \cdot \overline{\underset{\sim}{E}} = -\frac{(\mu\varepsilon - 1)}{\mu} \overline{\underset{\sim}{\nabla}} \cdot [(\ell^2 - 1)\, \overline{\underset{\sim}{E}}_{\perp} + \ell^2 \underset{\sim}{V} \times \overline{\underset{\sim}{B}}] \qquad (61b) \end{array} \right.$$

and equation 56 just gives the ordinary equations.

$$\text{all } R : \left\{ \begin{array}{l} \overline{\underset{\sim}{\nabla}} \times \overline{\underset{\sim}{E}} = -\frac{\partial}{\partial T} \overline{\underset{\sim}{B}} \qquad (62a) \\ \\ \overline{\underset{\sim}{\nabla}} \cdot \overline{\underset{\sim}{B}} = 0 \qquad (62b) \end{array} \right.$$

as it should. Since no coefficients are time dependent, we can put $e^{-i\omega T}$ time dependence to $\overline{\underset{\sim}{E}}, \overline{\underset{\sim}{B}}$ and then obtain from equations 61a, 62a the wave equation for $\overline{\underset{\sim}{E}}$

$$\overline{\underset{\sim}{\nabla}} \times \overline{\underset{\sim}{\nabla}} \times \overline{\underset{\sim}{E}} - k^2 \overline{\underset{\sim}{E}} = i\omega \Big\{ [\underset{\sim}{V}\, \overline{\underset{\sim}{\nabla}} \cdot \overline{\underset{\sim}{E}} + (\mu\varepsilon - 1)\Omega \frac{\partial}{\partial \Phi} \overline{\underset{\sim}{E}}]$$

$$+ (\mu\varepsilon - 1)(-i\omega + \Omega \frac{\partial}{\partial \Phi})[(\ell^2 - 1)\overline{\underset{\sim}{E}}_{\perp} + \frac{\ell^2}{i\omega} \underset{\sim}{V} \times \overline{\underset{\sim}{\nabla}} \times \overline{\underset{\sim}{E}}]\Big\}, \quad R < a \qquad (63)$$

---

*The $\partial/\partial\Phi$ here does not operate on the coordinate unit vectors.

or simply rewriting as

$$\bar{\nabla} \times \bar{\nabla} \times \bar{E} - k^2\bar{E} = i\omega \, S \cdot \bar{E} \quad , \quad R < a \tag{64}$$

where

$$S \equiv [V\,\bar{\nabla} + (\mu\varepsilon - 1)\Omega \frac{\partial}{\partial\Phi} U] + (\mu\varepsilon-1)(-i\omega + \Omega\frac{\partial}{\partial\Phi})[(\ell^2-1)(U - \frac{VV}{V^2}) + \frac{\ell^2\Omega}{\omega i}\lambda\,]$$

$$U \equiv \text{unit dyadic} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

$$\lambda \equiv \begin{pmatrix} -\partial/\partial\Phi & 0 & \sin\Theta \frac{\partial}{\partial R} \\ 0 & -\partial/\partial\Phi & \partial/\partial\Theta(\sin\Theta) \\ 0 & 0 & 0 \end{pmatrix}, \quad k^2 \equiv \omega^2\mu\varepsilon \tag{65}$$

Compared to the wave equation in simple medium at rest, now outside the rotating sphere in $R > a$ region

$$\bar{\nabla} \times \bar{\nabla} \times \bar{E} - k_o^{\,2}\,\bar{E} = 0 \;, \; R > a \;; \;\; k_o^2 \equiv \omega^2\mu_o\varepsilon_o \tag{66}$$

we see that the S in equation 64 is zero either when the sphere is not rotating $\Omega = 0$ or there is nothing but $\mu = 1 = \varepsilon$ vacuum being rotated, thus S is purely a medium rotating effect.

Now the d.e.'s (equations 64 and 66) with outward-going radiation condition on the scattered field at $R = \infty$ and B.C.'s at $R = a$ which are now implied by equation II.69 in rotating K and laboratory $\bar{K}$

$$\left\{\begin{array}{l} B^{(R)},\ E^{(\Theta)},\ E^{(\Phi)} = \text{continuous across } R = a \\ \frac{1}{\mu_o} B^{(\Phi)}\Big)_{R=a^+} = \frac{1}{\mu} B^{(\Phi)}\Big)_{R=a^-} \\ \frac{1}{\mu_o} B^{(\Theta)}\Big)_{R=a^+} = \ell^2\left[B^{(\Theta)}\left(\frac{1}{\mu} - \varepsilon V^2\right) + E^{(R)} V\left(\varepsilon - \frac{1}{\mu}\right)\right]_{R=a^-} \\ \varepsilon_o E^{(R)}\Big)_{R=a^+} = \ell^2\left[E^{(R)}\left(\varepsilon - \frac{V^2}{\mu}\right) - B^{(\Theta)}\, V\left(\varepsilon - \frac{1}{\mu}\right)\right]_{R=a^-} \end{array}\right. \tag{67}$$

form a B.V.P. It is difficult to solve in a closed form because the inside wave cannot be expanded as simple sum of spherical partial waves with arbitrary constant coefficients. We solve it by using an integral iteration method.

First combine equations 64 and 66 to be

$$\overline{\nabla} \times \overline{\nabla} \times \overline{E} - k_o^{\,2}\, \overline{E} = p(R)[(k^2 - k_o^{\,2})\underset{\approx}{U} + i\omega\, \underset{\approx}{S}(\underset{\sim}{R})] \cdot \underset{\sim}{E} \tag{68}$$

where $p(R) \equiv 1$ for $R < a$, $\equiv 0$ for $R > a$. Now, making use of the well-known dyadic radiation-condition-included Green's function $\underset{\approx}{\Gamma}(\underset{\sim}{R},\underset{\sim}{R}')$ of equation 68 (Ref. 19, p.32)

$$\underset{\approx}{\Gamma}(\underset{\sim}{R},\underset{\sim}{R}') = (\underset{\approx}{U} + \frac{1}{k_o^{\,2}}\, \overline{\nabla}\, \overline{\nabla})\, \frac{e^{ik|\underset{\sim}{R} - \underset{\sim}{R}'|}}{4\pi|\underset{\sim}{R} - \underset{\sim}{R}'|} \tag{69}$$

we can change equation 68 into an integral equation

$$\underset{\sim}{\overline{E}}(\underset{\sim}{R}) = \underset{\sim}{\overline{E}}^{homo}(\underset{\sim}{R}) + \int_{R'<a} d^3R' \; \underset{\approx}{\Gamma}(\underset{\sim}{R},\underset{\sim}{R}') \cdot [(k^2 - k_o^2)\underset{\approx}{U} + i\omega \underset{\approx}{S}(\underset{\sim}{R}')] \cdot \underset{\sim}{\overline{E}}(\underset{\sim}{R}') \qquad \text{all } \underset{\sim}{R} \tag{70}$$

The $\underset{\sim}{\overline{E}}^{homo}$ is the wave propagated in the μ_o, ε_o medium without the scatterer rotating μ, ε dielectric sphere, so $\underset{\sim}{\overline{E}}^{homo} = \underset{\sim}{\overline{E}}^{inc}$. Thus equation 70 is an integral equation for $R < a$, the solution of which then serves as a source of current density

$$\underset{\sim}{\overline{J}}^{equivalent}(\underset{\sim}{R}) = \frac{1}{i\omega\mu_o} [(k^2 - k_o^2) \underset{\approx}{U} + i\omega \underset{\approx}{S}(\underset{\sim}{R})] \cdot \underset{\sim}{\overline{E}}(\underset{\sim}{R}), \; R < a \tag{71}$$

to determine the scattered radiation field at any $R > a$. The B.C. in equation 67 implied by the d.e.'s is included in solving the integral equation.

Now equation 70 can be solved using iteration approximation. First, roughly approximate the inside $\underset{\sim}{\overline{E}}(\underset{\sim}{R})$ by incident wave and get first order total solution, then put it inside the integration to get the next order solution, and so on, assuming the iterating series converges. This gives

$$\underset{\sim}{\overline{E}}(R) = \underset{\sim}{\overline{E}}^{inc}(\underset{\sim}{R}) + \underset{\sim}{\overline{E}}^{sc}_{Mie} + \underset{\sim}{\overline{E}}^{sc}_{Mixed} + \underset{\sim}{\overline{E}}^{sc}_{rotating\ medium} \tag{72}$$

where we have split the scattered field due to different effects; namely, the difference of media, the rotation of the spherical medium, and their mixed effect as follows

$$\underset{\sim}{\overline{E}}^{sc}_{Mie} = (k^2 - k_o^2) \int_{R'<a} d^3R' \; \underset{\approx}{\Gamma}(\underset{\sim}{R},\underset{\sim}{R}') \cdot \underset{\sim}{\overline{E}}^{inc}(\underset{\sim}{R}') + (k^2 - k_o^2)^2$$

$$\times \iint_{R',R''<a} d^3R' d^3R'' \; \underset{\approx}{\Gamma}(\underset{\sim}{R},\underset{\sim}{R}') \cdot \underset{\approx}{\Gamma}(\underset{\sim}{R}',\underset{\sim}{R}'') \cdot \underset{\sim}{\overline{E}}^{inc}(\underset{\sim}{R}'') + \cdots\cdot \qquad (73a)$$

$$\underset{\sim}{\overline{E}}^{sc}_{mixed} = i\omega(k^2 - k_o^2) \iint_{R',R''<a} d^3\underset{\sim}{R}' d^3\underset{\sim}{R}'' \; \underset{\approx}{\Gamma}(\underset{\sim}{R},\underset{\sim}{R}')$$

$$\cdot \; [\underset{\approx}{\Gamma}(\underset{\sim}{R}',\underset{\sim}{R}'') \cdot \underset{\approx}{S}(\underset{\sim}{R}'') + \underset{\approx}{S}(\underset{\sim}{R}') \cdot \underset{\approx}{\Gamma}(\underset{\sim}{R}',\underset{\sim}{R}'')] \cdot \underset{\sim}{\overline{E}}^{inc}(\underset{\sim}{R}'') + \cdots\cdot \qquad (73b)$$

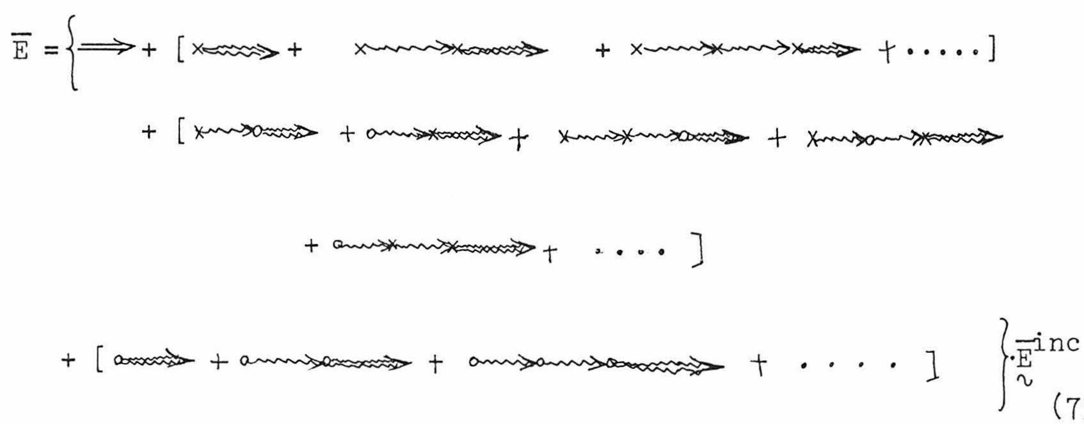
$$\underset{\sim}{\overline{E}}^{sc}_{rotation} = i\omega \int_{R'<a} d^3R' \; \underset{\approx}{\Gamma}(\underset{\sim}{R},\underset{\sim}{R}') \cdot \underset{\approx}{S}(\underset{\sim}{R}') \cdot \underset{\sim}{\overline{E}}^{inc}(\underset{\sim}{R}')$$

$$+ (i\omega)^2 \iint_{R',R''<a} d^3R' d^3R'' \; \underset{\approx}{\Gamma}(\underset{\sim}{R},\underset{\sim}{R}') \cdot \underset{\approx}{S}(\underset{\sim}{R}') \cdot \underset{\approx}{\Gamma}(\underset{\sim}{R}',\underset{\sim}{R}'') \cdot \underset{\approx}{\dot{S}}(\underset{\sim}{R}'') \cdot \underset{\sim}{E}^{inc}(\underset{\sim}{R}'')$$

$$+ \cdots\cdots \qquad (74)$$

The $\underset{\sim}{\overline{E}}^{sc}_{Mie}$ is the well-known (35) Mie scattering of a plane wave propagating in a simple medium hitting a sphere of different medium expressed in different form from the Mie's spherical partial Hertz wave expansion. The $\underset{\sim}{\overline{E}}^{sc}_{rotating}$ is the scattering caused purely by the "something rotating" which occurs even if the rotating sphere is made of the same $\mu_o; \varepsilon_o$ medium as its surrouhdings and the Mie scatter does not exist. $\underset{\sim}{\overline{E}}^{sc}_{mixed}$ is the scattered field caused by the mixed effects of both the "medium rotation" and the "medium difference" which is a second-order effect.

We can graphically represent equations 72, 73 and 74 as follows. Draw $\mathbf{R}' \rightsquigarrow \mathbf{R}$ for $\boldsymbol{\Gamma}(\mathbf{R},\mathbf{R}')$ as a propagator for waves going from $\mathbf{R}'$ to $\mathbf{R}$; draw × for $(k^2 - k_o^2)\,\mathbf{U}$ as a Mie scatterer at $\mathbf{R}'$; draw o for $i\omega\,\mathbf{S}(\mathbf{R}')$ as a rotating-medium scatterer at $\mathbf{R}'$; and draw $\longrightarrow$ as a propagator for direct propagation. Then equations 72 , 73 and 74 can be represented as, with integration understood,



(75)

where only the double drawn propagator can propagate to all $\mathbf{R}$. Now obviously we can interpret the total field at any $\mathbf{R}$ as the sum of these incident waves which directly go through and hit nothing; which are Mie scattered in the sphere once and propagate there; which are rotationally scattered once and propagate out; which are rotationally scattered, propagate to other points in the sphere, Mie scattered and then propagate out; etc.

From equations 64 and 66 the order of magnitude of the ratio

$$\left| \frac{E^{sc}_{Rot}}{E^{sc}_{Mie}} \right| \sim \frac{(\mu\varepsilon - 1)\sqrt{\mu_o \varepsilon_o}\; a\Omega}{(\mu\varepsilon - \mu_o \varepsilon_o)} \tag{76}$$

is small if $a\Omega << 1$, unless $\mu\varepsilon \sim \mu_o\varepsilon_o$. In either case the second order mix-scattering can be neglected and the first order $\bar{\underset{\sim}{E}}^{sc}_{Rot}$ will give a physical picture on rotational scattering. Now consider two kinds of $\bar{\underset{\sim}{E}}^{inc}$ polarizations as in Fig. 5.

