THE THEORY OF A MODIFIED
ASTON-TYPE MASS-SPECTROMETER

Thesis by
George C. Munro

In Partial Fulfillment of
the Requirements for the
Degree of Doctor of Philosophy,

California Institute of Technology,
Pasadena, California,
1933.

TABLE OF CONTENTS

SUMMARY

The theory of an Aston-type mass-spectrometer using a cylindrical condenser and bringing all ions of the same specific mass to a point focus by variation of the magnetic field alone is developed. The theory of the motion of an ion in an inverse first-power electrostatic field is studied under the particular set of conditions where all ions regardless of their energy enter the condenser normal to its leading edge and at the median position in the gap. Taylor series expressions have been obtained which give the exit position, the square of the exit velocity, and the tangent of the angle of deflection in the field. The existence, uniqueness, continuity, and uniform convergence of these series solutions are established. The equation of the family of straight lines which the ions follow beyond the electrostatic field and before the magnetic field are obtained. The existence, location, and width of a virtual source are established for this system under the above conditions of collimation. Expressions for the coordinates of the faces of the magnetic pole pieces are obtained for the case where the leading and trailing faces are mirror images of one another. in the line containing the centers of curvature of the paths of the ions in the magnetic field. The maximum mass-resolution is obtained for a magnetic deflection of about one radian when the electric deflection is one fourth of a radian. A study is made of the corrections for various edge effects in the magnetic field. Tables are given of the numerical results of this investigation.

I. INTRODUCTION

An Aston-type mass-spectrograph may be characterized by a certain combination of electrostatic and magnetic fields. A well collimated plane-parallel beam of accelerated positive ions is first resolved by the electrostatic field into an energy spectrum (or better, into a spectrum dependent on mv^2/e where m, e, and v are respectively the mass, charge, and initial velocity of an ion as it enters the electrostatic field). This energy spectrum is further analyzed by the magnetic field in terms of the momentum of an ion (or better, in terms of the mV/e for an ion where V is the final velocity of the ion as it leaves the electrostatic field). These two fields are arranged in such a way as to bring all ions of the same ratio of mass to charge to a focus dependent on this ratio but independent of the velocity.

This type of mass-spectrograph may be characterized in a somewhat different way which is significant for the purposes of design. The electrostatic field breaks up a well collimated plane-parallel beam of accelerated positive ions into a spectrum which is strictly equivalent to a point virtual source, the same for all ions, together with an angular spread about a certain median ray and dependent on mv^2/e. The magnetic field then converts the point virtual source into real point foci independent of the velocity of an ion but dependent on its ratio of mass to charge. The system thus displays analogies with an optical system, and the language of optics may well be used in its description.

In the following presentation there has been developed the theory of a modified Aston-type mass-spectrograph which uses a cylindrical section condenser in place of Aston's plane-parallel plate condenser, and which uses in place of the several foci employed simultaneously by Aston only one focus. This focus is then obtained in turn for each individual value of m/e by variation of the magnetic field alone.

The Significant Parts of the Mass-Spectrograph.

As a preface to the derivation of the theory of the mass-spectrograph a qualitative account of the several integral parts of the apparatus which are important in the theory of focussing will be given. Also a discussion will be given here of the arguments in favor of the various modifications which are being incorporated into the apparatus as against those of other possible alternatives.

It is virtually impossible to obtain a theory of the mass-spectrograph without the use of a considerable number of approximations. In order to insure in the highest degree that the approximations involved in the following calculations are legitimate and that no untoward assumptions have been introduced, the conditions and some of the more important dimensions of the apparatus under consideration will be given.

The parts of the apparatus which are directly important for the theory of focussing may be enumerated as follows:

(1) Source of ions. This type of mass-spectrograph places severe restrictions on the arrangements possible for the production of positive ions. Aston has come more and more to believe that the constancy of the positive ion source determines the ultimate limit to the usefulness of the mass-spectrograph (F:W. Aston, Journal of the Chemical Society, February, 1933). With the present method of measuring the relative amounts of ions with different ratios of mass to charge it becomes of paramount importance that the source of ions display only very slight fluctuations. Otherwise relative intensities have no meaning.

A nearly constant source of ions can be obtained by the following method. A stream of vapor or gas is introduced by means of a capillary into the discharge tube exactly opposite the collimating system. The pressure in the capillary can be maintained quite constant over a considerable period of time. Concentric with the mouth of the capillary there is located a filament for the production of electrons and, at a smaller radius, a grid for the acceleration of these electrons. The neutral molecules coming from the capillary are thus ionized in a space of small dimensions and of small potential variation. Such methods as a heater type filament and evaporation require extreme precautions in the preparation and maintenance of the surface before constancy can be approximated.

In order that the number of ions beyond the electrostatic field be an appreciable fraction of those formed at the source it is essential that the source of ions be restricted to a rather small region exactly opposite the collimating slits and to a region of nearly constant potential after the accelerating field has been applied across the discharge tube. The former precaution assures that a fair proportion of the ions will get through the collimating slits and the latter that these ions will not be drawn to the lower condenser plate and thus removed from the measurements.

(2) The discharge tube. The discharge tube contains at one end the anode in the immediate neighborhood of the source and at the other end, the cathode. The anode may conveniently be the grid of the electron bombardment arrangement. The cathode may be either simply the first slit of the collimator or a separate arrangement in the neighborhood of this slit. Between the anode and cathode an accelerating field having a potential drop of the order of 1500 volts is applied.

It is essential that the pressure be maintained at such a value as to insure that the kinetic theory mean-free-path is several times larger than the greatest linear dimension of the tube. This will prevent, in a sufficient measure, collisions between the anode to cathode ion stream and any gas molecules which may be present in the tube.

(3) The collimating system. The ion beam is made effectively plane-parallel by means of two horizontal slits about .04 by 2.5 millimeters separated by a distance of 40 centimeters. The important quantity from this system in so far as the theory of focussing is concerned is the angular spread of the ion

beam evaluated at the center of the system. This quantity is .04/200 = .0002 radians approximately. The tube containing the collimating system is 2 inches in diameter and is connected to ground.

The pressure in all parts of the apparatus beyond the cathode slit must be maintained at 10^{-5} millimeters of mercury or less as collisions of the ions with neutrals might very well invalidate some or all of the subsequent conclusions.

(4) The electrostatic field condenser. The plane-parallel plate condenser of Aston's design has been replaced by a cylindrical section condenser. This innovation deserves some comment. The advantages of this form are as follows:

(a) Since the potential of the condenser can be adjusted so that those ions possessing a median energy will be bent in a true circle concentric with the axis of the two cylindrical plates, the whole useable portion of the beam of ions will have nearly the same curvature as this median ray. Hence the condenser gap and the applied condenser potential can be reduced to the order of a third or fourth of their values for a corresponding plane-parallel plate condenser. The potential, of the order of 50 volts in this case, is easier to maintain constant.

(b)The errors due to edge effects are materially reduced. It is impossible theoretically to correct rigorously for edge effects for a condenser incorporated into a vacuum apparatus. Since the vacuum system is in large part of metal at the potential of the grounded condenser plate it in effect forms part of the condenser. For an isolated condenser with a small gap the edge effects are directly proportional to the gap width. For larger gap widths the edge effects depend on higher powers of the width. Therefore an arrangement which allows the gap to be narrowed by an appreciable amount produces a more than corresponding improvement in the accuracy of the energy analyzer.

(c) It turns out from the analysis that the field due to a cylindrical section condenser not only separates a linearly collimated beam of positive ions into an energy spectrum but also effectively concentrates that beam about the median ray (which is the ray following a true circle in the condenser gap). Ions with higher energy than the median find that they must oppose a potential gradient and thus lose energy, while those with lower energy than the median go with the potential gradient and thus gain energy. The same effect in a smaller degree is shown by the plane-parallel plate condenser. It may be shown that both of these systems can be replaced by point virtual sources together with certain energy spreads. (See F.W. Aston and R.H. Fowler, Phil. Mag., 1922 for the theory in the case of the plane-parallel plate condenser). It is further shown here that the cylindrical section condenser has a larger concentrating effect on the ionic beam because its point virtual image lies considerably behind the corresponding point for the plane-parallel plate condenser.

To offset, in part, these advantages the theory of the inverse first power field is very much more complicated than that of the uniform field. The complexity of the results also restrict the shape of the magnetic pole faces to rather specialized forms. It so happens however that the corrections for edge effects in the magnetic field also demand these very same forms.

The form of the condenser plates may be expressed either as that of an axial sector of a cylindrical shell or of a sector of an annular pipe. The height of the cylinder is 1/2 inch. The sector subtends an angle of 1/4 radian from the center of the cylinder. The outer and inner radii of the outer plate are 12.9 and 12.1 centimeters respectively. The outer and inner radii of the inner plate are 11.9 and 10.3 centimeters respectively. The gap between the plates is thus 2 millimeters.

The inner plate is connected directly to the body of the apparatus and hence is at ground potential. The outer plate is supported by means of insulating spacers and can be maintained at a fixed positive potential. The constants of the condenser, such as its capacity, must be determined, if needed, by experiment since their values when incorporated into the apparatus may be slightly different from their values when the condenser is isolated from other bodies. It is assumed that the spacers are very accurately machined and are not subject to shrinkage in the vacuum. However a small error in the spacing should not greatly affect the uniformity in the field between the condenser plates. The radii of the condenser plates enter into the equations only through the logarithm of their ratio so that the equations will be very insensitive to slight changes in the radii.

The edge errors are of two kinds. First, there is an error made in assuming no field outside the curved portions of the gap. The ions which reach the focus never come near enough to these edges to be appreciably affected by the non-uniformity of field there. These edges may cause a discrepancy between the effective potential of the field and the actual potential applied; but as the effective potential is to be determined in the last analysis by experiment, it will cause no difficulty. Second, there is an edge error made by assuming that the field is strictly radial at the straight edges of the gap and vanishes outside the gap. This error affects the ion stream more directly than the first but certainly has only a small effect even when magnified by the shrinkage in the gap. In any case the effect can be calculated approximately by the use of a Schwarz transformation. The fact that the outer and inner plates of the condenser are respectively 4 and 8 gap widths in thickness makes it legitimate to use the Schwarz transformation corresponding to infinitely thick plates.

Still a further complication arise from the fact that at the trailing edge of the condenser only ions traveling along the median ray have their direction of motion always perpendicular to the field. At both leading and trailing edges the lines of force outside the condenser gap have atendency to bow away from the gap. At the leading edge this causes no complication for the beam is very narrowly confined and all ions move perpendicular to the field. At the trailing edge the part of the beam which comes close to the plates will be given an added and unpredicted deflection. This condition is remedied by the inception of a diaphragm before the magnetic field to remove these border ions from the measurements.

Since the apparatus possesses a vertical plane of symmetry containing the center of the thin ribbon of the positive ion stream in the collimating system, all calculations can be made in terms of the two polar or cartesian coordinates of this plane. Furthermore the collimation will be considered as perfect at the leading edge of the electric field; i.e. the positive ion stream will be represented as a mathematical straight line normal to the line of the leading edge of the sector and 12 centimeters removed from the axis of the sector. The path of an ion which travels through the electric field on a circle having this radius is, by definition, the median ray. This term will also be used for the corresponding straight line in the region of separation of the energy spectrum beyond the electric field. The deflection of the median ray from the direction of motion in the collimating system is therefore equal to the angle of the sector or 1/4 radian. The finite angular spread of the ionic beam in the collimating system will be taken into account in the subsequent discussion of the width of the virtual point source.

(5) Beyond the electric field the positive ion stream diverges as though it came from a point source located about 1.5 centimeters back of the trailing edge of the condenser as measured along the median ray. The divergence of a particular ion in the beam is determined by the value of mv^2/e for that

ion. The location of this virtual source and the direction of the extension of the median ray which passes through it determine a unique reference system for all subsequent considerations since a point and a direction determine a line uniquely in a two dimensional region.

In this same space before the magnetic field and after the electric field there has been incorporated into the apparatus a diaphragm which may be varied both as to width and to position in the ionic beam from outside the vacuum. This arrangement makes it possible to ascertain for what value of the condenser potential the maximum number of ions follow the median ray. It also enables one to test the theoretical conclusions from the study of the inverse first power field. In the final arrangement it will be used to prevent those ions which have passed too close to the condenser plates and thus have an undetermined deflection from continuing into the magnetic field.

(6) The magnetic pole pieces. The purpose of the magnetic field in an Aston-type mass-spectrograph is to convert the virtual point source of the electrostatic field which is independent of the specific mass into a real point focus dependent on the specific mass. No attempt is made or need be made in this design to have the focus sharp for other than that value of the specific mass which will be brought to a focus for a particular value of the magnetic field strength.

It is essential to this method of measuring the comparative amounts of ions with different specific masses that the magnetic field intensity be measured precisely each time the field is changed. A graphite crystal suspended between the magnetic poles from one arm of a torsion balance is used for this purpose. The crystal, which is diamagnetic, gives a force directly proportional to the square of the field intensity. A mirror mounted on the balance, symmetrical with its axis, reflects a beam of light to the same scale as is used for the deflections from the galvanometer in the recording circuit. Thus for each measurement of positive ion current a simultaneous value for the field intensity is obtained without disturbing the remainder of the apparatus. This arrangement establishes the necessary relation between field intensity and specific mass.

In a calculation of the shape of pole pieces and of the mass-dispersion the assumption is always made at first that the magnetic field is uniform inside the pole gap and cuts off sharply at the pole faces. Under this assumption the radius of the path of an ion is a constant for constant field strength and is in fact given by the product of the specific mass of the ion and the ratio of the velocity of the ion to the field strength. The purpose of this assumption is to so simplify the problem of focussing as to make it comparatively easy to calculate the shape of pole faces for various conditions which may appear desirable for experimental reasons.

The symmetry between the virtual source and the real focus suggests that the trailing edge of the magnetic pole pieces may well be taken to be simply the mirror image of the leading edge in the perpendicular bisector of the line joining the source with the focus. This bisector then contains the centers of curvature of the paths of the ions in the magnetic field. This arrangement very greatly simplifies the calculation of the shape required for the pole faces.

The magnetic median ray is, by definition, that extension of the electric median ray, in the magnetic field and in the space beyond the magnetic field, which will arrive at the focus for a fixed value of the magnetic field intensity. If the straight portion of the electric median ray is extended forward until it meets the backward extension of the magnetic median ray an unique reference point for the magnetic field is obtained which will be designated as <u>the center of magnetic field</u>. This center lies on the line of centers of curvature of the ions in the magnetic field and is 40 centimeters removed from both the virtual source and the focus.

The angle of deflection of the median ray in the magnetic field together with the magnetic field strength are the variables which may be adjusted so as to give the maximum mass- resolution. This angle of deflection may be chosen either inthe same direction as or opposite to the deflection in the electrostatic field and the pole faces will then be shaped so as to bring all ions with the same specific mass to the focus. The radius of curvature of the median ray is first chosen to correspond to such a value of the field intensity as to ensure that the maximum field strength derivable from the magnet will bring those ions having the maximum specific mass to the focus. After this radius is fixed a few points on each of the pole faces corresponding to several values both positive and negative for the deflection of the median ray in the magnetic field are calculated. Then graphically the paths of ions arriving at the leading edge with the same energies and positions as the ones which reach the focus but having specific masses ten percent greater and ten percent less than these are mapped out through the magnetic field and extended until they form their respective foci. The distance between these foci and the real focus is taken as a measure of the mass-resolution. It is thus found that for an electric deflection of 1/4 radian the mass-resolution is a maximum for a magnetic deflection of about 1 radian.in the opposite direction. Hence this value of 1 radian which is the same as for Aston's design will be maintained. The diagram of this work is not given in the subsequent theory.

A large number of points are then calculated on the curve for the pole faces for a magnetic deflection of one radian. This curve which still represents the situation for sharp cut off at the pole faces must then be altered to correspond to the actual condition of gradual cut off. In so doing corrections must be made for first and second order edge effects.

The first order edge effect is due simply to the fact that the magnetic field does not cut off sharply at the pole faces. If we started at the center of the gap and explored the field as we move away from the center it would be found that the field remained almost constant until within a gap width of the pole faces and that the field rapidly decreased for a short distance beyond the faces and then more slowly until the field is zero only at a very large distance. Hence the failure in sharp cut off gives an effective addition to the field. By the use of a Schwarz transformation the curve of field strength against distance measured along the normal to one of the pole faces can be obtained for essentially plane pole faces. From this curve the position can be determined for a hypothetical pole face which would for sharp cut off give the same integrated field as does the actual pole face with gradual cut off. Then by cutting from the actual pole face a depth equal to this distance along the normal to the pole face the first order edge effect can be fully corrected. A certain amount of judgment and pious hope is involved in this process because at a sufficiently large distance from the pole faces their slight curvature and the corners of the magnet may seriously alter the field from the idealized situation for which the calculations have been made. In this case the field beyond 20 centimeters from the pole faces has been simply neglected.

From the behavior of the first order edge effects for the magnetic field it can be seen that whereas here only a reduction in the pole faces will correct these effects the corresponding effects in the electrostatic field may be amply corrected by a reduction of the plate potential by a suitable amount as determined by experiment. The whole distinction arises from the fact that there is only a very slight variation in the lengths of the ionic paths in the electrostatic field while in the magnetic field there is a large variation in the lengths. This correction for the edge effects of the electric field by a change in potential is certainly subject to no more serious objections than would be the case if the corrections were done by reducing the length of the condenser by an equal amount on either edge and it materially improves the accuracy which can be obtained in the alignment of the condenser plates.

With sharp cut off it has been assumed that the path of the ion will be straight until the ion reaches the leading pole face and will then be a true circle until the ion reaches the trailing pole face when the path will again be straight until the focus is reached. In actuality the path of the ion will be curved from the time it leaves the condenser gap. A short distance behind the leading pole face the ion will follow a circular path of the desired radius but on account of the stray field outside the gap the path will be displaced both in angle and position. This condition would cause no difficulty if the effect before and near the leading edge were exactly cancelled by a similar effect near and after the trailing edge. A theoretical investigation was made of the arrangement necessary to effect this cancellation and it was found that this second order edge effect vanishes identically with the correction for the first order edge effect when the pole faces are mirror images of one another in the symmetry line and are essentially plane.

(7) The ions after leaving the magnetic field follow, neglecting edge effects, straight line trajectories tangent to their paths in the magnetic field at the trailing edge. The edge corrections for the pole faces are such as to insure that an ion after traversing a small distance beyond the trailing face will be following the same path as it would normally with no edge effects.

At a distance of 40 centimeters from the center of magnetic field as measured along the magnetic median ray a variable slit has been incorporated into the apparatus. Thus this slit occupies exactly the position obtained by reflecting the virtual source of the electrostatic field in the line of centers of the ionic paths in the magnetic field. Hence the virtual source, the magnetic field, and the focus form a symmetrical system.

(8) The collecting and measuring system. The ions which pass the variable slit are caught in a Faraday cage placed immediately behind the slit. This collector must have a small electrostatic capacity since any charging up of the collector would result in an increase in the background and in a time lag in the galvanometer response.

The positive ion current is amplified by means of a balanced Pliotron (General Electric FP-54) circuit which has a very high current amplification factor and a voltage amplification factor of unity. An Ayrton shunt is used in the circuit so that the smaller peaks and background may be examined very carefully at the highest sensitivity while a lower sensitivity may be used to record the larger peaks. The amplified current is recorded visually from the circuit galvanometer.

Thus it is possible to obviate very effectively the difficulty in determining the relation between the number of positive ions collected and the blackening of the photographic plate which Aston has found to be inherent in photographic methods of recording mass-spectra. With proper precautions it seems impossible that any method of recording mass-spectra could be freer from objections than that which measures the amplified positive ion current itself.

A complete description of the actual apparatus using the following theory will be found elsewhere. (See Dissertation of Dwight D. Taylor). The above description is adequate for the exigencies of the theory of this mass-spectrograph. The considerations here given will be taken into account in their proper sequence and without further specific mention.

