LORENTZ-ZYGMUND SPACES

AND

INTERPOLATION OF WEAK TYPE OPERATORS

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To Jill (and Thor)

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ABSTRACT

The Lorentz-Zygmund spaces $L^{pa}(\log L)^{\alpha}$ are a class of function spaces containing as special cases the classical Lebesgue spaces L^p, the Lorentz spaces L^{pa} and the Zygmund spaces $L^p(\log L)^{\alpha}$. It is shown here that the Lorentz-Zygmund spaces provide the correct framework for the interpolation theory of weak type operators. The interpolation principles established here unify many classical results in harmonic analysis. In particular, there are applications to the Fourier transform, the Hardy-Littlewood maximal operator, the Hilbert transform, and the Weyl fractional integrals.

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CHAPTER I

INTRODUCTION

1. Introduction

The principle of operator interpolation has extensive applications in harmonic analysis. To illustrate this principle let us consider a special case of the classical interpolation theorem of Riesz-Thorin [43, Chapter XII]. Let T be a linear operator defined on the Lebesgue space $L^1(\mathbb{T})$, the class of Lebesgue integrable functions on the unit circle $\mathbb{T}$. Suppose T has the following properties:

$$T : L^1 \to L^1 \quad ({}^1)$$

and

$$T : L^\infty \to L^\infty;$$

equivalently, T is strong type (1,1) and strong type (∞,∞), respectively (cf. (1.4)). Since the L^p spaces on the circle $\mathbb{T}$ satisfy the inclusions

$$L^\infty \subset L^p \subset L^1, \qquad 1 < p < \infty,$$

it is then natural to ask whether T is bounded on the L^p spaces, $1 < p < \infty$. The Riesz-Thorin theorem provides the affirmative answer:

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(1) The notation $T : X \to Y$, where X and Y are (quasi) normed spaces, signifies that T is a continuous map of X into Y.

$$T : L^p \to L^p, \qquad 1 < p < \infty. \tag{1.1}$$

Thus, the Riesz-Thorin theorem takes the strong type hypotheses on the "endpoint spaces" L^1 and L^∞ and "interpolates" to establish the intermediate result (1.1).

We shall see that, for the applications, the conditions strong type (1,1) and (∞, ∞) are much too stringent. Marcinkiewicz [28] was able to relax these conditions by introducing the notion of weak type (p,q) (cf. (1.5)). Under the weaker assumptions that the operator T is of weak types (1,1) and (∞,∞), Marcinkiewicz was still able to establish the desired interpolation result (1.1). Calderón [9] and Hunt [21] have shown that the Lorentz spaces L^{pa} form the natural setting for Marcinkiewicz's interpolation theorem.

Of course, the L^p spaces are not the only function spaces intermediate between L^1 and L^∞ for which we may want to interpolate operators of weak types (1,1) and (∞,∞). For instance, let L log L denote the class of all functions on the unit circle $\mathbb{T}$ for which

$$\frac{1}{2\pi}\int_0^{2\pi} |f(t)| \, \log^+|f(t)|\,dt$$

is finite. The space L log L is related to the L^p spaces by the following inclusions:

$$L^p \subset L \log L \subset L^1, \qquad 1 < p \leq \infty.$$

Hence, L log L is very "close" to the "endpoint space" L^1. If T is a linear operator of weak types (1,1) and (∞,∞), we have the following result due to Zygmund [43, p. 119] :

$$T : L \log L \to L^1. \qquad (1.2)$$

The classical theory contains many results of the types described above. It was shown in [6] that the natural setting for all of these results is a class of function spaces $L^{pa}(\log L)^{\alpha}$ called the Lorentz-Zygmund spaces. They contain as special cases both the Lorentz spaces L^{pa} (take $\alpha = 0$) and the Zygmund spaces $L^{p}(\log L)^{\alpha}$ (take $a = p$). Furthermore, the interpolation theorems for these spaces produce easily the various classical estimates. These theorems were established in [6] in the context of the unit circle $\mathbb{T}$.

The purpose of this dissertation is to extend the results of [6] to arbitrary measure spaces. This more general setting requires a more complex technical machinery. However, we show that the simple structure of the results themselves remains intact.

The following operators arise naturally in classical harmonic analysis: the Fourier transform $\mathcal{F}$; the Hardy-Littlewood maximal operator M; the Hilbert transform H (conjugate-function operator); and the fractional integrals I_{λ}, $0 < \lambda < 1$. Each of these operators has a definition on functions on the unit circle $\mathbb{T}$, the integers $\mathbb{Z}$, or euclidean space $\mathbb{R}^n$ (cf. [36, 42, 43]). There are many classical results concerning the mapping properties of these operators on the Lebesgue spaces L^p, the Zygmund spaces $L^p(\log L)^{\alpha}$ and the Lorentz spaces L^{pa}. Each of these mapping properties is intrinsically interesting but, in their existing form, they are apparently unrelated. However, in the setting of the Lorentz-Zygmund spaces, all of these classical results are unified by a natural, comprehensive interpolation theory.

Let $(\mathfrak{X},\mu)$ be any measure space and let f^* denote the decreasing rearrangement of a measurable function f on $\mathfrak{X}$ (cf. (3.2)). For $0 < p, a \leq \infty$, the Lorentz space $L^{pa}(\mathfrak{X}) \equiv L^{pa}$ consists of all (classes of) measurable functions f for which the quasinorm

$$\|f\|_{pa} = \begin{cases} \left(\int_0^\infty [t^{1/p} f^*(t)]^a \, dt/t\right)^{1/a}, & 0 < a < \infty, \\ \sup_{0<t<\infty} t^{1/p} f^*(t), & a = \infty, \end{cases} \tag{1.3}$$

is finite.

A quasilinear operator T mapping measurable functions on a measure space $(\mathfrak{X},\mu)$ into measurable functions on a measure space $(\mathcal{Y},\nu)$ is <u>strong type (p,q)</u> if

$$T : L^p \to L^q, \qquad 0 < p,q \leq \infty, \tag{1.4}$$

and is <u>weak type (p,q)</u> if

$$T : L^{p1} \to L^{q\infty}, \qquad 0 < p < \infty, \quad 0 < q \leq \infty, \tag{1.5.i}$$

$$T : L^{\infty} \to L^{q\infty}, \qquad p = \infty, \quad 0 < q \leq \infty. \tag{1.5.ii}$$

For $p \geq 1$, the notion of weak type (p,q) is indeed weaker than the notion of strong type (p,q) (cf. Section 5).

The definition (1.3) shows that the Lorentz space $L^{\infty 1} = \{0\}$. Hence, the definition (1.5.i) is of no interest when $p = \infty$. The definition (1.5.ii) is somewhat artificially made to accommodate this situation. The definitions (1.4) and (1.5.ii) show that weak type (∞,∞) is equivalent to strong type (∞,∞). Thus, the condition weak type (∞,∞) is much too restrictive as we shall see later.

The two theorems cited below give the known weak type and strong type mapping properties for the aforementioned operators.

THEOREM 1.1 (a) : The Fourier transform $\mathcal{F}$ is strong (hence weak) types $(1,\infty)$ and $(2,2)$.

(b) (Hardy-Littlewood [15]). The Hardy-Littlewood maximal operator M is weak types $(1,1)$ and (∞,∞).

(c) (Zygmund [41]). The fractional integral operator I_λ, $0 < \lambda < 1$, is weak types $(1, (1-\lambda)^{-1})$ and (λ^{-1},∞).

(d) (Kolmogorov [23]). The Hilbert transform H is weak type $(1,1)$.

THEOREM 1.2 (a) (Hausdorff-Young [19, 43, p. 101]) : The Fourier transform $\mathcal{F}$ has the property:

$$\mathcal{F} : L^p \to L^q, \qquad 1 < p < 2, \quad 1/p + 1/q = 1.$$

(b) (Hardy-Littlewood [15, 42, p. 32]). The Hardy-Littlewood maximal operator M has the property:

$$M : L^p \to L^p, \qquad 1 < p < \infty.$$

(c) (Hardy-Littlewood [14, 43, p. 142]). The fractional integral operator I_λ, $0 < \lambda < 1$, has the property:

$$I_\lambda : L^p \to L^q, \qquad 1 < p < q < \infty, \quad 1/p - 1/q = \lambda.$$

(d) (M. Riesz [34, 42, p. 254]). The Hilbert transform H has the property:

$$H : L^p \to L^p, \qquad 1 < p < \infty .$$

Theorems 1.1.d and 1.2.d show one might expect (by "extrapolation") some form of a weak type estimate for the Hilbert transform H at the "endpoint" ∞ . However, H is not bounded on L^∞; in fact, H is not even bounded on characteristic functions (cf. [37]). (Actually H maps L^∞ into the larger class of functions of bounded mean oscillation, BMO (cf. [12]).) Hence, H is not weak type (∞,∞)(= strong type (∞,∞)). This difficulty is overcome by introducing the notion of weak type $(p,q;r,s)$ ($p < r$, and $q \neq s$; see Section 13 for the definition and details). The notion of weak type $(p,q;r,s)$ was first introduced in [4] when $p = q$ and $r = s$, and in general was first presented in [6]. If $r < \infty$, a quasilinear operator T is of weak type $(p,q;r,s)$ if and only if T is weak types (p,q) and (r,s). The crucial difference occurs when $r = \infty$. The following theorem summarizes the weak type estimates for the operators $\mathcal{F}$, M, H and I_λ relative to the notion of weak type $(p,q;r,s)$.

THEOREM 1.3 ([6, Part IV]):

(a) The Fourier transform $\mathcal{F}$ is weak type $(1,\infty;2,2)$.

(b) The Hardy-Littlewood maximal operator M is weak type $(1,1;\infty,\infty)$.

(c) The fractional integral operator I_λ, $0 < \lambda < 1$, is weak type $(1, (1 - \lambda)^{-1}; \lambda^{-1}, \infty)$.

(d) The Hilbert transform H is weak type $(1,1;\infty,\infty)$.

Theorem 1.3.d shows that we now have weak type estimates at both "endpoints" 1 and ∞ for the Hilbert transform H. This fact, in conjunction with the interpolation theorems cited below, gives a direct verification of the strong type mapping properties for H (cf. Theorem 1.2.d) in addition to establishing a variety of other mapping properties.

Suppose $0 < p, a \leq \infty$, $-\infty < \alpha < \infty$. The <u>Lorentz-Zygmund space $L^{pa}(\log L)^{\alpha}$</u> defined on any measure space $(\mathfrak{X},\mu)$ is the set of measurable functions for which the quasinorm

$$\|f\|_{pa;\alpha} = \begin{cases} \left(\int_0^\infty [t^{1/p}(1+|\log t|)^{\alpha} f^*(t)]^a \, dt/t\right)^{1/a}, & 0 < a < \infty, \\ \sup_{0<t<\infty} t^{1/p}(1+|\log t|)^{\alpha} f^*(t), & a = \infty, \end{cases} \tag{1.6}$$

is finite. When $\alpha = 0$, the space $L^{pa}(\log L)^0$ reduces to the Lorentz space L^{pa}. When the underlying measure space is the circle $\mathbb{T}$ and $a = p$, the Lorentz-Zygmund space $L^{pp}(\log L)^{\alpha}$ reduces to the Zygmund space $L^p(\log L)^{\alpha}$ (cf. [6, Section 10]). With respect to these Lorentz-Zygmund spaces and operators of weak type $(p,q;r,s)$, we shall prove the following generalization of the fundamental Marcinkiewicz interpolation theorem.

<u>THEOREM A</u>: Let $0 < p < r \leq \infty$ and $0 < q,s \leq \infty$, with $q \neq s$. Suppose T is a quasilinear operator of weak type $(p,q;r,s)$. Suppose $0 < \theta < 1$ and let

$$\frac{1}{u} = \frac{\theta}{p} + \frac{1-\theta}{r} \quad ; \quad \frac{1}{v} = \frac{\theta}{q} + \frac{1-\theta}{s} \quad . \tag{1.7}$$

Let $0 < a \leq \infty$ and $-\infty < \alpha < \infty$. Then

$$T : L^{ua}(\log L)^{\alpha} \to L^{va}(\log L)^{\alpha} .$$

Using Theorem A and the weak type estimates of Theorem 1.3, we now have a simple proof of Theorem 1.2.

Next we choose θ' such that $0 < \theta' < \theta < 1$, and we define u' and v' so that u', v' and θ' are related as in (1.7). Then Theorem A also shows that (for $0 < a' \leq \infty$, $-\infty < \alpha' < \infty$)

$$T : L^{ua}(\log L)^{\alpha} + L^{u'a'}(\log L)^{\alpha'} \to L^{va}(\log L)^{\alpha} + L^{v'a'}(\log L)^{\alpha'}, \qquad (1.8)$$

and

$$T : L^{ua}(\log L)^{\alpha} \cap L^{u'a'}(\log L)^{\alpha'} \to L^{va}(\log L)^{\alpha} \cap L^{v'a'}(\log L)^{\alpha'}. \qquad (1.9)$$

The next theorem deals with the limiting case of these results where we let $\theta \to 1$ and $\theta' \to 0$ (that is, $u = p$, $v = q$, $u' = r$ and $v' = s$).