A. Incident $\bar{\underset{\sim}{E}}$ is parallel to axis of rotation; then from equation 74 and Fig. 5a

$$\bar{\underset{\sim}{E}}^{inc} = \underset{\sim}{e}_Z\, e^{i(k_o Y - \omega T)}$$

$$\bar{\underset{\sim}{E}}^{sc}_{Rot,1st\ order} = i\omega \int_{R'<a} d^3R'\ \underset{\approx}{\Gamma}(\underset{\sim}{R},\underset{\sim}{R}') \cdot \underset{\approx}{S}(\underset{\sim}{R}') \cdot \underline{e}_Z e^{i(k_o Y' - \omega T)} \tag{77}$$

But for $R >> R'$

$$\underset{\approx}{\Gamma}(\underset{\sim}{R},\underset{\sim}{R}') \simeq (\underset{\approx}{U} - \underset{\sim}{e}_R\,\underset{\sim}{e}_R)\, \frac{e^{ik_o R}}{4\pi R}\, e^{-ik_o\, \underset{\sim}{e}_R \cdot \underset{\sim}{R}'} \tag{78}$$

Thus the first order far zone rotational scattered field is

$$\bar{\underset{\sim}{E}}^{sc}_{Rot,1st,Far\ zone} = \frac{e^{ik_o R}}{4\pi R}\, i\omega(\mu\varepsilon-1)(\underset{\approx}{U} - \underset{\sim}{e}_R\,\underset{\sim}{e}_R) \cdot \underset{\sim}{e}_Z \int_{R'<a} d^3R'$$

$$\times \left\{-i\omega(\ell^2-1) + \ell^2 ik_o V'[2\cos\Phi' + \frac{1}{i\omega}(\Omega \sin\Phi' - ik_o V' \cos^2\Phi')]\right\} e^{ik_o(Y' - \underset{\sim}{e}_R \cdot \underset{\sim}{R}')} \tag{79}$$

Evaluating the terms of order Ω gives (Appen. XIII), neglecting (ℓ^2-1)

$$\bar{\underset{\sim}{E}}^{sc}_{Rot.,far\ zone,1st} = i8\pi\, \frac{e^{i(k_o R - \omega T)}}{4\pi R}\, (\mu\varepsilon-1)\, \frac{\Omega a^2}{\sqrt{\mu_o\varepsilon_o}}\, f(\Theta,\Phi)\underset{\sim}{e}_{(\Theta)} + O(\Omega^2)$$

$$f(\Theta,\Phi) = \frac{(k_o a)^3 \sin^2\Theta \cos\Phi \left[(3-\delta^2)\sin\delta - 3\delta\cos\delta\right]}{\delta^5} \tag{80}$$

where

$$\delta \equiv k_o a \sqrt{2(1 - \sin\Theta \sin\Phi)}$$

The antisymmetry of this first order rotational scattering with respect to the Y-Z plane is a result of opposite rotational motion of the sphere as seen at the -Y axis of incidence; its symmetry about the $\Theta = \pi/2$ plane results from the fact that the upper and lower halves of the sphere are in identical motion with respect to the incident wave. Plots of equation 80 show that (Fig. 6A) on the $\Theta = \pi/2$ plane it has a resemblance to quadrupole radiation such that it can be simply interpreted as radiation from successive electric quadrupole sheaths at $|y| = \text{const} = a$ caused by induced $\underset{\sim}{V} \times \underset{\sim}{B}^{inc}$ electric polarization at the sphere, with forward bending lobes as the effect from traveling wave antennas which is caused by the traveling of the inducing incident wave. There is no scattering at backward $\Phi = -\pi/2$ and forward $\Phi = \pi/2$, but the main lobes bend from side-ends toward forward direction more as $k_o a \to$ larger. Equation 80 has only a Θ-component as the first order Mie scattering, although the latter has a dipole pattern for scatter amplitude. Also, there is no Doppler frequency shift since the motion of the scatterer is tangential to its boundary and the scattered wave is in the same medium as the incident wave.

B. Consider the incident wave as in Fig. 5b, from equation 74 neglecting ℓ^2-1 ; we have

$$\begin{cases} \overline{\underset{\sim}{E}}^{inc} = \underset{\sim}{e}_X \, e^{i(k_o Z - \omega T)} \\ \overline{\underset{\sim}{E}}^{sc}_{Rot,1^{st},far\ zone} = i\omega(\mu\varepsilon-1) \dfrac{e^{ik_o R}}{4\pi R} \displaystyle\int_{R'<a} d^3R' \, e^{-ik_o \underset{\sim}{R}' \cdot \underset{\sim}{e}_R} (\underset{\approx}{U} - \underset{\sim}{e}_R \underset{\sim}{e}_R) \end{cases}$$

$$\cdot \left\{ - \underset{\sim}{e}_Y \Omega + \frac{k_o V'}{\omega} (i\omega \sin \Phi' - \Omega \cos \Phi') \underset{\sim}{e}_Z \right\} e^{i(k_o Z' - \omega T)} \tag{81}$$

Now consider $\Omega/\omega << 1$ and keep order Ω term only, the integral gives (Appen. XIV)

$$\overline{\underset{\sim}{E}}^{sc}_{Rot,1^{st},far\ zone} = \frac{e^{i(k_o R - \omega T)}}{4\pi R} 4\pi a^3 (\mu\varepsilon - 1) \, i\omega\Omega (\underset{\approx}{U} - \underset{\sim}{e}_R \underset{\approx}{e}_R) \cdot$$

$$\left\{ -\underset{\sim}{e}_Y \frac{\sin \delta_1 - \delta_1 \cos \delta_1}{\delta_1^{\ 3}} + \underset{\sim}{e}_Z (k_o a)^2 \sin \Theta \sin \Phi \frac{(3-\delta_2^{\ 2}) \sin \delta_2 - 3\delta_2 \cos \delta_2}{\delta_2^{\ 5}} \right\}$$

$$+ \, O(\Omega^2) \tag{82}$$

where $\delta_1 \equiv 2k_o a \sin \frac{\Theta}{2}$, $\delta_2 \equiv k_o a \sqrt{2(1-\cos \Theta)}$. Expressed in spherical components this rotational scattering has an amplitude, with a constant factor $a^3(\mu\varepsilon-1) i\omega\Omega$

$$f(\Theta,\Phi) = -\underset{\sim}{e}_{(\Theta)} \cdot \sin \Phi [\cos \Theta \frac{\sin \delta_1 - \delta_1 \cos \delta_1}{\delta_1^3}$$

$$+ (k_o a)^2 \sin^2 \Theta \frac{(3-\delta_2^2) \sin \delta_2 - 3\delta_2 \cos \delta_2}{\delta_2^5}]$$

$$- \underset{\sim}{e}_{(\Phi)} \cos \Phi \frac{\sin \delta_1 - \delta_1 \cos \delta_1}{\delta_1^3} \tag{83}$$

which is plotted in Figs. 6B. The $E^{(\Theta)} \sim \sin \Phi$ and $E^{(\Phi)} \sim \cos \Phi$ Φ-dependences which are just opposite to those of the corresponding Mie scattering (35), change the polarization of the total scattering. Unlike the previous case where the induced $\underset{\sim}{\nabla} \times \underset{\sim}{M} \sim \underset{\sim}{\nabla} \times (\underset{\sim}{V} \times \underset{\sim}{E}) \sim 0$, this scattering has contributions from both the P-like and M-like induced polarizations $\underset{\sim}{P} \sim VB \sin \Phi \, \underset{\sim}{e}_Z$ and $\underset{\sim}{\nabla} \times \underset{\sim}{M} \sim \Omega E \, \underset{\sim}{e}_Y$ and can be interpreted accordingly. The plots show on the $\Theta = \pi/2$ plane a $E^{(\theta)} \sim \sin \Phi$ from $\underset{\sim}{P}$ and a $E^{(\Phi)} \sim \cos \Phi$ from $\underset{\sim}{M}$, on the X-Z plane only a $E^{(\Phi)} \sim$ constant from $\underset{\sim}{M}$, on the Y-Z plane a $E^{(\Theta)} \sim \sin \Theta$ from $\underset{\sim}{P}$ and a $E^{(\Theta)} \sim \cos \Theta$ from $\underset{\sim}{M}$; all are forward drifted or bent from the traveling wave effect as before. The forward and backward rotational scattering which is 90° rotated with respect to the E^{inc} and in Y-direction are perpendicular to the X-polarized Mie scattering and elliptically polarize the total scattered field.

III.3 Summary

Applying the theory of Part II to media in linear acceleration and rotation finds electrodynamic equations for local observable EM fields and sources with physical constitutive relations in these co-moving non-inertial frames. Studying in detail wave propagation in a simple accelerated medium reveals that EM wave is dragged by the medium with its amplitude changed, frequency shifted and phase velocity dragged along reasonably; its path is neither null nor a geodesic. Scattering of a plane EM wave by a rotating sphere is solved by an integral iteration in the laboratory frame. The rotational scattering is separated from the Mie scattering, and its first order scattering for incidences parallel and perpendicular to the rotation axis are evaluated and plotted which agree with intuition and can be interpreted simply as radiation from properly induced traveling electric and magnetic polarization sheaths.

Also as the simplest case of II.19, it turns out that the local physical tetrads for co-accelerating and co-rotating observers chosen as the instantaneous Lorentz transforms of the basis of the natural laboratory coordinates are just the unit tangents and Frenet normals of these observers along their world lines. One can easily show that for $\{e_{(i)}\}$ of equation 5

$$a_1 = a \qquad a_2 = 0 \qquad a_3 = 0 \tag{84a}$$

and for $\{\underline{e}_{(i)}\}$ of equation 45

$$a_1 = \frac{-r_c \Omega^2}{1-r_c^2 \Omega^2} \qquad a_2 = \frac{\Omega}{1-r_c^2 \Omega^2} \qquad a_3 = 0 \tag{84b}$$

where a_i are the curvatures defined in equation II.20.

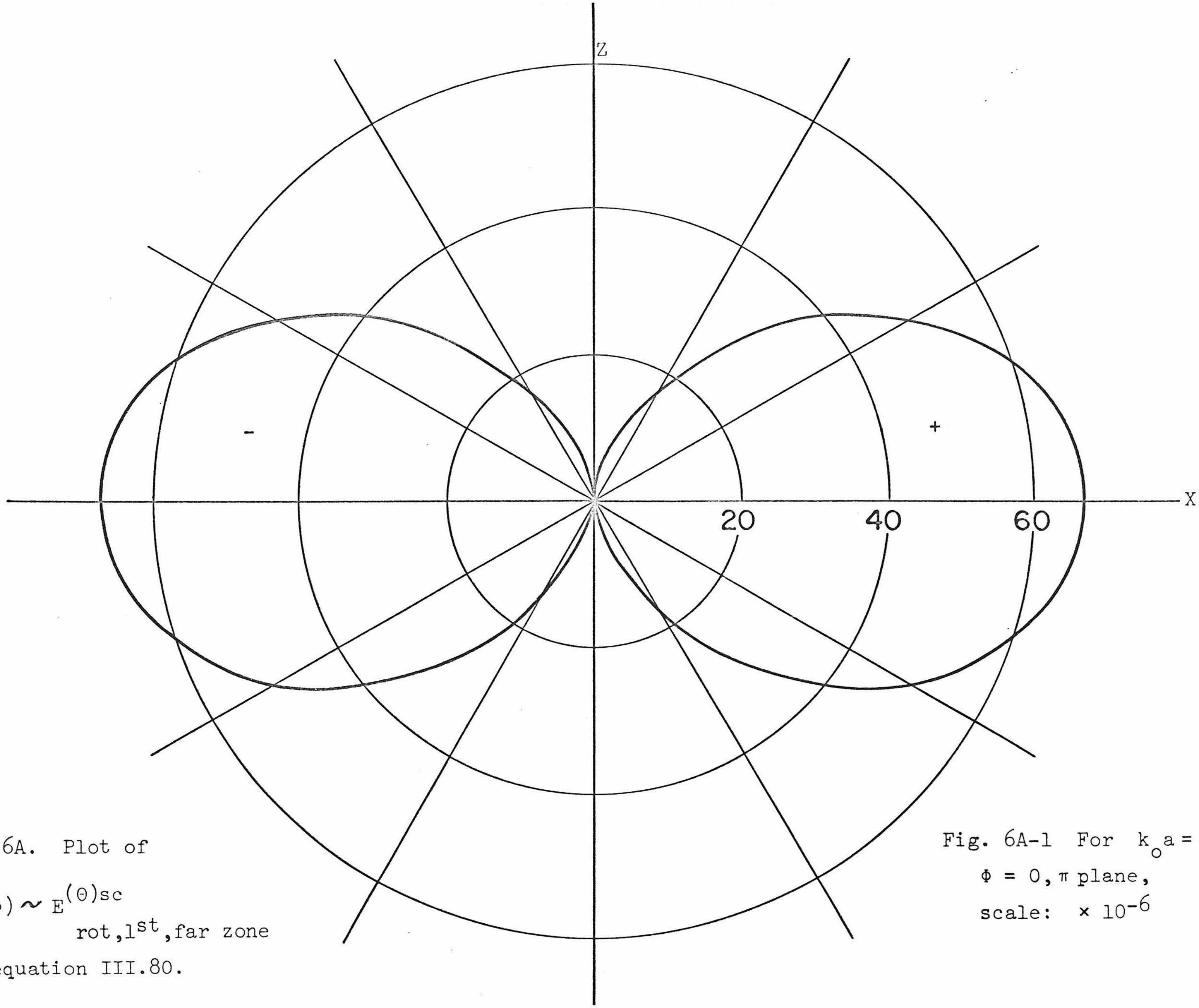


Fig. 6A. Plot of

$f(\Theta,\Phi) \sim E^{(\Theta)sc}_{rot,1^{st},far\ zone}$

of equation III.80.