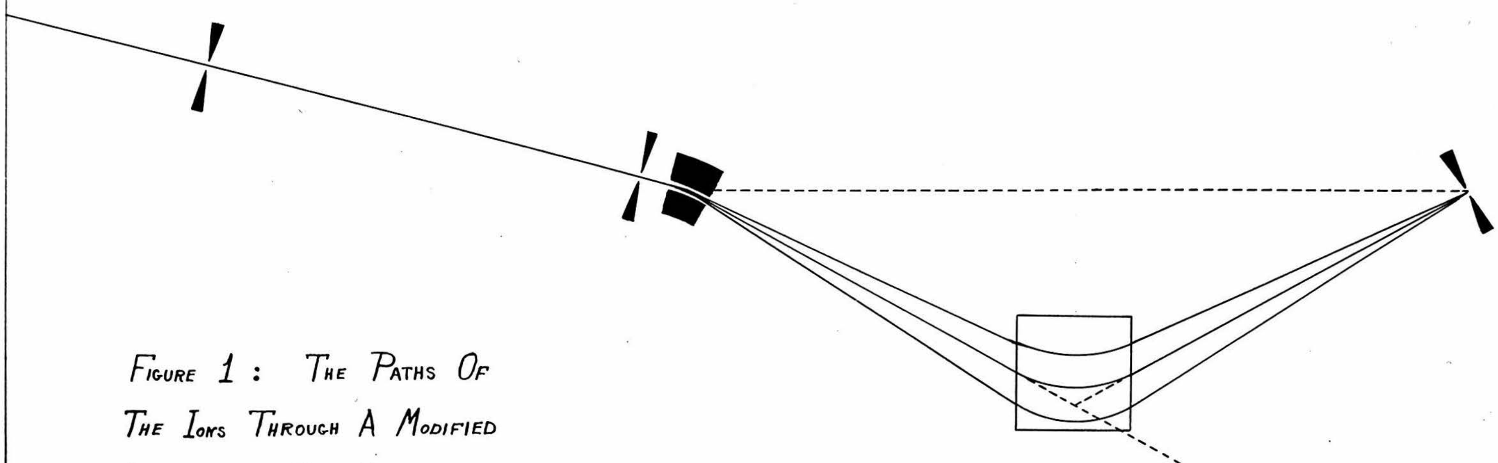

Figure 1: The Paths Of The Ions Through A Modified Aston-Type Mass-Spectrometer

II: THE SOLUTION IN THE ELECTROSTATIC FIELD

The center of the sector of the electrostatic field condenser will be chosen as the origin of a set of polar coordinates in the vertical symmetry plane, and the leading edge of the sector will be taken as the original line. The radius vector to any point in the gap will be designated by ρ, and φ will be the angle measured from the original line to the radius vector in a clockwise direction. The outer radius of the inner plate and the inner radius of the outer plate will be designated by ρ_1 and ρ_2 respectively. By a we shall mean the radius of the median ray. The angle subtended by the sector at the origin will be written as δ and the notation $(\varphi=\delta)$, written as a subscript, will indicate quantities which are to be evaluated at the trailing edge of the electric field.

If we may anticipate some results to appear in the following analysis, we may state that V, the velocity of emergence of the ions from the electric field, $\rho_{(\varphi=\delta)}$, and $\left(\frac{1}{\rho}\frac{d\rho}{d\varphi}\right)_{(\varphi=\delta)}$ are the most important quantities to be obtained for the subsequent developments. These quantities will be determined in terms of v, the initial velocity, m/e, the specific mass of an ion, Φ_0, the potential applied across the condenser plates , and of δ, ρ_1, ρ_2, and a, which have already been defined.

A negative line charge of E units per unit length gives rise to a radial field whose intensity I at a distance ρ from the line charge is given by $I = 2E/\rho$. The potential Φ of this field is

$$\Phi = -\int I \, d\rho = -\int \frac{2E}{\rho} d\rho = C - 2E \lg \rho$$

where C is an undetermined constant. The equipotentials of this field are given by $\rho =$ constant. If Φ_2 and Φ_1 are two values of the potential corresponding respectively to radii ρ_2 and ρ_1 we have since $\Phi_0 = \Phi_2 - \Phi_1$

$$\Phi_2 = C - 2E \lg \rho_2,$$

$$\Phi_1 = C - 2E \lg \rho_1,$$

$$\Phi_0 = \Phi_2 - \Phi_1 = -2E \lg \frac{\rho_2}{\rho_1};$$

hence

$$\Phi = \frac{\Phi_0}{\lg \frac{\rho_2}{\rho_1}} \lg \rho + C.$$

The potential energy P of a positive ion of charge e is given by

$$P = e\Phi = \frac{e\Phi_0}{\lg \frac{\rho_2}{\rho_1}} \lg \rho + C'$$

where C' is a constant depending on the zero of potential energy.

The kinetic energy T of an ion of mass m is given by

$$T = \frac{m}{2}\left\{\left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\varphi}{dt}\right)^2\right\}.$$

The constant total energy $\mathcal{H}$ in this conservative field is

$$\mathcal{H} = T + P = \frac{m}{2}\left\{\left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\varphi}{dt}\right)^2\right\} + \frac{e\Phi_0}{\lg \frac{\rho_2}{\rho_1}} \lg \rho + C'.$$

When $\varphi = 0$ and $\rho = a$, $\frac{d\rho}{dt} = 0$ and $\rho \frac{d\varphi}{dt} = v$, the velocity of the ion before the electric field. Hence we have

$$\mathcal{H} = \frac{m}{2} v^2 + \frac{e\Phi_0}{\lg \frac{\rho_2}{\rho_1}} \lg a + C'.$$

Therefore

$$\left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\varphi}{dt}\right)^2 - v^2 + \frac{2e\Phi_0}{m \lg \frac{\rho_2}{\rho_1}} \lg\frac{\rho}{a} = 0.$$

We shall make use of Lagrange's equations for the motion of the ions. The Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = T - P = \frac{m}{2}\left\{\left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\varphi}{dt}\right)^2\right\} - \frac{e\Phi_0}{\lg\frac{\rho_2}{\rho_1}} \lg\rho - C'.$$

The equation for ρ is, where dots designate differentiations with respect to the time,

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\rho}}\right) - \frac{\partial\mathcal{L}}{\partial\rho} = 0 \quad ;$$

$$m\frac{d^2\rho}{dt^2} - m\rho\left(\frac{d\varphi}{dt}\right)^2 + \frac{e\Phi_0}{\lg\frac{\rho_2}{\rho_1}} \cdot \frac{1}{\rho} = 0 \; ;$$

$$\frac{d^2\rho}{dt^2} = \rho\left(\frac{d\varphi}{dt}\right)^2 - \frac{e\Phi_0}{m \lg\frac{\rho_2}{\rho_1}} \frac{1}{\rho} . \tag{1}$$

The equation for φ is

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = 0 \; ;$$

$$\frac{d}{dt}\left(m\rho^2\frac{d\varphi}{dt}\right) = 0 \quad ;$$

$$m\rho^2\frac{d\varphi}{dt} = h$$

where h is the constant of angular momentum. When $\rho = a$ and $\varphi = 0$, $\frac{d\varphi}{dt} = \frac{v}{a}$ and hence $h = mav$. We are thus lead to

$$\rho^2\frac{d\varphi}{dt} = av. \tag{2}$$

The differential equation of the trajectory of the ion can be obtained as follows:

$$\frac{d\rho}{d\varphi} = \frac{\frac{d\rho}{dt}}{\frac{d\varphi}{dt}} \quad \text{or} \quad \frac{d\rho}{dt} = \frac{d\rho}{d\varphi}\frac{d\varphi}{dt} \; ; \tag{3}$$

$$\frac{d^2\rho}{d\varphi^2} = \frac{\frac{d}{dt}\left(\frac{d\rho}{d\varphi}\right)}{\frac{d\varphi}{dt}} = \frac{\frac{d^2\rho}{dt^2}\frac{d\varphi}{dt} - \frac{d\rho}{dt}\frac{d^2\varphi}{dt^2}}{\left(\frac{d\varphi}{dt}\right)^3} .$$

From (2) we have $2\rho \frac{d\rho}{dt} \frac{d\varphi}{dt} + \rho^2 \frac{d^2\varphi}{dt^2} = 0;$

$$\frac{d^2\varphi}{dt^2} = -\frac{2}{\rho}\frac{d\rho}{dt}\frac{d\varphi}{dt} = -\frac{2}{\rho}\frac{d\rho}{d\varphi}\left(\frac{d\varphi}{dt}\right)^2 .$$

We have $\frac{d^2\rho}{dt^2}$ given by (1) and $\frac{d\rho}{dt}$, by (3). Hence

$$\frac{d^2\rho}{d\varphi^2} = \frac{\rho\left(\frac{d\varphi}{dt}\right)^3 - \frac{e\Phi_0}{m \lg\frac{\rho_2}{\rho_1}} \frac{\left(\frac{d\varphi}{dt}\right)}{\rho} + \frac{2}{\rho}\left(\frac{d\rho}{d\varphi}\right)^2\left(\frac{d\varphi}{dt}\right)^3}{\left(\frac{d\varphi}{dt}\right)^3};$$

$$\frac{d^2\rho}{d\varphi^2} = \rho - \frac{e\Phi_0}{m \lg\frac{\rho_2}{\rho_1}} \frac{1}{\rho\left(\frac{d\varphi}{dt}\right)^2} + \frac{2}{\rho}\left(\frac{d\rho}{d\varphi}\right)^2 .$$

By (2) we have $\left(\frac{d\varphi}{dt}\right)^2 = \frac{a^2v^2}{\rho^4}$.

Therefore $$\frac{d^2\rho}{d\varphi^2} = \rho - \frac{e\Phi_0}{mv^2 \lg\frac{\rho_2}{\rho_1}} \frac{\rho^3}{a^2} + \frac{2}{\rho}\left(\frac{d\rho}{d\varphi}\right)^2. \quad (4)$$

Substitute $u = \frac{\rho}{a}$ in (4) and let $\kappa = \frac{e\Phi_0}{mv^2 \lg\frac{\rho_2}{\rho_1}}$;

then $$\frac{d^2u}{d\varphi^2} = u - \kappa u^3 + \frac{2}{u}\left(\frac{du}{d\varphi}\right)^2 . \quad (5)$$

This is <u>the differential equation of the trajectories of the ions in the electrostatic field</u>. It has no solution in terms of a finite number of known functions. Although it may have apparently simpler forms, it will be found on setting the boundary conditions to the approximations that these simplifications are illusory. For example, by the substitution $u = \frac{1}{\sqrt{\kappa}\, y}$ we obtain the equation

$$\frac{d^2y}{d\varphi^2} = \frac{1}{y} - y .$$

κ no longer appears in the differential equation, but if we approximate to the solution by means of Taylor's series it will immediately reappear in the resulting expression. Hence it is probably best to use (5) as it stands and to attempt the solution by means of a Taylor's series subject to the boundary conditions that at $\rho = a$ or $u = 1$, and $\varphi = 0$, $\frac{du}{d\varphi} = 0$. For this purpose it will be necessary to calculate the derivatives of u with respect to φ up to a sufficient order for our approximations. It is to be noticed that κ is a dimensionless quantity which will enter into our equations as the parameter of the family of trajectories in the electrostatic field. It will also occur as the parameter in the energy spectrum beyond the electric field and in the calculations for the magnetic field.

Before giving the successive derivatives a method of solution of the differential equation will be outlined which is so appealing to one approaching the equation for the first time as to be well nigh irresistible. The equation (5) can be written (if the values $u = \infty$ and $u = 0$ are excluded from the domain of u)

$$\frac{d}{d\varphi}\left(\frac{1}{u^2}\frac{du}{d\varphi}\right) = \frac{1}{u} - \kappa u .$$

The substitution $u = 1/v$ yields the equation

$$\frac{d^2v}{d\varphi^2} = \frac{\kappa}{v} - v .$$

Multiplying this equation by $2\frac{dv}{d\varphi}$ gives

$$2\frac{dv}{d\varphi}\frac{d^2v}{d\varphi^2} = 2\left(\frac{\kappa}{v} - v\right)\frac{dv}{d\varphi}$$

which on integration becomes

$$\left(\frac{dv}{d\varphi}\right)^2 = 2\kappa \lg v - v^2 + C.$$

The boundary conditions give C = 1 and we have

$$\frac{dv}{d\varphi} = \pm\sqrt{1 - v^2 + 2\kappa \lg v} .$$

Integrating again we obtain

$$\varphi - \varphi_0 = \pm\int_{v_0}^{v}\frac{dv}{\sqrt{1 - v^2 + 2\kappa \lg v}}$$

which evaluated at ($\varphi = \delta$) and under the above boundary conditions becomes

$$\delta = \pm \int_{1}^{v(\varphi=\delta)} \frac{dv}{\sqrt{1 - v^2 + 2\kappa \lg v}} .$$

This expression defines v as a function of δ and κ in a way similar to the definition of an elliptic function as the result of an inversion of an elliptic integral. In this case we are only interested in v for $\delta = 1/4$ radian and hence v is a function of κ alone. To solve the differential equation by means of this expression it would be necessary to plot the function under the integral sign as a function of v for a great number of values of κ. Then by means of a mechanical integration the value of v such as to give an area under the curve of 1/4 could be determined. This would be an exceedingly long and monotonous process requiring tremendous care to obtain a reasonable degree of accuracy. Moreover the exit position of an ion is not the most important quantity to be determined from the differential equation. Both the exit velocity and the angle of deflection of an ion in the field must be found from the analysis by a further calculation. The errors made in the extension of the calculations would be accumulative with the errors made in the calculation of the exit position. It is for this reason that the series method has been chosen for the solution of the equation.

The first 13 derivatives of u with respect to φ have been calculated and simplified by means of the original differential equation so that the expressions for them involve only u, κ, and the first derivative of u with respect to φ.

$$u' = u'$$

$$u'' = (u - \kappa u^3) + \frac{2}{u}(u')^2 \tag{5}$$

$$u''' = (5 - 7\kappa u^2)\, u' + \frac{6}{u^2}(u')^3 \tag{6}$$

$$u^{IV} = (5u - 12\kappa u^3 + 7\kappa^2 u^5) + \left(\frac{28}{u} - 46\kappa u\right)(u')^2 + \frac{24}{u^3}(u')^4 \tag{7}$$

$$u^{V} = (61 - 184\kappa u^2 + 127\kappa^2 u^4)\, u' + \left(\frac{180}{u^2} - 326\kappa\right)(u')^3 + \frac{120}{u^4}(u')^5 \tag{8}$$

$$u^{VI} = (61u - 245\kappa u^3 + 311\kappa^2 u^5 - 127\kappa^3 u^7) + \left(\frac{662}{u} - 2{,}254\kappa u + 1{,}740\kappa^2 u^3\right)(u')^2$$
$$+ \left(\frac{1{,}320}{u^3} - \frac{2{,}556}{u}\kappa\right)(u')^4 + \frac{720}{u^5}(u')^6 \tag{9}$$

$$u^{VII} = (1{,}385 - 6{,}567\kappa u^2 + 9{,}543\kappa^2 u^4 - 4{,}369\kappa^3 u^6)\, u' + \left(\frac{7{,}266}{u^2} - 26{,}774\kappa + 22{,}404\kappa^2 u^2\right)(u')^3$$
$$+ \left(\frac{10{,}920}{u^4} - \frac{22{,}212}{u^2}\kappa\right)(u')^5 + \frac{5{,}040}{u^6}(u')^7 \tag{10}$$

$$u^{VIII} = (1{,}385u - 7{,}952\kappa u^3 + 16{,}110\kappa^2 u^5 - 13{,}912\kappa^3 u^7 + 4{,}369\kappa^4 u^9)$$
$$+ \left(\frac{24{,}568}{u} - 128{,}388\kappa u + 204{,}792\kappa^2 u^3 - 102{,}164\kappa^3 u^5\right)(u')^2 + \left(\frac{83{,}664}{u^3} - \frac{326{,}304}{u}\kappa + 290{,}292\kappa^2 u\right)(u')^4$$
$$+ \left(\frac{100{,}800}{u^5} - \frac{212{,}976}{u^3}\kappa\right)(u')^6 + \frac{40{,}320}{u^7}(u')^8 \tag{11}$$

$$u^{IX} = (50{,}521 - 329{,}768\kappa u^2 + 746{,}910\kappa^2 u^4 - 711{,}296\kappa^3 u^6 + 243{,}649\kappa^4 u^8)\, u'$$
$$+ \left(\frac{408{,}360}{u^2} - 2{,}281{,}812\kappa + 3{,}899{,}928\kappa^2 u^2 - 2{,}080{,}644\kappa^3 u^4\right)(u')^3 + \left(\frac{1{,}023{,}120}{u^4} - \frac{4{,}166{,}784}{u^2}\kappa + 3{,}890{,}484\kappa^2\right)(u')$$
$$+ \left(\frac{1{,}028{,}160}{u^6} - \frac{2{,}239{,}344}{u^4}\kappa\right)(u')^7 + \frac{362{,}880}{u^8}(u')^9 \tag{12}$$

$$u^{X} = (50{,}521u - 380{,}289\kappa u^3 + 1{,}076{,}678\kappa^2 u^5 - 1{,}458{,}206\kappa^3 u^7 + 954{,}945\kappa^4 u^9 - 243{,}649\kappa^5 u^{11})$$
$$+ \left(\frac{1{,}326{,}122}{u} - 9{,}389{,}588\kappa u + 23{,}026{,}680\kappa^2 u^3 - 23{,}632{,}084\kappa^3 u^5 + 8{,}678{,}422\kappa^4 u^7\right)(u')^2$$
$$+ \left(\frac{6{,}749{,}040}{u^3} - \frac{39{,}640{,}392}{u}\kappa + 71{,}485{,}764\kappa^2 u - 40{,}258{,}860\kappa^3 u^3\right)(u')^4 \tag{13}$$
$$+ \left(\frac{13{,}335{,}840}{u^5} - \frac{56{,}206{,}800}{u^3}\kappa + \frac{54{,}580{,}248}{u}\kappa^2\right)(u')^6 + \left(\frac{11{,}491{,}200}{u^7} - \frac{25{,}659{,}360}{u^5}\kappa\right)(u')^8 + \frac{3{,}628{,}800}{u^9}(u')^{10}$$

$$u^{XI} = (2{,}702{,}765 - 22{,}572{,}287\kappa u^2 + 70{,}215{,}926\kappa^2 u^4 - 103{,}524{,}970\kappa^3 u^6 + 73{,}215{,}517\kappa^4 u^8 - 20{,}036{,}983\kappa^5 u^{10})(u')$$
$$+ \left(\frac{30{,}974{,}526}{u^2} - 232{,}505{,}668\kappa + 605{,}691{,}384\kappa^2 u^2 - 659{,}667{,}252\kappa^3 u^4 + 256{,}498{,}082\kappa^4 u^6\right)(u')^3$$
$$+ \left(\frac{113{,}760{,}240}{u^4} - \frac{694{,}738{,}584}{u^2}\kappa + 1{,}308{,}094{,}164\kappa^2 - 770{,}328{,}948\kappa^3 u^2\right)(u')^5 \tag{14}$$
$$+ \left(\frac{185{,}280{,}480}{u^6} - \frac{803{,}065{,}680}{u^4}\kappa + \frac{805{,}657{,}608}{u^2}\kappa^2\right)(u')^7 + \left(\frac{139{,}708{,}800}{u^8} - \frac{318{,}540{,}960}{u^6}\kappa\right)(u')^9 + \frac{39{,}916{,}800}{u^{10}}(u')$$

$$u^{XII} = \left(2{,}702{,}765u - 25{,}275{,}052Ku^3 + 92{,}788{,}213K^2u^5 - 173{,}740{,}896K^3u^7 + 176{,}740{,}487K^4u^9 - 93{,}252{,}500K^5u^{11}\right.$$
$$\left. + 20{,}036{,}983K^6u^{13}\right) + \left(\frac{98{,}329{,}108}{u} - 880{,}729{,}730Ku + 2{,}935{,}886{,}712K^2u^3 - 4{,}624{,}275{,}668K^3u^5\right.$$
$$\left. + 3{,}480{,}651{,}172K^4u^7 - 1{,}009{,}938{,}042K^5u^9\right)(u')^2 + \left(\frac{692{,}699{,}304}{u^3} - \frac{5{,}437{,}528{,}128}{u}K + 14{,}859{,}694{,}812K^2u\right.$$
$$\left. - 16{,}988{,}788{,}080K^3u^3 + 6{,}929{,}621{,}724K^4u^5\right)(u')^4 + \left(\frac{1{,}979{,}524{,}800}{u^5} - \frac{12{,}476{,}331{,}792}{u^3}K + \frac{24{,}342{,}004{,}656}{u}K^2\right.$$
$$\left. - 14{,}883{,}550{,}632K^3u\right)(u')^6 + \left(\frac{2{,}739{,}623{,}040}{u^7} - \frac{12{,}154{,}904{,}640}{u^5}K + \frac{12{,}534{,}759{,}936}{u^3}K^2\right)(u')^8$$
$$+ \left(\frac{1{,}836{,}172{,}800}{u^9} - \frac{4{,}261{,}576{,}320}{u^7}K\right)(u')^{10} + \frac{479{,}001{,}600}{u^{11}}(u')^{12} \qquad (15)$$

$$u^{XIII} = \left(199{,}360{,}981 - 2{,}033{,}942{,}832Ku^2 + 8{,}097{,}173{,}949K^2u^4 - 16{,}336{,}511{,}032K^3u^6 + 17{,}800{,}518{,}063K^4u^8\right.$$
$$\left. - 10{,}006{,}955{,}928K^5u^{10} + 2{,}280{,}356{,}863K^6u^{12}\right)u' + \left(\frac{3{,}065{,}784{,}540}{u^2} - 28{,}943{,}234{,}378K + 101{,}740{,}098{,}744K^2u^2\right.$$
$$\left. - 169{,}012{,}412{,}580K^3u^4 + 133{,}960{,}802{,}108K^4u^6 - 40{,}847{,}681{,}442K^5u^8\right)(u')^3 + \left(\frac{15{,}340{,}645{,}320}{u^4} - \frac{124{,}797{,}836{,}448}{u^2}K\right.$$
$$\left. + 354{,}647{,}271{,}996K^2 - 422{,}230{,}000{,}608K^3u^2 + 179{,}386{,}386{,}204K^4u^4\right)(u')^5 + \left(\frac{35{,}773{,}657{,}920}{u^6} - \frac{231{,}443{,}207{,}568}{u^4}K\right.$$
$$\left. + \frac{465{,}279{,}367{,}824}{u^2}K^2 - 293{,}764{,}237{,}704K^3\right)(u')^7 + \left(\frac{43{,}018{,}335{,}360}{u^8} - \frac{194{,}681{,}442{,}240}{u^6}K + \frac{205{,}567{,}642{,}368}{u^4}K^2\right)(u')$$
$$+ \left(\frac{25{,}945{,}920{,}000}{u^{10}} - \frac{61{,}148{,}511{,}360}{u^8}K\right)(u')^{11} + \frac{6{,}227{,}020{,}800}{u^{12}}(u')^{13} \qquad (16)$$

The first twelve derivatives have been carefully verified by calculating them from the original differential equation (5) without using this equation to reduce the order of the derivatives on the right. Then for each derivative the order of derivatives on the right is reduced to the first order by means of re-substitution of the preceding derivatives. The derivatives are thus verified by a method which is as free as possible from the bias of known results for the calculations. The results of this calculation are very much more involved than the above derivatives and will not be given.

