ASPECTS OF THE THEORY OF NORMED SPACES

Thesis by

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For Ronelle, my parents, Paul, Vicky, Ryan and David

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ABSTRACT

The dissertation will be divided into two parts. The first part will, in essence, be a study of weak compactness in a variety of families of normed spaces. Included in this study will be general characterizations of weak compactness in spaces of vector measures and tensor products that contain all known results of this nature as special cases (in particular, we do not need to restrict attention to only those range spaces with strong geometric properties such as, for example, the Radon-Nikodym property). The methods of Nonstandard Analysis constitute a fundamental tool in these investigations.

The second part of the dissertation will contain a discussion and a study of Model theoretic aspects of categories of normed spaces. We will introduce multi-sorted formal languages that enable us to view various subcategories of the category of normed spaces as being equivalent to categories of set-valued models of coherent theories in these languages. We see, in particular, that the category of real normed spaces is equivalent to the category of set-valued models of a lim-theory, and that, for instance, the category of L-spaces is equivalent to the category of set-valued models of a coherent extension of this lim-theory. These considerations allow for proofs of existence of 2-adjoints to inclusion functors from some 2-categories into the 2-category of Topos-valued normed spaces, and the study of the elementary properties of these adjoints.

The coherent theory of Hilbert spaces gives rise to interesting spatial Toposes when the appropriate "adjoint functor theorems" are

proved. The sites of these toposes are spectral spaces (in the sense of Algebraic geometry) with interesting cohomological properties.

PART I

ASPECTS OF WEAK COMPACTNESS IN CERTAIN NORMED SPACES

Introduction

As a labor saving device and in order to facilitate communication we introduce the following notation and definitions which will serve unchanged throughout Part I.

$(X, \| \|_X)$ will denote a Banach space; as is customary, we will often use $(X, \| \|)$ or X as "equivalent" notation. Σ will denote a σ-algebra of subsets of the set Ω and μ will denote a finite, non-negative σ-additive measure defined on Σ. The space of Bochner integrable functions on Ω with values in X, $L_1(\Omega, X)$, is defined as follows: $f : \Omega \to X$ is Bochner integrable if and only if f is μ-measurable (i.e., f is the μ-almost everywhere limit of simple functions) and there exists a sequence of simple functions $(f_n)_{n\epsilon}$ such that the sequence $\int_\Omega \| f_n - f \| d_\mu$ converges to zero. For each $E \epsilon \Sigma$, $\int_E f d\mu$ denotes the limit, $\lim_{n\to\infty} \int \chi_E f_n d\mu$, where the (f_n) are as above and χ_E denotes the characteristic function of E. A vector valued measure with values in X is a function $G : \Sigma \to X$ with the property that $G(E_1 \cup E_2) = G(E_1) + G(E_2)$ for disjoint $E_1, E_2 \epsilon \Sigma$. If G satisfies: $G(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} G(E_i)$ for every pairwise disjoint sequence in Σ, then G is called a σ-additive measure. The variation of G, $|G|$, is the function $|G| : \Sigma \to \mathcal{R} \cup \{\infty\}$ defined by: $|G|(E) = \sup_\pi \sum_{A\epsilon\pi} \| G(A) \|$ where the supremum is taken over all partitions π of E consisting of a finite number of members of Σ.

if $|G| < \infty$, then G is said to be of bounded variation. We denote by $BM(\Sigma, X)$ (respectively $\sigma BM(\Sigma, X)$) the space of all bounded X-valued measures $G : \Sigma \to X$ (respectively all vector measures $G : \Sigma \to X$ of bounded variation which are also σ-additive) normed with the variation norm $|G| =: |G|(\Omega)$.

One of the central problems that we will study in this part of the report, is a generalization of the following well known and well understood problem: find properties of subsets $K \subseteq L_1(\Sigma, \mathcal{R})$ that characterize the subsets K with weakly compact closure; we should clearly only be interested in "decent" properties, where "decent" is interpreted in popular systems of mathematical aesthetics and mathematical ethics.

The following theorem of Dunford is considered to be an acceptable solution to the above mentioned classical problem.

<u>0.1 Theorem</u> (Dunford [1]). A subset $K \subseteq L_1(\Sigma, \mathcal{R})$ is weakly sequentially compact if and only if it is bounded and the countable additivity of the integrals $\int_E f\, d\mu$ is uniform with respect to f in K.

<u>0.2 Remarks:</u> (1) The integrals $\int_E f\, d\mu$ have uniform countable additivity with respect to f in K precisely when for every sequence $\{E_n\}$ in Σ with $\bigcap_{n=1}^{\infty} E_n = \phi$, $\lim_{n\to\infty} \int_{E_n} f\, d\mu = 0$ uniformly in K.

(2) According to the Eberlein-Smulyan theorem, Dunford's theorem actually characterizes the relatively weakly compact subsets of $L_1(\Sigma, \mathcal{R})$ i.e., the subsets of $L_1(\Sigma, \mathcal{R})$ that have weakly compact weak closures.

The question then arises: to what extent can this type of characterization of relative weak compactness by generalized to the situation where $\mathcal{R}$ is replaced by a non-trivial Banach space X? It turns out that one can replace $\mathcal{R}$ by any reflexive space X if we change "$\lim_{n\to\infty} \int_{E_n} f\,d\mu = 0$", to "$\lim_{n\to\infty} \int_{E_n} \|f\|\,d\mu = 0$" in (0.1). The proof of this generalized version of (0.1) depends crucially on the fact that every reflexive space X possesses the so-called Radon-Nikodym property.

0.3 Definition. X has the Radon-Nikdym property with respect to (Ω, Σ, μ) if for each μ-continuous vector measure $G : \Sigma \to X$ of bounded variation there exists a $g \in L_1(\Sigma, X)$ such that $G(E) = \int_E g\,d\mu$ for all $E \in \Sigma$ (G is μ-continuous if $\lim_{\mu(E)\to 0} G(E) = 0$).

One can generalize Dunford's theorem even further to get the following result.

0.4 Theorem [2]. If both X and X^* (the dual of X) have the Radon-Nikodym property and $K \subseteq L_1(\Sigma, X)$ then K is relatively weakly compact if and only if

(i) K is bounded

(ii) K is uniformly integrable (i.e.,

$$\lim_{\mu(E)\to 0} \int_E \|f\|\,d\mu = 0 \text{ uniformly in K)}$$

(iii) for each $E \in \Sigma$, the set $\{\int_E f\,d\mu / f \in K\}$ is relatively weakly compact. The question that we then address in Chapter 2 is the following: To what extent can (0.4) be generalized?; in particular, what happens if it is not true thtt X and X^* both have the Radon-Nikodym property?

0.5 Remarks. (i) The Radon-Nikodym property can be viewed as a geometric property (cf. the notion of "dentability") and is currently one of the most intensely studied geometric properties of normed spaces ([2], [3]-[9]).

(ii) The following is a quotation from [2]: "The problem of characterizing the relatively weakly compact subsets of $L_1(\Sigma, X)$ for general X, remains one of the most elusive problems in the theory of vector measures." One must add to this, as was suggested previously, that it is not at all clear at this stage when a set of criteria for relative weak compactness will be an acceptable set.

0.6 A notable extension of (0.4) appears in a paper by Brooks and Dinculeanu [10]; they prove that conditions (i)-(iii) (0.4) with the following:

(iv) For every countable subset $K_0 \subseteq K$ there exists a sequence (π_n) of μ-partitions such that f_{π_n} converges strongly to f, uniformly for $f \in K_0$ (where $f_{\pi_n} =: \sum_{A \in \pi_n} [(\mu(A))^{-1} \int_A f\,d\mu)\,\chi_A]$ imply that K is relatively weakly compact. If it is assumed that X^* has the Radon-Nikodym property, then, given that K is relatively weakly compact, (iv) with "converges strongly" replaced by "converges weakly" holds true.

0.7. In Chapter 1 we study sets $K \subseteq L_1(\mu, X)$ such that the union of the ranges of the members of K is well behaved; in particular, using the main result of Chapter 1, (1.10) we prove that if this common range is relatively weakly compact, then conditions (i)-(iii) of (0.4) are sufficient to ensure that K is relatively weakly compact. This result was independently discovered by J. Diestel [11] -- his methods of proof are entirely different from ours (he uses a factorization result that follows from a lemma of Davis, Figiel, Johnson, Pelczynski (cf. [2])), and do not appear to be amenable to sufficient generalization to yield our results.

0.8. The fact that $L_1(\Sigma, X)$ embeds isometrically in $\sigma BM(\Sigma, X)$ suggests that results similar to (0.4) may be obtained in this more general context. When it is assumed that X and X^* both have the Radon-Nikodym property, then the following result is easily obtained.

Theorem [2] $K \subseteq \sigma BM(\Sigma, X)$ is relatively weakly compact if and only if

(i) K is bounded;

(ii) there is a μ on Σ such that $\lim_{\mu(E)\to 0} |G|(E) = 0$ uniformly in $G \in K$;

(iii) $\{G(E)/G \in K\}$ is relatively weakly compact for every $E \in \Sigma$.

In fact, the three conditions of the theorem imply that K is contained in $L_1(\Sigma, X)$ and the result follows from (0.4).

In Chapter 3 we drop the assumption that X and X^* have the Radon-Nikodym property and obtain a general characterization of weak compactness in $\sigma BM(\Sigma, X)$. Using this characterization, the main result of Chapter II can be partly recovered.

0.9. The last chapter of Part I contains some applications of the methods of the previous chapters to the study of weak compactness in spaces of operators.

0.10. A recurrent theme throughout all of Chapter 1 is the following: dual spaces can often be fairly well understood if they are restricted to operate on well behaved, "small" sets of vectors.

CHAPTER 1

The main result of this chapter is theorem 1.10. On our way to the proof of this theorem, we gather together a sequence of preliminary results that will be relevant to the proof of (1.11).

Our first result is well known in the case $X = \mathcal{R}$ (cf. [12]) -- however, we could not find any reference in the literature to the more general situation that is of importance to us; hence we present here our elementary proof (which is proably not the most elegant one).

1.1 Proposition. Let $f \in L_1(\Sigma, X)$ and let $S \subseteq X$ be weakly compact and convex. If $\mu(E)^{-1} \int_E f\,d\mu$ is an element of S $\forall$ $E \in \Sigma^+$ (Σ^+ will from now on denote the set of all members of Σ with positive μ-measure), then $f(\omega) \in S$ for almost every $\omega \in \Omega$.

Remark. This proposition is a converse to "the mean value theorem for the Bochner integral" (cf. [2] p. 48).

Proof. Pick $a \in X \setminus S$ and let $B(a,r)$ $(=\{x \in X / \|x-a\| < r\})$ be contained in $X \setminus S$. $B(a,r)$ is convex and open; hence we can find $\alpha \in \mathcal{R}$, $x^* \in X^*$ such that $x^*[B(a,r)] < \alpha \leq x^*[S]$.

As $x^*[S]$ is closed in $\mathcal{R}$ and $(\int_E x^* f\,d\mu)(\mu(E))^{-1} = x^*((\int_E f\,d\mu)(\mu(E))^{-1}$ is in $x^*[S]$ $\forall E \in \Sigma^+$, $x^*(f(\omega)) \in x^*[S]$ for almost every $\omega \in \Omega$, by the scalar value case of our proposition. As $f^{-1}[B(a,r)] \subseteq (x^* f)^{-1}[x^* B(a,r)]$. it follows that $f^{-1}[B(a,r)]$ is contained in a null set.

According to the Pettis measurability theorem ([2], p. 42), we can find $E \in \Sigma$ such that $f[E]$ is norm separable and $\mu(\Omega \setminus E) = 0$. Let $D \leq X$ be countable and norm dense in $f[E]$. For every $x \in F[E] \setminus S$, let $d_x = \inf_{s \in S} ||x - s||$ and let $B_x = B(x, \frac{1}{2}d_x)$. Then $D \cap (\bigcup_{x \in F[E]} B_x)$ (let us denote this set by $\hat{D}$) is norm dense in $f[E] \cap [X \setminus S]$. If $y \in \hat{D}$, then $B(y, d_y) \subseteq X \setminus S$ and $f[E] \cap [X\ S] \subseteq \bigcup_{y \in \hat{D}} B(y, d_y)$. As $\hat{D}$ is countable, $f^{-1}[\bigcup_{\hat{D}} B(y, d_y)]$ is contained in a null set which in turn implies that $f^{-1}[f(E) \cap [X \setminus S]]$ is contained in a null set. Therefore, as

$$f^{-1}[X \setminus S] = f^{-1}[(f[E] \cap [X \setminus S]) \cup ([X \setminus f[E]] \cap [X \setminus S])]$$

$$\subseteq (f^{-1}[f[E] \cap [X \setminus S]]) \cup (\Omega \setminus E)$$

we have the desired result.

1.2 Proposition. Let $f : \Omega \to X$ be measurable, let $S \subseteq X$ be weakly compact and let $C(S)$ denote the space of all continuous real valued functions on S. Say $f[\Omega] \subseteq S \subseteq X$; then for every measurable function $g : \Omega \to C(S)$, the function $h_g : \Omega \to \mathcal{R}$ defined by $h_g(\omega) = g(\omega)(f(\omega))$ is measurable.

Proof. If g is of the form $k\chi_\Omega$, then $h_g = (k \circ f)\, \chi_\Omega = k \circ f$ which is measurable (use the fact that $g : \Omega \to \mathcal{R}$ is measurable if and only if the inverse image $g^{-1}[U]$ is in the Lebesque extension of Σ for every open U). It follows that simple functions g give rise to measurable h_g.

If g is not simple, let $g_i \to g$ outside a null set where the g_n are simple. Then $|g(\omega)(f(\omega)) - g_n(\omega)(f(\omega)| \leq \sup_S |g(\omega) - g_n(\omega)| = ||g(\omega) - g_n(\omega)|| \to 0$. Hence h_g, being the almost everywhere limit of measurable functions, is measurable.

1.3 Notation. If $f : \Omega \to X$ and $g : \Omega \to C(S)$ with $f[\Omega] \subseteq S \subseteq X$, then we denote by $\langle f, g \rangle$ the function $\omega \mapsto g(\omega)(f(\omega))$.

We now introduce the notion of quasi-equicontinuity -- a notion that will play a fundamental role in what follows.

1.4 Definition. Let S be a Hausdorff space. A family $F \subseteq C(S)$ is said to be quasi-equicontinuous on S if the convergence of a net S_α to S_0 in S implies that the convergence $f(S_\alpha) \to f(S_0)$ is quasi-uniform on F. That is, given $\epsilon > 0$ and α_0, there exists a finite set $\alpha_i \geqslant \alpha_0$ $(i = 1, \cdots, n)$ such that for each $f \in F$,

$$\min_{1 \leq i \leq n} |f(S_{\alpha_i}) - f(S_0)| < \epsilon .$$

Quasi-equicontinuity was first studied by Arzela in connection with the properties of the pointwise limits of sets of continuous functions ([1]).

1.5 Lemma. Fix $k \in \mathcal{R}^+$ and let $F \subseteq \{n^* \in X^* / ||x^*|| \leqslant k\}$. Then F, considered as a family of continuous real valued functions on X with the weak topology, is quasi-equicontinuous on X.

Proof. $D = \{x^* \in X^* / ||x^*|| \leqslant k\}$ is weak* compact according to the Banach-Alaoglu theorem. Let $\{S_\alpha\}$ be a net in X converging weakly to S_0, and pick $\epsilon > 0$ and α_0. Let $F_\alpha = \{x^* \in D / |x^*(S_\alpha) - x^*(S_0)| < \epsilon\}$, for $\alpha \geqslant \alpha_0$. We know that $x^*(S_\alpha) \to x^*(S_0)$ $\forall x^* \in X^*$, hence $D = \bigcup_{\alpha \in A} F_\alpha$. But $F_\alpha = \{x^* \in D / |f_{(S_\alpha - S_0)}(x^*)| < \epsilon\}$, where $F_{(S_\alpha - S_0)}(x^*) =: x^*(S_\alpha - S_0)$, is open in the weak* topology induced on D. Hence there exists $\alpha_1, \cdots, \alpha_n \in A$, $\alpha_i \geqslant \alpha_0$ $(i = 1, \cdots, n)$ such that $D = \bigcup_{i=1}^{n} F_{\alpha_i}$.

1.6 Definition. A vector measure $G : \Sigma \to X$ is (Σ, μ)-representable if there exists a Bochner integrable function $g : \Omega \to X$ such that

$G(E) = \int_E g\,d\mu$ for every $E \in \Sigma$.

1.7 Theorem ([2]) ("utility grade Radon-Nikodym theorem").
If $G: \Sigma \to X$ is μ-continuous and for each $E_1 \in \Sigma^+$ there is an $E_2 \in \Sigma^+$ with $E_2 \leq E_1$ such that $\{\mu(E)^{-1}G(E)/E \in \Sigma^+,\ E \subseteq E_2\}$ is relatively weakly compact, then G is (Σ, μ)-representable.

1.8 Lemma. Let ℓ be in the dual of $L_1(\Sigma, X)$ and let $S \subseteq X$ be weakly compact. Define $G: \Sigma \to C(S)$ by $G(E)(S) = \ell(S\chi_E)$; then G is representable by a function $g: \Omega \to$ closure$\{f \in C(S)/f =$ the restriction to S of some $x^* \in X^*\}$ (denote this closure by $\bar{B}_S$).

Proof. Define $\bar{G}: \Sigma \to x^*$ by $G(E)(x) = \ell(x\chi_E)$; $\bar{G}(E)$ is linear on X and $|\bar{G}(E)(x)| = |\ell(x\chi_E)| \leq ||\ell||\ ||x\chi_E||_1 \leq ||\ell||\ ||x||\ \mu(E)$. We observe that $G(E)$ is the restriction of $\bar{G}(E)$ to S $\forall E \in \Sigma$. It is clear that G is a measure; furthermore we see that

$$||G(E)||_\infty = \sup_S |\ell(x\chi_E)| \leq ||\ell||_1 \mu(E) \sup_S ||s||$$

in particular it follows that G is μ-continuous and $|G|(\Omega) < \infty$. $\{\mu(E)^{-1}G(E)/E \in \Sigma^+\}$ is sup norm bounded by $||\ell||_1$ where $K = \sup_S ||s||$ and $\{\mu(E)^{-1}\bar{G}(E)/E \in \Sigma^+\} \subseteq \{x^* \in X^*/\ ||x^*|| \leq ||\ell||\}$.

Let $s_2 \to s$ in S with its induced weak topology; then $s_\alpha \to s$ in X with the weak topology. According to (1.5) we can find $\alpha_1, \cdots, \alpha_n$ for every α_0, $\epsilon > 0$ such that $\alpha_i \geq \alpha_0 (i = 1, \cdots, n)$ and

$$\min_i |\mu(E)^{-1}(G(E)(s_{\alpha_i}) - G(E)(s_0))| = \min_i |\mu(E)(\bar{G}(E)(s_{\alpha_i}) - \bar{G}(E)(s_0))| < \epsilon$$

It follows that $\{\mu(E)^{-1}G(E)/E \in \Sigma^{+}\}$ is relatively weakly compact in

However, the weak topology of $\overline{B}_S$ is the topology induced by the weak topology of C(S) (use the Hahn-Banach theorem to verify this). Hence $\{\mu(E)^{-1}G(E)/E \in \Sigma^{+}\}$ is relatively weakly compact in $\overline{B}_S$: as B_S is convex, $\overline{B}_S$ is the weak closure of B_S in C(S); hence the weak closure of $\{\mu(E)^{-1}G(E)/E \in \Sigma^{+}\}$ in C(S) is the same as its weak closure in $\overline{B}_S$. Before we can state and prove our first main result, we need an integral representation result for members of $(L,(\Sigma,X))^{*}$.

<u>1.9 Lemma</u>. If ℓ is an element of the dual of $L_1(\Sigma,X)$ and $f:\Omega \to S$ is measurable, where $S \subseteq X$ is weakly compact, then there is a measurable function $g:\Omega \to \overline{B}_S$ (notation as in 1.8) such that

$$\ell(f) = \int < f, g > d\mu .$$

<u>Proof</u>. Let P be the class of all partitions $\pi \leq \Sigma$ of Ω and direct this class by refinement. Let $\overline{g}_{\pi} = \sum_{E\in\pi} \mu(E)^{-1}\overline{G}(E)$ for every $\pi \in P$ ($\overline{G}$ is as defined in 1.13 and $\mu(E)^{-1}G(E)$ is taken to be equal to zero if $\mu(E) = 0$); then the $\overline{g}_{\pi}$ define, in the obvious fashion, a martingale in $L_1(\Sigma,\overline{B}_S)$ (cf. [2] p. 128).