Fig. 6A-1 For $k_o a = 0.1$,

$\Phi = 0, \pi$ plane,

scale: $\times 10^{-6}$

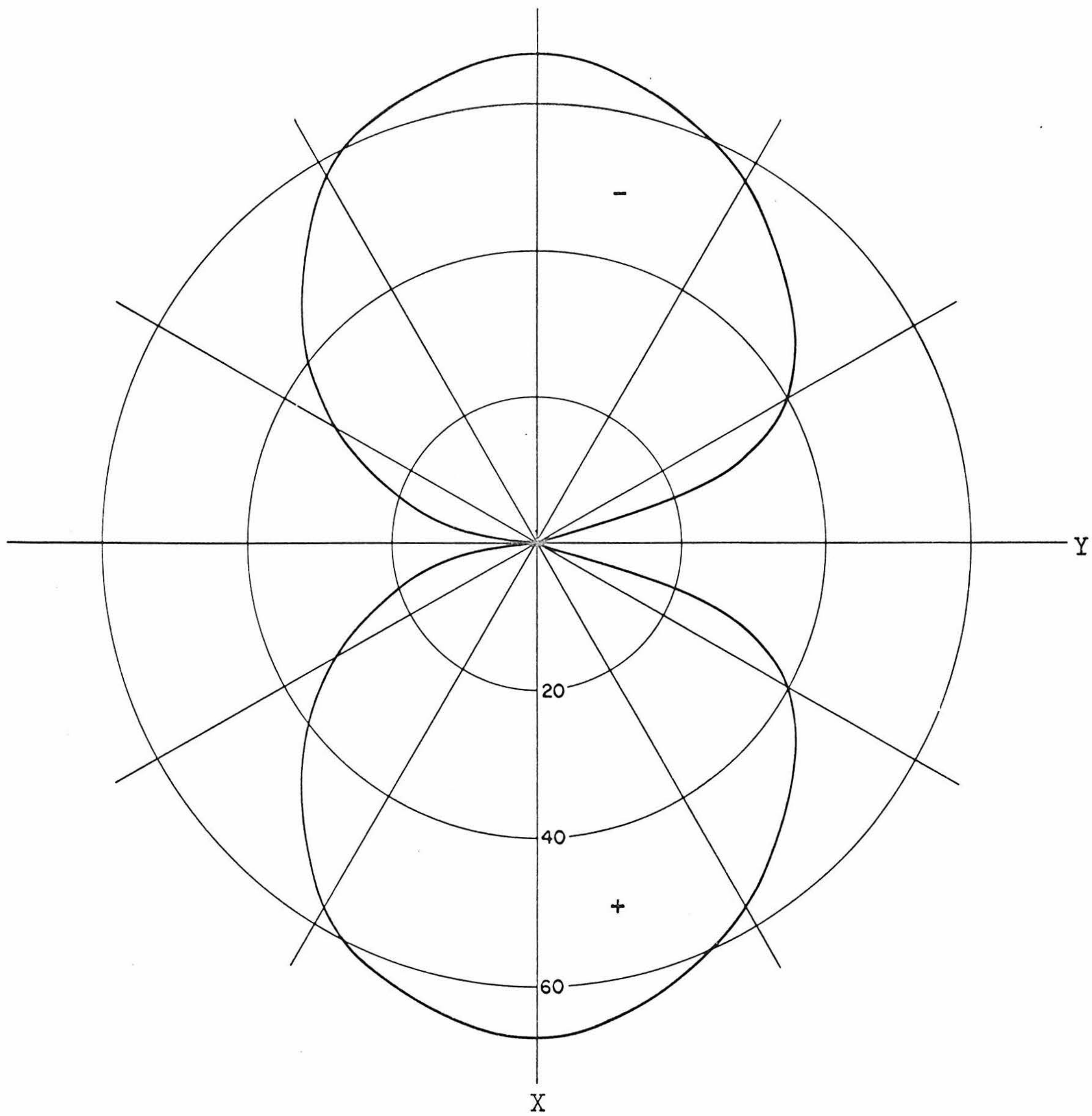


Fig. 6A-2 $k_o a = 0.1$, $\Theta = \frac{\pi}{2}$, scale: $\times 10^{-6}$

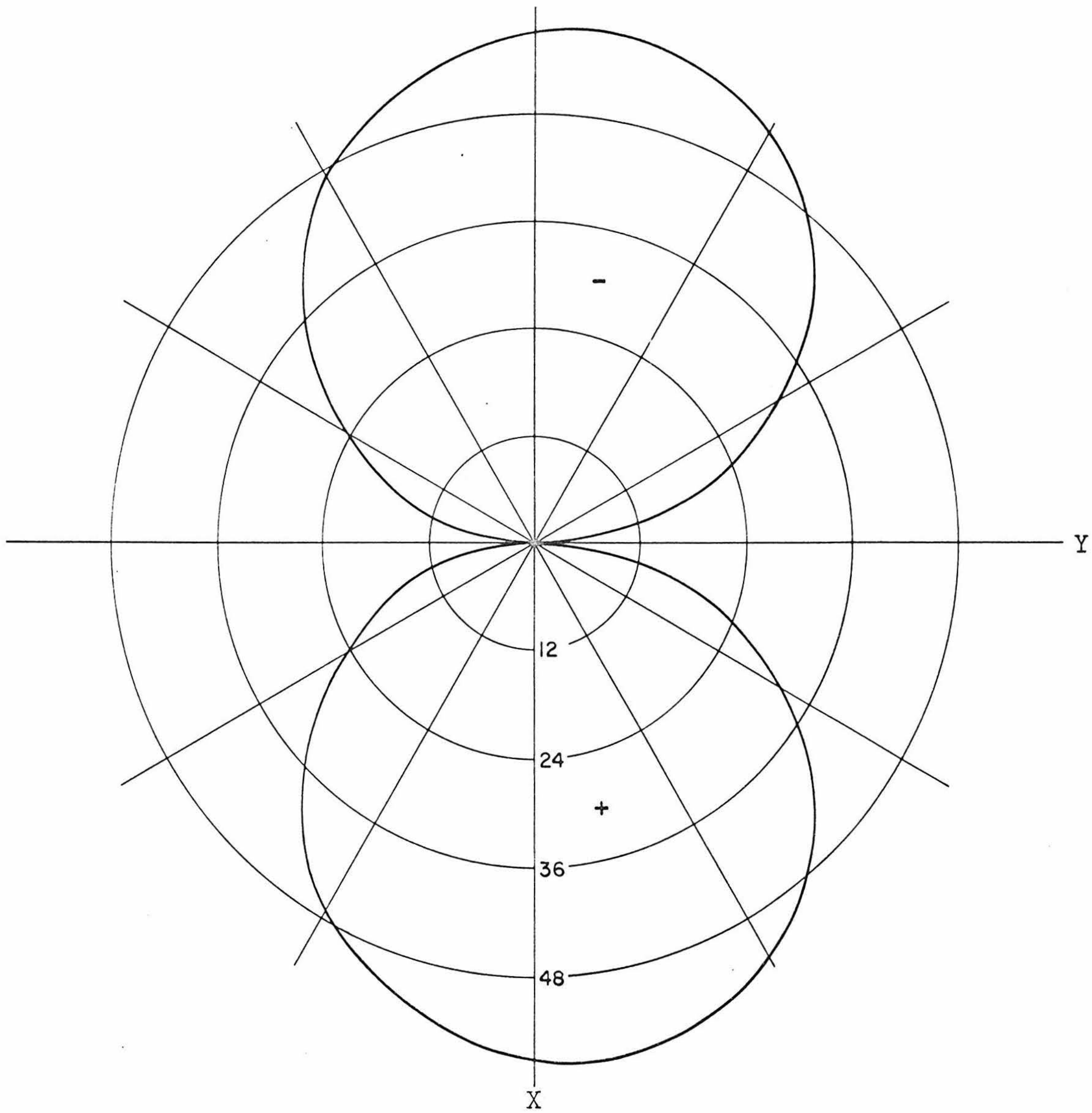


Fig. 6A-3 ka = 1 , $\Theta = \frac{\pi}{2}$, scale: $\times 10^{-3}$

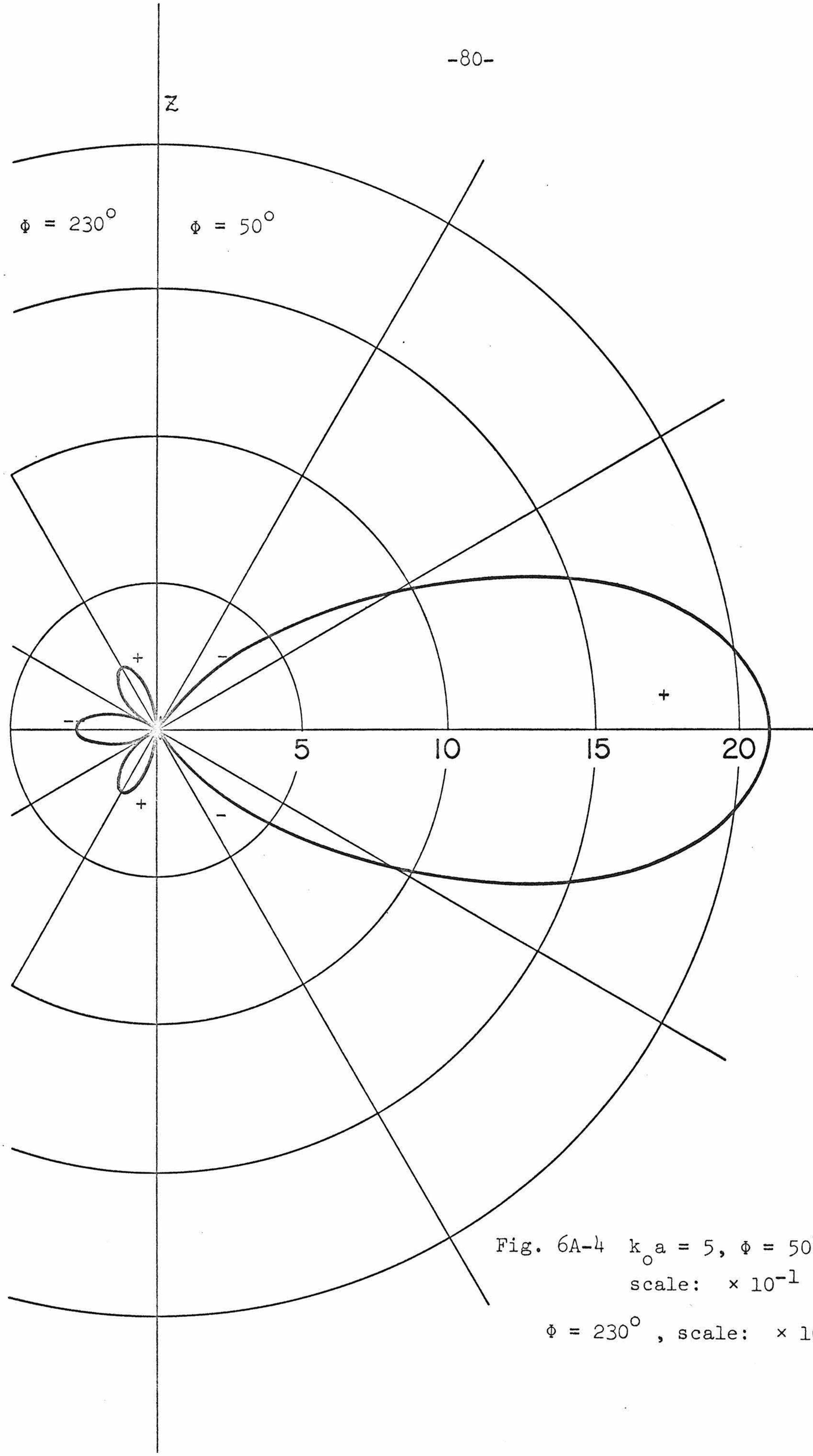


Fig. 6A-4 $k_o a = 5$, $\Phi = 50^{\circ}$, scale: $\times 10^{-1}$

$\Phi = 230^{\circ}$, scale: $\times 10^{-2}$

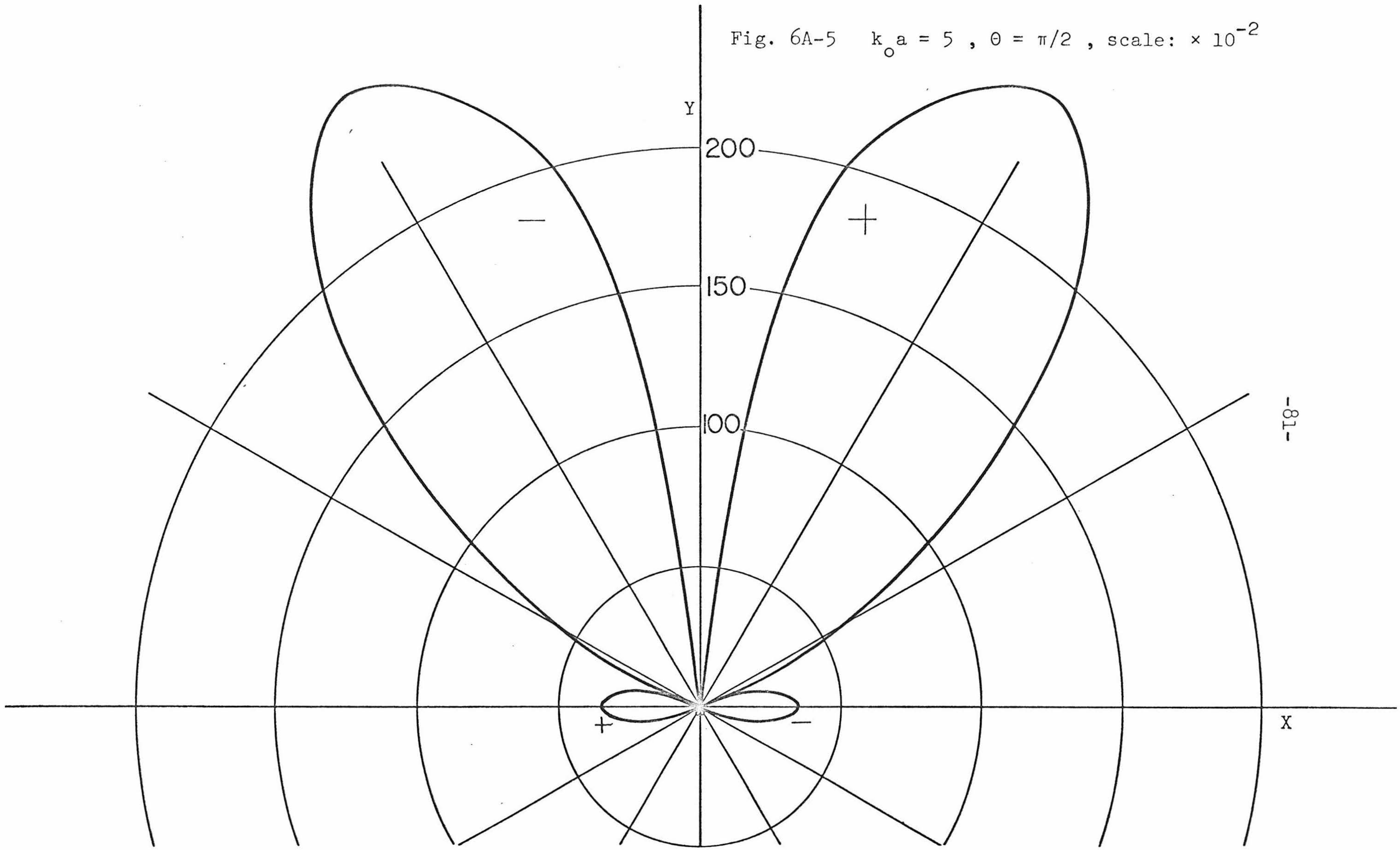


Fig. 6A-5 $k_o a = 5$, $\Theta = \pi/2$, scale: $\times 10^{-2}$