It is to be noted that only the first parenthesis of the even-order derivatives remain after the boundary condition $\frac{du}{d\varphi} = 0$ for $\varphi = 0$ has been applied and that for $u = 1$ these first terms can be obtained in each case by multiplying the first parenthesis of the previous odd-order derivative by $(1-\mathcal{K})$. Hence the 13th derivative gives quite straight-forwardly the contribution of the 14th to the Taylor's series.

The series expansion for u is of the form

$$u = (u)_{(\varphi=0)} + \sum_{n=1}^{n=14} \left(\frac{d^n u}{d\varphi^n}\right)_{(\varphi=0)} \cdot \frac{\varphi^n}{n!} + R$$

which becomes, for $\frac{du}{d\varphi} = 0$ at $\varphi = 0$ and u = 1, the series

$$\begin{aligned} u = 1 + \frac{(1-\mathcal{K})}{2}\varphi^2 \Big\{ & 1 + \frac{(5-7\mathcal{K})}{2^2 \cdot 3}\varphi^2 + \frac{(61-184\mathcal{K}+127\mathcal{K}^2)}{2^3 \cdot 3^2 \cdot 5}\varphi^4 + \frac{(1{,}385-6{,}567\mathcal{K}+9{,}543\mathcal{K}^2-4{,}369\mathcal{K}^3)}{2^6 \cdot 3^2 \cdot 5 \cdot 7}\varphi^6 \\ & + \frac{(50{,}521-329{,}768\mathcal{K}+746{,}910\mathcal{K}^2-711{,}296\mathcal{K}^3+243{,}649\mathcal{K}^4)}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7}\varphi^8 + \frac{(2{,}702{,}765-22{,}572{,}287\mathcal{K}+70{,}215{,}926\mathcal{K}^2-103{,}524{,}970\mathcal{K}^3+73{,}215{,}517\mathcal{K}^4-20{,}036{,}983\mathcal{K}^5)}{2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11}\varphi^{10} \\ & + \frac{(199{,}360{,}981-2{,}033{,}942{,}832\mathcal{K}+8{,}097{,}173{,}949\mathcal{K}^2-16{,}336{,}511{,}032\mathcal{K}^3+17{,}800{,}518{,}063\mathcal{K}^4-10{,}006{,}955{,}928\mathcal{K}^5+2{,}280{,}356{,}863\mathcal{K}^6)}{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13}\varphi^{12} \Big\} + R \end{aligned} \tag{17}$$

where in each case R is the remainder after 14 terms.

For the purposes of the following calculations we are interested in the value of u only for $(\varphi = \delta)$ at the trailing edge of the electrostatic field condenser. We are interested in u as a power series in $\mathcal{K}$ rather than as a power series in φ. The denominators of the terms in (17) and (18) have been factored into powers of their prime number constituents for ease in cancellations. These operations lead to

$$\begin{aligned} (u)_{(\varphi=\delta)} = 1 + (1-\mathcal{K}) \Big\{ & \left[\frac{\delta^2}{2} + \frac{5\delta^4}{2^3 \cdot 3} + \frac{61\delta^6}{2^4 \cdot 3^2 \cdot 5} + \frac{277\delta^8}{2^7 \cdot 3^2 \cdot 7} + \frac{50{,}521\delta^{10}}{2^8 \cdot 3^4 \cdot 5^2 \cdot 7} + \frac{540{,}553\delta^{12}}{2^{10} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11} + \frac{199{,}360{,}981\delta^{14}}{2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} + \cdots\right] \\ & - \delta^4\left[\frac{7}{2^3 \cdot 3} + \frac{23\delta^2}{2 \cdot 3^2 \cdot 5} + \frac{2{,}189\delta^4}{2^7 \cdot 3 \cdot 5 \cdot 7} + \frac{41{,}221\delta^6}{2^5 \cdot 3^4 \cdot 5^2 \cdot 7} + \frac{22{,}572{,}287\delta^8}{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11} + \frac{4{,}708{,}201\delta^{10}}{2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} + \cdots\right]\mathcal{K} \\ & + \delta^6\left[\frac{127}{2^4 \cdot 3^2 \cdot 5} + \frac{3{,}181\delta^2}{2^7 \cdot 3^2 \cdot 5 \cdot 7} + \frac{8{,}299\delta^4}{2^7 \cdot 3^2 \cdot 5 \cdot 7} + \frac{3{,}191{,}633\delta^6}{2^9 \cdot 3^5 \cdot 5^2 \cdot 7} + \frac{2{,}699{,}057{,}983\delta^8}{2^{11} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} + \cdots\right]\mathcal{K}^2 \\ & - \delta^8\left[\frac{4{,}369}{2^7 \cdot 3^2 \cdot 5 \cdot 7} + \frac{5{,}557\delta^2}{2 \cdot 3^4 \cdot 5^2 \cdot 7} + \frac{10{,}352{,}497\delta^4}{2^9 \cdot 3^5 \cdot 5 \cdot 7 \cdot 11} + \frac{2{,}042{,}063{,}879\delta^6}{2^8 \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} + \cdots\right]\mathcal{K}^3 \\ & + \delta^{10}\left[\frac{34{,}807}{2^8 \cdot 3^4 \cdot 5^2} + \frac{73{,}215{,}517\delta^2}{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11} + \frac{5{,}933{,}506{,}021\delta^4}{2^{11} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} + \cdots\right]\mathcal{K}^4 \\ & - \delta^{12}\left[\frac{20{,}036{,}983}{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11} + \frac{138{,}985{,}499\delta^2}{2^8 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} + \cdots\right]\mathcal{K}^5 \\ & + \delta^{14}\left[\frac{2{,}280{,}356{,}863}{2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} + \cdots\right]\mathcal{K}^6 - \cdots \Big\} \end{aligned} \tag{18}$$

where all the simple cancellations possible have been made. In the further use of this equation it will always be written

$$u = 1 + (1-\kappa)\cdot f(\delta,\kappa)$$

which in the case $\delta = 1/4$ will be abbreviated to

$$u = 1 + (1-\kappa)\cdot f(\kappa). \qquad (19)$$

The velocity of exit of the ions from the electrostatic field may be obtained as follows: If we let **V** be the velocity of an ion at any point in the electrostatic field then by the use of the first equation on page 20. we obtain the relation

$$\mathbf{V}^2 = \left\{\left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\varphi}{dt}\right)^2\right\} = v^2\left\{1 - \frac{ze\Phi_0}{mv^2 \lg\frac{\rho_2}{\rho_1}} \lg u\right\}$$

or

$$\mathbf{V}^2 = v^2\left\{1 - 2\kappa \lg u\right\} . \qquad (20)$$

We have thus reduced the problem to a determination of lg u. A fastly converging series for lg u in the neighborhood of u = 1 is the series

$$\lg u = \sum_{n=1}^{n=\infty} \frac{(-1)^{n-1}}{n}(u-1)^n \qquad 2>u>0 .$$

Equation (17) gives us directly the value of (u–1) to be substituted in this equation. If we retain only those terms in this expansion which contain powers of φ up to and including the 14th power the resulting expression for $\mathbf{V}^2$ is

$$\begin{aligned}\mathbf{V}^2 = v^2 \Big\{ 1 - \kappa(1-\kappa)\varphi^2 \Big[1 &+ \frac{(1-2\kappa)}{2\cdot 3}\varphi^2 + \frac{(4-16\kappa+13\kappa^2)}{2\cdot 3^2\cdot 5}\varphi^4 + \frac{(34-204\kappa+345\kappa^2-176\kappa^3)}{2^3\cdot 3^2\cdot 5\cdot 7}\varphi^6 \qquad (21)\\ &+ \frac{(496-3{,}968\kappa+10{,}350\kappa^2-10{,}946\kappa^3+4{,}069\kappa^4)}{2^3\cdot 3^4\cdot 5^2\cdot 7}\varphi^8 + \frac{(11{,}056-110{,}560\kappa+392{,}326\kappa^2-639{,}926\kappa^3+490{,}511\kappa^4-143{,}408\kappa^5)}{2^4\cdot 3^5\cdot 5^2\cdot 7\cdot 11}\varphi^{10}\\ &+ \frac{(349{,}504-4{,}194{,}048\kappa+18{,}886{,}536\kappa^2-41{,}988{,}508\kappa^3+49{,}501{,}332\kappa^4-29{,}712{,}792\kappa^5+7{,}157{,}977\kappa^6)}{2^4\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13}\varphi^{12}\Big]\Big\} + R\end{aligned}$$

If we now rearrange this series in ascending powers of κ rather than of φ, set $(\varphi = \delta)$, and make all possible cancellations we obtain the square of the exit velocity in the form

$$
\begin{aligned}
V^2_{(\varphi=\delta)} = v^2 \Bigg\{ 1 - \kappa(1-\kappa) \Bigg[& \left\{ \delta^2 + \frac{\delta^4}{2\cdot 3} + \frac{2\delta^6}{3^2\cdot 5} + \frac{17\delta^8}{2^2\cdot 3^2\cdot 5\cdot 7} + \frac{2\cdot 31\,\delta^{10}}{3^4\cdot 5^2\cdot 7} + \frac{691\,\delta^{12}}{3^5 5^2\cdot 7\cdot 11} + \frac{2^2\cdot 5{,}461\,\delta^{14}}{3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13} + \cdots \right\} \\
& - \delta^4 \left\{ \frac{1}{3} + \frac{2^3\delta^2}{3^2\cdot 5} + \frac{17\delta^4}{2\cdot 3\cdot 5\cdot 7} + \frac{2^4\cdot 31\,\delta^6}{3^4\cdot 5^2\cdot 7} + \frac{2\cdot 691\,\delta^8}{3^5\cdot 5\cdot 7\cdot 11} + \frac{2^4\cdot 5{,}461\,\delta^{10}}{3^4\cdot 5^2\cdot 7^2\cdot 11\cdot 13} + \cdots \right\} \kappa \\
& + \delta^6 \left\{ \frac{13}{2\cdot 3^2\cdot 5} + \frac{23\delta^2}{2^3\cdot 3\cdot 7} + \frac{23\delta^4}{2^2\cdot 3^2\cdot 7} + \frac{193{,}163\,\delta^6}{2^2\cdot 3^5\cdot 5^2\cdot 7\cdot 11} + \frac{262{,}313\,\delta^8}{2\cdot 3^3\cdot 5^2\cdot 7^2\cdot 11\cdot 13} + \cdots \right\} \kappa^2 \\
& - \delta^8 \left\{ \frac{2\cdot 11}{3^2\cdot 5\cdot 7} + \frac{5{,}473\,\delta^2}{2\cdot 3^4\cdot 5^2\cdot 7} + \frac{45{,}709\,\delta^4}{2^3\cdot 3^5\cdot 5^2\cdot 11} + \frac{10{,}497{,}127\,\delta^6}{2^2\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13} + \cdots \right\} \kappa^3 \\
& + \delta^{10} \left\{ \frac{4{,}069}{2^3\cdot 3^4\cdot 5^2\cdot 7} + \frac{70{,}073\,\delta^2}{2^4\cdot 3^5\cdot 5^2\cdot 11} + \frac{1{,}375{,}037\,\delta^4}{2^2\cdot 3^3\cdot 5^2\cdot 7^2\cdot 11\cdot 13} + \cdots \right\} \kappa^4 \\
& - \delta^{12} \left\{ \frac{8{,}963}{3^5\cdot 5^2\cdot 7\cdot 11} + \frac{1{,}238{,}033\,\delta^2}{2\cdot 3^4\cdot 5^2\cdot 7^2\cdot 11\cdot 13} + \cdots \right\} \kappa^5 \\
& + \delta^{14} \left\{ \frac{7{,}157{,}977}{2^4\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13} + \cdots \right\} \kappa^6 - \cdots \Bigg] \Bigg\}
\end{aligned}
\tag{22}
$$

For all future discussion this expression will be written either as

$$
\left.
\begin{aligned}
V^2 &= v^2 \left\{ 1 - \kappa(1-\kappa)\, \Phi(\delta,\kappa) \right\} \quad \text{for } \varphi = \delta \\
V^2 &= v^2 \left\{ 1 - \kappa(1-\kappa)\, \Phi(\kappa) \right\} \quad \text{for } \delta = 1/4.
\end{aligned}
\right\} \tag{23}
$$

or as (second line above), where the function Φ is defined by this equation and is not to be confused with the electrostatic potential.

There are at least three ways in which we may calculate the quantity $\frac{1}{\rho}\frac{d\rho}{d\varphi}$ or its equal $\frac{1}{u}\frac{du}{d\varphi}$. (1) The simplest way would be to simply differentiate the expression we have obtained for lg u in the determination of V^2. This avenue has not been used since, after $\frac{1}{u}\frac{du}{d\varphi}$ is calculated in another way, we may use the integration of this expression as a check on the correctness of the expression for lg u. The expression for V^2 has been checked in this way. (2) The quantity $\frac{1}{u}\frac{du}{d\varphi}$ may be calculated from the successive differentiation of the quantity itself and the use of an appropriate Taylor's series subject to the same boundary conditions as were used in the calculation of u. This method has been used. The intermediate steps, which are very complicated, will not be given. (3) The first equation on page 20., taken in conjunction with equations (2) and (3) of that page for the elimination of $\left(\frac{d\varphi}{dt}\right)$, leads to

$$\left(\frac{d\rho}{dt}\right)^2 = v^2\left\{\left(1-\frac{a^2}{\rho^2}\right) - 2\,\frac{e\Phi_0}{mv^2 \lg\frac{\rho_2}{\rho_1}}\lg\frac{\rho}{a}\right\} = v^2\left\{\left(1-\frac{1}{u^2}\right) - 2\kappa\lg u\right\}$$

But

$$\left(\frac{d\rho}{dt}\right) = \left(\frac{d\rho}{d\varphi}\right)\left(\frac{d\varphi}{dt}\right) = \left(\frac{d\rho}{d\varphi}\right)\frac{av}{\rho^2} = \frac{v}{au^2}\left(\frac{d\rho}{d\varphi}\right) = \frac{v}{u^2}\left(\frac{du}{d\varphi}\right)$$

$$\frac{v^2}{u^4}\left(\frac{du}{d\varphi}\right)^2 = v^2\left\{\left(1-\frac{1}{u^2}\right) - 2\kappa\lg u\right\}$$

$$\frac{1}{\rho^2}\left(\frac{d\rho}{d\varphi}\right)^2 = \frac{1}{u^2}\left(\frac{du}{d\varphi}\right)^2 = \left\{u^2 - 1 - 2\kappa u^2\lg u\right\}$$

and finally we have

$$\frac{1}{u}\frac{du}{d\varphi} = \left\{u^2 - 1 - 2\kappa u^2\lg u\right\}^{1/2} \tag{24}$$

From equations (17) and (21) we have expressions for u and $2\kappa\lg u$. Hence we may calculate u^2-1 and $2\kappa u^2\lg u$ from these and substitute into the equation for $\frac{1}{u}\frac{du}{d\varphi}$. The extraction of the square root by means of the binomial theorem then yields the expression

$$\begin{aligned}\frac{1}{u}\frac{du}{d\varphi} = (1-\kappa)\varphi\Big\{1 &+ \frac{(1-2\kappa)}{3}\varphi^2 + \frac{(4-16\kappa+13\kappa^2)}{2\cdot3\cdot5}\varphi^4 + \frac{(34-204\kappa+345\kappa^2-176\kappa^3)}{2\cdot3^2\cdot5\cdot7}\varphi^6 \\ &+ \frac{(496-3{,}968\kappa+10{,}350\kappa^2-10{,}946\kappa^3+4{,}069\kappa^4)}{2^3\cdot3^4\cdot5\cdot7}\varphi^8 + \frac{(11{,}056-110{,}560\kappa+392{,}326\kappa^2-639{,}926\kappa^3+490{,}511\kappa^4-143{,}408\kappa^5)}{2^3\cdot3^4\cdot5^2\cdot7\cdot11}\varphi^{10} \\ &+ \frac{(349{,}504-4{,}194{,}048\kappa+18{,}886{,}536\kappa^2-41{,}988{,}508\kappa^3+49{,}501{,}332\kappa^4-29{,}712{,}792\kappa^5+7{,}157{,}977\kappa^6)}{2^4\cdot3^5\cdot5^2\cdot7\cdot11\cdot13}\varphi^{12}\Big\} + R\end{aligned} \tag{25}$$

Again we are only interested in this series for the trailing edge of the electrostatic field condenser where $(\varphi=\delta)$. After this series has been rearranged in powers of κ rather than of φ and all simple cancellations made, the resulting expression is

$$\left(\frac{1}{u}\frac{du}{d\varphi}\right)_{(\varphi=\delta)} = (1-\mathcal{K})\Bigg\{\left[\delta + \frac{\delta^3}{3} + \frac{2\delta^5}{3\cdot 5} + \frac{17\delta^7}{3^2\cdot 5\cdot 7} + \frac{2\cdot 31\,\delta^9}{3^4\cdot 5\cdot 7} + \frac{2\cdot 691\,\delta^{11}}{3^4\cdot 5^2\cdot 7\cdot 11} + \frac{2^2\cdot 43\cdot 127\,\delta^{13}}{3^5\cdot 5^2\cdot 7\cdot 11\cdot 13} + \cdots\right]$$
$$- \delta^3\left[\frac{2}{3} + \frac{2^3\delta^2}{3\cdot 5} + \frac{2\cdot 17\,\delta^4}{3\cdot 5\cdot 7} + \frac{2^4\cdot 31\,\delta^6}{3^4\cdot 5\cdot 7} + \frac{2^2\cdot 691\,\delta^8}{3^4\cdot 5\cdot 7\cdot 11} + \frac{2^4\cdot 43\cdot 127\,\delta^{10}}{3^4\cdot 5^2\cdot 7\cdot 11\cdot 13} + \cdots\right]\mathcal{K}$$
$$+ \delta^5\left[\frac{13}{2\cdot 3\cdot 5} + \frac{23\,\delta^2}{2\cdot 3\cdot 7} + \frac{5\cdot 23\,\delta^4}{2^2\cdot 3^2\cdot 7} + \frac{17{,}833\,\delta^6}{2^2\cdot 3^4\cdot 5^2\cdot 7} + \frac{262{,}313\,\delta^8}{2\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 13} + \cdots\right]\mathcal{K}^2$$
$$- \delta^7\left[\frac{2^3\cdot 11}{3^2\cdot 5\cdot 7} + \frac{5{,}473\,\delta^2}{2^2\cdot 3^4\cdot 5\cdot 7} + \frac{45{,}709\,\delta^4}{2^2\cdot 3^4\cdot 5^2\cdot 11} + \frac{10{,}497{,}129\,\delta^6}{2^2\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13} + \cdots\right]\mathcal{K}^3$$
$$+ \delta^9\left[\frac{4{,}069}{2^3\cdot 3^4\cdot 5\cdot 7} + \frac{70{,}073\,\delta^2}{2^3\cdot 3^4\cdot 5^2\cdot 11} + \frac{1{,}375{,}037\,\delta^4}{2^2\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 13} + \cdots\right]\mathcal{K}^4$$
$$- \delta^{11}\left[\frac{2\cdot 8{,}963}{3^4\cdot 5^2\cdot 7\cdot 11} + \frac{1{,}238{,}033\,\delta^2}{2\cdot 3^4\cdot 5^2\cdot 7\cdot 11\cdot 13} + \cdots\right]\mathcal{K}^5$$
$$+ \delta^{13}\left[\frac{7{,}157{,}977}{2^4\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13} + \cdots\right]\mathcal{K}^6 - \cdots\Bigg\} \qquad (26)$$

which will be written in the subsequent discussion as either

$$\frac{1}{u}\frac{du}{d\varphi} = (1-\mathcal{K})\,F(\delta,\mathcal{K}) \qquad \text{for } \varphi = \delta$$

or

$$\frac{1}{u}\frac{du}{d\varphi} = (1-\mathcal{K})\,F(\mathcal{K}) \qquad \text{for } \delta = 1/4. \qquad (27)$$

The quantity $\frac{1}{u}\frac{du}{d\varphi}$ is, with the appropriate convention as to its sign, exactly the tangent of the angle which the tangent to the path of any ion at the trailing edge of the electrostatic field makes with the tangent to the median ray taken at the same edge.