<u>THEOREM B</u>: Let $0 < p < r \leq \infty$, and $0 < q, s \leq \infty$, with $q \neq s$. Suppose T is a quasilinear operator of weak type $(p,q;r,s)$. Suppose $1 \leq a \leq b \leq \infty$, $1 \leq c \leq d \leq \infty$ and $-\infty < \alpha, \beta, \gamma, \delta < \infty$. Then,

(a) if $\alpha + 1/a = \beta + 1/b > 0$ and $\gamma + 1/c = \delta + 1/d > 0$, we have

$$T : L^{pa}(\log L)^{\alpha+1} + L^{rc}(\log L)^{\gamma+1} \to L^{qb}(\log L)^{\beta} + L^{sd}(\log L)^{\delta};$$

and

(b) if $\alpha + 1/a = \beta + 1/b < 0$ and $\gamma + 1/c = \delta + 1/d < 0$, we have

$$T : L^{pa}(\log L)^{\alpha+1} \cap L^{rc}(\log L)^{\gamma+1} \to L^{qb}(\log L)^{\beta} \cap L^{sd}(\log L)^{\delta} .$$

The sums appearing in Theorem B are defined as follows. If $p < q$, the space $L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta}$ is generated by the quasinorm

$$\left(\int_0^1 [t^{1/p}(1-\log t)^{\alpha} f^*(t)]^a dt/t\right)^{1/a} \qquad (1.10)$$
$$+ \left(\int_1^{\infty} [t^{1/q}(1+\log t)^{\beta} f^*(t)]^b dt/t\right)^{1/b},$$

and if $p > q$, the generating quasinorm is

$$\left(\int_0^1 [t^{1/q}(1-\log t)^{\beta} f^*(t)]^b dt/t\right)^{1/b} \qquad (1.11)$$
$$+ \left(\int_1^{\infty} [t^{1/p}(1+\log t)^{\alpha} f^*(t)]^a dt/t\right)^{1/a}.$$

These spaces are the usual algebraic sums of the spaces $L^{pa}(\log L)^{\alpha}$ and $L^{qb}(\log L)^{\beta}$ in all instances except when one of the spaces is trivial (cf. Section 9).

The function space generated by (1.10) when $p > q$, or by (1.11) when $p < q$ is just the usual set theoretic intersection $L^{pa}(\log L)^{\alpha} \cap L^{qb}(\log L)^{\beta}$ (cf. Section 9).

In the case where the underlying measure space is the unit circle $\mathbb{T}$ (or any finite measure space), the sums and intersections in Theorem B reduce to a single Lorentz-Zygmund space (cf. Section 7). When the underlying measure space is the integers $\mathbb{Z}$, the sums and intersections now reduce to a Lorentz-Zygmund sequence space $\ell^{pa}(\log \ell)^{\alpha}$ (cf. Section 9), consisting of sequences $\{c_n\}$ for which the quasinorm

$$\|\{c_n\}\|_{\ell^{pa}(\log \ell)^\alpha} = \left(\sum_{n=1}^{\infty}[n^{1/p}(1+\log n)^\alpha c_n^*]^a n^{-1}\right)^{1/a}$$

is finite (here $\{c_n^*\}_{n=1}^{\infty}$ is the decreasing rearrangement of the sequence $\{c_n\}$).

In view of the above remarks, when T maps function spaces on $\mathbb{T}$ or $\mathbb{Z}$ into function spaces on $\mathbb{T}$ or $\mathbb{Z}$, the statement of Theorem B is simplified. Restating Theorem B in these special instances, we have the following theorems.

<u>THEOREM B1</u> (Measure spaces $\mathbb{T}$, $\mathbb{T}$) : Suppose $0 < p < r \leq \infty$ and $0 < q < s \leq \infty$. Then

(a) $T : L^{pa}(\log L)^{\alpha+1} \to L^{qb}(\log L)^{\beta}$, if $\alpha + 1/a = \beta + 1/b > 0$;

and

(b) $T : L^{ra}(\log L)^{\alpha+1} \to L^{sb}(\log L)^{\beta}$, if $\alpha + 1/a = \beta + 1/b < 0$.

<u>THEOREM B2</u> (Measure spaces $\mathbb{T}$, $\mathbb{Z}$) : Suppose $0 < p < r < \infty$ and $0 < s < q \leq \infty$. Then

(a) $T : L^{pa}(\log L)^{\alpha+1} \to \ell^{qb}(\log \ell)^{\beta}$, if $\alpha + 1/a = \beta + 1/b > 0$;

and

(b) $T : L^{ra}(\log L)^{\alpha+1} \to \ell^{sb}(\log \ell)^{\beta}$, if $\alpha + 1/a = \beta + 1/b < 0$.

Similar restatements of Theorem B can be obtained by other combinations of the measure spaces $\mathbb{T}$ and $\mathbb{Z}$.

We may now present some classical results in harmonic analysis which, after reformulation in terms of Lorentz-Zygmund spaces, are direct consequences of Theorems B1 or B2. To apply Theorem B1 or B2 we restrict our attention to operators defined on the circle $\mathbb{T}$.

THEOREM 1.4 (Hardy-Littlewood [15, 43, pp. 158-159]): The Hardy-Littlewood maximal operator M has the property:

$$M : L(\log L) \to L^1.$$

THEOREM 1.5 (Zygmund [39, 41, 42, 43]):

(a) For the conjugate-function operator H, we have

$$H : L(\log L) \to L^1.$$

(b) If $|f| \leq 1$ a.e., then

$$\int_0^{2\pi} \exp(\gamma |Hf|) \leq C < \infty,$$

for some positive constants γ and C independent of f.

(c) H also has the property:

$$H : L(\log L)^{\alpha} \to L(\log L)^{\alpha-1}, \qquad \alpha > 0.$$

THEOREM 1.6: If $0 < \lambda < 1$, the Weyl fractional integral operator I_λ has the following properties:

(a) (Zygmund [40]). $I_\lambda : L(\log L)^{1-\lambda} \to L^{1/(1-\lambda)}$.

(b) (Zygmund [43, pp. 158-159]). If $\|f\|_{L^{1/\lambda}} \leq 1$, then

$$\int_0^{2\pi} \exp(\gamma |I_\lambda f|^{1/(1-\lambda)}) \leq C < \infty,$$

where γ and C are positive constants independent of f.

(c) (O'Neil [31]). Let $p = (1-\lambda)^{-1}$. Then

(i) $I_\lambda : L(\log L)^\alpha \to K^p(\log^+ K)^{p(\alpha-1)}, \qquad \alpha \geq 1,$

and

(ii) $I_\lambda : L(\log L)^\alpha \to L^{p\alpha^{-1}}, \qquad 0 < \alpha < 1.$

THEOREM 1.7 (Hardy-Littlewood [16,17], Zygmund [40]): Let $\mathcal{F}(f) = \{c_n\}_{n=-\infty}^{\infty}$ be the sequence of Fourier coefficients of f with respect to the orthonormal system e^{int}, $n = 0, \pm 1, \pm 2, \ldots$. Let $\{c_n^*\}_{n=1}^{\infty}$ denote the decreasing rearrangement of $\{c_n\}_{n=-\infty}^{\infty}$. Let $f \in L(\log L)^\alpha$, $\alpha > 0$.

(a) For some constants A_α and B_α independent of f, we have

$$\sum_{n=1}^{\infty} n^{-1}(\log n)^{\alpha-1} c_n^* \leq A_\alpha \int_0^{2\pi} |f|\,(\log^+|f|)^\alpha + B_\alpha .$$

(b) If $0 < \alpha \leq 1$, then

$$\sum_{n=1}^{\infty} \frac{|c_n^*|^{1/\alpha}}{n} < \infty .$$

We have already remarked that the spaces L^{pa} and $L^p(\log L)^\alpha$ are the Lorentz-Zygmund spaces $L^{pa}(\log L)^0$ and $L^{pp}(\log L)^\alpha$, respectively. The following theorem, proved in [6, Section 10], shows that the other classes of functions mentioned in the four previous theorems are also Lorentz-Zygmund spaces.

THEOREM 1.8:

(a) If $\alpha > 0$, the Lorentz-Zygmund space $L^{\infty\infty}(\log L)^{-\alpha}$ is the Zygmund space consisting of those functions f (on $\mathbb{T}$) for which

$$\int_0^{2\pi} \exp(\lambda|f|^{1/\alpha}) < \infty,$$

for some positive constant $\lambda = \lambda(f)$.

(b) If $0 < \alpha, p < \infty$, the Lorentz-Zygmund space $L^{p1}(\log L)^{\alpha}$ coincides with O'Neil's space $K^p(\log^+ K)^{\alpha p}$. (The space $K^p(\log^+ K)^{\alpha}$ consists of all f for which

$$\int_1^{\infty} D_f(y)^{1/p}(\log y)^{\alpha/p} dy < \infty,$$

where D_f is the distribution function of f (cf. (3.1))).

(c) For sequences $\{c_n\}_{n=-\infty}^{\infty}$ of complex numbers and $\alpha > 0$, we have that

$$\text{(i)} \quad \left\{\{c_n\}_{n=-\infty}^{\infty} : \sum_{n=1}^{\infty} n^{-1}(\log n)^{\alpha} c_n^* < \infty\right\} = \ell^{\infty 1}(\log \ell)^{\alpha};$$

and (ii) if $0 < \alpha \leq 1$, we have

$$\left\{\{c_n\}_{n=-\infty}^{\infty} : \sum_{n=1}^{\infty} \frac{|c_n^*|^{1/\alpha}}{n} < \infty\right\} = \ell^{\infty, 1/\alpha}(\log \ell)^{0}.$$

Applying Theorem 1.8 to reformulate Theorems 1.4 through 1.7, these theorems now reflect that the various operators M, H, I_λ and $\mathcal{F}$ are continuous transformations of Lorentz-Zygmund spaces.

<u>THEOREM 1.4'</u> (Hardy-Littlewood):

$$M : L^{11}(\log L)^1 \to L^{11}(\log L)^0 .$$

<u>THEOREM 1.5'</u> (Zygmund):

(a) $$H : L^{11}(\log L)^1 \to L^{11}(\log L)^0.$$

(b) $H : L^{\infty\infty}(\log L)^0 \to L^{\infty\infty}(\log L)^{-1}$.

(c) $H : L^{11}(\log L)^\alpha \to L^{11}(\log L)^{\alpha-1}, \qquad \alpha > 0.$

<u>THEOREM 1.6′</u> (Zygmund, O'Neil): Let $p = (1-\lambda)^{-1}$.

(a) $I_\lambda : L^{11}(\log L)^{1-\lambda} \to L^{pp}(\log L)^0$.

(b) $I_\lambda : L^{\lambda^{-1}\lambda^{-1}}(\log L)^0 \to L^{\infty\infty}(\log L)^{\lambda-1}$.

(c) (i) $I_\lambda : L^{11}(\log L)^\alpha \to L^{p1}(\log L)^{\alpha-1}, \qquad \alpha \geq 1,$

and

(ii) $I_\lambda : L^{11}(\log L)^\alpha \to L^{p,1/\alpha}(\log L)^0, \qquad 0 < \alpha \leq 1.$

<u>THEOREM 1.7′</u> (Hardy-Littlewood, Zygmund):

(a) $\mathcal{F} : L^{11}(\log L)^\alpha \to \ell^{\infty 1}(\log \ell)^{\alpha-1}, \qquad \alpha > 0.$

(b) $\mathcal{F} : L^{11}(\log L)^\alpha \to \ell^{\infty,1/\alpha}(\log \ell)^0, \qquad 0 < \alpha \leq 1.$

It is clear that the weak type estimates of Theorem 1.3 in conjunction with the interpolation Theorems B1 and B2 give simple proofs of Theorems 1.4′ through 1.7′. Let us call the number $\alpha + 1/a$ an <u>index of smoothness</u> for the Lorentz-Zygmund space $L^{pa}(\log L)^\alpha$ (or $\ell^{pa}(\log \ell)^\alpha$). The common feature of all the results in Theorems 1.4′ through 1.7′ is that the index of smoothness of the domain is always 1 greater than the index of smoothness of the indicated image space. This is the simple essence of Theorem B.

We conclude this section with a discussion of the Hilbert transform H on the real line $\mathbb{R}$ and the fractional integrals I_λ (Riesz potentials) on euclidean space $\mathbb{R}^n$. These results are implicit in the work of Calderón and Zygmund [10, 11] and are also discussed in the work of Torchinsky [38] and Koizumi [22].

THEOREM 1.9: For the Hilbert transform H on the real line, we have that

$$H : L(\log L)^{\alpha} + L^{p} \to L(\log L)^{\alpha-1} + L^{p}, \qquad (1.12)$$

for $\alpha > 0$ and $1 < p < \infty$.

THEOREM 1.10: Let $0 < \lambda < 1$ and $1 < p < q < \infty$, with $1/p - 1/q = \lambda$. Then the fractional integral operator I_λ (Riesz potential) has the property:

$$I_\lambda : L(\log L)^{1-\lambda} + L^{p} \to L^{1/(1-\lambda)} + L^{q}. \qquad (1.13)$$

The space $L(\log L)^{\alpha} + L^{p}$ appearing in (1.12) has the quasinorm (cf. (1.10)

$$\int_0^1 (1-\log t)^{\alpha} f^*(t)dt + \left(\int_1^{\infty} [f^*(t)]^p dt\right)^{1/p} .$$

Hence, Theorem 1.9 is really just a combination of the local result in Theorem 1.5.c with the strong type result of Theorem 1.2.d for the Hilbert transform H. Similarly, we see that the result (1.13) for I_λ is a combination of the local result in Theorem 1.6.a with the strong type result in Theorem 1.2.c. Both Theorems 1.9 and 1.10 are instances of the following theorem, which is the limiting case of the results (1.8) and (1.9) where we let either $\theta \to 1$ or $\theta' \to 0$.

THEOREM C: Let $0 < p < r \leq \infty$, and $0 < q,s \leq \infty$, with $q \neq s$. Suppose T is a quasilinear operator of weak type $(p,q;r,s)$. Suppose

$1 \leq a \leq b \leq \infty$, $0 < c \leq \infty$, $-\infty < \alpha, \beta, \gamma < \infty$ and $0 < \theta < 1$. Let

$$\frac{1}{u} = \frac{\theta}{p} + \frac{1-\theta}{r} \quad ; \quad \frac{1}{v} = \frac{\theta}{q} + \frac{1-\theta}{s} .$$

(a) If $\alpha + 1/a = \beta + 1/b > 0$, we have

(i) $T : L^{pa}(\log L)^{\alpha+1} + L^{uc}(\log L)^{\gamma} \to L^{qb}(\log L)^{\beta} + L^{vc}(\log L)^{\gamma}$,

and

(ii) $T : L^{uc}(\log L)^{\gamma} + L^{ra}(\log L)^{\alpha+1} \to L^{vc}(\log L)^{\gamma} + L^{sb}(\log L)^{\beta}$.

