DISJOINTNESS PRESERVING OPERATORS

Thesis by

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ABSTRACT

Let E and F be Archimedian Riesz spaces. A linear operator $T : E \to F$ is called disjointness preserving if $|f| \wedge |g| = 0$ in E implies $|Tf| \wedge |Tg| = 0$ in F. An order continuous disjointness preserving operator $T : E \to E$ is called bi-disjointness preserving if the order closure of $|T|E$ is an ideal in E. If the order dual of E separates the points of E, then every order continuous disjointness preserving operator whose adjoint is disjointness preserving is bi-disjointness preserving. If E is in addition Dedekind complete, then the converse holds.

DEFINITION. *Let* $T : E \to E$ *be a bi-disjointness preserving operator. We say that* T *is:*

(i) quasi-invertible if T *is injective and* $\{TE\}^{dd} = E$.

(ii) of forward shift type if T *is injective and* $\bigcap_{n=1}^{\infty} \{T^n E\}^{dd} = \{0\}$.

(iii) of backward shift type if $\bigvee_{n=1}^{\infty} \operatorname{Ker} T^n = E$ *and* $\{TE\}^{dd} = E$.

(iv) hypernilpotent if $\bigvee_{n=1}^{\infty} \operatorname{Ker} T^n = E$ *and* $\bigcap_{n=1}^{\infty} \{T^n E\}^{dd} = \{0\}$.

The supremum in (iii) and (iv) is taken in the Boolean algebra of bands.

The following decomposition theorem is proved.

THEOREM. *Let* $T : E \to E$ *be a bi-disjointness preserving operator on a Dedekind complete Riesz space* E. *Then there exist* T*-reducing bands* E_i $(i = 1,2,3,4)$ *such that* $\bigoplus_{i=1}^{4} E_i = E$ *and the restriction of* T *to* E_i *satisfies the ith property listed in the preceding definition.*

Quasi-invertible operators can be decomposed further in the following way. Set $\mathrm{Orth}(E) := \{T \in \mathcal{L}_b(E) : TB \subset B$ for every band $B\}$. We say that a quasi-invertible operator T has strict period n ($n \in \mathbb{N}$) if $T^n \in \mathrm{Orth}(E)$ and for every non-zero band $B \subset E$, there exists a band A s.t. $\{0\} \neq A \subset B$ and $A, \{TA\}^{dd}, \ldots, \{T^{n-1}A\}^{dd}$ are mutually disjoint. A quasi-invertible operator is called aperiodic if for every $n \in \mathbb{N}$ and every non-zero band $B \subset E$, there exists a band A s.t. $\{0\} \neq A \subset B$ and $A, \{TA\}^{dd}, \ldots, \{T^nA\}^{dd}$ are mutually disjoint.

THEOREM. *Let* $T : E \to E$ *be a quasi-invertible operator on a Dedekind complete Riesz space* E. *Then there exist* T*-reducing bands* E_n ($n \in \mathbb{N} \cup \{\infty\}$) *such that the restriction of* T *to* E_n ($n \in \mathbb{N}$) *has strict period* n, *the restriction of* T *to* E_∞ *is aperiodic and* $E = \bigoplus_{n \in \mathbb{N} \cup \{\infty\}} E_n$.

Finally, the spectrum of bi-disjointness preserving operators is considered.

THEOREM. *Let* E *be a Banach lattice which is either Dedekind complete or has a weak Fatou norm. Let* $T : E \to E$ *be a bi-disjointness preserving operator. If* T *is either of forward shift type, of backward shift type, hypernilpotent or aperiodic quasi-invertible, then the spectrum of* T *is rotationally invariant. If* T *is quasi-invertible with strict period* n, *then* $\lambda \in \sigma(T)$ *implies* $\lambda\alpha \in \sigma(T)$ *for any nth root of unity* α.

The above theorems can be combined to deduce results concerning the spectrum of arbitrary bi-disjointness preserving operators. One such result is given below.

THEOREM. *Let* $T : E \to E$ *be a bi-disjointness preserving operator on a Dedekind complete Banach lattice* E. *Suppose, for each*

$0 < r \in \mathbb{R}$, $\{z \in \mathbb{C} : |z| = r\} \cap \sigma(T)$ *lies in an open half plane. Then there exists* T*-reducing bands* E_1 *and* E_2 *such that* $E = E_1 \oplus E_2$, $T|_{E_1}$ *is an abstract multiplication operator (i.e. is in the center of* E*) and* $T|_{E_2}$ *is quasi-nilpotent.*

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TERMINOLOGY AND NOTATION

For terminology and the general theory of Riesz spaces and Banach lattices, we refer to the standard treatises of Luxemburg-Zaanen [LZ], Zaanen [Z] and Schaefer [S]. We have attempted to make this thesis reasonably self-contained except for material in the above works and other well known results. All Riesz spaces considered herein will be assumed to satisfy the Archimedian axiom. Unless otherwise stated, the results in chapters 1-4 are valid for both real and complex Riesz spaces (see [S] II § 11); all spaces considered in chapter 5 are complex.

We now give a partial list of notation.

E, F	(Archimedian) Riesz spaces.
E_+	$\{f \in E : f \geq 0\}$.
E^*	The order dual of E.
E_n^*	The order continuous dual of E.
$\hat{E}$	The Dedekind completion of E.
$\mathcal{B}(E)$	The Boolean algebra of bands of E.
$\mathcal{P}(E)$	The Boolean algebra of all band projections of E.
$\mathcal{L}_b(E,F)$	The collection of all order bounded operators from E to F.
$\mathcal{L}_b(E)$	$\mathcal{L}_b(E,E)$
J_f, J_S	The ideal generated by an element $f \in E$ or a subset $S \subset E$.
$\overline{J}_f$ $\overline{J}_S$	The uniformly closed ideal generated by $f \in E$ or $S \subset E$.
$\{S\}^d$	$\{f \in E : \|f\| \wedge \|g\| = 0$ for all $g \in S\}$.
$C(X)$	The collection of all continuous functions defined on a topological space X.

T^*	The order adjoint of T.
T_n^*	The restriction of T^* to E_n^*.
$\tilde{T}$	The associated operator of T (see def. 3.3 and pg 38).
$t_T = t$	The Luxemburg "t" map (see def. 3.15).
$\{S\}^0$	The annihilator of a subspace $S \subset E$.
${}^0\{S\}$	The pre-annihilator of a subspace of E^* or E_n^*.
$\sigma(T)$	The spectrum of T.
$P\sigma(T)$	The point spectrum of T.
$A\sigma(T)$	The approximate point spectrum of T.
$R\sigma(T)$	The residual spectrum of T $(= \sigma(T) \setminus A\sigma(T))$.
$r(T)$	The spectral radius of T.
D_r	$\{z \in \mathbb{C} : \|z\| \leq r\}$
C_r	$\{z \in \mathbb{C} : \|z\| = r\}$
D, C	D_1, C_1
$\mathcal{L}(B)$	The space of all (norm) bounded operators from a Banach space B to itself.

INTRODUCTION

Two elements f and g of a Riesz space E are called disjoint if $|f| \wedge |g| = 0$. This thesis studies disjointness preserving operators; in other words operators between two Riesz spaces which take disjoint elements to disjoint elements. On concrete function spaces (such as C(X) and L^p spaces) this means that the operator takes functions of (essentially) disjoint support to functions of (essentially) disjoint support. It is not difficult to see that bounded disjointness preserving operators correspond to weighted composition operators on such concrete function spaces, that is, to operators of the form $Tf(x) = h(x)f(\varphi(x))$ (see propositions 1.3 and 1.4).

Disjointness preserving operators are of interest in many different contexts. First of all, along with the kernel operators, they form one of the two major classes of concrete bounded operators and thus supply a rich source of examples. The classical theorems of Banach-Stone [B] [St] and Banach-Lamperti [B] [La] (see also [Ro] 15.8) show that the isometries between C(X) spaces (where X is compact Hausdorff) and L^p spaces $(0 < p < \infty, p \neq 2)$ on a σ-finite measure space are disjointness preserving. Disjointness preserving operators are naturally of considerable interest in the theory of dynamical systems. Such theories study mappings φ of a set X carrying some additional structure into itself, which preserves the structure of X. The composition operator $Sf(x) = f(\varphi(x))$ and weighted composition operators $Tf(x) = h(x)f(\varphi(x))$ defined on function spaces over X are important tools which are used to understand the properties of the mapping φ. Finally, Riesz homomorphisms, which are disjointness preserving, are of obvious interest in the study of abstract Riesz spaces.

The notion of a disjointness preserving operator was first introduced by B. Vulikh [V], though positive disjointness preserving operators (Riesz homomorphisms) were studied considerably earlier. His primary interest was to find conditions on a disjointness preserving operator defined on certain function spaces which will make them multiplicative, and so of the form $Tf(x) = f(\varphi(x))$. More general representations of the form $Tf(x) = h(x) f(\varphi(x))$ have been given by several authors on various spaces and in varying degrees of generality, see [Kp] [La] [Wo] [M 3] [Kn] [AVK] [Ab].

The abstract theory of disjointness preserving operators (also sometimes called disjunctive operators, d-homomorphisms or Lamperti operators) has been studied only in the past decade (see [M 1] [M 3] [AVK] [Ki 2] [M 4] [Ar 2] [Ab] [dP 2]). The primary motivation for this research was to extend both the theory of orthomorphisms (i.e. abstract multiplication operators, see ch. 2) and the theory of Riesz homomorphisms. An exposition of much of this work is contained in this thesis (see especially chapter 1). We now give a summary of each chapter of this thesis.

Chapter 1 discusses the basic properties of disjointness preserving operators. These results will be used repeatedly in the later chapters.

Chapter 2 studies a special class of disjointness preserving operators known as orthomorphisms. We first give a survey of the most important properties of such operators. We then use a known result about orthomorphisms to show that if F has the σ-interpolation property (this holds in particular when F is Dedekind σ-complete), then the range of every disjointness preserving operator from E to F is a Riesz subspace of F. The chapter concludes with a discussion of some extension properties of certain types of orthomorphisms.

Chapter 3 discusses two auxiliary maps associated with a disjointness preserving operator. In concrete situations where the operator is of the form $Tf(x) = h(x)\,f(\varphi(x))$, the two associated maps roughly correspond to the composition operator $\tilde{T}f(x) = f(\varphi(x))$ and the underlying map φ. Most of the chapter is devoted to discussing the relationship between a disjointness preserving operator and its associated maps.

In chapter 4 we discuss bi-disjointness preserving operators. Under certain circumstances, bi-disjointness preserving operators are exactly those order continuous disjointness preserving operators whose adjoint is also disjointness preserving. The main result of this chapter is to show that a bi-disjointness preserving operator on a Dedekind complete Riesz space may be decomposed into a direct sum of simple components, each of which can be easily analyzed.

Chapter 5 discusses the spectrum of disjointness preserving operators. The first half of the chapter is devoted to computing the spectrum of the simple bi-disjointness preserving operators which make up the "blocks" in the decomposition theorem proved in chapter 4. We then use this decomposition theorem in the second half of the chapter to derive various properties of the spectrum of an arbitrary bi-disjointness preserving operator.

Chapter 1

BASIC PROPERTIES

This chapter gives some examples of disjointness preserving operators and discusses their basic properties. The central result is the characterization of disjointness preserving operators given in theorem 1.5. All results in this chapter are essentially known.

DEFINITION 1.1. *Two elements* f *and* g *of a Riesz space are said to be* disjoint *if* $|f| \wedge |g| = 0$. *This relation will be denoted by* $f \perp g$.

An operator $T : E \to F$ *between Riesz spaces* E *and* F *is called* disjointness preserving *if* $f, g \in E$ *and* $f \perp g$ *imply* $Tf \perp Tg$.

We begin by characterizing bounded disjointness preserving operators on C(X) spaces.

PROPOSITION 1.2. *Let* X *be a compact Hausdorff space and* ψ *be a non-zero bounded linear functional on* C(X). *The following are equivalent:*

a) ψ *is disjointness preserving.*

b) There exists a unique point $x \in X$ *and a unique non-zero scalar* c *such that* $\psi(f) = cf(x)$ *for all* $f \in C(X)$.

Proof. b) $\Rightarrow$ a) is obvious.

a) $\Rightarrow$ b): By the Riesz representation theorem, there exists a finite (complex) Borel measure μ on X s.t. $\psi(f) = \int_X f\,d\mu$ for all $f \in C(X)$. Suppose A and B are disjoint closed sets in X. By Urysohn's lemma, there exist functions $e_1, e_2 \in C(X)$ s.t. $e_1(x) = 1$ for all $x \in A$, $e_2(x) = 1$ for all $x \in B$ and $|e_1| \wedge |e_2| = 0$. Since ψ is disjointness preserving,

$|\psi(e_1)| \wedge |\psi(e_2)| = 0$, and so either $\psi(e_1) = 0$ or $\psi(e_2) = 0$. Hence, either $\mu(A) = 0$ or $\mu(B) = 0$. It follows easily that μ must be concentrated at a single point, i.e. there exists a point $x \in X$ s.t. $\mu\{x\} = \mu(X)$. Hence $\psi(f) = \int_X f\,d\mu = \mu\{x\} \cdot f(x) = cf(x)$, where $c = \mu\{x\}$.

PROPOSITION 1.3. *Let* X *and* Y *be compact Hausdorff spaces and let* $T : C(X) \to C(Y)$ *be a (norm) bounded operator. The following are equivalent:*

a) T *is disjointness preserving.*

b) There exists a continuous function $h \in C(Y)$ *and a continuous map* $\varphi : \mathrm{Coz}(h) \to X$ (where $\mathrm{Coz}(h) := \{y \in Y : h(y) \neq 0\}$) such that

$$Tf(y) = \begin{cases} h(y)\ f(\varphi(y)) & y \in \mathrm{Coz}(h) \\ 0 & \text{otherwise} . \end{cases}$$

Proof. b) $\Rightarrow$ a) is clear. To show that a) $\Rightarrow$ b), pick $y \in Y$ and let δ_y be the disjointness preserving linear functional given by $\delta_y(f) = f(y)$ for all $f \in C(Y)$. Since the composition of two disjointness preserving operators is disjointness preserving, by proposition 1.2 there exists a scalar (depending on y) $h(y)$ and a point $x \in X$ s.t. $\delta_y \circ T = h(y)\ \delta_x$. If $h(y) \neq 0$, the point x depends uniquely on y, so there is a function $\varphi : \mathrm{Coz}(h) \to X$ s.t. $\delta_y \circ T = h(y)\ \delta_{\varphi(y)}$. In other words,

$$Tf(y) = \delta_y \circ T(f) = \begin{cases} h(y) \circ \delta_{\varphi(y)}(f) & \text{if } y \in \mathrm{Coz}(h) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} h(y)\ f(\varphi(y)) & y \in \mathrm{Coz}(h) \\ 0 & \text{otherwise} . \end{cases}$$

It remains to show that h and φ are continuous. Let e be the function which is identically one on X. Then $Te(y) = h(y)\, e(\varphi(y)) = h(y)$. Hence $h = Te \in C(Y)$. To show that φ is continuous, pick an open set $U \subset X$ and suppose $w \in \varphi^{-1}(U)$. By Urysohn's lemma, there exists a function $g \in C(X)$ s.t. $g(\varphi(w)) = 1$ and $g(x) = 0$ if $x \notin U$. Then $Tg \in C(Y)$, $Tg(w) = h(w)g(\varphi(w)) = h(w) \neq 0$ since $w \in \varphi^{-1}(X) = \mathrm{Coz}(h)$. Furthermore, if $x \notin \varphi^{-1}(U)$, it is clear that $Tg(x) = 0$. Hence $w \in \mathrm{Coz}(Tg) \subset \varphi^{-1}(U)$. Since $Tg \in C(Y)$, $\mathrm{Coz}(Tg)$ is open, so $\varphi^{-1}(U)$ is open which shows that φ is continuous.

We now wish to discuss disjointness preserving operators on L^p spaces. Due to possible measure-theoretic pathologies, a disjointness preserving operator between L^p spaces need not induce a "point map" between the spaces as was the case for $C(X)$ spaces (see [W] pg. 54 for an example of such pathologies). For this reason a slightly weaker concept must be introduced.

By a measure space (X, Σ, μ), we mean a set X together with a σ-algebra of subsets Σ and a σ-additive measure μ. The measure algebra of (X, Σ, μ) will be denoted by $\hat{\Sigma}$. Via the map $A \to 1_A$, the measure algebra of (X, Σ, μ) can be identified with the characteristic functions in $L^\infty(X, \Sigma, \mu)$. Thus, if (X, Σ, μ) and (Y, Λ, ν) are measure spaces, a (Boolean) σ-homomorphism $t : \hat{\Sigma} \to \hat{\Lambda}$ induces a map S from the characteristic functions of $L^\infty(X, \Sigma, \mu)$ to those of $L^\infty(Y, \Lambda, \nu)$. It is not difficult to see that S can be extended to simple functions, and then to all $L^\infty(X, \Sigma, \mu)$, and that the map $S : L^\infty(X, \Sigma, \mu) \to L^\infty(Y, \Lambda, \nu)$ is a lattice and algebra homomorphism (see [F] sec. 45). We will call S the operator induced by the σ-homomorphism t. In non-pathological cases, there exists a map $\varphi : Y \to X$, defined except possibly on a set of measure zero in Y such that $Sf(y) = f(\varphi(y))$ for all $f \in L^{\infty}(X, \Sigma, \mu)$ and almost all $y \in Y$ (see [Ro] pg. 329).

PROPOSITION 1.4. *Let* (X, Σ, μ) *and* (Y, Λ, ν) *be finite measure spaces and* $T : L^p(X, \Sigma, \mu) \to L^q(Y, \Lambda, \nu)$ $(1 \leq p,q < \infty)$ *be a norm bounded disjointness preserving operator such that* $\{T(1_X)\}^d = \{0\}$. *Then there exists a function* $h \in L^{\infty}(Y, \Lambda, \nu)$ *and a σ-homomorphism* $t : \hat{\Sigma} \to \hat{\Lambda}$ *such that* $Tf = h \cdot Sf$, *for all* $f \in L^{\infty}(X, \Sigma, \mu)$, *where* S *denotes the operator induced by* t.

Proof. For any element $A \in \hat{\Sigma}$, define $t : \hat{\Sigma} \to \hat{\Lambda}$ by $t(A) = \mathrm{supp}(T1_A)$. If A and B are disjoint in $\hat{\Sigma}$, since T is disjointness preserving we have $t(A \cap B) = \emptyset = t(A) \cap t(B)$. It follows that $t(A \cap B) = t(A) \cap t(B)$ for arbitrary $A,B \in \hat{\Sigma}$. Moreover, since $\mathrm{supp}(T1_X) = Y$ and T is disjointness preserving we have $t(A^c) = \mathrm{supp}(T1_{A^c}) = \{\mathrm{supp}(T1_A)\}^c$, where the "c" denotes the complement of the set. It follows from the fact that T is order continuous that t is a σ-homomorphism. Set $h = T1_X$ and let S be the operator induced by t. It is easy to see that $Tf = h \cdot Sf$ holds for characteristic functions and hence for arbitrary functions $f \in L^p(X, \Sigma, \mu)$.

Remarks: 1) The above argument was first used by Lamperti [La] in his discussion of isometries of L^p spaces. See also [B], [Ro] pg. 333, [Kn].

2) With appropriate modifications, the above theorem can be extended to Banach function spaces with order continuous norm, defined on σ-finite measure spaces.

We now give an important characterization of order bounded disjointness preserving operators, due to M. Meyer ([M1], [M3], [M4], see also [Ar 2]). In particular it states that every disjointness preserving operator which is order bounded is already regular.

THEOREM 1.5. *Let* $T \in \mathcal{L}_b(E,F)$ *be an order bounded operator between Riesz spaces* E *and* F. *The following are equivalent:*

a) T *is disjointness preserving*

b) $|Tf| = |T|f||$ *for all* $f \in E$.

c) $|T|$ *exists and satisfies* $|T||f| = |Tf|$ *for all* $f \in E$.

Proof. a) ⇒ c): By the Yosida representation theorem ([LZ] 45.3), there exist compact Hausdorff spaces X and Y such that the principal ideals generated by f and Tf, J_f and J_{Tf} can be identified with uniformly dense subsets of $C(X)$ and $C(Y)$ respectively. Since T is order bounded, $T|_{J_f}$ can be uniquely extended under the above identification to a disjointness preserving operator $T_f : C(X) \to C(Y)$. By proposition 1.3, there exists a function $h \in C(Y)$ and a continuous map $\varphi : Y \to X$ such that $T_f f(y) = h(y)\, f(\varphi(y))$ for all $y \in Y$. It is clear that $|T_f|$ exists and $|T_f| f(y) = |h(y)|\, f(\varphi(y))$. Moreover $|T_f f|(y) = |h(y)|\, |f(\varphi(y))| = |T_f|\, |f|(y)$. It follows that $|T|$ exists, since it is defined on the positive cone of E by the formula $|T|f := \sup_{|g| \leq f} |Tg| = \sup_{|g| \leq f} |T_f g| = |T_f| f$

and it can be extended linearly to all of E. Furthermore, for any $f \in E$, $|T||f| = |T_f||f| = |T_f f| = |Tf|$ which gives c).

c) ⇒ b): Applying c) first to $|f|$ and then to f yields $|T|f|| = |T||f| = |Tf|$.

b) ⇒ a): It is easy to see that two elements f and g of a Riesz space are disjoint iff $|f + \lambda g| = |f - \lambda g|$ for all scalars λ (c.f. [LZ] 14.4, 14.5 in the real case). Hence, $f \perp g$ in $E \Rightarrow |f + \lambda g| = |f - \lambda g|$ for all scalars $\lambda \Rightarrow |Tf + \lambda Tg| = |T(f + \lambda g)| = |T||f + \lambda g|| = |T||f - \lambda g|| = |T(f - \lambda g)| = |Tf - \lambda Tg|$ for all scalars $\lambda \Rightarrow Tf \perp Tg$ in F.

The remainder of this chapter will be primarily devoted to deriving various consequences of theorem 1.5. We begin with an easy corollary.

COROLLARY 1.6. *Suppose* E *and* F *are Riesz spaces and* $T \in \mathcal{L}_b(E,F)$ *is disjointness preserving.*

i) If $|g| \leq |f|$ *in* E, *then* $|Tg| \leq |Tf|$ *in* F.

ii) If J *is an ideal in* F *then* $T^{-1}(J)$ *is an ideal in* E. *In particular,* Ker T *is an ideal in* E.