The existence, uniqueness, continuity, and uniform convergence of these series expressions for u, V^2, and $\frac{1}{u}\frac{du}{d\varphi}$ have been thoroughly investigated and are given in the Mathematical Appendix to this thesis. The results of this investigation have been incorporated into the theory to follow in their appropriate places.

THE VIRTUAL POINT SOURCE AND THE ENERGY SPECTRUM BEYOND THE ELECTROSTATIC FIELD

Consider for a moment the original differential equation

$$\frac{d^2u}{d\varphi^2} = u - \kappa u^3 + \frac{2}{u}\left(\frac{du}{d\varphi}\right)^2 \qquad (5)$$

When $\kappa = 0$ this differential equation can be written in the form

$$\frac{d}{d\varphi}\left[\frac{1}{u^2}\frac{du}{d\varphi}\right] = \frac{1}{u}$$

Now let $v = 1/u$; $\frac{dv}{d\varphi} = -\frac{1}{u^2}\frac{du}{d\varphi}$. The equation now takes the form

$$\frac{d^2v}{d\varphi^2} = -v$$

which has for its solution

$$v = C\cos(\varphi+\alpha) .$$

For $\varphi = 0$ and $u=v=1$, $\frac{du}{d\varphi} = \frac{dv}{d\varphi} = 0$, we have

$$v = \cos\varphi .$$

Therefore we have the solutions for our three quantities

$$u = 1/v = \sec\varphi ;$$

$$\lg u = \lg \sec\varphi ;$$

$$\frac{1}{u}\frac{du}{d\varphi} = \tan\varphi .$$

Hence for $\kappa = 0$ and $\varphi = \delta$, we will have

$$(u)_{(\varphi=\delta)} = \sec\delta ;$$

$$(\lg u)_{(\varphi=\delta)} = \lg \sec\delta ;$$

$$\left(\frac{1}{u}\frac{du}{d\varphi}\right)_{(\varphi=\delta)} = \tan\delta .$$

When the series for u, lg u, and $\frac{1}{u}\frac{du}{d\varphi}$ are found in terms of ascending powers of κ, they must reduce to the above values for $\kappa = 0$. It will develop subsequently that it is this term, which remains when $\kappa = 0$, which contributes most to the series so long as κ is positive and not very large (say $\kappa < 2$).

Hence we are justified in taking some pains to calculate sec δ, lg sec δ, and tan δ to a rather high order of accuracy for the value $\delta = 1/4$ radian. For this purpose we will use the appropriate power series. According to Peirce, A Short Table of Integrals, formulae 776, 783, and 774 the desired series are respectively

$$\sec \delta = \sum_{m=1}^{m=\infty} \frac{B_{2m}\delta^{2m}}{(2m)!} + 1 \qquad \left[\delta^2 < \frac{\pi^2}{4}\right]$$

$$\lg \sec \delta = \sum_{m=1}^{m=\infty} \frac{2^{2m-1}(2^{2m}-1)B_{2m-1}}{m(2m)!}\delta^{2m} \qquad \left[\delta^2 < \frac{\pi^2}{4}\right]$$

$$\tan \delta = \sum_{m=1}^{m=\infty} \frac{2^{2m}(2^{2m}-1)B_{2m-1}}{(2m)!}\delta^{2m-1} \qquad \left[\delta^2 < \frac{\pi^2}{4}\right]$$

The B's of odd order occurring in the last two expressions are known as Bernoulli's Numbers. A relation is known which will yield B_{2m+1} provided B_{2m-1} and the B's of lower indices are known. The first 62 have been calculated by J.C. Adams in the Journal für die reine und angewandte Mathematik, Volume 85, (1878). The first 10 which will be used in these calculations are as follows: 1/6; 1/30; 1/42; 1/30; 5/66; 691/2,730; 7/6; 3,617/510; 43,867/798; 174,611/330.

The B's of even order occurring in the first expression are known as Euler's Numbers. A great many of them have been calculated by W. Scherk and published in Mathematische Abhandlung, Berlin, 1825. This journal is not within my reach at present. The first 6 are given in Peirce, page 90, together with a formula for obtaining the higher members. The next 4 have been calculated by means of this formula. The first 10 Euler's Numbers are as follows: 1; 5; 61; 1,385; 50,521; 2,702,765; 199,360,981; 19,391,512,145; 2,404,879,675,441; 370,371,188,237,525.

If we now calculate the coefficient of the powers of δ for each term in each of these series, simplify the coefficients by factoring them into their prime constituents, and cancell wherever possible we will obtain the following three series:

$$\sec\delta = 1 + \frac{\delta^2}{2} + \frac{5\delta^4}{2^3 \cdot 3} + \frac{61\delta^6}{2^4 \cdot 3^2 \cdot 5} + \frac{277\delta^8}{2^7 \cdot 3^2 \cdot 7} + \frac{50{,}521\delta^{10}}{2^8 \cdot 3^4 \cdot 5^2 \cdot 7} + \frac{540{,}553\delta^{12}}{2^{10} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11} + \frac{199{,}360{,}981\delta^{14}}{2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13}$$
$$+ \frac{17 \cdot 228{,}135{,}437\delta^{16}}{2^{15} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} + \frac{2{,}404{,}879{,}675{,}441\delta^{18}}{2^{16} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17} + \frac{14{,}814{,}847{,}529{,}501\delta^{20}}{2^{18} \cdot 3^8 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19} + \cdots$$

$$\lg\sec\delta = \frac{\delta^2}{2} + \frac{\delta^4}{2^2 \cdot 3} + \frac{\delta^6}{3^2 \cdot 5} + \frac{17\delta^8}{2^3 \cdot 3^2 \cdot 5 \cdot 7} + \frac{31\delta^{10}}{3^4 \cdot 5^2 \cdot 7} + \frac{691\delta^{12}}{2 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11} + \frac{2 \cdot 43 \cdot 127\delta^{14}}{3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13}$$
$$+ \frac{257 \cdot 3{,}617\delta^{16}}{2^4 \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13} + \frac{73 \cdot 43{,}867\delta^{18}}{3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17} + \frac{31 \cdot 41 \cdot 174{,}611\delta^{20}}{2 \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19} + \cdots$$

$$\tan\delta = \delta + \frac{\delta^3}{3} + \frac{2\delta^5}{3 \cdot 5} + \frac{17\delta^7}{3^2 \cdot 5 \cdot 7} + \frac{2 \cdot 31\delta^9}{3^4 \cdot 5 \cdot 7} + \frac{2 \cdot 691\delta^{11}}{3^4 \cdot 5^2 \cdot 7 \cdot 11} + \frac{2^2 \cdot 43 \cdot 127\delta^{13}}{3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13}$$
$$+ \frac{257 \cdot 3{,}617\delta^{15}}{3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13} + \frac{2 \cdot 73 \cdot 43{,}867\delta^{17}}{3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17} + \frac{2 \cdot 31 \cdot 41 \cdot 174{,}611\delta^{19}}{3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19} + \cdots$$

For the case $\delta = 1/4$ these series become

$$\sec\left(\tfrac{1}{4}\right) = 1 + \frac{1}{32} + \frac{5}{6{,}144} + \frac{61}{294{,}9120} + \frac{277}{528{,}482{,}304} + \frac{50{,}521}{3{,}805{,}072{,}588{,}800} + \frac{540{,}553}{1{,}607{,}262{,}661{,}509{,}120}$$
$$+ \frac{199{,}360{,}981}{23{,}401{,}744{,}351{,}572{,}787{,}200} + \frac{3{,}878{,}302{,}429}{89{,}862{,}698{,}310{,}039{,}502{,}848{,}000} + \frac{2{,}404{,}879{,}675{,}441}{439{,}967{,}770{,}925{,}953{,}405{,}943{,}808{,}000} + \cdots$$

$$\lg\sec\left(\tfrac{1}{4}\right) = \frac{1}{32} + \frac{1}{3{,}072} + \frac{1}{184{,}320} + \frac{17}{165{,}150{,}720} + \frac{31}{14{,}863{,}564{,}800} + \frac{691}{15{,}695{,}924{,}428{,}800} + \frac{5{,}461}{5{,}713{,}316{,}492{,}083{,}200}$$
$$+ \frac{929{,}569}{43{,}878{,}270{,}659{,}198{,}976{,}000} + \frac{3{,}202{,}291}{6{,}713{,}375{,}410{,}857{,}443{,}328{,}000} + \frac{221{,}930{,}581}{18{,}561{,}569{,}276{,}250 \cdot 1{,}099{,}511{,}627{,}776} + \cdots$$

$$\tan\left(\tfrac{1}{4}\right) = \frac{1}{4} + \frac{1}{192} + \frac{1}{7{,}680} + \frac{17}{5{,}160{,}960} + \frac{31}{371{,}589{,}120} + \frac{691}{326{,}998{,}425{,}600} + \frac{5{,}461}{102{,}023{,}508{,}787{,}200}$$
$$+ \frac{929{,}569}{685{,}597{,}979{,}049{,}984{,}000} + \frac{242{,}185}{1{,}771{,}585{,}177{,}865{,}158{,}656{,}000} + \frac{221{,}930{,}581}{1{,}801{,}296{,}492{,}172{,}582{,}846{,}464{,}000} + \cdots$$

which reduce to

$$\sec\left(\tfrac{1}{4}\right) = 1.032{,}085{,}023{,}984{,}212$$

$$\lg\sec\left(\tfrac{1}{4}\right) = .031{,}581{,}051{,}247{,}468$$

$$\tan\left(\tfrac{1}{4}\right) = .255{,}341{,}921{,}221{,}138$$

For the case $\delta = 1/4$ the three series for $f(\kappa)$, $\Phi(\kappa)$, and $F(\kappa)$ are

$$f(\kappa) = \Bigg\{ \left[\sec\left(\tfrac{1}{4}\right) - 1\right] - \left[\frac{7}{6{,}144} + \frac{23}{368{,}640} + \frac{2{,}189}{880{,}803{,}840} + \frac{41{,}221}{475{,}634{,}073{,}600} + \frac{22{,}572{,}287}{8{,}036{,}313{,}307{,}545{,}600} + \frac{4{,}708{,}201}{54{,}170{,}704{,}517{,}529{,}600} + \cdots\right]\kappa$$
$$+ \left[\frac{127}{2{,}949{,}120} + \frac{3{,}181}{880{,}803{,}840} + \frac{8{,}299}{42{,}278{,}584{,}320} + \frac{3{,}191{,}633}{365{,}286{,}968{,}524{,}800} + \frac{2{,}699{,}057{,}983}{7{,}800{,}581{,}450{,}524{,}262{,}400} + \cdots\right]\kappa^2$$
$$- \left[\frac{4{,}369}{2{,}642{,}411{,}520} + \frac{5{,}557}{29{,}727{,}129{,}600} + \frac{10{,}352{,}497}{803{,}631{,}330{,}754{,}560} + \frac{2{,}042{,}063{,}879}{2{,}925{,}218{,}043{,}946{,}598{,}400} + \cdots\right]\kappa^3$$
$$+ \left[\frac{34{,}807}{543{,}581{,}798{,}400} + \frac{73{,}215{,}517}{8{,}036{,}313{,}307{,}545{,}600} + \frac{5{,}933{,}506{,}021}{7{,}800{,}581{,}450{,}524{,}262{,}400} + \cdots\right]\kappa^4$$
$$- \left[\frac{20{,}036{,}983}{8{,}036{,}313{,}307{,}545{,}600} + \frac{138{,}985{,}499}{325{,}024{,}227{,}105{,}177{,}600} + \cdots\right]\kappa^5 + \left[\frac{2{,}280{,}356{,}863}{23{,}401{,}744{,}351{,}572{,}787{,}200} + \cdots\right]\kappa^6 - \cdots \Bigg\}$$

$$\Phi(\mathcal{K}) = 2\left\{\left[\lg\sec\left(\tfrac{1}{4}\right)\right] - \left[\frac{1}{1,536} + \frac{1}{46,080} + \frac{17}{27,525,120} + \frac{31}{1,857,945,600} + \frac{691}{1,569,592,442,880} + \frac{5,461}{476,109,707,673,600} + \cdots\right]\mathcal{K}\right.$$
$$+ \left[\frac{13}{737,280} + \frac{23}{22,020,096} + \frac{23}{528,482,304} + \frac{193,163}{125,567,395,430,400} + \frac{262,313}{5,078,503,548,518,400} + \cdots\right]\mathcal{K}^2$$
$$- \left[\frac{11}{20,643,840} + \frac{5,473}{59,454,259,200} + \frac{45,709}{17,938,199,347,200} + \frac{10,497,127}{91,413,063,873,331,200} + \cdots\right]\mathcal{K}^3$$
$$+ \left[\frac{4,069}{237,817,036,800} + \frac{70,073}{35,876,398,694,400} + \frac{1,375,037}{10,157,007,097,036,800} + \cdots\right]\mathcal{K}^4$$
$$\left. - \left[\frac{8,963}{15,695,924,428,800} + \frac{1,238,033}{15,235,510,645,555,200} + \cdots\right]\mathcal{K}^5 + \left[\frac{7,157,977}{365,652,255,493,324,800} + \cdots\right]\mathcal{K}^6\right\}$$

$$F(\mathcal{K}) = \left\{\left[\tan\left(\tfrac{1}{4}\right)\right] - \left[\frac{1}{96} + \frac{1}{1,920} + \frac{17}{860,160} + \frac{31}{46,448,640} + \frac{691}{32,699,842,560} + \frac{5,461}{8,501,959,065,600} + \cdots\right]\mathcal{K}\right.$$
$$+ \left[\frac{13}{30,720} + \frac{23}{688,128} + \frac{115}{66,060,288} + \frac{17,833}{237,817,036,800} + \frac{262,313}{90,687,563,366,400} + \cdots\right]\mathcal{K}^2$$
$$- \left[\frac{11}{645,120} + \frac{5,473}{2,972,712,960} + \frac{45,709}{373,712,486,400} + \frac{10,497,127}{1,632,376,140,595,200} + \cdots\right]\mathcal{K}^3$$
$$+ \left[\frac{4,069}{5,945,425,920} + \frac{70,073}{747,424,972,800} + \frac{1,375,037}{181,375,126,732,800} + \cdots\right]\mathcal{K}^4$$
$$\left. - \left[\frac{8,963}{326,998,425,600} + \frac{1,238,033}{272,062,690,099,200} + \cdots\right]\mathcal{K}^5 + \left[\frac{7,157,977}{6,529,504,562,380,800} + \cdots\right]\mathcal{K}^6 - \cdots\right\}$$

which reduce to the approximations,

$$f(\mathcal{K}) \cong \left\{[.032,085,023,984,212] - [.001,204,289,203]\mathcal{K} + [.000,046,880,55]\mathcal{K}^2 - [.000,001,853,96]\mathcal{K}^3 + [.000,000,073,95]\mathcal{K}^4 - [.000,000,002,97]\mathcal{K}^5 + [.000,000,000,11]\mathcal{K}^6\right\} \qquad (28)$$

$$\Phi(\mathcal{K}) \cong 2\left\{[.031,581,051,247,468] - [.000,673,377,81]\mathcal{K} + [.000,018,721,99]\mathcal{K}^2 - [.000,000,627,57]\mathcal{K}^3 + [.000,000,019,2]\mathcal{K}^4 - [.000,000,000,65]\mathcal{K}^5 + [.000,000,000,02]\mathcal{K}^6\right\} \qquad (29)$$

$$F(\mathcal{K}) \cong \left\{[.255,341,921,221,138] - [.010,957,952,9]\mathcal{K} + [.000,458,419,9]\mathcal{K}^2 - [.000,019,040,1]\mathcal{K}^3 + [.000,000,790,8]\mathcal{K}^4 - [.000,000,032,5]\mathcal{K}^5 + [.000,000,001,3]\mathcal{K}^6\right\} \qquad (30)$$

In a few cases the uncertainty in these coefficients is in the next to last place, but for most of them it lies in the last digit. In every case the numerical results calculated from these expressions have their uncertainty in the last decimal place.

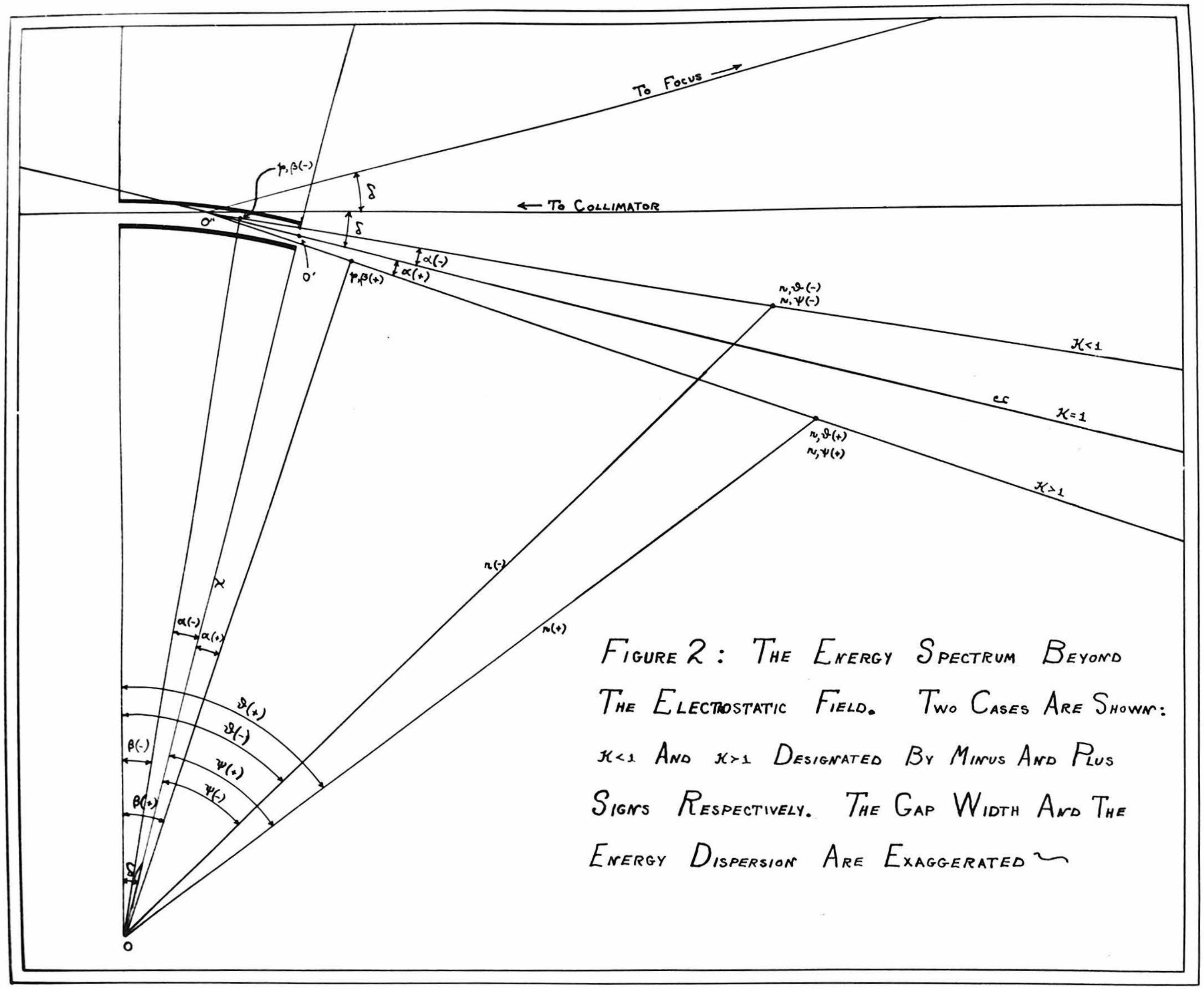

Figure 2: The Energy Spectrum Beyond The Electrostatic Field. Two Cases Are Shown: K<1 And K>1 Designated By Minus And Plus Signs Respectively. The Gap Width And The Energy Dispersion Are Exaggerated ~

After the ions leave the electrostatic field they will travel along straight trajectories. Since it requires two conditions to determine a straight line we will choose the two which are most accessible in this case. These are the position and direction of the line at the trailing edge of the condenser. We shall use the same origin of coordinates for the spreading straight lines as we did for the trajectories in the electrostatic field. The equations of these lines in polar coordinates will be of the form

$$r \cos(\vartheta - \beta) = p$$

where r and ϑ are the coordinates of any point on a particular one of the lines, and p and β are the coordinates of the point on a particular line at which the radius vector is perpendicular to the line and hence determine with which one of the lines we are dealing. The center of the sector is now the origin, the leading edge of the sector is the original line, and angles are positive in a clock-wise direction.