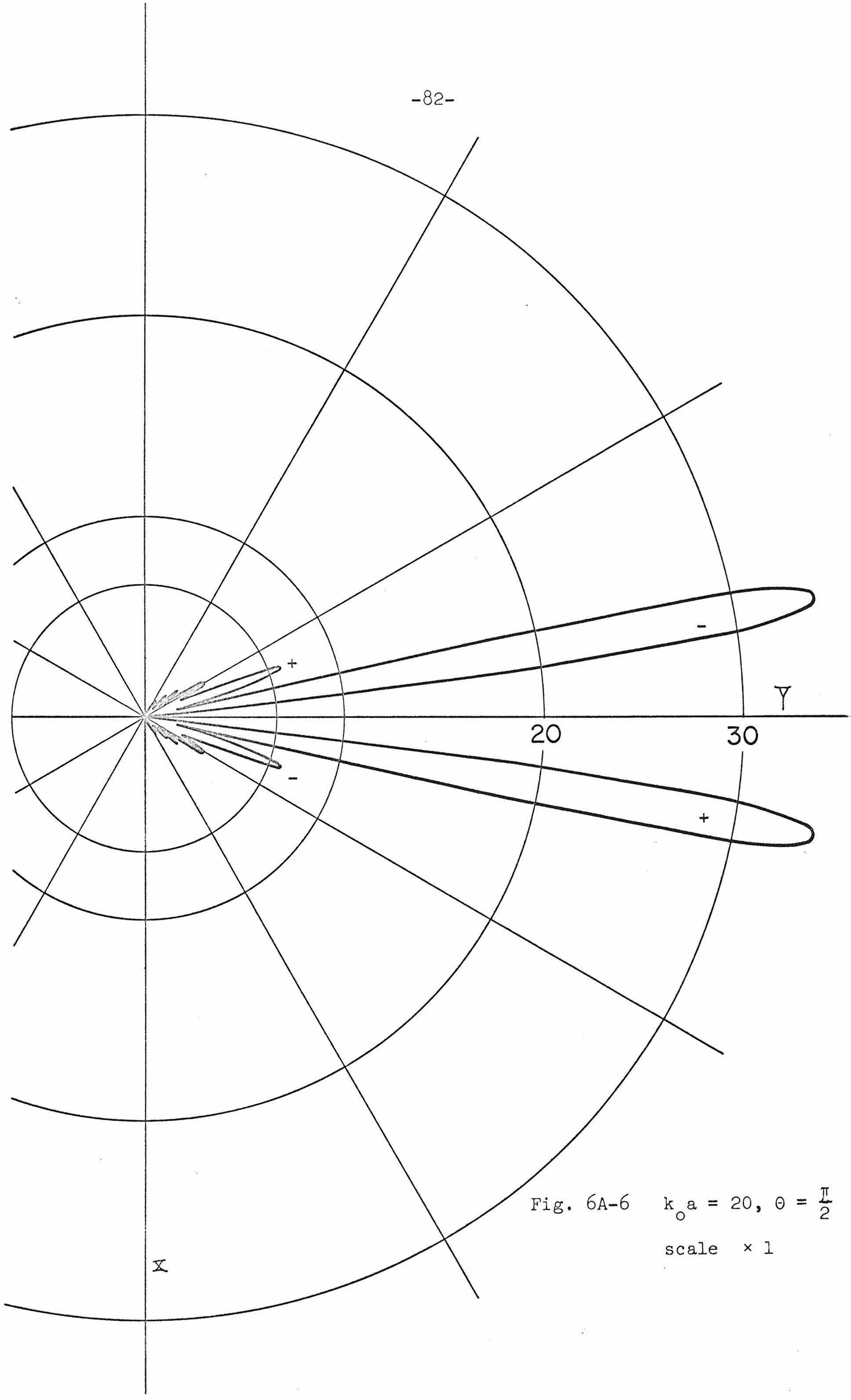


Fig. 6A-6 $k_o a = 20$, $\Theta = \frac{\pi}{2}$

scale × 1

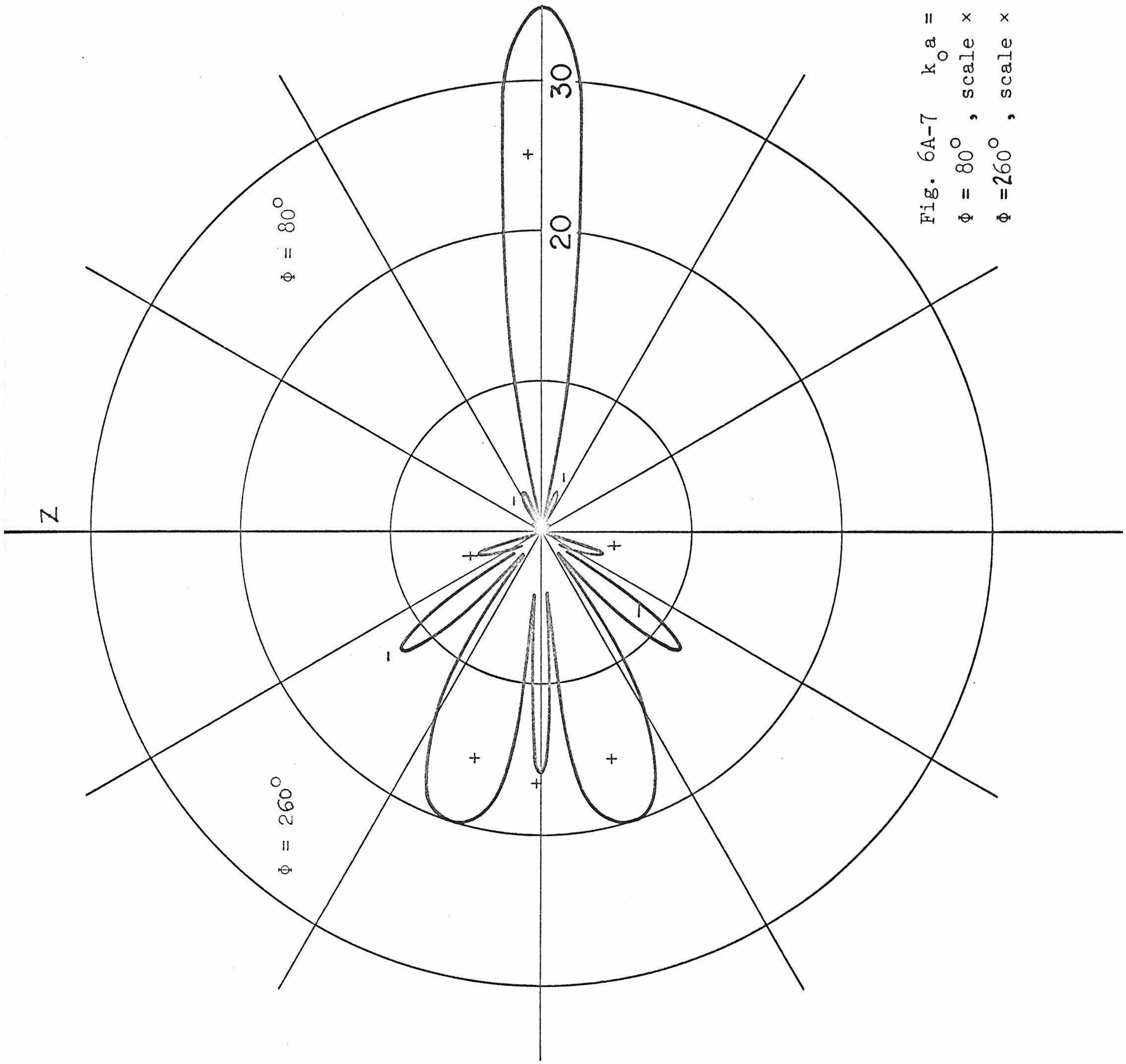


Fig. 6A-7 $k_o a = 20$
$\Phi = 80^\circ$, scale × 1
$\Phi = 260^\circ$, scale × 10^{-3}

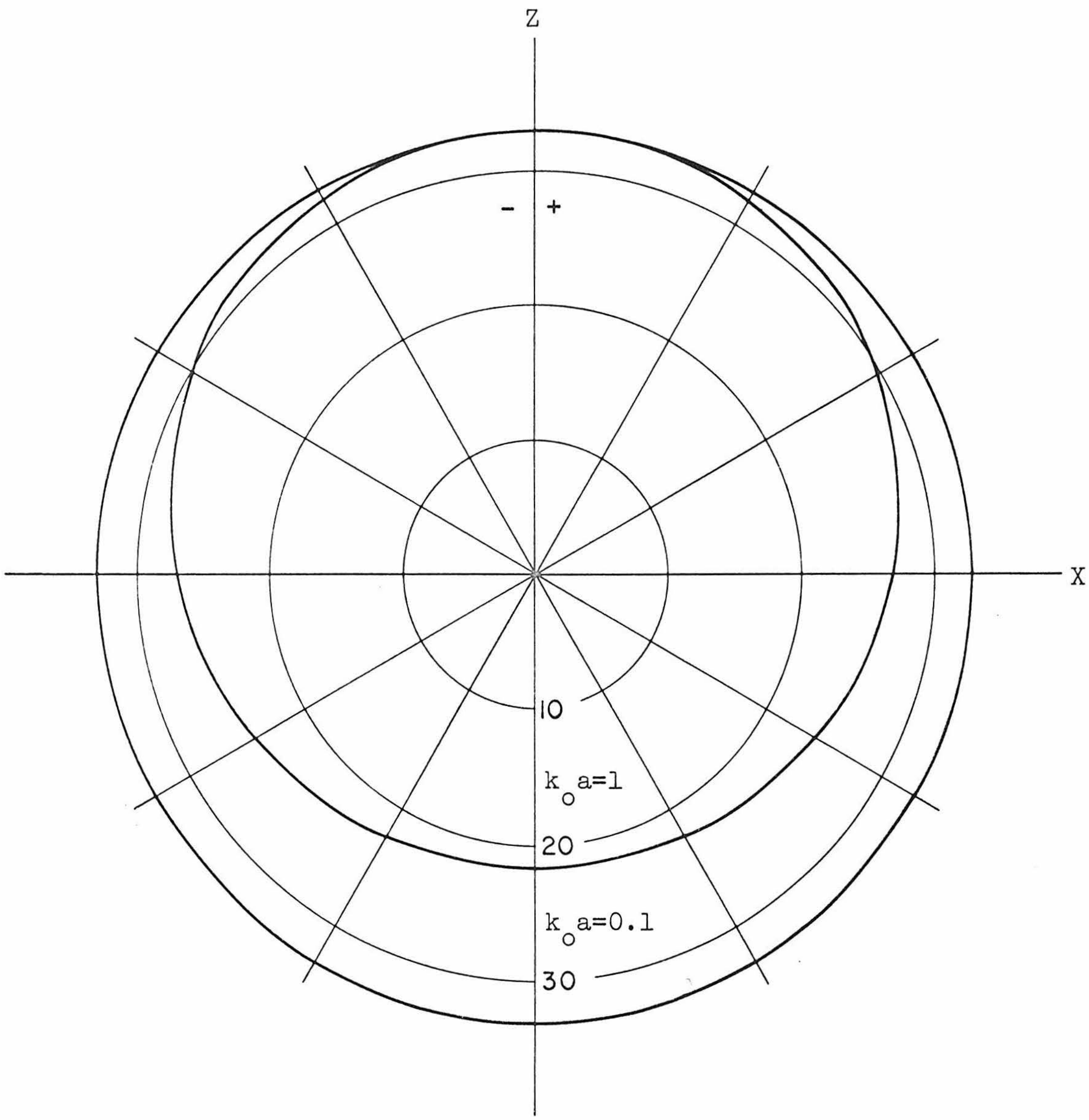


Fig. 6B Plots of II.83

Fig. 6B.1 $f^{(\Phi)}$ at $\Phi = 0,\pi$, $k_o a' = 0.1, 1$
scale: $\times 10^{-2}$

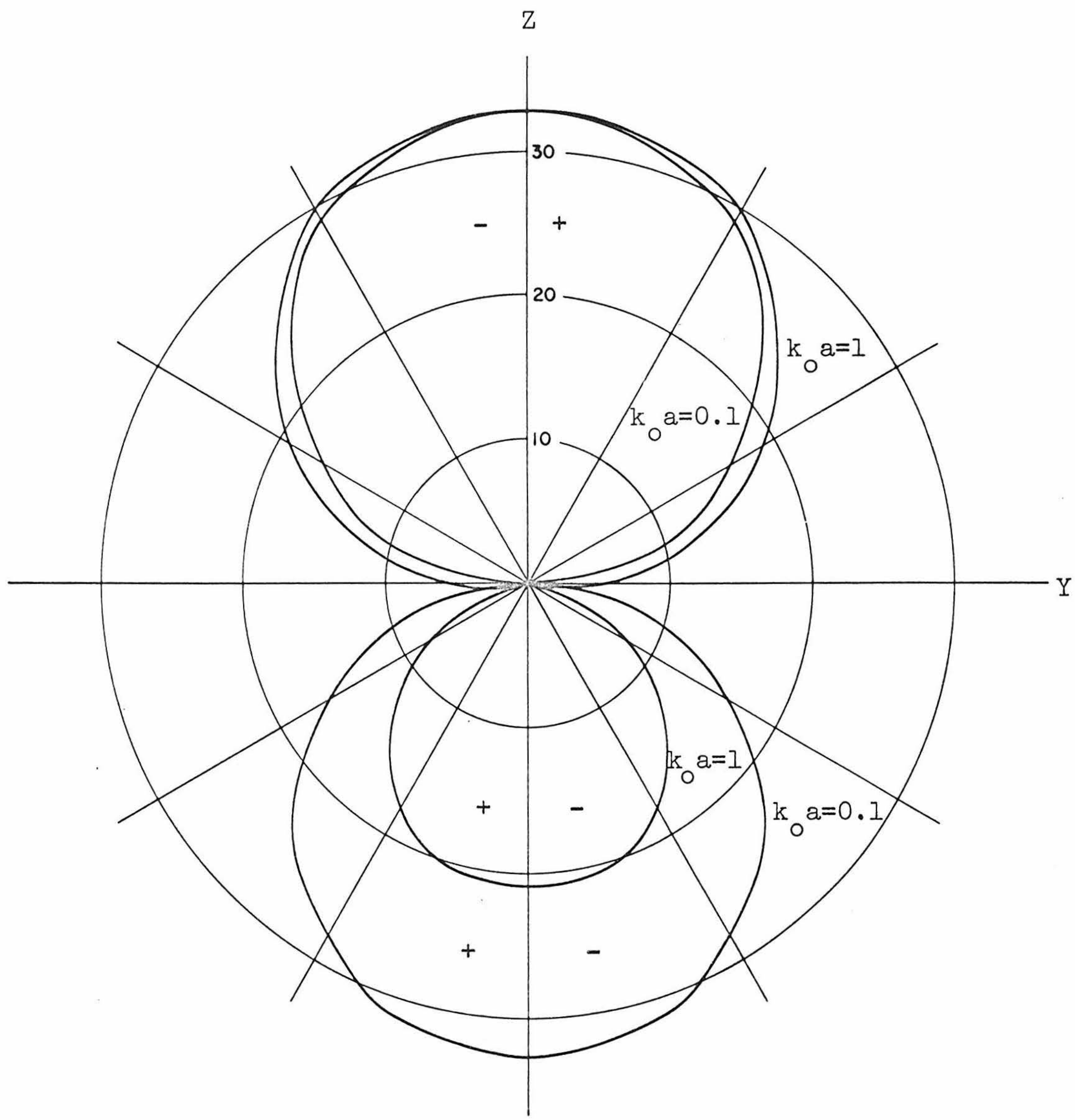


Fig. 6B.2 $f^{(\Theta)}$ at $\Phi = \frac{\pi}{2}, \frac{3\pi}{2}$, $k_o a = 0.1, 1$