Proof. i): If $|g| \leq |f|$ in E, then by theorem 1.5, $|Tg| = |T||g| \leq |T||f| = |Tf|$.

ii): Suppose $Tf \in J$ and $|g| \leq |f|$. Then by i), $|Tg| \leq |Tf| \in J$, so $Tg \in J$ since J is an ideal, which shows that $T^{-1}(J)$ is an ideal.

Recall that an operator $T: E \to F$ between Riesz spaces E and F is called a *Riesz homomorphism* if $|Tf| = T|f|$ for all $f \in E$. It is easy to see that T is a Riesz homomorphism iff $Tf \vee Tg = T(f \vee g)$ iff $Tf \wedge Tg = T(f \wedge g)$ for all $f,g \in \text{Re } E$ (c.f. [LZ] 18.3). There is a simple relationship between Riesz homomorphisms and disjointness preserving operators.

PROPOSITION 1.7. *An operator* $T : E \to F$ *between Riesz spaces* E *and* F *is a Riesz homomorphism iff it is positive and disjointness preserving.*

Proof. If T is a Riesz homomorphism, it is clearly positive and hence it follows from the definition of a Riesz homomorphism that T satisfies condition c) of theorem 1.5, so T is disjointness preserving.

Conversely, if T is positive and disjointness preserving, then by c) of theorem 1.4 for all $f \in E$, $|Tf| = |T||f| = T|f|$, whence T is a Riesz homomorphism.

The next result is another version of Meyer's theorem ([M1], [M3], see also [dP 2]), which is occasionally useful.

THEOREM 1.8. *Let* E *and* F *be real Riesz spaces and let* $T \in \mathcal{L}_b(E,F)$ *be disjointness preserving. Then there exist Riesz homomorphisms* $T^+, T^- \in \mathcal{L}_b(E,F)$ *such that* $T = T^+ - T^-$ *and* $T^+f \wedge T^-f = 0$ *for all* $f \in E$.

Proof. Put $T^+ = \frac{1}{2}(|T| + T)$ and $T^- = \frac{1}{2}(|T| - T)$. T^+ and T^- are clearly positive and disjointness preserving, and hence are Riesz homomorphisms. Furthermore, for any $0 \leq f \in E$, $T^+f = \frac{1}{2}(|T|f + Tf) = \frac{1}{2}(|Tf| + Tf) = (Tf)^+$. Similarly, $T^-f = (Tf)^-$. Therefore, $T^+f \wedge T^-f = (Tf)^+ \wedge (Tf)^- = 0$ and the proof is complete.

THEOREM 1.9. *Let* E *and* F *be Riesz spaces.*
Then any disjointness preserving operator $T \in \mathcal{L}_b(E,F)$ *can be extended to a disjointness preserving operator* $\hat{T} \in \mathcal{L}_b(\hat{E},\hat{F})$, *where* $\hat{E}$ *and* $\hat{F}$ *denote the Dededind completion of* E *and* F *respectively. If* T *is order continuous, then this extension is unique.*

Proof. The complex case can be reduced to the real case by considering the real and imaginary parts of T. Hence, it can be assumed that T is real. Then by theorem 1.8, $T = T^+ - T^-$ where T^+ and T^- are Riesz homomorphisms. It follows from the results of [LS 2] that T^+ and T^- can be extended to Riesz homomorphisms $\hat{T}^+$ and $\hat{T}^-$ on $\hat{E}$ into $\hat{F}$. It now follows easily that $\hat{T} := \hat{T}^+ - \hat{T}^-$ is the desired extension. The assertion about uniqueness is obvious.

We conclude this chapter by saying a few words about an important recent discovery in the theory of disjointness preserving operators. In the special cases discussed in propositions 1.2 - 1.4, it is easy to see using the characterizations proved there that every norm bounded disjointness preserving operator is order bounded and in fact is regular (theorem 1.5). It is a remarkable fact, due to Abramovich [Ab], that even a slightly more general result is true.

THEOREM 1.10. *Let* $T : E \to F$ *be a disjointness preserving operator between Riesz spaces* E *and* F. *Suppose that* $\inf_n(|Tf_n| + |Tg_n|) = 0$ *whenever* $f_n, g_n \to 0$ *relative uniformly. Then* T *is order bounded.*

A simple proof of this result can be found in [dP2]. If E and F are normed Riesz spaces, then theorem 1.10 immediately implies that every norm bounded disjointness preserving operator is order bounded.

Chapter 2

ORTHOMORPHISMS

In this chapter, we shall study a special class of disjointness preserving operators known as orthomorphisms. On concrete function lattices, orthomorphisms correspond to multiplication operators.

In the first part of the chapter, we state for future reference the basic properties of these operators. As these results are well known, most of the proofs will be omitted. For proofs and further properties of orthomorphisms, we refer to [Z], ch. 20. Other references include [L], [dP1], [AB1], [F1] and [BKW].

In the second part of the chapter, it is shown that every disjointness preserving operator between uniformly complete Riesz spaces has a local "polar decomposition" (theorem 2.9) into a product of a Riesz homomorphism and an orthomorphism. Using this result, we prove that under certain conditions, the range of a disjointness preserving operator is a Riesz subspace (theorem 2.10). Finally, we prove some extension theorems for orthomorphisms which will play an important role in the next chapter.

DEFINITION 2.1. *Let* E *be a Riesz space and let* $\pi \in \mathcal{L}_b(E)$.

i) We shall call π *an* orthomorphism *if for any band* $B \in \mathcal{B}(E)$, $\pi(B) \subset B$.

ii) We say that π *is a* contractor *if for all ideals* $J \subset E$, $\pi(J) \subset J$.

iii) We say that π *is in the* center *of* E *if there exists a positive real number* λ *such that* $|\pi f| \leq \lambda |f|$ *for all* $f \in E$.

We denote the collection of all orthomorphisms on E *by* Orth(E), *the collection of all contractors on* E *by* Con(E) *and the center of* E *by* Z(E).

It is easy to see that $\pi \in$ Orth(E) iff $\pi \in \mathcal{L}_b(E)$ and $\pi f \perp g$ for all $f, g \in E$ such that $f \perp g$. It follows from this that every orthomorphism is disjointness preserving. It is also clear that $\mathcal{P}(E) \subset Z(E) \subset \text{Con}(E) \subset \text{Orth}(E)$, where $\mathcal{P}(E)$ denotes the collection of all band projections on E.

Recall that an *f-algebra* A is a Riesz space which is also an algebra that satisfies the following conditions:

1) If $a, b \in A_+$, then $ab \in A_+$.

2) Multiplication by an element $a \in A$ is an orthomorphism. More precisely, if $a, b, c \in A$ and $b \perp c$, then $ca \perp b$ and $ac \perp b$.

A well-known theorem of Birkhoff and Peirce states that an (Archimedean) f-algebra is necessarily commutative.

If A and B are f-algebras, a linear operator $T: A \to B$ is called an *f-algebra homomorphism* if T is both an algebra homomorphism and a Riesz homomorphism.

We are now ready to state the basic theorem about orthomorphisms.

THEOREM 2.2. *For any Riesz space* E, Orth(E) *is an f-algebra, where multiplication is defined by composition and addition and the lattice operations are defined pointwise. In other words, for all* $\pi_1, \pi_2 \in \text{Re Orth}(E)$. *and* $f \in E_+$, $(\pi_1 \vee \pi_2)f = \pi_1 f \vee \pi_2 f$, *and* $(\pi_1 \wedge \pi_2)f = \pi_1 f \wedge \pi_2 f$. Z(E) *and* Con(E) *are f-subalgebras of* Orth(E).

If E *is Dedekind complete, then* Orth(E) *and* Z(E) *are, respectively,* *the band and ideal generated by the identity operator in* $\mathcal{L}_b(E)$. *If* E *is Dedekind (σ-Dedekind, uniformly) complete, then* Orth(E) *and* Z(E) *are Dedekind (resp. σ-Dedekind, uniformly) complete.*

If E *is a Banach lattice, then* Z(E) = Orth(E), Z(E) *is a Banach lattice under the operator norm, which is given by* $\|\pi\| = \inf\{\lambda \in \mathbb{R}_+ : |\pi| \leq \lambda I\}$, *and* Z(E) *is isometrically and f-algebraically isomorphic to a space of type* C(X) *where* X *is a compact Hausdorff space.*

If E is a uniformly complete (real) Riesz space, and $E_{\mathbb{C}}$ and $\mathrm{Orth}_{\mathbb{C}}(E)$ denote the complexification of E and Orth(E) respectively, then it is easy to see that $\mathrm{Orth}_{\mathbb{C}}(E) = \mathrm{Orth}(E_{\mathbb{C}})$. Most of the following results have been proven for real Riesz spaces; complex versions follow immediately from the above observation.

It follows from Theorem 2.2 that Orth(E) is commutative. If E has the principal projection property, then the converse holds in the sense that if $T \in \mathcal{L}_b(E)$ and $T\pi = \pi T$ for all $\pi \in \mathrm{Orth}(E)$, then $T \in \mathrm{Orth}(E)$. In fact, the following slightly stronger result can be proven.

THEOREM 2.3. *Suppose* E *is a Riesz space which has the principal projection property, and* $T \in \mathcal{L}_b(E)$ *satisfies* $TP = PT$ *for all projections* $P \in \mathcal{P}(E)$. *Then* T *is an orthomorphism.*

Another important result is given in the following theorem.

THEOREM 2.4. *Every orthomorphism* π *on a Riesz space* E *is order continuous. Furthermore, if* S *is a subset of* E *and* $\pi_1, \pi_2 \in \mathrm{Orth}(E)$ *satisfy* $\pi_1 f = \pi_2 f$ *for all* $f \in S$, *then* $\pi_1 f = \pi_2 f$ *for all* $f \in \{S\}^{dd}$.

We next discuss some results about Riesz spaces whose centers possess one of the following important properties.

DEFINITION 2.5.

i) A Riesz space E *is said to have an* algebraically rich center *if for any* f, g $\in E_+$ *satisfying* $0 \leq g \leq f$, *there exists an operator* $\pi \in Z(E)_+$ *such that* $0 < \pi f < g$ *and* $\pi h = 0$ *for all* $h \in \{f\}^d$.

ii) A Riesz space E *is said to have a* transitive center *if for all* f, g $\in E_+$ *satisfying* $0 \leq g \leq f$, *there exists an operator* $\pi \in Z(R)_+$ *such that* $\pi f = g$ *and* $\pi h = 0$ *for all* $h \in \{f\}^d$.

Obviously every Riesz space with transitive center has an algebraically rich center. Also, if E has a transitive center, f, g $\in$ E and $|g| \leq |f|$, it is easy to see that there exists an operator $\pi \in Z(E)$ such that $g = \pi f$.

It follows from [M2], 1.13 that E has algebraically rich center iff for every band $B \subset E$, there exists an operator $0 \neq \pi \in Z(E)_+$ such that $\pi(E) \subset B$. Some other characterizations can also be found in [M2].

Our next two results, which are well known, give examples of spaces with these two properties.

THEOREM 2.6. *Let* E *be a Banach lattice with a quasi-interior point* e. *Then* E *has an algebraically rich center.*

Proof. Let K be the structure space of E (see [S] III, §4). E may be identified with a Riesz space of continuous functions $\hat{E}$ on K, which are infinite on at most a rare subset of K. For every band $B \subset \hat{E}$, there exists an open set $\theta \subset K$ s.t. $B = \{f \in \hat{E}: f(x) \neq 0 \text{ for all } x \in \theta\}$. By Urysohn's lemma, there exists a function $g \in C(K)$ s.t..

$0 \leq g(x) \leq 1$ for all $x \in K$ and $g(x) = 0$ for all $x \notin \theta$. Define $\pi\colon \hat{E} \to \hat{E}$ by $\pi h = g \cdot h$. It is evident that $\pi \in Z(\hat{E})$ and $\pi(\hat{E}) \subset B$, which completes the proof.

The next result is due to Luxemburg([L] ch. 3; 7.6).

THEOREM 2.7. *Every Dedekind σ-complete Riesz space* E *has a transitive center.*

Proof. Suppose $f, g \in E$ and $0 \leq g \leq f$. Then by Freudenthal's spectral theorem ([LZ] 40.2), for each $j \in \mathbb{N}$, there exists a natural number n_j, constants α_{ij} and projections P_{ij} ($i = 1, 2, \ldots, n_j$) s.t. the elements $g_j := \sum_{i=1}^{n_j} \alpha_{ij} P_{ij} f$ satisfy $0 \leq g_j \uparrow g$ f-uniformly in E. Since E has the principal projection property, the P_{ij} may be taken to satisfy $P_{ij}h = 0$ for all $h \in \{f\}^d$. Define $\pi_j = \sum_{i=1}^{n_j} \alpha_{ij} P_{ij}$. We have $\pi_j h = 0$ for all $j \in \mathbb{N}$ and $h \in \{f\}^d$, $\pi_j \in Z(E)$ and $0 \leq \pi_j \uparrow$. Since E and hence Z(E) is Dedekind σ-complete, there exists an element $\pi \in Z(E)$ s.t. $\pi_j \uparrow \pi$ in Z(E). It is easy to see that $\pi h = 0$ for all $h \in \{f\}^d$ and $(\pi - \pi_j)f \downarrow 0$ in E. Thus, $|\pi f - g| \leq (|\pi f - \pi_j f| + |\pi_j f - g|) \downarrow 0$ in E, which shows that $\pi f = g$ and proves the theorem.

While a uniformly complete Riesz space need not have a transitive center (e.g. C[0,1]), the next result says that "locally" this is the case.

PROPOSITION 2.8. *Suppose* E *is a uniformly complete Riesz space. Then for all* $f, g \in E$ *such that* $0 \leq g \leq f$, *there exists an element* $\pi \in Z(J_f)$ *such that* $g = \pi f$.

Proof. By the Yosida representation theorem ([LZ] 45.4), J_f can be identified with a $C(X)$ space, where X is a compact Hausdorff space, such that f is identified with the constant function 1. Define $\pi \in Z(C(X))$ by $\pi h = g \cdot h$. It is obvious that $\pi f = g$.

Remark: The above proposition can also be proven without representation theory, see [dP1] 19.5.

Using a similar argument as above, we now prove a local polar decomposition theorem for disjointness preserving operators.

THEOREM 2.9. *Let* E *and* F *be Riesz spaces with* F *uniformly complete. Suppose that* $T \in \mathcal{L}_b(E,F)$ *is disjointness preserving. Then for all* $f \in E$, *there exists an operator* $\pi \in Z(J_{Tf})$ *such that* $Tg = \pi|T|g$ *for all* $g \in J_f$.

Proof. By the Yosida representation theorem ([LZ] §45), there exist compact Hausdorff spaces X and Y s.t. J_f can be identified with a Riesz subspace $\hat{E}$ of $C(X)$ and J_{Tf} can be identified with $C(Y)$. Then $T|_{J_f}$ can be considered to be a map $\hat{T}: \hat{E} \to C(Y)$. Hence, since T is disjointness preserving, there exists a continuous function $h \in C(Y)$ and a continuous map $\varphi: Y \to X$ s.t. $h(y) \neq 0$ and $\hat{T}g(y) = h(y)g(\varphi(y))$ for all $g \in \hat{E}$ and $y \in Y$. Define $\pi \in Z(C(Y))$ to be multiplication by $\frac{h}{|h|}$. It is clear that $\hat{T} = \pi|\hat{T}|$, which proves the theorem.

Remark: If F is Dedekind complete, a global polar decomposition can be proved. Since we will not need this result, the proof will be omitted; the real case follows easily from theorem 1.8.

Recall that a Riesz space E is said to have the *σ-interpolation property* if for any sequences $\{f_m\}$, $\{g_n\}$ in E which satisfy $f_m \uparrow \leq g_n \downarrow$, there exists in element $h \in E$ such that $f_m \leq h \leq g_n$ for all m and n.

Every Dedekind σ-complete Riesz space has the σ-interpolation property, and the σ-interpolation property, in turn, implies uniform completeness.

In contrast to Riesz homomorphisms, the range of a disjointness preserving operator T need not be a Riesz subspace. Indeed, this may not hold even if T is an orthomorphism. However, if E has the σ-interpolation property, then the range of every orthomorphism on E is a Riesz subspace (in fact, it is even an ideal, see [Z] 146.7, [dP1] 16.4, [HP]). The next theorem generalizes this to disjointness preserving operators.

THEOREM 2.10. *Let* E *and* F *be Riesz spaces and suppose* F *has the* σ*-interpolation property. Then the range of every disjointness preserving operator* $T \in \mathcal{L}_b(E,F)$ *is a Riesz subspace of* F.

Proof. It suffices to show that $f \in TE$ implies $|f| \in TE$. To this end, suppose $f \in TE$. By theorem 2.9, there exists $\pi \in Z(J_{Tf})$ s.t. $T_g = \pi|T|g$ for all $g \in J_{Tf}$. Since $|T|$ is a Riesz homomorphism and J_f is a Riesz subspace of E, $|T|J_f$ is a Riesz subspace of F. Thus, $\pi|T|J_f$ is a Riesz subspace, since F and hence J_{Tf} has the σ-interpolation property. It follows that $|f| \in \pi\,|T|J_f \subset TE$, so TE is a Riesz subspace.

We conclude this chapter by discussing some extension properties of contractors. Let S and V be subspaces of a Riesz space E and suppose $S \subset V$. Let $T: S \to S$ be an operator on S. T is said to have an *extension* to V if there exists an operator $\overline{T}: V \to V$ such that $\overline{T}|_S = T$.

THEOREM 2.11. *Let* R *be a Riesz subspace of a uniformly complete Riesz space* E, *and let* J *be the ideal generated by* R. *Then every* $S \in \mathrm{Con}(R)$ *has a (unique) extension* $\overline{S} \in \mathrm{Con}(J)$.

Proof. Suppose $g \in J$. Then there is an element $f \in R$ which satisfies $|g| \leq |f|$. By proposition 2.8, there exists an operator $\pi \in Z(J_f)$ s.t. $\pi f = g$. Define $\overline{S}: J \to J$ by $\overline{S}g := \pi Sf$. Note that this definition makes sense since S preserves ideals. It remains to show that the definition is independent of the choice of f. Suppose $f' \in R$ satisfies $|g| \leq |f'|$. Let $h = |f| \vee |f'|$. There exist elements $\pi' \in Z(J_{f'})$ and $\pi_f, \pi_{f'}, \pi_g \in Z(J_h)$ s.t. $\pi' f' = g$, $\pi_f h = f$, $\pi_{f'} h = f'$ and $\pi_g h = g$. We have $\pi Sf = \pi S \pi_f h = \pi \pi_f Sh = \pi_g Sh = \pi' \pi_{f'} Sh = \pi' S \pi_{f'} h = \pi' S f'$. This shows that $\overline{S}$ is well defined; it is clear that $\overline{S} \in Con(J)$ and that this extension is unique.

It has been shown by B. de Pagter ([dP1], 20.1) that elements of the center of an ideal in a uniformly complete Riesz space can be extended to the uniform closure of that ideal. Our next theorem follows from this result and theorem 2.11. For the sake of completeness, we present a proof (see also [Wi 1]).

THEOREM 2.12. *Let* R *be a Riesz subspace of a uniformly complete Riesz space* E. *Then every* $\pi \in Z(R)$ *has a unique extension* $\overline{\pi} \in Z(\overline{J})$ *to the uniformly closed ideal* $\overline{J}$ *generated by* R.

Proof. By the last theorem, we only have to show that every $\pi \in Z(J)$ can be extended to $\overline{J}$, where J denotes the ideal generated by R. It can be assumed that $0 \leq \pi \leq I$. Define $\mathcal{J}$ to be the collection of all ideals A s.t. $J \subset A \subset \overline{J}$ and π has an extension π_A to A satisfying $0 \leq \pi_A \leq I$. Define $H := \bigcup_{A_\alpha \in \mathcal{J}} A_\alpha$. It is clear that H is solid. Suppose $A, B \in \mathcal{J}$ and denote the extensions by π_A and π_B respectively. Since J is order dense in $A \cap B$, π_A and π_B agree on $A \cap B$. It follows that π can be extended to $A + B$ by defining $\pi_{A+B}(f+g) = \pi_A f + \pi_B g$ $(f \in A, g \in B)$. This shows that H is an ideal, so $H \in \mathcal{J}$. Let $\overline{\pi}$ be the extension of π to H, and let H' be the relative uniform pseudo-closure of H. It follows from $0 \leq \overline{\pi} \leq I$ and the r.u. completeness of E that if $f_n \to f$ r.u.$(f_n \in H, f \in E)$ then $\overline{\pi} f_n$ converges relative uniformly. Hence, π can be extended to H'. It follows that H = H', so H is r.u. closed ([LZ] 16.6), which shows that $H = \overline{J}$ and the proof is complete.

THEOREM 2.13. *Let* R *be a Riesz subspace of a Dedekind complete Riesz space* E *and suppose* $\pi \in Z(R)$. *Then* π *can be extended uniquely to an element* $\overline{\pi} \in Z(\{R\}^{dd})$. *Hence,* π *can be extended to an element* $\pi' \in Z(E)$ *such that* $\pi'(E) \subset \{R\}^{dd}$ *and* $\pi' f = \overline{\pi} f$ *for all* $f \in \{R\}^{dd}$.

Proof. By theorem 2.11, it can be assumed that R is an ideal. It also may be assumed that π is positive. Suppose $0 \leq f \in \{R\}^{dd}$. Then there exists a net $\{f_\alpha\}$ in R s.t. $f_\alpha \uparrow f$. Define $\overline{\pi} f = \sup_\alpha \pi f_\alpha$. Since π is order continuous, $\overline{\pi}$ is well defined on $\{R\}^{dd}_+$ and hence has a unique extension to $\{R\}^{dd}$. It is immediate that $\overline{\pi} \in Z(\{R\}^{dd})$. The last statement follows by defining $\pi' f = \overline{\pi} P f$, where $P: E \to \{R\}^{dd}$ is the band projection onto $\{R\}^{dd}$.

Remark: The last theorem can also be obtained by using the vector-valued Hahn-Banach theorem (see [LS1], [Wi 4]). Our approach has the advantage of actually constructing the extension, as well as yielding the intermediate results 2.11 and 2.12. Some other extension properties of orthomorphisms can be found in [dP1] and [Wi 4].

The proof of the next theorem follows immediately from 2.4, 2.11, 2.12 and 2.13.