We now define an angle α by means of the equation

$$\tan\alpha = -\left(\frac{1}{u}\frac{du}{d\varphi}\right)_{(\varphi=\delta)} = (\kappa - 1)\, F(\delta, \kappa). \qquad (31)$$

This angle α is the acute angle between the line for an ion of given energy and the median ray.

It is apparent from figure 2 that $\beta = \delta + \alpha$ and hence the equation of the energy spectrum becomes

$$r \cos(\vartheta - \delta - \alpha) = p \quad .$$

If we now replace ϑ by $\psi = \vartheta - \delta$, the trailing edge of the sector becomes the original line and the equation of the energy spectrum becomes

$$r \cos(\psi - \alpha) = p \quad ;$$

or

$$r \left\{ \cos\psi \cdot \cos\alpha + \sin\psi \cdot \sin\alpha \right\} = p . \qquad (32)$$

In any field for which the law of the conservation of angular momentum holds we may write the following relations:

$$V^2 = \left\{ \left(\frac{d\rho}{dt}\right)^2 + \rho^2 \left(\frac{d\varphi}{dt}\right)^2 \right\} = \left\{ \left(\frac{d\rho}{d\varphi}\right)^2 + \rho^2 \right\} \left(\frac{d\varphi}{dt}\right)^2 = \left\{ \left(\frac{d\rho}{d\varphi}\right)^2 + \rho^2 \right\} \frac{a^2 v^2}{\rho^4}$$

$$V^2 = \left\{ \left(\frac{1}{\rho}\frac{d\rho}{d\varphi}\right)^2 + 1 \right\} \frac{a^2 v^2}{\rho^2} = \left\{ \left(\frac{1}{u}\frac{du}{d\varphi}\right)^2 + 1 \right\} \frac{v^2}{u^2}$$

For this problem we have

$$\sqrt{1 + \left\{ \left(\frac{1}{u}\frac{du}{d\varphi}\right)_{(\varphi=\delta)} \right\}^2} = \frac{V_{(\varphi=\delta)}}{v} u_{(\varphi=\delta)}$$

From this we determine

$$\sin\alpha = - \frac{\left(\frac{1}{u}\frac{du}{d\varphi}\right)_{(\varphi=\delta)}}{\left[1 + \left\{ \left(\frac{1}{u}\frac{du}{d\varphi}\right)_{(\varphi=\delta)} \right\}^2 \right]^{1/2}} = + \frac{v}{V_{(\varphi=\delta)}} \cdot \frac{\tan\alpha}{u_{(\varphi=\delta)}}$$

$$\cos\alpha = + \frac{1}{\left[1 + \left\{ \left(\frac{1}{u}\frac{du}{d\varphi}\right)_{(\varphi=\delta)} \right\}^2 \right]^{1/2}} = + \frac{v}{V_{(\varphi=\delta)}} \cdot \frac{1}{u_{(\varphi=\delta}}$$

On substituting these expressions for $\sin\alpha$ and $\cos\alpha$ into equation (32) we obtain

$$r \frac{v}{V_{(\varphi=\delta)} u_{(\varphi=\delta)}} \left\{ \cos\psi + \tan\alpha \cdot \sin\psi \right\} = p .$$

For $\psi = 0$ we must have

$$r_{(\psi=0)} = \rho_{(\varphi=\delta)} = a\, u_{(\varphi=\delta)}$$

and we are thus lead to the value of p,

$$p = a \frac{v}{V_{(\varphi=\delta)}}$$

which is a direct consequence of the law of the conservation of moment of momentum. Equation (32) now becomes

$$r \left\{ \cos\psi + \tan\alpha \cdot \sin\psi \right\} = a u_{(\varphi=\delta)} ;$$

$$r \left\{ \cos\psi - (1-\kappa) F(\delta,\kappa) \sin\psi \right\} = a \left\{ 1 + (1-\kappa) f(\delta,\kappa) \right\}$$

$$r \cos\psi - a = (1-\kappa) \left\{ r \sin\psi \, F(\delta,\kappa) + a \, f(\delta,\kappa) \right\}$$

It now seems desirable to make a transformation to a set of rectangular coordinates for which the median ray will be the y-axis and the trailing edge of the sector will be the x-axis. We take y to be positive in the direction away from the condenser, and x to be positive in the direction from the origin to the center of the sector. Angles are measured from the y-axis in a clock-wise direction. The desired transformation is

$$x = a - r \cos\psi \quad ; \quad y = r \sin\psi .$$

We now have

$$x = (\kappa-1) \left\{ y \, F(\delta,\kappa) + a \, f(\delta,\kappa) \right\} . \qquad (33)$$

This is <u>the equation of the energy spectrum beyond the electrostatic field</u>. It is valid for any value of δ so long as the two series, $F(\delta,\kappa)$ and $f(\delta,\kappa)$, are convergent. It is to be noted that $x = 0$ for the y value,

$$y_0 = a \frac{f(\delta,\kappa)}{F(\delta,\kappa)} . \qquad (34)$$

For $\delta = 1/4$ this expression for y_0 becomes

$$y_0 = a \frac{f(1/4,\kappa)}{F(1/4,\kappa)} = a \frac{f(\kappa)}{F(\kappa)} . \qquad (34a)$$

Hence y_0 is a constant, independent of $\varkappa$, only in the case where $f(\varkappa)$ is exactly a constant times $F(\varkappa)$. This is obviously not exactly true in this case. However, as can be seen from the tables of page 39 which give $f(\varkappa)$, $F(\varkappa)$, and their quotient for a variety of values of $\varkappa$, the quotient varies surprisingly little in the pertinent range. This range in $\varkappa$ is from $\varkappa = .74$ to $\varkappa = 1.27$ as determined by excluding all values of $\varkappa$ corresponding to ions which would strike either of the condenser plates. For $a = 12$ centimeters and $\varkappa = .74$, 1.00, and 1.27 we obtain for y_0:

$$y_0 = 1.5137816364 \text{ cms. for } \varkappa = .74$$

$$y_0 = 1.5158229024 \text{ cms. for } \varkappa = 1.00$$

$$y_0 = 1.5179195436 \text{ cms. for } \varkappa = 1.27$$

This represents an overall variation of y_0 of about 1 part in 366. Hence to 1 part in 732 on each side we have firmly <u>established the existence of a point virtual source</u> at $y_0^1 = 1.5158229$ cms. This lack of constancy in the position of the source is only important for our purposes in that it gives a finite width to the source. It is not possible to give this width exactly but we may place a generous upper limit on it as follows: Let ξ be the absolute value of the variation in y_0; then

$$\xi = .0020412660 \text{ cms. for } \varkappa = .74$$

$$\xi = .0020966412 \text{ cms. for } \varkappa = 1.27 \quad .$$

Let ς be the absolute value of the upper limit of the width of the virtual source; then

$$\varsigma < \xi \left\{ |\tan\alpha| + .0002 \right\}$$

where .0002 is the contribution from the lack of perfect collimation in the collimator. From the table we calculate

$$|\tan\alpha| = .064{,}343{,}911 \quad \text{for } \varkappa = .74$$

$$|\tan\alpha| = .065{,}374{,}479 \quad \text{for } \varkappa = 1.27 \;.$$

Hence

$$\rho < .000{,}131{,}751 \text{ cms.} \quad \text{for } \varkappa = .74$$

$$\rho < .000{,}137{,}486 \text{ cms.} \quad \text{for } \varkappa = 1.27 \;.$$

Hence we may feel confident that the latter figure sets the upper limit to the width of the point virtual source.

The time has come to say a word in justification of the number of figures which have been carried in the calculations. The calculations are only accurate to the accuracy of $F(\delta,\varkappa)$ which has its uncertainty in the 9th or 10th figure. The variation in y_0 was in the 4th figure. This requires carrying 9 figures to get 6 figure accuracy beyond the electrostatic field. The magnification of errors in the magnetic field cannot reduce this 6 figure accuracy to less than 4 figure accuracy. It is possible to approach an accuracy of .001 of an inch on the milling machine for the cutting of the theoretical shape onto the pole faces. Hence we have this accuracy theoretically plus a margin of safety of 1 figure. The accuracy while more than adequate is not superfluous.

x	$f(x)$	$F(x)$	$\frac{f(x)}{F(x)}$
.50	.03149437231	.2499752182	.1259899783
.55	.03143654447	.2494506221	.1260231151
.60	.03137893636	.2489281681	.1260561896
.62	.03135595438	.2487197832	.1260694022
.64	.03133300726	.2485117382	.1260826047
.66	.03131009494	.2483040321	.1260957975
.68	.03128721734	.2480966643	.1261089803
.70	.03126437437	.2478896338	.1261221532
.72	.03124156596	.2476829397	.1261353164
.74	.03121879204	.2474765814	.1261484697
.75	.03120741796	.2473735279	.1261550426
.76	.03119605249	.2472705581	.1261616131
.78	.03117334726	.2470648687	.1261747468
.80	.03115067629	.2468595126	.1261878708
.82	.03112803946	.2466544891	.1262009849
.84	.03110543675	.2464497974	.1262140894
.85	.03109414813	.2463475755	.1262206379
.86	.03108286798	.2462454363	.1262271839
.88	.03106033319	.2460414053	.1262402690
.90	.03103783224	.2458377038	.1262533442
.92	.03101536505	.2456343307	.1262664098
.94	.03099293156	.2454312853	.1262794656
.95	.03098172742	.2453298854	.1262859899
.96	.03097053167	.2452285669	.1262925118
.97	.03095934433	.2451273301	.1262990313
.98	.03094816534	.2450261747	.1263055483
.99	.03093699472	.2449251004	.1263120630
1.00	.03092583246	.2448241077	.1263185752
1.01	.03091467856	.2447231960	.1263250851
1.02	.03090353297	.2446223654	.1263315925
1.03	.03089239572	.2445216158	.1263380974
1.04	.03088126480	.2444209469	.1263445919
1.05	.03087014620	.2443203587	.1263511005
1.06	.03085903387	.2442198513	.1263575983
1.08	.03083683408	.2440190783	.1263705867
1.10	.03081466739	.2438186265	.1263835657
1.12	.03079253371	.2436184955	.1263965351
1.14	.03077043295	.2434186846	.1264094948
1.15	.03075939492	.2433188988	.1264159713
1.16	.03074836523	.2432191928	.1264224458
1.18	.03072633013	.2430200195	.1264353867
1.20	.03070432757	.2428211639	.1264483173
1.22	.03068235797	.2426226254	.1264612396
1.24	.03066042079	.2424244027	.1264741521
1.25	.03064946440	.2423254100	.1264806047
1.26	.03063841609	.2422264957	.1264866422
1.28	.03061664381	.2420289035	.1264999484
1.30	.03059480371	.2418316251	.1265128318
1.32	.03057299620	.2416346601	.1265257070
1.34	.03055122076	.2414380074	.1265385723
1.36	.03052947740	.2412416665	.1265514280
1.38	.03050776613	.2410456366	.1265642746
1.40	.03048608666	.2408499170	.1265771109
1.45	.03043202803	.2403619707	.1266091634
1.50	.03037816738	.2398759478	.1266411562

IV THE SOLUTION IN THE MAGNETIC FIELD AND EDGE EFFECT CORRECTIONS

The path of an ion in an uniform magnetic field of strength H is a circle of radius R. The equilibrium between the d'Alembert and magnetic forces on the ion may be expressed by

$$\frac{mV^2}{R} = HeV$$

which reduces to

$$R = \frac{m}{e}\frac{V}{H} . \qquad (35)$$

V can be obtained from equation (23) which is

$$V^2 = v^2 \left\{ 1 - \kappa(1-\kappa)\, \Phi(\delta, \kappa) \right\} . \qquad (23)$$

The definition of κ was

$$\kappa = \frac{e\Phi_0}{mv^2 \lg(\rho_2/\rho_1)} :$$

If we denote the velocity of an ion following the median ray by v_0 or V_0, since it is unchanged in the electrostatic field, and let κ_0 be the corresponding value of κ we have

$$\kappa_0 = \frac{e\Phi_0}{mv^2 \lg(\rho_2/\rho_1)} = \frac{e\Phi_0}{mV_0^2 \lg(\rho_2/\rho_1)} = 1$$

and hence

$$\kappa = v_0^2/v^2 = V_0^2/v^2 \quad ; \quad v^2 = V_0^2/\kappa .$$

We are thus lead to

$$V^2 = \frac{V_0^2}{\kappa}\left\{1 - \kappa(1-\kappa)\, \Phi(\delta, \kappa)\right\}$$

or

$$V/V_0 = \left\{ \frac{1}{\kappa} - (1-\kappa)\, \Phi(\delta, \kappa) \right\}^{1/2}$$

κ	$\Phi(\kappa)$	$\left\{\frac{1}{\kappa}-(1-\kappa)\Phi(\kappa)\right\}$	$\frac{R}{R_0}=\left\{\frac{1}{\kappa}-(1-\kappa)\Phi(\kappa)\right\}^{1/2}$
.50	.062497931146	1.968751034427	1.403121889
.55	.062432508464	1.790087189373	1.337941400
.60	.062367262722	1.641719761578	1.281296126
.62	.062341213898	1.589213564584	1.260640141
.64	.062315193230	1.540066530434	1.240994170
.66	.062289200660	1.493973186928	1.222281959
.68	.062263236138	1.450663999730	1.204435137
.70	.062237299602	1.409900238690	1.187392201
.72	.062211391004	1.371469699408	1.171097647
.74	.062185510282	1.335183118678	1.155501241
.75	.062172580354	1.317790188244	1.147950429
.76	.062159656978	1.300871156009	1.140557388
.78	.062133832246	1.268381838957	1.126224595
.80	.062108034826	1.237578393035	1.112465007
.82	.062082265064	1.208337387410	1.099244007
.84	.062056522908	1.180547146811	1.086529865
.85	.062043662164	1.167164038910	1.080353664
.86	.062030808300	1.154106384512	1.074293435
.88	.062005121186	1.128923021822	1.062507892
.90	.061979461508	1.104913164960	1.051148498
.92	.061953829222	1.082000215401	1.040192393
.94	.061928224266	1.060114093778	1.029618421
.95	.061915432018	1.049535807346	1.024468549
.96	.061902646586	1.039190560804	1.019406965
.97	.061889867960	1.029071139013	1.014431436
.98	.061877096134	1.019170621342	1.009539807
.99	.061864331098	1.009482366790	1.004729997
1.	.061851572852	1.	1.
1.01	.061838821382	.990717398115	.995347877
1.02	.061826076686	.981628678397	.990771758
1.03	.061813338756	.972728186571	.986269834
1.04	.061800607586	.964010485841	.981840356
1.05	.061787883164	.955470346539	.977481635
1.06	.061775165494	.947102736345	.9731920347
1.08	.061756950358	.930866481954	.9648142215
1.10	.061724362138	.915263345304	.9566939664
1.12	.061699000750	.900261022947	.9488208592
1.14	.061673666172	.885827295720	.9411839861
1.15	.061661008916	.878814368728	.9374509954
1.16	.061648358336	.871932702851	.9337733680
1.18	.061623077198	.858549781015	.9265796140
1.20	.061597822700	.845652897873	.9195938766
1.22	.061572594788	.833218102002	.9128078122
1.24	.061547393416	.821222987323	.9062135440
1.25	.061534802662	.815383700666	.9029859914
1.26	.061522218524	.809646570467	.8998036288
1.28	.061497070046	.798469179613	.8935710266
1.30	.061471947992	.787672353628	.8875090724
1.32	.061446852246	.777238750295	.8816114509
1.34	.061421782778	.767152062861	.8758721727
1.36	.061396739530	.757396943878	.8702855531
1.38	.061371722462	.747958935695	.8648461919
1.40	.061346731516	.738824406892	.8595489555
1.45	.061284368108	.717233138063	.8468961790
1.50	.061222166856	.697277750095	.8350315863

Furthermore, if R_0 denote the radius of the median ray in the magnetic field we have

$$R_0 = \frac{m}{e}\frac{V_0}{H}$$

and equation (35) can be written in the form

$$\frac{R}{R_0} = \frac{V}{V_0} = \left\{\frac{1}{\kappa} - (1-\kappa)\ \Phi(\delta,\kappa)\right\}^{1/2} \tag{36}$$

We are now in position to calculate the radius R for any ion in the magnetic field as soon as R_0 is fixed and the value of κ is known. On page 41 the values of $\Phi(\kappa)$, of R^2/R_0^2, and of R/R_0 have been calculated for a variety of values of κ in the pertinent range. The effect of the change in velocity of an ion in the electrostatic field on its radius of curvature in the magnetic field can be readily determined from the last column. This effect, although small, is important for accuracy in focusing.

The symmetry which exists between the virtual point source and the real focus suggests at once that the leading and trailing edges of the magnetic pole pieces should be mirror images of one another. In this case there will exist a line of symmetry which will contain the centers of curvature of the paths of the ions in the magnetic field. This scheme reduces the theory of focussing to the problem of determining under what conditions the tangents to the paths of the ions as the ions cross the line of symmetry will be perpendicular to the line of symmetry. It is apparent that if the theory can be made to represent the experimental situation accurately and obtain the condition that all ions of the desired specific mass cross the symmetry

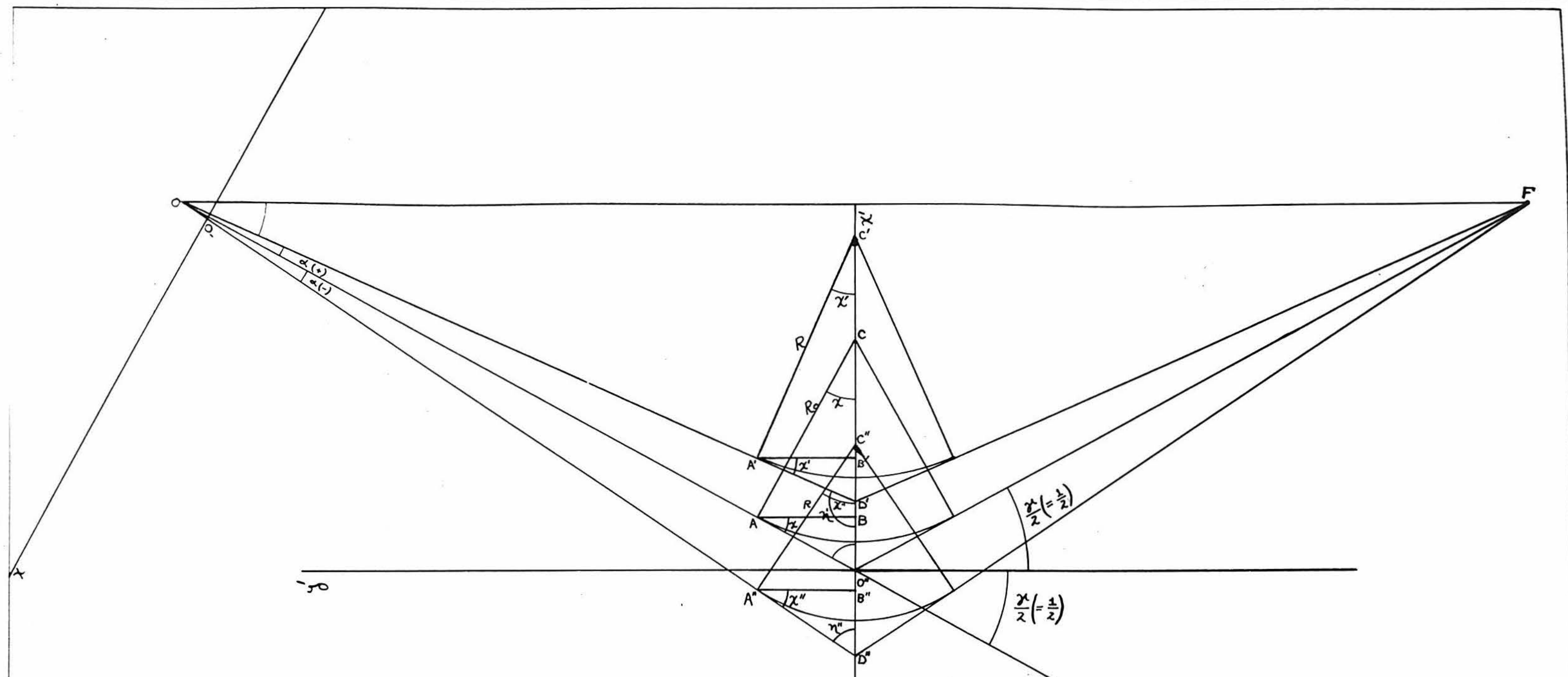

Figure 3: The Solution In The Magnetic Field ~ Unprimed Letters Refer To The Median Ray; Singly Primed Letters, To α Positive; Doubly Primed Letters, To α Negative ~ $\chi = \frac{\gamma}{2} - \alpha$; $\eta = \frac{\pi}{2} \pm \left(\frac{\gamma}{2} - \alpha\right)$ ~

line perpendicular to that line the most favorable condition for focussing has been found. All of the following theory is grounded upon this argument.