scale: $\times 10^{-2}$

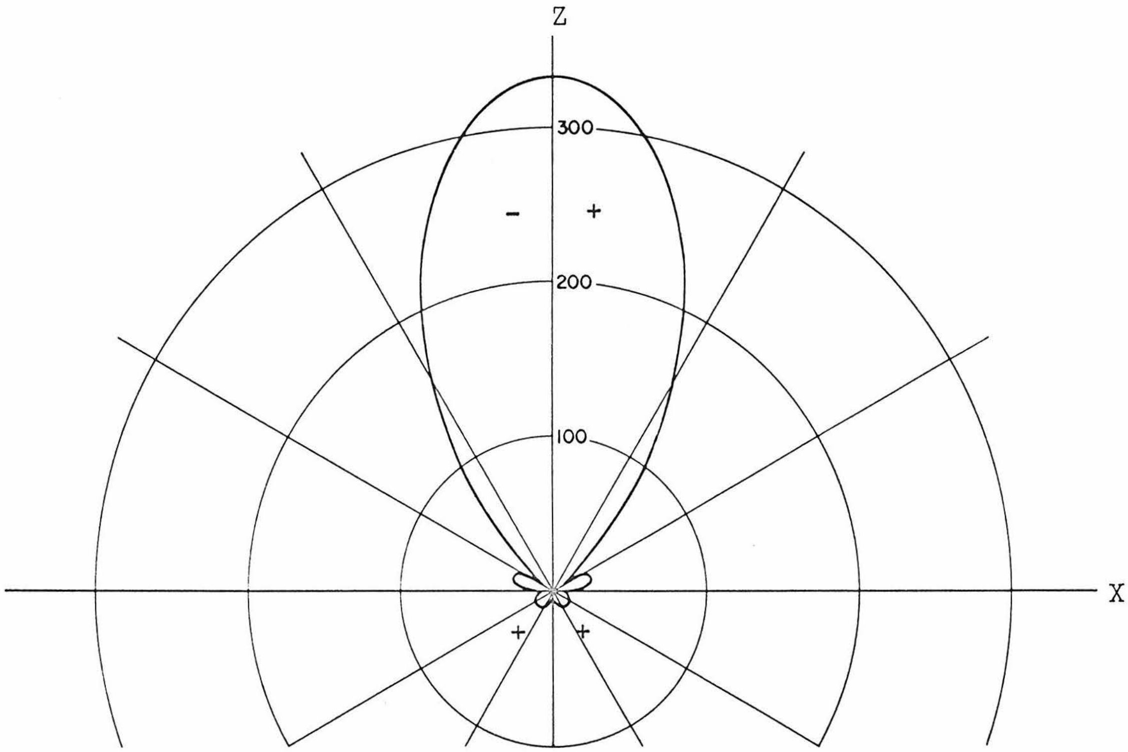


Fig. 6B.3 $f^{(\Phi)}$ at $\Phi = 0$,

$k_o a = 5$, scale: $\times 10^{-3}$

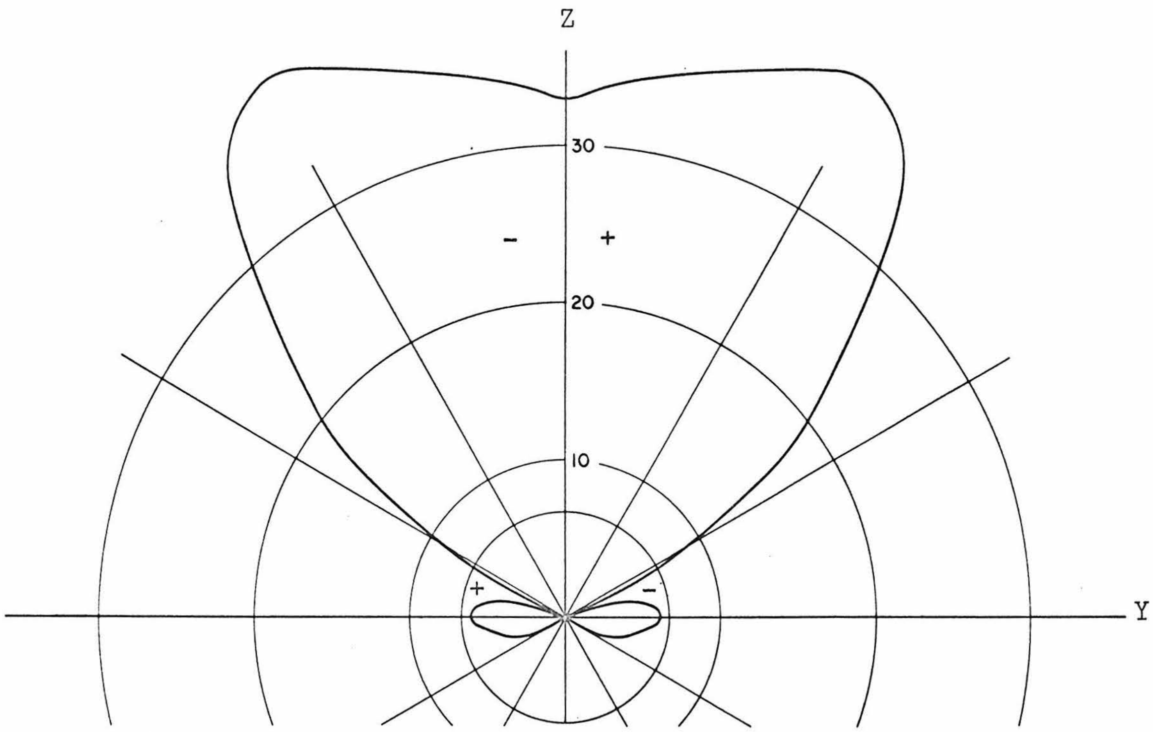


Fig. 6B.4 $f^{(\theta)}$ at $\Phi = \frac{\pi}{2}, \frac{3\pi}{2}$

$k_o a = 5$, scale: $\times 10^{-2}$

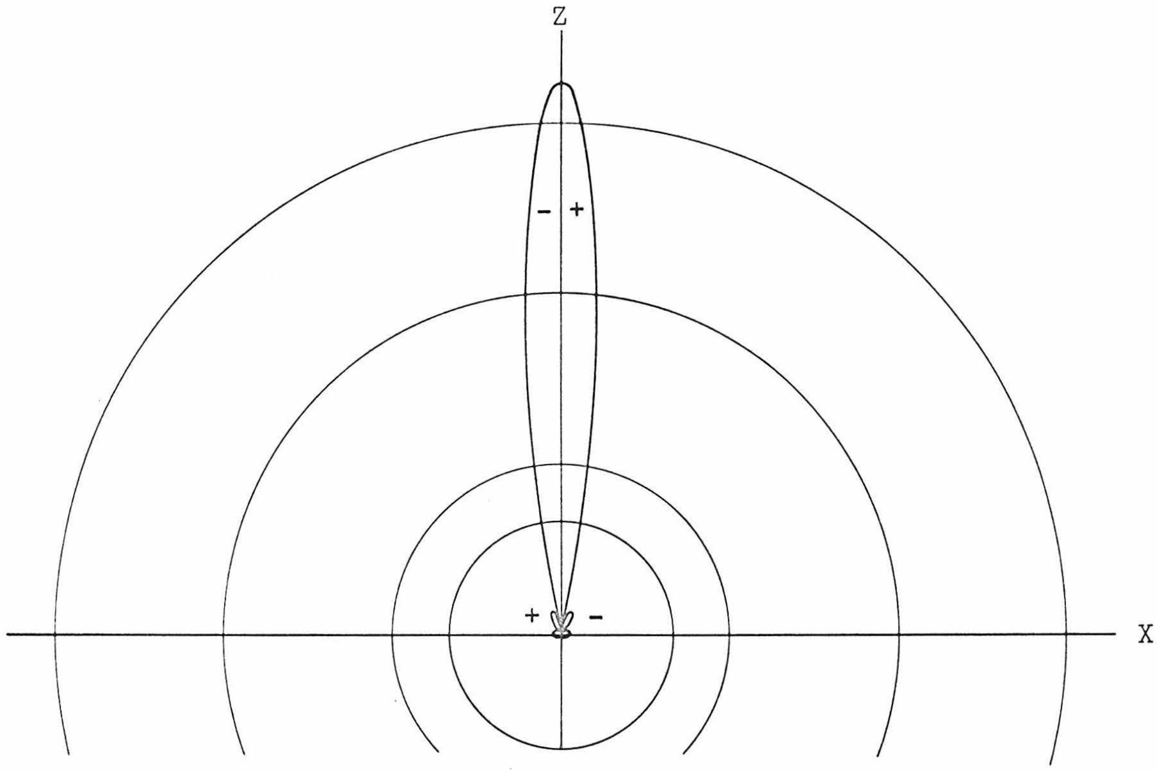
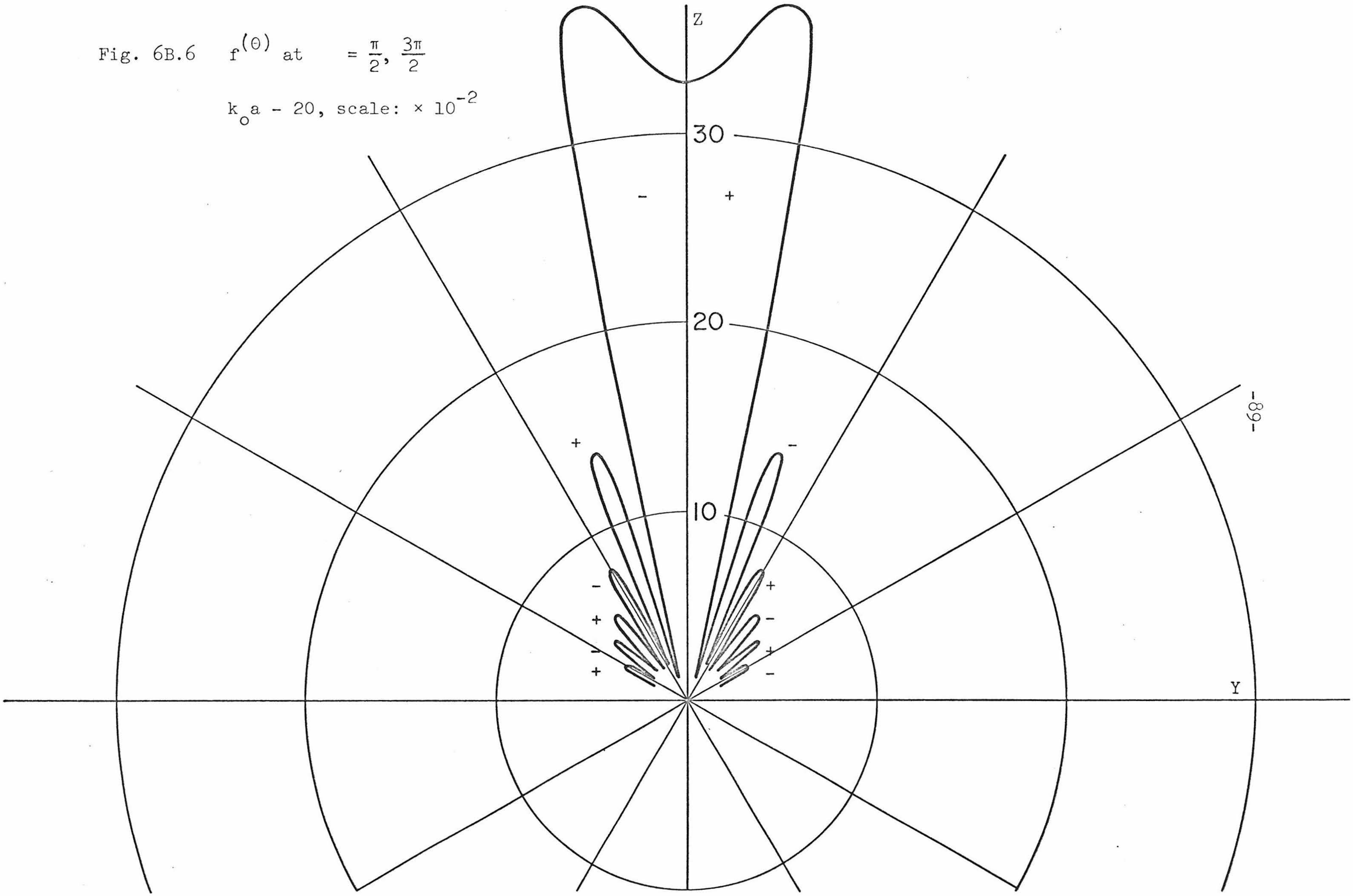


Fig. 6B.5 $f^{(\Phi)}$ at $\Phi = 0, \pi$

$k_o a = 20$, scale: $\times 10^{-3}$

Fig. 6B.6 $f^{(\Theta)}$ at $= \frac{\pi}{2}, \frac{3\pi}{2}$

$k_o a$ - 20, scale: × 10^{-2}



APPENDIX I

CONVERSION TABLE FROM GEOMETRIZED TO mks UNITS**

M = Meter, Kg = Kilogram, S = Second, Q = Coulomb, in geometrized unit c = 1, G = 1, k = 1; in mks units $G = 6.67 \times 10^{-11}$, $k = 1.3805 \times 10^{-23}$

QUANTITY	mks	DIMENSION	GEOMETRIZED	DIMENSION
length	ℓ	M	$\ell^* = \ell$	M
time	t	S	$t^* = ct$ $= 2.997925 \times 10^{8} t$	M
mass	m	Kg	$m^* = \frac{G}{c^2} m$ $= 0.742 \times 10^{-27} m$	M
charge	q	Q	$q^* = q \frac{1}{c^2}\sqrt{\frac{G}{\varepsilon_o}}$ $= 3.042 \times 10^{-17} q$	M
temperature	T	^oK	$T^* = \frac{Gk}{c^4} T$ $= 1.14 \times 10^{-67} T$	M
force	F	$\frac{Kg - M}{S^2}$ (Newton)	$F^* = \frac{G}{c^4} F$ $= 0.826 \times 10^{-44} F$	dimen-sionless
pressure	p	$\frac{Kg}{MS^2}$	$p^* = \frac{G}{c^4} p$	M^{-2}
energy	$\mathcal{E}$	$\frac{Kg - M^2}{S^2}$ (Joule)	$\mathcal{E}^* = \frac{G}{c^4} \mathcal{E}$	M
frequency	f	$\frac{1}{S}$	$f^* = f/c$	M^{-1}

**Mechanical parts from Ref. 26.

QUANTITY	mks	DIMENSION	GEOMETRIZED	DIMENSION
mass density	ρ_m	$\frac{Kg}{M^3}$	$\rho^*_m = \frac{G}{c^2}$	M^{-2}
entropy	S	$\frac{Kg\text{-}M^2}{S^2\text{-}{}^oK}$ (Joule/^{o}K)	$S^* = S/k$ $= 7.2435 \times 10^{22} S$	dimen- sionless
Planck constant	$\hbar$	$\frac{Kg\text{-}M^2}{S}$	$\hbar^* = \frac{G}{c^3}\hbar$ $= 2.61 \times 10^{-70}$	M^2
velocity	$\underset{\sim}{V}$	$\frac{M}{S}$	$\underset{\sim}{V}^* = \underset{\sim}{V}/c$	dimen- sionless
power	W	$\frac{Kg\text{-}M^2}{S^3}$ (Watt)	$W^* = \frac{G}{c^5} W$ $= 0.275 \times 10^{-52} W$	dimen- sionless
current	i	$\frac{Q}{S}$ (Amp)	$i^* = i \frac{1}{c^3}\sqrt{\frac{G}{\varepsilon_o}}$ $= 1.014 \times 10^{-25} i$	dimen- sionless
current density	$\underset{\sim}{J}$	$\frac{Q}{S - M^2}$	$\underset{\sim}{J}^* = \underset{\sim}{J} \cdot \frac{1}{c^3}\sqrt{\frac{G}{\varepsilon_o}}$	M^{-2}
volume electric charge density	ρ_q	$\frac{Q}{M^3}$	$\rho^*_q = \rho \frac{1}{c^2}\sqrt{\frac{G}{\varepsilon_o}}$	M^{-2}
surface electric charge density	σ_q	$\frac{Q}{M^2}$	$\sigma^*_q = \sigma_q \frac{1}{c^2}\sqrt{\frac{G}{\varepsilon_o}}$	M^{-1}
electric field	$\underset{\sim}{E}$	$\frac{Kg\text{-}M}{Q\ \text{-}S^2}$ (Newton/ Coulomb)	$\underset{\sim}{E}^* = \underset{\sim}{E}\frac{\sqrt{\varepsilon_o G}}{c^2}$ $= 2.68 \times 10^{-28} \underset{\sim}{E}$	M^{-1} M^{-1}

QUANTITY	mks	DIMENSION	GEOMETRIZED	DIMENSION
magnetic flux density	$\underset{\sim}{B}$	$\frac{Kg}{S - Q}$ (Weber/M^2)	$\underset{\sim}{B}^* = \underset{\sim}{B}\frac{\sqrt{\varepsilon_o G}}{c}$ $= 0.811 \times 10^{-19}\underset{\sim}{B}$	M^{-1}
electric displacement	$\underset{\sim}{D}$	$\frac{Q}{M^2}$	$\underset{\sim}{D}^* = \underset{\sim}{D}\frac{1}{c^2}\sqrt{\frac{G}{\varepsilon_o}}$	M^{-1}
magnetic intensity	$\underset{\sim}{H}$	$\frac{Q}{S - M}$	$\underset{\sim}{H}^* = \underset{\sim}{H}\frac{\mu_o}{c}\sqrt{G\varepsilon_o}$ $= \underset{\sim}{H}\frac{1}{c^3}\sqrt{\frac{G}{\varepsilon_o}}$	M^{-1}
volt	v	$\frac{Kg\text{-}M^2}{Q - S^2}$ (Joule/ Coulomb)	$v^* = v\frac{1}{c^2}\sqrt{\varepsilon_o G}$	dimen-sionless
conductivity	σ	$\frac{Q^2 - S}{M^3 - Kg}$	$\sigma^* = \sigma\frac{1}{c\varepsilon_o}$ $= 3.768 \times 10^2\sigma$	M^{-1}
dielectric constant	ε	$\frac{Q^2 - S^2}{Kg - M^3}$ (Farad/M)	$\varepsilon^* = \frac{\varepsilon}{\varepsilon_o} = 36\pi \times 10^9\varepsilon$	dimen-sionless
permeability	μ	$\frac{Kg - M}{Q^2}$ (henry/M)	$\mu^* = \frac{\mu}{\mu_o} = \frac{10^7}{4\pi}\mu$	

APPENDIX II

PROOF OF TENSOR PROPERTIES OF $\underline{e}_\mu$, $\underline{e}^\mu$ AND DERIVATION OF II-18

By definition, II-1 implies

$$\underline{e}_\mu = \frac{\partial x^{\bar{\alpha}}}{\partial x^\mu} \underline{e}_{\bar{\alpha}} \tag{AII-1}$$

Now the contravariant components of $\underline{e}_o$ in $\{x^\mu\}$ are $(\underline{e}_o)^\alpha \equiv (1,0,0,0)$. Tensor transforms these components to $\{x^{\bar{\mu}}\}$ and we get

$$\frac{\partial x^{\bar{0}}}{\partial x^o} , \frac{\partial x^{\bar{1}}}{\partial x^o} , \frac{\partial x^{\bar{2}}}{\partial x^o} , \frac{\partial x^{\bar{3}}}{\partial x^o}$$

which, by AII-1 are indeed the contravariant components of $\underline{e}_o$ in $\{x^{\bar{\mu}}\}$. So $\underline{e}_o$ is a vector. Thus $\underline{e}_\mu$ for all μ are vectors, and so are all $\underline{e}^\mu$. Their components are:

$$(\underline{e}_\mu)^\nu = \underline{e}_\mu \cdot \underline{e}^\nu = \delta_\mu{}^\nu$$

$$(\underline{e}_\mu)_\nu = \underline{e}_\mu \cdot \underline{e}_\nu = g_{\mu\nu} \tag{AII-2}$$

At a fixed point $(x^i \equiv \text{fixed})$ in $\{x^\mu\}$, the local spatial (time orthogonal) coordinate vectors are

$$\underline{d}_i \equiv \underline{e}_i - \frac{g_{oi}}{g_{oo}} \underline{e}_o$$

If the coordinate is rigid, then $\{\underline{d}_i\}$ do not change direction with respect to each other, i.e., they can only change their magnitudes and adjust their contravariant time components to keep the time orthogonality, but

$$\frac{\underline{d}_i}{\sqrt{\underline{d}_i \cdot \underline{d}_i}} \cdot \frac{\underline{d}_j}{\sqrt{\underline{d}_j \cdot \underline{d}_j}} \equiv \frac{\gamma_{ij}}{\sqrt{\gamma_{ii}\ \gamma_{jj}}} = \text{constant in time} \tag{AII-3}$$

APPENDIX III

DERIVATION AND LOCAL APPROXIMATION OF II-49

From tensor equation II-46a we have

$$(\sqrt{-g}\, G^{\mu\nu})_{,\nu} = -\sqrt{-g}\, J^{\mu}$$

or

$$(\sqrt{-g}\, e^{\mu}{}_{(\alpha)}\, e^{\nu}{}_{(\beta)}\, G^{(\alpha)(\beta)})_{,\nu} = -\sqrt{-g}\, e^{\mu}{}_{(\alpha)}\, J^{(\alpha)} \qquad \text{(AIII-1)}$$

Thus we obtain for $\mu = 0$

$$(\sqrt{-g}\, e^{o}{}_{(\alpha)}\, e^{\nu}{}_{(\beta)}\, G^{(\alpha)(\beta)}]_{,\nu} = -\sqrt{-g}\, [J^{(o)} \frac{1}{\sqrt{g_{oo}}} + J^{(\nu)}\, e^{o}{}_{(i)} \qquad \text{(AIII-2)}$$

but the left side with the help of II-48 and the dual of $G^{(\alpha)(\beta)}$ is

$$[\sqrt{-g}\, e^{o}{}_{(\alpha)}\, e^{\ell}{}_{(\beta)}\, G^{(\alpha)(\beta)}]_{,\ell}$$

$$= -[\sqrt{-g}\, \frac{1}{\sqrt{g_{oo}}}\, e_{(i)}{}^{\ell}\, D^{(i)}]_{,\ell} - [\sqrt{-g}\, e_{(i)}{}^{o}\, e_{(j)}{}^{\ell}\, \eta^{oijk}\, H_{(k)}]_{,\ell}$$

thus AIII-2 becomes II-49a

$$[\sqrt{-g}\, \frac{e_{(i)}{}^{\ell}}{\sqrt{g_{oo}}}\, D^{(i)}]_{,\ell} + [\sqrt{-g}\, \eta^{oijk}\, e_{(i)}{}^{o}\, e_{(j)}{}^{\ell}\, H_{(k)}]_{,\ell}$$

$$= +\sqrt{-g}\, [\frac{1}{\sqrt{g_{oo}}}\, \rho + e^{o}{}_{(i)} J^{(i)}] \qquad \text{(AIII-3)}$$