THEOREM 2.14. *Suppose* E *is a uniformly complete Riesz space and* R *is a Riesz subspace of* E. *Let* J *and* $\overline{J}$ *denote the ideal and closed ideal generated by* R, *respectively. Then* $\mathrm{Con}(R) \simeq \mathrm{Con}(J)$ *and* $Z(R) \simeq Z(\overline{J})$. *If* E *is Dedekind complete, then* $Z(R) \simeq Z(\{R\}^{dd})$ *and there exists a unique embedding* $i: Z(R) \to Z(E)$ *such that* $i(\pi)f = 0$ *for all* $\pi \in Z(R)$ *and* $f \in \{R\}^d$.

Chapter 3

THE ASSOCIATED HOMOMORPHISMS

It was shown in proposition 1.4 that every bounded disjointness preserving operator between L^p spaces has associated with it a Boolean homomorphism between the underlying measure algebras and an f-algebra homomorphism between the corresponding L^∞ spaces. The purpose of this chapter is to generalize these results to arbitrary Riesz spaces and to relate the properties of a disjointness preserving operator with those of its associated homomorphisms.

If $E = L^p(X, \Sigma, \mu)$ $(1 \leq p \leq \infty)$, then its underlying measure algebra is isomorphic to the Boolean algebra of bands of E, $\mathcal{B}(E)$, and $L^\infty(X, \Sigma, \mu)$ is isomorphic to $Z(E) = \mathrm{Orth}(E)$. Thus, the associated homomorphisms of a disjointness preserving operator $T \in \mathcal{L}_b(E,F)$ between arbitrary Riesz spaces will be formulated under appropriate conditions in terms of a Boolean homomorphism defined on $\mathcal{B}(E)$ and an f-algebra homomorphism defined on Z(E) (or Con(E) or, under certain restrictions, Orth(E)).

We first discuss the associated f-algebra homomorphism. This type of construction was first introduced by A. W. Wickstead [Wi 1] for a lattice homomorphism on a Banach lattice; several authors then gave some variations on this idea (see [M3] [Sd 2] [Ar 2]). Theorem 3.1 and corollary 3.2 give generalizations of these results. We then show that the properties of a disjointness preserving operator are closely linked with those of its associated f-algebra homomorphism on the center, provided that the centers of the spaces have sufficiently many elements.

The associated Boolean homomorphism defined on the bands is discussed in the last part of the chapter, using a construction introduced by

W.A.J. Luxemburg [L] (definition 3.15). We then discuss the relationship between a disjointness preserving operator and its two associated homomorphisms. The advantage of the associated Boolean homomorphism over the f-algebra homomorphism is that its properties closely reflect those of the original operator even without any assumptions on the spaces. However, it is often more difficult to work with than the f-algebra homomorphism. Both associated homomorphisms will be used frequently in the last two chapters.

THEOREM 3.1. *Let* E *and* F *be Riesz spaces. Suppose* $T \in \mathcal{L}_b(E,F)$ *is disjointness preserving and* TE *is a Riesz subspace of* F. *Then there exists an f-algebra homomorphism* $\tilde{T}$: Orth(E) $\to$ Orth(TE) *defined by* $\tilde{T}(\pi)Tf = T\pi f$ *for all* $\pi \in$ Orth(E) *and* $f \in E$. *Furthermore,* $\tilde{T} = |T|^{\sim}$, $\tilde{T}$ (Con(E)) $\subset$ Con(TE) *and* $\tilde{T}$(Z(E)) $\subset$ Z(TE).

Remark: The hypothesis that TE is a Riesz subspace holds in particular whcn T is positive or when F has the σ-interpolation property (thm. 2.10).

Proof. Since Ker T is a uniformly closed ideal and each $\pi \in$ Orth(E) leaves such ideals invariant ([dP 1] 15.2), $Tf = 0$ implies $T\pi f = 0$. Hence, $Tg = Th$ implies $T\pi g = T\pi h$, which shows that $\tilde{T}(\pi)$ is well defined. It is clearly linear.

Now suppose $0 \leq Tg \leq Tf$. Then by thm. 1.5 and [LZ] 59.1, there exists an element $k \in$ Ker T s.t. $|g| \leq |f| + k$. Hence if $\pi \in$ Orth(E), $|\tilde{T}(\pi)Tg| = |T\pi g| \leq |T\pi(|f| + k)| = |\tilde{T}(\pi)Tf|$ so $\tilde{T}(\pi)$ is order bounded.

Next, suppose $Tf \perp Tg$. By replacing f and g by their absolute values if necessary, it can be assumed that $f,g \geq 0$. Define $f' = f - f \wedge g$

and $g' = g - f \wedge g$. Since $|T(f \wedge g)| = |Tf| \wedge |Tg| = 0$, it follows that $Tf = Tf'$ and $Tg = Tg'$. As it is clear that $f' \perp g'$, for any $\pi \in Orth(E)$ we have $\pi f' \perp g'$. Thus, since T is disjointness preserving, $0 = |T\pi f'| \wedge |Tg'| = |\tilde{T}(\pi)Tf'| \wedge |Tg'| = |\tilde{T}(\pi)Tf| \wedge |Tg|$. We have therefore shown that $Tf \perp Tg$ implies $\tilde{T}(\pi)Tf \perp Tg$, so $\tilde{T}(\pi) \in Orth(E)$.

For any $\pi \in Orth(E)$ and $f \in E$ s.t. $Tf \geq 0$ we have $|\tilde{T}(\pi)|Tf = |\tilde{T}(\pi)Tf| = |T\pi f| = |(T|\pi|f)| = \tilde{T}(|\pi|)Tf$, so $\tilde{T}$ is a Riesz homomorphism. Moreover, for any $\pi_1, \pi_2 \in Orth(E)$ and $f \in E$, $\tilde{T}(\pi_1\pi_2)Tf = T\pi_1\pi_2 f = \tilde{T}(\pi_1)T\pi_2 f = \tilde{T}(\pi_1)\tilde{T}(\pi_2)Tf$, which shows that $\tilde{T}$ is an f-algebra homomorphism.

Next, we show that $\tilde{T} = |T|^{\sim}$. Note that this makes sense because $|TE| = ||T|E|$, so $TE = |T|E$ since they are both Riesz subspaces of F. For any $\pi \in Orth(E)_+$ and $f \in E$ s.t. $Tf \geq 0$, we have $\tilde{T}(\pi)Tf = T\pi f = |T|\pi|f| = |T|^{\sim}(\pi)\ |T||f| = |T|^{\sim}(\pi)Tf$.

It follows that $\tilde{T}(\pi) = |T|^{\sim}(\pi)$, whence $\tilde{T} = |T|^{\sim}$, since they agree on the positive cone of Orth(E).

Now suppose that $\pi \in Con(E)$ and J is an ideal in TE. Then $T^{-1}(J)$ is an ideal in E by corollary 1.6. Therefore $\tilde{T}(\pi)J = \tilde{T}(\pi)T(T^{-1}(J)) = T\pi(T^{-1}(J)) \subset T(T^{-1}(J)) = J$. Hence $\tilde{T}(\pi) \in Con(TE)$.

Finally, suppose $\pi \in Z(E)$, so there exists a positive real number λ s.t. $|\pi f| \leq \lambda|f|$ for all $f \in E$. Then for any $f \in E$, $|\tilde{T}(\pi)Tf| = |T\pi f| = |T||\pi f| \leq \lambda|T||f| = \lambda|Tf|$, which shows that $\tilde{T}(\pi) \in Z(E)$. This completes the proof.

Remark: Theorem 3.1 has an interesting application to the theory of f-algebras. If A is an f-algebra, A can be canonically embedded in Orth(A) by the map $i : f \mapsto \pi_f$, where π_f is defined as $\pi_f(g) = f \cdot g$ for all $g \in A$.

Now suppose A and B are f-algebras, and $T: A \to B$ is a surjective f-algebra homomorphism. It is easy to see that $\tilde{T}(\pi_f) = \pi_{Tf}$ for all $f \in A$, which gives the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Orth}(A) & \xrightarrow{\tilde{T}} & \mathrm{Orth}(B) \\ i \uparrow & & \uparrow i \\ A & \xrightarrow[T]{} & B \end{array}$$

In other words, every surjective f-algebra homomorphism $T: A \to B$ can be extended (under the canonical embedding) to an f-algebra homomorphism $\tilde{T}: \mathrm{Orth}(A) \to \mathrm{Orth}(B)$.

If we restrict our attention to Con(E) and Z(E), we can obtain a better result than theorem 3.1 by applying the extension theorems proved in the last chapter.

COROLLARY 3.2. (c.f. [Wi 1], [M3], [Sd 2], [Ar 2])

Let E *and* F *be Riesz spaces with* F *uniformly complete, and suppose* $T \in \mathcal{L}_b(E,F)$ *is disjointness preserving. Denote the ideal and uniformly closed ideal generated by* TE *in* F *by* J *and* $\overline{J}$ *respectively.*

i) There exists a (unique) f-algebra homomorphism $\tilde{T}: \mathrm{Con}(E) \to \mathrm{Con}(J)$ *such that* $\tilde{T}(\pi)T = T\pi$ *for all* $\pi \in \mathrm{Con}(E)$.

ii) There exists a (unique) f-algebra homomorphism $\tilde{T}: Z(E) \to Z(\overline{J})$ *such that* $\tilde{T}(\pi)T = T\pi$ *for all* $\pi \in Z(E)$.

Furthermore, in both i) and ii) we have $\tilde{T} = \widetilde{|T|}$.

Proof. If T is positive, i) follows immediately from theorems 3.1 and 2.11. For an arbitrary disjointness preserving operator T, define $\tilde{T} = |T|^{\sim}$ (note that the ideals generated by TE and |T|E in F are the same). We must show that $\tilde{T}(\pi)T = \pi T$ for all $\pi \in \mathrm{Con}(E)$. By theorem 2.9, for every $f \in E$ there exists an operator $\pi_f \in Z(J_{Tf})$ such that $Tf = \pi_f |T| f$. Therefore, $T\pi f = \pi_f |T| \pi f = \pi_f \tilde{T}(\pi)|T|f = \tilde{T}(\pi)\pi_f |T| f = \tilde{T}(\pi)Tf$. This proves i). The second statement follows from the first and from theorem 2.12.

We now consider the relationship between a disjointness preserving operator T and its associated f-algebra homomorphism $\tilde{T}$ defined in the theorems given above. Since we are primarily interested in the $\tilde{T}$ map defined on the center, we single it out in the following definition.

DEFINITION 3.3. *Let* E *and* F *be Riesz spaces with* F *uniformly complete and let* $T \in \mathcal{L}_b(E,F)$ *be disjointness preserving. We shall call the f-algebra homomorphism* $\tilde{T} : Z(E) \to Z(\overline{J_{TE}})$ *defined in corollary 3.2 the* associated operator *of* T.

Warning: In the end of this chapter, we will slightly modify the definition of the associated operator. See page 38.

We now wish to relate the properties of a disjointness preserving operator T with those of its associated operator $\tilde{T}$. However, if the center of the domain or range of $\tilde{T}$ is trivial (i.e. consists of scalar multiples of the identity operator only) then it is clear that the properties of T cannot be reflected accurately in $\tilde{T}$. Hence, in the following we will typically need a condition which connects a Riesz space with its center. Two such conditions have already been defined (definition 2.5).

We give two more such conditions below.

DEFINITION 3.4. *Let* E *be a Riesz space.*

i) We say that E *has a* regular center *if it follows from* $\pi_\alpha \downarrow 0$ *in* Z(E) *that* $\pi_\alpha f \downarrow 0$ *in* E *for all* $f \in E_+$.

ii) We say that E *has a* uniformly rich center *if for every uniformly closed ideal* $J \subset E$ *there exists a non-zero element* $\pi \in Z(E)$ *such that* $\pi(E) \subset J$.

The condition that a Riesz space has regular center is very weak. In fact, the author does not know of an example of a Riesz space which does *not* have a regular center. It is clear that every Riesz space whose center is trivial has regular center, and our next result, due to Meyer [M2], shows the same is true when the center is algebraically rich.

PROPOSITION 3.5. *Every Riesz space* E *whose center is algebraically rich has a regular center.*

Proof. Suppose $\pi_\alpha \downarrow 0$ in Z(E) but $\pi_\alpha e \geq f > 0$ for some $e,f \in E_+$ and all α. Since Z(E) is algebraically rich, there exists an element $\pi \in Z(E)$ such that $0 < \pi e \leq f$ and $\pi h = 0$ for all $h \in \{e\}^d$. By theorem 2.4, $\pi_\alpha g \geq \pi g \geq 0$ for all $g \in \{e\}^{dd}$. Therefore, since $\pi h = 0$ for all $h \in \{f\}^d$, $\pi_\alpha \geq \pi > 0$, a contradiction.

In constrast to regularity of the center, the condition that a space has uniformly rich center is quite strong. Any Banach lattice with order continuous norm as well as any Dedekind complete Banach lattice with a quasi-interior point has uniformly rich center. It seems to be unknown whether or not every Dedekind complete Riesz space (or Banach lattice) has uniformly rich center.

THEOREM 3.6. *Let* E *and* F *be Riesz spaces and let* $T \in \mathcal{L}_b(E,F)$ *be disjointness preserving. Suppose* E *has a transitive center and* $\overline{J_{TE}}$ *has a regular center. Then* T *is order continuous iff its associated operator* $\tilde{T}$ *is order continuous.*

It should be remarked that the theorem holds in particular when E and F are Dedekind σ-complete.

Proof. Suppose T is order continuous and $\pi_\alpha \downarrow 0$ in Z(E). Since the center of E is transitive, it is algebraically rich. Thus, by proposition 3.5, $\pi_\alpha f \downarrow 0$ for all $f \in E_+$. Hence, since T is order continuous, $\tilde{T}(\pi_\alpha)Tf = T\pi_\alpha f \downarrow 0$. This shows that if $0 \le \pi \le \tilde{T}(\pi_\alpha)$ for all α, then $\pi = 0$ on TE. It now follows from theorem 2.4 that $\pi = 0$ on $\overline{J_{TE}}$, whence $\tilde{T}$ is order continuous.

Conversely, suppose $\tilde{T}$ is order continuous, and $f \ge f_\alpha \downarrow 0$ in E. Since E has transitive center there exist elements $0 \le \pi_\alpha \in Z(E)$ such that $\pi_\alpha f = f_\alpha$ and $\pi_\alpha h = 0$ for all $h \in \{f\}^d$. Then $\pi_\alpha g$ order converges to zero for all g in the order dense set $\{f\} \cup \{f\}^d$, which shows that $\pi_\alpha \downarrow 0$ in Z(E). Since $\tilde{T}$ is order continuous $\tilde{T}(\pi_\alpha) \downarrow 0$ in $Z(\overline{J_{TE}})$. Therefore, since $\overline{J_{TE}}$ has regular center, $Tf_\alpha = T\pi_\alpha f = \tilde{T}(\pi_\alpha)Tf \to 0$ in order, which shows that T is order continuous.

Example. This example shows that the assumption on E given in the last theorem is essential. Let E be the Riesz space of all continuous, piecewise linear functions on [0,1] and let F be the Riesz space of all bounded functions on [0,1]. Let $T: E \to F$ be the natural embedding. It is clear that T is not order continuous. However, Z(E) consists of scalar multiples of the identity operator (see [L] ch. 3, thm. 8.2), so $\tilde{T}$ is order continuous.

THEOREM 3.7. *Let* E *and* F *be uniformly complete Riesz spaces. Suppose* $T \in \mathcal{L}_b(E,F)$ *is disjointness preserving with associated operator* $\tilde{T}$. *Consider the following statements:*

a) T *is injective.*

b) $\tilde{T}$ *is injective.*

Then a) ⇒ *b). If the center of* E *is algebraically rich and any one of the following conditions are satisfied, then b)* ⇒ *a), so a) and b) are equivalent.*

i) T *is order continuous.*

ii) $\tilde{T}$ *is surjective and* E *has transitive center.*

iii) E *has uniformly rich center.*

iv) E *and* F *are Banach lattices and* E *has a quasi-interior point.*

Proof. Suppose $\tilde{T}$ is not injective. Pick $0 \neq \pi \in \operatorname{Ker} \tilde{T}$. Then there exists an element $0 \neq f \in E$ s.t. $\pi f \neq 0$. Since $T\pi f = \tilde{T}(\pi)\, Tf = 0$, T is not injective.

Now suppose that the center of E is algebraically rich and that T is not injective. If i) holds, then Ker T is a band, so there exists an element $0 \neq \pi \in Z(E)$ such that $\pi(E) \subset \operatorname{Ker} T$ since Z(E) is algebraically rich. Similarly, if (iii) holds, then there is an element $0 \neq \pi \in Z(E)$ such that $\pi(E) \subset \operatorname{Ker} T$ since Ker T is a uniformly closed ideal. In either case we have $\tilde{T}(\pi)Tf = T\pi f = 0$ for all $f \in E$, so $\tilde{T}(\pi) = 0$ and hence $\tilde{T}$ is not injective.

If ii) holds and $\tilde{T}$ is injective, then $\tilde{T}$ is invertible and hence order continuous. By theorem 3.6 and part i) proved above, T is injective.

Finally, suppose E and F are Banach lattices and E has quasi-interior point e. Suppose T is not injective so $Tf = 0$ for some non-zero element $f \in E$. Since Ker T is an ideal, it can be assumed that $0 < f < e$. By theorem 2.6 there exists an operator $\pi \in Z(E)_+$ such that $\pi e \leq f$ and $\pi g = 0$ for all $g \in \{f\}^d$. Hence, $0 = |T|f \geq |T|\pi e = \tilde{T}(\pi)|T|e \geq 0$, from which it follows that $\tilde{T}(\pi) = 0$, since Te is quasi-interior to $\overline{J_{TE}}$.

Remark: The equivalence of a) and b) was first observed for a lattice homomorphism between Banach lattices with quasi-interior points by Wickstead [Wi 1].

We next wish to investigate the duality relationships of disjointness preserving operators. If E is a Riesz space, we will denote the order dual of E (i.e. the Riesz space of all order bounded linear functionals on E) by E^*. If $T \in \mathcal{L}_b(E,F)$, where E and F are Riesz spaces, we will denote the order adjoint of T by T^*(i.e $T^* \in \mathcal{L}_b(F^*,E^*)$ is the restriction of the algebraic adjoint to F^*). The following result is due to W. Arendt [Ar 2].

PROPOSITION 3.8. *Let* $T \in \mathcal{L}_b(E,F)$ *be a disjointness preserving operator between Riesz spaces* E *and* F. *Then* $|T^*| = |T|^*$.

Proof. For any $0 \leq f \in E$ and $0 \leq \mu \in E^*$ we have

$\langle f, |T^*|\mu\rangle = \langle f, \sup_{|\nu|\leq\mu} |T^*\nu|\rangle \geq \sup_{|\nu|\leq\mu} |\langle f, T^*\nu\rangle| = \sup_{|\nu|\leq\mu} |\langle Tf, \nu\rangle|$

$= \langle |Tf|, \mu\rangle = \langle |T|f, \mu\rangle = \langle f, |T|^*\mu\rangle \geq \sup_{|\nu|\leq\mu} \langle f, |T^*\nu|\rangle$. Since the second and the last expression are the same, equality must hold throughout. It follows that $|T|^* = |T^*|$.

DEFINITION 3.9. *A positive operator* $T \in \mathcal{L}_b(E,F)$ *where* E *and* F *are Riesz spaces, is said to be* interval preserving *(or has the* Maharam property*) if for all* $f \in E_+$ *and all* $g \in F$ *satisfying* $0 < g < Tf$, *there exists an element* $f' \in E$ *such that* $0 \leq f' \leq f$ *and* $Tf' = g$.

Our next result is essentially a special case of the Luxemburg-Schep "Radon-Nikodym" theorem (see [LS 1] or [L] ch 4, 4.1). We include an alternate proof of this result.

PROPOSITION 3.10. *Let* E *and* F *be Riesz spaces, and suppose that* E *is Dedekind complete and* F *has a transitive center. Let* $T \in \mathcal{L}_b(E,F)$ *be an order continuous positive operator. Then* T *is interval preserving iff for every* $\pi \in Z(F)$, *there exists an element* $\pi' \in Z(E)$ *such that* $\pi T f = T\pi' f$ *for all* $f \in E$. *In particular, an order continuous Riesz homomorphism* T *is interval preserving iff its associated operator* $\tilde{T}$ *is surjective.*

Proof. Suppose that for all $\pi \in Z(F)$, there exists $\pi' \in Z(E)$ s.t. $\pi T = T\pi'$, and let $0 < g < Tf$ $(f \in E_+, g \in F_+)$. Then there exists an operator $\pi \in Z(F)$ s.t. $g = \pi T f$, and hence an operator $\pi' \in Z(E)$ s.t. $g = T\pi f$, so T is interval preserving.

Conversely, suppose T is interval preserving. Pick a disjoint order basis $\{f_\sigma\}_{\sigma\in S}$ for E (see [LZ] pg. 163). It can be assumed that $f_\sigma \geq 0$ for all $\sigma \in S$. Pick $\pi \in Z(F)$ and, without loss of generality, it can be assumed that $0 \leq \pi \leq I$. Then for each $\sigma \in S$ we have $0 \leq \pi T f_\sigma \leq T f_\sigma$. Since T is interval preserving, there exists an element $g_\sigma \in E$ s.t. $0 \leq g_\sigma \leq f_\sigma$ and $Tg_\sigma = \pi T f_\sigma$. By theorem 2.7 there exists an operator $\pi_\sigma \in Z(E)$ s.t. $\pi_\sigma f_\sigma = g_\sigma$ and $\pi_\sigma h = 0$ for all $h \in \{f_\sigma\}^d$. Now suppose $0 < x < f_\sigma$ for some $\sigma \in S$. Then there exists an operator $\pi_x \in Z(E)$ s.t. $\pi_x f_\sigma = x$. We have $T\pi_\sigma x = T\pi_\sigma \pi_x f_\sigma = \tilde{T}(\pi_x)T\pi_\sigma f_\sigma = \tilde{T}(\pi_x)Tg_\sigma = \tilde{T}(\pi_x)\pi T f_\sigma = \pi T \pi_x f_\sigma = \pi Tx$. Thus, $T\pi_\sigma x = \pi T x$ for all $x \in J_{f_\sigma}$. Define $\pi' = \sup_{\sigma\in S} \pi_\sigma$. It follows from the above that $\pi T x = T\pi' x$ for all x in the (order dense) ideal generated by the f_σ. Since T, π and π' are order continuous, it follows that $\pi T y = T\pi' y$ for all $y \in E$.