In Figure 3. this situation is shown for the median ray and for two rays corresponding to a larger (primed letters) and a smaller (double-primed letters) value of the kinetic energy than that for the median ray. The angle α is now taken to be positive in an anti-clockwise direction from the y-axis. O' is the origin of the x,y-coordinate system. O and F are the virtual source and the real focus respectively.

The calculations for the shape of the leading and trailing edges of the magnetic pole pieces can be most easily made by the use of a system of coordinates x',y' which has for its origin O'', the center in the magnetic field, and for its x'-axis, the line of symmetry as shown in the figure. C is the center of curvature of the path of an ion. A is the point on the leading edge of the pole pieces where an ion of given energy would enter the magnetic field with sharp cutoff in the field at the pole face. B is the projection of A on the x'-axis. R is the radius in the magnetic field. D is the intersection of the extension of the path of an ion beyond the electrostatic field with the line of symmetry. γ is the angle of deflection of the median ray in the magnetic field. χ is the angle between R and the x'-axis. η is the internal angle at D in the triangle ODO''. Primes correspond to energies greater than that of the median ray and double-primes, to energies less than that of the median ray. If lines are drawn from C' and C'' perpendicular to OO'' these lines will form at C' and C'' an angle with the x'-axis of exactly $\gamma/2$, and an angle with R of exactly α. Hence $\chi = \gamma/2 - \alpha$. Also it can be seen that $\eta' = \pi/2 + (\gamma/2 - \alpha)$ and $\eta'' = \pi/2 - (\gamma/2 - \alpha)$ since α is essentially negative in the latter case.

We may now determine the coordinates of the leading edge in parametric form. We have

$$y' = BA = R \sin\chi = R \sin(\gamma/2 - \alpha) \qquad (37)$$

and

$$x' = O''B = O''D + DB .$$

From the figure it can be seen that the angle BAD $= \chi$ and that

$$DB = BA \tan(\gamma/2 - \alpha) = R \sin(\gamma/2 - \alpha) \tan(\gamma/2 - \alpha) .$$

Also using the law of sines on the triangles D'OO'' and D''OO'', we have

$$\frac{|OO''|}{\sin\eta'} = \frac{|OO''|}{\sin\{\pi/2 + (\gamma/2 - \alpha)\}} = \frac{|OO''|}{\cos(\gamma/2 - \alpha)} = \frac{O''D'}{\sin\alpha}$$

$$\frac{|OO''|}{\sin\eta''} = \frac{|OO''|}{\sin\{\pi/2 - (\gamma/2 - \alpha)\}} = \frac{|OO''|}{\cos(\gamma/2 - \alpha)} = \frac{O''D''}{\sin\alpha}$$

And hence for both cases we have

$$O''D = |OO''| \sin\alpha \sec(\gamma/2 - \alpha) .$$

Therefore

$$x' = |OO''| \sin\alpha \sec(\gamma/2 - \alpha) + R \sin(\gamma/2 - \alpha) \tan(\gamma/2 - \alpha). \qquad (38)$$

In this case we know α only through $\tan\alpha$. Therefore since

$$\cos\alpha = \frac{1}{\sqrt{1 + \tan^2\alpha}} \quad \text{and} \quad \sin\alpha = \frac{\tan\alpha}{\sqrt{1 + \tan^2\alpha}},$$

we may rearrange as follows:

$$y' = R\left\{\sin\gamma/2 \cos\alpha - \cos\gamma/2 \sin\alpha\right\}$$

$$y' = \frac{R}{\sqrt{1 + \tan^2\alpha}}\left\{\sin\gamma/2 - \cos\gamma/2 \tan\alpha\right\}$$

$$y' = \frac{R_0\left\{1/\kappa - (1-\kappa)\,\Phi(\delta,\kappa)\right\}^{1/2}}{\sqrt{1 + (1-\kappa)^2\left\{F(\delta,\kappa)\right\}^2}}\left\{\sin\gamma/2 - (1-\kappa)\,F(\delta,\kappa)\cos\gamma/2\right\} \qquad (39)$$

$$x' = \frac{|OO''|\ \sin\alpha}{\left\{\cos\gamma/2\cos\alpha + \sin\gamma/2\sin\alpha\right\}} + \frac{R\left\{\sin\gamma/2\cos\alpha - \cos\gamma/2\sin\alpha\right\}^2}{\left\{\cos\gamma/2\cos\alpha + \sin\gamma/2\sin\alpha\right\}}$$

$$x' = \frac{|OO''|\ \tan\alpha}{\left\{\cos\gamma/2 + \sin\gamma/2\tan\alpha\right\}} + \frac{R\,(\tan\gamma/2 - \tan\alpha)^2\cos\gamma/2}{\left\{1 + \tan\gamma/2\tan\alpha\right\}\sqrt{1 + \tan^2\alpha}}$$

$$x' = \frac{|OO''|\ (1-\kappa)\,F(\delta,\kappa)}{\left\{\cos\gamma/2 + (1-\kappa)\,F(\delta,\kappa)\sin\gamma/2\right\}} \qquad (40)$$

$$+ \frac{R_0\left\{1/\kappa - (1-\kappa)\,\Phi(\delta,\kappa)\right\}^{1/2}\left\{\tan\gamma/2 - (1-\kappa)F(\delta,\kappa)\right\}^2\cos\gamma/2}{\left\{1 + (1-\kappa)F(\delta,\kappa)\tan\gamma/2\right\}\sqrt{1 + (1-\kappa)^2\left\{F(\delta,\kappa)\right\}^2}}$$

Equations (39) and (40) are the expressions for the coordinates of the leading edge of the magnetic pole pieces. The trailing edge is obtained by simply reversing the sign of y' leaving everything else the same. In our case $\delta = 1/4$ radian, $\gamma = 1$ radian, $OO'' = 40$ centimeters, and $R_0 = 10$ centimeters.

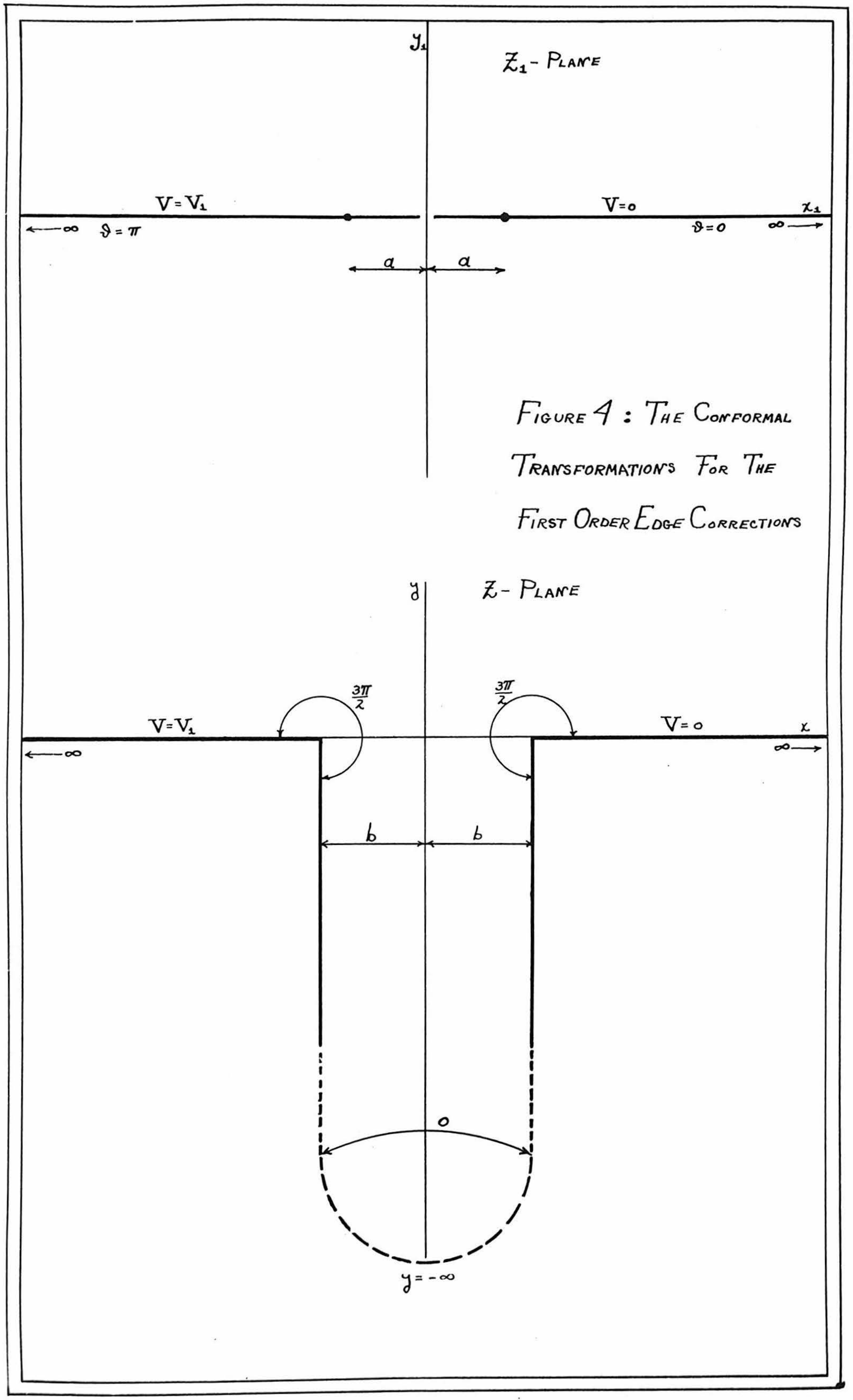

Figure 4: The Conformal Transformations For The First Order Edge Corrections

The Derivation of the Conformal Transformation for the Pole Faces.

The solution of a difficult problem in potential theory can frequently be reduced to the solution of a much simpler problem by means of a conformal transformation in the complex domain. In the case where the transformation function is analytic both the real and imaginary parts of the function separately satisfy La Place's equation and the pair may be used to represent the lines of force and the equipotentials respectively of the field. The notations R(z) and I(z) will be used to denote the real and imaginary parts of the complex variable Z.

If we use the transformation $W(z_1) = A \lg z_1$ and take $R(W)$ = constant to be the lines of force and $I(W)$ = constant to be the equipotentials we can show that as a special case this transformation gives the field when the two halves of a plane are raised to different potentials. Let

$$W(z_1) = U(z_1) + iV(z_1) = R(z_1) + iI(z_1)$$

and $z_1 = r e^{i\vartheta}$. Then

$$W(z_1) = U(z_1) + iV(z_1) = A \lg z_1 = A(\lg r + i\vartheta).$$

Let $V = 0$ for $\vartheta = 0$ and $V = V_1$ for $\vartheta = \pi$. Then $A = V_1/\pi$ and

$$U(r) = \frac{V_1}{\pi} \lg r \; ; \quad V(\vartheta) = \frac{V_1}{\pi} \vartheta$$

(See Jeans P318, page 268). Hence we have the field due to the plane $\vartheta = 0$ being at zero potential and the plane $\vartheta = \pi$, at an arbitrary potential V_1. Figure 4 shows the result of this transformation. The two half planes extend to infinity in the directions away from the origin.

We now desire that transformation which will bend these planes at the points (or better, lines), (a,0) and (-a,0) through a right angle and extend the bent edges on to infinity in the direction of the negative y-axis. It is well known that a Schwarz transformation is the appropriate transformation for a sharp corner. We will call this new plane the z-plane. We bend the plane on the left through an angle of $3\pi/2$ radians. Then we bend the planes through what apparently is two right angles consequetively at $(0,-\infty)$ but may as well be taken as a zero angle. This equivalence can be seen either from the fact that the bending is done at infinity or that the form of a Schwarz transformation is the same for a bend of two right angles as for a zero bend.at the same position. According to the theory in Jeans, Electricity and Magnetism, P322, page 271 the correct transformation is

$$\frac{dz}{dz_1} = \frac{\sqrt{z_1^2 - a^2}}{z_1}$$

It is by no means a surety that this transformation will leave the axes located in the same position as regards the unbent portion of the plane as they were before the transformation. The axes are almost certain to be translated <u>but they will not be rotated</u>. Also the scale may be expanded or contracted. These effects must be found by a process resembling the fixing of boundary conditions. The integration of this equation involves a rather subtle point. The function possesses an infinite number of Riemann sheets such that its sign alternates between any two consequetive sheets. Fortunately one of the signs gives a function which is physically reasonable while the other gives a function which makes it impossible to calculate the field for any value but y = negative infinity. By the following process the correct function may be obtained:

$$z = \int \frac{\sqrt{z_1^2 - a^2}}{z_1}\, dz_1 = i \int \frac{\sqrt{a^2 - z_1^2}}{z_1}\, dz_1$$

$$z = i \left\{ \sqrt{a^2 - z_1^2} - a \lg \frac{a + \sqrt{a^2 - z_1^2}}{z_1} + \text{const.} \right\}$$

We now desire that z be purely imaginary when z_1 is purely imaginary. This is equivalent to making the y-axis of $z = x + iy$ correspond to the y_1-axis of $z_1 = x_1 + iy_1$. This condition is

$$iy = i \left\{ \sqrt{a^2 + y_1^2} - a \lg \frac{a + \sqrt{a^2 + y_1^2}}{iy_1} + \text{const.} \right\}$$

It is well known that the logarithm of i is $i\pi/2$. Hence in order for both sides of the above equation to be purely imaginary the constant must equal $-a \lg i = -ia\pi/2$. This leads us to

$$z = i \left\{ \sqrt{a^2 - z_1^2} - a \left[\lg \frac{a + \sqrt{a^2 - z_1^2}}{z_1} + \frac{i\pi}{2} \right] \right\}$$

When $y = y_1 = 0$ we must have $x = b$ for $x_1 = a$ and $x = -b$ for $x_1 = -a$. Therefore

$$b = i \left\{ -a \left[\lg 1 + i\pi/2 \right] \right\} = a\pi/2$$

$$-b = i \left\{ -a \left[\lg(1/-1) + i\pi/2 \right] \right\} = i \left\{ -a \left[-i\pi + i\pi/2 \right] \right\} = -a\pi/2$$

Hence in either case $a = 2b/\pi$. If we now substitute $\chi = \pi y/2b$ and $\omega = \pi y_1/2b$ we have

$$\chi = \left\{ \sqrt{\omega^2 + 1} - \lg \frac{1 + \sqrt{1 + \omega^2}}{\omega} \right\} \qquad (41)$$

The magnetic field H is given by

$$H = \left|\frac{dW}{dz}\right| = \left|\frac{dW}{dz_1}\right| \cdot \left|\frac{dz_1}{dz}\right| = \left|\frac{V_1}{\pi z_1}\right|\left|\frac{z_1}{[z_1^2 - a^2]^{1/2}}\right| = \left|\frac{V_1}{\pi [z_1^2 - a^2]^{1/2}}\right|$$

For $x_1 = 0$ we have

$$H = \left|\frac{V_1}{\pi [y_1^2 + a^2]^{1/2}}\right| = \left|\frac{V_1}{\pi [y_1^2 + 4b^2/\pi^2]^{1/2}}\right| = \left|\frac{V_1}{2b[\pi^2 y_1^2/4b^2 + 1]^{1/2}}\right|$$

If we let $\mathcal{H} = 2bH/V_1$ and $\omega = \pi y_1/2b$ as before we obtain

$$\mathcal{H} = \frac{1}{[\omega^2 + 1]^{1/2}}. \qquad (42)$$

On pages 50 and 51, $\mathcal{H}$ and χ are given for a wide range of values of ω. Figure 5 gives a plot of $\mathcal{H}$ against χ for these values. It is unfortunate that the relation between the two is only obtained in parametric form. However it can be seen that χ and ω are asymtotically equal and this fact may be used in extrapolating for the effect beyond that here calculated. This edge correction must at best be taken with a grain of salt. At some distance from the pole face the effect of the slight curvature of the face and of its finite extent will dominate the situation.

By a process of counting squares under the $\mathcal{H}$-χ curve we have found that the area under the curve is the same as the area under the curve $\mathcal{H} = 1$ extended to where $\chi = 3.6036$. For a gap width of 3mm. or .118 inch this corresponds to a correction of .1353 inch to be cut from the pole face normal to the face.

I wish to thank Professor William R. Smythe for setting up these conformal transformations for me.