(b) If $\alpha + 1/a = \beta + 1/b < 0$, we have

(i) $T : L^{pa}(\log L)^{\alpha+1} \cap L^{uc}(\log L)^{\gamma} \to L^{qb}(\log L)^{\beta} \cap L^{vc}(\log L)^{\gamma}$,

and

(ii) $T : L^{uc}(\log L)^{\gamma} \cap L^{ra}(\log L)^{\alpha+1} \to L^{vc}(\log L)^{\gamma} \cap L^{sb}(\log L)^{\beta}$.

CHAPTER II

INEQUALITIES AND PRELIMINARIES

2. Generalized Hardy inequalities

The inequalities established in this section form the foundation of the subsequent development and will be appealed to frequently. We begin with two modest technical lemmas.

LEMMA 2.1: Let λ,μ and a be positive real numbers. Then, we have

$$1/2(\lambda^a+\mu^a) \leq (\lambda+\mu)^a \leq 2^a(\lambda^a+\mu^a). \tag{2.1}$$

Proof: Clearly,

$$\lambda \leq (\lambda^a + \mu^a)^{1/a}$$

and

$$\mu \leq (\lambda^a + \mu^a)^{1/a} .$$

Adding the two inequalities and taking a^{th} powers, we have the right-hand inequality in (2.1). Replacing λ by $\lambda^{1/a}$ and μ by $\mu^{1/a}$ in the inequality just established, then taking a^{th} roots, we obtain

$$1/2(\lambda^{1/a} + \mu^{1/a}) \leq (\lambda + \mu)^{1/a}.$$

This is precisely the left-hand inequality in (2.1) with a replaced by 1/a.

LEMMA 2.2: Let $\beta > 0$, α real. Then there is $N = N(\alpha,\beta) \geq 1$ such that

(a) $t^{-\beta}(N + |\log t|)^{\alpha}$ is decreasing for $t \in (0,\infty)$;

(b) $t^{\beta}(N + |\log t|)^{\alpha}$ is increasing for $t \in (0,\infty)$.

<u>Proof</u>: An examination of the derivative shows that $N = 1 + |\alpha|/\beta$ will suffice.

The following two lemmas were established in [6, Lemmas 6.1 and 6.2].

<u>LEMMA 2.3</u>: Suppose $0 < a \leq \infty$, and $0 \leq \nu < \infty$. Let φ be a nonnegative decreasing function on $(0,\infty)$. Then, for each $0 < t < \infty$, we have

$$\sup_{0<s\leq t} s^{\nu}\varphi(s) \leq c\left(\int_0^t [s^{\nu}\varphi(s)]^a ds/s\right)^{1/a} \quad (^2) \tag{2.2}$$

and

$$\sup_{t\leq s<\infty} s^{\nu}\varphi(s) \leq c\left(\int_{t/2}^{\infty} [s^{\nu}\varphi(s)]^a ds/s\right)^{1/a}, \tag{2.3}$$

where c is a constant independent of φ and t.

<u>LEMMA 2.4</u>: Let $0 < \nu < \infty$ and let φ be a nonnegative decreasing function on $(0,\infty)$. Then, for $0 < t < \infty$,

(a) if $0 < a \leq 1$, then

$$\int_0^t s^{\nu}\varphi(s)ds/s \leq c\left(\int_0^t [s^{\nu}\varphi(s)]^a ds/s\right)^{1/a}, \tag{2.4}$$

and

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(2) Throughout this paper, when $a = \infty$, an integral $\left(\int_c^d [\psi(t)]^a dt/t\right)^{1/a}$ is to be interpreted as $\operatorname{ess\,sup}_{c\leq t\leq d} \psi(t)$.

$$\int_t^\infty s^\nu \varphi(s)ds/s \le c\left(\int_{t/2}^\infty [s^\nu \varphi(s)]^a ds/s\right)^{1/a} . \tag{2.5}$$

(b) If $1 \le a \le \infty$, then

$$\left(\int_0^t [s^\nu \varphi(s)]^a ds/s\right)^{1/a} \le c\int_0^t s^\nu \varphi(s)ds/s, \tag{2.6}$$

and

$$\left(\int_t^\infty [s^\nu \varphi(s)]^a ds/s\right)^{1/a} \le c\int_{t/2}^\infty s^\nu \varphi(s)ds/s. \tag{2.7}$$

The next theorem is a variant of the classical Hardy inequalities (cf. [21, p. 256]). For measurable functions on (0,1), it was first proved in [6, Theorem 6.4].

THEOREM 2.5: Suppose $\lambda > 0$, $1 \le a \le \infty$ and $-\infty < \alpha < \infty$. Let ψ be a nonnegative measurable function on $(0,\infty)$. Then the following four inequalities hold:

(a)
$$\left(\int_0^1 [t^{-\lambda}(1-\log t)^\alpha \int_0^t \psi(s)ds]^a dt/t\right)^{1/a} \le c\left(\int_0^1 [t^{-\lambda+1}(1-\log t)^\alpha \psi(t)]^a dt/t\right)^{1/a}; \tag{2.8}$$

(b)
$$\left(\int_0^1 [t^{\lambda}(1-\log t)^\alpha \int_t^1 \psi(s)ds]^a dt/t\right)^{1/a} \le c\left(\int_0^1 [t^{\lambda+1}(1-\log t)^\alpha \psi(t)]^a dt/t\right)^{1/a}; \tag{2.9}$$

(c)
$$\left(\int_1^\infty [t^{-\lambda}(1+\log t)^\alpha \int_1^t \psi(s)ds]^a dt/t\right)^{1/a} \le c\left(\int_1^\infty [t^{-\lambda+1}(1+\log t)^\alpha \psi(t)]^a dt/t\right)^{1/a}; \tag{2.10}$$

$$\text{(d)}\qquad \left(\int_1^\infty [t^\lambda(1+\log t)^\alpha \int_t^\infty \psi(s)ds]^a dt/t\right)^{1/a}$$
$$\leq c\left(\int_1^\infty [t^{\lambda+1}(1+\log t)^\alpha \psi(t)]^a dt/t\right)^{1/a}. \qquad (2.11)$$

Furthermore, suppose $0 < a < 1$ and $\psi(t) = t^{\mu-1}\varphi(t)$, where $\mu > 0$ and φ is a nonnegative decreasing function. Then (2.8) and (2.9) remain valid while (2.10) and (2.11) are replaced by

$$\text{(c}'\text{)}\qquad \left(\int_1^\infty [t^{-\lambda}(1+\log t)^\alpha \int_1^t \psi(s)ds]^a dt/t\right)^{1/a}$$
$$\leq c\left(\int_1^\infty [t^{-\lambda+1}(1+\log t)^\alpha \psi(t/2)]^a dt/t\right)^{1/a}; \qquad (2.10')$$

$$\text{(d}'\text{)}\qquad \left(\int_1^\infty [t^\lambda(1+\log t)^\alpha \int_t^\infty \psi(s)ds]^a dt/t\right)^{1/a}$$
$$\leq c\left(\int_1^\infty [t^{\lambda+1}(1+\log t)^\alpha \psi(t/2)]^a dt/t\right)^{1/a}. \qquad (2.11')$$

The constant c depends only on λ, a, α (and μ when $0 < a < 1$).

Proof: The proof given here for (2.10) and (2.11) is a modification of that given for the classical Hardy inequalities in [21, p. 256]. The inequalities (2.8) and (2.9) are the content of Theorem 6.4 in [6] and so we omit their proofs.

First, we note that when $N \geq 1$,

$$1 + |\log t| \leq N + |\log t| \leq N(1+|\log t|). \qquad (2.12)$$

Hence, it will suffice to prove the inequalities (2.10) and (2.11) with $(1 + |\log t|)$ replaced by $(N + |\log t|)$.

Let $1 \leq a \leq \infty$. To prove (2.10) choose γ so that

$$1-\lambda < \gamma < 1. \tag{2.13}$$

We next write, for $t > 1$,

$$\int_1^t \psi(s)ds = \int_1^t [s^\gamma \psi(s)][s^{1-\gamma}]ds/s.$$

Now we apply Hölder's inequality (with respect to the measure ds/s) to the expression on the right to obtain the inequality

$$\int_1^t \psi(s)ds \leq ct^{1-\gamma}\left(\int_1^t [s^\gamma \psi(s)]^a ds/s\right)^{1/a}. \tag{2.14}$$

Inequality (2.14), plus a change in the order of integration yields

$$\begin{aligned} &\int_1^\infty [t^{-\lambda}(N+\log t)^\alpha \int_1^t \psi(s)ds]^a dt/t \\ &\leq c\int_1^\infty [s^\gamma \psi(s)]^a \left(\int_s^\infty t^{-(\lambda+\gamma-1)a}(N+\log t)^{\alpha a} dt/t\right) ds/s. \quad (^3) \end{aligned} \tag{2.15}$$

Let $N = N(\alpha, 1/2(\lambda+\gamma-1))$ as in Lemma 2.2.a so that $t^{-(\lambda+\gamma-1)/2}(N+\log t)^\alpha$ decreases for $t \in [1,\infty)$. Thus, on $[s,\infty)$, $t^{-(\lambda+\gamma-1)/2}(N+\log t)^\alpha$ is largest when $t = s$. Letting I denote the right-hand side of (2.15), we have

$$I \leq c\int_1^\infty [s^\gamma \psi(s) \cdot s^{-(\lambda+\gamma-1)/2}(N+\log s)^\alpha]^a \left(\int_s^\infty t^{-(\lambda+\gamma-1)a/2} dt/t\right) ds/s.$$

The choice of γ, (2.13), allows us to evaluate the integral on (s,∞), giving

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(3) The constant c may change from line to line.

$$I \leq \frac{2c}{a(\lambda+\gamma-1)} \int_1^\infty [s^{-\lambda+1}(N+\log s)^\alpha \psi(s)]^a ds/s. \tag{2.16}$$

The estimates (2.15) and (2.16) produce (2.10) (modulo (2.12)) for $1 \leq a < \infty$.

For the case $a = \infty$, (2.14) reads

$$\int_1^t \psi(s)ds \leq ct^{1-\gamma} \sup_{1<s<t} s^\gamma \psi(s).$$

By Lemma 2.2.a, we choose $N = N(\alpha, \lambda + \gamma - 1)$ so that $t^{1-\gamma-\lambda}(N+\log t)^\alpha$ decreases. Let $1 < t < \infty$. then

$$t^{-\lambda}(N+\log t)^\alpha \int_1^t \psi(s)ds \leq ct^{1-\gamma-\lambda}(N+\log t)^\alpha \sup_{1<s<t} s^\gamma \psi(s)$$

$$\leq c \sup_{1<s<t} s^{-\lambda+1}(N+\log s)^\alpha \psi(s).$$

Replacing the supremum over $(1,t)$ by the supremum over $(1,\infty)$, and taking the supremum of the left-hand side over $1 < t < \infty$, we have (2.10) for this case, too.

Now suppose $0 < a < 1$. We then have $\psi(s) = s^{\mu-1}\varphi(s)$ where $\mu > 0$ and φ is decreasing. In addition to (2.13), we choose γ so that

$$\gamma > 1 - \mu . \tag{2.17}$$

Now we write, for $1 < t < \infty$,

$$\int_1^t \psi(s)ds = \int_1^t s^{1-\gamma}[s^{\mu+\gamma-1}\varphi(s)]ds/s \leq t^{1-\gamma}\int_1^t s^{\mu+\gamma-1}\varphi(s)ds/s.$$

By (2.17), $\mu + \gamma - 1 > 0$, so we may apply (2.5) (with $t = 1$) to obtain

$$\int_1^t \psi(s)ds \leq ct^{1-\gamma}\left(\int_{1/2}^t [s^\gamma \psi(s)]^a ds/s\right)^{1/a} .$$

This is the analogue of the crucial inequality (2.14). We then have the estimate

$$J \equiv \int_1^\infty [t^{-\lambda}(N+\log\, t)^\alpha \int_1^t \psi(s)ds]^a dt/t$$

$$\leq c\int_1^\infty [t^{-(\lambda+\gamma-1)}(N+\log\, t)^\alpha]^a\Big(\int_{1/2}^t [s^\gamma \psi(s)]^a ds/s\Big)dt/t.$$

Enlarging the range of integration $(1,\infty)$ to $(1/2,\infty)$ and then interchanging the order of integration, we obtain,

$$J \leq c\int_{1/2}^\infty [s^\gamma\psi(s)]^a\Big(\int_s^\infty t^{-(\lambda+\gamma-1)a}(N+|\log\, t|)^{\alpha a}dt/t\Big)ds/s.$$

Proceeding as before, we have the estimate

$$J \leq c\int_{1/2}^\infty [s^{-\lambda+1}(N+|\log\, s|)^\alpha \psi(s)]^a ds/s.$$

Changing variables (let $s = t/2$), we have

$$J \leq c\int_1^\infty [t^{-\lambda+1}(N+\log\, t)^\alpha \psi(t/2)]^a dt/t,$$

which is (2.10′) (modulo (2.12)).

The inequalities (2.11) and (2.11′) are proved quite similarly. We choose γ now so that

$$1 < \gamma < 1 + \lambda\,. \tag{2.18}$$

For $1 \leq a \leq \infty$, the analogue of the crucial inequality (2.14) is

$$\int_t^\infty \psi(s)ds \leq ct^{1-\gamma}\Big(\int_t^\infty [s^\gamma\psi(s)]^a ds/s\Big)^{1/a}. \tag{2.19}$$

By (2.18), $1 - \gamma + \lambda > 0$, so we choose N by Lemma 2.2.b so that $t^{(1-\gamma+\lambda)/2}(N+\log t)^{\alpha}$ increases. Applying (2.19), we have the estimate

$$\int_1^\infty [t^\lambda (N+\log t)^\alpha \int_t^\infty \psi(s)ds]^a dt/t$$

$$\le c\int_1^\infty [t^{1-\gamma+\lambda}(N+\log t)^\alpha]^a \Big(\int_t^\infty [s^\gamma \psi(s)]^a ds/s\Big) dt/t$$

$$= c\int_1^\infty [s^\gamma \psi(s)]^a \Big(\int_1^s [t^{1-\gamma+\lambda}(N+\log t)^\alpha]^a dt/t\Big) ds/s$$

$$\le c\int_1^\infty [s^\gamma \psi(s) s^{(1-\gamma+\lambda)/2}(N+\log s)^\alpha]^a \Big(\int_1^s t^{(1-\gamma+\lambda)a/2} dt/t\Big) ds/s$$

$$\le c\int_1^\infty [s^{\lambda+1}(N+\log s)^\alpha \psi(s)]^a ds/s,$$

which is the desired inequality (2.11). As before, the case $a = \infty$ is easy so we omit it.