It was first observed by T. Ando that the property of being interval preserving is "almost" dual to the property of being a Riesz homomorphism (see [Lo2], [LS1], [S] III, prob. 24). We wish to give a variation of this result. To do so, we need to introduce the notion of the "absolute weak topology."

Suppose that E is a Riesz space whose order dual E^* separates the points of E. Recall that the *absolute weak topology* on E is the locally convex-solid Hausdorff topology generated by the Riesz seminorms $\{\rho_\mu : \mu \in E^*\}$ where $\rho_\mu(x) := |\mu|(|x|)$ for $x \in E$. A linear functional μ on E is continuous with respect to the absolute weak topology iff μ is order bounded. For proofs and further information, see [AB] pg. 40-41.

PROPOSITION 3.11. *Let* E *and* F *be Riesz spaces and suppose the order dual* F^* *separates the points of* F. *Let* $T \in \mathcal{L}_b(E,F)$ *be a positive operator.*

Then the order adjoint T^* *is a Riesz homomorphism iff for all positive elements* $e \in E_+$ $\overline{T[0,e]}^{|\sigma|} = [0,Te]$, *where* $\overline{T[0,e]}^{|\sigma|}$ *denotes the closure of* $T[0,e]$ *in the absolute weak topology.*

Proof. For simplicity, we will assume that E and F are real Riesz spaces. Suppose T^* is a Riesz homomorphism. Pick $f \in F_+$ and suppose $f \notin \overline{T[0,e]}^{|\sigma|}$. By the geometric Hahn-Banach theorem (c.f. [Ru] thm. 3.4), there exists a linear functional $\mu \in F^*$ and a real number α such that $\mu(f) > \alpha > \mu(g)$ for all $g \in T[0,e]$. Since T^* is a homomorphism and $f \geq 0$,
$\mu^+(f) \geq \mu(f) > \alpha > \sup_{h\in[0,e]} T^*\mu(h) = (T^*\mu)^+(e) = T^*\mu^+(e) = \mu^+(Te)$, which shows that $f \notin [0,Te]$.

Conversely, suppose that $\overline{T[0,e]}^{|\sigma|} = [0,Te]$ for all $e \in E_+$. Then for any $\mu \in F^*$ we have $(T^*\mu)^+(e) = \sup_{h\in[0,e]} T^*\mu(h) = \sup_{h\in[0,e]} \mu(Th) = \sup_{g\in[0,Te]} \mu(g) = \mu^+(Te) = T^*\mu^+(e)$ which shows that T^* is a homomorphism.

PROPOSITION 3.12. *Let* E *and* F *be Dedekind complete Riesz spaces and suppose* $T \in \mathcal{L}_b(E,F)$ *is an order continuous Riesz homomorphism. Consider the following two statements:*

a) T *is interval preserving*

b) The order adjoint T^* *is a Riesz homomorphism.*

Then a) ⇒ *b). If the order dual* F^* *separates the points of* F, *then b)* ⇒ *a) so the two statements are equivalent.*

Proof. a) ⇒ b) follows from proposition 3.11. Conversely, suppose F^* separates the points of F and that T^* is a Riesz homomorphism. Pick $e \in E_+$ and suppose $0 < g < Te$ for some $g \in F$. By proposition 3.11 there exists a net $\{g_\alpha\}_{\alpha\in A}$ s.t. $g_\alpha \to g$ $|\sigma|(E,E^*)$, and for each $\alpha \in A$ there exists an element $h_\alpha \in E$ s.t. $0 \le h_\alpha \le e$ and $T h_\alpha = g_\alpha$. For any $\alpha \in A$, define $f_\alpha = \inf_{\beta>\alpha} h_\beta$. Note that $e \ge f_\alpha \uparrow$, so there exists an element $f \in E_+$ s.t. $f_\alpha \uparrow f$ in E because E is Dedekind complete. Since T is an order continuous Riesz homomorphism, $T f = T \sup_\alpha \inf_{\beta>\alpha} h_\alpha = \sup_\alpha \inf_{\beta>\alpha} T h_\alpha = \lim \inf_\alpha g_\alpha = g$, which shows that T is interval preserving.

We collect the preceding results in the following theorem.

THEOREM 3.13. *Let* E *and* F *be Dedekind complete Riesz spaces and suppose* $T \in \mathcal{L}_b(E,F)$ *is an order continuous disjointness preserving operator. Consider the following statements:*

a) $|T|$ *is interval preserving*

b) TE *is an ideal in* F.

c) The associated operator $\tilde{T}$ *is surjective.*

d) The order adjoint $|T|^*$ *is a Riesz homomorphism.*

e) T^* *is disjointness preserving*

Then a), b) and c) are equivalent, as are d) and e). Furthermore a) $\Rightarrow$ *d) and if the order dual* F^* *separates the points of* F *then d)* $\Rightarrow$ *a) so all five statements are equivalent.*

Proof. a) $\Leftrightarrow$ c) is proposition 3.10. a) $\Rightarrow$ b): Suppose $f \in E$ and $g \in F$ satisfy $|g| \le |T f|$. Then there exists an element $h \in E_+$ s.t. $T h = |g|$. By theorem 2.7, there exists an operator $\pi \in Z(F)$ s.t. $g = \pi|g| = \pi T h$. By c), there exists an operator $\pi' \in Z(E)$ s.t. $g = \pi T h = T \pi' h$, which shows that TE is an ideal.

b) $\Rightarrow$ a) follows from $|Tf| = |T||f|$. d) $\Leftrightarrow$ e) is obtained by applying proposition 3.8. a) $\Rightarrow$ d) follows from proposition 3.12 as does d) $\Rightarrow$ a) when F^* separates the points of F.

Remark: The assumption in theorem 3.13 that F is Dedekind complete is essential. Indeed, if $E = C[0,1]$ there are elements of $Z(E)$ (whence their adjoints are in $Z(E^*)$) whose ranges are not even Riesz subspaces.

If E is a uniformly complete Riesz space, its center is uniformly complete and has a strong order and algebra unit I. Hence $Z(E)$ is f-algebraically isomorphic to a space of the type $C(K)$, where K is a compact Hausdorff space. If $T \in \mathcal{L}_b(E,F)$ is disjointness preserving, then its associated operator $\tilde{T}$ can thus be considered as an f-algebra homomorphism $\tilde{T}: C(X) \to C(Y)$, where X and Y are compact Hausdorff spaces, $C(X) \simeq Z(E)$ and $C(Y) \simeq Z(\overline{J_{TE}})$. Hence, by proposition 1.3, there is a continuous map $\varphi_T : Y \to X$ s.t. $\tilde{T}f(x) = f(\varphi_T(x))$ for all $f \in C(X)$. We now investigate the relationship between T and φ_T.

THEOREM 3.14. *Let* E *and* F *be Dedekind complete Riesz spaces and suppose* $T \in \mathcal{L}_b(E,F)$ *is disjointness preserving. Identify* $Z(E) \simeq C(X)$ *and* $Z(\{TE\}^{dd}) \simeq Z(\overline{J_{TE}}) \simeq C(Y)$ *and let* $\varphi_T : Y \to X$ *be the continuous map as defined above.*

a) T *is order continuous iff* φ_T *is an open mapping.*

b) If T *is injective then* φ_T *is surjective. Conversely, if* φ_T *is surjective and any of conditions i) - iv) listed in theorem 3.7 is satisfied, then* T *is injective.*

c) If T *is order continuous, then* TE *is an ideal in* F *iff* φ_T *is injective.*

Proof. Statement a) follows immediately from theorem 3.6 and [S] III 9.3; b) follows from theorem 3.7 and [S] III 9.3.

To prove c), first suppose that φ_T is not injective, so for some $y_1, y_2 \in Y$, $\varphi_T(y_1) = \varphi_T(y_2)$. Then for all $f \in C(Y)$, $\tilde{T}f(y_1) = \tilde{T}f(y_2)$. It follows from Urysohn's lemma that $\tilde{T}$ is not surjective, and hence by theorem 3.13 TE is not an ideal.

Conversely, suppose that φ_T is injective. By factoring out the kernel of T if necessary, it can be assumed that $\tilde{T}$ is injective and hence φ_T is surjective ([S] III 9.3), i.e. φ_T is a homeomorphism. This says that $\tilde{T}$ is invertible and hence surjective. By theorem 3.13, TE is an ideal.

Remark: The previously mentioned result [S] III 9.3 was proved by Nagel [N]. Some other related results can be found in [Wi 2].

Let E and F be Dedekind complete Riesz spaces and let $T \in \mathcal{L}_b(E)$ be disjointness preserving. Then by theorem 2.14, the associated operator can be considered as a map $\tilde{T}: Z(E) \to Z(F)$ such that $\tilde{T}(\pi)f = 0$ for all $\pi \in Z(E)$ and $f \in \{TE\}^d$. *From this point on, the associated operator* $\tilde{T}$ *will always mean the map* $\tilde{T}: Z(E) \to Z(F)$ *constructed in this manner.*

Since $P \in \mathcal{P}(E)$ iff $P \in Z(E)$ and $P^2 = P$ (where $\mathcal{P}(E)$ denotes the collection of all band projections on E), for any $P \in \mathcal{P}(E)$, $\tilde{T}(P) \in \mathcal{P}(F)$. In fact, since $\tilde{T}$ is an f-algebra homomorphism, the restriction of $\tilde{T}$ to $\mathcal{P}(E)$ is a Boolean homomorphism from $\mathcal{P}(E)$ into $\mathcal{P}(F)$. By identifying $\mathcal{P}(E)$ and $\mathcal{P}(F)$ with their respective Boolean algebras of bands, $\tilde{T}$ induces a Boolean homomorphism from $\mathcal{B}(E)$ to $\mathcal{B}(F)$, which will be denoted by T_*.

There is another way for T to induce a map from $\mathcal{B}(E)$ into $\mathcal{B}(F)$ which requires no completeness assumption. It was introduced for Riesz homomorphisms by Luxemburg ([L] ch 3, sec. 3).

DEFINITION 3.15. *Let* $T \in \mathcal{L}_b(E,F)$ *be a disjointness preserving operator between Riesz spaces* E *and* F. *We define* $t_T : \mathcal{B}(E) \to \mathcal{B}(F)$ *by* $t_T(B) = \{TB\}^{dd}$ $(B \in \mathcal{B}(E))$.

If no ambiguity will arise, we will denote t_T simply by t. It should be remarked that t_T is not always a Boolean homomorphism (for examples see [L]). The final goal of this chapter is to investigate the relationship of $T, T_\star$ and t_T.

PROPOSITION 3.16. *Let* E *and* F *be Dedekind complete Riesz spaces, and suppose* $T \in \mathcal{L}_b(E,F)$ *is disjointness preserving. Then* $T_\star = t_T$ *so in particular* t_T *is a Boolean homomorphism.*

Proof. Let $B \in \mathcal{B}(E)$ with corresponding projection P. Then
$t_T(B) = \{TB\}^{dd} = \{TPE\}^{dd} = \{\tilde{T}(P)TE\}^{dd} = \tilde{T}(P)\{TE\}^{dd} = \tilde{T}(P)E = T_\star(B)$.

PROPOSITION 3.17. *Let* $T \in \mathcal{L}_b(E,F)$ *be an order continuous disjointness preserving operator between Riesz spaces* E *and* F. *Let* $\hat{E}$ *and* $\hat{F}$ *be the Dedekind completions of* E *and* F *respectively and let* $\hat{T}$ *be the unique extension to* $\hat{E}$ *of* T *(theorem 1.9). Let* $\hat{T}_\star : \mathcal{B}(\hat{E}) \to \mathcal{B}(\hat{F})$ *be the Boolean homomorphism induced by the associated operator of* $\hat{T}$ *as defined above and* $t_T : \mathcal{B}(E) \to \mathcal{B}(F)$ *be the map given in definition* 3.14. *Then* t_T *is a Boolean homomorphism which corresponds, under the identification of* $\mathcal{B}(E)$ *and* $\mathcal{B}(F)$ *with* $\mathcal{B}(\hat{E})$ *and* $\mathcal{B}(\hat{F})$ *to* $\hat{T}_\star$.

Proof. Since T and hence $\hat{T}$ is order continuous, it is clear that the bands generated by TE and $\hat{T}\hat{E}$ in $\hat{F}$ are the same. Therefore, t_T can be identified with $t_{\hat{T}}$ under the canonical identification on $\mathcal{B}(E)$ and $\mathcal{B}(F)$

with $\mathcal{B}(\hat{E})$ and $\mathcal{B}(\hat{F})$. Applying proposition 3.16 to $t_{\hat{T}}$ yields the result.

LEMMA 3.18. *Let* E *and* F *be Riesz spaces,* $T \in \mathcal{L}_b(E,F)$ *be disjointness preserving and suppose* E *has the principal projection property. Then the associated operator* $\tilde{T}$ *is order continuous iff the restriction of* $\tilde{T}$ *to the order projections* $\mathcal{P}(E)$ *is order continuous.*

Proof. If $\tilde{T}$ is order continuous, then obviously $\tilde{T}|_{P(E)}$ is order continuous.

Conversely, suppose $\tilde{T}|_{\mathcal{P}(E)}$ is order continuous and suppose $I \geq \pi_\alpha \downarrow 0$ in Z(E). Suppose there exists an element $\pi_0 \in Z(E)$ s.t. $\tilde{T}(\pi_\alpha) \geq \pi_0 \geq 0$. Pick $\varepsilon > 0$ and let $P_{\varepsilon,\alpha}$ be the projections onto the carrier bands of $(\pi - \varepsilon I)^+$. Note that $P_{\varepsilon,\alpha} \downarrow_\alpha 0$ in Z(E) and that $\varepsilon I + P_{\varepsilon,\alpha} \geq \pi_\alpha$ for all α. Hence, since $\tilde{T}(P_{\varepsilon,\alpha}) \downarrow_\alpha 0$, $\varepsilon I \geq \pi_0 \geq 0$. As ε is arbitrary, this implies that $\pi_0 = 0$ and completes the proof.

The final result of this chapter was obtained, with a different proof, by Luxemburg [L].

THEOREM 3.19. *Let* E *and* F *be Riesz spaces and suppose* $T \in \mathcal{L}_b(E,F)$ *is disjointness preserving. Then* T *is order continuous iff* t_T *is an order continuous Boolean homomorphism.*

Proof. If T is order continuous, so is its extension $\hat{T} \in \mathcal{L}_b(\hat{E},\hat{F})$ to the Dedekind completion of E. Hence, $\hat{T}_*$ is order continuous by theorem 3.8 and the lemma. It follows immediately from propositions 3.16 and 3.17 that t_T is an order continuous Boolean homomorphism.

Conversely, suppose t_T is an order continuous Boolean homomorphism and let $\hat{T}$ be any extension of T to $\hat{E}$. Define $\hat{t} : \mathcal{B}(E) \to \mathcal{B}(F)$ by

$\hat{t}(\mathcal{B}) = \{TB\}^{dd}$, where the "dd" is taken in $\hat{F}$. Since t is an order continuous Boolean homomorphism, so is $\hat{t}$. Furthermore, since $TB \subset \hat{T}\hat{B}$ $\hat{t}(B) \subset t_{\hat{T}}(\hat{B})$ for all $B \in \mathcal{B}(E)$ (where $\hat{B}$ is defined to be $\{B\}^{dd}(\hat{E})$). By proposition 3.16 $t_{\hat{T}}$ is a Boolean homomorphism. Hence, $\hat{t}(B^{d(E)}) = (\hat{t}(B))^{d(\hat{F})} \supset t_{\hat{T}}(B)^{d(\hat{F})} = t_{\hat{T}}(B^{d(\hat{E})})$ for all $B \in \mathcal{B}(E)$. Combining the two inclusions yields $\hat{t} = t_{\hat{T}}$. Since $t_{\hat{T}} = \hat{T}_{*}$ and $\hat{t}$ is order continuous, the restriction of the associated operator of $\hat{T}$ to $\mathcal{P}(\hat{E})$ is order continuous. By lemma 3.18 and theorem 3.6 $\hat{T}$ is order continuous, and hence T is order continuous as well.

Remark: If T is order continuous, analogues of theorems 3.7 and 3.13 can easily be obtained for t_T in place of $\hat{T}$.

Chapter 4

BI-DISJOINTNESS PRESERVING OPERATORS

This chapter studies a special type of disjointness preserving operator which is given in definition 4.1 below. The main object of this chapter is to decompose such operators into simple components whose properties can be easily analyzed (see theorems 4.13 and 4.19).

DEFINITION 4.1. *Let* E *be a Riesz space. We will say that an order continuous Riesz homomorphism* $T \in \mathcal{L}_b(E)$ *is* bi-disjointness preserving *if for every* $f, g \in E_+$ *satisfying* $0 < g < Tf$, *there exist a net* $\{h_\alpha\}$ *such that* $Th_\alpha \to g$ *in order.*

A disjointness preserving operator will be called bi-disjointness preserving if its absolute value is bi-disjointness preserving.

It is clear that the h_α in the preceding definition can be taken to satisfy $0 \le h_\alpha \le f$. Our first goal is to show that under certain conditions an order continuous disjointness preserving operator is bi-disjointness preserving iff its adjoint is disjointness preserving. This observation justifies the definition given these operators. We first give an important, though somewhat technical, characterization of bi-disjointness preserving operators.

PROPOSITION 4.2. *Let* $T \in \mathcal{L}_b(E)$ *be an order continuous disjointness preserving operator on a Riesz space* E. *Let* $\hat{E}$ *be the Dedekind completion of* E *and let* $\hat{T}$ *be the (unique) extension of* T *to* $\hat{E}$. *Then* T *is bi-disjointness preserving iff* $\hat{T}\hat{E}$ *is an ideal in* $\hat{E}$.

Proof. It can be assumed that T is a Riesz homomorphism. Suppose $\hat{T}$ is bi-disjointness preserving and $0 < \hat{f}, \hat{g} \in \hat{E}$ satisfy

$0 < \hat{g} < \hat{T}\hat{f}$. Then there exists an element f and a net $\{g_\alpha\}$ in E such that $g_\alpha \uparrow \hat{g} \leq Tf$. Since T is bi-disjointness preserving, for each α there exists a net $\{h_{\alpha\beta}\}_{\beta\in B}$ in E such that $0 \leq h_{\alpha\beta} \leq f$ and $Th_{\alpha\beta} \to g_\alpha$ in order. Define $\hat{h} = \sup_{\alpha,\beta}\{\inf_{\substack{\gamma\geq\beta \\ \beta,\gamma\in B}} h_{\alpha\gamma}\}$. Since T is an order continuous Riesz homomorphism, $\hat{T}\hat{h} = \hat{g}$, so $\hat{T}\hat{E}$ is an ideal.

Conversely, suppose $\hat{T}\hat{E}$ is an ideal and $0 < g < Tf$ in E. Then there exists an element $\hat{h} \in \hat{E}$ such that $\hat{T}\hat{h} = g$. Hence, there exist elements $h_\alpha \in E$ such that $0 \leq h_\alpha \uparrow \hat{h}$ and thus $g = T \sup h_\alpha = \sup Th_\alpha$, which shows that T is bi-disjointness preserving.

COROLLARY 4.3. *Let* E *be a Dedekind complete Riesz space whose order dual* E^* *separates the points of* E. *Let* $T \in \mathcal{L}_b(E)$ *be an order continuous disjointness preserving operator. The following are equivalent:*

a) T *is bi-disjointness preserving,*

b) TE *is an ideal in* E.

c) T^* *is disjointness preserving.*

Proof. The corollary follows immediately from theorem 3.13 and proposition 4.2.

Example. Let X be an extremely disconnected compact Hausdorff space, and let $E = C(X)$. Then every disjointness preserving operator $T \in \mathcal{L}_b(E)$ is of the form $Tf(x) = h(x)f(\varphi(x))$, where $h \in C(X)$ and φ: $\mathrm{Coz}(h) \to X$. It follows from theorem 3.14 and corollary 4.3 that T is bi-disjointness preserving iff φ is injective and an open mapping.

If E is not Dedekind complete, we can at least obtain the following result.

PROPOSITION 4.4. *Let* E *be a Riesz space such that* E^* *separates the points of* E. *If* $T \in \mathcal{L}_b(E)$ *is an order continuous disjointness preserving operator and* T^* *is also disjointness preserving then* T *is bi-disjointness preserving.*

Proof. Since E^* separates the points, every absolute weakly convergent net converges in order. The result now follows from propositions 3.11, 3.12 and the definition of bi-disjointness preserving operators.

Besides the Dedekind complete case, there is one other situation where a dual formulation of bi-disjointness preserving operators can be given. Recall that a *normal integral* of a Riesz space E is an order continuous linear functional on E. We shall denote the collection of all normal integrals by E_n^*. It is well known that E_n^* is a band in E^* and hence is itself a Dedekind complete Riesz space. If $T \in \mathcal{L}_b(E)$ is order continuous, then we will denote the restriction of T^* to E_n^* by T_n^*. Note that $T_n^* \in \mathcal{L}_b(E_n^*)$.

PROPOSITION 4.5. *Suppose* E *is a Riesz space such that* E_n^* *separates the points of* E. *Then an order continuous operator* $T \in \mathcal{L}_b(E)$ *is bi-disjointness preserving iff* T *and* T_n^* *are disjointness preserving.*

Proof. Let $T \in \mathcal{L}_b(E)$ be an order continuous disjointness preserving operator and let $\hat{T}$ be its extension to $\hat{E}$, the Dedekind completion of E. It is easy to see that each element of E_n^* can be extended uniquely to $\hat{E}$ so that E_n^* and $\hat{E}_n^*$ are Riesz isomorphic. It follows that T_n^* is disjointness preserving iff $\hat{T}_n^*$ is disjointness preserving. Therefore, by theorem 3.13 and [LS1] 4.1, $\hat{T}\hat{E}$ is an ideal iff $\hat{T}_n^*$ is disjointness preserving iff T_n^* is disjointness preserving, which proves the result.

We now introduce the four basic types of bi-disjointness preserving operators.

DEFINITION 4.6. *Let* $T \in \mathcal{L}_b(E)$ *be a bi-disjointness preserving operator on a Riesz space* E.

i) T *is said to be* quasi-invertible *if* T *is injective and* $\{TE\}^{dd} = E$.

ii) T *is said to be of* forward shift type *if* T *is injective and* $\bigcap_{n=1}^{\infty} \{T^n E\}^{dd} = \{0\}$.

iii) T *is said to be of* backward shift type *if* $\bigvee_{n=1} \operatorname{Ker} T^n = E$ *and* $\{TE\}^{dd} = E$.

iv) T *is said to be* hypernilpotent *if* $\bigvee_{n=1}^{\infty} \operatorname{Ker} T^n = E$ *and* $\bigcap_{n=1}^{\infty} \{T^n E\}^{dd} = \{0\}$.