Similarly, for $\mu = i$ in AIII-2 we have

$$(\sqrt{g}\; e^{i}{}_{(\alpha)}\; e^{\nu}{}_{(\beta)}\; G^{(\alpha)(\beta)})_{,\nu} = -\sqrt{-g}\; e^{i}{}_{(j)}\; J^{(j)} \qquad \text{(AIII-4)}$$

and the left can be simplified as

$$(\sqrt{-g}\; e^{i}{}_{(j)}\; e^{\nu}{}_{(o)}\; G^{(j)(o)})_{,\nu} + (\sqrt{-g}\; e^{i}{}_{(j)}\; e^{\nu}{}_{(k)}\; G^{(j)(k)})_{,\nu}$$

$$= (\sqrt{-g}\; e^{i}{}_{(j)}\; e^{o}{}_{(o)}\; G^{(j)(o)})_{,o} - [\sqrt{-g}\; e^{i}{}_{(j)}\; e^{\nu}{}_{(k)}\; \eta^{jko\ell}\; ({}^{*}G_{(o)(\ell)}]_{,\nu}$$

$$= +(\sqrt{-g}\; e^{i}{}_{(j)}\; \frac{1}{\sqrt{g_{oo}}}\; D^{(j)})_{,o} - [\sqrt{-g}\; e^{i}{}_{(j)}\; e^{\nu}{}_{(k)}\; \eta^{ojk\ell}\; H_{(\ell)}]_{,\nu}$$

Thus AIII-4 becomes II-49b

$$[\sqrt{-g}\; e^{j}{}_{(i)}\; e^{\nu}{}_{(k)}\; \eta^{oik\ell}\; H_{(\ell)}]_{,\nu} = [\sqrt{-g}\; e^{j}{}_{(i)}\; \frac{1}{\sqrt{g_{oo}}}\; D^{(i)}]_{,o} + \sqrt{-g}\; e^{j}{}_{(i)}\; J^{(i)} \qquad \text{(AIII-5)}$$

Equations II-49c,d are derived similarly.

Now express the coordinate differential operators locally with

$$\underline{dx} \equiv dx^{\mu}\, \underline{e}_{\mu} \equiv dX^{(\mu)}\, \underline{e}_{(\mu)}$$

$$\equiv dx_{\mu}\, \underline{e}^{\mu} \equiv dX_{(\mu)}\, \underline{e}^{(\mu)}$$

then

$$\underline{e}_{(\mu)} = \frac{\partial x^{\alpha}}{\partial X^{(\mu)}}\, \underline{e}_{\alpha} \quad , \quad \underline{e}_{\alpha} = \frac{\partial X^{(\mu)}}{\partial x^{\alpha}}\, \underline{e}_{(\mu)}$$

$$\underline{e}^{(\mu)} = \frac{\partial X^{(\mu)}}{\partial x^{\alpha}}\, \underline{e}^{\alpha} \quad , \quad \underline{e}^{\alpha} = \frac{\partial x^{\alpha}}{\partial X^{(\mu)}}\, \underline{e}^{(\mu)} \qquad \text{(AIII-6)}$$

and

$$e_{(\mu)}{}^{\nu} \equiv \underline{e}_{(\mu)} \cdot \underline{e}^{\nu} = \frac{\partial x^{\nu}}{\partial X^{(\mu)}} \qquad \text{(AIII-7)}$$

Then in the left side of II-49a

$$\left(\sqrt{-g}\, e^{j}{}_{(i)} \frac{1}{\sqrt{g_{oo}}}\right)_{,j} D^{(i)} + \underline{\sqrt{-g} \frac{1}{\sqrt{g_{oo}}} e^{j}{}_{(i)} \frac{\partial}{\partial x^{j}} D^{(i)}}$$

$$+ \left(\sqrt{-g}\, \eta^{ojik}\, e^{o}{}_{(j)}\, e^{\ell}{}_{(i)}\right)_{,\ell} H_{(k)} + \underline{\sqrt{-g}\, \eta^{ojik}\, e_{(j)}{}^{o}\, e_{(i)}{}^{\ell} \frac{\partial}{\partial x^{\ell}} H_{(k)}}$$

the underlined terms become

$$\sqrt{-g} \frac{1}{\sqrt{g_{oo}}} \left(e^{\mu}{}_{(i)} \frac{\partial}{\partial x^{\mu}} D^{(i)} - e^{o}{}_{(i)} \frac{\partial}{\partial x^{o}} D^{(i)}\right) + \sqrt{-g}\, \eta^{ojik}\, e^{o}{}_{(j)} e^{\mu}{}_{(i)} \frac{\partial}{\partial x^{\mu}} H_{(k)}$$

$$= \sqrt{-g} \frac{1}{\sqrt{g_{oo}}} \left(\frac{\partial}{\partial X^{(i)}} D^{(i)} - e^{o}{}_{(i)} \frac{\partial}{\partial x^{o}} D^{(i)}\right) + \sqrt{-g}\, \eta^{ojik}\, e^{o}{}_{(j)} \frac{\partial}{\partial X^{(i)}} H_{(k)}$$

Then II-49a can be rewritten as

$$\sqrt{-g} \frac{1}{\sqrt{g_{oo}}} \left[\frac{\partial D^{(i)}}{\partial X^{(i)}} - \rho\right] + \sqrt{-g}\, e^{o}{}_{(j)} \left[\eta^{ojik} \frac{\partial}{\partial X^{(i)}} H_{(k)} - \frac{\partial}{\partial X^{o}} D^{(j)} - J^{(j)}\right]$$

$$+ \left(\sqrt{-g}\, e^{j}{}_{(i)} \frac{1}{\sqrt{g_{oo}}}\right)_{,j} D^{(i)} + \left(\sqrt{-g}\, \eta^{ojik}\, e^{o}{}_{(j)}\, e^{\ell}{}_{(i)}\right)_{,\ell} H_{(k)} = 0 \qquad \text{(AIII-8)}$$

Similarly, II-49b can be rewritten as

$$e^{j}{}_{(i)} \sqrt{-g}\, [\eta^{oijk} \frac{\partial}{\partial X^{(j)}} H_{(k)} - J^{(i)} - \frac{\partial}{\partial X^{o}} D^{(i)}]$$

$$- (\sqrt{-g}\, e^{j}{}_{(i)} \frac{1}{\sqrt{g_{oo}}})_{,o} D^{(i)} + (\sqrt{-g}\, \eta^{oik\ell} e^{j}{}_{(i)} e^{\nu}{}_{(k)})_{,\nu} H_{(\ell)} = 0 \qquad \text{(AIII.9)}$$

Then we can clearly see that the local Cartesian equations II-50a are approximations when the change of $g_{\mu\nu}$ in a small enough neighborhood is neglected. The same remarks apply for II-50b.

APPENDIX IV

FERMI TRANSPORT OF THE $\{\underline{e}_{(i)}\} \perp \underline{u}$ ALONG Γ

$$\frac{D}{Ds} (\underline{e}_{(i)})^{\mu} = -u^{\mu} (\underline{e}_{(i)})_{\lambda} \frac{Du^{\lambda}}{Ds}$$

then the preservation of keeping $\perp \underline{u}$ is achieved by adjusting their orientations with respect to the time direction $\underline{u}$; but spatially they are parallel transported, i.e., spatial directions are fixed with respect to distant stars.

APPENDIX V

LOCAL EM ENERGY DENSITY AND

SYMMETRY PROPERTIES OF $\varepsilon^{\mu\nu}$, $K^{\mu\nu}$, $C^{\mu\nu\alpha\beta}$ FOR LOSSLESS AND LOSSY MEDIA

From the local energy balance in SRT, if the medium is lossy, then the average loss per cycle of field is

$$\delta W_{loss} = \oint \underset{\sim}{E} \cdot d\underset{\sim}{D} + \oint \underset{\sim}{H} \cdot d\underset{\sim}{B}$$

$$\equiv \oint \underset{\sim}{E} \cdot d(\underset{S}{\underset{\sim}{D}} + \underset{A}{\underset{\sim}{D}}) + \oint (\underset{S}{\underset{\sim}{H}} + \underset{A}{\underset{\sim}{H}}) \cdot d\underset{\sim}{B} \qquad \text{(AV.1)}$$

where

$$\underset{S}{D}^{(i)} \equiv \underset{S}{\varepsilon}^{(i)}{}_{(j)}\, E^{(j)} \quad , \quad \underset{S}{\varepsilon}^{(i)}{}_{(j)} = \underset{S}{\varepsilon}{}_{(j)}{}^{(i)}$$

$$\underset{A}{D}^{(i)} \equiv \underset{A}{\varepsilon}^{(i)}{}_{(j)}\, E^{(j)} \quad , \quad \underset{A}{\varepsilon}^{(i)}{}_{(j)} = -\underset{A}{\varepsilon}{}_{(j)}{}^{(i)} \qquad \text{(AV.2)}$$

Similar definitions apply for H, in that we separate the symmetric and antisymmetric part of $K^{(i)}{}_{(j)}$. Then, for time-independent linear medium

$$(\delta W_{loss})_{\underset{\sim}{E}\ part} = \oint E^{(i)} \underset{S}{\varepsilon}_{(i)(j)}\, dE^{(j)} + \oint E^{(i)} \underset{A}{\varepsilon}_{(i)(j)}\, dE^{(j)}$$

$$= \oint \underset{S}{\underset{\sim}{D}} \cdot d\underset{\sim}{E} - \oint \underset{A}{\underset{\sim}{D}} \cdot d\underset{\sim}{E}$$

$$= \int_{a_E} \underset{\sim}{\nabla}_E \times (\underset{S}{\underset{\sim}{D}} - \underset{A}{\underset{\sim}{D}}) \cdot d\underset{\sim}{a}_E$$

$$= \int_{a_E} \eta^{ijk} \frac{\partial}{\partial E^{(j)}} \left(\underset{S}{\varepsilon}_{(k)(\ell)} E^{(\ell)} - \underset{A}{\varepsilon}_{(k)(\ell)} E^{(\ell)} \right) \cdot d\underset{\sim}{a}_E$$

$$= + \int_{a_E} \eta^{ijk} \underset{A}{\varepsilon}_{(j)(k)} \cdot d\underset{\sim}{a}_E$$

$$\Rightarrow (\delta W_{loss})_{\underset{\sim}{E}\ part} = - \oint \underset{A}{\underset{\sim}{D}} \cdot d\underset{\sim}{E} \qquad (AV.3)$$

Similarly, we have

$$(\delta W_{loss})_{\underset{\sim}{B}\ part} = \oint \underset{A}{\underset{\sim}{H}} \cdot dB \qquad (AV.4)$$

Thus the loss to media for $\delta\underset{\sim}{B}$, $\delta\underset{\sim}{E}$ change

$$\delta W_{loss} = \underset{A}{\underset{\sim}{H}} \cdot \delta\underset{\sim}{B} - \underset{A}{\underset{\sim}{D}} \cdot \delta\underset{\sim}{E} \equiv \frac{1}{2} \underset{A}{G}^{(\mu)(\nu)} \delta F_{(\mu)(\nu)} = \frac{1}{2} \underset{A}{G}^{\mu\nu} \delta F_{\mu\nu} \qquad (AV.5)$$

is totally due to the antisymmetric part of $\varepsilon^{(i)(j)}$, $K^{(i)(j)}$, thus due to the antisymmetric part of $\varepsilon^{\mu\nu}$, $K^{\mu\nu}$, or the $(\mu\nu) \rightleftarrows (\alpha\beta)$ antisymmetric part of $c^{\mu\nu\alpha\beta}$.

Now for a lossless medium. $\delta W_{loss} \equiv 0$, then $\underset{A}{\varepsilon}^{(i)}{}_{(j)} \equiv 0$, $\underset{A}{K}^{(i)}{}_{(j)} \equiv 0$ and the stored energy is, from AIV-1

$$U = \int_0^T dT \left(\underset{\sim}{E} \cdot \frac{\partial \underset{\sim}{D}}{\partial T} + \underset{\sim}{H} \cdot \frac{\partial \underset{\sim}{E}}{\partial T} \right) \qquad (AV.6)$$

Consider the $\underset{\sim}{E}, \underset{\sim}{B}$ being built from 0 to $\underset{\sim}{E}, \underset{\sim}{B}$ during time 0 to T; also suppose that $\varepsilon^{(i)}{}_{(j)}$, $K^{(i)}{}_{(j)}$ are time independent. Being lossless, the EM energy storage is only a function of final field states and we can break AIV-6 into three parts:

1. $E^{(1)}$ is built from 0 to $E^{(1)}$, with $E^{(2)} = 0 = E^{(3)}$.
2. $E^{(2)}$ is built from 0 to $E^{(2)}$, with $E^{(1)}$ = const., $E^{(3)} = 0$.
3. $E^{(3)}$ is built from 0 to $E^{(3)}$, with $E^{(1)}$ = const., $E^{(2)}$=const.

Then the electric part of AIV-6 is

$$U_e = \int_0^{E^{(1)}} E^{(i)} \varepsilon_{(i)(1)} \, dE^{(1)} + \int_0^{E^{(2)}} E^{(i)} \varepsilon_{(i)(2)} \, dE^{(2)} + \int_0^{E^{(3)}} E^{(i)} \varepsilon_{(i)(3)} \, dE^{(3)}$$

$$= \frac{1}{2} \varepsilon_{(1)(1)} (E^{(1)})^2 + [\frac{1}{2} \varepsilon_{(2)(2)} (E^{(2)})^2 + \varepsilon_{(1)(2)} E^{(1)} E^{(2)}]$$

$$+ [\frac{1}{2} \varepsilon_{(3)(3)} (E^{(3)})^2 + \varepsilon_{(1)(3)} E^{(1)} E^{(3)} + \varepsilon_{(2)(3)} E^{(2)} E^{(3)}]$$

$$= \frac{1}{2} \varepsilon_{(i)(j)} E^{(i)} E^{(j)} = \frac{1}{2} \underset{\sim}{D} \cdot \underset{\sim}{E} \qquad \text{(AV.7)}$$

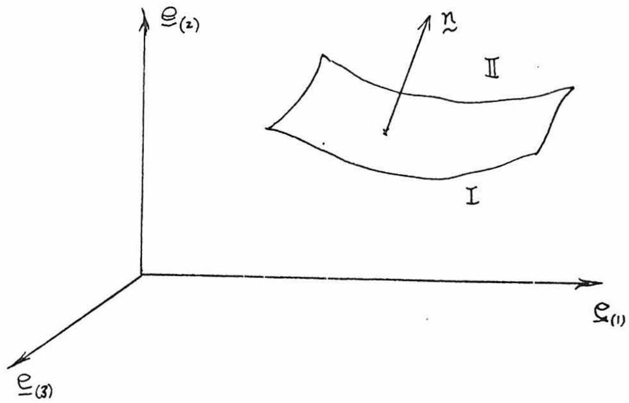
and similar result holds for the magnetic part. Thus

$$U = \frac{1}{2} (\underset{\sim}{D} \cdot \underset{\sim}{E} + \underset{\sim}{H} \cdot \underset{\sim}{B}) \equiv \frac{1}{2} (\underset{S}{\underset{\sim}{D}} \cdot \underset{\sim}{E} + \underset{S}{\underset{\sim}{H}} \cdot \underset{\sim}{B}) \qquad \text{(AV.8)}$$

stands for local stored EM energy density in lossless media. For lossy media it stands for the stored part of the energy, since the losses corresponding to $\underset{A}{\underset{\sim}{D}}$, $\underset{A}{\underset{\sim}{H}}$ are deleted in AIV-8.