When $0 < a < 1$, we write

$$\int_t^\infty \psi(s)ds = \int_t^\infty s^{1-\gamma}[s^{\mu+\gamma-1}\varphi(s)]ds/s \le t^{1-\gamma}\int_t^\infty s^{\mu+\gamma-1}\varphi(s)ds/s.$$

The inequality (2.5) now gives the slightly different version of the crucial inequality (2.19):

$$\int_t^\infty \psi(s)ds \le ct^{1-\gamma}\Big(\int_{t/2}^\infty [s^\gamma \psi(s)]^a ds/s\Big)^{1/a}.$$

Proceeding as before, we obtain (2.11′). This completes the proof.

The following theorem is a generalization of the classical Hardy inequalities (they may be obtained by taking $\alpha = 0$). The proof is entirely similar to the proof of Theorem 2.5 so we omit it.

THEOREM 2.6: Let $\lambda > 0$, $0 < a \leq \infty$ and $-\infty < \alpha < \infty$. If either

(a) $1 \leq a \leq \infty$ and ψ is a nonnegative measurable function on $(0,\infty)$;

or

(b) $0 < a < 1$ and $\psi(t) = t^{\mu-1}\varphi(t)$, where $\mu > 0$ and φ is a nonnegative, measurable decreasing function;

then

$$\text{(i)} \quad \left(\int_0^\infty [t^{-\lambda}(1+|\log t|)^\alpha \int_0^t \psi(s)ds]^a dt/t\right)^{1/a} \leq c\left(\int_0^\infty [t^{-\lambda+1}(1+|\log t|)^\alpha \psi(t)]^a dt/t\right)^{1/a}, \tag{2.20}$$

and

$$\text{(ii)} \quad \left(\int_0^\infty [t^{\lambda}(1+|\log t|)^\alpha \int_t^\infty \psi(s)ds]^a dt/t\right)^{1/a} \leq c\left(\int_0^\infty [t^{\lambda+1}(1+|\log t|)^\alpha \psi(t)]^a dt/t\right)^{1/a}. \tag{2.21}$$

The following variants of Hardy's inequalities correspond to the limiting cases $\lambda = 0$ in Theorem 2.5. In contrast to Theorem 2.5, the order of the logarithmic term now increases by a factor of one from left to right. The inequalities (2.22) and (2.24) were established in [1, Theorem 6.2].

THEOREM 2.7: Suppose $1 \leq a \leq \infty$ and $\alpha + 1/a \neq 0$. Let ψ be a nonnegative measurable function on $(0,\infty)$.

(a) If $\alpha + 1/a > 0$, then

$$\text{(i)}\quad \left(\int_0^1 [(1-\log t)^{\alpha}\int_0^t \psi(s)ds]^a dt/t\right)^{1/a} \le c\left(\int_0^1 [t(1-\log t)^{\alpha+1}\psi(t)]^a dt/t\right)^{1/a}, \tag{2.22}$$

$$\text{(ii)}\quad \left(\int_1^\infty [(1+\log t)^{\alpha}\int_t^\infty \psi(s)ds]^a dt/t\right)^{1/a} \le c\left(\int_1^\infty [t(1+\log t)^{\alpha+1}\psi(t)]^a dt/t\right)^{1/a}. \tag{2.23}$$

(b) If $\alpha + 1/a < 0$, then

$$\text{(iii)}\quad \left(\int_0^1 [(1-\log t)^{\alpha}\int_t^1 \psi(s)ds]^a dt/t\right)^{1/a} \le c\left(\int_0^1 [t(1-\log t)^{\alpha+1}\psi(t)]^a dt/t\right)^{1/a}, \tag{2.24}$$

$$\text{(iv)}\quad \left(\int_1^\infty [(1+\log t)^{\alpha}\int_1^t \psi(s)ds]^a dt/t\right)^{1/a} \le c\left(\int_1^\infty [t(1+\log t)^{\alpha+1}\psi(t)]^a dt/t\right)^{1/a}. \tag{2.25}$$

Proof: We shall prove (2.23). Let a' satisfy

$$1/a + 1/a' = 1.$$

Since $\alpha + 1/a > 0$, we have $\alpha + 1 > 1/a'$. Thus, we may choose γ so that

$$1/a' < \gamma < \alpha + 1. \tag{2.26}$$

We now write

$$\int_t^\infty \psi(s)ds = \int_t^\infty [s(1+\log s)^{\gamma}\psi(s)](1+\log s)^{-\gamma}ds/s.$$

Then, we apply Hölder's inequality and (2.26) to obtain

$$\int_t^\infty \psi(s)ds \le c(1+\log t)^{-\gamma+1/a'}\left(\int_t^\infty [s(1+\log s)^\gamma \psi(s)]^a ds/s\right)^{1/a}. \qquad (2.27)$$

For $1 \le a < \infty$, we can now make the estimate

$$\int_1^\infty [(1+\log t)^\alpha \int_t^\infty \psi(s)ds]^a dt/t$$

$$\le c\int_1^\infty [(1+\log t)^{\alpha-\gamma+1/a'}]^a\left(\int_t^\infty [s(1+\log s)^\gamma \psi(s)]^a ds/s\right)dt/t$$

$$= c\int_1^\infty [s(1+\log s)^\gamma \psi(s)]^a\left(\int_1^s (1+\log t)^{\alpha a-\gamma a+a-1}dt/t\right)ds/s$$

$$\le c\int_1^\infty [s(1+\log s)^{\alpha+1}\psi(s)]^a ds/s,$$

(the last inequality is available by (2.26)). The above estimate gives (2.23) for $1 \le a < \infty$. The case $a = \infty$ is easier and so we omit it.

The inequality (2.25) is proved in much the same way. Since $\alpha + 1/a < 0$, we now choose γ so that

$$\alpha + 1 < \gamma < 1/a'.$$

The crucial inequality (2.27) is replaced by

$$\int_1^t \psi(s)ds \le c(1+\log t)^{-\gamma+1/a'}\left(\int_1^t [s(1+\log s)^\gamma \psi(s)]^a ds/s\right)^{1/a},$$

and we proceed as before. This completes the proof.

3. Decreasing Rearrangements of Functions

Let $(\mathfrak{X},\mu)$ be any measure space. For each complex-valued measurable function f on $\mathfrak{X}$ we have its distribution function

$$D_f(y) = \mu\{x : |f(x)| > y\}, \qquad 0 < y < \infty. \quad (3.1)$$

The function D_f is decreasing and right continuous. Hence, it has a right continuous inverse

$$f^*(t) = \inf\{y : D_f(y) \le t\}, \qquad 0 < t < \infty. \quad (3.2)$$

We call f^* the decreasing rearrangement of f. The mapping $f \to f^*$ is not subadditive, but we do have

$$(f+g)^*(t) \le f^*(t/2) + g^*(t/2), \qquad 0 < t < \infty. \quad (3.3)$$

However, for the averaged rearrangement

$$f^{**}(t) = t^{-1}\int_0^t f^*(s)ds, \qquad 0 < t < \infty, \quad (3.4)$$

the mapping $f \to f^{**}$ is subadditive. That is,

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t), \qquad 0 < t < \infty. \quad (3.5)$$

More generally, if φ is a nonnegative decreasing function on $(0,\infty)$, then

$$\int_0^t \varphi(s)(f+g)^*(s)ds \le \int_0^t \varphi(s)f^*(s)ds + \int_0^t \varphi(s)g^*(s)ds, \qquad 0 < t < \infty. \quad (3.6)$$

Another useful fact is: given two measurable functions f and g and a measurable subset E of $\mathfrak{X}$, we have

$$\int_E |fg|d\mu \le \int_0^{\mu(E)} f^*(t)g^*(t)dt. \quad (3.7)$$

Discussion and proofs of these results may be found, for instance, in the work of Luxemburg [27], Lorentz [24], and Hunt [21].

4. Averaging Operators

For $0 < p \leq \infty$, the averaging operators A_p, B_p, C_p and D_p are defined as follows: Let f be a measurable function on $(\mathfrak{X},\mu)$ and f^* its decreasing rearrangement. For $0 < t < \infty$, define

$$(A_p f^*)(t) = t^{-1/p}\int_0^t s^{1/p} f^*(s)ds/s; \tag{4.1}$$

$$(B_p f^*)(t) = t^{-1/p}\int_t^\infty s^{1/p} f^*(s)ds/s; \tag{4.2}$$

$$(C_p f^*)(t) = t^{-1/p} \sup_{0<s\leq t} s^{1/p} f^*(s) \; ; \tag{4.3}$$

$$(D_p f^*)(t) = t^{-1/p} \sup_{t\leq s<\infty} s^{1/p} f^*(s) \; . \tag{4.4}$$

As seen in the work of Calderón [9] and Boyd [7], the operators A_p and B_p play an important role in the theory of weak type interpolation. The operators C_p and D_p first appear in [6] and play an equally prominent role (cf. Section 13).

Each of the functions $A_p f^*$, $B_p f^*$, $C_p f^*$ and $D_p f^*$ is decreasing and right continuous; hence, each is equal to its own decreasing rearrangement.

Note that $A_1 f^* = f^{**}$ and so A_1 is subadditive by (3.5). The relation (3.6) implies

$$A_p(f+g)^* \leq A_p f^* + A_p g^*, \qquad 1 < p \leq \infty. \tag{4.5}$$

The following relations among the operators A_p, B_p, C_p and D_p will be useful later.

LEMMA 4.1: Let $0 < p \leq \infty$. Then

$$f^*(t) \leq (C_p f^*)(t) \leq c(A_p f^*)(t), \tag{4.6}$$

and

$$f^*(t) \leq (D_p f^*)(t) \leq c(B_p f^*)(t/2), \tag{4.7}$$

for all $t > 0$. The constant c is independent of f and t.

Proof: The left-hand inequalities in (4.6) and (4.7) are obvious. The right-hand inequalities in (4.6) and (4.7) are precisely the inequalities (2.2) and (2.3), respectively (letting $a = 1$ and $\nu = 1/p$).

5. Lorentz Spaces L^{pa}

This section is devoted to a brief discussion of the Lorentz spaces $L^{pa} \equiv L^{pa}(\mathfrak{X})$ (recall (1.3)). (A complete discussion of the Lorentz spaces may be found, for instance, in [21].)

The Lorentz spaces are complete linear spaces, but the functional $\|\cdot\|_{pa}$ defined by (1.3) is in general only a quasinorm. The spaces L^{11} and L^{pa}, $1 < p \leq \infty$, $1 \leq a \leq \infty$ are (equivalent to) Banach spaces. For $1 < p \leq \infty$, $1 \leq a \leq \infty$, the equivalent norm is obtained by replacing f^* by f^{**}(cf. (3.4)) in (1.3). The quasinorm $\|f\|_{pa}$ actually is a norm in the case $0 < a < p \leq \infty$ (cf. [25]). The Lorentz spaces L^{pa} generalize the classical Lebesgue spaces L^p since $L^{pp} = L^p$.

We have the following inclusion relations among Lorentz spaces with the same primary indices p:

$$L^{pa} \subseteq L^{pb}, \qquad 0 < a \leq b \leq \infty. \tag{5.1}$$

In particular, we have

$$L^{p1} \subseteq L^{p}, \qquad 1 \leq p \leq \infty, \qquad (5.2)$$

and

$$L^{q} \subseteq L^{q\infty}, \qquad 0 < q \leq \infty. \qquad (5.3)$$

Note that if T is a quasilinear operator such that

$$T : L^{p} \to L^{q}, \qquad 1 \leq p \leq \infty,$$

then (5.2) and (5.3) show that

$$T : L^{p1} \to L^{q\infty}.$$

Thus, for $p \geq 1$, the notion of weak type (p,q) is indeed weaker than the notion of strong type (p,q) (recall (1.4) and (1.5)).

Lorentz spaces L^{pa} with different primary indices are related in special circumstances. For example, if $\mu(\mathfrak{X}) < \infty$, we have

$$L^{qb} \subseteq L^{q\infty} \subseteq L^{pa}, \qquad 0 < p < q \leq \infty,\ 0 < a, b \leq \infty. \qquad (5.4)$$

If $\mu(E) \geq 1$ for every set E of positive measure (which happens when $\mathfrak{X} = \mathbb{Z}$, the integers), then

$$L^{pa} \subseteq L^{p\infty} \subseteq L^{qb}, \qquad 0 < p < q < \infty,\ 0 < a, b \leq \infty \text{ or } q = b = \infty. \qquad (5.5)$$

<u>REMARK 5.1</u>: Throughout this thesis, the notation $X \subseteq Y$, where X and Y are quasinormed linear spaces, will signify continuous embedding.

CHAPTER III

THE LORENTZ-ZYGMUND SPACES

6. The Lorentz-Zygmund Spaces $L^{pa}(\log L)^{\alpha}$

Let $0 < p, a \leq \infty$ and $-\infty < \alpha < \infty$. The Lorentz-Zygmund space $L^{pa}(\log L)^{\alpha}$ on $(\mathfrak{X},\mu)$ consists of all (classes of) measurable functions f for which the quasinorm

$$\|f\|_{pa;\alpha} = \left(\int_0^{\infty}[t^{1/p}(1+|\log t|)^{\alpha}f^*(t)]^a dt/t\right)^{1/a} \tag{6.1}$$

is finite.