ω	ℋ	χ	ω	ℋ	χ
.00	1.	- ∞	.8	.78087	.23303
.01	.99995	- 4.2983	.9	.7433	.38754
.02	.9998	- 3.60507	1.	.70711	.53284
.03	.99955	- 3.1995	1.1	.67267	.67103
.04	.9992	- 2.91162	1.2	.64018	.80356
.05	.99875	- 2.68828	1.3	.60971	.93168
.06	.9982	- 2.50567	1.4	.58124	1.05614
.07	.99756	- 2.35118	1.5	.5547	1.17764
.08	.99682	- 2.21728	1.6	.53	1.29665
.09	.99598	- 2.09907	1.7	.50702	1.41361
.1	.99503	- 1.99324	1.8	.48564	1.52879
.11	.99401	- 1.89742	1.9	.46575	1.64247
.12	.99288	- 1.80982	2.	.44721	1.75488
.13	.99166	- 1.72914	2.1	.42993	1.86614
.14	.99034	- 1.65435	2.2	.4138	1.97642
.15	.98893	- 1.5921	2.3	.39873	2.08587
.16	.98744	- 1.51932	2.4	.38462	2.19453
.17	.98585	- 1.45789	2.5	.37139	2.30255
.18	.98418	- 1.39985	2.6	.35898	2.40997
.19	.98242	- 1.34484	2.7	.34731	2.51687
.2	.98058	- 1.29267	2.8	.33634	2.62326
.22	.97665	- 1.19526	2.9	.32599	2.72925
.24	.97238	- 1.10596	3.	.31623	2.83486
.26	.96782	- 1.02347	3.1	.307	2.94006
.28	.96296	- .94669	3.2	.29828	3.04498
.3	.95783	- .87485	3.3	.29001	3.14961
.32	.95243	- .80732	3.4	.28217	3.25396
.34	.94677	- .74346	3.5	.27472	3.35807
.36	.94088	- .6829	3.6	.26764	3.462
.38	.93478	- .62526	3.7	.26091	3.56567
.4	.92848	- .57018	3.8	.25449	3.66918
.42	.92198	- .51747	3.9	.24838	3.77247
.44	.91532	- .46683	4.	.24254	3.87564
.46	.90849	- .41809	4.1	.23696	3.97864
.48	.90153	- .37106	4.2	.23162	4.08151
.5	.89443	- .3256	4.3	.22651	4.18423
.52	.88722	- .28157	4.4	.22162	4.28685
.54	.8799	- .23885	4.5	.21693	4.38933
.56	.87251	- .19736	4.6	.21243	4.49173
.58	.86503	- .15697	4.7	.20811	4.59402
.6	.85749	- .11753	4.8	.20395	4.69621
.62	.8499	- .07919	4.9	.19996	4.79831
.64	.84227	- .04167	5.	.19612	4.90034
.66	.83461	- .00497	5.1	.19241	5.00227
.68	.82692	+ .03096	5.2	.18885	5.10415
.7	.81923	+ .06622	5.3	.18541	5.20592

ω	$\mathcal{H}$	χ
5.4	.18209	5.30766
5.5	.17889	5.40934
5.6	.17578	5.51112
5.7	.1728	5.6125
5.8	.16991	5.71401
5.9	.16711	5.81547
6.	.1644	5.91686
6.1	.16178	6.01821
6.2	.15923	6.11953
6.3	.15677	6.2208
6.4	.15438	6.32203
6.5	.15206	6.42323
6.6	.14981	6.52439
6.7	.14762	6.62552
6.8	.14549	6.72661
6.9	.14343	6.82767
7.	.14142	6.92869
7.5	.13216	7.43343
8.	.12403	7.93758
8.5	.11684	8.44125
9.	.11043	8.9445
9.5	.10468	9.44731
10.	.099504	9.95004
11.	.090536	10.95458
12	.083046	11.95835
13	.076697	12.96155
14	.071247	13.9643
15	.066519	14.96668
16	.062378	15.96876
17	.058722	16.9706
18	.05547	17.97224
19	.052559	18.97369
20	.049938	19.975
21	.047565	20.9762
22	.045408	21.97728
23	.043437	22.97827
24	.041631	23.97917
25	.039968	24.98
26	.038433	25.98077
27	.037012	26.98148

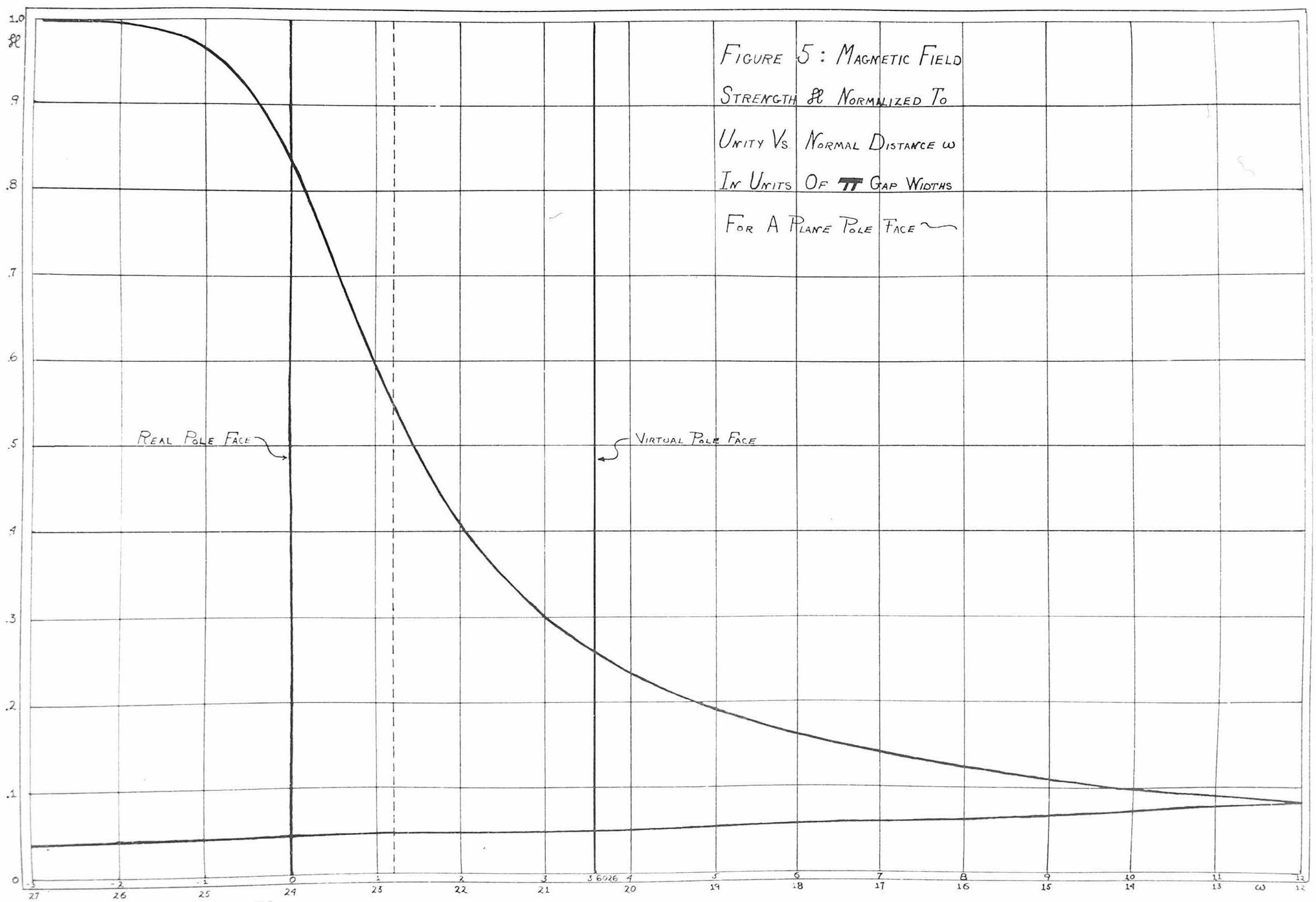

Figure 5: Magnetic Field Strength ℋ Normalized To Unity Vs. Normal Distance ω In Units Of π Gap Widths For A Plane Pole Face

The Investigation of Second-Order Edge Effects.

The appropriate conformal transformations which we have just given enable one to plot the magnetic field H as a function of the normal distance n from the pole faces so long as the latter are considered to be plane which is true, to a good order of approximation, in this case. The ions will suffer a change in direction in passing through the variable magnetic field in the neighborhood of the pole faces. This angular deflection and the consequent linear deflection may be formulated as follows:

Let V be the velocity of the ion, m and e, its mass and charge respectively, and R the variable radius in the variable magnetic field. Then as previously in the consideration of a field with a sharp jump from zero magnetic field outside to full magnetic field inside the gap, we have

$$mV/R = He .$$

Now let φ be the angle which the tangent to the path of an ion at any point makes with the normal to the pole faces. Since the pole faces in this problem are not accurately planes we must make some specification as to which point of the surface we are considering the normal to be drawn outward from before we can give a meaning to the term. We will consider the normal as drawn outward from the point where an ion would pass into the air gap with zero magnetic field. Let ds be an element of length measured along the path of the ion and let dn be a corresponding element along the normal. Then by

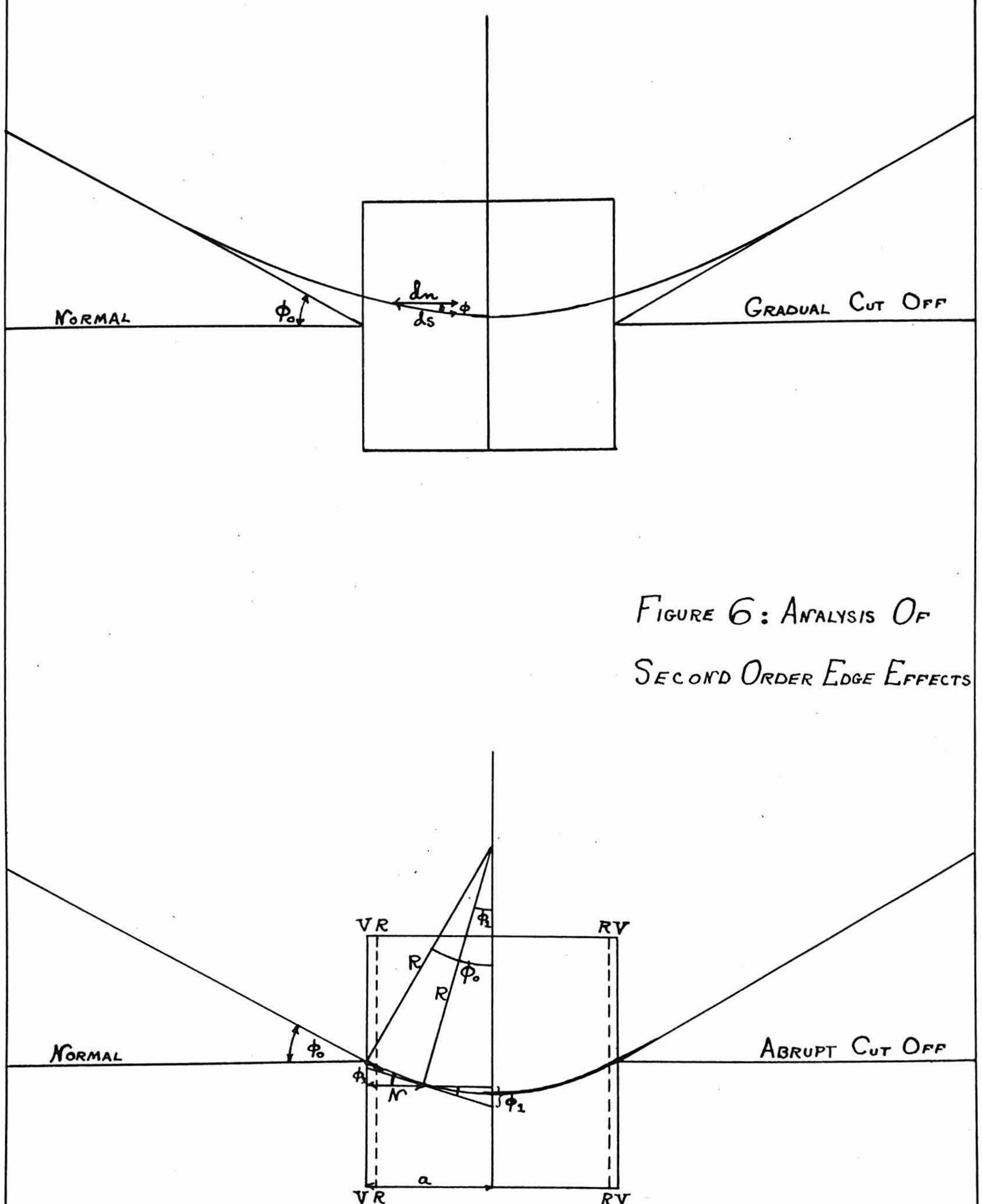

Figure 6: Analysis Of Second Order Edge Effects

simple geometry

$$dn = \cos\varphi \; ds \; . \qquad \text{(See Figure 6)}.$$

Furthermore $\frac{ds}{d\varphi}$ is the radius of curvature of the path of an ion. Hence we have

$$R = \frac{ds}{d\varphi} \qquad \text{and} \qquad \frac{m\,V}{eH(n)} = \frac{dn}{d\varphi} \sec\varphi \, .$$

Hence we have the relations

$$\cos\varphi \; d\varphi = \frac{e}{mV} \mathbf{H}(n) \; dn \quad ;$$

$$\int_{\phi_0}^{\phi} \cos\varphi \; d\varphi = \frac{e}{mV} \int_{\infty}^{N} \mathbf{H}(n) \; dn \quad ;$$

$$\sin\phi_0 - \sin\phi = \frac{e}{mV} \int_{N}^{\infty} \mathbf{H}(n) \; dn \quad .$$

In this equation ϕ_0 is the angle which the path makes with the normal in the absence of the magnetic field and N is the normal distance at a particular point on the path. The signs have been chosen so that for $\phi_0 > \phi$, which is physically the case, we will have the integral positive. The infinite limit will be replaced in practice by a suitable approximation. Likewise N will be such a value as to ensure that H has arrived very nearly to its full value. This can be expressed as

$$\mathbf{H(N)} \fallingdotseq \mathbf{H(-\infty)} \; .$$

Thus we have found the change in direction of an ion traveling in the variable field near the pole faces. The second order correction to the pole faces however involves only the difference between this deflection and the corresponding deflection with sharp cutoff of the field at the pole faces.

We shall now calculate the deflection for the idealized case of sharp cutoff of the field at the pole faces. Let ϕ_0 be the angle made by the path of an ion with the normal to the pole face at the point of incidence. Let R be the radius of the path of an ion in the constant field. Let ϕ_1 and N be respectively the angle made by the tangent to this arc at any point with the normal and the distance measured from the pole faces along the normal. Also let a be the distance from the point taken on the pole faces to the plane of symmetry as measured along the normal. See Figure 5 for the graphical representation of this situation.

From the figure we have that

$$a = N + R \sin \phi_1 = R \sin \phi_0$$

or

$$R(\sin \phi_0 - \sin \phi_1) = N \quad ;$$

$$\sin \phi_0 - \sin \phi_1 = N/R = eNH_0/mV$$

where H_0 is the constant strength of magnetic field between the pole faces.

We have previously found that

$$\sin \phi_0 - \sin \phi = \frac{e}{mV} \int_N^\infty H(n)\, dn.$$

Eliminating ϕ_0 from these two equations gives

$$\sin \phi_1 - \sin \phi = \frac{e}{mV} \left\{ \int_N^\infty H(n)\, dn - H_0 N \right\}$$

Hence $\sin \phi_1 = \sin \phi$ providing that

$$\int_N^\infty H(n)\, dn = H_0 N \quad .$$

In particular on the plane of symmetry of the pole faces where N = a the condition is that

$$\int_a^\infty H(n)\, dn = H_0 a \quad .$$

From Figure 5. it is seen that the magnetic field attains its full value very soon after the pole faces are passed by an ion coming in from outside the gap. Hence the second order edge effects vanish with the accurate correction for the first order effects. It is to be particularly noted that all that is required for the correction of second order effects is that conditions be exactly symmetrical for the trailing edge with those for the leading edge.

One further consideration need be made and that is for the directional effect due to the fact that the angle of incidence which the path of an ion makes with the normal to the pole faces is a function of the energy. An elementary analysis of this situation leads to the new coordinates

$$y' - y'' = \frac{.118 \cos(i + \beta)}{\cos i} \qquad (43)$$

$$x' - x'' = \frac{.118 \sin(i + \beta)}{\cos i} \qquad (44)$$

where x' and y' are the coordinates given by (40) and (39), x'' and y'' are the corrected coordinates after the edge corrections have been taken into account, and i is the angle of incidence. On page 56. tables are given of these four coordinates for the same range of values of κ as has been used throughout this investigation. For the purposes of design that page represents the final result of all this theory.

κ	x′	y′	x″	y″
.5	2.89903	2.0267		
.55	2.71363	1.99393	2.6614	1.8661
.6	2.52858	1.96793	2.4752	1.8416
.62	2.45458	1.95906	2.4009	1.8342
.64	2.38056	1.951	2.3265	1.8267
.66	2.30649	1.94361	2.2519	1.8199
.68	2.23238	1.93689	2.1773	1.8137
.7	2.15821	1.93078	2.1025	1.8081
.72	2.08395	1.92522	2.0278	1.8029
.74	2.0096	1.92014	1.9528	1.7981
.75	1.97239	1.9178	1.9154	1.7961
.76	1.93515	1.91556	1.8778	1.7941
.78	1.8606	1.91142	1.8027	1.7902
.8	1.78592	1.90768	1.7274	1.7868
.82	1.71112	1.90432	1.652	1.7837
.84	1.63619	1.90131	1.5764	1.7809
.85	1.59868	1.89994	1.5387	1.7797
.86	1.5611	1.8986	1.5008	1.7786
.88	1.48589	1.89626	1.425	1.7765
.9	1.41052	1.89419	1.3489	1.7746
.92	1.33499	1.89238	1.2728	1.773
.94	1.25929	1.89081	1.1964	1.7715
.95	1.22137	1.89011	1.1582	1.771
.96	1.18341	1.88948	1.1199	1.7705
.97	1.14541	1.88891	1.0816	1.77
.98	1.10737	1.88839	1.0432	1.7696
.99	1.06928	1.88792	1.0048	1.7691
1.	1.03115	1.8875	.9664	1.7688
1.01	.99296	1.88713	.9278	1.7685
1.02	.95474	1.88681	.8892	1.7683
1.03	.91647	1.88652	.8507	1.7681
1.04	.87814	1.88629	.8119	1.768
1.05	.83978	1.88612	.7733	1.7679
1.06	.80135	1.886	.7345	1.7678
1.08	.72437	1.8858	.6569	1.7677
1.1	.64719	1.88578	.579	1.7679
1.12	.5698	1.88592	.5009	1.7681
1.14	.49221	1.8862	.4226	1.7685
1.15	.45334	1.88639	.3834	1.7688
1.16	.4144	1.88661	.3441	1.7691
1.18	.33639	1.88715	.2654	1.7696
1.2	.25816	1.8878	.1865	1.7704
1.22	.17972	1.88858	.1073	1.7714
1.24	.10104	1.88945	.0278	1.7722
1.25	.06163	1.88993	- .0119	1.7728
1.26	.02214	1.89043	- .0517	1.7734
1.28	- .05698	1.8915	- .1316	1.7745
1.3	- .13635	1.89266	- .2117	1.7758
1.32	- .21593	1.89393	- .2919	1.7771
1.34	- .29576	1.89526	- .3726	1.7786
1.36	- .37583	1.89667	- .4533	1.7801
1.38	- .45614	1.89814	- .5344	1.7816
1.4	- .53668	1.89971	- .6157	1.7833
1.45	- .73915	1.90391	- .8201	1.7876
1.5	- .94319	1.90847		

MATHEMATICAL APPENDIX

In order to establish the existence, uniqueness, and uniform convergence of the solutions of the original inverse first-power field differential equation I would like to revamp and prove an existence and uniqueness theorem due to Birkhoff (George D. Birkhoff, Dynamical Systems, American Mathematical Society Colloquium Publications, Volume IX, pages 1-6 inclusive). I shall not attempt to make these theorems more general than is actually necessary for the particular case at hand. A more general proof may restrict the range of validity of the theorems to a very bothersome degree. Existence and uniqueness theorems are much easier to establish for a system of first order differential equations than for a single higher order equation. Hence the present second order equation will be reduced to two first order equations which will in no way invalidate the theorems established as applying either to the system or to the single equation.

The original differential equation is

$$u'' = u - k u^3 + \frac{2}{u}[u']^2 \tag{I}$$

where φ is the independent variable and primes denote differentiation with respect to φ. An interval of the type $0 \leqq \varphi \leqq b$ will be referred to as a <u>neighborhood</u> of $\varphi = 0$. Since we are concerned with the solution of the above differential equation which takes on the boundary values

$$u'(0) = 0 \quad \text{and} \quad u(0) = 1,$$

we shall use these same boundary conditions for all solutions of differential equations treated in this discussion. Thus when we speak of a solution of a differential equation we shall mean a solution of this differential equation which satisfies the above initial conditions.

Let us make the substitutions, $y = k\,u^2$ and $z = 2u'/u$.

$$y' = 2kuu' = ku^2 \cdot \frac{2u'}{u} = y\;z$$

$$z' = \frac{2u''}{u} - 2\frac{(u')^2}{u^2} = 2\left\{1 - ku^2 + 2\frac{(u')^2}{u^2}\right\} - 2\frac{(u')^2}{u^2} = z^2/2 + 2(1-y)$$

Hence we can replace the second order differential equation by the system of first order equations

$$\left\{\begin{array}{l} y' = y\;z \\ z' = z^2/2 + 2(1-y) \end{array}\right\} \qquad \text{(II)}$$

with the boundary conditions $y(0) = k$ and $z(0) = 0$. Since (I) and (II) are strictly equivalent except possibly for $u = 0$ and $u = \infty$, values which are not involved in the problem, an existence or uniqueness theorem for (II) immediately gives an entirely equivalent theorem for (I).

Birkhoff's Existence Theorem. We are here dealing with a physical system which is completely determined for any φ by the values of y and z corresponding to φ. Hence the system is such that the rates of change of y and z with φ depend merely upon the values of y and z themselves. The laws of motion can be expressed by the first order differential equations,

$$\left.\begin{aligned} \frac{dy}{d\varphi} &= X_1(y,z) = yz \\ \frac{dz}{d\varphi} &= X_2(y,z) = z^2/2 + 2(1-y) \end{aligned}\right\} \qquad (II)$$

The set of two functions X_i will be assumed to be real and uniformly continuous in some open finite two dimensional continuum R in the 'space' with rectangular coordinates y and z. A 'solution' $y(\varphi)$, $z(\varphi)$ of the equations (II) in the open interval $\varphi' < \varphi < \varphi''$ is defined to be a set of two functions $y(\varphi)$ and $z(\varphi)$, both continuous together with their first derivatives and represented for any such φ by a point (y,z) in R such that the differential equations are satisfied by this set of functions.