APPENDIX VI

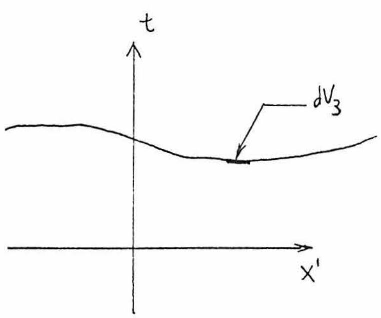
DERIVATION OF EQUATION II.69

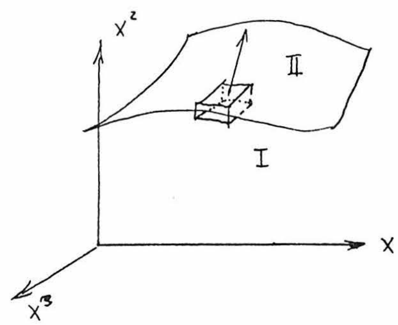


Insert $(\sqrt{-g}\, G^{\mu\nu})_{,\nu} = -\sqrt{-g}\, J^{\mu}$ into equation II.68:

$$\int_{V_3} (-\sqrt{-g}\, J^{\mu}) d\Sigma^*_{\mu} = \frac{1}{2} \oint_{V_2} (\sqrt{-g}\, G^{\mu\nu})\, d\Sigma^*_{\mu\nu} \qquad \text{(AVI.1)}$$

Now choose a small spatial 3-volume, as shown below, at the boundary





$\ni$ its normal $\underset{\sim}{n}$ from I to II points in dx^2 direction, then

$$\begin{cases} d\Sigma^*_{\mu} = -\eta_{\mu 123}\, dx^1 dx^2 dx^3 \\ \text{for } dV_3 \end{cases} , \qquad \begin{cases} d\Sigma^*_{\mu\nu} = \pm\, \eta_{\mu\nu 31}\, dx^3 dx^1 \\ + \text{ at bottom and } - \text{ at top surface} \end{cases}$$

and

$$\int_{V_3} (-\sqrt{-g}\, J^o)\, dx^1 dx^2 dx^3 = - \int_{V_2\ \text{top}} G^{20} \sqrt{-g}\, dx^1 dx^3 - \int_{V_2 \text{bottom}} G^{20} \sqrt{-g}\, dx^1 dx^3 \tag{AVI.2}$$

where the side integrals are shrunk to vanish first. Shrink the end surfaces, then

$$J^o dx^2 = (\underset{II}{G^{20}} - \underset{I}{G^{20}}) \tag{AVI.3}$$

then obviously if $\underset{\sim}{n}$ points in general direction, we have

$$J^o\, \underset{\sim}{n} \cdot \underset{\sim}{dx} = (\underset{II}{G^{io}} - \underset{I}{G^{io}})\, n_i \tag{AVI.4}$$

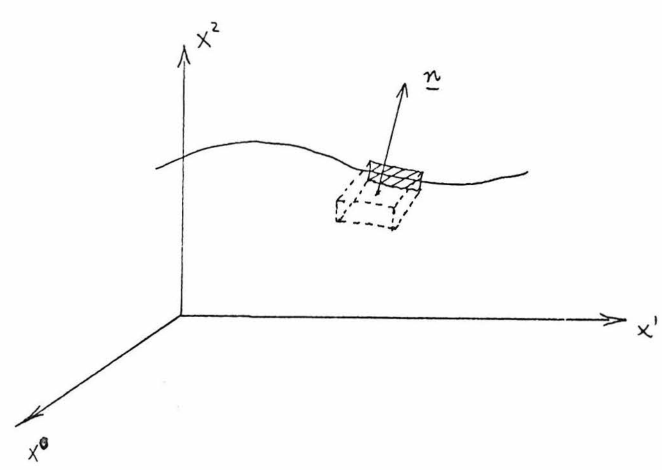
which, expressed locally in a time orthogonal frame, is

$$\sigma \equiv \rho\, \underset{\sim}{n} \cdot \underset{\sim}{dx} = (\underset{II}{\underset{\sim}{D}} - \underset{I}{\underset{\sim}{D}}) \cdot \underset{\sim}{n}\ , \qquad \sigma \equiv \text{local surface charge density on the spatial surface} \perp \underset{\sim}{n} \tag{AVI.5}$$

Similarly, we have

$$0 = (\underset{II}{\underset{\sim}{B}} - \underset{I}{\underset{\sim}{B}}) \cdot \underset{\sim}{n} \tag{AVI.6}$$

Now choose another small space-time V_3 as



with

$$d\Sigma^*_{\mu} = -\eta_{\mu 012}\, dx^o dx^1 dx^2 \qquad \text{for} \quad dV_3$$

$$d\Sigma^*_{\mu\nu} = \mp\, \eta_{\mu\nu 01}\, dx^o dx^1 \qquad \begin{array}{l} - : \text{top} \\ + : \text{bottom} \end{array}$$

Perform similarly to first case, but shrink the dummy dx^o after shrinking the thickness of the small volume

$$J^3 dx^2 = (\underset{II}{G^{23}} - \underset{I}{G^{23}}) \tag{AVI.7}$$

Then obviously for a general surface direction $\underset{\sim}{n}$ we have

$$(\underset{\sim}{j} - \frac{\underset{\sim}{n} \cdot \underset{\sim}{j}}{|\underset{\sim}{n} \cdot \underset{\sim}{n}|}\underset{\sim}{n})\, \underset{\sim}{n} \cdot \underset{\sim}{dx} = \underset{\sim}{n} x (\underset{II}{\underset{\sim}{h}} - \underset{I}{\underset{\sim}{h}}), \quad \begin{array}{l} \underset{\sim}{h} \equiv (-G^{23}, G^{13}, -G^{12}) \\ \underset{\sim}{j} \equiv (J^1,\ J^2,\ J^3) \end{array} \tag{AVI.8}$$

which, in time orthogonal frame, locally is

$$\underset{\sim}{k} \equiv (\underset{\sim}{J} - \frac{\underset{\sim}{n} \cdot \underset{\sim}{J}}{\underset{\sim}{n} \cdot \underset{\sim}{n}}\underset{\sim}{n})\, \underset{\sim}{n} \cdot \underset{\sim}{dx} = \underset{\sim}{n} \times (\underset{II}{\underset{\sim}{H}} - \underset{I}{\underset{\sim}{H}}), \tag{AVI.9}$$

$\underset{\sim}{k} \equiv$ local surface current density on spatial surface $\perp \underset{\sim}{n}$

Similarly, we have

$$0 = \underset{\sim}{n} \times (\underset{\sim}{E}_{II} - \underset{\sim}{E}_{I}) \tag{AVI.10}$$

APPENDIX VII

DERIVATION OF EQUATION II.73

Physically as shown by the graph we have

$$\frac{1}{2}\,\delta \int d^3x\,(\underset{\sim}{E} \cdot \underset{\sim}{D}) = \frac{1}{2}\int d^3x\,(\delta\underset{\sim}{E} \cdot \underset{\sim}{D} + \delta\underset{\sim}{D} \cdot \underset{\sim}{E})$$

$$= \frac{1}{2}\int d^3x\,(2\underset{\sim}{D} \cdot \delta\underset{\sim}{E} + \underset{\sim}{E}\,\underset{\sim}{E} \cdot \delta\underset{\approx}{\varepsilon})$$

$$= \int d^3x\,(\underset{\sim}{\nabla}_E(\underset{\sim}{E} \cdot \underset{\sim}{D}) + \frac{1}{2}\,E_iE_j\,\underset{\sim}{\nabla}\varepsilon_{ij}) \cdot \delta\underset{\sim}{x}$$

$$= \int d^3x\,[\underset{\sim}{D} \cdot \underset{\sim}{\nabla}\underset{\sim}{E} + \underset{\sim}{D} \times \underset{\sim}{\nabla} \times \underset{\sim}{E} + \frac{1}{2}\,E_iE_j\,\underset{\sim}{\nabla}\varepsilon_{ij}] \cdot \delta\underset{\sim}{x}$$

$$= \int d^3x\,[-(\underset{\sim}{\nabla} \cdot \underset{\sim}{D})\,\underset{\sim}{E} - \underset{\sim}{D} \times \frac{\partial\underset{\sim}{B}}{\partial T} + \frac{1}{2}\,E_iE_j\,\underset{\sim}{\nabla}\varepsilon_{ij}] \cdot \delta\underset{\sim}{x}$$

$$= \int d^3x\,(-\rho\,\underset{\sim}{E} - \underset{\sim}{D} \times \frac{\partial\underset{\sim}{B}}{\partial T} + \frac{1}{2}\,E_iE_j\,\underset{\sim}{\nabla}\varepsilon_{ij}) \cdot \delta\underset{\sim}{x} \qquad \text{AVII.1)}$$

Similarly,

$$\frac{1}{2}\,\delta \int d^3x\;\underset{\sim}{H} \cdot \underset{\sim}{B}$$

$$= \int d^3x\,(-\underset{\sim}{J} \times \underset{\sim}{B} - \frac{\partial\underset{\sim}{D}}{\partial T} \times \underset{\sim}{B} - \frac{1}{2}\,B_iB_j\;\underset{\sim}{\nabla}K_{ij}) \cdot \delta\underset{\sim}{x} \qquad \text{(AVII.2)}$$

Thus we get

$$\underset{\sim}{f} = \rho\underset{\sim}{E} + \underset{\sim}{J} \times \underset{\sim}{B} + \frac{\partial}{\partial T}\,(\underset{\sim}{D} \times \underset{\sim}{B}) + \frac{1}{2}\,(B_iB_j\underset{\sim}{\nabla}K_{ij} - E_iE_j\,\underset{\sim}{\nabla}\varepsilon_{ij}) \qquad \text{(AVII.3)}$$

APPENDIX VIII

ENERGY MOMENTUM TENSOR OF E.M. FIELD IN A MEDIUM

1° Consider a Lorentz force

$$F^{\mu\lambda}J_{\lambda} \equiv f^{\mu} = (F^{\mu\lambda}G^{\nu}{}_{\lambda} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta}G_{\alpha\beta})_{;\nu} + \frac{1}{4} g^{\mu\nu}(G_{\alpha\beta;\nu}F^{\alpha\beta} - F^{\alpha\beta}{}_{;\nu} G_{\alpha\beta})$$

then

$$+(-F^{\mu\lambda} G^{\nu}{}_{\lambda} + \frac{1}{4} g^{\mu\lambda} F^{\alpha\beta} G_{\alpha\beta})_{;\nu}$$

$$= -F^{\mu\lambda}J_{\lambda} + \frac{1}{4} g^{\mu\nu} c^{\alpha\beta\gamma\delta}{}_{;\nu} F_{\alpha\beta} F_{\gamma\delta} \qquad \text{(AVIII.1)}$$

in which the last term is zero for vacuum or lossless media, so we identify the expression in parenthesis as $T^{\mu\nu}$.

2° In SRT consider

$$\mathcal{L}_1 \equiv -\frac{1}{4} G^{\alpha\beta} F_{\alpha\beta} - J_{\mu} A^{\mu} \qquad \text{of equation II.55}$$

and from*

$$S^{\lambda\alpha} \equiv -\eta^{\lambda\alpha}\mathcal{L}_1 + A_{\mu}{}^{,\lambda} \frac{\partial \mathcal{L}_1}{\partial A_{\mu,\alpha}} + f^{\lambda\alpha\beta}{}_{,\beta} \qquad S^{\lambda\alpha}{}_{,\alpha} \equiv 0 \qquad \text{(AVIII.2)}$$

where $f^{\alpha\beta\gamma} \equiv -f^{\alpha\gamma\beta}$, we obtain

$$S^{\lambda\alpha} = [-F^{\lambda\mu} G^{\alpha}{}_{\mu} + \frac{1}{4} \eta^{\lambda\alpha} G^{\beta\gamma} F_{\beta\gamma}] + [-A^{\gamma}J^{\alpha} + \eta^{\lambda\alpha} A_{\mu}J^{\mu}] \qquad \text{(AVIII.3)}$$

Now we can just arbitrarily interpret the first bracket as $T_{(em)}$ and the second bracket as that of explicit interaction $T_{(int)}$ which

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*See Landau, (27), p. 87.

obeys the conservation law as a whole.

3° Consider expression*

$$u_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \left[- \frac{\partial}{\partial x^\lambda} \frac{\partial \sqrt{-g}\mathcal{L}}{\partial g^{\mu\nu}_{\ ,\lambda}} + \frac{\partial \sqrt{-g}\mathcal{L}}{\partial g^{\mu\nu}} \right] \tag{AVIII.4}$$

as energy momentum tensor of a system with Lagrangian $\mathcal{L}$; substitute $\mathcal{L}$ to be $\mathcal{L}_{em} + \mathcal{L}_{int}$, then we obtain

$$u_{\mu\nu} = [- \frac{1}{2} G_{\mu\lambda} F_\nu{}^\lambda + G_{\nu\lambda} F_\mu{}^\lambda) + \frac{1}{4} g_{\mu\nu} F^{\alpha\beta}G_{\alpha\beta}] \tag{AVIII.5}$$

$$+ [-(A_\mu J_\nu + A_\nu J_\mu) + g_{\mu\nu}A_\alpha J^\alpha]$$

and its physical significance comes at

$$[u^{\mu\nu} + t^{\mu\nu}]_{;\nu} = 0 \tag{AVIII.6}$$

where $t_{\mu\nu}$ is defined by equation 4 with $\mathcal{L}$ for the whole system except $\mathcal{L}_{em} + \mathcal{L}_{int}$ and $\mathcal{L}_{(g)}$ ($\equiv$ R) . This first bracket in equation 5 can also be called the energy momentum tensor of EM field in media, which is just the symmetrization of equation II.75 .

In general for a system, if an energy momentum tensor which obeys conservation law $T^{\mu\nu}{}_{;\nu} = 0$ or expresses a physical meaningful power-force throughout a volume on the enclosing boundary cannot be obtained, then the concept of the energy momentum tensor is arbitrary and has only limited physical meaning.

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* See Landau, (27), p. 311.

APPENDIX IX

DERIVATION OF EQUATION II.77

Consider $\delta I \equiv +\delta \int [-m ds - q \, A_\mu dx^\mu] = 0$ (AIX.1)

Here λ is an arbitrary parameter along the world line. Now

$$\delta \int ds \equiv \int \frac{d\lambda}{2\sqrt{u^\delta u_\delta}} (g_{\mu\nu,\alpha} \, u^\mu u^\nu \delta x^\alpha + 2 g_{\mu\nu} u^\mu \delta \, u^\nu)$$

$$= \int \frac{d(u_\nu \delta x^\nu)}{\sqrt{u^\delta u_\delta}} + \frac{d\lambda}{2\sqrt{u^\delta u_\delta}} [g_{\mu\nu,\alpha} \, u^\mu u^\nu - 2 \frac{d}{d\lambda} u_\alpha] \delta x^\alpha \qquad \text{(AIX.2)}$$

$$\delta \int A_\mu dx^\mu = \int \delta A_\mu \, dx^\mu + A_\mu \delta dx^\mu$$

$$= A_\alpha \delta x^\alpha \Big)_{x^\alpha(\lambda_1)}^{x_2^\alpha(\lambda_2)} + \int (A_{\mu,\alpha} - A_{\alpha,\mu}) \, \delta x^\alpha \, dx^\mu \qquad \text{(AIX.3)}$$

thus choosing λ to be arc length such that $u^\mu u_\mu \equiv 1$, then equations 1, 2, and 3 give

$$m(\frac{d}{ds} u_\alpha - \frac{1}{2} g_{\mu\nu,\alpha} \, u^\mu u^\nu) - q F_{\alpha\mu} u^\nu = 0 \qquad \text{(AIX.4)}$$

or

$$\frac{Du^\alpha}{Ds} = \frac{q}{m} F^{\alpha\mu} \, u_\mu \qquad \text{(AIX.5)}$$

APPENDIX X

$\varepsilon^{\mu\nu}$, $K^{\mu\nu}$ EXPRESSED BY $C^{\mu\nu\alpha\beta}$

If we begin formally with equation II.32, then no mix of $\underset{\sim}{B}$ in $\underset{\sim}{D}$ and $\underset{\sim}{E}$ in $\underset{\sim}{H}$ implies

$$\text{in the medium-rest frame} \quad \begin{cases} C^{(o)(\mu)(i)(j)} \equiv 0 & \text{(AX.1a)} \\ C^{(i)(j)(o)(\mu)} \equiv 0 & \text{(AX.1b)} \end{cases}$$

Then compare equation II.32 to II.27 in co-moving frame and de-localize as before; we can get

$$\varepsilon^{\alpha\beta} = 2C^{\mu\alpha\nu\beta} u_\mu u_\nu \qquad \text{(AX.2a)}$$

$$K^{\gamma\delta} = (*u)^{\gamma\mu\nu} (*u)^{\alpha\beta\delta} C_{\mu\nu\alpha\beta} \qquad \text{(AX.2b)}$$

such that $G^{\mu\nu} = C^{\mu\nu\alpha\beta} F_{\alpha\beta}$ will give

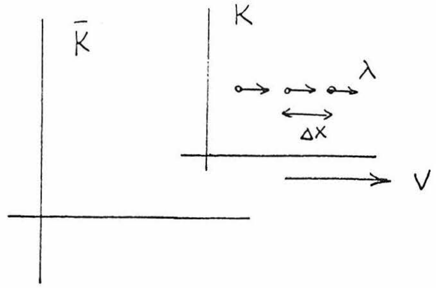
$$\text{(1a) \& (2a)} \implies G^{\mu\nu}u_\nu \equiv \varepsilon^{\mu\lambda} F_{\lambda\nu}u^\nu$$

$$\text{(1b) \& (2b)} \implies (*G)^{\mu\nu}u_\nu \equiv K^{\mu\lambda}(*F)_{\lambda\nu}u^\nu \qquad \text{(AX.3)}$$

Thus in any frame equation 2 effectively serves as a formula expressing the reverse of equation II.37. The non-consistency of equation 2 and II.37 just shows the freedom of assigning convenient values to the non-physics dummy elements in respective expressions.