The inequalities (3.3) and (2.1) show that $\|f\|_{pa;\alpha}$ is indeed a quasinorm. As for the Lorentz spaces L^{pa} (cf. [21]), the space $L^{pa}(\log L)^{\alpha}$ is complete with respect to the quasinorm $\|\cdot\|_{pa;\alpha}$. When $\alpha = 0$, the Lorentz-Zygmund space $L^{pa}(\log L)^{0}$ is just the Lorentz space L^{pa}. When $a = p$ and the underlying measure space is the unit circle $\mathbb{T}$, the Lorentz-Zygmund space $L^{pp}(\log L)^{\alpha}$ is the Zygmund space $L^{p}(\log L)^{\alpha}$ consisting of all functions f for which

$$\frac{1}{2\pi}\int_0^{2\pi}[|f(t)|\ \log^{\alpha}(2+|f(t)|)]^p dt$$

is finite. In view of these last remarks, we will make the following abbreviations:

$$L^{pp}(\log L)^0 = L^p; \quad L^{pa}(\log L)^0 = L^{pa};$$

$$L^{pp}(\log L)^\alpha = L^p(\log L)^\alpha; \; L^{11}(\log L)^\alpha = L(\log L)^\alpha;$$

$$L^{11}(\log L)^1 = L \log L.$$

The next theorem describes the action of the averaging operators A_p, B_p, C_p and D_p (cf. Section 4) on the Lorentz-Zygmund spaces $L^{pa}(\log L)^\alpha$. In the context of this section, Theorem 6.1 will aid in classifying which Lorentz-Zygmund spaces are in fact Banach spaces. Specifically, Theorem 6.1 shows that the operators A_p and C_p act on $L^{qa}(\log L)^\alpha$ as the identity provided $p < q$; if $p > q$, B_p and D_p act on $L^{qa}(\log L)^\alpha$ as the identity.

<u>THEOREM 6.1</u>: Let $0 < q, a \leq \infty$ and $-\infty < \alpha < \infty$. Then we have

$$\|A_p f^*\|_{qa;\alpha} \sim \|f^*\|_{qa;\alpha}, \; ({}^4) \tag{6.3}$$

provided $0 < p < q$. The result (6.3) is also valid for C_p. Similarly, we have

$$\|B_p f^*\|_{qa;\alpha} \sim \|f^*\|_{qa;\alpha}, \tag{6.4}$$

provided $0 < q < p \leq \infty$. The result (6.4) is also valid for D_p.

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(4) Since $A_p f^*$ and f^* are defined on the interval $(0,\infty)$ it is implicit in (6.3) and (6.4) that the measure space underlying the quasinorm $\|\cdot\|_{pa;\alpha}$ is the interval $(0,\infty)$. The symbol "$\sim$" denotes equivalence; i.e. there are positive constants c_1 and c_2 independent of f such that

$$c_1\|f^*\|_{qa;\alpha} \leq \|A_p f^*\|_{qa;\alpha} \leq c_2\|f^*\|_{qa;\alpha}.$$

Proof: Suppose $0 < p < q$. By the relations (4.6) and the definition (6.1), we clearly have

$$\|A_p f^*\|_{qa;\alpha} \geq \|f^*\|_{qa;\alpha},$$

and

$$\|C_p f^*\|_{qa;\alpha} \geq \|f^*\|_{qa;\alpha} .$$

To obtain the converse inequality for A_p, we apply the Hardy inequality (2.20) with $\lambda = 1/p - 1/q > 0$ and $\psi(t) = t^{1/p-1} f^*(t)$ (in which case Theorem 2.6 applies for all $0 < a \leq \infty$). To obtain the reverse inequality for C_p, we use the result for A_p and the relation (4.6).

The proof of (6.4) is similar, now utilizing the relation (4.7) and the Hardy inequality (2.21).

COROLLARY 6.2: Let $1 < p \leq \infty$, $1 \leq a \leq \infty$ and $-\infty < \alpha < \infty$. Then the functional

$$\|f\|^{**}_{pa;\alpha} = \left(\int_0^\infty [t^{1/p}(1+|\log t|)^\alpha f^{**}(t)]^a dt/t\right)^{1/a} \qquad (6.5)$$

defines an equivalent norm on the Lorentz-Zygmund space $L^{pa}(\log L)^\alpha$. Hence, for these choices of p, a and α, $L^{pa}(\log L)^\alpha$ is (equivalent to) a Banach space.

Proof: We have that $A_1 f^* = f^{**}$ (cf. (3.4)) and so the equivalence (6.3) shows that

$$\|f\|^{**}_{pa;\alpha} \sim \|f\|_{pa;\alpha}$$

because $p > 1$. The mapping $f \to f^{**}$ is subadditive (cf. (3.5)); hence, by Minkowski's inequality, $\|f\|^{**}_{pa;\alpha}$ is indeed a norm.

At times the expression $(1+|\log t|)^{\alpha}$ is awkward in computations. We have that

$$\|f\|_{pa;\alpha} \sim \left(\int_0^{\infty}[t^{1/p}|\log t|^{\alpha}f^*(t)]^a dt/t\right)^{1/a}, \quad \alpha+1/a > 0. \tag{6.6}$$

The equivalence (6.6) is valid because $1 + |\log t|$ and $|\log t|$ are asymptotically the same at 0 and ∞, and the condition $\alpha + 1/a > 0$ assures that the integral in (6.6) converges at $t = 1$.

7. Lorentz-Zygmund Spaces on the Unit Circle

In this section let $(\mathfrak{X},\mu)$ be the unit circle $\mathbb{T}$ with normalized Lebesgue measure $dt/2\pi$. In addition to Corollary 6.2 we have:

<u>THEOREM 7.1</u>: The Lorentz-Zygmund space $L^{11}(\log L)^{\alpha}$ on the unit circle $\mathbb{T}$ is a Banach space whenever $\alpha \geq 0$.

<u>Proof</u>: In this situation (6.1) becomes

$$\|f\|_{11;\alpha} = \int_0^1 (1-\log t)^{\alpha} f^*(t)dt. \tag{7.1}$$

Since $\alpha \geq 0$, the function $(1-\log t)^{\alpha}$ is decreasing and this is necessary and sufficient that the quasinorm (7.1) defines a norm (cf. [25]). This completes the proof.

As for the Lorentz spaces L^{pa} on a finite measure space (cf. (5.4)), we have the following inclusion relations when the primary indices differ.

THEOREM 7.2: Let $0 < p < q \leq \infty$, $0 < a, b \leq \infty$ and $-\infty < \alpha, \beta < \infty$. Then for the Lorentz-Zygmund spaces on $\mathbb{T}$ (or any finite measure space),

$$L^{qb}(\log L)^{\beta} \subseteq L^{pa}(\log L)^{\alpha}. \tag{7.2}$$

Proof: Choose r so that

$$p < r < q. \tag{7.3}$$

It will suffice to establish that

$$\|f\|_{pa;\alpha} \leq c_1 \|f\|_{r\infty;0} \leq c_2 \|f\|_{qb;\beta} . \tag{7.4}$$

We first make the estimate:

$$\begin{aligned}\Big(\int_0^1 [t^{1/p}(1-\log t)^{\alpha} f^*(t)]^a dt/t\Big)^{1/a} &= \Big(\int_0^1 [t^{1/r} f^*(t)]^a [t^{1/p-1/r}(1-\log t)^{\alpha}]^a dt/t\Big)^{1/a} \\ &\leq \sup_{0<t<1} t^{1/r} f^*(t) \Big(\int_0^1 [t^{1/p-1/r}(1-\log t)^{\alpha}]^a dt/t\Big)^{1/a} .\end{aligned}$$

The supremum is, by (6.1), the quasinorm $\|f\|_{r\infty;0}$. The last integral is finite by (7.3). This proves the first inequality in (7.4).

To verify the second inequality in (7.4) we apply (2.2) (with $\nu = 1/r$, $\varphi = f^*$) to get

$$\sup_{0<t\leq 1} t^{1/r} f^*(t) \leq c\Big(\int_0^1 [t^{1/r} f^*(t)]^b dt/t\Big)^{1/b}. \tag{7.5}$$

If $\beta \geq 0$, (7.3) implies

$$t^{1/r} \leq t^{1/q}(1-\log t)^{\beta}, \qquad 0 < t \leq 1.$$

Inserting this inequality into (7.5) we have the second inequality in (7.4) for $\beta \geq 0$. If $\beta < 0$, choose $\epsilon > 0$ so that

$$1/q < 1/q + \epsilon < 1/r.$$

Notice that

$$t^{-\epsilon}(1-\log t)^{\beta} \geq c > 0, \qquad 0 < t \leq 1,$$

where c is a constant independent of t. Hence,

$$t^{1/q}(1-\log t)^{\beta} = t^{1/q+\epsilon}(t^{-\epsilon}(1-\log t)^{\beta}) \geq ct^{1/r}, \quad 0 < t \leq 1. \qquad (7.6)$$

This inequality, in conjunction with (7.5), gives the second inequality in (7.4) when $\beta < 0$.

An extensive discussion of the Lorentz-Zygmund spaces $L^{pa}(\log L)^{\alpha}$ on the unit circle $\mathbb{T}$ may be found in [6].

8. Lorentz-Zygmund Spaces on the Integers

We open this section with a couple of results valid for any measure space $(\mathfrak{X},\mu)$. It is of interest to know when the spaces $L^{pa}(\log L)^{\alpha}$ are trivial.

<u>LEMMA 8.1</u>: Suppose $p = \infty$. If either (i) $0 < a < \infty$ and $\alpha + 1/a \geq 0$, or (ii) $a = \infty$ and $\alpha > 0$, then

$$L^{\infty a}(\log L)^{\alpha} = \{0\}.$$

Proof: If $0 < a < \infty$, then

$$\|f\|^a_{\infty a;\alpha} = \int_0^\infty (1+|\log t|)^{\alpha a}(f^*(t))^a dt/t,$$

and $\int_0^\infty (1+|\log t|)^{\alpha a} dt/t$ is finite if and only if $\alpha + 1/a < 0$.

If $a = \infty$, then

$$\|f\|_{\infty\infty;\alpha} = \sup_{0<t<\infty} (1+|\log t|)^{\alpha} f^*(t),$$

which can be finite only if $\alpha \leq 0$.

The next technical lemma is very important to the rest of the development.

LEMMA 8.2: Suppose $0 < p < r < q \leq \infty$, $0 < a,b \leq \infty$ and $-\infty < \alpha, \beta < \infty$. Then for any measurable function f on $\mathfrak{X}$,

$$\Big(\int_1^\infty [t^{1/q}(1+\log t)^{\beta} f^*(t)]^b dt/t\Big)^{1/b} \leq c_1 \sup_{1\leq t<\infty} t^{1/r} f^*(t)$$

$$\leq c_2\Big[\Big(\int_{1/4}^1 [t^{1/p} f^*(t)]^a dt/t\Big)^{1/a} + \Big(\int_1^\infty [t^{1/p}(1+\log t)^{\alpha} f^*(t)]^a dt/t\Big)^{1/a}\Big]. \tag{8.1}$$

Proof: We rewrite the left-most expression in (8.1) as

$$\Big(\int_1^\infty [t^{1/r} f^*(t)]^b [t^{1/q-1/r}(1+\log t)^{\beta}]^b dt/t\Big)^{1/b}.$$

This is clearly dominated by

$$\Big(\sup_{1\le t<\infty} t^{1/r}f^*(t)\Big)\Big(\int_1^\infty [t^{1/q-1/r}(1+\log t)^\beta]^b dt/t\Big)^{1/b} .$$

The last integral is finite since $r < q$, proving the first inequality in (8.1).

To prove the second inequality in (8.1), we fix t, $1 \le t < \infty$, and write

$$t^{1/r} = \frac{t^{1/r}}{r(t^{1/r}-1/2^r)} \int_{1/2}^t s^{1/r} ds/s \le c\int_{1/2}^t s^{1/r} ds/s .$$

Therefore, since f^* is decreasing, we have

$$t^{1/r}f^*(t) \le c\int_{1/2}^t s^{1/r}f^*(s)ds/s \le c\int_{1/2}^\infty s^{1/r}f^*(s)ds/s. \tag{8.2}$$

We rewrite the last integral as

$$\int_{1/2}^\infty [s^{1/p}(1+|\log s|)^\alpha f^*(s)][s^{1/r-1/p}(1+|\log s|)^{-\alpha}]ds/s$$

When $1 \le a \le \infty$, we apply Hölder's inequality to get (by (8.2))

$$t^{1/r}f^*(t) \le c\Big(\int_{1/2}^\infty [s^{1/p}(1+|\log s|)^\alpha f^*(s)]^a ds/s\Big)^{1/a}. \tag{8.3}$$

The right-hand side of (8.3) is dominated by

$$c\Big[\Big(\int_{1/2}^1 [s^{1/p}f^*(s)]^a ds/s\Big)^{1/a} + \Big(\int_1^\infty [s^{1/p}(1+\log s)^\alpha f^*(s)]^a ds/s\Big)^{1/a}\Big] .$$

This proves the second inequality in (8.1) when $1 \leq a \leq \infty$. When $0 < a < 1$, we apply (2.5) to (8.2) to give

$$t^{1/r}f^*(t) \leq c\Big(\int_{1/4}^{\infty} [s^{1/r}f^*(s)]^a ds/s\Big)^{1/a}. \tag{8.4}$$

Since $p < r$ it is easy (cf. (7.6)) to show that

$$s^{1/r} \leq cs^{1/p}(1+\log s)^{\alpha}, \qquad 1 \leq s < \infty. \tag{8.5}$$

The inequalities (8.4) and (8.5), together with (2.1), imply the second inequality in (8.1) when $0 < a < 1$. The proof is now complete.

For the remainder of this section, the underlying measure space $(\mathfrak{X},\mu)$ for the Lorentz-Zygmund spaces will be the integers $\mathbb{Z}$. In addition to Corollary 6.2 we have:

THEOREM 8.3: The Lorentz-Zygmund space $L^{11}(\log L)^{\alpha}$ on the integers $\mathbb{Z}$ is a Banach space whenever $\alpha \leq 0$.