Let $f \in L_1(\Sigma,X)$ and let $\{f_n\} \leq L_1(\Sigma,X)$ be a norm approximating sequence of simple function for f. Then

$$|\ell(f) - \int \langle f, \overline{g}_{\pi} \rangle d\mu| \leq |\ell(f) - \ell(f_n)| + |\ell(f_n) - \int \langle f_n, \overline{g}_{\pi} \rangle d\mu|$$

$$= |\ell(f) - \ell(f_n)| + |\ell(f_n) - \int \sum_{A\in\pi_n, E\in\pi} \langle x_A \chi_A, \mu(E)^{-1}\overline{G}(E)\chi_E \rangle \chi_{A\cap E} d\mu|$$

(where $f_n = \sum_{\pi_n} x_A \chi_A$)

$$= |\ell(f) - \ell(f_n)| + |\ell(\Sigma x_A \chi_A) - \sum_{A,E} \ell(x_A)\chi_{A\cap E}\cdot\mu(E)^{-1}\mu(A\cap E)|,$$

which is small when π and n are large.

Let $g_\pi = \sum_\pi \mu(E)^{-1} G(E)\chi_E$; then if f takes its values in S, we have $\langle f, \bar{g}_\pi\rangle = \langle f, g_\pi\rangle$. We claim that g_π converges as martingale to g -- in fact, as is easily seen,

$$G(E) = \lim_P \int_E g_\pi \, d\mu;$$

according to a martingale convergence theorem ([2], p. 125), (which says the following: "A martingale in $L_\rho(\Sigma, X)$ converges in $L_\rho(\Sigma, X)$-norm if and only if there exists $f \in L_\rho(\Sigma, X)$ such that $\int_E f\,d\mu = \lim \int_E f_\tau \, d\mu$) we have the required result as G is represented by g. By examining the proof of the above-mentioned martingale theorem, we see that we can also say that $||g_\pi - g||_1 \to 0$.

Now

$$|\int\langle f, g_\pi\rangle d\mu - \int\langle f, g\rangle d\mu| = |\int\langle f, g - g_\pi\rangle d\mu|$$

$$\leq \int \|g - g_\pi\| d\mu \to 0.$$

Hence $\ell(f) = \int\langle f, g\rangle d\mu$ if f takes its values in S.

We state our main results in Non-Standard Analysis terminology; the reader should have no difficulty in providing the statement of the corresponding standard result.

<u>1.10 Theorem.</u> $f \in {}^*L_1(\Sigma, X)$ is weakly near-standard if

(i) $^\circ\|f\| < \infty$ ($^\circ$ denotes the standard part map);

(ii) $\int_E \|f\| d\mu \sim 0$ for every $E \in {}^*\Sigma$ such that $\mu(E) \sim 0$ (two non-standard reals satisfy the relation $\sim$ precisely when their difference is an infinitesimal);

(iii) $\int_{*E} f d\mu$ is weakly near-standard $\forall E \in \Sigma$

(iv) There is a weakly compact set $S \subseteq X$ such that $f[{}^*\Omega] \subseteq {}^*S$.

<u>Remark.</u> It is clear that we may assume that S in condition (iv) is convex (take the closed convex hull of S in (iv)).

<u>Proof.</u> Define $G : \Sigma \to X$ as follows:

$$G(E) = \text{weak standard part of } \int_{*E} f d\mu \text{ for } E \in \Sigma.$$

Then for every $x^* \in X^*$ we know that $x^*(\mu(E)^{-1} G(E)) \sim x^*(\mu(E)^{-1} \int_E f d\mu)$ for every $E \in \Sigma^+$.

According to the mean-value theorem for the Bochner integral (cf. the introduction), $x^*(\mu(E)^{-1} \int_{*E} f d\mu)$ and therefore also $x^*(\mu(E)^{-1} G(E))$ lies in $x^*[{}^*S]$. If $\mu(E)^{-1} G(E)$ is not in S, then we can find x^* such that $x^*(\mu(E)^{-1} G(E)) < \alpha < x^*[S]$ for some $\alpha \in \mathcal{R}$; this, however, is impossible as we know that $x^*(\mu(E)^{-1} G(E))$ is in $x^*[S]$. Hence $\{\mu(E)^{-1} G(E) / E \in \Sigma^+\}$ is relatively weakly compact.

According to 1.7 we therefore knew that there is a $\rho \in L_1(\Sigma, X)$ such that $G(E) = \int_E \rho d\mu \ \forall E \in \Sigma$. [(1.10)(ii) ensures that G is μ-continuous.] Proposition (1.1) allows us to assume that ρ takes its values in S.

What remains to be shown is that p is the weak standard part of f. We know that for $\ell \in (L_1(\Sigma, X))^*$ there is a g such that $\ell(h) = \int \langle h, g \rangle$ provided that h takes its values in S. So it will suffice to show that $\int \langle f - p, {}^*g \rangle \sim 0$ for every $g \in L_1(\Sigma, \bar{B}_S)$.

If g is of the form $h\chi_\Omega$, $h \in \bar{B}_S$, let $(g_n) \subseteq B_S$ be such that $\|h - g_n\|_S \to 0$. Then

$$|\int \langle f - p, h\chi_\Omega \rangle d\mu| \leq$$

$$\leq |\int \langle f, (h - g_n)\chi_\Omega \rangle d\mu| + |\int \langle f, g_n\chi_\Omega \rangle d\mu - \int \langle p, g_n\chi_\Omega \rangle d\mu| + |\int \langle p, (g_n - h)\chi_\Omega \rangle d\mu|$$

$$\leq \|h - g_n\|_S \mu(\Omega) + \text{infinitesimal} + \|h - g_n\|_S \mu(\Omega).$$

Hence $|\int \langle f - p, h\chi_\Omega \rangle d\mu| \sim 0$.

If g is of the form $\Sigma h_E \chi_E$, then we clearly get the desired result.

Let $g \in L_1(\Sigma, \bar{B}_S)$ be a general function and let $(g_n) \subseteq L_1(\Sigma, \bar{B}_S)$ approximate g almost everywhere. For every $\delta > 0$ choose $\Omega_\delta \in \Sigma$ such that $\mu(\Omega \backslash \Omega_\delta) < \delta$ and $g_n \to g$ uniformly on Ω_δ. Then

$$|\int_\Omega \langle f - p, g \rangle d\mu| \leq |\int_{\Omega_\delta} \langle f - p, g \rangle d\mu| + \int_{\Omega \backslash \Omega_\delta} |\langle f, g \rangle| d\mu + \int_{\Omega \backslash \Omega_\delta} |\langle p, g \rangle| d\mu$$

Denote $\int_{\Omega \backslash \Omega_\delta} (|\langle f, g \rangle| + |\langle p, g \rangle|) d\mu$ by η. Then

$$|\int_{\Omega_\delta} \langle f - p, g \rangle d\mu| + \eta \leq$$

$$\leq \left|\int_{\Omega_\delta} \langle f, g - g_n\rangle d\mu\right| + \left|\int_{\Omega_\delta} (\langle f, g_n\rangle - \langle p, g_n\rangle) d\mu\right| + \eta + \left|\int \langle p, g_n - g\rangle d\mu\right|$$

$$\leq 2\mu(\Omega_\delta) \sup_{\Omega_\delta} \| g_n(\omega) - g(\omega)\| + \text{infinitesimal} + \eta.$$

We only need to show that g is bounded on Ω to finish the proof (for then $\eta \leqslant \delta \sup_{\Omega} \|g(w)\|$). We have seen before that the measure F defined by $F(E)(S) = \ell(s\chi_E)$, for a fixed ℓ in $(L_1(\Sigma,X))^*$, is representable and $|F|(E) = \int_E \|g\| d\mu$ for every g that represents F; we also know that $|F|(E) \leq \|\ell\| \sup_S \|s\| \mu(E)$; hence $\|g\|(\omega) \leq \|\ell\| \sup_S \|s\|$ almost always. But we are only interested in g's that represent members of $(L_1(\Sigma,X))^*$; hence we have the desired result.

<u>1.11 Theorem.</u> Let $f \in {}^*L_1(\Sigma,X)$; then f is weakly near-standard if the following conditions are satisfied:

(i) $f[{}^*\Omega]$ is bounded;

(ii) $\int_E \|f\| d\mu \sim 0 \quad \forall E \in {}^*\Sigma$ such that $\mu(E) \sim 0$;

(iii) $\int_{^*E} f d\mu$ is weakly near-standard $\forall E \in \Sigma$;

(iv) $\forall E_1 \in \Sigma^+ \ \exists\, E_2 \in \Sigma^+$ such that $\{\mu(E)^{-1}(\text{weak-}^\circ)(\int_E f d\mu)/E \leq E_2, E \in \Sigma^+\}$ is relatively weakly compact;

(v) there is a set $T \subseteq X$ such that T is bounded and for every ℓ in the dual of $L_1(\Sigma,X)$, $\exists\, g_\ell \in L_1(\Sigma, \bar{B}_f)$($\bar{B}_f$ =: the closure of X^* restricted to T, in the set of bounded continuous functions on T with the weak topology) such that g_ℓ represents the measure $G_\ell : \Sigma \to \bar{B}_f$; $G(E)(x) =: \ell(x)(\chi_E)$; T contains 0 and the closed convex hulls of the sets $\{(\mu(E))^{-1}(\text{weak-}^\circ)(\int_E f d\mu)/E \leq E_2, E \in \Sigma^+\}$ of (iv).

Remarks. (i) Any closed, convex T such that ${}^*T$ contains $f[{}^*\Omega]$, will contain the closed convex hulls of the sets of (iv); (ii) if f is the non-standard representation of a family $(f_\alpha)_A$ of integrable maps and S is the smallest set (we assume it is bounded) containing the range of f_α $(\alpha \in A)$, then we can phrase (v) in the following way: given any bounded subset D^* of X^*, and $\delta > 0$, then if $x^* \in D^*$ has the form $\sum_{i=1}^{\infty} \alpha_i x_i^*$ with $x_i^* \in D^*$ and $\Sigma \alpha_i = 1$, $\alpha_i > 0$ $(i = 1, \cdots)$, we can find an index i such that $x^* - x_i^*$ is, in modulus, less than δ on every finite convex combination of elements of S. That this version of (v) implies the original one, can be seen by using the methods discussed in ([2], Chapter 5). It is possible to get a version of (v) in which we don't refer to infinite convex sums $\Sigma \alpha_i x_i^*$, but only to finite ones (cf. the methods discussed in [2], Chapter 7).

Proof. According to (1.10) (ii)-(iv), $\exists p \in L_1(\Sigma, X)$ such that $x^* \int_E p \, d\mu \sim x^* \int_E f \, d\mu$ for every $E \in \Sigma$ (cf. Chapter 2).

I claim that $p(\omega) \in T$ (T as in (1.10) (v)) for almost every $\omega \in \Omega$: as $\mu(E)^{-1}(\text{weak-}^\circ \int_E f \, d\mu) \in$ closed convex hull of $f[\Omega]$ (by the mean value theorem for the Bochner integral) an application of (1.1) together with (1.10) (iv) shows that the following condition is satisfied:

$$(**): \quad E_1 \in \Sigma^+ \; \forall E_2 \in \Sigma^+, \exists E_2 \leq E_1 \text{ such that } p\chi_{E_2} \text{ almost maps } \Omega \text{ into } T;$$

(**) together with a typical "exhaustion" argument then imply that the claim is true (the exhaustion argument goes as follows: Let $\mathcal{B} = \{E \in \Sigma / p\chi_E$ maps almost every $\omega \in \Omega$ into $T\}$ and let $c = \sup \mu(A)$;

choose (B_r) from $\mathscr{B}$ such that $\lim \mu(B_r) = c$; let $E_n = \bigcup_{k=1}^{n} B_k$; then each $E_n \in \mathscr{B}$ and $\lim \mu(E_n) = c$; if $\mu(\Omega \setminus \bigcup_{n=1}^{\infty} E_n) > 0$, then (**) implies that $\exists B \in \mathscr{B}$ with $\mu(B) > 0$ such that $B \leq \Omega \setminus \bigcup_{n=1}^{\infty} E_n$; but $\lim \mu(A \cup E_n) = \lim \mu(A) + \lim \mu(E_n) = \mu(A) + c > c$, which is impossible as $(A \cup E_n) \subseteq \mathscr{B}$; hence $p\chi_{E_k}$ maps Ω almost into T and we can conclude that ρ maps Ω almost into T (compare this version of "exhaustion" to the proof of ([2] lemma 4, p. 70)).

If we now examine the proof of (1.10), then we see that it can be used to complete the proof of (1.11).

1.12 Proposition. If X^* has the Radon-Nikodym property, then (1.10) (v) is satisfied. If $f[\Omega] \subseteq S$, where S is relatively weakly compact, then (1.10) (iv), (v) are satisfied.

Proof. If X^* has the Radon-Nikodym property, then, according to ([2], p. 98), every $\ell \in L_1(\Sigma, X)^*$ has an integral representation: $\exists$ g in $L_1(\Sigma, X^*)$ such that $\ell(f) = \int \langle f, g \rangle \, d\mu$. We can view these g's as being in $L_1(\Sigma, \bar{B}_T)$ (notation as in (1.10) and we get the required result.

1.13 Proposition (1.12) can be used to show that (1.11) is strictly stronger than (1.10). Diestel shows in [11] that the conditions of (1.10) are not necessary for relative weak compactness in $L_1(\Sigma, X)$ -- he gives the following counterexample: (x_n) is any bounded sequence in X where X^* is assumed to have the Radon-Nikodym property; (r_n) is the sequence of Rademacher functions $[r_n(t) = \text{sign}(\sin(2\pi t))]$ on $[0,1]$; then $(x_n r_n)$ converges weakly to zero. According to the results of chapter 2, (1.11) (i)-(iv) are necessary, while, according to (1.12), (1.11) (v) is satisfied when X^* has the Radon-Nikodym property; hence the example introduced above does not violate any of the conditions of (1.11).

CHAPTER 2

In this chapter we extend some of the results of Dunford, Bartle and Schwartz about weak compactness in spaces of Bochner integrable functions. We rely heavily on the results and methods of Non-Standard Analysis.

It is assumed throughout that, where appropriate, we have at our disposal a superstructure $V(\overline{X})$ on some set $\overline{X}$ such that $\overline{X}$ and $V(\overline{X})$ are "large enough" to accommodate all the relevant entities under consideration; all our non-standard action will take place in some appropriate "highly" saturated non-standard extension, $*M$, of $V(\overline{X})$ - we may on occasion assume something more about the structure of such an enlargement.

2.1 Proposition. There is a $*$-finite partition $\overline{\pi} \leq *\Sigma$ of $*\Omega$ which is finer than every finite, standard partition $\pi \leq \Sigma$ of Ω.

Proof. $R = \{(\pi, q)/\pi, q \leq \Sigma; \pi, q \text{ finite}; q \text{ finer than } \pi\}$ is a concurrent binary relation.

2.2 Definition. If $\overline{\pi}$ is as in (2.1), then $\overline{\pi}$ is called infinite (all partitions that we will consider will be $*$-finite; hence the terminology should not cause confusion).

2.3 Proposition. If $K \leq L_1(\Sigma, X)$ is uniformly integrable, then $\mu(E) \sim 0$ implies that $\int_E \|f\| d\mu \sim 0$ for every $f \in *K$.

Proof. The argument is the usual sort of one. One might for instance assume that $*M$ is an ultraproduct and argue on representatives.

When X and X* both have the Radon-Nikodym property, then the dual of $L_1(\Sigma, X)$ is $L_\infty(\Sigma, X^*)$; by using this representation of the dual, one can, in many instances, reduce questions of weak convergence in $L_1(\Sigma, X)$ to questions of convergence with respect to topologies induced by linear functionals of the form $\hat{\sum_{i=1}} x_i^* \chi_{E_i}$, where $E_i \in \Sigma$. Crucial use is made of this reduction in the proof of (0.4). The following result gives a description of weak convergence in general $L_1(\Sigma, X)$ spaces and provides a tool that will allow us to give a complete characterization of relative weak compactness in $L_1(\Sigma, X)$.

2.4 Proposition. Let $(h_\lambda)_{\lambda \in \Lambda} \leq L_1(\Sigma, X)$ be a net and let $f \in L_1(\Sigma, X)$. Then (h_λ) converges to h in the weak topology of $L_1(\Sigma, X)$ if and only if

$$^\circ\!\int \langle {}^*h_\lambda, g\rangle \, d\mu \to {}^\circ\!\int \langle {}^*h, g \rangle \, d\mu$$

for every *-simple function $g \in {}^*L_\infty(\Sigma, X^*)$ with finite L_∞-norm.

Proof. Say $h_\lambda \to h$ in the weak topology. Let g be a *-finite simple function in $L_\infty(\Sigma, {}^*X)$ with finite norm. Define $\ell_g : L_1(\Sigma, X) \to \mathscr{R}$ by $\ell(f) = {}^\circ\!\int \langle {}^*f, g \rangle \, d\mu$; then ℓ_g is clearly linear and bounded with norm $\leq {}^\circ\|g\|_\infty \|f\|_1$ and consequently ${}^\circ\!\int \langle {}^*h_\lambda, g\rangle d\mu \to {}^*\!\int \langle {}^*h, g \rangle d\mu$.

Say conversely that ${}^\circ\!\int \langle {}^*h_\lambda, g\rangle d\mu \to {}^\circ\!\int \langle {}^*h, g\rangle d\mu$ for all g as described in the statement of (2.4). Let ℓ be a bounded linear functional on $L_1(\Sigma, X)$ and define $G : \Sigma \to X^*$ by $G(E)(x) = \ell(x\chi_E)$. Then G is a vector measure that satisfies

$$\|G(E)\| \leq \|\ell\| \mu(E).$$

Let $\bar{\pi}$ be an infinite partition. Then

$$G(E) = G(\bigcup_{\substack{A\in\bar{\pi},\\ A\le{}^*E}} A) = \sum_{\substack{A\in\bar{\pi}\\ A\le{}^*E}} G(A)$$

$$= \sum_{\substack{A\in\bar{\pi}\\ A\le{}^*E}} \int \mu(A)^{-1} G(A) d\mu = \int_{{}^*E} \sum_{A\in\bar{\pi}} \mu(A)^{-1} G(A)\, \chi_A d\mu.$$

Let $\pi \le \Sigma$ be a finite partition of Ω and let $f = \sum_{E\in\pi} x_E \chi_E$ be a simple function with values in X. Then

$$\ell(f) = \sum_{\pi} \ell(x_E \chi_E) = \sum_{\pi} G(E)(x_E)$$

and

$$\int \langle {}^*f, \bar{\pi}(G)\rangle\, d\mu =: \int \sum_{\pi,\bar{\pi}} \mu(A)^{-1} G(A)(x_E)\chi_{A\ E} d\mu$$

$$= \sum_{\pi,\bar{\pi}} \mu(A)^{-1} G(A)(xE) = \sum_{\pi} G(E)(x_E).$$

Now let $f \in L_1(\Sigma, X)$ be arbitrary and approximate f in L_1 norm by (f_n). Then

$$|\ell(f) - \int \langle {}^*f, \bar{\pi}(G)\rangle\, d\mu \le |\ell(f) - \ell(f_n)| + |\ell(f_n) - \int \langle {}^*f, \bar{\pi}(G)\rangle\, d\mu|$$

$$\le \|\ell\|\, \|f - f_n\|_1 + |\ell(f_n) - \int \langle {}^*f_n, \bar{\pi}(G)\rangle\, d\mu| + |\int \langle {}^*f_n - {}^*f, \bar{\pi}(G)\rangle\, d\mu|$$

$$\le \|\ell\|\, \|f - f_n\|_1 + \int \|\bar{\pi}(G)\|\, \|f_n - f_n\|\, d\mu \le \|f - f_n\|_1 (\|\ell\| + \|\bar{\pi}(G)\|_\infty)$$

$$\le 2\|f - f_n\|_1 \to 0.$$

Hence $\ell(f) = {}^\circ\int \langle {}^*f, \bar{\pi}(G)\rangle\, d\mu$ for all $f \in L_1(\Sigma, X)$; our reigning assumption then shows that $\ell(h_\lambda) \to \ell(h)$.