EXISTENCE THEOREM: If the point (k,0) is in R at a distance at least D from the boundary of R, and if M is an upper bound for the functions $|X_i|$ in R, there exists a solution $y(\varphi)$, $z(\varphi)$ of the functions (II) defined in the interval

$$|\varphi - \varphi_0| < \frac{D}{\sqrt{2}\,M} \qquad (\varphi_0 = 0)$$

and for which $y(0) = k$ and $z(0) = 0$.

To establish this theorem, we observe first that, for any solution of the type sought, the two equations

$$S_1 \equiv y - k - \int_0^{\varphi} yz\, d\varphi = 0 \;;$$

$$S_2 \equiv z - 0 - \int_0^{\varphi} (z^2/2 + 2(1-y))\, d\varphi = 0,$$

hold. Conversely any set of continuous functions $y(\varphi)$, $z(\varphi)$ in R which make the expressions S_i vanish in the interval containing $\varphi = 0$ as an interior point will obviously reduce to $y^0(=k)$, $z^0(=0)$ for $\varphi = 0$ and will satisfy the differential equations in question, as follows by direct differentiation.

Now define the set of infinitely multiple-valued functions $X_i^m(y,z)$ as that given by any set $X_i(s,t)$ taken at a point (s,t) whose various coordinates differ from those of the point (y,z) by not more than $1/m$ in numerical value. It is evident that with this definition the two components of X^m may be chosen as constant in any rectangular domain

$$|y - a_1| \leqq 1/m , \quad |z - a_2| \leqq 1/m ,$$

namely as the component parts of $X(a_1,a_2)$.

If the functions X_i be replaced by X_i^m and the functions y, z by y^m, z^m the expressions for S_i become

$$S_1^m \equiv y^m - k - \int_0^\varphi y^m z^m \, d\varphi$$

$$S_2^m \equiv z^m - \int_0^\varphi (z^2/2 + 2 - 2y)^m \, d\varphi .$$

We propose to show that these expressions can be made to vanish.

Choose X^m as $X(k,0)$ in the rectangular domain

$$|y - k| < 1/m \quad \text{and} \quad |z| < 1/m .$$

The integrals in the above expressions for S_i^m will then be linear functions of φ and hence y^m and z^m may be defined as

$$y^m = k + X_1(k,0)\varphi = k$$

$$z^m = X_2(k,0)\varphi = 2\,(1 - k)\varphi$$

as long as the point (y^m, z^m) continues to be in this domain. In geometrical terms, the expressions for $y^m(\varphi)$ and $z^m(\varphi)$ yield the coordinates of a straight line with φ as parameter, which passes through the center of the domain for $\varphi = 0$. If the functions X_i^m happen to vanish, the line reduces to the point $(k,0)$.

In case the line emerges from the domain for $\varphi = \varphi_1 > 0$ at a point (s^0, t^0) we can take this point as the center of a second like rectangular domain of the same dimensions, and take

$$y^m = s^0 + s^0 t^0 (\varphi - \varphi_1)$$

$$z^m = t^0 + \left\{ {t^0}^2/2 + 2(1-s^0) \right\} (\varphi - \varphi_1)$$

in this second domain. The expressions S_1^m and S_2^m will then continue to vanish for $\varphi \geqq \varphi_1$ until the point (y^m, z^m) leaves this second domain at a point (p^0, q^0).

Thus, by a succession of steps, the expressions S_1^m and S_2^m can be made to vanish for $\varphi > 0$ and likewise for $\varphi < 0$ (although we are not interested in the latter case). The process can only terminate in case the broken line representing $y^m(\varphi)$, $z^m(\varphi)$ passes a boundary point of R.

If y^m and z^m be taken as the coordinates of a point then the quantity

$$\left\{ (X_1^m)^2 + (X_2^m)^2 \right\}^{1/2} \leqq \sqrt{2}\, M$$

since

$$|X_i^m| \leqq M$$

by the definition of X_i^m and M . Hence the point must remain inside of R at least in the interval $|\varphi| < D/\sqrt{2}\,M$. It will develop subsequently that $D/\sqrt{2}\,M$ is the lesser of the two quantities $\pi/2$ and $\sqrt{2/k}$. Both functions y^m and z^m are defined in this fixed φ interval whatever be the value of m. (Note: m is a notation for a very specialized operation and is not an exponent.)

As m takes on the values 1, 2, 3, ..., there arises an infinite sequence of sets $y^m(\varphi)$, $z^m(\varphi)$ of functions defined in this interval. All of these sets lie in R, and so are uniformly bounded. Furthermore since $S_1{}^m$ and $S_2{}^m$ vanish for all m, the inequalities

$$\left| y^m(\varphi+\epsilon) - y^m(\varphi)\right| = \left| \int_\varphi^{\varphi+\epsilon} X_1{}^m(y^m, z^m)\, d\varphi \right| \leqq M\epsilon ,$$

$$\left| z^m(\varphi+\epsilon) - z^m(\varphi)\right| = \left| \int_\varphi^{\varphi+\epsilon} X_2{}^m(y^m, z^m)\, d\varphi \right| \leqq M\epsilon$$

obtain. We now need a special theorem due to Ascoli.

Osgood's Statement of Ascoli's Theorem on Uniform Convergence. Theorem: Let $f_n(x)$ be a real function of the positive integer n and the real variable x in the finite closed interval $a \leqq x \leqq b$; and let (a) $f_n(x)$ regarded as a function of x and n, be finite:

$$|f_n(x)| < M; \quad a \leqq x \leqq b , \quad n = 1, 2, 3, \ldots,$$

M being a positive constant; (b) let the difference quotient also remain finite

$$\left| \frac{f_n(x') - f_n(x'')}{x' - x''} \right| < M',$$

where x' and x'' are any two distinct points of the above interval, and n is arbitrary; M' being a positive constant.

Comment: From (b) it follows that $f_n(x)$ is a continuous function of x in the closed interval (a,b). Moreover, Condition (b) will always be fulfilled when $f_n(x)$ possesses a derivative which, regarded as a function of x and n, remains finite.

Then it is possible to choose from the functions $f_n(x)$ a set

$$f_{n_1}(x),\ f_{n_2}(x),\ \dots,$$

which converges uniformly in the above interval (a,b).

The statement and proof of this theorem is given by William F. Osgood, The Uniformization of Algebraic Functions, Annals of Mathematics, vol. 14, series 2, pages 152-153 (1912). References are also given there to the earlier work, in particular to that of Paul Koebe.

Hence as a special case of Ascoli's theorem there exists an infinite sequence of values of m for which both elements of the set y^m, z^m approaches a function $\bar{y}$, $\bar{z}$ of the set $(\bar{y},\bar{z})$ uniformly; these functions being themselves continuous.

It is easy to prove that the functions $\bar{y}$, $\bar{z}$ so obtained satisfy the integral form of the differential equations. In fact since S_1^m and S_2^m vanish for all m, we have

$$\bar{S}_1 = \bar{S}_1 - S_1^m = (\bar{y} - y^m) - \int_0^\varphi \left[X_1(\bar{y},\bar{z}) - X_1^m(y^m,z^m)\right] d\varphi \ ;$$

$$\bar{S}_2 = \bar{S}_2 - S_2^m = (\bar{z} - z^m) - \int_0^\varphi \left[X_2(\bar{y},\bar{z}) - X_2^m(y^m,z^m)\right] d\varphi \ .$$

For m sufficiently large, the first term on the right becomes uniformly small inasmuch as $\bar{y}$, $\bar{z}$ are approached uniformly by the corresponding y^m, z^m over the sequence under consideration. Also $X_1(\bar{y},\bar{z})$ and $X_2(\bar{y},\bar{z})$ will differ respectively from $X_1(y^m,z^m)$ and $X_2(y^m,z^m)$ by an uniformly small quantity, since X_1 and X_2 are uniformly continuous in R by hypothesis; and $X_1(y^m,z^m)$ and $X_2(y^m,z^m)$ in turn will differ from $X_1^m(y^m,z^m)$ and $X_2^m(y^m,z^m)$ by a uniformly small quantity, in virtue of the definitions of the functions X_1^m and X_2^m. Hence the quantity under the integral sign on the right also becomes uniformly small as m increases and the expressions $\bar{S}_1$ and $\bar{S}_2$, which are independent of m, must vanish as stated, so that $\bar{y}(\varphi)$, $\bar{z}(\varphi)$ yield the required solutions of the original system of differential equations.

By repeated use of the existence theorem, the given solutions $y(\varphi)$, $z(\varphi)$ may be extended beyond their interval of definition unless as φ approaches either end of the interval, the corresponding point $y(\varphi)$, $z(\varphi)$ approaches the boundary of R.

Birkhoff's Uniqueness Theorem. It may now be proved that there is only one solution of the type described in the existence theorem, in case the functions X_1 and X_2 possess continuous first partial derivatives as they do here. This last requirement may be lightened to a well known form given by Lipschitz.

Uniqueness Theorem: If for both i's and for every pair of points (y,z), (s,t) in R the functions X_i satisfy a Lipschitz condition

$$\left| X_i(y,z) - X_i(s,t) \right| \leqq L_1|y-s| + L_2|z-t| ,$$

the quantities L_1 and L_2 being fixed positive quantities, then there is only one solution $y(\varphi)$, $z(\varphi)$ such that $y(0) = k$, $z(0) = 0$.

For if two distinct solutions $y(\varphi)$, $z(\varphi)$ and $s(\varphi)$, $t(\varphi)$ respectively have the same values k,0 for $\varphi = 0$, the corresponding integral forms of the differential equations give at once

$$y - s - \int_0^\varphi [X_1(y,z) - X_1(s,t)]\, d\varphi = 0$$
$$z - t - \int_0^\varphi [X_2(y,z) - X_2(s,t)]\, d\varphi = 0 ,$$

and thence by the Lipschitz condition imposed,

$$y - s \leqq \int_0^\varphi \{L_1 |y - s| + L_2 |z - t|\}\, d\varphi$$
$$z - t \leqq \int_0^\varphi \{L_1 |y - s| + L_2 |z - t|\}\, d\varphi$$

Let L be the greater of the two positive constants L_1 and L_2, and let Q be the greater of the quantities $|y - s|$ and $|z - t|$ in any closed interval within the interval $|\varphi| \leqq 1/4L$. The maximum Q must be attained for some value of φ, say φ^*, and for either $|y-s|$ or $|z-t|$. If we insert the value φ^* of φ in the corresponding inequality above, and apply the mean value theorem to the right-hand member, there results

$$Q \leqq 2LQ\,|\varphi^* - \varphi_0| \leqq Q/2.$$

This proves that Q must be zero. Hence the two solutions $y(\varphi)$, $z(\varphi)$ and $s(\varphi)$, $t(\varphi)$ respectively which coincide for $\varphi = 0$ will continue to do so in any such interval. The theorem follows by repeated application of this result.

Theorem I. Let Φ denote the shorter of the two intervals: $0 \leqq \varphi < \pi/2$, $0 \leqq \varphi < \sqrt{2/k}$. There exists a solution of

$$u'' = u - ku^3 + \frac{2}{u}[u']^2 \qquad \text{(I)}$$

that is continuous together with its derivatives of all orders and satisfies (I) on Φ. Furthermore, if $u(\varphi)$ denotes this solution, then

$$\left.\begin{cases} 0 < u(\varphi) \leqq \text{secant } \varphi \\ 1 - k\varphi^2/2 \leqq u(\varphi) \end{cases}\right\} \text{at every point of } \Phi .$$

Proof: The equation (I) with $u'(0) = 0$ and $u(0) = 1$ is equivalent to the system

$$\left.\begin{cases} y' = yz \\ z' = z^2/2 + 2(1 - y) \end{cases}\right\} \qquad \text{(II)}$$

with $y(0) = k$ and $z(0) = 0$. A direct application of Birkhoff's existence and uniqueness theorems shows that system (II) has an unique solution in a neighborhood of $\varphi = 0$ that takes on the desired initial values. It follows from this that system (I) has an unique solution in a certain neighborhood of $\varphi = 0$.

Consider the equations,

$$w'' = w + \frac{2}{w}[w']^2 , \qquad w(0) = 1 , \qquad w'(0) = 0 \qquad \text{(III)}$$

$$v'' = -kv^3 , \qquad v(0) = 1 , \qquad v'(0) = 0 \qquad \text{(IV)}$$

For (III) we have, as was obtained in the calculations,

$$w = \sec \varphi .$$

For (IV) we have

$$\int_1^v \frac{dv}{\sqrt{1 - v^4}} = \pm \sqrt{k/2}\,\varphi.$$

Since the integral may be written in the form

$$\int_1^v \frac{dv}{\sqrt{1 - v^4}} \equiv \int_1^v \frac{dv}{\sqrt{(1+v^2)(1+v)(1-v)}}$$

its convergence for v = 1 is assured since the integral

$$\int_1^v \frac{dv}{\sqrt{1 - v}}$$

converges. Hence the inverse function $v \equiv v(\varphi)$ obtained from inverting the equation

$$\int_1^v \frac{dv}{\sqrt{1 - v^4}} = \pm \sqrt{k/2}\,\varphi$$

will converge. The integral $\int_1^v \frac{dv}{\sqrt{1 - v^4}}$ is not nearly so well known as $\int_1^v \frac{dv}{\sqrt{1 - v}}$. However since we are here interested in v as a member of an inequality so long as we are certain that v converges and serves as a lower bound to $u(\varphi)$ we are satisfied.

In the caption to Theorem I the second equality is derived from the use of

$$v'' = -k, \quad v'(0) = 0, \quad v(0) = 1,$$

which gives the unique solution $v(\varphi) = 1 - k\varphi^2/2$ in place of the (IV) given above. The (IV) above is a somewhat better lower bound for $u(\varphi)$ than this one. In case complications occur in the inversion of the integral we may revert to this form. Hence equation (III) has the unique solution $w(\varphi) = \sec\varphi$ while equation (IV) has in one form the unique solution

$v(\varphi) = 1 - k\varphi^2/2$ and in the other, the function obtained by the inversion of the integral above. These solution exist and are positive for φ on the interval Φ. We now show that for each φ on Φ, the solution $u(\varphi)$ of (I) exists and that

$$v(\varphi) \leqq u(\varphi) \leqq w(\varphi) . \qquad \text{(V)}$$

We shall now compare the right-hand sides of equations (I), (III), and (IV). It will be observed that if $u(\varphi)$ were identical with $w(\varphi)$ at some point then the right-hand side of (III) would exceed the right-hand side of (I) since the neglected term is actually negative on the range considered. The fact that $w(\varphi) > u(\varphi)$ for $\varphi \neq 0$ serves to strengthen this inequality $w'' > u''$. Exactly similar reasoning holds in the comparision of (I) and (IV) except for the reversal of the inequality signs. In connection with (IV) it is to be noted that $v(\varphi)$ is actually less than unity when $\varphi > 0$.

We have now established that $v''(0) < u''(0) < w''(0)$ and this inequality holds throughout some neighborhood of $\varphi = 0$. This follows from the continuity of these second derivatives and the above established existence. It follows immediately from

$$\begin{aligned} v(\varphi) &= 1 + \int_0^\varphi \int_0^t v'' \, ds \, dt , \\ u(\varphi) &= 1 + \int_0^\varphi \int_0^t u'' \, ds \, dt , \qquad \text{(VI)} \\ w(\varphi) &= 1 + \int_0^\varphi \int_0^t w'' \, ds \, dt , \end{aligned}$$

that the inequalities,

$$v(\varphi) < u(\varphi) < w(\varphi) \qquad \text{(VII)}$$

hold on this same sub-interval (where $v'' < u'' < w''$) with the exception of the point $\varphi = 0$. Let us assume a point q $(q > 0)$ of the interval Φ at which one of the inequalities (VII) fails. Also let q be the smallest positive number for which (VII) fails. (Such a smallest number exists from the closure of the point sets on which $v(\varphi) = u(\varphi)$, $u(\varphi) = w(\varphi)$, these functions being continuous, and the above established fact that $\varphi = 0$ is not a limit point of either of these point sets).

Since (VII) holds interior to $0 < \varphi < q$, it follows from (I), (III), and (IV) that $v'' < u'' < w''$ at each point of $0 < \varphi < q$. Hence equations (VI) yield $v(q) < u(q) < w(q)$. It follows from this that the inequalities (VII) hold on Φ provided the solution $u(\varphi)$ can be extended throughout this interval. Such extension is immediately accomplished by the usual procedure. Assume a point $\varphi = q$ on Φ such that this point is either the last point of Φ for which the solution $u(\varphi)$ exists or else it is the first point for which $u(\varphi)$ fails to exist. (Thus we follow the idea of the Dedekind Cut). Let M be the bound of $w(\varphi) = \sec\varphi$ on the interval $0 \leqq \varphi \leqq q$. The inequality (VII) holds at all points of a neighborhood of q where the solution $u(\varphi)$ exists: If we use the bound M + 1 and apply Birkhoff's existence theorem we prove that $u(\varphi)$ exists at all points of an interval of positive length that has q for an interior point. Hence such a point q does not exist and the solution $u(\varphi)$ exists on Φ. We have thus established the existence of $u(\varphi)$ on Φ and shown that it satisfies the inequalities

$$\text{or}\quad \begin{Bmatrix} 0 \leqq 1 - k\varphi^2/2 \leqq u(\varphi) \leqq \sec\varphi \\ 0 \leqq v(\varphi) \leqq u(\varphi) \leqq \sec\varphi \end{Bmatrix} \qquad \text{(VIII)}$$

where $v(\varphi)$ is given in the second case by the inversion of an integral. These inequalities hold on the shorter of the two intervals $0 \leqq \varphi \leqq \pi/2$, $0 \leqq \varphi \leqq \sqrt{2/k}$. (IX)

The existence and continuity of all derivatives follow from the continuity of u and u' as given by the existence theorem and the fact that u'' is a continuous function of these (by virtue of the original differential equation). Higher derivatives are calculated by differentiating the equation (I).

<u>Theorem II</u>: On the interval Φ described in (IX) above we have

(A) When $k > 1$, $\quad 1 - k\varphi^2/2 \leqq u(\varphi) \leqq 1$;

(B) When $k = 1$, $\quad u(\varphi) \equiv 1$;

(C) When $k < 1$, $\quad 1 \leqq u(\varphi) \leqq \sec\varphi$;

for every φ on the interval Φ .

Proof: Theorem I establishes the existence of $u(\varphi)$ and also some of the inequalities desired in Theorem II. If we write equation (I) in the form

$$u'' = u - u^3 + \frac{2}{u}[u']^2 + (1-k)u^3 \qquad \text{(X)}$$

and repeat the arguments of the proof of Theorem I using the equation

$$s'' = s - s^3 + \frac{2}{s}[s']^2 \qquad \text{(XI)}$$

instead of equation (III) in the case (A) and using this equation instead of equation (IV) in case (C), the remaining inequalities are established. Case (B) is established by solving equation (I) for this value of k .

The series for $w(\varphi) = \sec\varphi$ converges uniformly on the range Φ and represents a solution of (III) on this interval. It is shown above that $w(\varphi)$ marjorates the solution $u(\varphi)$ on this interval. It has also been shown in the discussion of the virtual source that the series for $\sec\varphi$ marjorates the series for $u(\varphi)$. Hence the series for $u(\varphi)$ is uniformly convergent and represents the solution of the differential equation (I) on the interval Φ . By Theorem I we have obtained existence and uniqueness on the range of φ and k that is pertinent to the problem. Note that a lower bound for the solution $u(\varphi)$ has been established by Theorem II. The equalities of (A) and (C) of Theorem II hold only at $\varphi = 0$.

The series for lg u given in the calculations is absolutely and uniformly convergent on the range shown there. Hence the substitution of u as a series in φ and k into the series for lg u gives a series representation of lg u which converges absolutely and uniformly on the double range of φ and u, namely on Φ and $0 < u < 2$. Hence the series for lg u can be differentiated term by term and gives a series representation of $\frac{1}{u}\frac{du}{d\varphi}$ which converges in the above ranges. Hence existence, uniqueness, continuity, and uniform convergence have been established for the series representation of the three quantities u, lg u, and $\frac{1}{u}\frac{du}{d\varphi}$ on the interval Φ .

Appreciation: I am deeply indebted to Professor William M. Whyburn for his detailed counsel on the problems attacked on pages 66-72 inclusive. It was only through his help that it was possible to bring this part of the work to such a high degree of completion.