Proof: Let

$$\varphi(t) = \begin{cases} \int_0^1 (1-\log s)^{\alpha} ds, & 0 < t < 1, \\ (1+\log t)^{\alpha}, & 1 \leq t < \infty. \end{cases} \tag{8.6}$$

If f is a function on $\mathbb{Z}$, then

$$f^*(t) = f^*(n-), \qquad n-1 \leq t < n, \ n = 1, 2, \ldots . \tag{8.7}$$

Therefore, by (6.1), we have

$$\|f\|_{11;\alpha} = f^*(1-)\Big(\int_0^1 (1-\log t)^{\alpha} dt\Big) + \int_1^{\infty} (1+\log t)^{\alpha} f^*(t)dt.$$

Hence, by (8.6), we have

$$\|f\|_{11;\alpha} = \int_0^\infty \varphi(t)f^*(t)dt.$$

The definition (8.6) and the fact $\alpha \leq 0$ show that φ is a decreasing function on $(0,\infty)$. Hence, the functional $\|f\|_{11;\alpha}$ satisfies the triangle inequality (cf. (3.6)).

We can now establish inclusion relations analogous to (5.5) for the spaces $L^{pa}(\log L)^{\alpha}(\mathbb{Z})$ with distinct primary indices.

<u>THEOREM 8.4</u>: Let $0 < p < q \leq \infty$, $0 < a, b \leq \infty$, $-\infty < \alpha, \beta < \infty$ and suppose $L^{qb}(\log L)^{\beta} \neq \{0\}$ (cf. Lemma 8.1). Then, for the Lorentz-Zygmund spaces on $\mathbb{Z}$,

$$L^{pa}(\log L)^{\alpha} \subseteq L^{qb}(\log L)^{\beta}. \tag{8.8}$$

<u>Proof</u>: Let $f \in L^{pa}(\log L)^{\alpha}$. Since $(\mathfrak{X},\mu) = \mathbb{Z}$, the relation (8.7) is valid. Let us first suppose that either

$$q < \infty; \quad \text{or} \quad q = \infty, \quad 0 < b < \infty \quad \text{and} \quad \beta + 1/b < 0 \tag{8.9}$$

holds. By the inequalities (2.1) and (8.1) and the relation (8.7), we have that

$$\begin{aligned}\|f\|_{qb;\beta} &= \left(\int_0^\infty [t^{1/q}(1+|\log t|)^{\beta} f^*(t)]^b dt/t\right)^{1/b} \\ &\leq c\Big\{f^*(1-)\Big[\Big(\int_0^1 [t^{1/q}(1-\log t)^{\beta}]^b dt/t\Big)^{1/b} + \Big(\int_{1/4}^1 t^{a/p} dt/t\Big)^{1/a}\Big] \\ &\qquad + \Big(\int_1^\infty [t^{1/p}(1+\log t)^{\alpha} f^*(t)]^a dt/t\Big)^{1/a}\Big\}.\end{aligned} \tag{8.10}$$

The coefficient of $f^*(1-)$ in (8.10) is a constant independent of f by (8.9). Hence, we have

$$\|f\|_{qb;\beta} \le c\Big\{\Big(\int_0^1 [t^{1/p}(1-\log t)^{\alpha} f^*(t)]^a dt/t\Big)^{1/a} + \Big(\int_1^{\infty} [t^{1/p}(1+\log t)^{\alpha} f^*(t)]^a dt/t\Big)^{1/a}\Big\} . \tag{8.11}$$

For $0 < a < \infty$, apply (2.1) to give (8.8). If $a = \infty$, we use the following analogue of (2.1) to produce (8.8):

$$1/2\Big(\sup_{0<t\le 1} h(t) + \sup_{1\le t<\infty} h(t)\Big) \le \sup_{0<t<\infty} h(t) \le \sup_{0<t<1} h(t) + \sup_{1\le t<\infty} h(t), \tag{8.12}$$

where h is any nonnegative function on $(0,\infty)$.

The only other case when $L^{qb}(\log L)^{\beta} \ne \{0\}$ is when $q = b = \infty$ and $\beta \le 0$. The proof is similar so we omit it.

For functions on the integers, i.e. sequences, it is often desirable (and necessary to accommodate the classical theory) to attempt to express (6.1) in terms of series. We make the following definition: If $\{c_n\}$ is a sequence of complex numbers, let $\{c_n^*\}_{n=1}^{\infty}$ denote its nonnegative decreasing rearrangement (if $c_n = c(n)$, then $c_n^* = c^*(n-)$). The Lorentz-Zygmund sequence space $\ell^{pa}(\log \ell)^{\alpha}$ for $0 < p, a \le \infty$, $-\infty < \alpha < \infty$ is the set of sequences for which

$$\Big(\sum_{n=1}^{\infty} [n^{1/p}(1+\log n)^{\alpha} c_n^*]^a n^{-1}\Big)^{1/a} \quad (5) \tag{8.13}$$

(5) When $a = \infty$, this is to be interpreted in the obvious way as $\sup_{1\le n<\infty} n^{1/p}(1+\log n)^{\alpha} c_n^*$.

is finite. The following theorem shows that $L^{pa}(\log L)^{\alpha} = \ell^{pa}(\log \ell)^{\alpha}$ (provided $L^{pa}(\log L)^{\alpha}$ is non-trivial) when the underlying measure space is the integers $\mathbb{Z}$.

THEOREM 8.5: Suppose $0 < p, a \leq \infty$, $-\infty < \alpha < \infty$ and $L^{pa}(\log L)^{\alpha} \neq \{0\}$. Then,

$$L^{pa}(\log L)^{\alpha} = \ell^{pa}(\log \ell)^{\alpha} \tag{8.14}$$

Proof: We will only consider the case $0 < p, a < \infty$ and $\alpha \geq 0$. The other cases are proved similarly. First, we observe that

$$\begin{aligned}\sum_{n=1}^{\infty} [n^{1/p}(1+\log n)^{\alpha} c_n^*]^a n^{-1} \\ = c_1^* + \sum_{n=2}^{\infty} \int_{n-1}^{n} [n^{1/p}(1+\log n)^{\alpha} c^*(t)]^a dt/n .\end{aligned} \tag{8.15}$$

Next, we notice there are constants k_1 and k_2 such that

$$\begin{aligned}k_1[t^{1/p}(1+\log t)^{\alpha}]^a t^{-1} \leq [n^{1/p}(1+\log n)^{\alpha}]^a n^{-1} \\ \leq k_2[t^{1/p}(1+\log t)^{\alpha}]^a t^{-1},\end{aligned} \tag{8.16}$$

for $n-1 \leq t \leq n$, $n = 2, 3, \ldots$. Since $p < \infty$, the integral

$$\int_0^1 [t^{1/p}(1-\log t)^{\alpha}]^a dt/t$$

is finite. Using this, (8.15) and the second inequality in (8.16), we have

$$\sum_{n=1}^{\infty} [n^{1/p}(1+\log n)^{\alpha} c_n^*]^a n^{-1} \leq k \sum_{n=1}^{\infty} \int_{n-1}^{n} [t^{1/p}(1+\log t)^{\alpha} c^*(t)]^a dt/t$$

$$= k \, \|c(t)\|_{pa;\alpha} .$$

The reverse inequality follows from the first inequality in (8.16).

<u>REMARK 8.6</u>: The spaces $L^{\infty a}(\log L)^{\alpha}$, when (i) $0 < a < \infty$ and $\alpha + 1/a \geq 0$ or (ii) $a = \infty$ and $\alpha > 0$ are all trivial (cf. Lemma 8.1). However, the corresponding spaces $\ell^{\infty a}(\log \ell)^{\alpha}$ generated by the quasinorm

$$\left(\sum_{n=1}^{\infty} [(1+\log n)^{\alpha} c_n^*]^a n^{-1} \right)^{1/a}$$

are clearly non-trivial.

For the Lorentz-Zygmund sequence spaces $\ell^{pa}(\log \ell)^{\alpha}$ we have the inclusions (8.8) with <u>no</u> restrictions on the parameters q, b and β .

<u>THEOREM 8.7</u>: Suppose $0 < p < q \leq \infty$, $0 < a, b \leq \infty$ and $-\infty < \alpha, \beta < \infty$. Then

$$\ell^{pa}(\log \ell)^{\alpha} \subseteq \ell^{qb}(\log \ell)^{\beta} . \tag{8.17}$$

<u>Proof</u>: The inclusion (8.17) is precisely the content of Lemma 8.2 when the underlying measure space is the integers $\mathbb{Z}$.

9. The Spaces $L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta}$ and $L^{pa}(\log L)^{\alpha} \cap L^{qb}(\log L)^{\beta}$

Suppose $0 < p, q \leq \infty$, $0 < a, b \leq \infty$ and $-\infty < \alpha, \beta < \infty$. The Lorentz-Zygmund space $L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta}$ consists of all functions for which the quasinorm (1.10) is finite when $p < q$; when

$p > q$, $L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta}$ is generated by the quasinorm (1.11). First, we note that

$$L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta} = L^{qb}(\log L)^{\beta} + L^{pa}(\log L)^{\alpha}.$$

To justify the notation "+", we will show that $L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta}$ is just the usual algebraic sum of the Lorentz-Zygmund spaces $L^{pa}(\log L)^{\alpha}$ and $L^{qb}(\log L)^{\beta}$; the only exception occurs when either space is trivial (cf. Lemma 8.1).

The space $L^{pa}(\log L)^{\alpha} \cap L^{qb}(\log L)^{\beta}$ is generated by (1.11) when $p < q$, and is generated by (1.10) when $p > q$. For all values of the parameters we will show that $L^{pa}(\log L)^{\alpha} \cap L^{qb}(\log L)^{\beta}$ is the usual set-theoretic intersection of the Lorentz-Zygmund spaces $L^{pa}(\log L)^{\alpha}$ and $L^{qb}(\log L)^{\beta}$.

The quasinorms (1.10) and (1.11) are rather unwieldy, so we shall adopt the following notation convention: If h is a nonnegative measurable function on $(0,\infty)$ and $0 < p, a \leq \infty$, $-\infty < \alpha < \infty$, we define

$$I_{pa;\alpha}h = \left(\int_0^1 [t^{1/p}(1-\log t)^{\alpha}h(t)]^a dt/t\right)^{1/a}, \tag{9.1}$$

and

$$J_{pa;\alpha}h = \left(\int_1^{\infty} [t^{1/p}(1+\log t)^{\alpha}h(t)]^a dt/t\right)^{1/a}. \tag{9.2}$$

We thus have (cf. (1.10) and (1.11))

$$\|f\|_{L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta}} = \begin{cases} I_{pa;\alpha}f^* + J_{qb;\beta}f^*, & p < q, \\ I_{qb;\beta}f^* + J_{pa;\alpha}f^*, & p > q; \end{cases} \tag{9.3}$$

and

$$\|f\|_{L^{pa}(\log L)^{\alpha} \cap L^{qb}(\log L)^{\beta}} = \begin{cases} I_{qb;\beta}f^* + J_{pa;\alpha}f^*, & p < q, \\ I_{pa;\alpha}f^* + J_{qb;\beta}f^*, & p > q. \end{cases} \tag{9.4}$$

If $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are two quasinormed spaces continuously embedded in a larger topological vector space, we define two new quasinormed spaces $X_1 + X_2$ and $X_1 \cap X_2$ (cf. [8]). For each $f \in X_1 + X_2$, the quasinorm of f is given by

$$\|f\|_{X_1+X_2} = \inf(\|f_1\|_1 + \|f_2\|_2), \tag{9.5}$$

where the infimum is taken over all decompositions $f = f_1 + f_2$, $f_1 \in X_1$ and $f_2 \in X_2$. For $f \in X_1 \cap X_2$, the quasinorm is given by

$$\|f\|_{X_1 \cap X_2} = \max(\|f\|_1, \|f\|_2). \tag{9.6}$$

<u>THEOREM 9.1</u>: Let $0 < p < q \leq \infty$, $0 < a, b \leq \infty$ and $-\infty < \alpha, \beta < \infty$. If the space $L^{qb}(\log L)^{\beta} \neq \{0\}$, then the quasinorm defined by (9.5) on the algebraic sum $L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta}$ is equivalent to the quasinorm (9.3). Hence, the space defined by (9.3) is the usual algebraic sum provided neither space is trivial.