<u>2.5 Remark.</u> "finite L_∞-norm" in the statement of (2.4) can be replaced by "L_∞-norm ≤ 1": divide by $^\circ\|\bar{\pi}(G)\|_\infty + 1$.

<u>2.6 Proposition.</u> Let $\int(*)$ denote the topology induced on $L_1(\Sigma, X)$ by the family $\{f \mapsto x^* \int_E f\, d\mu / x^* \in X^*,\ E \in \Sigma\}$ of bounded linear maps; then $\int(*)$ is Hausdorff and if $s \in L_\infty(\Sigma, {}^*X)$ is simple, then the map F_s: $(L_1(\Sigma, X), \int(*)) \mapsto \mathcal{R}$; $f \to \int\langle f, s\rangle\, d\mu$ is continuous.

<u>Proof.</u> Straightforward.

<u>2.7 Proposition.</u> Let $K \leq L_1(\Sigma, X)$ satisfy the following:

(i) K is bounded;

(ii) K is uniformly integrable;

(iii) $\{\int_E f\, d\mu / f \in K\}$ is relatively weakly compact $\forall E \in \Sigma$;

(iv) $\forall f \in {}^*K$, $\forall E \in \Sigma^+\ \exists E_1 \in \Sigma^+$ such that $E_1 \leq E$ and $\{\text{weak-}^\circ(\mu(F)^{-1} \int_{*F} f\, d\mu) / F \in \Sigma^+,\ F \leq E_1\}$ is relatively weakly compact.

Then K is relatively compact in the topology $\int(*)$ introduced in (2.6).

<u>Proof.</u> For every $f \in {}^*K$, define $G_f : \Sigma \to X$ by $G(E) = \text{weak-}^\circ\int_{*E} f\, d\mu$. Then G is obviously an additive vector measure. Let $E \in \Sigma$ and let $x_E^* \in X^*$ be such that $x_E^*(G(E)) = \|G(E)\|$ and $\|x_E^*\| \leq 1$; there exists an infinitesimal η_E such that $x_E^*(G(E)) = \eta_E + x_E^* \int_E f\, d\mu$; we have

$$\|G(E)\| = x_E^*(G(E)) = \eta_E + x^* \int_E f\, d\mu \leq \eta_E + \int_E \|f\|\, d\mu \leq 1 + \|f\|_1 < \infty .$$

Let $\{E_1, \cdots E_n\} \leq \Sigma$ be a partition π of Ω; then

$$\hat{\sum_{c=1}} \|G(E_i)\| = \sum x^*_{E_i}(G(E_i)) = \sum (\eta_{E_i} + x^*_{E_i} \int_{E_i} f\, d\mu)$$

$$\leq \sum (\eta_{E_i} + \int_{E_i} \|f\|\, d\mu) \leq \dot{1} + \|f\|_1 ;$$

hence $|G|(\Omega) \leq 1 + {}^\circ\|f\|$.

Let (E_n) be a sequence of pairwise disjoint members of Σ and let $E = \cup E_n$; then for every $x^* \in X^*$ we see that

$$|x^* G(E) - \Sigma x^* G(E_n)| = |x^* (G(E) - G(\hat{\bigcup_{i=1}} E_i)|$$

$$= |x^* G(\bigcup_{i=n+1}^{\infty} E_i)| = |x^* \int_{\bigcup_{i=n+1}^{\infty} E_i} f\, d\mu + \eta| ;$$

let $A_n = \bigcup_{n+1}^{\infty} E_i$; then $\mu(A_n) \to 0$ and consequently we know that $\int_{A_\omega} \|f\|\, d\mu \sim 0$ as K is uniformly integrable, where ω is infinite; it follows that $|x^* G(E) - \Sigma x^* G(E_n)| \to 0$. This shows that G is weakly σ-additive. It is clear that $G \ll \mu$.

Condition (2.7) (iv) shows that we can apply (1.7) to G to conclude that $G(E) = \int_E g\, d\mu \;\; \forall E \in \Sigma$ for some $g \in L_1(\Sigma, X)$. But, if $x^* \in X^*$, then

$$x^* \int_E g\, d\mu = x^* G(E) \sim x^* \int_{*E} f\, d\mu, \qquad \text{i.e.,}$$

f is in the $\int(*)$-monad of g.

2.8 Proposition. Define two sets $\mathcal{E}$ and $\bar{\mathcal{E}}$ as follows:

$\bar{\epsilon}$ $=:\{\mathcal{G}_s : K \to \mathcal{R} / \mathcal{F}\, s \in {}^*L_\infty(\Sigma, X^*)[(s$ is $*$-simple, ${}^\circ\|s\|_\infty \leq 1)$ and

$\mathcal{G}_s(f) = {}^\circ\int\langle {}^*f, s\rangle\, d\mu\}$

ϵ $=:\{F_s : K \to \mathcal{R} / \mathcal{F}\, s \in L_\infty(\Sigma, X^*)$ (s is simple and $\|s\|_\infty \leq 1)\}$.

then $\bar{\mathcal{E}}$ is the closure of $\mathcal{E}$ in the product topology of $\mathcal{R}^k$.

Proof. $F \in \mathcal{R}^k$ is in the closure of $\mathcal{E}$ in the product monad if and only if the product monad of F intersects $*\mathcal{E}$. Now $G \in *(\mathcal{R}^k)$ is in the product monad of F if and only if $G(f) \sim F(f)$ for every standard $F \in *K$. The rest of the proof is now quite straightforward.

2.9. We return now to our dicussion of quasi-equicontinuity and introduce the notion of quasi-uniform convergence.

Definition. A set $(f_\alpha)_A$ in $\mathcal{R}^S$ converges quasi-uniformly to $f \in \mathcal{R}^S$ if and only if $\forall \delta \in \mathcal{R}^+$, $\forall \alpha_0 \mathcal{F}$ a finite subset of A such that: every $\alpha \in B_{\alpha_0}$ is bigger than α_0 and $\forall s \in S(\min_{\alpha \in B_{\alpha_0}} |f_\alpha(s) - f(s)| < \delta)$.

2.10. Theorem. (Grothendieck-Bartle) ([1])

Let S be a compact Hausdorff space and let $F \subseteq C(S)$.

Then the following conditions are equivalent:

(1) F is relatively weakly compact in C(S);

(2) F is bounded and its closure in the product topology is compact and consists of continuous functions;

(3) F is bounded and quasi-equicontinuous on S;

(4) F is bounded and if F_0 is a denumerable subset of F and $\{s_0, s, \dots\}$ is a sequence in S for which $f(S_n) \to f(S_0)$, $f \in F_0$, then $s_n \to s_0$ quasi-uniformly on F.

(5) F is weakly sequentially compact;

(the sequence $\{s_0, s_1, \cdots\}$ in condition (4) can be chosen with $s_1, \cdots, s_n, \cdots$ in a preassigned dense subset of S).

<u>2.11 Theorem.</u> $K \leq L_1(\Sigma, X)$ is relatively weakly compact if and only if the following conditions are satisfied:

(i) K is bounded;

(ii) K is uniformly integrable;

(iii) $\{\int_E f\,d\mu / f \in K\}$ is relatively weakly compact $\forall E \in \Sigma$;

(iv) For every $f \in {}^*K$, $\forall E \in \Sigma^+$ $\exists E_1 \in \Sigma^+$ such that $E_1 \leq E$ and $B_{f,E} = \{(\text{weak-}^\circ)(\mu(F)^{-1} \int_{{}^*F} f\,d\mu) / F \in \Sigma^+, F \leq E_1\}$ is relatively weakly compact;

(v) or (v′) where (v) is: the family $\mathscr{F} = \{F_s :$ closure of K in $\int(*)$-topology $\to \mathscr{R} / s \in L_\infty(\Sigma, X^*)$, s simple, $\|s\|_\infty \leq 1$; $F_s(f) = \int \langle f, s\rangle\,d\mu\}$ satisfies any of the equivalent statements of (2.10) where S of (2.10) is taken to be the closure of K in the $\int(*)$-topology;

and (v′) is: given any sequence of partitions $(\pi_n = \{E_1^n, \cdots, E_{K_n}^n\})_{n=1}^\infty$ of Ω consisting of measurable sets and given any sequence of sets $(\{n^{*n}, \cdots, n_{k_n}^{*n}\})_{n=1}^\infty$ in the unit ball of X^* and any sequence $(f_1, \cdots, f_n, \cdots)$ in K such that $(x^* \int_E f_n\,d\mu)_{n=1}^\infty$ is a Cauchy sequence of real numbers for each $E \in \Sigma$ and each $n^* \in X^*$, then there exists, for each $\epsilon > 0$ and $n_0 \in \mathscr{N}$, $n_1, \cdots, n_k \in \mathscr{N}$ bigger than n_0 such that

$$\inf_{i=1,\cdots,k} \left(\left|\left(\sum_{j=1}^{k_n} {}^*n_j^n \int_{E_j^n} f_i\, d\mu\right) - \sum_{j=1}^{k_n} \lim_{\ell\to\infty} {}^*n_j^n \int_{E_j^n} f_\ell\, d\mu\right|\right) < \epsilon$$

for every $n \in \mathcal{N}$.

If X has the Radon-Nikodym property, then condition (iv) is redundant.

2.12 Remarks. Condition (iv) of (2.11), which is somewhat of an oddity, is in one sense superfluous: we will show in Chapter 3 that (2.11) remains true if we drop (iv) and (v). We also see (as we already know from (1.9)) that if all of the members of K map into some fixed weakly compact, convex set S, then, by the mean value theorem for the Bochner integral, the $B_{f,E}$ of (2.11) (iv) are rel. weakly compact.

2.13 Proof of (2.11). We first give a proof of the last statement of the theorem. If X has the Radon-Nikodym property, then according to the proof of (2.7), $G(E) =: (\text{weak-}^\circ)(\int_{*E} f\, d\mu)$ is a representable measure for every $f \in {}^*K$. Hence the sets $\{\mu(F)^{-1}G(F)/F \in \Sigma^+,\ F \leq E_1\} = B_{f,E}$ are relatively weakly compact for appropriate choices of E_1 (cf. [2] p. 72); hence we have the desired result.

Proof of the sufficiency of (2.11) (i)-(v). Denote by $\bar{\bar{K}}$ the closure of K in the $\int(*)$-topology. According to (2.7), $\bar{\bar{K}}$ is $\int(*)$-compact. So if we use (2.6), (2.10) and (2.11) (v) we can conclude that the closure of $\mathcal{F}$ in the product topology is compact (in the product topology) and consists of $\int(*)$-continuous functions; hence by (2.8), the functions $\bar{F}_s : \bar{\bar{K}} \to \mathcal{R}$ ($s \in {}^*L_\infty(\Sigma, X^*)$, s-*simple, $\|s\|_\infty \leq 1$) defined by: $\bar{F}_s(f) = {}^\circ\int \langle {}^*f, s\rangle\, d\mu$, are $\int$*-continuous. Let (f_λ) be a net in K and let f be a $\int(*)$-cluster point (recall that we know that k is relatively $\int(*)$-compact); let s be a $*$-simple member of the unit ball of $*L_\infty(\Sigma,X^*)$; then if (f_{λ_μ}) is a subset of (f_λ) that converges in the $\int(*)$-topology to f, $^\circ\int\langle f_{\lambda_\mu}, s\rangle d\mu$ converges to $^\circ\int\langle f, s\rangle d\mu$ as $\bar{F}_s$ is $\int(*)$-continuous but then, according to (2.4) (f_{λ_μ}) converges weakly to f. Hence (i)-(v) imply that k is relatively weakly compact.

Proof of the necessity of (2.11) (i)-(v). (i) is satisfied because K is weakly, and therefore strongly, bounded.

That (iii) is satisfied is well known: use the fact that $f \mapsto \int_E f\, d\mu$ is a bounded linear operator $\forall E \in \Sigma$.

Let $f \in *k$; as f is weakly near-standard, there is a $g \in L_1(\Sigma,X)$ such that f is in the weak monad of g. Hence in particular, we see that

$$\int_E g\, d\mu = (\text{weak-}^\circ)\left(\int_{*E} f\, d\mu\right) \forall E \in \Sigma.$$

But according to [2] p. 72, $\forall E \in \Sigma^+ \mathcal{F} E_1 \in \Sigma^+$ such that $E_1 \subseteq E$ and such that $\{\mu(F)^{-1} \int_F g\, d\mu / F \in \Sigma^+, F \subseteq E_1\}$ is relatively weakly compact. Hence (iii) is satisfied.

We defer the proof that (ii) is satisfied to Chapter 3 where a more general result will be proved --the informed reader will realize that the necessity of (ii) is a classical result.

The family $F_s : f \mapsto \int\langle f, s\rangle d\mu$; s simple and in the unit ball of $L_\infty(\Sigma,X^*)$, is pointwise bounded: as $\int(*)$ is weaker than the weak topology on $L_1(\Sigma,X)$, the weak closure of K is $\int(*)$-compact; hence $\bar{\bar{K}}$ is strongly bounded and so we have

$$(**) \qquad |F_s(f)| = |\int\langle f, s\rangle d\mu| \leq \|S\|_\infty \|f\|_1 \leq \sup_{f \in \bar{\bar{K}}} \|f\|_1 \quad \text{on } \bar{\bar{K}}\ .$$

Hence the family $\mathcal{F}$ is relatively compact in the product topology on $\mathcal{R}^{\bar{\bar{K}}}$. Let $s \in {}^*L_\infty(\Sigma, X^*)$ be in the unit ball and *-simple, then $\bar{F}_s$ is $\int$*-continuous: let (f_λ) be a net in $\bar{\bar{K}}$ that converges in $\int(*)$ to $f \in \bar{\bar{K}}$; then as $\bar{\bar{K}}$ = weak closure of K (we know that $\bar{\bar{K}} \leq$ weak closure of K, and because the weak closure of K is compact, we get the reverse inclusion) $\exists$ a subnet (f_{λ_α}) of (f_λ) that converges weakly to f; hence by (2.4), $\bar{F}_s(f_{\lambda_\alpha}) \to \bar{F}_s(f)$; if $\bar{F}(f_\lambda)$ does not converge to $\bar{F}(f)$, then we can find $\epsilon > 0$ such that $|\bar{F}_s(f_{\lambda_\delta}) - \bar{F}(f)| > \epsilon$ for some subnet (f_{λ_δ}); but then, by the argument given above, there is a subnet $(f_{\lambda_{\delta_\eta}})$ of (f_{λ_δ}) that converges weakly to f and so $|\bar{F}_s(f_{\lambda_{\delta_\eta}}) - \bar{F}_s(f)| < \epsilon$ for big λ, which is a contradiction. Hence, by (2.8), the closure of $\mathcal{F}$ in the product topology is compact and consists of continuous functions and by (**) we see that $\mathcal{F}$ is bounded in $C(\bar{\bar{k}})$.

Proof of the sufficiency of (v′), (i)-(iv). It is clear that (2.11) (v′) is a weakened version of (2.10) (iv) interpreted in the proper context. In particular: we first show that (2.11)(ii)-(iv) imply that $\mathcal{F}$ is a bounded family of continuous functions and then we show that there is no need to refer to $\bar{\bar{K}} \setminus K$.

If $h \in \bar{\bar{K}}$, then there is an f in *K such that h is in the $\int$*-monad of f; let G_f be as in (2.7) (proof); then for every $x^* \in X^*$ and $E \in \Sigma$,

$$x^* \int_E h \, d\mu \sim x^* \int_{*E} f \, d\mu \sim x^* G(E)$$

in particular we see that $x^* \int_E h \, d\mu = x^* G(E)$ $\forall x^* \in X^*$ $\forall E \in \Sigma$; hence $G(E) = \int_E h \, d\mu$ and, as we know from [2], thm. 4, p. 46), $|G|(E) = \int_E \|h\| \, d\mu$ -- but from the proof of (2.7) we know that

$$|G|(\Omega) \leq 1 + {}^\circ\|f\| \leq 1 + \sup_{*K}{}^\circ\|f\| < \infty; \text{ hence } \|h\|_1 \leq 1 + \sup_{f\in *k}\|f\|_1$$

But then

$$|F_s(h)| = |\int \langle h, s\rangle \, d\mu| \leq \|h\|_1 \|s\|_\infty \leq \|h\|_1 \leq 1 + \sup_{f\in *k}{}^\circ\|f\|.$$

Using the remark following the 5th condition of (2.10), and translating to get into our context, we see that (2.11)(i-iv) imply that (2.11)(v) and the following condition:

($\otimes$) If $\mathscr{F}_0 \leq \mathscr{F}$ is denumerable and if $(g_1, \cdots, g_i \cdots)$ is a Cauchy sequence in $\int(x)$, then given $\epsilon > 0$, $n_0 \in \mathscr{N}$, there exist $n, \cdots, n_k$ all bigger than n_0 such that $\lim_{i=1,\cdots,k} |f(g_{n_i}) - \lim_{n\to\infty} f(g_i)| < \epsilon$ for every $f \in \mathscr{F}_0$ are equivalent ($\otimes \Rightarrow$(v) is obvious, while (v) $\Rightarrow \otimes$ uses the fact that $\bar{\bar{K}}$ is $\int$*-compact).

The proof of (2.11) is now complete.

<u>2.12 Remarks</u>. Theorem (2.11) is a proper extension of theorem (0.4): if the assumption that both X and X* have the Radon-Nikodym property is dropped, then (0.4) is no longer true, We refer the interested reader to [2] where the appropriate counterexamples are given.

We already know from (0.4) that (2.11)(v) becomes superfluous when the condition that both X and X* have the Radon-Nikodym property is imposed, and we have seen in (2.11) that there is no need for (2.11)(iv) when X has the Radon-Nikodym property. In the following result we show how the assumption that X* has the Radon-Nikodym property allows us to drop (2.11)(v).

<u>2.13 Proposition.</u> If X* has the Radon-Nikodym property, then for every $s \in SL_\infty(\Sigma, X^*)$, $\|s\|_\infty \leq 1$, the function $f \mapsto {}^\circ\int\langle f, s\rangle \, d\mu$ is

CHAPTER 3

In what follows we prepare the way for the statement and proof of a theorem that characterizes weak compactness in $\sigma BM(\Sigma, X)$. Using the embedding $L_1(\Sigma, X) \vdash \sigma BM(\Sigma, X)$; $f \to (E \to \int_E f\, d\mu)$, we see that part of (2.11) can be recovered.

Our approach to the proof of the main result of this chapter is very similar to the approach that yielded (2.11). The main obstacle to overcome is the fact that not much is known about the dual of $\sigma BM(\Sigma, X)$ --even when X and X^* have strong geometric properties; we will rely on a Helly-type extension to overcome this problem.

3.1 Lemma. There is an isometric isomorphism from $\sigma BM(\Sigma, X)$ into the dual of the normed space of all simple functions in $L_\infty(\Sigma, X^*)$.

Proof. The proof is straightforward--we will give it in full for the reader's convenience.

Let $G \in \sigma BM(\Sigma, X)$ and let $\int f dG$ denote the Bartle integral of f with respect to the bilinear pairing $X^* \times X \to \mathcal{R}$, $(x^*, x) \mapsto x^*(x)$ (it is clear how this integral is defined for simple f in $L_\infty(\Sigma, X^*)$, if f is an arbitrary member of $L_\infty(\Sigma, X^*)$, then we approximate it with simple functions and define $\int f dG$ as the limit of the integrals of the simple functions). Any member f of $L_\infty(\Sigma, X^*)$ is Bartle integrable and

$$\left| \int f\, dG \right| \leq \|f\|_\infty |G| \quad \text{for such f.}$$

Define $\Phi : \sigma BM(\Sigma, X) \to (L_\infty(\Sigma, \mu, X^*))^*$ by

$$\Phi(G)(f) = \int f\, dG.$$

Let $\{E_1, \cdots, E_n\}$ be a partition of Ω and let $\{x_1^*, \cdots, x_n^*\} \leq X^*$ satisfy $\|x_i^*\| \leq 1 (i = 1, \cdots, n)$; $x_i^*(G(E_i)) = \|G(E_i)\|$ $(i = 1, \cdots, n)$, then $\Sigma \|G(E_i)\| = \Sigma x_i^*(G(E_i)) = \int (\Sigma x_i^* \chi_{E_i}) dG = \Phi(G)(\Sigma x_i^* \chi_{E_i})$

$$\leq \|\Phi(G)\| \, \|\Sigma x_i^* \chi_{E_i}\|_\infty \leq \|\Phi(G)\|;$$

hence $|G|(\Omega) \leq \|\Phi(G)\| \leq |G|(\Omega)$ which shows that Φ is an isometric isomorphism onto its range.