Remark: It is easy to see that a bi-disjointness preserving operator has one of the four properties listed in the above definition iff its extention to the Dedekind completion of E has the same property (c.f. propositions 3.16 and 3.17).

Example. Let $E = \ell^\infty(\mathbb{Z})$ be the Riesz space of all bounded, doubly infinite sequences. Pick a weight sequence $\{w_n\}_{n=-\infty}^{\infty} \in E$ and let T be the weighted bilateral shift operator $T\{x_n\}_{n=-\infty}^{\infty} = \{w_n x_{n+1}\}_{n=-\infty}^{\infty}$. It is clear that T is bi-disjointness preserving. Then T is quasi-invertible iff $w_n \neq 0$ for all integers $n \in \mathbb{Z}$ and invertible iff there exists a constant $c > 0$ s.t. $w_n > c$ for all $n \in \mathbb{Z}$. T is hypernilpotent iff for each integer n there exists integers n_1, n_2 s.t. $n_1 < n < n_2$ and $w_{n_1} = w_{n_2} = 0$. Let $A = \{\{x_n\} \in E : x_n = 0 \ \forall n \leq 0\}$ and $B = A^d$. The restriction of T to A is of forward shift type iff $w_n \neq 0$ for all $n \in \mathbb{N}$. The restriction of T to B is of backward shift type iff $w_n \neq 0$ for all integers $n \leq 0$.

We begin our discussion of these four classes by describing the duality relationship between them. Let S be a subspace of a Riesz space E. Then the set $S^0 := \{\Psi \in E_n^* : \Psi(f) = 0$ for all $f \in S\}$ will be called the *annihilator* of S. Similarly, if A is a subspace of E_n^*, ${}^0A := \{f \in E : \Psi(f) = 0$ for all $\Psi \in A\}$ will be called the *pre-annihilator* of A. If S is an ideal in E and A is an ideal in E_n^*, then S^0 and ${}^0A$ are bands in E_n^* and E, respectively. The following simple relationship between the ideals and the annihilators was first observed by Luxemburg and Zaanen [LZ1].

PROPOSITION 4.7. *Let* E *be a Riesz space and suppose* E_n^* *separates the points of* E. *Then for any ideal* S *in* E *and any ideal* A *in* E_n^*, ${}^0\{S^0\} = S^{dd}$ *and* $\{{}^0A\}^0 = A^{dd}$.

Proof. Since ${}^0\{S^0\}$ is a band which contains S, $S^{dd} \subset {}^0\{S^0\}$. On the other hand, by the bipolar theorem ([S1] IV; 1.5), ${}^0\{S^0\}$ is the $\sigma(E,E_n^*)$ closure of S. But since E_n^* separates the points of E, every band in E is $\sigma(E,E_n^*)$ closed ([Z] 106.1). Hence ${}^0\{S^0\} \subset S^{dd}$, so ${}^0\{S^0\} = S^{dd}$. The second statement is proved similarly (c.f. [Z] 106.2).

If E_n^* separates the points of E and $T \in \mathcal{L}_b(E)$ is order continuous, then it is easy to see that Ker T $= {}^0\{T^*E_n^*\}$ and Ker $T_n^* = \{TE\}^0$ (c.f. [Ru] 4.12 and [S1] IV 2.3). Using these facts, we can easily prove the following duality relationships.

THEOREM 4.8. *Let* $T \in \mathcal{L}_b(E)$ *be a bi-disjointness preserving operator on a Riesz space* E *and suppose* E_n^* *separates the points of* E. *Then* T_n^* *is bi-disjointness preserving. Furthermore:*

i) T *is quasi-invertible iff* T_n^* *is quasi-invertible*

ii) T *is of forward shift type iff* T_n^* *is of backward shift type*

iii) T *is of backward shift type iff* T_n^* *is of forward shift type*

iv) T *is hypernilpotent iff* T_n^* *is hypernilpotent.*

Proof. It follows from [LS1] 4.1 that T_n^* is bi-disjointness preserving. By proposition 4.4 and the above remarks we have

(1) $\operatorname{Ker} T_n^* = \{TE\}^0 = \{^0\{\{TE\}^0\}\}^0 = \{\{TE\}^{dd}\}^0$ and

(2) $\{T_n^* E_n^*\}^{dd} = \{^0\{T_n^* E_n^*\}\}^0 = \{\operatorname{Ker} T\}^0$.

Hence, $\{TE\}^{dd} = E$ iff $\operatorname{Ker} T_n^* = \{0\}$ and $\operatorname{Ker} T = \{0\}$ iff $\{T_n^* E_n^*\}^{dd} = E_n^*$. This proves i). By [Ko] pg. 247 (6), proposition 4.7 and formulas (1) and (2) we have:

(3) $$\left\{\bigvee_{k=1}^{\infty} \operatorname{Ker} T^k\right\}^0 = \bigcap_{k=1}^{\infty} \{\operatorname{Ker} T^k\}^0 = \bigcap_{k=1}^{\infty} \{(T_n^*)^k E_n^*\}^{dd}$$

and

(4) $$\left\{\bigcap_{k=1}^{\infty} \{T^k E\}^{dd}\right\}^0 = \bigvee_{k=1}^{\infty} \{\{T^k E\}^{dd}\}^0 = \bigvee_{k=1}^{\infty} \operatorname{Ker} (T_n^*)^k.$$

Statements ii), iii) and iv) now follow immediately from the definitions and formulas (1) through (4).

Quasi-invertible operators can be characterized as precisely those disjointness preserving operators whose associated operator is an isomorphism.

PROPOSITION 4.9. *Let* E *be a Dedekind complete Riesz space and let* $T \in \mathcal{L}_b(E)$ *be disjointness preserving. The following are equivalent:*

a) T *is quasi-invertible.*

b) The associated operator $\tilde{T}$ *is an f-algebra isomorphism of* Z(E) *onto itself.*

c) The restriction of $\tilde{T}$ *to the projection bands is a Boolean isomorphism of* $\mathcal{P}(E)$ *onto itself.*

Proof. a) ⇒ b). Since $\{TE\}^{dd} = E$ and TE is an ideal, $\tilde{T}$ is surjective by theorems 2.14 and 3.13, and injective by theorem 3.7.

b) ⇒ a). Since $\tilde{T}$ is surjective, the projection P onto $\{TE\}^d$ is in the range of $\tilde{T}$. But it follows immediately from the definition of $\tilde{T}$ that its range must be disjoint from P, whence $\{TE\}^{dd} = E$. By theorem 3.13, TE is an ideal. Since $\tilde{T}$ is invertible, it is order continuous, so T is order continuous by theorem 3.6. It therefore follows from theorem 3.7 that T is injective and hence is quasi-invertible.

b) ⇒ c). If $P \in \mathcal{P}(E)$, then $(\tilde{T}(P))^2 = \tilde{T}(P^2) = \tilde{T}(P)$. Since $\tilde{T}(P) \in Z(E)$, this shows that $\tilde{T}(P) \in \mathcal{P}(E)$. Similarly, $\tilde{T}^{-1}(P) \in \mathcal{P}(E)$, so $\tilde{T}$ is a Boolean isomorphism.

c) ⇒ b). By lemma 3.18, $\tilde{T}$ is order continuous and hence its kernel is a band. Thus, Ker $\tilde{T} = P\,Z(E)$ for some $P \in \mathcal{P}(E)$. Since $\tilde{T}(P) = 0$, $P = 0$ by c). Therefore $\tilde{T}$ is injective. On the other hand, by c), $\tilde{T}(Z(E))$ contains $\mathcal{P}(E)$, and hence all linear combinations of band projections. It follows easily from Freudenthal's spectral theorem ([LZ] 40.2) that $\tilde{T}$ is surjective, which proves b).

A similar result can also be obtained for the associated Boolean homomorphism. It is an immediate consequence of propositions 3.17 and 4.9 and theorem 3.19.

COROLLARY 4.10. *Let* $T \in \mathcal{L}_b(E)$ *be a disjointness preserving operator on a Riesz space* T. *The following are equivalent:*

a) T *is quasi-invertible.*

b) t_T *is a Boolean isomorphism of* $\mathcal{B}(E)$ *onto itself.*

We wish to show that a bi-disjointness preserving operator on a Dedekind complete Riesz space can be decomposed into components satisfying one of the four properties listed in definition 4.6. We will need two lemmas.

LEMMA 4.11. *Let* E *be a Dedekind complete Riesz space and suppose* $T \in \mathcal{L}_b(E)$ *is bi-disjointness preserving. Define* $K \in \mathcal{B}(E)$ *by* $K = \bigvee_{n=1}^{\infty} \operatorname{Ker} T^n$. *Then* K *is a reducing band for* T*; in other words, if* $P \in \mathcal{P}(E)$ *is the projection onto* K, *then* $TP = PT$.

Proof. It can be assumed the T is positive. For each natural number n, let P_n be the projection onto $\operatorname{Ker} T^n$, and let P_0 be the zero operator. Now $TP_nE = T \operatorname{Ker} T^n = TE \cap \operatorname{Ker} T^{n-1} = P_{n-1}TE$. Thus, if Q denotes the projection onto $\{TE\}^{dd}$, we have $\tilde{T}(P_n) = QP_{n-1}$. Hence, for any $f \in E$,

$$TPf = T\bigvee_{n=1}^{\infty}(P_nf) = \bigvee_{n=1}^{\infty}(TP_nf) = \bigvee_{n=1}^{\infty}(QP_{n-1}Tf) = Q(\bigvee_{n=1}^{\infty}P_n)Tf =$$

$QPTf = PTf$, and the proof is complete.

LEMMA 4.12. *Suppose* E *is a Dedekind complete Riesz space and* $T \in \mathcal{L}_b(E)$ *is bi-disjointness preserving. Define* $A = \bigcap_{n=1}^{\infty} \{T^n E\}^{dd}$. *Then* A *is a reducing band for* T.

Proof. For each natural number n, let Q_n be the band projection onto $\{T^n E\}^{dd}$, let $Q_0 = I$ and $Q = \bigwedge_{n=1}^{\infty} Q_n$. We must show that $TQ = QT$.

First of all, by theorem 3.13, for each natural number n, there exists a projection $R_n \in \mathcal{P}(E)$ s.t. $\tilde{T}(R_n) = Q_n$. Denote the projection onto Ker T by P. We claim that $P \vee R_n = P \vee Q_{n-1}$. To show this, suppose $f \in (I-P)T^{n-1}E$. Then by the definition of Q_n, $Tf = TQ_{n-1}f = Q_n Tf = TR_n f$. Since $f \in (I-P)E = \{\text{Ker } T\}^d$, this implies that $R_n f = f$. It follows that $R_n \vee P \geq Q_{n-1} \vee P$. On the other hand, suppose $f \in E$ satisfies $f \perp (T^{n-1}E \cup \text{Ker } T)$. Then $Tf \perp T^n E$ and hence $0 = Q_n Tf = TR_n f$. Since $f \perp \text{Ker } T$, this implies that $R_n f = 0$. Thus $(Q_{n-1} \vee P)g = 0$ implies $(R_n \vee P)g = 0$. i.e. $R_n \vee P \leq Q_{n-1} \vee P$. Combining the two inequalities yields the claim.

Finally, since $\tilde{T}$ is order continuous, $TQ = \tilde{T}(Q)T = \tilde{T}\left(\bigwedge_{n=1}^{\infty} Q_n\right)T = \left(\bigwedge_{n=1}^{\infty} \tilde{T}(Q_n)\right)T = \left(\bigwedge_{n=1}^{\infty} Q_{n-1}\right)T = QT$, and the lemma is proved.

THEOREM 4.13. *Suppose* $T \in \mathcal{L}_b(E)$ *is a bi-disjointness preserving operator on a Dedekind complete Riesz space* E. *Then there exist T-reducing bands* $E_i (i = 1, 2, 3, 4)$ *such that* $E = \bigoplus_{i=1}^{4} E_i$ *and the restrictions of* T *to* E_i *are respectively quasi-invertible, of forward shift type, of backward shift type, and hypernilpotent.*

Proof. Let P be the projection onto $\bigvee_{n=1}^{\infty} \text{Ker } T^n$ and let Q be the projection onto $\bigcap_{n=1}^{\infty} \{T^n E\}^{dd}$. Define E_i $(i = 1, 2, 3, 4)$ by $E_1 = (I-P)QE$,

$E_2 = (I-P)(I-Q)E, E_3 = PQE$ and $E_4 = P(I-Q)E$. That the E_i are T-reducing bands and that $\bigoplus_{i=1}^{4} E_i = E$ follow immediately from the definitions and from lemmas 4.11 and 4.12.

For $i \in \{1,2,3,4\}$ set $T_i := T|_{E_i}$. It is clear that T_1 and T_2 are injective and that $\bigcap_{n=1}^{\infty} \{T_2^n E_2\}^{dd} = \{0\}$ and $\bigcap_{n=1}^{\infty} \{T_4^n E_4\}^{dd} = \{0\}$. Suppose $f \in QE$ but $f \perp TQE$. Then $f \perp QTE$ and hence $f \perp TE$ since $f \in QE$. This implies that $f = 0$ since $f \in QE = \bigcap_{n=1}^{\infty} \{T^n E\}^{dd}$. Therefore, $\{TQE\}^{dd} = QE$ and hence $\{TE_1\}^{dd} = E_1$ and $\{TE_3\}^{dd} = E_3$, which completes the proof.

The final goal of this chapter is to show that quasi-invertible operators can be further decomposed into components satisfying the following properties.

DEFINITION 4.14. *Let* E *be a Riesz space and let* $T \in \mathcal{L}_b(E)$ *be a quasi-invertible disjointness preserving operator.*

i) We shall say that T *has* strict period n *for some* $n \in \mathbb{N}$ *if* $T^n \in \mathrm{Orth}(E)$ *and for every band* $0 \neq B \in \mathcal{B}(E)$, *there exists a band* $0 \neq A \in \mathcal{B}(E)$ *such that* $A \subset B$ *and* $A, \{T(A)\}^{dd}, \{T^2(A)\}^{dd}, \ldots, \{T^{n-1}(A)\}^{dd}$ *are mutually disjoint.*

ii) We say that T *is* aperiodic *if for every band* $0 \neq B \in \mathcal{B}(E)$ *and every natural number* n, *there exists a band* $0 \neq A_n \in \mathcal{B}(E)$ *such that* $A_n \subset B$ *and* $A_n, \{T(A_n)\}^{dd}, \ldots, \{T^n(A_n)\}^{dd}$ *are mutually disjoint.*

Example 1: Let X be compact Hausdorff and $E = C(X)$. Then every disjointness preserving operator $T : E \to E$ is of the form $Tf(x) = h(x)f(\varphi(x))$ for some $h \in E$ and some continuous map $\varphi : \mathrm{Coz}(h) \to X$ (thm 1.3). For any

$m, f \in C(X)$ we have $T(mf)(x) = h(x)\, m(\varphi(x))\, f(\varphi(x)) = m(\varphi(x))(Tf)(x)$. Under the identification of $C(X)$ with $Z(C(X))$, this shows that $\tilde{T}m(x) = m(\varphi(x))$ for all $m \in Z(C(X))$ and all $x \in \mathrm{Coz}(h)$.

Now suppose T is quasi-invertible. Then since $\{TE\}^{dd}$ is a band, h must be a weak order unit of E. This is equivalent to the statement that $Z(h) := \{x \in X : h(x) = 0\}$ has empty interior (see [JR] 12.9 or [LZ] 22.10). Since $Z(h)$ is closed, it must therefore be nowhere dense.

Let $\hat{E}$ and $Z(E)^{\wedge}$ denote the Dedekind completion of E and $Z(E)$ respectively. Since $Z(\hat{E}) \simeq Z(E)^{\wedge}$, it is easy to see that the associated operator of the extension of T to $\hat{E}$ can be identified with the extension of $\tilde{T}$ to $Z(E)^{\wedge}$. Thus, by proposition 4.9, the extension of $\tilde{T}$ to $Z(E)^{\wedge}$ is invertible, so $\tilde{T}$ must be invertible as well (and conversely if $\tilde{T}$ is invertible, then T is quasi-invertible by the same argument). This shows that $\varphi : \mathrm{Coz}(h) \to X$ can be extended continuously to all of X in such a way to make φ a homeomorphism. In conclusion, an operator $Tf(x) = h(x)\, f(\varphi(x))$ from $C(X)$ to itself is quasi-invertible iff $Z(h)$ is nowhere dense and φ can be extended to a homeomorphism of X onto X.

Now suppose $Tf(x) = h(x)\, f(\varphi(x))$, where $Z(h)$ is nowhere dense and φ is a homeomorphism of X onto itself. Set $F_n = \{x \in X : \varphi^n(x) = x\}$ $(n = 1, 2, \dots)$. The F_n are clearly closed sets. We claim that T has strict period n iff $F_1, F_2, \dots, F_{n-1}$ all are nowhere dense but $F_n = X$. To see this, first suppose T has strict period n. Then since $T^n \in \mathrm{Orth}(E) = Z(E)$, $\tilde{T}^n$ is the identity operator on $Z(E)$. Since $\tilde{T}^n m(x) = m(\varphi^n(x))$ for all $m \in Z(E) \simeq C(X)$, it follows that $F_n = X$. Suppose F_k contained a non-empty open set U for some $k \in \{1, 2, \dots, n-1\}$. Then by Urysohn's lemma, there exists a non-zero function f which is zero off U. Then for any $g \in \{f\}^{dd}, T^k g = g$, whence $B = \{T^k B\}^{dd}$ for any band $B \subset \{f\}^{dd}$. This contradicts the assumption that T has strict period n, so the F_k must all be nowhere dense.

Conversely, suppose $F_n = X$ and $F_1, \ldots, F_{n-1}$ are nowhere dense. Then obviously $T^n \in Z(E)$. Let U be any non-empty regularly open set in X and pick a point $x \in U \setminus (\bigcup_{k=1}^{n-1} F_k)$. Then $x, \varphi(x), \ldots, \varphi^{n-1}(x)$ are distinct points, so by the continuity of φ, there exists a regularly open neighborhood θ of x such that $\theta, \varphi(\theta), \ldots, \varphi^{n-1}(\theta)$ are mutually disjoint. Since bands in E correspond to regularly open sets in X ([JR] 12.9 or [LZ] 22.10), it follows that if A,B are the bands corresponding to θ and U respectively, then $\{0\} \neq A \subset B$ and $A, \{T(A)\}^{dd}, \ldots, \{T^n(A)\}^{dd}$ are mutually disjoint. Thus, T has strict period n.

Using the same reasoning as above, it can be shown that T is aperiodic iff F_n is nowhere dense for all natural numbers n.

Remark: The only property of E in the above example, which was used to show that T is quasi-invertible iff $\tilde{T}$ is invertible, is that $Z(\hat{E}) = Z(E)^{\wedge}$. It is not difficult to see that this condition is equivalent to the property that E has algebraically rich center. Thus, the equivalence of a) and b) in proposition 4.9 remains valid for non-Dedekind complete spaces whenever the space has an algebraically rich center.

Example 2: Results similar to example 1 can be obtained for quasi-invertible operators on $L^p[0,1]$ ($1 \leq p < \infty$) with Lebesgue measure (or more generally, $L^p(X, \Sigma, \mu)$, where (X, Σ, μ) is a finite Lebesgue space). Firstly, T is quasi-invertible iff $Tf(x) = h(x) f(\varphi(x))$, where $h \in L^{\infty}[0,1]$ is non-zero almost everywhere and $\varphi : A \to B$ is an invertible bi-measurable map (where A,B are sets of measure 1 in [0,1]) such that $f \mapsto h \cdot f \circ \varphi$

defines a bounded operator on $L^p[0,1]$. Such a quasi-invertible operator T has strict period n iff $\{x \in [0,1] : \varphi^k(x) = x\}$ has measure zero for each $k = 1, 2, \ldots, n-1$ and $\varphi^n(x) = x$ for almost every $x \in [0,1]$. T is aperiodic iff $\{x \in [0,1] : \varphi^k(x) = x\}$ has measure zero for all natural numbers k. The proofs of these facts are straightforward and will be omitted.

The two examples given above show that the definitions of strict period n and aperiodicity given in 4.14 agrees in the concrete case with the usual definitions given these concepts in the theory of dynamical systems (c.f. [Fr] pg. 102, [Rn], [HS]).

We next wish to give a characterization of those operators which have strict period n or are aperiodic when the space is Dedekind complete (c.f. [Ar1]; [L] ch 3., 1.7).

LEMMA 4.15. *Let* E *be a Dedekind complete Riesz space and let* $T \in \mathcal{L}_b(E)$ *be a quasi-invertible disjointness preserving operator. The following are equivalent:*

i) For every projection $0 \neq P \in \mathcal{P}(E)$, *there exists a projection* $0 \neq P' \in \mathcal{P}(E)$ *such that* $P' \leq P$ *and* $P'\tilde{T}(P') = 0$.

ii) $I \wedge |T| = 0$

Proof. i) $\Rightarrow$ ii): Let P be the projection onto $\{\mathrm{Ker}(I \wedge |T|)\}^d$ and suppose that $P \neq 0$. Then by i), there exists a projection $0 \neq P' \in \mathcal{P}(E)$ s.t. $P' \leq P$ and $P'\ \tilde{T}(P') = 0$. We have $0 < (I \wedge |T|)P' = (I \wedge |T|)(P')^2 = P'(I \wedge |T|)P' \leq P'|T|P' = P'\tilde{T}(P')|T| = 0$, a contradiction. Therefore $P = 0$, whence $I \wedge |T| = 0$.

ii) ⇒ i): Suppose i) does not hold for some projection $0 \neq P \in \mathcal{B}(E)$. Then for every $0 \neq P' \in \mathcal{P}(E)$ s.t. $P' \leq P$, we have $P'\tilde{T}(P') \neq 0$. Fix $0 \neq P' \in \mathcal{P}(E)$ with $P' \leq P$ and define $Q = P' - P'\tilde{T}(P')$ and $R = P' - P'\tilde{T}^{-1}(P')$. We have $Q\tilde{T}(Q) = (P' - \tilde{T}(P')P')(\tilde{T}(P') - \tilde{T}(P')\tilde{T}^2(P'))$ $\leq (P' - \tilde{T}(P')P')\tilde{T}(P') = \tilde{T}(P')P' - \tilde{T}(P')P' = 0$. Similarly, $R\tilde{T}(R) = 0$ Since $Q,R \in \mathcal{P}(E)$ and $0 \leq Q \leq P$, $0 \leq R \leq P$, this implies that $Q = R = 0$, by assumption. Now $Q = 0$ implies $P' \leq \tilde{T}(P')$ and $R = 0$ implies $P' \leq \tilde{T}^{-1}(P')$ and hence $\tilde{T}(P') \leq P'$. Therefore, $P' = \tilde{T}(P')$, which shows that $TP' = P'T$ for all $P' \in \mathcal{P}(E)$ satisfying $P' \leq P$. By theorem 2.3, this implies that $T|_{PE} \in \mathrm{Orth}(PE)$. Hence, TP is non-zero and in the band generated by $I \wedge |T|$ in $\mathcal{L}_b(E)$. This contradicts ii).