Proof: (i) Let us first assume $0 < q < \infty$ so that $L^{qb}(\log L)^{\beta} \neq \{0\}$. We pick f in the algebraic sum $L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta}$. We may assume that f is real-valued. Define functions f_1 and f_2 as follows:

$$f_1(x) = \begin{cases} f(x) - f^*(1), & \text{if } f(x) \geq f^*(1), \\ f(x) + f^*(1), & \text{if } f(x) \leq -f^*(1), \\ 0, & \text{otherwise;} \end{cases} \tag{9.7}$$

$$f_2(x) = f(x) - f_1(x). \tag{9.8}$$

With this choice of f_1 and f_2 we have

$$f_1^*(t) = \begin{cases} f^*(t) - f^*(1), & 0 < t < 1, \\ 0, & 1 \leq t < \infty; \end{cases} \tag{9.9}$$

and

$$f_2^*(t) = \begin{cases} f^*(1), & 0 < t \leq 1 \\ f^*(t), & 1 \leq t < \infty. \end{cases} \tag{9.10}$$

We thus have for $x \in \mathfrak{X}$ and $0 < t < \infty$,

$$f(x) = f_1(x) + f_2(x); \quad f^*(t) = f_1^*(t) + f_2^*(t). \tag{9.11}$$

We now must show (9.11) is a proper decomposition, i.e., $f_1 \in L^{pa}(\log L)^{\alpha}$ and $f_2 \in L^{qb}(\log L)^{\beta}$. To this end, we let $f = g + h$ where $g \in L^{pa}(\log L)^{\alpha}$ and $h \in L^{qb}(\log L)^{\beta}$. By (3.3), we have

$$f^*(t) \leq g^*(t/2) + h^*(t/2). \tag{9.12}$$

The description (9.9) shows that (cf. (6.1) and (9.1))

$$\|f_1\|_{pa;\alpha} \le I_{pa;\alpha} f^*. \tag{9.13}$$

This and (9.12) show that

$$\|f_1\|_{pa;\alpha} \le I_{pa;\alpha}(g^*(t/2)+h^*(t/2))$$
$$\le c(I_{pa;\alpha}(g^*(t/2)) + I_{pa;\alpha}(h^*(t/2)).$$

Now $I_{pa;\alpha}(g^*(t/2))$ is finite since $g \in L^{pa}(\log L)^{\alpha}$. Also, $I_{pa;\alpha}(h^*(t/2))$ is finite because $I_{pa;\alpha} \le I_{qb;\beta}$ (cf.(7.2)) and $h \in L^{qb}(\log L)^{\beta}$. Hence, $\|f_1\|_{pa;\alpha}$ is finite and so $f_1 \in L^{pa}(\log L)^{\alpha}$. The description (9.10) shows that (cf. (6.1), (9.2))

$$\|f_2\|_{qb;\beta} \le c\left\{f^*(1)\left(\int_0^1 [t^{1/q}(1-\log t)^{\beta}]^b dt/t\right)^{1/b} + J_{qb;\beta} f^*\right\}. \tag{9.14}$$

The first integral is finite since $0 < q < \infty$. By (9.12), the second integral is dominated by (cf. (9.2))

$$c\left(J_{qb;\beta}(g^*(t/2)) + J_{qb;\beta}\left(h^*(t/2)\right)\right).$$

By Lemma 8.2, we have

$$J_{qb;\beta}(g^*(t/2)) \le c\left\{\left(\int_{1/4}^1 [t^{1/p} g^*(t/2)]^a dt/t\right)^{1/a} + J_{pa;\alpha}(g^*(t/2))\right\},$$

which is finite because $g \in L^{pa}(\log L)^{\alpha}$. The expression $J_{qb;\beta}(h^*(t/2))$ is clearly finite since $h \in L^{qb}(\log L)^{\beta}$. These estimates show that $f_2 \in L^{qb}(\log L)^{\beta}$, as we wished.

To show that (9.3) dominates (9.5), we observe that (9.13) and (9.14) give

$$\|f_1\|_{pa;\alpha} + \|f_2\|_{qb;\beta} \le c\Big\{I_{pa;\alpha}f^* + J_{qb;\beta}f^* + f^*(1)\Big(\int_0^1 [t^{1/q}(1-\log t)^{\beta}]^b dt/t\Big)^{1/b}\Big\} . \tag{9.15}$$

We have

$$f^*(1)\Big(\int_0^1 [t^{1/q}(1-\log t)^{\beta}]^b dt/t\Big)^{1/b} \le cf^*(1)\Big(\int_0^1 [t^{1/p}(1-\log t)^{\alpha}]^a dt/t\Big)^{1/a}$$

because each of the integrals is finite. This last expression is dominated by

$$c\Big(\int_0^1 [t^{1/p}(1-\log t)^{\alpha} f^*(t)]^a dt/t\Big)^{1/a} = cI_{pa;\alpha}f^*$$

since f^* decreases. This estimate and (9.15) show

$$\|f_1\|_{pa;\alpha} + \|f_2\|_{qb;\beta} \le c\Big(I_{pa;\alpha}f^* + J_{qb;\beta}f^*\Big) .$$

Taking the infimum over all decompositions $f = f_1 + f_2$, we have that (9.3) dominates (9.5).

To prove the reverse inequality, let $f = f_1 + f_2$ where $f_1 \in L^{pa}(\log L)^{\alpha}$ and $f_2 \in L^{qb}(\log L)^{\beta}$. The inequality (3.3) shows that

$$I_{pa;\alpha}f^* \le \Big(\int_0^1 [t^{1/p}(1-\log t)^{\alpha}(f_1^*(t/2) + f_2^*(t/2))]^a dt/t\Big)^{1/a} .$$

Applying Minkowski's inequality (or (2.1) if $0 < a < 1$) and substituting $t/2 \to t$, we obtain

$$I_{pa;\alpha}f^* \le c\Big\{\Big(\int_0^{1/2}[t^{1/p}(1-\log t)^{\alpha}f_1^*(t)]^a dt/t\Big)^{1/a}$$

$$+\Big(\int_0^{1/2}[t^{1/p}(1-\log t)^{\alpha}f_2^*(t)]^a dt/t\Big)^{1/a}\Big\}.$$

$$\le c\Big(I_{pa;\alpha}f_1^* + I_{pa;\alpha}f_2^*\Big).$$

Hence, it follows from (7.2) that

$$I_{pa;\alpha}f^* \le c(I_{pa;\alpha}f_1^* + I_{qb;\beta}f_2^*). \tag{9.16}$$

Estimating $J_{qb;\beta}f^*$ similarly, only appealing to (8.1) instead of (7.2), we obtain

$$J_{qb;\beta}f^* \le c\Big\{I_{pa;\alpha}f_1^* + J_{pa;\alpha}f_1^* + I_{qb;\beta}f_2^* + J_{qb;\beta}f_2^*\Big\}. \tag{9.17}$$

Adding the inequalities (9.16) and (9.17), then using (2.1)(or (8.12)), we have the estimate

$$I_{pa;\alpha}f^* + J_{qb;\beta}f^* \le c(\|f_1\|_{pa;\alpha} + \|f_2\|_{qb;\beta}),$$

where c is a constant independent of f, f_1 and f_2. Taking the infimum over all decompositions $f = f_1 + f_2$, we have that (9.5) dominates (9.3). Hence, when $0 < q < \infty$, the quasinorms (9.3) and (9.5) are equivalent.

(ii) The other instances where $L^{qb}(\log L)^{\beta} \ne \{0\}$ may be treated similarly so their proofs are omitted.

<u>THEOREM 9.2</u>: Suppose $0 < p < q \le \infty$, $0 < a,b \le \infty$ and $-\infty < \alpha,\beta < \infty$. Then the quasinorm defined by (9.6) on $L^{pa}(\log L)^{\alpha} \cap L^{qb}(\log L)^{\beta}$ is equivalent to the quasinorm (9.4). Hence, the "intersection" defined

by (9.4) is the usual set-theorectic intersection.

Proof: Clearly, by (6.1), (9.1) and (9.2), we have that

$$I_{qb;\beta}f^* + J_{pa;\alpha}f^* \leq \|f\|_{qb;\beta} + \|f\|_{pa;\alpha} \leq 2 \max(\|f\|_{qb;\beta}, \|f\|_{pa;\alpha}),$$

proving that (9.6) dominates (9.4).

To prove the reverse inequality, (2.1) (or (8.12) if $a = \infty$) gives

$$\|f\|_{pa;\alpha} \leq c(I_{pa;\alpha}f^* + J_{pa;\alpha}f^*).$$

Applying (7.2) to $I_{pa;\alpha}f^*$, we obtain

$$\|f\|_{pa;\alpha} \leq c(I_{qb;\beta}f^* + J_{pa;\alpha}f^*). \tag{9.18}$$

Similarly,

$$\|f\|_{qb;\beta} \leq c(I_{qb;\beta}f^* + J_{qb;\beta}f^*),$$

and by Lemma 8.2, we have

$$\|f\|_{qb;\beta} \leq c(I_{qb;\beta}f^* + J_{pa;\alpha}f^*). \tag{9.19}$$

The estimates (9.18) and (9.19) show that (9.4) dominates (9.6), completing the proof.

REMARK 9.3: From now on, when we speak of sums or intersections of Lorentz-Zygmund spaces, we refer to the quasinorms (9.3) or (9.4). Theorem 9.2 shows that this is consistent for the intersections. Theorem 9.1 shows that the sum given by (9.3) is the usual algebraic sum provided neither of the spaces is trivial. The sum given by the quasinorm (9.3) is of greater interest for us. For example, the algebraic sum of the spaces L^1 and $L^{\infty 1}$ is just L^1. However, the space $L^1 + L^{\infty 1}$ given by (9.3) consists of the larger class of functions for which

$$\int_0^1 f^*(t)dt + \int_1^\infty f^*(t)dt/t \qquad (9.20)$$

is finite. This space is of interest for the Hilbert transform H. For example, it is well known that Hf(x) is defined a.e. for $f \in L^p$, $1 \leq p < \infty$ (cf. [36, 42]). It is shown in [6] that Hf(x) is actually defined a.e. for functions f in the larger space $L^1 + L^{\infty 1}$ given by (9.20).

THEOREM 9.4: Let $0 < p < q \leq \infty$, $0 < a,b,c,d \leq \infty$ and $-\infty < \alpha,\beta,\gamma,\delta < \infty$. Then

$$L^{pa}(\log L)^\alpha \cap L^{qc}(\log L)^\gamma \subseteq L^{pb}(\log L)^\beta + L^{qd}(\log L)^\delta. \qquad (9.21)$$

Proof: The (quasi)norm of a function f in $L^{pb}(\log L)^\beta + L^{qd}(\log L)^\delta$ is (cf. (9.3))

$$I_{pb;\beta}f^* + J_{qd;\delta}f^*. \qquad (9.22)$$

The norm of a function f in $L^{pa}(\log L)^\alpha \cap L^{qc}(\log L)^\gamma$ is (cf. (9.4))

$$I_{qc;\gamma}f^* + J_{pa;\alpha}f^*. \qquad (9.23)$$

The inequalities (7.2) and (8.1) show that (9.23) dominates (9.22). This completes the proof.

The next theorem characterizes the sums and intersections of Lorentz-Zygmund spaces when the underlying measure space is the unit circle $\mathbb{T}$ (or any finite measure space) or the integers $\mathbb{Z}$.

<u>THEOREM 9.5</u>: Let $0 < p < q \leq \infty$, $0 < a,b \leq \infty$ and $-\infty < \alpha,\beta < \infty$.

(i) If the underlying measure space is $\mathbb{T}$, then

(a) $L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta} = L^{pa}(\log L)^{\alpha}$;

(b) $L^{pa}(\log L)^{\alpha} \cap L^{qb}(\log L)^{\beta} = L^{qb}(\log L)^{\beta}$.

(ii) If the underlying measure space is $\mathbb{Z}$, then (cf. (8.13))

(c) $L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta} = \ell^{qb}(\log \ell)^{\beta}$;

(d) if, in addition, $L^{qb}(\log L)^{\beta} \neq \{0\}$,

$L^{pa}(\log L)^{\alpha} \cap L^{qb}(\log L)^{\beta} = \ell^{pa}(\log \ell)^{\alpha}$.

<u>Proof</u>: (a). By (9.3) it is clear that

$$\|f\|_{L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta}} = \|f\|_{pa;\alpha} .$$

(b). By Theorem 9.2, the intersection in part (b) is the usual set-theoretic intersection, and

$$L^{qb}(\log L)^{\beta} \subseteq L^{pa}(\log L)^{\alpha}$$

by (7.2).

(c). If $L^{qb}(\log L)^{\beta} \neq \{0\}$, Theorem 9.1 shows that the sum in (c) is the usual algebraic sum of the two spaces. So, for the integers $\mathbb{Z}$, the inclusion (8.8) shows that

$$L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta} = L^{qb}(\log L)^{\beta} .$$

The statement (c) now follows in this case by (8.14).

If $q = \infty$ (which encompasses the situation $L^{qb}(\log L)^{\beta} = \{0\}$), then (9.3) becomes (cf. (8.7))

$$f^*(1-)\Big(\int_0^1 [t^{1/p}(1-\log t)^{\alpha}]^a dt/t\Big)^{1/a}$$
$$+ \Big(\sum_{n=2}^{\infty} (f^*(n-))^b \int_{n-1}^{n} (1+\log t)^{\beta b} dt/t\Big)^{1/b} .$$

By the inequalities (8.16), this quasinorm is equivalent to

$$\Big(\sum_{n=1}^{\infty} [(1+\log n)^{\beta} f^*(n-)]^b n^{-1}\Big)^{1/b},$$

which is the quasinorm (8.13) on $\ell^{\infty b}(\log \ell)^{\beta}$.

(d). Theorem 8.4 applies, so that

$$L^{pa}(\log L)^{\alpha} \cap L^{qb}(\log L)^{\beta} = L^{pa}(\log L)^{\alpha}.$$

Since $L^{pa}(\log L)^{\alpha} \neq \{0\}$ $(p<\infty)$, the statement (d) follows by (8.14).

10. Inclusion Relations

Theorems 7.2 and 8.4 exhibit inclusion relations for the spaces $L^{pa}(\log L)^{\alpha}$ when the primary indices are distinct and the underlying measure space is the unit circle $\mathbb{T}$ or the integers $\mathbb{Z}$. It is also of interest to study the inclusion relations when the primary indices are the same. The following theorem generalizes the facts that the spaces $L^{pa}(\log L)^{\alpha}$ decrease with increasing α, while the spaces L^{pa} increase with increasing a.

THEOREM 10.1 ([6, Theorem 9.3] (6)): Suppose $0 < p \leq \infty$, $0 < a,b \leq \infty$ and $-\infty < \alpha,\beta < \infty$. Then

$$L^{pa}(\log L)^{\alpha} \subseteq L^{pb}(\log L)^{\beta} \tag{10.1}$$

whenever either of the following holds:

(i) $a \leq b$ and $\alpha \geq \beta$, or

(ii) $a > b$ and $\alpha + 1/a > \beta + 1/b$.

REMARKS 10.2: (i) If $a > b$, the condition $\alpha + 1/a > \beta + 1/b$ in Theorem 10.1 cannot in general be relaxed to $\alpha + 1/a \geq \beta + 1/b$. Hence, the spaces $L^{pa}(\log L)^{\alpha}$ are not ordered along the "diagonals" where $\alpha + 1/a$ is constant.

(ii) We wish to establish similar inclusion relations for the sums and intersections defined in Section 9. Theorem 10.1 shows exactly what to expect (provided none of the spaces involved in the sums are trivial).