Before we state and prove our next theorem, we want to recall the following well known result.

3.2 Theorem [13]. Given x^{**} in X^{**} there exists an $x \in {}^*X$ such that $x^{**}(x^*) = x^*(x)$ for every $x^* \in X^*$, and ${}^\circ\|x\| = \|x^{**}\|$ and $\|x^{**}\| \leq \|x\|$. (For a more general version of (2.19), we refer the reader to [13] where the notion of ω-norm fundamental subspaces allows for extra generality.)

3.3 Theorem. A net $(G_\lambda) \subseteq \sigma BM(\Sigma, X)$ converges weakly to $G \in \sigma BM(\Sigma, X)$ if and only if ${}^\circ\int f dG_\lambda \to {}^\circ\int f dG$ for every *-simple function f in the unit ball of ${}^*L_\infty(\Sigma, X^*)$.

Proof. $G \to {}^\circ\int f dG$ is a bounded linear functional on $\sigma BM(\Sigma X)$ for every f in ${}^*L_\infty(\Sigma, X^*)$ with finite norm; hence if $G_\lambda \to G$ weakly, then ${}^\circ\int f dG \to {}^\circ\int f dG$.

Say conversely that ${}^\circ\int f dG \to {}^\circ\int f dG$ for all the appropriate f. Let G^* be in the dual of $\sigma BM(\Sigma, X)$. By (3.1) and the Hahn-Banach theorem, G^* extends to a member, $\overline{G}^*$, of the second dual of $SL_\infty(\Sigma, X^*)$ ($SL_\infty(\Sigma, X^*)$ denotes the normed space of simple functions in $L_\infty(\Sigma, X^*)$. According to (3.2) we can find a $g \in {}^*SL_\infty(\Sigma, X^*)$ of finite norm such that $\overline{G}^*(f) = {}^\circ(({}^*f)(g))$ for every f in $(SL_\infty(\Sigma, X^*))^*$; in particular we have

$G^*(G) = {}^{\circ}\!\int g dG$ for every G in $\sigma BM(\Sigma,X)$. Hence $G^*(G_\lambda) = {}^{\circ}\!\int g dG_\lambda$

$$= ({}^{\circ}\|g\|_\infty + 1) \int \frac{g}{{}^{\circ}\|g\|_\infty + 1} dG_\lambda \rightarrow ({}^{\circ}\|g\|_\infty + 1)\, {}^{\circ}\!\int \frac{gdG}{{}^{\circ}\|g\|_\infty + 1} = G^*(G).$$

Our main theorem now takes the following form.

3.3 Theorem. $\mathcal{G} \subseteq \sigma BM(\Sigma,X)$ is relatively weakly compact if and only if the following conditions are satisfied:

(i) $\mathcal{G}$ is bounded in variation norm;

(ii) $\{|G|/G \in \mathcal{G}\}$ is uniformly strongly additive (i.e., given any sequence (E_n) of pairwise disjoint members of Σ, then $\lim \| \sum_{m=n}^{\infty} G(E_m) \| = 0$ uniformly in $G \in \mathcal{G}$; as $\mathcal{G} \subseteq \sigma BM(\Sigma,X)$ is uniformly bounded and Σ is a σ-algebra, the uniform strong additivity of $\mathcal{G}$ is equivalent to the existence of a positive σ-additive measure $\bar{\mu}$ on Σ such that $\{|G|G \in \mathcal{G}\}$ is uniformly $\bar{\mu}$-continuous);

(iii) $\{G(E)/G \in \mathcal{G}\}$ is relatively weakly compact for every $E \in \Sigma$.

One of the conditions (iv), (iv′):

(iv) the family $\mathcal{F} = \{G \mapsto \Sigma x^*_{E_i}(G(E_i)/\{E_1, \cdots, E_n\}$ is a partition of Ω consisting of measurable sets; $\|x^*_i\| \le 1$ $(i, 1, \cdots, n)\}$ restricted to the closure of $\mathcal{G}$ in the topology τ defined as follows:

> τ is the topology on $\sigma BM(\Sigma,X)$ induced by the family $\{G \mapsto x^* G(E)/E \in \Sigma,\ x^* \in X^*\}$ of linear functionals

satisfies the equivalent conditions of (2.10);

(iv′) given any sequence of finite partitions $(\pi_n) \subseteq \Sigma$ of Ω and any sequence of sets $E_{\pi_n} = \{x^*_{E,n} | E \in \pi_n,\ x^*_{E,n} \in$ unit ball of $X^*\}$ and any sequence G_n in $\mathcal{G}$ such that $(x^* G_n(E))_n$ is a Cauchy sequence of reals sequence with pairwise disjoint members. As $|G|(\bigcup_{n\geq m} E_n) \sim 0$ for every infinite m and every $G \in {}^*\mathcal{G}$

$$v(\bigcup_{n\geq m} E_n) = \sum_{i=1}^{\omega} |G| 2^{-i}$$

$$\leq \max_{i=1,\cdots,\omega} |G_i|(\bigcup_{n\geq m} E_n) \sum_{i=1}^{\omega} 2^{-i} \leq \max_{i=1,\cdots,\omega} |G_i|(\bigcup_{n\geq m} E_n) \sim 0$$

for every infinite m. Hence $\overline{\mu}(\bigcup_{n\geq m} E_n) \to 0$ as $k \to \infty$.

Proof of (3.3) (1) sufficiency.

Let $G \in {}^*\mathcal{G}$ and define $H : \Sigma \to X$ by

$$H(E) = (\text{weak-}^{0})(G(E)), \quad E \in \Sigma .$$

Let $x_E^* \in X^*$ satisfy $x_E^*(H(E)) = \|H(E)\|$ and $\|x_E^*\| \leq 1$. Let η_E be an infinitesimal such that $x_E^*(G(E)) = x_E^*(H(E)) - \eta_E$. Then

$$\|H(E)\| = x_E^*(H(E)) = x_E^*(G(E)) + \eta_E \leq \|x_E^*\| \, \|G(E)\| + \eta_E \leq \|G(E)\| + \eta_E$$

Let $\{E_1, \cdots, E_n\} \subseteq \Sigma$ be a finite partition of Ω; then

$$\sum_{i=1}^{n} \|H(E_i)\| \leq \sum (\|G(E_i)\| + \eta_{E_i}) \leq |G|(\Omega) + 1; \text{ hence } |H| < \infty .$$

Fix $x^* \in X^*$ and let $\{E_n\}_{\mathcal{N}} \subseteq \Sigma$ be a countable partition of Ω. According to condition (ii), $\|G(\bigcup_{k\geq n} E_k)\| \sim 0$ for every infinite n. Hence $x^*H(\bigcup_{k\geq n} E_k) \to 0$ as $n \to \infty$. We conclude that G is strongly σ-additive (use the Orlics-Pettis theorem).

Therefore, under assumptions (3.3) (ii)-(iii), $\mathcal{G}$ is relatively weakly compact in the Hausdorff topology τ. By (3.3) (iv), the closure

for every $E \in \Sigma$, $x^* \in X^*$ of norm ≤ 1, then, given $\epsilon > 0$ and $n_0 \in$,
there is a finite set $n_1, \cdots, n_k$ bigger than n_0 such that

$$\min_{i=1,\cdots,k} \left| \sum_{E\in\pi_n} x^*_{E,n}(G(E_i)) - \sum_{\pi_i} \lim_{k\to\infty} x^*_{E,n}(G_k(E)\right| < \epsilon$$

for every n.

If conditions (i), (iii) and (iv′) are satisfied, then $\mathcal{G}$ is already relatively weakly compact.

3.4 Remarks. The last statement of (3.3) allows one to get the promised improvement of (2.11) (cf. (2.12)--in fact (2.11) [(i), (iii) and (v′)] will suffice to prove that K in (2.11) is relatively weakly compact.

Before we prove (3.3), we give a proof of the equivalences mentioned in (3.3) (ii); the proof we give is a substantial simplification of proof given in [2].

3.5 Proposition. Let $\mathcal{G} \subseteq \sigma BM(\Sigma, X)$ be bounded with $\{|G|/G \in \mathcal{G}\}$ uniformly σ-additive. Then there exists a positive $\bar{\mu} \in \sigma BM(\Sigma, \mathcal{R})$ such that $|G|(E) \leq \mu(E)$.

Proof. Let $R = \{(G, v)/G \in \mathcal{G},\ v \in \sigma BM(\Sigma, \mathcal{R}),\ v$ positive, $|G|(E) \leq v(E)\ \forall E \in \Sigma,\ |v| \leq k\}$ where $k = \sup_{\mathcal{G}} |G|$. Then R is concurrent on $\mathcal{G}$: let $G_1, \cdots, G_n \in \mathcal{G}$ and let $v = (\hat{\sum_{i=1}} |G_i| 2^{-i}) 2^{-1}$. Hence by saturation of our non-standard model, we can find v in $^*\sigma\ BM(\Sigma, \mathcal{R})$ satisfying: $|^*G|(E) \leq v(E)$ $(G \in \mathcal{G},\ E \in {}^*\Sigma)$, $|v| \leq k$ and $v \geq 0$. Define $\mu : \Sigma \to \mathcal{R}$ by: $\mu(E) = {}^0 v(E)$ $(E \in \Sigma)$.

We only have to verify that μ is σ-additive. Let $(E_n) \leq \Sigma$ be a of $P = \{F_s : \bar{\bar{\mathcal{G}}} \to \mathcal{R} / G_s(G) = \int s dG$ for some $s \in SL_\infty(\Sigma, X^*)$, $\|s\|_\infty \leq 1\}$ (let $\bar{\bar{\mathcal{G}}}$ denote the closure of $\mathcal{G}$ in τ) in the product topology is compact in the product topology and consists of continuous functions (notice that $\overline{K}^\tau$ is bounded by $\sup_{\mathcal{G}} |G| + 1$; hence P is a bounded family of functions).

As before we see that the closure of P in the product topology consists of the functions $G \to {}^0\int f dG$ where $f \in {}^*SL_\infty(\Sigma, X^*)$ has norm bounded by 1 (recall that $SL_\infty(\Sigma, X)$ denotes the simple functions in $L_\infty(\Sigma, X)$. Hence convergence in τ implies convergence of ${}^0\int fd(\cdot)$ for every $f \in {}^*SL(\Sigma, X)$, $\|f\| \leq 1$, and that in turn implies convergence in the weak topology of $\sigma BM(\Sigma, X)$. Hence if a net in $\mathcal{G}$ has a τ-convergent subnet, then it has a weakly convergent subnet.

Before we continue with the proof of (3.3), we first establish some notation and a simple result.

<u>3.6 Notation.</u> Let E, $F \in \Sigma$; denote by G_E, (respectively H_F) the measure $A \to G(A \cap E)$ (respectively $A \to H(A \cap F)$) where $G, H \in \sigma BM(\Sigma, X)$. Then G_E, H_F are in $\sigma BM(\Sigma, X)$ and $|G_E| = |G|(E)$ and $|G_E + H_F| = |G_E| + |H_F|$ when E and F are disjoint.

<u>Proof of necessity of conditions (3.3) [(i)-(iv), (iv')].</u>

As $\mathcal{G}$ is relatively weakly compact, it is weakly bounded and therefore strongly bounded. The map $G \to G(E)$ is a bounded linear map; hence weak compactness is preserved by it.

Say $\{|G| / G \in \mathcal{G}\}$ is not uniformly strongly additive; then there is a sequence $(G_n) \leq \mathcal{G}$ such that $|G_n|(E_n) > \epsilon$. We are now in a position to apply the so-called Rosenthal's Lemma (this lemma says the following: if $\mathcal{F}$ is a field of subsets of Ω, (μ_n) is a uniformly bounded sequence of finitely additive scalar measures on $\mathscr{F}$, then, if (E_n) is a disjoint sequence of $\mathscr{F}$ and $\epsilon > 0$, there is a subsequence (E_{n_j}) of (E_n) such that $|\mu_{n_j}|(\bigcup_{\substack{k\neq j\\ k\in\Delta}} E_{n_k}) < \epsilon$, for all finite subsets $\Delta \subseteq N$ and all $j = 1, 2, \cdots$; if $\mathscr{F}$ is a σ-field, then the subsequence may be chosen such that $|\mu_{n_j}|(\bigcup_{k\neq j} E_{n_k}) < \epsilon$ for all $j = 1, 2, \cdots$).

By this lemma we may assume that $|G_n|(\bigcup_{k\neq n} E_k) < \epsilon/2$. Let $\alpha = \sup|G|$ and let $(\beta_n) \in \ell_1$ (ℓ_1 =: space of all real integrable functions on $\mathscr{N}$ with the discrete measure). Then

$$|\Sigma \beta_n G_n| \leqslant \alpha \|(\beta_n)\|_{\ell_1} \quad \text{and}$$

$$|\Sigma \beta_n G_n| \geqslant |\Sigma \beta_n G_n|(\bigcup_{i=1}^{\infty} E_i) = |(\Sigma\beta_n G_n)_{\bigcup_{i=1}^{\infty} E_i}| \quad \text{(notation defined in (3.6))}$$

$$= |\Sigma \beta_n (G_{n\,\bigcup_{i=1}^{\infty} E_i})| = |\sum_{n=1}^{\infty} \beta_n (G_{n,E_n} + G_{n,\bigcup_{k\neq n} E_k})|$$

$$\geqslant |\Sigma \beta_n G_{n,E_n}| - |\sum_{n=1}^{\infty} \beta_n G_{n,\bigcup_{k\neq n} E_k}|$$

$$= \Sigma|\beta_n|\,|G_{n,E_n}| - |\sum_{n=1}^{\infty} \beta_n G_{n,\bigcup_{k\neq n} E_k}| \quad \text{by (3.6)}$$

$$= \Sigma|\beta_n|\,|G_n|(E_n) - |\sum_{n=1}^{\infty} \beta_n G_{n,\bigcup_{k\neq n} E_k}|$$

$$\geqslant \Sigma(\beta_n)|G_n|(E_n) - \Sigma|\beta_n|\,|G_n|(\bigcup_{k\neq n} E_k)$$

$$\geqslant \Sigma\|\beta_n\|_{\ell_1} - \frac{\epsilon}{2}\|(\beta_n)\|_{\ell_1} = \frac{\epsilon}{2}\|(\beta_n)\|_{\ell_1}.$$

Hence $\mathcal{G}$ contains a copy of the canonical basis of ℓ_1, a fact that precludes the possibility that $\mathcal{G}$ is relatively weakly compact. (We remark that the line of reasoning involving the use of Rosenthal's lemma is a generalization of the proof of ([2] theorem 4, p. 104).

As before we see that the weak and τ-closures of $\mathcal{G}$ coincide and that the weak and τ topologies agree on the weak closure of $\mathcal{G}$. Hence the functions $G \mapsto {}^0\!\int fdG$, $f \in {}^*SL_\infty(\Sigma, X^*)$, f bounded by 1, are all τ-continuous and hence that the closure of $\mathcal{F}$ in the product topology is compact and consists of continuous maps; the Grothendieck-Bartle theorem then gives the desired result.

CHAPTER 4

In this chapter we show that the techniques developed in the previous chapter can be extended to situations outside of the context of measure theory to characterize relative weak compactness. We concentrate our attention on the study of weak compactness in spaces of absolutely summing maps between Banach spaces.

We begin by recalling some definitions and results from the theory of tensor products of named spaces.

4.1 Definitions. Fix $k \in \mathcal{R}^+ \cup \{\infty\}$ such that $k \geq 1$ and k' solves the equation $x^{-1} + k^{-1} = 1$. If $(x_i)_I \leq X$ is a family of elements of X, then $N_k(x_i)$ is defined as follows:

$$N_k(x_i) = \begin{cases} (\sum_I \|x_i\|^k)^{k^{-1}} & \text{if } k \neq \infty \\ \sup_I \|x_i\| & \text{if } k = \infty \end{cases}$$

$M_k(x_i)$ is defined to be $\sup_{\|x^*\| \leq 1} N_k(x^*(x_i))$. If $u = \sum_{i=1}^{n} x_i \otimes y_i$ is in $X \otimes Y$, then $g_k(u) =: \inf N_k(x_i) M_{k'}(y_i)$ where the infimum is taken over the representations of u of the form $\Sigma x_i \otimes y_i$. We let $d_k(u) = \inf M_{k'}(x_i) N_k(y_i)$ where the infimum is once again taken over representatives of u.

If $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y, then $T \in \mathcal{L}(X, Y)$ is said to be k-absolutely summable if we can find a constant $A > 0$ such that for each finite sequence (x_i) in X, $N_k(Tx_i) \leq AM_k(x_i)$; $S^k(X, Y)$ denotes the space of all k-absolutely summable operators. We define a norm, π_k, on $S^k(X, Y)$ as follows: $\pi_k(T) = \inf A = \sup_{M_k(x_i) \leq 1} N_k(TX_i)$. (Notice that

$S^{\infty}X,Y) = \mathcal{L}(X,Y)$ and $\pi_{\infty}(T) = \|T\|$.)

<u>4.2 Theorem.</u> The dual of the normed space $(X \otimes Y, d_k)$ is isometrically isomorphic to $S^{k'}(X, Y^*)$.

<u>4.3 Remark.</u> All the material of (4.1), (4.2) can be found in [14] where some extensions of the results of Grothendieck [15], [16] appear.

<u>4.4 Proposition.</u> $T_\lambda \to T \in S^k(X,Y^*)$ in the weak topology of $S^k(X,Y^*)$ if and only if ${}^0(\Sigma T_\lambda(x_i)(y_i)) \to {}^0(\Sigma T(x_i)(y_i))$ for every $\Sigma x_i \otimes y_i$ in the unit ball of ${}^*(X \otimes Y, d_{k'})$.

<u>Proof.</u> With the relevant material of the previous two chapters in mind, the reader should have no difficulty completing the proof of (4.4)--however, for the sake of completeness, I will write down a proof.

$T_\lambda \to T$ in the weak topology of $S^k(X,Y^*)$ if and only if $B(T_\lambda) \to B(T)$ for every B in the unit ball of the dual of $S^k(X,Y^*)$; however, B is in the dual of $S^k(X,Y^*)$ if and only if there is a $u \in {}^*(X \otimes Y, d_{k'})$ such that $B(T) = T(u) = {}^0T(u)$ for every $T \in S^k(X,Y^*)$ where $T(u) = \Sigma(Tx_i)(y_i)$ if $\Sigma x_i \otimes y_i$ is a representation of u; furthermore we know that ${}^0\|u\|_{d_{k'}} = \|B\| \leq 1$.