PROPOSITION 4.16. *Let* E *be a Dedekind complete Riesz space and let* n *be any natural number. The following are equivalent:*

i) For every $0 \neq P \in \mathcal{P}(E)$, *there exists a projection* $P_n \in \mathcal{P}(E)$ *such that* $0 \neq P_n \leq P$ *and* $P_n, \tilde{T}(P_n), \ldots, \tilde{T}^n(P_n)$ *are disjoint.*

ii) $I \wedge |T|^k = 0$ *for each* $k \in \{1, 2, \ldots, n\}$.

Proof. i) ⇒ ii): Suppose i) holds for some natural number n. Then for any $0 \neq P \in \mathcal{P}(E)$, there exists a projection $0 \neq P_n \in \mathcal{P}(E)$ s.t. $P_n, \tilde{T}(P_n), \ldots, \tilde{T}^n(P_n)$ are disjoint. In particular, for each $k \in \{1, 2, \ldots, n\}$ we have $P_n\tilde{T}^k(P_n) = 0$, which implies that $I \wedge |T^k| = 0$ by lemma 4.15.

ii) ⇒ i): By induction. The case $n = 1$ is lemma 4.15. Suppose that ii) ⇒ i) holds for some $n \in \mathbb{N}$ and that $I \wedge |T|^k = 0$ for each $k \in \{1, 2, \ldots, n + 1\}$. By induction hypothesis, for any $0 \neq P \in \mathcal{P}(E)$, there exists a projection $0 \neq P_n \in \mathcal{P}(E)$ s.t. $P_n \leq P$ and

P_n, $\tilde{T}(P_n)$, ..., $\tilde{T}^n(P_n)$ are disjoint. By lemma 4.15 (applied to $\tilde{T}^{n+1}$ and P_n), there exists a projection $0 \neq P_{n+1} \in \mathcal{P}(E)$ s.t. $P_{n+1} \leq P_n$ and $P_{n+1}\, \tilde{T}^{n+1}(P_{n+1}) = 0$. Since $P_{n+1} \leq P_n$, we have P_{n+1}, $\tilde{T}(P_{n+1})$, ..., $\tilde{T}^n(P_{n+1})$ are disjoint, and hence $\tilde{T}(P_{n+1}), \tilde{T}^2(P_{n+1})$, ..., $\tilde{T}^{n+1}(P_{n+1})$ are disjoint as well. It herefore follows from $P_{n+1}\tilde{T}^{n+1}(P_{n+1}) = 0$ that P_{n+1}, $\tilde{T}(P_{n+1})$, ..., $\tilde{T}^{n+1}(P_{n+1})$ are disjoint, which proves i).

THEOREM 4.17. *Let* E *be a Dedekind complete Riesz space and let* $T \in \mathcal{L}_b(E)$ *be a quasi-invertible disjointness preserving operator.*

i) For each natural number n, *the following are equivalent:*

a) T *has strict period* n.

b) $T^n \in \mathrm{Orth}(E)$ *and for every projection* $0 \neq P \in \mathcal{P}(E)$, *there exists a projection* $0 \neq P' \in \mathcal{P}(E)$ *such that* $P' \leq P$ and P', $\tilde{T}(P')$, ..., $\tilde{T}^{n-1}(P')$ *are mutually disjoint.*

c) $T^n \in \mathrm{Orth}(E)$ *and for each* $k \in \{1, 2, ..., n-1\}$, $I \wedge |T|^k = 0$.

ii) The following are equivalent:

a) T *is aperiodic.*

b) For each natural number n *and each projection* $0 \neq P \in \mathcal{P}(E)$, *there exists a projection* $0 \neq P_n \in \mathcal{P}(E)$ *such that* $P' \leq P$ and $P_n, \tilde{T}(P_n)$, ..., $\tilde{T}^n(P_n)$ *are mutually disjoint.*

c) For every natural number n, $I \wedge |T|^n = 0$.

Proof. The theorem follows immediately from propositions 3.16 and 4.16.

We now give a decomposition theorem for quasi-invertible operators on Dedekind complete spaces. We first need a lemma.

LEMMA 4.18. *Let* E *be a Dedekind complete Riesz space and suppose* $T \in \mathcal{L}_b(E)$ *is quasi-invertible. Then there exists a unique projection* $P \in \mathcal{P}(E)$ *such that:*

i) $TP = PT$

ii) $T|_{PE} \in \mathrm{Orth}(PE)$.

iii) $(I \wedge |T|)|_{(I-P)E} = 0$.

Proof. Let $\mathcal{S}$ be the collection of all projections $R \in \mathcal{P}(E)$ such that $TR' = R'T$ for all projections $0 \leq R' \leq R$. Suppose $R_1, R_2 \in \mathcal{S}$ and $R' \leq R_1 \vee R_2$. Then $R' = R_1R' + (I-R_1)R_2R'$. The first term is dominated by R_1 and the second by R_2. It follows that $TR' = R'T$, and hence $R_1 \vee R_2 \in \mathcal{S}$. Therefore $\mathcal{S} \uparrow$ in $\mathcal{P}(E)$. Set $P := \sup\{R \in \mathcal{S}\}$ in $P(E)$. Since T is order continuous, $P \in \mathcal{S}$. Thus, P satisfies i) and also ii) by theorem 2.3.

Suppose $Q \in \mathcal{P}(E)$ and $Q \neq Q \leq I - P$. Then by the definition of P, there exists a projection $0 \leq Q' \leq Q$ s.t. $TQ' \neq Q'T$. Hence $Q_1 := Q' - Q'\tilde{T}(Q')$ or $Q_2 := Q' - Q' \tilde{T}^{-1}(Q)$ is non-zero; say $Q_1 \neq 0$. As in the proof of lemma 4.15, we have $Q_1\tilde{T}(Q_1) = 0$. Hence, by lemma 4.15 $I \wedge |T||_{(I-P)E} = 0$, which proves iii) and completes the proof.

THEOREM 4.19. *Let* $T \in \mathcal{L}_b(E)$ *be a quasi-invertible disjointness preserving operator on a Dedekind complete Riesz space* E. *Then for each* $n \in \mathbb{N} \cup \{\infty\}$, *there exists a T-reducing band* E_n *such that* *i)* $E = \bigoplus_{n \in \mathbb{N} \cup \{\infty\}} E_n$

ii) $T|_{E_n}$ *has strict period* n *if* $n \in \mathbb{N}$ *and* $T|_{E_\infty}$ *is aperiodic.*

Proof. For each natural number n, let P_n be the projection which is obtained by applying the previous lemma to T^n. We claim that $TP_n = P_nT$, or equivalently, $\tilde{T}(P_n) = P_n$. For any $Q \in \mathcal{P}(E)$ s.t. $\tilde{T}^{-1}(Q) \leq P_n$, it follows from the definition of P_n that $\tilde{T}^n(Q) = \tilde{T}(\tilde{T}^n(\tilde{T}^{-1}(Q))) = \tilde{T}(\tilde{T}^{-1}(Q)) = Q$. Therefore, for any $Q \leq \tilde{T}(P_n)$, we have $T^nQ = QT^n$, which shows that $T^n|_{\tilde{T}(P_n)} \in \mathrm{Orth}(P_nE)$ and hence $\tilde{T}(P_n) \leq P_n$. Similarly, for any $R \in \mathcal{P}(E)$ s.t. $\tilde{T}(R) \leq P_n$, $\tilde{T}^n(R) = R$ and hence $\tilde{T}^{-1}(P_n) \leq P_n$. Combining the two inequalities proves the claim.

Define $E_1 = P_1E$ and $E_n = \left[\prod_{k=1}^{n-1} (I-P_k)\right] P_nE$ for n = 2, 3, ... The E_n are clearly disjoint and the above argument shows that they are T-reducing bands. By ii) of lemma 4.18, $T^n|_{E_n} \in \mathrm{Orth}(E_n)$ for each $n \in \mathbb{N}$. Furthermore, by iii) of the same lemma, for each natural number $n \geq 2$ and each $k \in \{1, 2, \ldots, n-1\}$ we have $I \wedge |T|^k|_{E_n} = 0$. This shows that $T|_{E_n}$ has strict period n.

Finally, let $E_\infty = \left\{\bigvee_{n=1}^{\infty} E_n\right\}^d$. It follows from lemma 4.18 that $I \wedge |T|^n|_{E_\infty} = 0$ for all $n \in \mathbb{N}$ so $T|_{E_\infty}$ is aperiodic.

Remark: Theorem 4.19 was proven for invertible operators on Banach lattices by Arendt [Ar 1]. Analogues in ergodic theory of this theorem are well known, see [Rn] and [HS].

Chapter 5

THE SPECTRUM

This chapter discusses the spectrum of disjointness preserving operators. The main object of the chapter is to calculate the spectrum of bi-disjointness preserving operators. Our technique will be to first consider the spectrum of the simple bi-disjointness preserving operators given in definitions 4.6 and 4.14. Except for the periodic quasi-invertible case, it will be shown that the spectrum of all such simple operators is rotationally invariant. We then combine these results with those of chapter four to yield a general theorem valid for arbitrary bi-disjointness preserving operators (theorem 5.15). The same idea was used by Arendt [Ar 1], who obtained a few of our results in the case when the operator is invertible. Detailed bibliographical remarks are given following theorem 5.15. We conclude the chapter by giving various consequences of these results; perhaps the most important of these is a far-reaching generalization of a well known theorem of Schaefer, Wolff and Arendt [SWA]; see theorem 5.16 and corollary 5.17.

All spaces in this chapter will be taken to be complex. If $P_n \in \mathcal{P}(E)$ and $\alpha_n \in \mathbb{C}$, define $\sum_{n=1}^{\infty} \alpha_n P_n := \text{o-}\lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n P_n$ if this limit exists in $Z(E)$. We will always consider a Banach lattice E to be isometrically embedded in its second dual space E^{**} in the canonical way. We will denote the spectrum, approximate point spectrum and point spectrum of an operator T or a Banach space by $\sigma(T)$, $A\sigma(T)$ and $P\sigma(T)$, respectively. The spectral radius of T will be denoted by $r(T)$. We will denote the disk and circle of radius r about the orgin in the complex plane by D_r and C_r respectively. The unit disk and circle will be denoted simply as D and C. We will say that a subset S of the complex plane is *rotationally invariant* if $\lambda \in S$ implies $\lambda e^{i\theta} \in S$ for all $\theta \in [0,2\pi)$. Let $T \in \mathcal{L}(B)$ be a bounded operator on a Banach space B. The following simple fact will be used often: If $A\sigma(T)$ is rotationally invariant, then $\sigma(T)$ is rotationally invariant as well. To see this, suppose $\lambda \in \sigma(T)$. If for some $\theta \in [0,2\pi)$, $\lambda e^{i\theta} \notin \sigma(T)$, then there must be a complex number $\theta_0 \in [0,2\pi)$ such that $\lambda e^{i\theta_0} \in \partial\sigma(T) \subset A\sigma(T)$, where $\partial\sigma(T)$ denotes the boundary of $\sigma(T)$. Since $A\sigma(T)$ is rotationally invariant, $\lambda e^{i\varphi} \in A\sigma(T) \subset \sigma(T)$ for all $\varphi \in [0,2\pi)$, a contradiction.

A set S in the complex plane is called *cyclic* if $re^{i\theta} \in S$ ($r > 0$, $\theta \in [0,2\pi)$) implies that $re^{in\theta} \in S$ for all integers n. A well known result of Lotz([Lo 1] [S]V §4) states that if T is a lattice homomorphism on a Banach lattice, then $P\sigma(T)$ and $A\sigma(T)$ are cyclic. This result was used by Scheffold ([Sd 1][S]V §4) to show that $\sigma(T)$ is cyclic. Lotz' proof is indirect, using among other things an ultraproduct (non-standard hull) construction. We now give a simple constructive proof of Lotz' result. This proof illustrates well the type of argument we will use throughout this chapter.

THEOREM 5.1. *Let E be a Banach lattice and suppose $T \in \mathcal{L}_b(E)$ is a lattice homomorphism. Then the point and approximate point spectrum of T are cyclic.*

Proof. Suppose $re^{i\theta} \in A\sigma(T)$ ($r > 0$, $\theta \in [0,2\pi)$). Then for any $\varepsilon > 0$, there exists an element $f \in E$ s.t. $\|f\| = 1$ and $\|Tf - re^{i\theta}f\| < \varepsilon$. Note that $\|T|f| - r|f|\| = \||Tf| - |rf|\| \leq \|Tf - re^{i\theta}f\| < \varepsilon$. By proposition 2.8, there exists an element $M \in Z(J_f)$ s.t. $f = M|f|$. By theorem 2.13, M can be extended uniquely to an operator $\hat{M} \in Z(E^{**})$ s.t. $M = \hat{M}$ on J_f and $\hat{M}h = 0$ for all $h \in \{f\}^d$, where the "d" operation is taken in E^{**}. Note that $\|\hat{M}\| = \|M\| = 1$. Denote by $\tilde{M}$ the image of $\hat{M}$ under the associated operator of T^{**} (i.e. $\tilde{M} = \tilde{T}^{**}(\hat{M})$). We have

$$(1) \quad \|\tilde{M}|f| - e^{i\theta}M|f|\| \le \|\tilde{M}|f| - \tfrac{1}{r}\tilde{M}T|f|\| + \|\tfrac{1}{r}\tilde{M}T|f| - e^{i\theta}M|f|\|$$

$$\le \frac{\|\tilde{M}\|}{r}\|r|f| - T|f|\| + \frac{1}{r}\|Tf - rf\| < \frac{2\varepsilon}{r}.$$

We claim that for all natural numbers n,

$$(2) \quad \|TM^n|f| - re^{in\theta}M^n|f|\| < (2n-1)\varepsilon.$$

(2) holds trivially for $n = 1$. Suppose it holds for some $n \in \mathbb{N}$. Then, by (1), $\|TM^{n+1}|f| - re^{i(n+1)\theta}M^{n+1}|f|\|$

$$\le \|\tilde{M}TM^n|f| - re^{in\theta}\tilde{M}M^n|f|\| + \|re^{in\theta}\tilde{M}M^n|f| - re^{i(n+1)\theta}M^{n+1}|f|\|$$

$$\le \|\tilde{M}\|\,\|TM^n|f| - re^{in\theta}M^n|f|\| + r\,\|M^n\|\,\|\tilde{M}|f| - e^{i\theta}M|f|\|$$

$\le (2n-1)\varepsilon + r\frac{2\varepsilon}{r} = (2n+1)\varepsilon$, which proves (2) by induction. The cyclicity of $A\sigma(T)$ follows immediately from (2); the cyclicity of $P\sigma(T)$ can be obtained by putting $\varepsilon = 0$ in the above proof.

Remark: A similar argument to that given above was recently used by Greiner and Groh [GG] to prove that the spectrum of a positive representation of a compact Abelian group is cyclic.

We begin the main part of this chapter by giving some preliminary results which will be used repeatedly.

Let B be any Banach space and $T: B \to B$ be a bounded operator. Suppose that $\sigma(T)$ can be separated by a Jordan curve γ into two disjoint parts; a bounded part σ_1 and an unbounded part σ_2. Then the *spectral projection* induced by γ is defined by $P_1 = \frac{1}{2\pi i}\int_\gamma (\lambda I - T)^{-1} d\lambda$, where the integration is taken counterclockwise. It is well known that P_1 and $P_2 := I - P_1$ are idempotent and commute with T. Furthermore, $B = B_1 \oplus B_2$ where $B_1 := P_1 B$ and $B_2 := P_2 B$, the spectrum of $T|_{B_1} \in \mathcal{L}(B_1)$ is σ_1 and the spectrum of $T|_{B_2} \in \mathcal{L}(B_2)$ is σ_2 (see [Do] 1.39). The following theorem is due to Arendt [Ar 2].

THEOREM 5.2. *Let* E *be a Banach lattice,* $T \in \mathcal{L}_b(E)$, *and suppose* T *and* T^* *are disjointness preserving. Suppose for some positive real number* s, $C_s \cap \sigma(T) = \emptyset$. *Let* P_1 *be the spectral projection induced by* C_s *as defined above. Then* P_1 *and* $P_2 := I - P_1$ *are band projections.*

Proof. Let T_1 be the restriction of T to $E_1 := P_1 E$. Since $r(T_1) < s$, the C. Neumann series $\sum_{n=0}^{\infty} \frac{T^n f}{s^{n+1}}$ converges uniformly for all $f \in E_1$. Now suppose $f \in E_1$ and $|g| \le |f|$. Then $|T^n g| \le |T^n f|$ for each $n \in \mathbb{N}$ by corollary 1.6, so $\|T^n g\| < \|T^n f\|$ and thus $\sum_{n=0}^{\infty} \frac{T^n g}{s^{n+1}}$ converges uniformly as well. It now can be easily verified that $(\lambda I - T)^{-1} g = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ for all g in the ideal generated by E_1 and all $\lambda \in C_s$. Thus, for any such g,

$$P_1 g = \frac{1}{2\pi i}\int_{C_s} \sum_{n=0}^{\infty} \frac{T^n g}{\lambda^{n+1}} d\lambda = \frac{1}{2\pi i}\sum_{n=0}^{\infty} T^n g \int_{C_s} \frac{d\lambda}{\lambda^{n+1}} = g.$$

This shows that E_1 is an ideal. Applying this argument to the adjoint T^* gives that $P_1^* E_1^*$ is an ideal in E_1^*. It follows that $P_2 E = \text{Ker } P_1 = {}^0(P_1^* E_1^*)$ is an ideal as well ([S] II 4.8, Cor.), where the pre-annihilator is taken with respect to the full topological dual. It therefore follows from [S] II 2.7 that P_1 and P_2 are band projections.

Recall that a Banach lattice is said to have a *weak Fatou norm* if there exists a constant $c > 0$ such that if $0 \leq f_\alpha \uparrow f$ in E then $\sup \|f_\alpha\| \geq c\,\|f\|$. Most non-pathological spaces have this property. For instance, if E is the Banach lattice of all bounded continuous functions on a completely regular space (under the usual "sup" norm), then E has a (weak) Fatou norm. Every dual Banach lattice as well as every Banach lattice with order continuous norm also has this property.

The following technical lemma will play a crucial role when the space is not Dedekind complete.

LEMMA 5.3. *Suppose* E *is a Banach lattice with a weak Fatou norm. Then there exists a constant* $c > 0$ *such that for every projection* $P \in \mathcal{P}(E^{**})$ such that $PE \cap E$ *is order dense in* E, $\|Pf\| \geq c\,\|f\|$ *for all* $f \in E$.

Proof. Let B be a band in E^{**} such that $E \cap B$ is order dense in E and pick $f \in E$. It may be assumed that $f \geq 0$. Since $E \cap B$ is an order dense ideal of E, there exist elements $f_\alpha \in E \cap B$ such that $0 \leq f_\alpha \uparrow f$ in E. Since E has a weak Fatou norm, there exists a constant $c > 0$ (independent of f) such that $\sup_\alpha \|f_\alpha\| \geq c\,\|f\|$. Let $P \in \mathcal{P}(E^{**})$ be the projection onto B. Then $\|Pf\| \geq \sup_\alpha \|Pf_\alpha\| = \sup_\alpha \|f_\alpha\| \geq c\,\|f\|$ and the proof is complete.

We will need one more technical lemma which is stated in terms of the Luxemburg "t" map given in definition 3.15.

LEMMA 5.4. *Let* $T \in \mathcal{L}_b(E)$ *be a bi-disjointness preserving operator on a Riesz space* E. *Then for any Riesz subspace* $S \subset E$, $\{T\{S\}^{dd}\}^{dd} = \{TS\}^{dd}$. In particular, $\{T\{TE\}^{dd}\}^{dd} = \{T^2E\}^{dd}$ *and hence* $t_T^n(B) = \{T^nB\}^{dd}$ *for any band* $B \in \mathcal{B}(E)$ *and any natural number* n.

Proof. By extending T to the Dedekind completion of E if necessary, it suffices to prove the lemma when E is Dedekind complete. Then TE is an ideal, so $T(J_S) = J_{TS}$. Since T is order continuous, $T(\{J_S\}^{dd}) = \{J_{TS}\}^{dd} = \{TS\}^{dd}$. Therefore, $\{T(\{S\}^{dd})\}^{dd} = \{T(\{J_S\}^{dd})\}^{dd} = \{TS\}^{dd}$.

We are now ready to compute the spectrum of the basic bi-disjointness preserving operators considered in the previous chapter. We begin our discussion with operators of forward shift type.

LEMMA 5.5. *Let* E *be a Riesz space and suppose* $T \in \mathcal{L}_b(E)$ *is bi-disjointness preserving and of forward shift type. For each natural number* n *define* $B_n = \{T^{n-1}(E)\}^{dd} \cap \{T^nE\}^d$. *Then the* B_n *are mutually disjoint bands,* $\bigvee_{n=1}^{\infty} B_n = E$ *and* $t(B_n) = B_{n+1}$ *for all natural numbers* n.

Proof. Since t is a Boolean homomorphism (theorem 3.19), we have by lemma 5.4, $t(B_n) = t(\{T^{n-1}E\}^{dd} \cap \{T^nE\}^d) = t(t^{n-1}(E) \cap \{t^n(E)\}^d) = t^n(E) \cap t^{n+1}(E) = \{T^nE\}^{dd} \cap \{T^{n+1}E\}^d = B_{n+1}$. Furthermore, since T is of forward shift type, $E \supset \bigvee_{n=1}^{\infty} B_n = \bigvee_{n=0}^{\infty}\left(\{T^nE\}^{dd} \cap \{T^{n+1}E\}^d\right) \supset \left\{\bigcap_{n=1}^{\infty} \{T^nE\}^{dd}\right\}^d = \{0\}^d = E$.