APPENDIX XI

INTERPRETATION OF EQUATION III.36



K,$\bar{K}$ inertial frames

If a series of particles spaced Δx apart and travelling with λ in K frame, then in a $\bar{K}$ with respect to which K moves with V in +X direction, the particles are travelling in +X with velocity $\bar{\lambda}$ and spaced $\Delta\bar{X}$ apart, where

$$\bar{\lambda} = \frac{\lambda + V}{1 + \lambda V} \quad , \qquad \Delta\bar{X} = \Delta x \, \frac{\sqrt{1 - V^2}}{1 + \lambda V}$$

Now think of these particles as photons in medium in K with $\lambda = 1/\sqrt{\mu\varepsilon}$, then in K

$$E \sim (\text{number of photons per unit length} \cdot \omega)^{\frac{1}{2}}$$

$$\sim \frac{1}{\Delta x} \sim \frac{1}{\Delta\bar{X}} \; \frac{\sqrt{1-V^2}}{1 + \frac{1}{\sqrt{\mu\varepsilon}} V} \sim \bar{E} \cdot \frac{\sqrt{1-V^2}}{1 + \frac{1}{\sqrt{\mu\varepsilon}} V}$$

Applying these to instantaneous local quantities in accelerated-medium frame K gives equation III.36

$$E(t) \sim E(t=0) \; \frac{1}{\cosh at + \frac{1}{\sqrt{\mu\varepsilon}} \sinh at}$$

APPENDIX XII

DERIVATION OF EQUATIONS III.38 AND III.38'

$$E^Y = \frac{\sqrt{\mu\varepsilon}(1 + \sqrt{\mu\varepsilon}\tanh at)}{\sqrt{\mu\varepsilon} + \tanh at} \cdot e^{ik_o(X - \frac{1}{a}\cosh at)}\, e^{-ik_o A(t)}$$

is obtained by Lorentz transform III.31 instantaneously from $\{0\}$ to $\{\overline{0}\}$ of $\{\overline{X^\mu}\}$ where

$$A(t) \equiv \begin{cases} \frac{2}{a}\frac{1}{\sqrt{1-\mu\varepsilon}}\left[\tanh^{-1}\left(\frac{\sqrt{1-\mu\varepsilon}}{1+\sqrt{\mu\varepsilon}}\right) - \tanh^{-1}\left(\frac{\sqrt{1-\mu\varepsilon}}{1+\sqrt{\mu\varepsilon}}\, e^{-at}\right)\right] & \text{if } \mu\varepsilon < 1 \\ \frac{2}{a}\frac{1}{\sqrt{\mu\varepsilon-1}}\left[\tan^{-1}\left(\frac{\sqrt{\mu\varepsilon-1}}{1+\sqrt{\mu\varepsilon}}\right) - \tan^{-1}\left(\frac{\sqrt{\mu\varepsilon-1}}{1+\sqrt{\mu\varepsilon}}\, e^{-at}\right)\right] & \text{if } \mu\varepsilon > 1 \end{cases}$$

From this we have the expected

$$E^Y \begin{cases} \xrightarrow{\mu\varepsilon \to 1} e^{ik_o(X-T)} \\ \xrightarrow{a \to 0} e^{ik_o(X - \frac{T}{\sqrt{\mu\varepsilon}})} \end{cases}$$

as it should. The intantaneous phase velocity of E^Y in $\overline{K}$ is

$$\left.\frac{dX}{dT}\right)_{\text{phase constant}} = \frac{d}{dT}\left(\frac{1}{a}\cosh at + A(t)\right)$$

$$= \frac{d}{dT}\left(\frac{1}{a}\sqrt{1+a^2T^2} + A(t)\right)$$

Substituting $e^{-at} = \dfrac{1}{aT + \sqrt{1+a^2T^2}}$ and differentiating, we have equation III.38$'$.

APPENDIX XIII

MAXWELL EQUATIONS IN ROTATING SPHERICAL FRAME IN COMPONENT FORM

Re-interpret equations III.55 and III.56 in spherical rotating $\{t,r,\theta,\phi\}$ of equation III.58, then

$$\left\{\begin{array}{l}
\left[\frac{1}{\ell r^2}\frac{\partial}{\partial r}(\ell r^2 D^{(r)}) + \frac{1}{\ell r \sin\theta}\frac{\partial}{\partial\theta}(\ell \sin\theta\, D^{(\theta)}) + \frac{1}{\ell r \sin\theta}\frac{\partial}{\partial\phi} D^{(\phi)}\right] \\
\qquad + \Omega \sin\theta \frac{1}{\ell r^2}\frac{\partial}{\partial r}(\ell r^3 H^{(\theta)}) - \frac{\Omega}{\ell \sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta\, \ell\, H^{(r)}) \\
\qquad\qquad = \rho + r\Omega \sin\theta\, J^{(\phi)} \\
\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, H^{(\phi)}) - \frac{1}{\ell}\frac{\partial}{\partial\phi} H^{(\theta)}\right] = J^{(r)} + \ell\frac{\partial}{\partial t} D^{(r)} + \ell\, r\Omega \sin\theta \frac{\partial}{\partial t} H^{(\theta)} \\
\frac{1}{r}\left[\frac{1}{\ell \sin\theta}\frac{\partial}{\partial\phi} H^{(r)} - \frac{\partial}{\partial r}(rH^{(\phi)})\right] = J^{(\theta)} + \ell\frac{\partial}{\partial t} D^{(\theta)} - \ell r\Omega \sin\theta \frac{\partial}{\partial t} H^{(r)} \\
\frac{\ell}{r}\left[\frac{\partial}{\partial r}\left(\frac{rH^{(\theta)}}{\ell}\right) - \frac{\partial}{\partial\theta}\left(\frac{H^{(r)}}{\ell}\right)\right] = J^{(\phi)} + \ell\frac{\partial}{\partial t} D^{(\phi)}
\end{array}\right.$$

$$\left\{\begin{array}{l}
\left[\frac{1}{\ell r^2}\frac{\partial}{\partial r}(\ell r^2 B^{(r)}) + \frac{1}{\ell r \sin\theta}\frac{\partial}{\partial\theta}(\ell \sin\theta\, B^{(\theta)}) + \frac{1}{\ell r \sin\theta}\frac{\partial}{\partial\phi} B^{(\phi)}\right] \\
\qquad - \frac{\Omega \sin\theta}{\ell r^2}\frac{\partial}{\partial r}(\ell r^3 E^{(\theta)}) + \frac{\Omega}{\ell \sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta\, \ell\, E^{(r)}) = 0 \\
\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, E^{(\phi)}) - \frac{1}{\ell}\frac{\partial}{\partial\phi} E^{(\theta)}\right] = -\frac{\partial}{\partial t} B^{(r)} + \ell r\Omega \sin\theta \frac{\partial}{\partial t} E^{(\theta)} \\
\frac{1}{r}\left[\frac{1}{\ell \sin\theta}\frac{\partial}{\partial\phi} E^{(r)} - \frac{\partial}{\partial r}(rE^{(\phi)})\right] = -\ell\frac{\partial}{\partial t} B^{(\theta)} - \ell r\Omega \sin\theta \frac{\partial}{\partial t} E^{(r)} \\
\frac{\ell}{r}\left[\frac{\partial}{\partial r}\left(\frac{r}{\ell} E^{(\theta)}\right) - \frac{\partial}{\partial\theta}\left(\frac{E^{(r)}}{\ell}\right)\right] = -\frac{\partial}{\partial t} B^{(\phi)}
\end{array}\right.$$

where the $\{\underline{e}_\mu\}$, $\{\underline{e}_{(\mu)}\}$ of $\{0\}$ are related to $\{\underline{e}_{\bar{\mu}}\} \equiv \{\underline{e}_T, \underline{e}_R, \underline{e}_\Theta, \underline{e}_\Phi\}$

of the inertial $\bar{K}$ by

$$\begin{cases} \underset{\sim}{e}_t = \underset{\sim}{e}_T + \Omega \underset{\sim}{e}_\Phi \\ \underset{\sim}{e}_i = \underset{\sim}{e}_{\bar{i}} \end{cases}$$

$$\ell \equiv \frac{1}{\sqrt{1 - r^2\Omega^2\sin^2\theta}}$$

$$\begin{cases} \underset{\sim}{e}_{(t)} = \ell \underset{\sim}{e}_t \\ \underset{\sim}{e}_{(r)} = \underset{\sim}{e}_r \\ \underset{\sim}{e}_{(\theta)} = \frac{1}{r} \underset{\sim}{e}_\theta \\ \underset{\sim}{e}_{(\phi)} = \ell r\Omega \sin\theta \, \underset{\sim}{e}_t + \frac{1}{\ell r \sin\theta} \underset{\sim}{e}_\phi \end{cases}$$

APPENDIX XIV

DERIVATION OF EQUATION III.80

From equation III.79 the integral is, neglecting the (ℓ^2-1) term,

$$I \equiv \int_{R'<a} d^3R' \, ik_oV'[2\cos\Phi' + \frac{1}{i\omega}(-ik_oV'\cos^2\Phi' + \Omega\sin\Phi')]e^{-ik_o(\underset{\sim}{R}'\cdot\underset{\sim}{e}_R - Y')}$$

Thus for $\frac{\Omega}{\omega} << 1$ the first order term in Ω is

$$2ik_o\Omega \int_0^a dR' \, R'^3 \int_0^\pi d\Theta' \sin^2\Theta' \, e^{-ik_oR'\cos\Theta\cos\Theta'}$$

$$\times \oint d\Phi' \cos\Phi' \, e^{-ik_oR'\sin\Theta'[\sin\Theta\cos(\Phi-\Phi') - \sin\Phi']}$$

Now integrate from rear

$$\oint d\Phi' = \oint d\Phi' \cos\Phi' \, e^{-i\lambda A\sin(\delta + \Phi')}$$

$$= \frac{1}{2}\, 2\pi [e^{-i\delta}\, J_1(\lambda A) + e^{i\delta}\, J_{-1}(\lambda A)]$$

$$= -2\pi i\; J_1(\lambda A)\; \frac{\sin\Theta\cos\Phi}{A}$$

where $\lambda \equiv k_o R' \sin\Theta'$, $A \equiv \sqrt{1+\sin^2\Theta - 2\sin\Theta\sin\Phi}$,

$$\tan\delta \equiv \frac{\sin\Theta\cos\Phi}{-1+\sin\Theta\sin\Phi}$$

then

$$\int_0^{\pi} d\Theta' = (-2\pi i)\, \frac{\sin\Theta\cos\Phi}{A} \int_0^{\pi} d\Theta'\, \sin^2\Theta'\, e^{-i\alpha\cos\Theta'}\; J_1(\beta\sin\Theta')$$

$$= (-4\pi i)\, \frac{\sin\Theta\cos\Phi}{A} \int_0^{\pi/2} d\Theta'\, \sin^2\Theta'\, J_1(\beta\sin\Theta')\, \cos(\alpha\cos\Theta')$$

$$= (-4\pi i)\sqrt{\frac{\pi}{2}}\, \sin\Theta\cos\Phi\; \frac{1}{\sqrt{2(1-\sin\Theta\sin\Phi)}}$$

$$\times\; \left(k_o R' \sqrt{2(1-\sin\Theta\sin\Phi)}\right)^{-1/2}\; J_{\frac{3}{2}}\!\left(k_o R' \sqrt{2(1-\sin\Theta\sin\Phi)}\right)$$

where $\alpha \equiv k_o R'\cos\Theta$, $\beta \equiv A k_o R'$ and page 743.2 of Ref.(28) is used;

then

$$\int dR' = -4\pi i\, \sqrt{\frac{\pi}{2}}\, \sin\Theta\cos\Phi\; \frac{1}{\sqrt{2(1-\sin\Theta\sin\Phi)}}\; \frac{1}{\eta^4} \int_0^{\eta a} du\, u^{\frac{5}{2}}\, J_{\frac{3}{2}}(u)$$

$$= -4\pi i\, \frac{k_o a^5 \sin\Theta\cos\Phi}{(\eta a)^5}\; \{(3-u^2)\sin u - 3u\cos u\}_{u=\eta a}$$

where $\eta \equiv k_o \sqrt{2(1 - \sin\Theta \sin\Phi)}$; then

$$I = -8\pi i(k_o a)\Omega\, a^4 \sin\Theta \cos\Phi \frac{(3-\delta^2)\sin\delta - 3\delta\cos\delta}{\delta^5}$$

where $\delta \equiv k_o a \sqrt{2(1 - \sin\Theta \sin\Phi)}$. Substitute this in equation III.79 and get equation III.80.

DERIVATION OF EQUATION III.82

From equation III.81 the integral is

$$I = \Omega \int d^3R'(-\underset{\sim}{e}_Y + ik_o R'\sin\Phi'\sin\Theta'\underset{\sim}{e}_Z)e^{-ik_o(\underset{\sim}{R}'\cdot\underset{\sim}{e}_R - Z')}$$

where

$$I_1 \equiv \Omega \int d^3R'(-\underset{\sim}{e}_Y)e^{-ik_o(\underset{\sim}{R}'\cdot\underset{\sim}{e}_R - Z')} = -\underset{\sim}{e}_Y \frac{4\pi a^3}{\delta_1^{\,3}}(\sin\delta_1 - \delta_1\cos\delta_1)$$

$$\delta_1 \equiv 2k_o a \sin\frac{\Theta}{2}$$

Now

$$I_2 \equiv \Omega\, ik_o \underset{\sim}{e}_z \int_0^a dR'R'^3 \int_0^\pi d\Theta'\sin^2\Theta' e^{-ik_o R'(\cos\Theta - 1)\cos\Theta'}$$

$$\times \oint d\Phi'\, e^{-ik_o R'\sin\Theta\sin\Theta'\cos(\Phi - \Phi')} \sin\Phi'$$

Integrate from rear as before

$$\oint d\Phi' = -2\pi i \sin\Phi\, J_1(\lambda) \qquad \lambda \equiv k_o R'\sin\Theta\sin\Theta'$$

then

$$\int d\Theta' = -2\pi i \sin\Phi \int_0^{\pi} d\Theta' \sin^2\Theta' e^{-i\alpha\cos\Theta'} J_1(\beta\sin\Theta')$$

$$= -4\pi i \sin\Phi \int_0^{\pi/2} d\Theta' \sin^2\Theta' \cos(\alpha\cos\Theta') J_1(\beta\sin\Theta')$$

$$= -4\pi i \sqrt{\frac{\pi}{2}} \frac{\sin\Theta\sin\Phi}{\sqrt{2(1-\cos\Theta)}} \cdot \eta^{-1/2} J_{\frac{3}{2}}(\eta)$$

where $\alpha \equiv k_o R'(\cos\Theta - 1)$, $\beta \equiv k_o R'\sin\Theta$, $\eta \equiv k_o R' \sqrt{2(1-\cos\Theta)}$

then $$\int dR' = -4\pi i \sqrt{\frac{\pi}{2}} \frac{\sin\Theta\sin\Phi}{\sqrt{2(1-\cos\Theta)}} \int_0^a dR' R'^3 \eta^{-1/2} J_{\frac{3}{2}}(\eta)$$

$$= -4\pi i \frac{k_o a^5 \sin\Theta\sin\Phi[(3-\delta_2{}^2)\sin\delta_2 - 3\delta_2\cos\delta_2]}{\delta_2{}^5}$$

where $\delta_2 \equiv k_o a \sqrt{2(1-\cos\Theta)}$. Thus combine $I_1 + I_2$ and get equation III.82.

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