So, all we have to check is that $T \mapsto {}^0(\Sigma(Tx_i)(y_i))$ defines a bounded linear functional in the dual of $S^k(X,Y^*)$ when $\Sigma x_i \otimes y_i$ is in the unit ball of ${}^*(X \otimes Y, d_k)$. However, that is immediate from the definition of the natural isometry $S^k(X,Y^*) \to (X \otimes Y, d_{k'})^*$

<u>4.5 Theorem.</u> $K \subseteq S^k(X,Y^*)$ is relatively weakly compact if and only if the following conditions are satisfied:

(i) K is bounded:

(ii) $\{Tx/T \in K\}$ is relatively weakly compact for every $x \in X$;

(iii) if F_0 is a denumerable set of members of the unit ball of $(X \otimes Y, d_k)$ and if $T_1, \cdots, T_n \cdots$ is a sequence in K such that $(T_n(x)(y))$ is Cauchy for every $(x,y) \in XxY$, then, given $\epsilon > 0, n_0 \in \mathcal{N}$, we can find $n_1, \cdots, n_k$, all bigger than n_0 such that

$$\min_{i=1,\cdots,k} (T_{n_i}(u) - \lim_{n\to\infty} T_n(u)| < \epsilon \text{ for every } u \in F_0$$

where, if $u = \Sigma x_i \otimes y_i$, $|T_{n_i}(u) - \lim_{n\to\infty} T_n(u)| =: |\sum_j T_{n_i}(x_j)(y_j)$

$$- \sum_j \lim_{n\to\infty} (T_n(x_j))(y_j)|.$$

Proof. Let $T \in {}^*K$ and define a linear map $L: X \to Y^*$ by $L(x) = (\text{weak-}^0)(T_x)$. Choose y^{**} in the unit ball of Y^{**} such that $\|Lx\| = y^{**}(Lx)$; then

$$\|Lx\| = y^{**}L(x) \sim y^{**}T(x) \leq \|Tx\| + 1 \quad \text{and}$$

$$\sum \|Lx_i\|^k = \sum \|y_i^{**}(Lx_i))^k = \sum (y_i^{**}Tx_i)^k + \eta$$

(η is infinitesimal)

$$\leqslant \sum \|Tx_i\|^k + \eta \leqslant \pi_k(T)\, M_k(x_i) + \eta \leqslant 1 + \sup_K \pi_k(T).$$

The set $\mathcal{F} =: \{L \mapsto \Sigma L(x_i)(y_i)/\Sigma x_i \otimes y_i \in \text{unit ball of } (X \otimes Y, d_{k'})\}$ consists of continuous maps in the topology τ induced by the family $\{L \mapsto L(x)(y)/(x,y) \text{ in } X \times Y\}$ of bounded linear functionals on $S^k(X, Y^*)$. As before, the closure of $\mathcal{F}$ in the product topology of $\mathcal{F}$ is the set $\bar{\mathcal{F}} =: \{L \to {}^0\Sigma L(x_i)(y_i)/\Sigma x_i \otimes y_i \in \text{unit ball } {}^*(X \otimes Y, d_{k'})\}$. Condition (iii) then ensures that $\mathcal{F}$ consists of continuous functions and hence, according to (4.2), we conclude that (4.5) ((i)-(iii)) are sufficient to ensure that K is relatively weakly compact.

We leave the rest of the proof in the capable hands of the interested reader.

4.6 Remarks.

(1) The reader naturally realizes that there is an expanded version of (4.5), i.e., conditions corresponding to the conditions of the Grothendieck-Bartle theorem can be added.

(2) Restricted versions of some of the results of Chapters 2 and 3 can be derived from (4.5) by using the following devices and results: (i) if Ω is a compact Hausdorff space and $T : C(\Omega) \to X$ is a bounded linear operator, then there exists a weak*-countably additive measure G on the Borel sets of Ω with values in X^{**} such that $x^*T(f) = \int f d(x^*G)$ for each $f \in C(\Omega)$ and each $x^* \in X^*$;

(ii) a bounded linear operator $T : C(\Omega) \to X$ is absolutely summable if and only if its representing measure G (as described in (4.6) (i)) is of bounded variation (in which case G maps into X) and $\pi_1(T) = |G|(\Omega)$. We refer the reader to [2] for more information about (i) and (ii).

$\int(*)$-continuous; it follows that, under this assumption on X^*, (2.11)(v) is superfluous.

Proof. Define a measure $G : \Sigma \to X^*$ as follows: $G(E)(x) = {}^\circ\int\langle x\chi_E, s\rangle d\mu$. Then $|G(E)(x)| \leq \|s\|_\infty \mu(E)\|x\| \leq \mu(E)\|x\|$, which shows that G is of bounded variation and μ-continuous (and σ-additive). Hence by the Radon-Nikodym theorem, $\exists$ a $g \in L_1(\Sigma, X^*)$ such that $G(E) = \int_E g\, d\mu$.

It is now fairly straightforward to show that ${}^\circ\int\langle f, s\rangle d\mu = \int\langle f, g\rangle d\mu \; \forall F \in L_1(\Sigma, X)$; by approximating g with simple functions and following the reasoning of the proof of (1.9), it follows that $f \mapsto {}^\circ\int\langle f, s\rangle d\mu$ is $\int(*)$-continuous.

2.14 Example. Let X have the Radon-Nikodym property and let $(x_n) \leq X$ be a bounded sequence; then $(x_n r_n)$ (r_n is the nth Rademacher function) is a sequence in $L_1(\Sigma, X)$, where Σ is the Lebesque measurable subsets of $[0,1]$, that satisfies (2.11) (i)-(iv). It is easy to verify that $x^* \int_E x_n r_n(t)\, dt \to 0 \;\; \forall E \in \Sigma, x^* \in X^*$; hence to check for weak compactness, we only have to study the action of certain finite linear combinations of elements of X^* on (s_n).

We can use the previous example to characterize dual spaces with the Radon-Nikodym property.

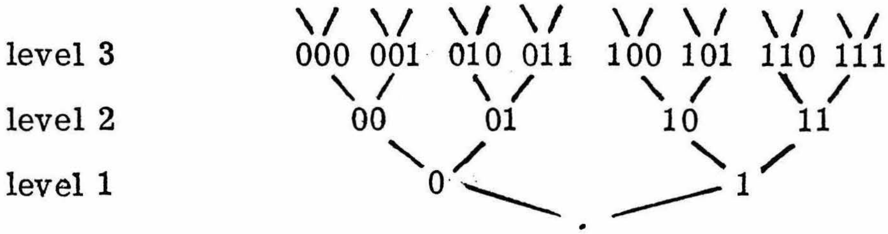
Recall that a δ-tree in X is a bounded set $\{x_{\Sigma_1 \cdots \Sigma_k} / k \geq 1\}$ where $\Sigma_i \in \{0, 1\}$ and

$$x_{\Sigma_1 \cdots \Sigma_k} = \tfrac{1}{2}(x_{\Sigma_1 \cdots \Sigma_k 0} + x_{\Sigma_1 \cdots \Sigma_k 1})$$

$$\|x_{\Sigma_1 \cdots \Sigma_k 1} - x_{\Sigma_1 \cdots \Sigma_k 0}\| \geq \delta$$

($\{\Sigma_1, \cdots, \Sigma_k\}$ may be empty).

A tree of sequences $\Sigma_1\cdots\Sigma_k$ ($\Sigma_i \in \{0, 1\}$ is associated with the class of all δ-trees and graphically represented as follows



We order each level linearly from left to right.

An $r(\delta)$-*tree is a sequence (x_n) in the unit ball of X together with a set $\{x^*_{\Sigma_1\cdots\Sigma_k}/k \geq 1\}$, ($\Sigma_i \in \{0, 1\}$), in the unit ball of X^* such that for $k \geq n$,

$$\left| \sum_{\Sigma} x^*_{\underbrace{000\cdots0}_{n}\,\Sigma_{n+1}\cdots\Sigma_k}(x_n) - \sum_{\Sigma} x^*_{\underbrace{00\cdots01}_{n}\,\Sigma_{n+1}\cdots\Sigma_k}(x_n) + \sum_{\Sigma}\cdots - \cdots \right| \geq \delta\cdot 2^{k-1}$$

with the sign of $\sum_{\Sigma}$ alternating as we go from left to right along the nth level.

Theorem. X^* lacks the Radon-Nikodym property if and only if one of the following equivalent conditions holds in X^*:

(i) X^* contains a δ-tree;

(ii) X^* contains an $r(\delta)$-*tree

(iii) there is a sequence (x_n) in X, with $\|x_n\| \leq 1$ ($n = 1, 2\cdots$), such that $(x_n r_n)$ is not relatively weakly compact in $L_1(\Sigma, X)$ where Σ is the σ-algebra of Lebesque measurable subset of $[0, 1]$ and μ in Lebesque measure.

Proof. That (i) is necessary and sufficient is a result of Stegall (cf. [2]). We already know that (iii) is sufficient to ensure that X^* lacks the Radon-Nikodym property.

Let $(x^*_{\Sigma_1 \cdots \Sigma_k})$ be a δ-tree in X^*, and let x^*_ϕ denote the vertex of this δ-tree indexed by the empty sequence. We may assume, without loss of generality, that $(x^*_{\Sigma_1 \cdots \Sigma_k})$ lies in a closed ball that does not contain the origin. Choose x_0^{**} such that $\|x_0^{**}\| \leq 1$ and $x_0^{**}(x^*_\phi) = 1$; according to Helly's principle, we can find $x_0 \in X$, with $\|x_0\| < 1 + \epsilon_0$ ($\epsilon_0 > 0$ is some fixed real number), such that $x^*_\phi(x_0) = 1$. By re-indexing the tree, we can arrange matters so that $\{(x^*_{\Sigma_1 \cdots \Sigma_k 0} - x^*_{\Sigma_1 \cdots \Sigma_k 1}) / \Sigma_i \in \{0,1\},\ i = 1, \cdots k\}$ generates a convex set that does not contain the origin (none of these differences in the original tree is 0; hence, by multiplying an appropriate subset of them with -1, we get a configuration of differences with convex hull staying away from 0). Choose x_1^{**} in the unit ball of X^{**} such that $x_1^{**}(x_0^* - x_1^*) > \delta$ and use Helly's principle to select x_1 in X with $\|x_1\| < 1 + \Sigma_1$ such that $x_1^{**}(x_0^* - x_1^*) = (x_0^* - x_1^*)(x_1)$. Continue this process to find (x_n) in X with $\|x_n\| < 1 + \Sigma_n$ such that $\min_{\Sigma_1, \cdots, \Sigma_{n-1}} \{(x^*_{\Sigma_1 \cdots \Sigma_{n-1} 0} - x^*_{\Sigma_1 \cdots \Sigma_{n-1} 1})(x_n) > \delta\}$.

We see that for $k > n$, $x^*_{\Sigma_1 \cdots \Sigma_n} \cdot 2^{-n} = \left(\sum_{\Sigma_{n+1}, \cdots, \Sigma_k} x^*_{\Sigma_1 \cdots \Sigma_k}\right) 2^{-k}$,

and it follows that by multiplying the (x_n) by a propriate constants, we get an $r(\delta)$-*tree.

If we have an $r(\delta)$-*tree, then, by taking the nth partition π_n in the fifth condition of our characterization of weak compactness to be the partition of $[0,1]$ in 2^n intervals of length 2^{-n}, we see that $x_n r_n$ is not relatively weakly compact.

Remarks. If there are sequences (x_n) and (x_n^*) respectively in the unit balls of X and X^* such that

$$(*) \qquad x_n^*(n_m) = \begin{cases} 1 \text{ if } n > m \\ -1 \text{ if } n \leqslant m \end{cases}$$

then we can easily construct an $r(\delta)$-*tree in X^*. A well-known characterization of non-reflexive spaces by R. C. James ([18]) takes the form of (*) with the exception that 0 replaces -1 in (*). It follows that dual spaces lacking the Radon-Nikodym property behave very much like non-reflexive spaces.

Bibliography

[1] N. Dunford and T. Schwartz; Linear operators, part I; Interscience Publishers, 1964.

[2] J. Diestel and J. J. Uhl, Jr.; Vector measures; A.M.S. Mathematical Surveys, number 15, 1977.

[3] J. Bourgain; On dentability and the Bishop-Phelps property; Israel Jounral of Math., 28, 1977, p. 265.

[4] C. Stegall; The duality between Asplund spaces and spaces with the Radon-Nikodym property; Israel Journal of Math., 30, 1978, p. 408.

[5] J. Bourgain; Dentability and finite dimensional decompositions; Studia Math., 67, 1980, p. 135.

[6] J. Bourgain; Sets with the Radon-Nikodym property in conjugate Banach spaces; Studia Math., 66, 1979, p. 291.

[7] M. Talagrand; Dual Banach lattices and Banach lattices with the Radon-Nikodym property; Israel Journal of Math., Vol. 38, no. 1, 1981, p. 46.

[8] J. Bourgain; Dunford-Pettis operators on L_1 and the Radon-Nikodym property; Israel Journal of Math., Vol. 37, no. 1, 1980, p. 34.

[9] J. Bourgain and H. P. Rosenthal; Martingales valued in certain subspaces of L'; Israel Journal of Math., Vol. 37, no. 2, 1980, p. 54.

[10] J. K. Brooks and N. Dinculeanu; Weak compactness in spaces of Bochner integrable functions and applications; Advances in Math., 24, 1977, p. 172.

Bibliography (continued)

[11] D. R. Lewis; Conditional weak compactness in certain inductive tensor products, Math. Ann. 201, 1973, p. 201.

[12] W. Rudin; Real and complex analysis; McGraw-Hill Book Company, 1974.

[13] W. A. J. Luxemburg; Contributions to Non Standard Analysis, North Holland, 1971.

[14] P. Saphar; Produits tensoriels d'espaces de Banach et classes d'applications linéaires; Studia Math., 40, 1971, p. 71.

[15] A. Grothendieck, Résumé de la théorie métrique des produits tensoriels topologiques; Boletin da sociedade de matemãtika de São Paulo, Vol. 8, fasc. 1, 1953.

[16] A. Grothendieck; Produits tensoriels topologiques et espaces nucléaires; Memoirs of the A.M.S., number 16.

[17] H. P. Rosenthal; A characterization of Banach spaces containing ℓ'; Proc. Nat. Acad. Sci. U.S.A., 71, 1974, p. 2411.

[18] R. C. James; Normed linear spaces; Symposium on infinite dimensional topology; Annals of Math. Studies, Princeton University Press, 1972.

PART II

Introduction

In this part of the thesis we study various aspects of model theory for normed spaces - in fact, much of the material discussed in this part is directly related to the discipline of categorical logic (cf. [20]).

We divide the introduction into three parts: the first part should serve as material of motivation; the second part contains some of the more pertinent necessary background material; in the third part we give a summary of the main body of results of chapter 1 for the readers who lack the desire to wade through the material of chapter 1.

0.1. The problem that originally served as motivation for the study undertaken in chapter 1, was the one of finding for each normed space, structures, sufficiently closely related to the given normed space, and with desirable properties, so that the study of these structures will throw some light on the behavior of the associated normed space. In what follows we will discuss this problem and show how, in pursuit of its solution, we were led to consider other (seemingly unrelated) problems.

0.1.1. To elucidate the meaning of the first sentence of 0.1, let's give a partial solution to the problem mentioned in it. Take a normed space X and let $^*M$ be an appropriately saturated non-standard model of a fragment of set-theory such that X is contained in $^*M$. Then inside $^*M$ we can find *-finite dimensional normed spaces that contain $^*X$. It is well known that these *-finite dimensional spaces can be viewed as structures of the nature described in 0.1. Other examples of such structures related to X are the various non-standard hulls, $\hat{X}_m$, of X, corresponding to the different non-standard models $^*M$ of set-theory; it is known that these hulls have, for example, the following "desirable" properties: super properties (super-reflexivity, super-Radon-Nikodym property, ...) and the corresponding properties (reflexicity, Radon-Nikodym property, ...) coincide in hulls; small classical spaces like c_0 and ℓ_ρ ($\rho \geq 1$) tend to "live" in hulls (cf. [4], [5], [6]), a fact that allows one to conclude, among other things, that stable X contain copies of ℓ_ρ and/or c_0 (Cf. [6]).

However, these structures (the ultraproducts and *-finite dimensional spaces) have, like most devices, limited use. The ultraproduct functor does not arise as solution to a universality problem (in the sense of category theory) (i.e., in some sense X has too many ultraproducts). Furthermore it is known that a nontrivial axiom of set theory must be invoked to ensure a nontrivial supply of ultrafilters; that means in particular that not much is known about the nature of ultraproducts. A substantial amount of information is lost in many instances where ultrafilters are employed: for instance, specific information about the behavior of Ramsey numbers cannot be obtained by proving Ramsey theorems using ultrafilters (cf. [24]); it is also well known that the proof, using isoperimetric inequalities, in ([7]) of Krivine's generalization of Dvoretzky's theorem, yields qualitative information that cannot be obtained by Krivine's methods (cf. [2]). This last theme then leads to the study of sheaflike models: the methods of homological algebra can be used, in certain instances, to "measure" the amount of information that is lost when we go from "local" to "global" situations.

0.1.2. Problem 2: find useful sheaf representations of normed spaces. The category of sheaves on a site is in a natural way a model for higher intuitionistic set theory (a listing of the axioms for higher order intuitionistic set theory, in the appropriate language, can be found in the article by Osius in [25]; a site is a "generalized" topological space and a sheaf on a site is the corresponding "general" version of a sheaf on a topological space (cf. [20], [8]; examples of categories of sheaves on sites include the following: the Boolean valued models of set theory, all categories of sheaves of sets on topological spaces, the étale topos and the crystalline topos). This fact was for the first time fully realized by Lawvere and Tierney (they were profoundly influenced by the work of the Grothendieck school ([8], [9]) and Scott and Solovay) and explains to some extent the success of the method of sheaf representation in various disciplines of mathematics. The category of sets is "the" final topos (as these categories of sheaves on sites are called) in in the sense that given any topos $\mathcal{E}$, then there exists a pair of functions $\Gamma : \mathcal{E} \to \mathrm{Set}$, $a : \mathrm{Set} \to \mathcal{E}$ such that a is left adjoint to Γ and a is left exact, and this pair is uniquely determined by these properties (such a pair of functions $\phi^* \vdash \phi_*$ (ϕ^* left adjoint to ϕ_*) and ϕ^* left exact is called a geometric morphism of toposes with domain the domain of ϕ_* and range the range of ϕ_*) (if $\mathcal{E}$ is the category of sheaves on a topological space, then Γ is the global sections functor and a is the associated sheaf functor).

When we talk of representing a category $\mathcal{A}$ as sheaves on sites we have the following in mind: associate with each $A \in \mathcal{A}$ a topos spec(A) and a sheaf $\overline{A}$ on "the" site of spec(A) in a functorial way such that $\Gamma(\overline{A})$ and A are closely related (hopefully isomorphic in $\mathcal{A}$). When such a sheaf representation exists, then we study $\{\overline{A}/A \in \mathcal{A}\}$ as "sets" in the spec(A) and hope that homological algebra will give us an indication to what extent the functor Γ preserve properties that $\overline{A}$ has in spec(A). This approach turned out to be enormously successful in algebraic geometry (it eventually led to proof of the truth of the "Weil conjectures") - largely because of the following fact: it is often possible to choose the pair (spec(A), $\overline{A}$) in such a fashion that $\overline{A}$ has stronger properties, as object of the category $\overline{\mathcal{A}}$ internal to spec(A) and corresponding to $\mathcal{A}$, than A as object of $\mathcal{A}$ internal to Set (e.g., if $\mathcal{A}$ is the category of rings and spec is the étale functor, then $\overline{A}$ is, as ring in spec(A), a strictly local ring).

Sheaf representation theory for normed spaces (in the sense as described above) is more or less nonexistent - if we restrict our attention to the category of Banach algebras, then the situation is totally different. There is at least one very natural association of toposes and normed spaces that jumps to mind: every normed space can be represented as subsheaf of the sheaf of continuous functions on the extreme points of the unit ball of the dual space with the weak-*topology (an interesting fact that we should point out in this context is that the sheaf of continuous functions on a topological space, acts like the "set" of Dedekind real numbers inside the associated topos; some interesting results about the continuous functions can therefore be obtained by doing elementary analysis on them inside the associated topos (cf. [10], [11], [12], [13], [14]). This representation has some defects: C(X), the topos associated to X in this representation, is not coherent (i.e., the corresponding topological space does not, in general, possess a subbase of open sets that are quasi-compact (i.e., every open cover of it has a finite subcover) and the intersection of two basic sets is quasi compact (cf. [8], Volume II, p. 207); coherent toposes are of fundamental importance in algebraic geometry (and in general topos theory) for various reasons: e.g., hom-functors on coherent toposes commute with filtered colimits; furthermore we know that C(X) does not localize well: if we restrict the sheaves to open subsets of the site of C(X), then we don't in general get C(Y) for some normed Y; the topology of C(X) does not uniquely determine X.

<u>0.1.3</u>. Problem 3: find normed spectra.