THEOREM 5.6. *Suppose* E *is a Banach lattice and* $T \in \mathcal{L}(E)$ *is bi-disjointness preserving and of forward shift type.*

i) The point spectrum of E *is empty*

ii) If E *is Dedekind complete, then the approximate point spectrum is rotationally invariant and* $\sigma(T) = D_{r(T)}$.

iii) If E *has a weak Fatou norm, then* $\sigma(T) = D_{r(T)}$.

Proof. Since Ker $T = \{0\}$, $0 \notin P\sigma(T)$. Suppose that, for some $0 \neq \lambda \in \mathbb{C}$ and $0 \neq f \in E$, $Tf = \lambda f$. Then clearly $\{f\}^{dd} \subset \{T^n E\}^{dd}$ for all natural numbers n, contradicting the assumption that T is of forward shift type. Thus, the point spectrum of T is empty.

Next suppose that $\lambda \in A\sigma(T)$. Then for each $\varepsilon > 0$, there exists an element $f \in E$ such that $\|f\| = 1$ and $\|Tf - \lambda f\| < \varepsilon$.

Now suppose that E is Dedekind complete. Let $B_n \in \mathcal{B}(E)$ be as in lemma 5.5 and let P_n be the band projection onto B_n. Since by lemma 5.5, $t_T(B_n) = B_{n+1}$, $\tilde{T}(P_n) = P_{n+1}$ by proposition 3.16. Pick $\alpha \in C$ and define $M = \sum_{n=1}^{\infty} \alpha^{-n} P_n$. Then $|M| = I$, so $\|Mf\| = 1$. Furthermore, since $\tilde{T}$ is order continuous by theorem 3.6, $T(M) = \sum_{n=1}^{\infty} \alpha^{-n} \tilde{T}(P_n) = \sum_{n=1}^{\infty} \alpha^{-n} P_{n+1} = \alpha M - \alpha^{-1} P_1$. Since $P_1 T = 0$, it follows that $TM = \alpha MT$. Therefore, $\|TMf - \alpha\lambda Mf\| = \|\alpha MTf - \alpha\lambda Mf\| \leq |\alpha|\ \|M\|\ \|Tf - \lambda f\| < \varepsilon$. Thus, $\alpha\lambda \in A\sigma(T)$, which shows that $A\sigma(T)$ and hence $\sigma(T)$ is rotationally invariant.

Now suppose E has a weak Fatou norm. For each natural number n, let $B_n \in \mathcal{B}(E)$ be the band defined in lemma 5.5 and define $A_n \in \mathcal{B}(E^{**})$ by $A_n = \{T^{n-1}E\}^{dd} \cap \{T^n E\}^d$ (unless otherwise specified, we will take the "d" operation in E^{**}). Note that $\{A_n \cap E\}^{dd(E)} = B_n$. Furthermore, $t_{T^{**}}(A_n) = t_{T^{**}}(\{T^{n-1}E\}^{dd}) \cap t_{T^{**}}(\{T^n E\}^d) = \{T^n E\}^{dd} \cap \{T^{n+1}E\}^d = A_{n+1}$ by lemma 5.4. Hence, if $P_n \in \mathcal{P}(E^{**})$ denotes the projection onto A_n, $\tilde{T}^{**}(P_n) = P_{n+1}$ (proposition 3.16).

Pick $\alpha \in C$ and define $M = \sum_{n=1}^{\infty} \alpha^{-n} P_n$. Since $\{E \cap A_n\}^{dd(E)} = B_n$, $E = \bigvee_{n=1}^{\infty} B_n = \{E \cap \bigvee_{n=1}^{\infty} A_n\}^{dd(E)}$. Hence $|M|\ E$ is order dense in E.

By lemma 5.3, there exists a constant $c > 0$ (independent of f) s.t.

$\|Mf\| \geq c\ \|f\| = c$. As in the Dedekind complete case, $T^{**}M = \alpha M T^{**}$ and hence $\|T^{**}\frac{Mf}{c} - \sigma\lambda\frac{Mf}{c}\| < \frac{\varepsilon}{c}$, which shows that $\alpha\lambda \in A\sigma(T^{**}) \subset \sigma(T)$. Therefore, $\sigma(T)$ is rotationally invariant.

Finally, let E be arbitrary and suppose $0 < s < r(T)$ but $se^{i\theta} \cap \sigma(T) = \emptyset$ for all $\theta \in [0,2\pi)$. Let P_1 be the spectral projection induced by C_s and let $E_2 = (I-P_1)E$. Then E_2 is a non-trivial reducing subspace and $T|_{E_2}$ is invertible. This clearly contradicts the assumption that T is of forward shift type. It follows that if E is Dedekind complete or has a weak Fatou norm $\sigma(T) = D_{r(T)}$.

The backward shift and hypernilpotent case can be considered together.

LEMMA 5.7. *Suppose* $T \in \mathcal{L}_b(E)$ *is a bi-disjointness preserving operator on a Riesz space* E *such that* $\bigvee_{n=1}^{\infty} \operatorname{Ker} T^n = E$. *For each natural number* n, *define* $B_n \in \mathcal{B}(E)$ *by* $B_n = \operatorname{Ker} T^n \cap \{\operatorname{Ker} T^{n-1}\}^d$. *Then the* B_n *are mutually disjoint* $\bigvee_{n=1}^{\infty} B_n = E$, $t_T(B_n) = B_{n-1} \cap \{TE\}^{dd}$ $(n = 2, 3, \ldots)$ *and* $t_T(B_1) = \{0\}$.

Proof. It is clear that the B_n are disjoint and that $\bigvee_{n=1}^{\infty} B_n = E$. Since t_T is a Boolean homomorphism, for any natural number $n \geq 2$, $t_T(B_n) = t_T(\operatorname{Ker} T^n) \cap \{t_T(\operatorname{Ker} T^{n-1})\}^d = \{T(\operatorname{Ker} T^n)\}^{dd} \cap \{T(\operatorname{Ker} T^{n-1})\}^d = \operatorname{Ker} T^{n-1} \cap \{TE\}^{dd} \cap (\{\operatorname{Ker} T^{n-2}\}^d \vee \{TE\}^{dd}) = \operatorname{Ker} T^{n-1} \cap \{\operatorname{Ker} T^{n-2}\}^d \cap \{TE\}^{dd} = B_{n-1} \cap \{TE\}^{dd}$. Furthermore, $t_T(B_1) = \{T(\operatorname{Ker} T)\}^{dd} = \{0\}$.

THEOREM 5.8. *Suppose* E *is a Banach lattice and* $T \in \mathcal{L}_b(E)$ *is a bi-disjointness preserving operator satisfying* $\bigvee_{n=1}^{\infty} \operatorname{Ker} T^n = E$.

i) If E *is Dedekind complete, then* $P\sigma(T)$ *is a (closed or open) disk and* $A\sigma(T)$ *is rotationally invariant.*

ii) If E *is either Dedekind complete or has a weak Fatou norm then* $\sigma(T) = D_{r(T)}$.

iii) If $\overline{\bigcup_{n=1}^{\infty} \text{Ker } T^n} = E$, *then* $\sigma(T) = A\sigma(T)$.

Proof. i): Let $B_n \in \mathcal{B}(E)$ be as in lemma 5.7 and let $P_n \in \mathcal{P}(E)$ be the projection onto B_n. Then the P_n are mutually disjoint and $\bigvee_{n=1}^{\infty} P_n = I$. Also, if Q denotes the projection onto $\{TE\}^{dd}$, $\tilde{T}(P_n) = P_{n-1}Q$. It follows that $TP_n = P_{n-1}QT = P_{n-1}T$.

Suppose $\lambda \in P\sigma(T)$, so that there exists an element $0 \neq f \in E$ such that $Tf = \lambda f$. Pick $0 \neq \alpha \in D$ and define $M \in Z(E)$ by $M = \sum_{n=1}^{\infty} \alpha^n P_n$. Then $\tilde{T}(M) = \sum_{n=1}^{\infty} \alpha^n \tilde{T}(P_n) = Q \sum_{n=2}^{\infty} \alpha^n P_{n-1} = \alpha QM$. Hence, $TMf = \alpha MTf = \alpha\lambda Mf$. Since $\bigvee_{n=1}^{\infty} P_n = I$, it is clear that $Mf \neq 0$, so $\alpha\lambda \in P\sigma(T)$. Since $0 \in P\sigma(T)$, it follows that $P\sigma(T)$ is a disk.

Next, suppose $\lambda \in A\sigma(T)$. Then for each $\varepsilon > 0$, there exists an element $f \in E$ s.t. $\|f\| = 1$ and $\|Tf - \lambda f\| < \varepsilon$. Pick $\alpha \in C$ and let $M = \sum_{n=1}^{\infty} \alpha^n P_n$ as above. Then $|M| = I$, so $\|Mf\| = 1$. Furthermore, $\|TMf - \alpha\lambda Mf\| = \|\alpha MQTf - \alpha\lambda Mf\| \leq |\alpha| \; \|M\| \; \|Tf - \lambda f\| < \varepsilon$. Hence, $\alpha\lambda \in A\sigma(T)$, so $A\sigma(T)$ and $\sigma(T)$ are rotationally invariant.

ii): Suppose E has a weak Fatou norm. Define $A_n \in \mathcal{B}(E^{**})$ by $A_n = \{\text{Ker } T^n\}^{dd} \cap \{\text{Ker } T^{n-1}\}^d$, $(n = 1, 2, \ldots)$ (unless otherwise specified, the "d" operation will be taken in E^{**}). Since $\{A_n \cap E\}^{dd(E)} = B_n$, it is clear that the A_n are mutually disjoint and that $E \cap \bigvee_{n=1}^{\infty} A_n$ is order dense in E. Furthermore, as in lemma 5.4.

$$t_{T^{**}}(A_n) = \{T\{\text{Ker } T^n\}\}^{dd} \cap \{T \text{ Ker } T^{n-1}\}\}^{dd}$$
$$= \{\text{Ker } T^{n-1}\}^{dd} \cap \{TE\}^{dd} \cap \{\text{Ker } T^n\}^{dd}$$
$$= \begin{cases} A_{n-1} \cap \{TE\}^{dd} & n = 2, 3, \ldots \\ 0 & n = 1 \end{cases}$$

Let $P_n \in \mathcal{P}(E^{**})$ be the projection onto A_n and let $Q \in \mathcal{P}(E^{**})$ be the projection onto $\{TE\}^{dd}$. By proposition 3.16, $\tilde{T}^{**}(P_n) = Q\,P_{n-1}$ for each $n \geq 2$ and $\tilde{T}^{**}(P_1) = 0$. Now pick $\alpha \in C$ and define $M = \sum_{n=1}^{\infty} \alpha^n P_n$. Suppose $\lambda \in A\sigma(T)$, so for each $\varepsilon > 0$, there exists an element $f \in E$ s.t. $\|f\| = 1$ and $\|Tf - \lambda f\| < \varepsilon$. Since $|M|E \cap E = \left(\bigvee_{n=1}^{\infty} A_n\right) \cap E$ is order dense in E, there exists a constant $c > 0$ such that $\|Mf\| \geq c$ by lemma 5.3. Moreover as in i) we have $\|TMf - \alpha\lambda Mf\| < \varepsilon$, which implies that $\alpha\lambda \in \sigma(T)$, so $\sigma(T)$ is rotationally invariant.

To show that $\sigma(T) = D_{r(T)}$ in the two given cases, it suffices to show that $\sigma(T)$ is connected. If not, since $\sigma(T)$ is rotationally invariant, there must be a positive real number $s < r(T)$ s.t. $C_s \cap \sigma(T) = \emptyset$. Let R be the spectral projection induced by C_s. Then by theorem 5.2, $B := (I-R)E$ is a non-trivial T-reducing band and $T|_B$ is invertible. But $B = \left(\bigvee_{n=1}^{\infty} \text{Ker } T^n\right) \cap B = \bigvee_{n=1}^{\infty} \text{Ker}(T^n|_B)$ which gives a contradiction. This proves ii).

iii): Suppose $\overline{\bigcup_{n=1}^{\infty} \text{Ker } T^n} = E$. It is easy to see that $(T^*)^n E^* \subset (\text{Ker } T^n)^0$, where the annihilator is taken with respect to E^*. By [Ko] pg. 247 (6),

$\{0\} = \left\{\bigvee_{n=1}^{\infty} \operatorname{Ker} T^n\right\}^0 = \bigcap_{n=1}^{\infty} (\operatorname{Ker} T^n)^0 \supset \bigcap_{n=1}^{\infty} \left\{(T^*)^n E^*\right\}$. It follows that $R\sigma(T) \subset P\sigma(T^*) \subset \{0\}$ which shows that $R\sigma(T) = \emptyset$, i.e. $A\sigma(T) = \sigma(T)$.

Remarks: 1) If T is hypernilpotent, then it is easy to see that $P\sigma(T) = \{0\}$.

2) The condition in iii) is always satisfied when E has order continuous norm.

We now turn to quasi-invertible operators.

LEMMA 5.9. *Let* E *be a Riesz space and suppose* $T \in \mathcal{L}_b(E)$ *is a quasi-invertible disjointness preserving operator with strict period* n *for some* $n \in \mathbb{N}$. *Then there exists a band* $B \in \mathcal{B}(E)$ *such that* $B, t(B), \ldots, t^{n-1}(B)$ *are mutually disjoint and* $E = \bigvee_{k=0}^{n-1} t^k(B)$.

Proof. Define $\mathcal{A} = \{A \in \mathcal{B}(E): A, t(A), \ldots, t^{n-1}(A) \text{ are disjoint}\}$. Let $\{B_\alpha\}$ be a chain in $\mathcal{A}$ under the ordering induced by $\mathcal{B}(E)$. Since t is an order continuous Boolean homomorphism (thm. 3.19), for any natural numbers i,j satisfying $0 \le i < j \le n - 1$ we have $t^i(\bigvee_\alpha B_\alpha) \wedge t^j(\bigvee_\alpha B_\alpha) = (\bigvee_\alpha t^i(B_\alpha)) \wedge (\bigvee_\alpha t^j(B_\alpha)) = \bigvee_\alpha (t^i(B_\alpha) \wedge t^j(B_\alpha)) = \{0\}$. Since $\mathcal{A}$ is clearly non-empty, this shows that $\mathcal{A}$ is inductively ordered and thus has a maximal element B by Zorn's lemma. Let $A = \left\{\bigvee_{k=0}^{n-1} t^k(B)\right\}^d$. If $A \neq \{0\}$, then there exists a band $A_0 \in \mathcal{B}(E)$ such that $\{0\} \neq A_0 \subset A$ and $A_0 \in \mathcal{A}$. Let $B_0 = B \vee A_0$. Since T has strict period n, it is clear that $t^i(B) \wedge t^j(A_0) = \{0\}$ for all $i,j \in \mathbb{N}$. It follows from this and $A_0, B \in \mathcal{A}$ that $B_0 \in \mathcal{A}$, which contradicts the maximality of B. Thus $A = \{0\}$, whence $\bigvee_{k=0}^{n-1} t^k(B) = E$ and the proof is complete.

THEOREM 5.10. *Let* E *be a Banach lattice and let* $T \in \mathcal{L}_b(E)$ *be a quasi-invertible disjointness preserving operator with strict period* n *for some* $n \in \mathbb{N}$.

i) $\sigma(T) = A\sigma(T)$.

ii) If E *is either Dedekind complete or has a weak Fatou norm, then* $\sigma(T) = \alpha\sigma(T)$ *for any nth root of unity* α.

iii) If E *is Dedekind complete, then* $P\sigma(T) = \alpha P\sigma(T)$ *for any nth root of unity* α.

Proof. We first show that $\sigma(T) = A\sigma(T)$. By the spectral mapping theorem (applied to both $\sigma(T)$ and $A\sigma(T)$) it suffices to show this when $T \in Z(E)$. In this case, under the identification $Z(E) \simeq C(X) \simeq Z(C(X))$, (where x is some compact Hausdorff space) T may be identified with a multiplication operator $T_f \in Z(C(X))$ ($f \in C(X)$) given by $T_f g = f \cdot g$ for all $g \in C(X)$. Suppose $\lambda \in \sigma(T)$. It follows from [Ar 2] 3.3 that $\lambda I - T$ is not invertible in $Z(E)$. Thus, $\lambda \cdot 1_X - T_f$ is not invertible in $Z(C(X))$. Since it is well known that $\sigma(T_f) = A\sigma(T_f)$, there exist functions $g_n \in C(X)$ ($n = 1, 2, \ldots$) such that $\|g_n\| = 1$ and $\|T_f g_n - \lambda g_n\| \to 0$ as $n \to \infty$. It follows that $\|T_f M_{g_n} - \lambda M_{g_n}\| \to 0$ as $n \to \infty$, where M_{g_n} denotes multiplication by g_n. Identify the M_{g_n} with elements $M_n \in Z(E)$. Since $\|M_n\| = 1$, there exist elements $h_n \in E$ s.t. $\|M_n h_n\| = 1$ and $\|h_n\| < 2$. It follows that $\|TM_n h_n - \lambda M_n h_n\| \to 0$ as $n \to \infty$, i.e. $\lambda \in A\sigma(T)$.

Next, suppose E has a weak Fatou norm. In the following, we will take "d" operation in E^{**}. Let $B \in \mathcal{B}(E)$ be the band defined in lemma 5.9. Then it is clear that $E \cap \bigvee_{k=0}^{n-1} \{t^k(B)\}^{dd}$ is order dense in E. Let

$P \in \mathcal{P}(E^{**})$ be the projection onto $\{B\}^{dd}$ and let α be any nth root of unity. Define $M = \sum_{k=0}^{n-1} \alpha^{-k}(\tilde{T}^{**})^k(P)$. Since $(\tilde{T}^{**})^n = \tilde{I}^{**}$,

$$\tilde{T}^{**}(M) = \sum_{k=0}^{n-1} \alpha^{-k}(\tilde{T}^{**})^{k+1}(P) = \alpha \sum_{k=1}^{n} \alpha^{-k}(\tilde{T}^{**})^k(P) = \alpha M.$$

Suppose $\lambda \in A\sigma(T)$, so for every $\varepsilon > 0$, there exists an element $f \in E$ s.t. $\|f\| = 1$ and $\|Tf - \lambda f\| < \varepsilon$. Then $\|T^{**}Mf - \alpha\lambda Mf\| = \|\tilde{T}^{**}(M)Tf - \alpha\lambda Mf\| = \|\alpha M(Tf - \lambda f)\| < \varepsilon$. Since $|M|\ E \cap E$ is order dense in E, by lemma 5.3 there exists a constant $c > 0$ s.t. $\|Mf\| > c$. It follows that $\alpha\lambda \in \sigma(T)$. The proof in the Dedekind complete case is similar, as is the proof of the assertion about $P\sigma(T)$.

COROLLARY 5.11. *Let* E *be a Banach lattice which is either Dedekind complete or has a weak Fatou norm. Suppose* $T \in \mathcal{L}_b(E)$ *is a quasi-invertible lattice homomorphism with strict period* n *for some* $n \in \mathbb{N}$. *Then* $\lambda \in \sigma(T)$ *iff* $|\lambda| \in \sigma(T)$ *and* $\lambda = |\lambda| . \alpha$ *where* α *is an nth root of unity.*

Proof. Since $T^n \in Z(E)$ and $T \geq 0$, $\sigma(T^n) \subset \mathbb{R}_+$. Hence, by the spectral mapping theorem, every $\lambda \in \sigma(T)$ is of the form $|\lambda|\alpha$, where a is an nth root of unity. The corollary now follows from theorem 5.10.

LEMMA 5.12. *Let* E *be a Riesz space and suppose* $T \in \mathcal{L}_b(E)$ *is an aperiodic quasi-invertible disjointness preserving operator. Then for any natural number* m, *there exists a band* B *such that* $B, t(B), \ldots, t^{m-1}(B)$ *are disjoint and* $\bigvee_{k=1}^{2m-1} t^k(B) = E$.

Proof. Fix a natural number m and define $\mathcal{A} = \{A \in \mathcal{B}(E) : A, t(A), \ldots, t^{m-1}(A)$ are mutually disjoint$\}$. As in the proof of lemma 5.9, $\mathcal{A}$ is inductively ordered and hence has a maximal element B. Let $G = \left\{ \bigvee_{k=0}^{2m-1} t^k(B) \right\}^d$ and $H = t^{-m}(G)$. If $G \neq \{0\}$, there exists a non-zero $H_0 \subset H$ such that $H_0, t(H_0), \ldots, t^{m-1}(H_0)$ are mutually disjoint. Define $B_0 = H_0 \vee B$. For all integers (j) such that $0 \leq i < j \leq m - 1$ we have

(*) $t^i(B_0) \cap t^j(B_0) = (t^i(H_0) \wedge t^j(H_0)) \vee (t^i(H_0) \wedge t^j(B)) \vee (t^i(B) \wedge t^j(H_0)) \vee (t^i(B) \wedge t^j(B))$. The first and last terms of the right hand side of (*) are zero by assumption. For any integers $k, \ell \in \{0, 1, \ldots, m - 1\}$ we have $t^k(B) \cap t^\ell(H_0) = t^{\ell-m}(t^{m+k-\ell}(B) \cap t^m(H_0)) \subset t^{\ell-m}\left(\left\{ \bigvee_{k=0}^{2m-1} t^k(B) \cap G \right\}\right) = \{0\}$. Hence the second and third terms of the right hand side of (*) are zero as well, which shows that $B_0 \in \mathcal{A}$. This contradicts the maximality of B which completes the proof.

LEMMA 5.13. *Let* $T \in \mathcal{L}_b(E)$ *be an aperiodic quasi-invertible disjointness preserving operator on a Banach lattice* E.

i) If E *is Dedekind complete, then for every element* $f \in E$ *and every natural number* n, *there exists a projection* $P \in \mathcal{P}(E)$ *such that* $P, \tilde{T}(P), \ldots, \tilde{T}^{2n-1}(P)$ *are disjoint and* $\left\| \sum_{k=0}^{n-1} \tilde{T}^k(P)f \right\| \geq \frac{1}{4} \|f\|$.

ii) If E *has a weak Fatou norm, then there exists a constant* $c > 0$ *such that for every element* $f \in E$ *and every natural number* n *there exists a projection* $P \in \mathcal{P}(E^{**})$ *such that* $P, \tilde{T}^{**}(P), \ldots, (\tilde{T}^{**})^{2n-1}(P)$ *are disjoint and* $\left\| \sum_{k=0}^{n-1} (\tilde{T}^{**})^k(P)f \right\| \geq c \|f\|$.

Proof. Fix n and f, and let B be a band satisfying lemma 5.12 with $m = 2n$. If E is Dedekind complete, let $Q \in \mathcal{P}(E)$ be the projection onto B. Then $\bigvee_{k=0}^{4n-1} T^K(Q) = I$. For each $j \in \{0, 1, 2, 3\}$ define $R_j = \sum_{k=0}^{n-1} T^{nj+k}(Q)$. Then $\|f\| = \|\bigvee_{k=0}^{4n-1} \tilde{T}^k(Q)|f|\| \leq \|\sum_{k=0}^{4n-1} (\tilde{T}^k(Q)|f|)\| = \|\sum_{j=0}^{3} R_j|f|\| \leq \sum_{j=0}^{3} \|R_j f\|$. Thus for at least one j, say j_0, we have $\|R_{j_0} f\| \geq \frac{1}{4}\|f\|$. Then $P := \tilde{T}^{nj_0}(Q)$ satisfies i). The second part of the lemma is proved similarly.

Remark: Lemma 5.13 is a somewhat modified weak functional analytic version of the well known "Rohlin-Halmos lemma" of ergodic theory (c.f. [H] pg. 71, [Rn], [Fr] §7]. A more conventional formulation would be that, under the assumptions of i) in the lemma, for every $f \in E$ and $n \in \mathbb{N}$ there exists a projection $P \in \mathcal{P}(E)$ such that $P, \tilde{T}(P), \ldots, \tilde{T}^n(P)$ are disjoint and $\|\sum_{k=0}^{n} \tilde{T}^k(P)f\| \geq \frac{1}{2}\|f\|$. If E has order continuous norm, it follows from [CF] (see also [Fr] 7.9) that the lower bound of $\frac{1}{2}\|f\|$ can be improved to the classical $(1-\varepsilon)\ \|f\|$ bound. The author does not know whether this is possible in general.

THEOREM 5.14. *Let* $T \in \mathcal{L}_b(E)$ *be an aperiodic quasi-invertible disjointness preserving operator on a Banach lattice* E.

i) If E is Dedekind complete then $\sigma(T)$ *and* $A\sigma(T)$ *are rotationally invariant.*

ii) If E *has a weak Fatou norm then* $\sigma(T)$ *is rotationally invariant.*

Proof. To prove i), it suffices to prove the statement about $A\sigma(T)$. Suppose $\lambda \in A\sigma(T)$. Then for each $n \in \mathbb{N}$, there exists an element $f_n \in E$ s.t. $\|f_n\| = 1$ and $\|Tf_n - \lambda f_n\| < \frac{1}{n}$. Fix $n \in \mathbb{N}$ and for simplicity, assume that n is odd. By lemma 5.13, there exists a projection $P_n \in \mathcal{P}(E)$ s.t. $P_n, \tilde{T}(P_n), \ldots, \tilde{T}^{2n+1}(P_n)$ are disjoint and $\|\sum_{k=0}^{n} \tilde{T}^k(P_n)f_n\| \geq \frac{1}{4}$. Pick α in the unit circle and define

$$M_n := \sum_{k=1}^{n}\left[k\,\alpha^{-k}\,\tilde{T}^{(k-\frac{n+1}{2})}(P_n)\right] + \alpha^{-n-1}\sum_{k=0}^{n-1}\left[(n-k)\alpha^{-k}\,\tilde{T}^{(k+\frac{n+1}{2})}(P_n)\right] \text{ and}$$

$N_n := \frac{M_n}{\|M_n f_n\|}$. Note that $nI \geq |M_n| \geq \frac{n+1}{2}\sum_{k=0}^{n}\tilde{T}^k(P_n)$. Hence, $\|M_n f_n\| \geq \frac{n+1}{2}\|\sum_{k=0}^{n}\tilde{T}^k(P_n)f_n\| \geq \frac{n+1}{8}$. Note also that $\|\tilde{T}(N_n)\| \leq \frac{\|nI\|}{\|M_n f_n\|} \leq \frac{n}{(n+1)/8} < 8$. Furthermore, $|\tilde{T}(M_n) - \alpha M_n| = \sum_{k=1}^{2n+1}\tilde{T}^{(k-\frac{n+1}{2})}(P_n) \leq I$. Therefore, $\|TN_n f_n - \lambda\alpha N_n f_n\| \leq \|\tilde{T}(N_n)Tf_n - \lambda\tilde{T}(N_n)f_n\| + \|\lambda\tilde{T}(N_n)f_n - \lambda\alpha N_n f_n\|$
$\leq \|\tilde{T}(N_n)\|\,\|Tf_n - \lambda f_n\| + |\lambda|\,\|\tilde{T}(N_n) - \alpha N_n\|\,\|f_n\| \leq \frac{8}{n} + \frac{|\lambda|}{\|M_n f_n\|}\|\tilde{T}(M_n) - \alpha M_n\|$
$\leq \frac{8}{n} + \frac{8|\lambda|}{n+1} \to 0$ as $n \to \infty$. Therefore, $\lambda\alpha \in A\sigma(T)$, which proves i). The second statement is proved similarly.

Via the decomposition theorems proved in chapter 4, the preceding results may be combined in the Dedekind complete case to yield a general theorem valid for arbitrary bi-disjointness preserving operators.

THEOREM 5.15. *Let* $T \in \mathcal{L}_b(E)$ *be a bi-disjointness preserving operator on a Dedekind complete Banach lattice* E. *Let* E_k $(k = 1, 2, \ldots)$ *be the bands on which* T *is quasi-invertible with strict period* k (*theorem* 4.19). $M = \{k \in \mathbb{N}: E_k \neq \{0\}\}$ and let $T_k \in \mathcal{L}_b(E_k)$ *be the restriction of* T *to* E_k.

Then $\sigma(T) = \left[\bigcup_{k\in M} \sigma(T_k)\right] \cup R$ *and* $A\sigma(T) = \left[\bigcup_{k\in M} A\sigma(T_K)\right] \cup S$, *where R and S are rotationally invariant subsets of the complex plane.*

Proof. By theorem 4.13 there exist T-reducing bands B_1, B_2, B_3, B_4 such that the restriction of T to B_i (i = 1, 2, 3, 4) is, respectively quasi-invertible, of forward shift type, of backward shift type, or is hyper-nilpotent. By theorems 5.6 and 5.8, the spectrum of T restricted to the latter three parts is rotationally invariant. It therefore suffices to prove the theorem when T is quasi-invertible. In this case, by theorem 4.19, $E = \left(\bigoplus_{k\in M} E_k\right) \oplus E_\infty$ where M, E_k and E_∞ are as in theorem 4.19. By theorem 5.14, $A\sigma(T_\infty)$ is rotationally invariant. Hence, it can be supposed that $E_\infty = \{0\}$. Clearly, $\bigcup_{k\in M} A\sigma(T_k) \subset A\sigma(T)$. Suppose $\lambda \in A\sigma(T) \setminus \bigcup_{k\in M} A\sigma(T_k)$. We must show that $|\lambda|\ \alpha \in \sigma(T)$ for all $\alpha \in C$. Now for every $n \in \mathbb{N}$, there exists an element $f_n \in E$ s.t. $\|f\| = 1$ and $\|Tf_n - \lambda f_n\| < 1/n$. Since $\lambda \notin A\sigma(T_k)$ for all k, there must be a subsequence $\left\{f_{n_j}\right\}_{j=1}^{\infty}$ and a sequence of integers $\{m_j\}_{j=1}^{\infty}$ such that $m_j \to \infty$ as $j \to \infty$ and $f_{n_j} \in E_{m_j}$.

Pick $\alpha \in C$. Then for any $j \in \mathbb{N}$, there exists an m_jth root of unity α_j s.t. $|\alpha - \alpha_j| < \frac{\pi}{m_j}$. Define M_j as in the proof of theorem 5.10, so that $|M_j| = P_{m_j}$, where P_{m_j} denotes the projection onto E_{m_j}, and $\tilde{T}(M_j) = \alpha_j M_j$. We have $\|T M_j f_{n_j} - \lambda\alpha M_j f_{n_j}\| = \|\alpha_j M_j T f_{n_j} - \lambda\alpha M_j f_{n_j}\|$

$\leq \|\alpha_j M_j Tf_{n_j} - \lambda\alpha_j M_j f_{n_j}\| + \|\lambda\alpha_j M_j f_{n_j} - \lambda\alpha_j M_j f_{n_j} - \lambda\alpha M_j f_{n_j}\|$

$\leq |\alpha_j|\,\|M_j\|\ \|T f_{n_j} - \lambda f_{n_j}\| + |\lambda|\,\|M_j\|\ \|f_{n_j}\|\ |\alpha_j - \alpha| < 1/n_j + \frac{|\lambda|}{m_j} \to 0$ as $j \to \infty$

Hence $\lambda\alpha \in A\sigma(T)$ which proves the second statement. The statement about

$\sigma(T)$ now follows easily from the one about $A\sigma(T)$.

Remark: Various special cases of the preceding results are known. Results similar to theorems 5.6, 5.8, 5.11 and 5.14 were obtained by Wolff [Wo] for Markov operators on a C(X) space. Results similar to theorem 5.6 and 5.8 were also proven by Ridge [Ri] for composition operators on L^p space. Theorem 5.14 is well known for weighted shift operators on various sequence spaces, see Shields [Sh] for a survey. Theorems 5.10 and 5.14 were obtained by Parrott [Pa] (see also [Pe]) for operators induced by measure preserving transformations on an L^p space. Kitover [Ki 1] generalized the second of these results to operators induced by a non-singular measurable transformation on Banach function spaces with order continuous norm. Kitover [Ki 2] also stated theorem 5.14 for invertible operators on an arbitrary Banach lattice, though no proof was given. Arendt [Ar 1] proved corollary 5.11 and theorem 5.15 for lattice isomorphisms with zero aperiodic component. He also obtained special cases of theorem 5.14.

The remainder of this chapter gives various applications of the preceding results.

THEOREM 5.16. *Let* E *be a Dedekind complete Banach lattice. Suppose that* $T \in \mathcal{L}_b(E)$ *is bi-disjointness preserving and that for every* $r > 0$, $C_r \cap \sigma(T)$ *lies in some open half-plane. Then there exists a projection* $P \in \mathcal{P}(E)$ *such that* $TP = PT$, $T|_{PE} \in Z(PE)$ *and* $T|_{(I-P)E}$ *is quasi-nilpotent.*

Proof. Let E_1 be as in theorem 5.15 and let P be the band projection onto E_1 . Then $TP = PT$ and $T|_{PE} \in Z(PE)$. Suppose $\lambda \in \sigma(T|_{(I-P)E})$. Then by theorem 5.15 either $\lambda \in \sigma(T|_{E_k})$ for some $k = 2, 3, \ldots,$ or $\lambda \in S$.

But if $\lambda \in S$, $\lambda e^{i\theta} \in S$ for all $\theta \in [-\pi,\pi)$ by theorem 5.15, which forces $\lambda = 0$, by assumption. Similarly, if $\lambda \in \sigma(T|_{E_k})$ for some $k = 2, 3, \ldots$ then there exists a kth root of unity $e^{i\theta}$ $(\theta \in [-\pi,\pi))$ s.t. $|\theta| \geq \pi/2$. By theorem 5.10 $\lambda e^{i\theta} \in \sigma(T)$ and $\lambda e^{-i\theta} \in \sigma(T)$. Since λ, $\lambda e^{i\theta}$, $\lambda e^{-i\theta}$ all lie in the same open half-plane, $\lambda = 0$ and the proof is complete.

If T is invertible or 0 is an isolated point in $\sigma(T)$, then the Dedekind completeness and order continuity assumptions in the last theorem can be dropped.

COROLLARY 5.17. *Let* E *be a Banach lattice, and let* $T \in \mathcal{L}_b(E)$ *be a disjointness preserving operator whose adjoint is also disjointness preserving. Suppose that for all* $r > 0$, $C_r \cap \sigma(T)$ *lies in some open half—plane. Suppose also that there exists a positive number* s *such that* $\{z \in \mathbb{C} : 0 < |z| \leq s\} \cap \sigma(T) = \emptyset$. *Then there exists a band projection* $P \in \mathcal{P}(E)$ *such that* $TP = PT$, $T|_{PE} \in Z(PE)$, *and* $T|_{(I-P)E}$ *is quasi-nilpotent.*

Proof. Let Q be the spectral projection induced by C_s and let $P = I - Q$. Then $TP = PT$, and by theorem 5.2, P is a band projection. Clearly, $T|_{(I-P)E}$ is quasi-nilpotent. Let $PE = E_1$ and $T|_{E_1} = T_1$. Since $\sigma(T_1) = \sigma(T_1^*)$, we can apply theorem 5.16 to $T_1^* \in \mathcal{L}_b(E_1^*)$. Since T_1^* is invertible, we therefore have $T_1^* \in Z(E_1^*)$ whence $T_1 \in Z(E_1)$, which completes the proof.

Remarks: 1. Corollary 5.17 generalizes the results of Schaefer-Wolff-Arendt [SWA] and Arendt [A 2] , who proved special cases for lattice isomorphisms and invertible disjointness preserving operators, respectively. A special case of corollary 5.17 was obtained by Wickstead [W 3], who needed the additional assumptions that $E = C_0(X)$ (where X is a locally compact Hausdorff space) and that T is a lattice homomorphism with finite spectrum.

2. Even quasi-invertible disjointness preserving operators may be quasi-nilpotent. Examples are given by Schaefer ([S 2] or [S] pg. 353, problem 9) and Wickstead [Wi 3] example 4.1.

3. The assumption in the last two results that the adjoint is disjointness preserving cannot be dropped. For example, take $E = \mathbb{R}^2$ with the usual ordering and norm. Let T be the operator whose matrix is $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ under the standard basis. Then $\sigma(T) = \{0, 1\}$ but T does not satisfy the conclusion of the theorem.

Recall that an operator $T \in \mathcal{L}_b(E)$ on a Riesz space E is called *band-irreducible* if the only T-reducing bands are $\{0\}$ and E. If T is an order continuous disjointness preserving operator, then T is band-irreducible iff its unique extension to the Dedekind completion of E (see thm. 1.9) is band-irreducible. If T is in addition quasi-invertible, then it follows from corollary 4.10 that T is band-irreducible iff $t_T(B) = B \, (B \in \mathcal{B}(E))$ implies $B = \{0\}$ or $B = E$.

THEOREM 5.18. *Let* $T \in \mathcal{L}_b(E)$ *be a band-irreducible bi-disjointness preserving operator on a Banach lattice* E. *If* E *is infinite dimensional and is either Dedekind complete or has a weak Fatou norm, then the spectrum of* T *is either an annulus or a disk. In other words, there exist real numbers* $0 \leq r_1 \leq r_2$ *such that* $\sigma(T) = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$.

Proof. Let $\hat{T}$ be the extension of T to the Dedekind completion of E.

Since $\hat{T}$ is band-irreducible, by theorem 4.13, $\hat{T}$ and hence T must be either quasi-invertible, of forward shift type, of backward shift type or hypernilpotent. If T is one of the latter three, then $\sigma(T)$ is rotationally invariant by theorems 5.6 and 5.8. If T is quasi-invertible, then we claim that T is aperiodic. If not, then by theorem 4.19 T has strict period n for some natural number n. Pick $0 \neq f \in E$ and let $B = \bigvee_{k=0}^{n-1} t^k(\{f\}^{dd})$. Since t is a Boolean isomorphism and T has strict period n, $t(B) = \bigvee_{k=0}^{n-1} t^{k+1}(\{f\}^{dd}) = B$. As T is band-irreducible and quasi-invertible, B = E. For any non-zero band $A \subset \{f\}^{dd}$, it is clear that $A, t(A), \ldots, t^{n-1}(A)$ are mutually disjoint. Since T is band-irreducible, it follows as above that $E = \bigvee_{k=0}^{n-1} t^k(A)$. Therefore $A = \{f\}^{dd}$, which shows that f is an atom. Hence dim E = n contrary to assumption, so T must be aperiodic as claimed. By theorem 5.15, $\sigma(T)$ is rotationally invariant.

Combining the above results shows that $\sigma(T)$ is always rotationally invariant. It now follows easily from theorem 5.2 (see [Ar 2] 4.6) that T is either an annulus or a disk.

Remark: For invertible disjointness preserving operators, a result similar to theorem 5.18 was stated without proof by Kitover [Ki 2]. It was already noted by Arendt [Ar 1] that a band-irreducible lattice isomorphism on an infinite-dimensional Dedekind complete Banach lattice is aperiodic. Some special cases of theorem 5.18 on concrete function spaces have been proved; see [Pa], [Ke], [Ar 1].

Let E be a Dedekind complete Banach lattice. Then $\mathcal{L}_b(E)$ is a Banach algebra under the r-norm $\|T\|_r := \||T|\|$ (see [S] IV §1). The *order spectrum* of an operator $T \in \mathcal{L}_b(E)$ is the spectrum of T with respect to $\mathcal{L}_b(E)$ and will be denoted by $\sigma_0(T)$. It is clear that $\sigma(T) \subset \sigma_0(T)$; this inclusion may be strict, see [S 3] for an example and further discussion. Our next result shows that equality does hold for bi-disjointness preserving operators.

THEOREM 5.19. *Let* $T \in \mathcal{L}_b(E)$ *be a bi-disjointness preserving operator on a Dedekind complete Banach lattice* E. *Then* $\sigma(T) = \sigma_0(T)$.

Proof. It follows from $|Tf| = ||T|f|$ that the spectral radius of T in $\mathcal{L}(E)$, $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$, is the same as the spectral radius in $\mathcal{L}_b(E)$, $r_0(T) = \lim_{n\to\infty} \||T^n|\|^{1/n}$.

Suppose for some $0 \leq s < r(T)$, $C_s \cap \sigma(T) = \emptyset$. Let P be the spectral projection induced by C_s. By theorem 5.2, PE and (I-P)E are T-invariant bands. Let T_1 and T_2 be the restrictions of T to PE and (I-P)E, respectively. Then $\sigma_0(T) = \sigma_0(T_1) \cup \sigma_0(T_2)$. By the above, $s > r(T_1) = r_0(T_1)$ and $s < \sigma(T_2^{-1})^{-1} = r_0(T_2^{-1})^{-1}$. It follows that $C_s \cap \sigma_0(T) = \emptyset$.

Now suppose $\lambda \in \sigma_0(T)$. If $\lambda \notin \sigma(T)$, then it follows from the preceding paragraph that there exists an element $\mu \in A\sigma(T)$ such that $|\mu| = |\lambda|$. Since $\lambda \notin \sigma(T)$, it follows as in theorem 5.15 that for some natural number n that the restriction T_0 of T to $\left\{\bigcup_{k=1}^{n} E_k\right\}^d$ (with notation as in theorems 4.19 and 5.15) satisfies $\sigma(T_0) \cap C_{|\mu|} = \emptyset = \sigma_0(T_0) \cap C_{|\mu|}$. It therefore may be assumed that $E = \bigcup_{k=1}^{n} E_k$. In this case, $T^{n!} \in Z(E)$. But it is easy to see that $\sigma(M) = \sigma_0(M)$ for $M \in Z(E)$ (see [Ar 2] 3.3). Thus, by the spectral mapping theorem, $\sigma(T) = \sigma_0(T)$, whence $\lambda \in \sigma(T)$ which completes the proof.

Remark: The argument given in the above theorem is essentially due to Arendt [Ar 1], who proved theorem 5.19 there for a lattice isomorphism with zero aperiodic component.

Our final application is an analogue to a well-known result for normal operators on a Hilbert space.

THEOREM 5.20. *Let* E *be a Dedekind complete Banach lattice and suppose* $T \in \mathcal{L}_b(E)$ *is a bi-disjointness preserving operator. Then non-zero isolated points of* $A\sigma(T)$ *are contained in* $P\sigma(T)$.

Proof. Suppose $\lambda \neq 0$ is an isolated point in $A\sigma(T)$. Then it follows from theorem 5.15 that $\lambda \in A\sigma(T|_{E_n})$ for some $n \in \mathbb{N}$, where E_n denotes the band in which T has strict period n as in theorem 5.15. Thus, it suffices to prove the theorem when T is quasi-invertible with strict period n. Since $P\sigma(T^n) = (P\sigma(T))^n$ and $A\sigma(T^n) = (A\sigma(T))^n$, it may be assumed that $T \in Z(E)$. Using the isometric isomorphism $Z(E) \simeq Z(Z(E)) \simeq Z(C(X))$ where X is some compact Hausdorff space, T may be identified with a multiplication operator $T_f \in Z(C(X))$ defined by $T_f g = f \cdot g$ $(g \in C(X))$ for some $f \in C(X)$. Since $\sigma(T) = \sigma(T_f)$ (see [Ar 2] 3.3), λ is an isolated point in $\sigma(T_f)$. Thus, since $\sigma(T_f)$ = range of f, there must be a non-empty open-closed set $U \subset X$ such that $f(x) = \lambda$ for all $x \in U$. By Urysohn's lemma, there exists a non-zero function $g \in C(X)$ such that $g(y) = 0$ for all $y \in X \setminus U$. Note that $T_f g = \lambda g$. Identify g with its corresponding element $M \in Z(E)$ under the same identification $C(X) \simeq Z(E)$ used before and pick $h \in E$ such that $Mh \neq 0$. Then $T(Mh) = (TM)h = \lambda M h$ which completes the proof.

Remarks: 1) Theorem 5.20 was proven for lattice homomorphism on $C_0(X)$ spaces, where X is a locally compact Hausdorff space, by Wickstead [Wi 3].

2) As was pointed out in [Wi 3], the exclusion of zero in theorem 5.20 is necessary. For an example, see [Wi 3] 4.1.

In conclusion, we leave as an open question whether the hypothesis used throughout this chapter that E is either Dedekind complete or has a weak Fatou norm can be dropped.

The assumption that E has a weak Fatou norm is only needed to apply the conclusion of lemma 5.3. Thus, the results in this chapter remain valid if the assumption that E has a weak Fatou norm is replaced by the conclusion of lemma 5.3. (In fact, inspection of the proofs of theorems 5.6 and 5.8 shows that these theorems remain valid even if the constant in lemma 5.3 depends on the projection). It is possible, though unlikely, that lemma 5.3 holds for an arbitrary Banach lattice. If this is the case, then the results of this chapter would be true for arbitrary Banach lattices. It is quite possible, however, that the results of this chapter are false without some condition on the Banach lattice.

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