Consider the following problem: given a category $\mathcal{A}$ and a subcategory $\mathcal{B}$, does the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$ have a left adjoint? (I.e., is there a functor $F: \mathcal{A} \to \mathcal{B}$ and for every $A \in \mathcal{A}$ a map $A \to FA$, natural in A, such that FA is the "best possible approximation" to A in $\mathcal{B}$ in the sense that given $B \in \mathcal{B}$ and $A \to B$, there is a unique map $FA \to B$ such that $A \to FA \to B = A \to B$?). It is well known that this problem does not always have a positive solution: let, for example, $\mathcal{A}$ be the category of commutative rings, denoted by LAnn, and let be the category of local commutative rings, LocAnn. However, as was noticed by Hakim (cf. [9]), this type of problem does have a positive solution in a less restrictive content: let TopLAnn be the 2-category (a 2-category is a category $\mathcal{A}$ such that for each pair $A, \overline{A} \in \mathcal{A}$, hom$(A, \overline{A})$ is a category, together with a functor $\mu_{A, \overline{A}, \overline{\overline{A}}}: \text{hom}(A, \overline{\overline{A}}) \times \text{hom}(\overline{A}, \overline{\overline{A}}) \to \text{hom}(A, \overline{\overline{A}})$, "composition", for each triple $(A, \overline{A}, \overline{\overline{A}})$, satisfying certain identity preserving and associativity conditions; the category of all categories is the archetypical example of a 2-category) having as objects pairs $(\mathcal{E}, E)$ such that $\mathcal{E}$ is a topos and $E \in \mathcal{E}$ is a commutative ring inside $\mathcal{E}$, while hom $((\mathcal{E}, E), (\mathcal{F}, F))$ is the following category: the objects are the pairs $(\phi, \gamma) : (\mathcal{E}, E) \to (\mathcal{F}, F)$ such that $\phi: \mathcal{F} \to \mathcal{E}$ is a geometric map and $\gamma: \phi^* E \to F$ is a map of $\mathcal{F}$, the maps are the natural transformation $\mu: \phi^* \to \underline{\phi}^*$ such that $\underline{\gamma} \circ \mu_E = \gamma$; and let TopAdLocAnn be TopLocAnn (which is defined in the obvious manner) with some restriction of the γ's (cf. 0.2 for a proper definition); then the inclusion map TopAdLocAnn $\hookrightarrow$ TopLAnn has a left 2-adjoint. This left 2-adjoint to TopAdLocAnn $\to$ TopLAnn assigns to every (Set, R) the Zariski spectrum of R; by considering problems of this kind, the étale and crystalline spectra can be found similarly (cf. [14]).

The (LAnn, LocAnn) situation can be generalized as follows: the members of LAnn are the set valued models of a so-called lim-theory in the language of rings (the formation of lim-formulas involves only $\wedge$ and $\exists!$ ("there exists a unique") among the connectives and quantifiers) whereas the members of LocAnn are the set-valued models for a proper extension of the above-mentioned lim-theory (we need the formula $\exists x(xy = 1) \vee \exists z[(1 - y) z = 1]$) - this extension is still coherent ($\forall x$, $\neg$ do not appear in formulas and $\wedge$ appears only finitely often); if we then properlydefine the notions of topos valued model of a coherent theory and admissible map, then we get a result for (models of lim theory, models of proper coherent extension of lim-theory) which is similar to the result for (LAnn, LocAnn) - in fact, the (LAnn, LocAnn) situation is a special case of the (lim, coherent) situation (cf. [14]).

An alternative way in which to generalize the problem of finding a left adjoint to $\mathcal{B} \hookrightarrow \mathcal{A}$, is to ask for the existence of a so-called multi-adjoint (cf. [26]); finding multi-adjoints is closely related to finding 2-adjoints in the context as described above.

If one can then find a language for the category of real normed spaces that properly fits in the framework described above, then one can use some of the procedures described above to generate spectra for normed spaces that will have, as sheaves in appropriate toposes, stronger properties (w.r.t. a coherent theory) than the set-valued normed spaces to which they are associated. Coherent formulas in this language will then refer to "local" properties of normed spaces and the study of these formulas will fit in with the philosophy of Lindenstrauss and Pelczynski that "local" properties determine the geometric properties of normed spaces. There are languages around for the study of normed spaces: Henson uses one in his study of nonstandard hulls; this language satisfies the requirements to qualify as "language" in the sense of modern model theory (cf. [1]; Krivine defined the notion of real-valued language and shows that a theory can be defined in it that has the ability to, e.g., characterize L_p-spaces (cf. [2],[3]. None of these languages is suitable for the kind of work that we want to do with it.

<u>0.2.1. Lim-theories.</u> L will denote a multisorted formal language (i.e., if R is an n-ary relation symbol and f is an n-ary function symbol then to each of the n places of R(, ···,) (resp. f(, ···,)) there

corresponds a unique sort; for each sort that occurs, we have a sequence of distinct variables of that sort; variables in terms can only occur in places with matching sorts; to each f(, ,...,) is assigned a sort). Formulas are formed in the usual way - sort compatibility should be kept in mind. Lim-formulas are those formulas that can be inductively built from the atomic formulas, and the formula "$\underline{t}$" (true) by only using Λ and $\exists!x$ ("there exists a unique x") a lim-theory T in L has only lim-formulas in its axioms; lim-theorems and the corresponding list of logical rules are given in the expected fashion (cf. [21] for more information); theorems are denoted as follows: $\vec{\phi} \vdash \psi$ where $\vec{\phi}$ is a sequence (finite) of formulae, ψ is a formula.

<u>0.2.2.</u> If C is a category with finite limits, then a realization of L in C is: for every sort $\bar{i}$ an object of C; if $\bar{i} = (i_1, \cdots, i_n)$, then $M(\bar{i}) = M(i_1) x \cdots x M(i_n)$; for every relational symbol R(, ,) with corresponding sorts $(i_1, \ldots, i_n)$, a subobject $M(R) \rightarrowtail M(i_1, \ldots, i_n)$ i for every function symbol with corresponding sorts $((i_1, \ldots, i_n), j)$ a morphism $M(\bar{i}) \to M(j)$.

We define the interpretation of a term inductively as follows: let t be a term with variables in $\bar{x}$; the interpretation $M(t, \bar{x}) : M(\bar{i}) \to M(j)$ where $\bar{i}$ are the sorts of $\bar{x}$ is: the canonical projection $M(\bar{i}) \to M(j)$ if t is the variable y, of sort j, in x; the composite $M(\bar{i}) \xrightarrow{M(\bar{u}, \bar{x})} M(\bar{k}) \xrightarrow{M(f)} M(j)$ if t is $f(\bar{u})$ with $(\bar{k}, j)$ the sorts belonging to f.

The interpretation $M(\phi, \bar{x})$ (if it exists) of ϕ w.r.t. $\bar{x}$, where the free variables of ϕ occur in $\bar{x}$, is a subobject of $M(\bar{i})$ where $\bar{i}$ are the sorts of $\bar{x}$; we define $M(\phi, \bar{x})$ inductively as follows:

If ϕ is $R(\overline{t})$, with $\overline{j}$ the sorts of R, then $M(\phi,\overline{x})$ is the pullback

$$\begin{array}{ccc} M(\phi,\overline{x}) & \longrightarrow & M(R) \\ \downarrow & & \downarrow \\ M(\overline{i}) & \xrightarrow{M(\overline{t},\overline{x})} & M(\overline{j}) \end{array}$$

$M(\uparrow;\overline{x})$ is the identity $M(i) \to M(\overline{i})$; $M(\phi \wedge \psi,\overline{x}) \cong M(\phi,\overline{x}) \wedge M(\psi,\overline{x})$ (the intersection of subobjects); $M(\exists ! y\, \phi(y);\overline{x}) \to M(\overline{i})$ is the composite $M(\phi;\overline{u},y) \to M(\overline{i},j) \to M(\overline{i})$, where the last map is the canonical projection provided that this last composition is mono.

A model of T is a realization M in which the axioms of T are valid ($\phi \vdash \psi$ is valid in M if $M(\phi)$ is a subobject of $M(\psi)$).

Mod(T, C) denotes the category of C-valued models of T.

<u>0.2.3.</u> Lim(T) is the category with objects the couples $(\Phi,\overline{i})$ where is a finite sequence of sorts that appear in $\phi(\overline{x})$ (the free variables of ϕ appear in $\overline{x}$); the morphisms from $(\phi,\overline{i})$ to $(\psi,\overline{j})$ are equivalence classes of formulas $\theta(\overline{x},\overline{y})$ such that $\theta(\overline{x},\overline{y}),\ \theta(\overline{x},\overline{y}') \vdash \overline{y} = \overline{y}'$, $\phi(\overline{u}) \dashv\vdash \exists\ \overline{y}\,\theta(\overline{x},\overline{y}),\ \theta(\overline{x},\overline{y}) \vdash \psi(\overline{y})$, under the relation $\theta \dashv\vdash \theta'$.

$(\mathrm{Lim}(T))^{op}$ is equivalent to the finitely presentable objects of the category of set-valued models of T.

<u>0.2.4.</u> Let $\mathcal{A}$ be a category of the form $\mathrm{Lex}(\mathcal{B}^{op}, \mathrm{Set})$ (the category of Set-valued, left exact contravariant functors on $\mathcal{B}$) (notice that the category of Set valued models of T is equivalent to Lex(lim(T), Set)). Let V be a set of morphisms of $\mathcal{B}$. A map $f: A \to B$ in $\mathcal{A}$ is V-admissible if for every commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & Q \\ g\downarrow & & \downarrow h \\ A & \xrightarrow{f} & B \end{array}$$

with $\alpha \in V$, $\exists ! k : Q \to A$ such that $k \circ \alpha = g$ and $f \circ k = h$ (cf. [21] for a discussion of admissible maps).

If V is as above, and $A \in \mathcal{A}$, then V_A denotes the full subcategory of $A/\mathcal{A}$ generated by the objects $A \to A_\ell$ obtained from pushout diagrams $\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & & \downarrow \\ A & \longrightarrow & A_\ell \end{array}$ with $P \to Q$ in V.

<u>Theorem</u>. ([21]) $f : A \to B$ in $\mathcal{A}$ has an initial factorization $f = h \circ g$ with h admissible and g extremal (i.e., g is a filtered colimit of members of V).

<u>Theorem</u>. Let T be a lim-theory, Λ a set of couples $(\phi(\bar{x}), \psi(\bar{x}, \bar{y}))$ of conjunctions of atomic formulas, T' a coherent extension of T in L given by axioms $\phi(\bar{x}) \vdash \bigvee_I \exists y_i \psi_i(\bar{x}, \bar{y}_i)$ with all $\phi(\bar{x})$, $\psi(\bar{x}, \bar{y}_i)$ in Λ. Then every $f : A \to B$ where $A \in \mathrm{Mod}(T, \mathrm{Set})$, $B \in \mathrm{Mod}(T', \mathrm{Set})$ has an initial factorization $A \xrightarrow{g} C \xrightarrow{h} B$ with C a model of T' and h admissible.

A triple (T, Λ, T') as in the previous theorem will be called a localization triple. $f : A \to B$, Λ-extremal with B a model of T', is called a localization of A.

<u>0.2.5</u>. TopModT is the 2-category of topos-valued models of T defined as in 0.1.3.. TopAdModT$'$ is the 2-category of T'-modelled toposes with morphisms (ϕ, f) where f is admissible (cf. [21] where it is shown how to define the notion of admissibility in toposes).

Theorem. Let (T, Λ, T') be a localization triple. The forgetful 2-functor TopAdModT′ → TopModT has a 2-left adjoint $\mathrm{Spec}_T^{T',\Lambda}$.

0.3. In 1.2 we introduce a multisorted language L (L has an infinite "sequence" of sorts) for normed spaces; we form a lim-theory T in this language in such a way that one can think of a set-valued model of T as being the set $\{B_r / r \in \mathcal{R}^+$, B_r = the set of vectors of norm bounded by $r\}$ for some normed space, together with maps $+_{\rho,q} : B_q \times B_\rho \to B_{\rho+q}$ $r\cdot_\rho : B_\rho \to B_{|r|\rho}$, $-_{\rho,q} : B_\rho \times B_q \to B_{\rho+q}$ which are just the usual vector space operations reduced to the balls B_r and their cartesian products. In 1.3 we show that the Set-valued models of T can in fact be canonically identified with the normed vector spaces. In 1.4 coherent extensions, T_{L^1} and T_{L^2}, of T are defined; we show that the Set-valued models of T_{L^1} (resp. T_{L^2}) are precisely the pre-L^1-spaces (resp. pre-Hilbert spaces) (cf. 1.5); the proof of the fact that T_{L^1} characterizes pre-L^1-spaces uses some of Krivine's ideas ([2]).

The fact that $\mathrm{Nor}_1(\mathcal{R})$, the category of normed real spaces and contractive linear maps, is equivalent to the category of Set-valued models of a lim-theory implies that it is complete, cocomplete and locally finitely presentable; we exhibit explicit constructions for limits and colimits (colimits in $\mathrm{Nor}_1(\mathcal{R})$ differ from the colimits in $\mathrm{Ban}_1(\mathcal{R})$, the full subcategory of $\mathrm{Nor}_1(\mathcal{R})$ generated by the Banach spaces and studied in [19]). We also give a direct proof that the finitely presented objects of $\mathrm{Nor}_1(\mathcal{R})$ are precisely quotients of the form $\ell_1(n)/N$ where N is a subspace of $\ell_1(n)$ - the corresponding result for universal algebras is apparently well known but the proof does not appear to be readily available in the literature (cf. the remarks in [20] p. 292).

In (1.8) we present an equivalence between the category of finitely presented normed spaces and a certain category of formulas of L (this category is closely related to lim(T) of (0.2.3).

In 1.11 we identify the T_{L^2}-extremal and admissible maps: the extremal maps with codomains contained in pre-Hilbert spaces are precisely the maps onto pre-Hilbert spaces, while the admissible maps are the into isometries. We use these identifications to associate a topological space with each $X \in \mathrm{Nor}_1(\mathcal{R})$, such that this topological space is the site of $\mathrm{Spec}L^2(X)$ (denote this topological space with $|X|$). It turns out that $|X|$ is coherent, a spectral space in the sense of Hochster ([22]) and irreducible (in fact, $|X|$ is sober and there is a ring R_X, such that $|X|$ is the Zariski spectrum of R_X.

In (1.12) we show that $|X|$ has trivial cohomology (which is not totally surprising as the abelian structure of X is not significant in $\mathrm{Nor}_1(\mathcal{R})$). In 1.14 we show that $|X|$ localizes well (cf. discussion in (0.1.2)) and in (1.15) we show that the spectrum functor is faithful on $\mathrm{Nor}_1(\mathcal{R})$.

The work of Chapter I shows that spect(2) is well behaved in many respects. It still remains to get a better understanding of the cohomological behavior of these spectra - the fact that a study of Čech cohomology will often suffice in this context (1.12), will be quite useful.

A basic unresolved problem is the following: what are the topological obstructions to the preservation of the validity of coherent formulas by Γ? Preliminary investigations indicate that the theory of homotopical algebra (Quillen [27]) may shed some light on this problem.

It is clear that (1.15) indicates that it may be fruitful to study the category of spatial toposes with Hilbert space structure sheaf (it will suffice to restrict attention to irreducible spectral spaces and the sheaves on them) as generalization of the category $\mathrm{Nor}_1(\mathcal{R})$: the study of $\mathrm{Nor}_1(\mathcal{R})$ becomes the study of toposes on spectral, irreducible topological spaces together with the study of Hilbert spaces and isometries between them in these models of Set theory; a very important point in this context is the following fact: the meta-theorem of Barr ([23]) says roughly the following: if a coherent statement is true for set-valued models, then it is true for topos valued models; this meta-theorem, together with the fact that coherent theories for normed spaces are very powerful ((1.5)), promises to be very fruitful.

CHAPTER 1

1.1. Notation. In what follows, K will denote either the field of rationals or the field of reals.

1.2. Definitions. Let L be the first order language with the following nonlogical entities:

Sorts: A sequence $(B_q)_{q \in K^+}$ indexed by the non-negative elements of K.

Function symbols: $+_{\rho,q} : B_\rho \times B_q \to B_{\rho+q}$ for every pair $(\rho, q) \in K^+ \times K^+$; $i_{\rho,q} : B_\rho \to B_q$ for every pair $0 \leqslant \rho \leqslant q$ in K; $q\cdot_\rho : B_\rho \to B_{\rho|q|}$ for $(q, \rho) \in K \times K^+$; $-_{p,q} : B_\rho \times B_q \to B_{\rho+q}$, $(\rho, q) \in K^+ \times K^+$.

Constants: one constant symbol 0.

Let T_L be the following L-theory:

$$i_{\rho,q}(x) = i_{\rho,q}(y) \vdash x = y \quad ; \quad t \vdash i_{\rho,q}(i_{\rho,r}(x)) = i_{r,q}(x)$$

$$t \vdash i_{\rho+q,r+s}(x +_{\rho,q} y) = i_{\rho,r}(x) +_{r,s} i_{q,s}(y); \; t \vdash q\cdot_r \, i_{\rho,r}(x) = i_{\rho|q|,|q|r}(q\cdot_\rho x)$$

$$t \vdash q\cdot_\rho \, i_{0,\rho}(0) = i_{0,\rho q}(0) \quad ; \quad t \vdash q\cdot_{\rho+r}(x +_{\rho,r} y) = q\cdot_\rho(x) +_{\rho|q|,r|q|} q\cdot_r(y)$$

$$t \vdash ((q+r)\cdot_\rho x) -_{|q+r|\rho,\rho|q|+\rho|r|} (q\cdot_\rho x +_{\rho|q|,\rho|r|} r\cdot_\rho x) = i_{0,|q+r|\rho+(|q|+|r|\rho}(0)$$

$$t \vdash i_{0,0}(x) = 0 \quad ; \quad t \vdash x -_{\rho,\rho} x = i_{0,\rho}(0)$$

$$t \vdash (x +_{\rho,q} y) -_{\rho+q,r} z = i_{\rho+r,\rho+r}(x -_{\rho,r} z) + i_{\rho+r,q}\, i_{q,q}(y) = i_{\rho,\rho}(x) + i_{q+r,q+r}(y-z)$$

$$t \vdash x +_{\rho,0} 0 = x \quad ; \quad t \vdash (x +_{\rho,q} y) +_{\rho+q,r} z = x +_{\rho,q+r}(y +_{q,r} z)$$

$$t \vdash i_{\rho,\rho+q}(x) - i_{q,\rho+q}(y) = i_{\rho+q,\rho+q}(x-y) \quad ; \quad t \vdash x +_{\rho,q} y = y +_{q,\rho} x$$

We will often omit subscripts when no confusion should arise.

1.3. Proposition. The category of Set-valued models of T is equivalent to the category of normed K-modules (the category of K-modules, $Nor_1(K)$, is the category of pairs $(M, \|\cdot\|$ where M is a K-module and $\|\cdot\| \subseteq M \times K$ satisfies the following axioms: $(x, q) \in \|\cdot\|$ only if $q \geq 0$; if $(x, q) \in \|\cdot\|$ and $(y, r) \in \|\cdot\|$, then $(x + y, q + r) \in \|\cdot\|$, if $(x, \Gamma) \in \|\cdot\|$ then $(x, |\alpha|\Gamma) \in \|\cdot\|$ for $\alpha \in K$, $x = 0$ if and only if $(x, \Gamma) \in \|\cdot\| \ \forall \Gamma \in K^+$. Notice that if $K = \mathcal{R}$, then $Nor_1(K)$ is equivalent to the category of normed spaces and contractive linear maps).

Proof. Define a functor from the category of set valued models, $Mod_{Set}(T_L)$, of T_L to $Nor_1(K)$ by taking the filtered colimit of $(B_\rho, i_{\rho,q})$ and inducing the appropriate relations and operations using those of the set valued model $[(B_q), (i_{\rho,q})(+_{\rho,q}), (-_{\rho,q}), (q._\rho)]$.

1.4. Definitions. $T_{L'}$ is the L-theory that we get from T_L by adding the following axioms: for all $\mu_{\ell,j,k} \in K^+$, $\mu_{\ell,j,k} \neq 0$ $(\ell, j, k = 1, 2, \cdots, m)$; $\sigma_{\ell,j,k} \in K^+$, $\sigma_{\ell,j,k} \geq 0$ $(\ell, j, k = 1, 2, \cdots, n)$; $\tau_{\ell,j} \in K^+$ $(\ell = 1, \cdots, m;$ $j = 1, \cdots, n)$ and for every n terms $t_i(\beta_1, \cdots, \beta_{\underline{n}})$ of the language of K-vector lattices such that

$$(*)\ \forall \beta_1 \forall \beta_2 \cdots \forall \beta_n [\sum_{\ell,j,k} \mu_{\ell,j,k} |\beta_\ell + \beta_j - \beta_k| \geq \sum_{\ell,j,k} \sigma_{\ell,j,k} |t_\ell(\beta_1, \cdots, \beta_m)$$

$$+ |t_j(\beta_1, \cdots, \beta_n) - t_k(\beta_1, \cdots, \beta_m)| + \sum_{\ell,j} \tau_{\ell,j} |\beta_\ell - t_j(\beta_1, \cdots, \beta_m)||]$$

holds in K, the axioms $A_1(\alpha, x_\ell, x_i, x_k, z_{\ell,j,k})$ where

$$A_1(\alpha, x_\ell, x_i, x_k, z_{\ell,j,k}) =:$$

$$\bigwedge_{\ell,j,k} (i_{q_{\ell,j,k},(\mu_{\ell,j,k})(\rho_\ell+\rho_j+\rho_k)})(z_{\ell,j,k} = \mu_{\ell,j,k}(x_\ell + x_j + x_k)) \vdash$$

$$\bigvee_{\substack{(r_{\ell,j})_{\ell,j},(s_{\ell,j,k})_{\ell,j,k} \\ \in D_\alpha(q_{\ell,j,k})}} \exists y_1 \cdots \exists y_i \quad (u_{\ell,j})_{\ell,j} \; \exists (w_{\ell,j,k})_{\ell,j,k}$$

$$\times ([\bigwedge_{\ell,j,k} (i_{q_{\ell,j,k},(\mu_{\ell,j,k})(\rho_\ell}\; (z_{\ell,j,k}) =$$

$$= \mu_{\ell,j,k}(x_\ell + x_j + x_k)] \wedge [\bigwedge_{\ell,j} (i_{r_{\ell,j},(r+\rho_\ell)\tau_{\ell,j}}(u_{\ell,j}) = \tau_{\ell,j}(x_\ell - y_i))]$$

$$\wedge[\bigwedge_{\ell,j,k} (i_{s_{\ell,j,k},3r\sigma_{\ell,j,k}})(\omega_{\ell,j,k}) = \sigma_{\ell,j,k}(y_\ell + y_j + y_k))]) \text{ where } x_\ell$$

has type ρ_ℓ, and the other variables have the (unique) types that will make the axioms well-formed axioms: $D_\alpha(q_{i,j,k})$ is the subset of $\{(\sum_{\ell,j,k} q_{\ell,j,k})\alpha^{-1}, (\sum_{\ell,j,k} q_{\ell,j,k}) 2\alpha^{-1}, \cdots, \sum_{\ell,j,k} q_{\ell,j,k}(1+\alpha^{-1})\}^{m \cdot n^4}$ consisting of those $m \cdot n^4$-uples summing to $\sum_{q_{\ell,j,k}} (1+\alpha^{-1})$

and $r = (\sum_{\ell=1}^{m} \rho_\ell)(1 + (\sum_{\ell,j,k} \mu_{\ell,j,k})(\min_{\ell,j}(\tau_{\ell,j}))$.

T_{L^2} is the theory that we get from T_L by adding the following axioms: $A_2(x,y,n,r,s,u,v) =: (x \underset{\rho,q}{+} y = i_{r,\rho+q}(u)) \wedge (x - y = i_{s,\rho+q}(v))$

$$\vdash \bigvee_{(k,\ell)\in B^n_{r,s}} \exists \bar{x} \exists \bar{y} (x+y = i_{r,\rho+q}(u) \wedge (x-y = i(v)) \wedge (i_{k,\rho}(\bar{x}) = x)$$

$$\wedge (i_{\ell,q}(\bar{y}) = y))$$

where $r \in [0, \rho + q]$, $s \in [0, \rho + q]$ and $B^n_{r,s} = \{(k, \ell) \in K^2 / k^2, \ell^2 \in \{\frac{r^2+s^2}{2n}, \cdots (2n)^{-1} k(r^2+s^2), \cdots, 2^{-1}(r^2+s^2)(1+n^{-1})\}, k^2+\ell^2 = r^2 + s^2(1+n^{-1}) 2^{-1}\}$

$$B_2(x, y, n, r, s, \bar{x}, \bar{y}) =: (i_{r,\rho}(\bar{x}) = x) \wedge (i_{s,q}(\bar{y}) = y) \vdash$$

$$\bigvee_{(k,\ell) \in (B^n_{2r,2s})} \exists u\, \exists v\, (i_{r,\rho}(\bar{x}) = x) \wedge (i_{s,q}(\bar{y}) = y) \wedge (i_{k,\rho+q}(u) = x \underset{\rho,q}{+} y) \wedge$$

$$(i_{\ell,\rho+q}(v) = x - y) \text{ where } (r, s) \in [0, \rho] \times [0, q].$$

<u>1.5. Proposition.</u> If $K = \mathbb{R}$, then the category of set-valued models of T_{L^1} (respectively T_{L^2}) is equivalent to the category of normed spaces having their completions isometrically isomorphic to L_1-spaces (respectively L_2-spaces).

<u>Proof.</u> The assertion for T_{L^2} is clearly true; the proof of the remaining assertion is more involved, and we give a brief outline of it (the proof involves the notion of a "real-valued" language and certain results related to it; to make the proof self-contained will take us too far afield and we therefore refer the reader to [2] for the relevant information).

In [2], we find the following theorem: "soient E un espace de Banach et deux réels $n \geq 1$, $\rho > 1$. Pour qu'il existe un espace $L^\rho(\Omega, \ , \mu)$ et deux applications linéaires $\varphi : E \to L^\rho(\Omega, u, \mu)$, $\psi : L^\rho(\Omega, u, \mu) \to E$ telles que $\|\varphi\| \leq M$, $\|\psi\| \leq 1$, $\psi \cdot \varphi$ etant l'identité sur E, il faut et il suffit que E satisfasse les formules

$$\text{(i) } \forall x_1 \ldots \forall x_n \exists y_1, \cdots, \exists y_n [M^\rho \sum_{1 \leq i,j,k \leq n} \rho_{ijk} \|x_i + x_j - x\ \|^\rho \geq \sum_{1 \leq i,j,k \leq n} \sigma_{ijk} \|y_i + y_j - y_k\|^\rho + \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \tau_{ij} |x_i - x_j\|^\rho]$$

où ρ_{ijk}, δ_{ijk}, τ_{ij} sont des réels $\geqslant 0$ que la formule

$$\text{(ii) } \forall x_1 \ldots \forall x_n \left[\sum_{1 \leqslant i,j,k \leqslant n} \rho_{ijk} |x_i + x_i - x_k|^{\rho} \geqslant \sum_{1 \leqslant i,j,k \leqslant n} \sigma_{ijk} |t_i(x_1, \cdots, x_n) \right.$$

$$\left. + t_j(x_1, \cdots, x_m) - t_k(x_1, \cdots, x_m)|^{\rho} + \sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} \tau_{ij} |x_i - t_j(x_1, \cdots, x_m)|^{\rho} \right]$$

soit vraie dans $\mathcal{R}$, pour certaines termes $t_i(x_1, \cdots, x_m)$ $(1 \leqslant i \leqslant n)$ du langage $\mathcal{L}$, de la théorie des espaces vectoriels réticulés (c'est-a-dire formes avec les symboles $0, \lambda, +, \Lambda$)". We then notice, as is done in [2] that the previous theorem actually characterizes pre-L^{ρ}-spaces (the reader should convince himself at this stage that T_{L^1} and T_{L^2} are "finite approximations" to the corresponding real-valued theories of Krivine).

The device that we employ to deduce (1.5) from the theorem of Krivine's, is the ultraproduct: an appropriately chosen ultraproduct allows one to "glue together" the axioms of (1.4) to get equivalent axioms in a real valued language; using this fact together with the fact that Krivine's theorem remains true if we replace "E" in its statement by "$\hat{E}$", an ultrapower of E, "ca...que E satisfasse les formules..." we find that (1.5) is true (use the general version of théorème V.3 [2] and the fact that an ultraproduct of normed spaces is a universal extension of the normed spaces that constitute this ultraproduct to prove this more general version of Krivine's theorem along the lines of the proof of the original theorem).

1.6. We collect together some elementary facts about the behavior of

$\mathrm{Nor}_1(\mathbb{R})$ (we will, when appropriate, identify $\mathrm{Nor}_1(\mathbb{R})$ with the category of real normed spaces and contractive linear maps).

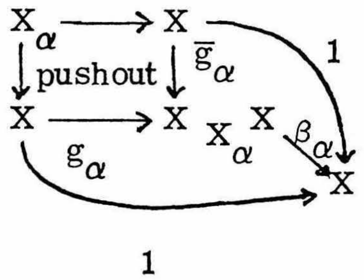
$\mathrm{Nor}_1(\mathbb{R})$ is complete and cocomplete (it is well known and well documented (cf. [19]) that $\mathrm{Ban}_1(\mathbb{R})$, the full subcategory of $\mathrm{Nor}_1(\mathbb{R})$ consisting of the real Banach spaces is complete and cocomplete). The limits of $\mathrm{Nor}_1(\mathbb{R})$ are defined as in $\mathrm{Ban}_1(\mathbb{R})$; $\{0\}$ is the zero object; coequalizers are defined as in $\mathrm{Ban}_1(\mathbb{R})$ whereas coproducts are defined as follows: $\oplus (X_i, \|\cdot\|_i) =: (\oplus X_i, \|\cdot\|_\oplus)$ where $\oplus X_i$ is the vector space coproduct of (X_i) and $\|\Sigma \alpha_k x_k\|_\oplus = \Sigma |\alpha_k| \, \|x_k\|$.

$\mathrm{Nor}_1(\mathbb{R})$ is locally finitely presentable (i.e., $\mathrm{Nor}_1(\mathbb{R})$ is cocomplete and there is a family of objects $(X_j)_J$ in $\mathrm{Nor}_1(\mathbb{R})$ such that $\hom(X_j,-)$ commutes with all filtered colimits $\forall_j \in J$ and a map $L: X \to Y$ in $\mathrm{Nor}_1(\mathbb{R})$ is an isomorphism if and only if $\hom(X_j, L)$ is an isomorphism in Set) because $\mathrm{Nor}_1(\mathbb{R})$ is equivalent to the category of Set-valued models for a lim-theory (cf. [21]).

We show now that the finitely presented normed spaces are precisely the quotients of finite-dimensional L_1-spaces (notice that not every finite dimensional normed space is such a quotient).

The fact that finitely presented spaces are quotients of the form described above, is proved as follows: we first show that if X is finitely presented and if X is the filtered colimit of finite dimensional spaces (X_α) such that the canonical maps $X_\alpha \to X$ are monomorphisms, then $X \cong X_\alpha$ for some α; then we use the fact that there is such a diagram (X_α) for X with every X_α a quotient of the form described above.

Say X is finitely presented and $X = \varinjlim X_\alpha$ with $X_\alpha \to X$ mono. Let Y_α be the pushout of $\begin{array}{c} X_\alpha \to X \\ \downarrow \\ X \end{array}$; then (Y_α) is an inductive diagram and $Y = \varinjlim Y_\alpha = \varinjlim (X \amalg_{X_\alpha} X) = X \amalg_{\varinjlim X_\alpha} X = X \amalg_X X = X$; we have the following diagram



(Y_α) induces a diagram $(\hom(X, X_\alpha) \to \hom(X, Y))_{\alpha \in A}$ and the image of g_α is $\hom(X, Y)$, $\beta_\alpha \circ g_\alpha$, is the same as the image of $\bar{g}_\alpha$ in $\hom(X, Y_\alpha)$, $\beta_\alpha \bar{g}_\alpha$; hence as $\hom(X, Y) = \hom(X, \varinjlim X_\alpha) = \varinjlim \hom(X, Y_\lambda)$ and from the canonical construction of the colimit of a diagram of sets, it follows that there is a $\bar{\alpha} \geqslant \alpha$ such that $g_{\bar{\alpha}} = \bar{g}_{\bar{\alpha}}$, i.e., X is the equalizer of $g_{\bar{\lambda}}, \bar{g}_{\bar{\lambda}}$; but it is easy to verify that $X_{\bar{\alpha}}$ is the equalizer of $\bar{g}_{\bar{\alpha}}$ and $g_{\bar{\alpha}}$; hence $X = X_{\bar{\alpha}}$ (this proof was inspired by a proof of Grothendieck ([8], Volume II, p. 196).

If X is finitely generated, then X is the colimit of the forgetful functor $\mathrm{Nor}_1(R)/_X \to \mathrm{Nor}_1(\mathcal{R})$ restricted to the full subcategory of $\mathrm{Nor}_1(\mathcal{R})/X$ having as objects the maps $Y \to X$ with Y a quotient of a finite dimensional L_1-space.

<u>1.7</u>. We have the following result which throws more light on the relationship between the finite dimensional normed spaces and the theory T_L.

<u>Theorem</u>. (Cf. [20], p. 292). If L is a language without relation symbols and T_0 is any equational theory, i.e., the axioms of T_0 are coherent sequents of the form $\vdash t = t'$ where t, t' are terms of L,

then C_0, the dual of the category of finitely presented set-valued models of T_0 (i.e., those set-valued models X such that hom(X, -) preserves filtered colimits) is equivalent to the following category: $C(T_0)$ has as objects the finite sets of atomic formulas of L; the morphisms of $C(T_0)$ are defined as follows: a morphism $\Phi(x_1, \cdots, x_n) \to \psi(y_1, \cdots, y_m)$ is an equivalence class of m-tuples of terms $(t_1, \cdots, t_m)$ with free variables among $x_1, \cdots, x_n$ such that $T_0 \vdash \phi \Rightarrow \psi\ (t_1/y_1, \cdots, t_n/y_n)$ under the equivalence relation $(t_1, \cdots, t_n) \sim (s_1, \cdots, s_n)$ if and only if $T_0 \vdash t_i = s_i$ $(i = 1, \cdots, n)$.

1.8. We give a convenient full, faithful functor $\mathcal{F}$: f in $\mathrm{Nor}_1(\mathcal{R}) \to (C(T_L))^{op}$, that will be useful later on. (fin$\mathrm{Nor}_1(\mathcal{R})$ is the category of finitely presented and normed spaces.)

If $\{t_1 = \bar{t}_1, \cdots, t_n = \bar{t}_n\}$ is a finite set of terms of L and this set of terms has k distinct variables, then $\mathcal{F}(\{t_1 = \bar{t}_1, \cdots, t_n = \bar{t}_n\}) = \bigoplus_k \mathcal{R}/N_{t_1, \cdots, t_n}$ where $\bigoplus_k \mathcal{R}$ is the k-fold direct sum of $\mathcal{R}$ as normed space with its usual norm and $N_{t_1, \cdots, t_n}$ is the subspace of $\bigoplus_k \mathcal{R}$ generated by the vectors $r_i - \bar{r}_i$ where r_i (respectively $\bar{r}_i$) is defined as follows: take t_i (respectively $\bar{t}_i$), if x_j occurs free in t_i (resp. $\bar{t}_i$), then replace x_j with the jth natural basis vector of $\bigoplus_k \mathcal{R}$ multiplied with the subscript of the sort of x_j; the result of replacing all of the free variables of t_i in this way is r_i (respectively $\bar{r}_i$).

If $\Phi_{t_1, \cdots, t_\rho}(x_1, \cdots, x_n) = \{t_1 = \bar{t}_1, \cdots, t_\rho = \bar{t}\}$ with free variables $x_1, \cdots, x_n$ and $\Psi_{\underline{t}_1, \cdots, \underline{t}_q}(y_1, \cdots, y_n) = \{\underline{t}_1 = \bar{\underline{t}}_1, \cdots, \underline{t}_q = \bar{\underline{t}}_q\}$ with free variables $y_1, \cdots, y_n$, and $[(\underline{t}_1, \cdots, \underline{t}_n)]: \Phi \to \psi$ is a morphism, then $\mathcal{F}([\underline{t}_1, \cdots, \underline{t}_n])$ is defined as follows: $\Sigma \alpha_i \underline{e}_i \in \bigoplus_m \mathcal{R} \mapsto [\Sigma \alpha_i t_i(\bar{e}_1, \cdots, \bar{e}_n)]$

where $\underline{\ell}_i$ (resp. $\overline{\ell}_i$) is ℓ_i multiplied by the subscript of the sort of y_i (resp. x_i) and $[\cdots]$ denotes the equivalence class of $\cdots$. This map factors through $\bigoplus_m \mathcal{R}/N_\psi$.

1.9. Notation. The 2-left adjoint to the forgetful functor $\text{AdmTopModT}_{L^1} \to \text{TopModT}_L$ (respectively $\text{AdmTopModT}_{L^2} \to \text{TopModT}_L$) is denoted by Spec(1) (resp. Spec(2)). We write Spec(1)(X) (resp. Spec(2)(X)) for Spec(1)(Set, X) (resp. Spec(2)(Set, X)).

1.10. We are now going to focus our attention of Spec(2): T_{L^2} has certain special properties that makes the study of Spec(2) and its related (relevant) entities significantly easier than the corresponding study in the case of T_{L^1}.

We have the following result that will make the nature of Spec(2) clearer.

Proposition. The L^2-extremal maps with codomains contained in pre-Hilbert spaces are precisely the maps which map onto pre-Hilbert space. The L^2-admissible maps are precisely the isometries (not necessarily onto).

Proof. A couple in Λ_{L^2}, $(\varphi(\overline{x}), \psi(\overline{x},\overline{y}))$, satisfies $\phi(\overline{x},\overline{y}), \psi(\overline{x},\overline{\overline{y}}) \vdash \overline{y} = \overline{\overline{y}}$ if and only if the map that corresponds to it in $\text{Nor}_1(\mathcal{R})$ is epi. It follows that V_A, where $A \in \text{Nor}_1(\mathcal{R})$, consists of epimorphisms and hence that every extremal map is epi. However, $\text{Nor}_1(\mathcal{R})$ is epi-mono factorizable (i.e., every map (L:X → Y factors uniquely through $X \to Z \to Y$ such that X → Z is epi and Z → Y is mono) hence every epi is extremal: if L:X → Y is epi, then $X \xrightarrow{L} Y \xrightarrow{1} Y$ is an initial factorization through an admissible map, hence L is extremal (we actually need the second part of the proposition to proposition to justify the claims of the last part of the previous sentence).

To complete the proof, we need the following result (cf. [21]): $f: A \to B$ between models of T is V-admissible if and only if for every $(\phi(\bar{x}), \psi(\bar{x}, \bar{y}))$ in Λ, every $\bar{a}$ in A satisfying $\Phi(\bar{a})$, and every $\bar{b}$ in B such that $\psi(f(\bar{a}), \bar{b})$ holds, there exists a unique $\bar{c}$ in A satisfying $\psi(\bar{a}, \bar{c})$ such that $f(\bar{c}) = \bar{b}$. When this result is applied to L_2, we see that if T is admissible, then if Tx has norm less than r, then x has norm less than r; hence $\|Tx\| = \|x\| \ \forall_x$ and T is an isometry. Conversely it is clear that every isometry is admissible.

From the fact that V_X is made up out of epimorphisms we see that spec(2)(X) is actually a spatial topos (i.e., there is a topological space such that spec(2)(X) is equivalent to the category of sheaves on this space) (cf. [21] thm. 4.4.1) and for a given normed space X, one can explicitly describe the underlying topological space $|X|$ of spec(2)(X) as follows: the points of $|X|$ are the isomorphism classes of maps $X \xrightarrow{T} Y$ where T is epi and Y is a pre-Hilbert space ($\begin{array}{l} X \xrightarrow{T} Y \\ \ \ {}_{T'} \searrow Y' \end{array}$ are isomorphic if and only if there is an isometry $Y' \to Y$ such that $\begin{array}{l} X \xrightarrow{T} Y \\ \ \ {}_{T'} \searrow \uparrow \\ \qquad\ \ Y' \end{array}$ is commutative); for every $X \xrightarrow{\ell} X$ in V_X, let D_ℓ be the set of all points of $|X|$ such that there is a factorization of the representatives of the points through $X \xrightarrow{\ell} X_\ell$; the D_ℓ constitute a basis for the open set of $|X|$.

We collect together a number of properties of $|X|$ (cf. [21]). $|X|$ is a spectral space in the sense of Hochster ([22]) (it will not be without interest to find a characterization, along the lines of the work of Hochster, of the topological spaces M such that $M = |X|$ for some X in $\mathrm{Nor}_1(\mathcal{R})$), i.e., there is a ring R such that $|X|$ is the Zariski spectrum of R. In particular we know that $|X|$ is sober (i.e., given any closed irreducible set, then there is a unique point in this set such that the closure of the point is the whole set. As the closure of irreducible sets are irreducible and a points are irreducible, every closure of a point is irreducible; it follows that $|X|$ is irreducible with generic point the isomorphism class of $X \to 0$; in fact, every basic open set contains $[X \to 0]$ as $X \to 0$ factorizes through every $X \xrightarrow{\ell} X_\ell$.

1.11. Proposition. The cohomology of $|X|$ is trivial in all abelian sheaves that are filtered colimits of locally constant sheaves.

Proof. Constant sheaves on irreducible spaces are flasque (i.e., if $U \leqslant V$ are open subsets of $|X|$, and F is constant, then the restriction map $F(V) \to F(U)$ is epi) because open subsets of an irreducible space are dense in the space. We know that every pair of open sets of $|X|$ intersect nontrivially; hence every locally constant abelian sheaf is actually constant. As $|X|$ is coherent, cohomology commutes with filtered colimits (cf. [8]).

1.12. Proposition. If X is a pre-hilbert space, then $|X|$ has trivial cohomology in all abelian sheaves.

Proof. If (U_α) is an open cover of $|X|$ consisting of basic sets, then $|X| = U_\alpha$ for some α: the equivalence class of the identity map $X \xrightarrow{1} X$ must belong to U_α for some α; if U_α is the set of equivalence classes of maps factoring through $X \xrightarrow{\ell} X_\ell$, then 1 factors through ℓ and consequently ℓ is an isometry; it follows that $U_\alpha = |X|$.

The desired result about the cohomology of $|X|$ follows because Γ is exact: if $f: F \to G$ is epi in $Sh(|X|)$, then, given a global section of G, there is an open covering of $|X|$ and for each member of this covering there is a section in the restriction of F to this member that gets mapped onto the restriction of the given global section by f; but as we have seen before, the only covers of $|X|$ are the trivial ones.

1.13. Example. If $X = \mathcal{R}$, then $|X|$ can be described as follows: $|X|$ is the interval $[0,1] \leq \mathcal{R}$ with basic open sets the closed subintervals of $[0,1]$ with 0 as one endpoint. We see that the open sets of $|\mathcal{R}|$ are precisely the intervals with one endpoint at the origin.

1.14. A natural question that arises in the following: if U is an open set of $|X|$, is there an $Y \in Nor_1(X)$ such that $Spec(2)(X)/U$ is equivalent to $Spec(2)(Y)$. We show in the following result that the answer to this question is positive if we restrict our attention to basic open sets U.

Proposition. Let (T, Λ, T') be a localization triple and let $A \in Mod_T(Set)$ be such that all the members $A \xrightarrow{m} A_m$ of V_A are epimorphic maps. Then, for every $A \xrightarrow{\ell} A_\ell \in V_A$, spec$(Set, A_\ell)$ is equivalent to spec$(Set, A)/\ell$.

Proof. All we need to show is that any map$(Set, A_\ell) \xrightarrow{(\Gamma, g)} (\mathcal{E}, E)$ of TopModT factors uniquely (up to 2-isomorphism) through $(Set, A_\ell \xrightarrow{(\Gamma,h)}$ Spec$(Set, A)/\ell$ where $h: A_\ell \to \Gamma(\widetilde{A} \times \ell \to \ell)$ is the map $\eta_{PA}(\ell): A_\ell \to ia(PA)(\ell) = \widetilde{A}(\ell) = \Gamma(\ell^*(\widetilde{A}))$ (where $\eta: 1 \to ia$ is the unit of adjunction $a \vdash i$, a is the associated sheaf functor and $\widetilde{A}$ denotes the spectrum of A) and such that the appropriate maps are admissible.

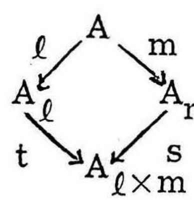
We will get the factorization result by showing that $(\mathrm{Set}, A) \xrightarrow{(1,\ell)} (\mathrm{Set}, A_\ell) \xrightarrow{(\Gamma, g)} (\mathcal{E}, E)$ factors in the appropriate manner through $\mathrm{Spec}(\mathrm{Set}, A) \to \mathrm{Spec}(\mathrm{Set}, A)/\ell$.

The map $(\mathrm{Set}, A) \xrightarrow{(\Gamma, g\ell)} (\mathcal{E}, F)$ factor into $(\mathrm{Set}, A) \xrightarrow{(\Gamma, \eta_A)}$ $\mathrm{Spec}(\mathrm{Set}, A) \xrightarrow{(\phi, f)}$ with f admissible (cf. [21] thm. 4.1.4). The assumption that the $A \to A_m$ are epi implies that all the objects of V_A^{op} are open in $\mathrm{Sh}(V_A^{op})$, and therefore, to show that $\phi: \epsilon \to \mathrm{Sh}(V_A^{op})$ factors through $\mathrm{Sh}(V_A^{op})/\ell$, it is enough to check that $\phi_* G \to (\phi_* G)^\ell$, the transpose of $\phi_* G \times \ell \xrightarrow{\pi} \phi_* G$, is an isomorphism (cf. [23] 3.54, 3.47, 3.52).

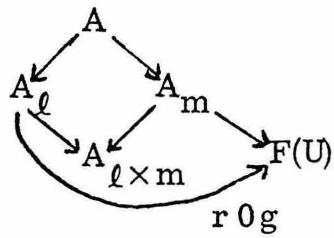
ϕ^* is induced by $\overline{\phi} : V_A^{op} \to \mathcal{E}$ defined as follows

$$\overline{\phi}(m)(u) = \begin{cases} \{k : A_m \to F(U) / k \circ m = \Gamma \circ g \circ \ell\} \text{ if } 1(U) \text{ is nonempty} \\ \text{the empty set otherwise} \end{cases}$$

Then $\phi_*(G)(m) = \hom(\overline{\phi}(m), G)$ (cf. [20] 1.3.4) and the map $\phi_*(G) \to \phi_*(G)^\ell$ is given by $\rho_m : \hom(\overline{\phi}(m), G) \to \hom(\phi(\ell \times m), G); \rho_m(\alpha) = \alpha \circ \phi(\ell \times m \xrightarrow{\pi} m)$. $\ell \times m$ is given by the pushout



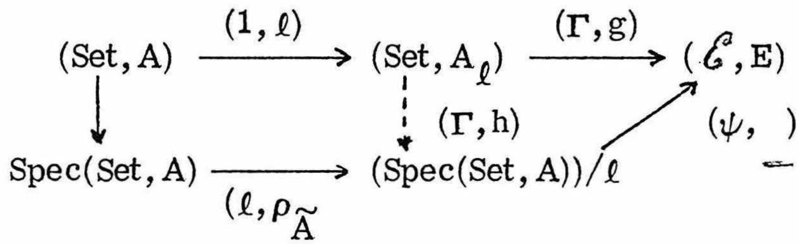
in A and $\phi(\pi : \ell \times m \to m)$ is the map $k \mapsto k \circ s$ (in the Uth coordinate). The fact that S is epi implies that $\phi(\pi)$ is mono. Let $\overline{K}$ be in $\phi(m)(U)$; then



is commutative and therefore $\exists\, k : A_{\ell \times m} \to F(U)$ such that $ks = \bar{k}$ (by the universal property of pushouts); this shows that $\phi(\pi)$ is epi. Hence $\phi(\pi)$ is an isomorphism and therefore ρ_m is invertible for all m.

According to the work in the previous paragraph we have $f : \widetilde{A} \to \phi_* F = \ell_* \psi_* F$ admissible, hence by adjunction $\exists\, \mu : \ell^* \widetilde{A} \to \psi_* F$ where $\phi = \ell$ is the factorization of ϕ found earlier.

We have the following commutative diagram (i.e., the solid diagram)



We want to check that the rectangle on the left commutes. It is clear that the underlying diagram of geometric maps is commutative because Set is the "final" topos. What remains to be shown is that $A \xrightarrow{PA^{(1)}} \Gamma\widetilde{A} \xrightarrow{\Gamma(P\widetilde{A})} \Gamma\pi_\ell \ell^* \widetilde{A}$ and $A \xrightarrow{\eta PA} \Gamma\widetilde{A} \xrightarrow{\widetilde{A}(\Delta)} \widetilde{A}(\ell)$ are, with the ranges appropriately identified, the same ($\widetilde{A}(\Delta)$ is the restriction map and the map $\widetilde{A}(\Delta)\eta_{PA}$ is the "same" as $A \xrightarrow{\ell} A_\ell \to \eta_{PA}(\ell) \to \Gamma(\widetilde{A} \times \ell \to \ell)$ via naturality). So, we make sure that

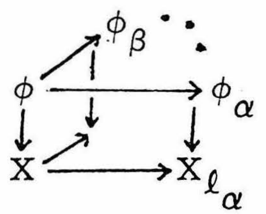
$$\begin{array}{ccc} \hom(1,A) & \longrightarrow & \hom(1,\pi_{\ell}\,\ell^*\tilde{A}) \\ \downarrow & & \downarrow \qquad \text{(adjunction)} \\ \hom(\ell,A) & \underset{\text{(adjunction)}}{\longrightarrow} & \hom(\ell_1^*,\ell^*\tilde{A}) \end{array}$$

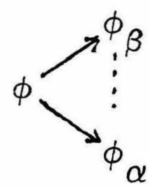
commutes - but $(1,\ell)$ is epi, and so we have the result.

To finish the proof, we show that μ is admissible. We know that the co-unit of adjunction, $\bar{\epsilon}:\ell^*\ell_* \to 1$ is an isomorphism (because $\ell \to 1$ is mono) (cf. [23] 4.4, 4.12); hence $\mu:\ell^*\tilde{A} \to \psi_*F = \ell^*\tilde{A} \xrightarrow{\ell^*(f)} \ell^*(\ell_*\psi_*F) \xrightarrow{\bar{\epsilon}} \psi_*F$ is admissible, as isomorphisms are admissible and the inverse images of admissible maps are admissible (cf. [21] 3.6.3).

<u>1.15. Proposition.</u> Let X be a normed space, and let ρX be the presheaf induced on $\rho(V_X^{op})$ by X. Then ρX is a separated presheaf and therefore the spectrum 2-functor restricted to set valued normed spaces is faithful.

<u>Proof</u>. The topology τ_X on the site V_X^{op} can be given as follows: push generating families of the axiom topology out along maps with X as codomain:





is a covering family in the axiom topology

The maps $X \to X_\alpha$ then induce maps $\begin{array}{ccc} X & \searrow & \\ \downarrow & & \\ X & \longrightarrow & X_\alpha \end{array}$ in V_X which are epi;

in V_X^{op} we therefore have monos $\begin{array}{ccc} X & & \\ \downarrow & \searrow & \\ X_\alpha & \longrightarrow & X \end{array}$ that cover the final object; push the basic cocoverings in V_X of the initial object out along the unique maps emanating from the initial object to the other objects of V_X; go to V_X^{op} with these pushed out maps and let them generate a Grothendieck topology τ_X.

To check that ρX is separated w.r.t. τ_X, we only need to make sure that it is separated w.r.t. the basic topology that induces τ_X. Let A be a finitely generated normed space. We want to show that the presheaf $\hom_{Nor_1}(A, \rho X(-))$ is separated; let $\left(\begin{array}{ccc} X & & \\ \downarrow & \searrow & \\ X & \xrightarrow{\ell_\alpha} & X_\alpha \end{array}\right)_\alpha$ be a basic cocover of $\begin{array}{c} X \\ \downarrow 1 \\ X \end{array}$ $\in \hom(A, \rho X(X \to X_\alpha)) = \hom(A, X_\alpha)$ be a compatible family; if $b \in \hom(A,X)$ restricts to b_α in each coordinate, then $\begin{array}{ccc} A & \xrightarrow{b} & X \\ {\scriptstyle b_\alpha}\downarrow & \swarrow {\scriptstyle \ell_\alpha} & \\ X_\alpha & & \end{array}$ commutes and therefore b is uniquely determined as ℓ_α is mono (look at the definition of ℓ_α and realize that it is a vector space isomorphism). It is clear that the same argument will work for general elements of V_X.

1.16. We know that every spatial topos has enough points, i.e., there is a family of geometric maps $\phi: \text{set} \to \mathcal{E}$, where $\mathcal{E}$ is a given spatial topos such that $f: A \to B$ in $\mathcal{E}$ is an isomorphism if and only if $\phi^*(f)$ is an isomorphism in Set for every ϕ; Deligne's theorem ([20], [8], [23]) also implies that every locally coherent topos (topos (for the general definition of a locally coherent topos cf. [8]) has enough points. It follows that Spec(1)(X) and Spec(2)(X) have enough points; seeing that

Deligne's theorem does not give an explicit construction of points for a locally coherent topos, we state and prove the following result.

Theorem. Let (T, Λ, T') be a localization triple and suppose that for every $A \in \text{Mod}(\text{Set}, T)$ there is a family $g_i : A \to B_i$ of morphisms of models with $B_i \in \text{Mod}(\text{Set}, T')$ such that the (g_i) form a monomorphic family. Then $\text{Spec}_T^{T', \Lambda}(A)$ has enough points for every $A \in \text{Mod}(\text{Set}, T)$.

Proof. Let $A \xrightarrow{g} B$ be a morphism of models with $A \in \text{Mod}(\text{Set}, T)$, $B \in \text{Mod}(\text{Set}, T')$. According to (0.2.5) $\exists$ a couple (ϕ^g, f^g) where $\phi^g : \text{Set} \to \text{Spec}(A)$, is a geometric map and $f^g : \phi^*(\tilde{A}) \to B$ is admissible such that $(\Gamma f)_{\tilde{A}} = g$ (cf. [21](4.1.4)) - in fact, the following explicit description of (ϕ, f) can be found: ϕ is induced by $\bar{\phi} : V_A^{op} \to \text{Set}$; $A \xrightarrow{\ell} A_\ell \longmapsto \{h : A_\ell \to B/g \text{ factors through } h\}$ and f is induced by the inclusion $\bar{\phi}(\ell) \hookrightarrow \hom(A_\ell, B)$.

Let $\alpha : F \to G$ be a map in Spec(A) and suppose that the images of α in $(\phi^g)^*$ are all mono. Say $\beta, \gamma : H \to F$ are coequalized by α; then $(\phi^g)^*(\beta) = (\phi^g)^*(\gamma)$ for every g as described in the statement of the theorem. Pick $\ell : A \to A_\ell$ in V_A^{op} and $x \in H(\ell)$. We know that

$$(\phi^g)^*(K)(M) = \varinjlim_{\substack{M \to \phi^g(m) \\ \in (D/\bar{\phi}^g)^{op}}} K(m) \quad \forall_{k,g,m}.$$

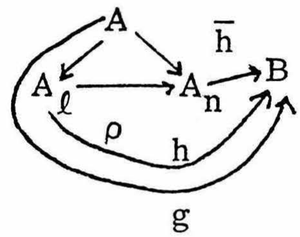
Let $[x]$ denote the equivalence class of x in $(\phi^{g^*})(H)(m)$; then $[\beta_\ell x] = [\gamma_\ell x]$ in $\phi^{g^*}(F)(M)$ and therefore there exists a diagram

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ \phi(n) & \xrightarrow[\phi(\beta)]{} & \phi(\ell) \end{array}$$

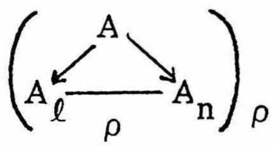
(we assume M is a final object of Set) and $\exists h : A_\ell \to B$, $\bar{h} : A_n \to B$ and

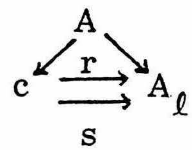
$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ A_n & \xleftarrow[\rho]{} & A_\ell \end{array}$$

such that

$F(\rho)\gamma_{\ell}x = K(\rho)\beta_{\ell}x$ and 

is commutative.

The claim is now that 

, where ρ varies with varying g, consitutes a monomorphic family: say the ρ's coequalize 

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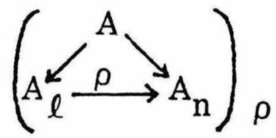
then the $\bar{h}\rho$ coequalize and therefore the h coequalize; now, the h can be chosen to constitute a monomorphic family; hence r = s. Hence we know that 

is cocovering and that $\gamma_{\ell}x$ and $\beta_{\ell}x$ and α is mono.

A similar proof shows that α is epi; hence α is an isomorphism and the proof is complete. (The proof for the corresponding result for the étale topos in [8] is quite similar to the proof given above).

BIBLIOGRAPHY

[1] C. W. Henson; Nonstandard hulls of Banach spaces; Israel Journal of Math., 25, p. 108.

[2] J.-L. Krivine, Langages à valeurs réelles et applications; Fundamenta Math., LXXXI, p. 213.

[3] J. Stern; Some applications of model theory in Banach space theory; Ann. Math. Logic, p. 49.

[4] D. Dacunha-Castelle, J.-L. Krivine, Sous-espaces de L^1; Is. J. Math., Vol. 26, no. 4, p. 320.

[5] J.-L. Krivine; Sous-espaces de dimension finie des espaces de Banach réticulés; Ann. Math., 104, p. 1.

[6] J.-L. Krivine, B. Maurey; Espaces de Banach stables; Is. J. Math., Vol. 39, no. 4, 1981, p. 273.

[7] D. Amir, V. D. Milman; Unconditional and symmetric sets in n-dimensional normed spaces; Is. J. Math., Vol. 37, no. 1, p. 3.

[8] M. Artin, A. Grothendieck, J.-L. Verdier; Theorie des Topos et cohomologie Etale des schemas; Vol. I, II, III, Springer Verlag, 1971.

[9] M. Hakim; Topos anneles et schemas relatifs; Springer Verlag, 1972.

[10] C. Mulvey; Intuitionistic algebra and representations of rings; A. M. S. Transactions, No. 148, p. 3.

[11] C. Rousseau; Topos theory and complex analysis; Applications of sheaves, p. 623; Lecture notes in math., no. 753, Springer Verlag, 1977.

BIBLIOGRAPHY

[12] G. Takeuti; Two applications of logic to mathematics; Pub. Math. Soc. Japan, No. 13, Iwanami Shoten and Princeton University Press.

[13] The work of Kock, Reyes and Wraith mentioned in the bibliography of [23].

[14] G. Wraith; Generic Galois theory of local rings; Appl. of sheaves, p. 739, Lecture notes in Math., no. 753, Springer Verlag, 1977.

[15] S. Szarek, N. Tomczak-Jaegermann; on nearly euclidean decomposition for some classes of Banach spaces.

[16] The papers by Cole, Tierney, Johnstone on spectra listed in the bibliography of [23].

[17] J. Linderstrauss, A. Pelczynski; Absolutely summing operators in L_p-spaces and their applications; Studia Math. 29, p. 275.

[18] R. Hartshorne; Algebraic Geometry; GTE, no. 52, Springer Verlag, 1977.

[19] J. Cigler <u>et al.</u>; Banach modules and functors on categories of Banach spaces; Lecture notes in pure and applied math., Vol. 46, Dekker, 1976.

[20] G. Reyes, M. Makkai, First order categorical logic; Lecture notes in math., no. 611, Springer Verlag.

[21] M. Coste; Localization, spectra and sheaf representation; Appl. of sheaves, p. 212, Lecture notes in math., no. 753, Springer Verlag.

[22] M. Hochster; Prime ideal structure in commutative rings, Trans. Amer. Math. Soc., 142, p. 43.

[23] P. T. Johnstone; Topos Theory; Academic Press, 1977.

[24] R. L. Graham et al.; Ramsey Theory; Wiley-Interscience, 1980.

[25] G. Osius; Logical and set theoretical tools in elementary Topoi; Lecture notes in math, 445, Springer Verlag, 1975.

[26] Y. Diers; Some spectra relative to functors; J. of Pure and Appl. Math., Vol. 22, No. 1, p. 57, 1981.

[27] D. Quillen; Homotopical algebra; Springer Lecture Notes in Mathematics, 43, Springer Verlag, 1967.