We shall need two technical lemmas; the first lemma shows that the integrals $I_{pa;\alpha}f^*$ and $J_{pa;\alpha}f^*$ (cf. (9.1) and (9.2)) satisfy a

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(6) This theorem was proved in [6] for the case where the underlying measure space is finite. However, the proof remains valid for any underlying measure space.

convexity property.

LEMMA 10.3: Let $0 < p \leq \infty$, $0 < a < b \leq \infty$ and $-\infty < \alpha, \beta < \infty$. Then we have

$$I_{pb;\beta} f^* \leq (I_{pa;\alpha} f^*)^{a/b} (I_{p\infty;\gamma})^{1-a/b}, \qquad (10.2)$$

and

$$J_{pb;\beta} f^* \leq (J_{pa;\alpha} f^*)^{a/b} (J_{p\infty;\gamma})^{1-a/b}, \qquad (10.3)$$

where
$$\gamma = \frac{b\beta - a\alpha}{b-a} = (\beta, \quad \text{if} \quad b = \infty). \qquad (10.4)$$

Proof: The case $b = \infty$ is obvious, so we assume $b < \infty$. We note that

$$[t^{1/p}(1-\log t)^{\beta} f^*(t)]^b = [t^{1/p}(1-\log t)^{\alpha} f^*(t)]^a [t^{1/p}(1-\log t)^{\gamma} f^*(t)]^{b-a}.$$

Hence, by (9.1),

$$I_{pb;\beta} f^* \leq \Big(\int_0^1 [t^{1/p}(1-\log t)^{\alpha} f^*(t)]^a dt/t\Big)^{1/b} \Big(\sup_{0<t\leq 1} [t^{1/p}(1-\log t)^{\gamma} f^*(t)]^{b-a}\Big)^{1/b}$$
$$= (I_{pa;\alpha} f^*)^{a/b} (I_{p\infty;\gamma} f^*)^{1-a/b}.$$

The inequality (10.3) is proved similarly.

LEMMA 10.4: Let $0 < p \leq \infty$, $0 < a < \infty$ and $-\infty < \alpha < \infty$. Then we have the inequality

$$t^{1/p}(1+\log t)^{\alpha} \leq c\Big(1 + \int_1^t [s^{1/p}(1+\log s)^{\alpha}]^a ds/s\Big)^{1/a}, \quad 1 \leq t < \infty. \qquad (10.5)$$

Proof: We choose N so that $s^{-1}(N+\log s)^{\alpha a}$ is decreasing on $(1,\infty)$ (cf. Lemma 2.2). Then,

$$\int_1^t s^{a/p}(1+\log s)^{\alpha a}ds/s \geq c\int_1^t s^{a/p}(N+\log s)^{\alpha a}ds/s$$

$$\geq ct^{-1}(N+\log t)^{\alpha a}\int_1^t s^{a/p}ds$$

$$\geq c\left(t^{a/p}(N+\log t)^{\alpha a} - N^{\alpha a}\right).$$

We add $cN^{\alpha a}$ to both sides and take a^{th} roots to prove (10.5).

THEOREM 10.5: Let $0 < p < q \leq \infty$, $0 < a,b,c,d \leq \infty$ and $-\infty < \alpha,\beta,\gamma,\delta < \infty$. Then

$$L^{pa}(\log L)^{\alpha} + L^{qc}(\log L)^{\gamma} \subseteq L^{pb}(\log L)^{\beta} + L^{qd}(\log L)^{\delta}, \tag{10.6}$$

and

$$L^{pa}(\log L)^{\alpha} \cap L^{qc}(\log L)^{\gamma} \subseteq L^{pb}(\log L)^{\beta} \cap L^{qd}(\log L)^{\delta}, \tag{10.7}$$

whenever one of the following conditions holds:

(10.8)

(i) $a \leq b$ and $\alpha \geq \beta$; $c \leq d$ and $\gamma \geq \delta$,

(ii) $a \leq b$ and $\alpha \geq \beta$; $c > d$ and $\gamma + 1/c > \delta + 1/d$,

(iii) $a > b$ and $\alpha + 1/a > \beta + 1/b$; $c \leq d$ and $\gamma \geq \delta$,

(iv) $a > b$ and $\alpha + 1/a > \beta + 1/b$; $c > d$ and $\gamma + 1/c > \delta + 1/d$.

Proof: By the Remark (10.2.ii), the inclusion (10.7) is a direct consequence of (10.1) and Theorem 9.2. The inclusion (10.6) follows from (10.1) and Theorem 9.1 provided $L^{qb}(\log L)^{\beta} \neq \{0\}$. It will therefore suffice to prove only (10.6) when $q = \infty$ (cf. Lemma 8.1).

We first assume that (10.8.i) holds. Since the space $L^{pa}(\log L)^{\alpha} + L^{\infty c}(\log L)^{\gamma}$ clearly decreases with increasing α and γ, we may assume $\alpha = \beta$ and $\gamma = \delta$. We therefore need to show

$$I_{pb;\alpha}f^* + J_{\infty d;\gamma}f^* \leq k(I_{pa;\alpha}f^* + J_{\infty c;\gamma}f^*). \tag{10.9}$$

The inclusion (10.1) (for a finite measure space) shows that

$$I_{pb;\alpha}f^* \leq kI_{pa;\alpha}f^*. \tag{10.10}$$

If $c = d$, the inequality (10.10) implies (10.9), so assume $c < d$. Utilizing (10.3), we obtain

$$J_{\infty d;\gamma}f^* \leq (J_{\infty c;\gamma}f^*)^{c/d}(J_{\infty\infty;\gamma}f^*)^{1-c/d}. \tag{10.11}$$

To estimate $J_{\infty\infty;\gamma}f^*$, we use (10.5) and the fact that f^* decreases to get

$$(1+\log t)^{\gamma}f^*(t) \leq k\Big(f^*(1)^c + \int_1^t [(1+\log s)^{\gamma}f^*(s)]^c ds/s\Big)^{1/c}, \quad 1 \leq t < \infty .$$

Replacing the range of integration $(1,t)$ by the range $(1,\infty)$ and taking the supremum over $1 \leq t < \infty$, we have

$$J_{\infty\infty;\gamma}f^* \leq k(f^*(1)^c + J_{\infty c;\gamma}f^*). \tag{10.12}$$

The estimates (10.10), (10.11) and (10.12) show that if the right-hand side of (10.9) is finite, then the left-hand side of (10.9) is also finite. This shows that the (set-theoretic) inclusion (10.6) holds. An appeal to the closed graph theorem shows that the inequality (10.9) also holds, i.e., the inclusion map is continuous.

Now assume the condition (10.8.ii) holds. We wish to show that

$$I_{pb;\beta}f^* + J_{\infty d;\delta}f^* \leq k(I_{pa;\alpha}f^* + J_{\infty c;\gamma}f^*). \tag{10.13}$$

The inequality (10.10) still remains valid. We assume that $c < \infty$, and write

$$J_{qd;\delta}f^* = \left(\int_1^\infty [t^{1/q}(1+\log t)^\gamma f^*(t)]^d [1+\log t]^{d(\delta-\gamma)} dt/t\right)^{1/d} .$$

Applying Hölders inequality with the conjugate exponents c/d and $c/(c-d)$, we obtain

$$J_{qd;\delta}f^* \leq \left(J_{qc;\gamma}f^*\right)\left(\int_1^\infty [1+\log t]^\eta dt/t\right)^{\frac{c-d}{cd}} , \tag{10.14}$$

where $\eta = \dfrac{cd(\delta-\gamma)}{c-d}$. The conditions $c > d$ and $\gamma + 1/c > \delta + 1/d$ (cf. (10.8.ii)) imply $\eta < -1$, and so the integral in (10.14) is finite and independent of f. Thus, the estimates (10.10) and (10.14) imply the required result (10.13).

The other parts of the theorem are proved in the same way. The details are omitted.

As noted in Remark 10.2.i, we still fail to have inclusions along the "diagonals" $\alpha + 1/a$ = constant and/or $\gamma + 1/c$ = constant.

11. The Auxiliary Lorentz-Zygmund Spaces

The averaging operators A_p, B_p, C_p and D_p of Section 4 are basic to the interpolation theory as we will show in Section 13. To study the action of these operators on the Lorentz-Zygmund spaces we shall now define auxiliary Lorentz-Zygmund spaces.

Let $0 < p < q \leq \infty$, $0 < a,b \leq \infty$, and $-\infty < \alpha, \beta < \infty$. The auxiliary Lorentz-Zygmund spaces $\mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} + \mathscr{L}^{qb}(\log \mathscr{L})^{\beta}$, $\mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} \cap \mathscr{L}^{qb}(\log \mathscr{L})^{\beta}$, $\mathscr{M}^{pa}(\log \mathscr{M})^{\alpha} + \mathscr{M}^{qb}(\log \mathscr{M})^{\beta}$ and $\mathscr{M}^{pa}(\log \mathscr{M})^{\alpha} \cap \mathscr{M}^{qb}(\log \mathscr{M})^{\beta}$ are generated by the following quasinorms (cf. (9.1) and (9.2)):

$$\|f\|_{\mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} + \mathscr{L}^{qb}(\log \mathscr{L})^{\beta}} = I_{pa;\alpha}(A_p f^*) + J_{qb;\beta}(B_q f^*), \qquad \alpha + 1/a > 0,\ \beta + 1/b > 0; \tag{11.1}$$

$$\|f\|_{\mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} \cap \mathscr{L}^{qb}(\log \mathscr{L})^{\beta}} = I_{qb;\beta}(B_q f^*) + J_{pa;\alpha}(A_p f^*), \qquad \alpha + 1/a < 0,\ \beta + 1/b < 0; \tag{11.2}$$

$$\|f\|_{\mathscr{M}^{pa}(\log \mathscr{M})^{\alpha} + \mathscr{M}^{qb}(\log \mathscr{M})^{\beta}} = I_{pa;\alpha}(C_p f^*) + J_{qb;\beta}(D_q f^*), \qquad \alpha + \frac{1}{a} > 0,\ \beta + \frac{1}{b} > 0; \tag{11.3}$$

$$\|f\|_{\mathscr{M}^{pa}(\log \mathscr{M})^{\alpha} \cap \mathscr{M}^{qb}(\log \mathscr{M})^{\beta}} = I_{qb;\beta}(D_q f^*) + J_{pa;\alpha}(C_p f^*), \qquad \alpha + \frac{1}{a} < 0,\ \beta + \frac{1}{b} < 0. \tag{11.4}$$

When $p > q$ we make the obvious definition so that

$$\mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} + \mathscr{L}^{qb}(\log \mathscr{L})^{\beta} = \mathscr{L}^{qb}(\log \mathscr{L})^{\beta} + \mathscr{L}^{pa}(\log \mathscr{L})^{\alpha}, \quad \text{etc.}$$

To avoid unnecessary confusion, we have defined these auxiliary spaces for only the values of the parameters a, b, α and β that we will ultimately be interested in.

By Lemma 4.1 we have the following inclusions:

$$\mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} + \mathscr{L}^{qb}(\log \mathscr{L})^{\beta} \subseteq \mathscr{M}^{pa}(\log \mathscr{M})^{\alpha} + \mathscr{M}^{qb}(\log \mathscr{M})^{\beta} \ (^{7}) \qquad (11.5)$$

and

$$\mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} \cap \mathscr{L}^{qb}(\log \mathscr{L})^{\beta} \subseteq \mathscr{M}^{pa}(\log \mathscr{M})^{\alpha} \cap \mathscr{M}^{qb}(\log \mathscr{M})^{\beta}. \qquad (11.6)$$

The next theorem gives important special cases when the auxiliary Lorentz-Zygmund spaces reduce to ordinary Lorentz-Zygmund spaces.

THEOREM 11.1: Suppose $0 < p,q < \infty$ with $p \neq q$. Then, if $\alpha+1, \beta+1 > 0$,

$$\mathscr{L}^{p1}(\log \mathscr{L})^{\alpha} + \mathscr{L}^{q1}(\log \mathscr{L})^{\beta} = L^{p1}(\log L)^{\alpha+1} + L^{q1}(\log L)^{\beta+1}, \qquad (11.7)$$

and if $\alpha, \beta > 0$

$$\mathscr{M}^{p\infty}(\log \mathscr{M})^{\alpha} + \mathscr{M}^{q\infty}(\log \mathscr{M})^{\beta} = L^{p\infty}(\log L)^{\alpha} + L^{q\infty}(\log L)^{\beta} . \qquad (11.8)$$

If $\alpha + 1, \beta + 1 < 0$

$$\mathscr{L}^{p1}(\log \mathscr{L})^{\alpha} \cap \mathscr{L}^{q1}(\log \mathscr{L})^{\beta} = L^{p1}(\log L)^{\alpha+1} \cap L^{q1}(\log L)^{\beta+1}, \qquad (11.9)$$

and if $\alpha, \beta < 0$,

$$\mathscr{M}^{p\infty}(\log \mathscr{M})^{\alpha} \cap \mathscr{M}^{q\infty}(\log \mathscr{M})^{\beta} = L^{p\infty}(\log L)^{\alpha} \cap L^{q\infty}(\log L)^{\beta} . \qquad (11.10)$$

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(7) We implicitly assume $\alpha+1/a > 0$, $\beta+1/b > 0$ so that the spaces are defined by (11.1) and (11.3). These implicit assumptions will be made throughout the remainder of the thesis without comment.

Proof: Let $p < q$. To prove (11.7), we apply the definitions (9.1), (9.2), (4.1) and (4.2) and interchange the order of integration to show that

$$I_{p1;\alpha}(A_p f^*) + J_{q1;\beta}(B_q f^*)$$

$$= \int_0^1 s^{1/p} f^*(s)\Big(\int_s^1 (1-\log t)^\alpha dt/t\Big) ds/s + \int_1^\infty s^{1/q} f^*(s)\Big(\int_1^s (1+\log t)^\beta dt/t\Big) ds/s$$

$$\sim \int_0^1 s^{1/p}(1-\log s)^{\alpha+1} f^*(s) ds/s + \int_1^\infty s^{1/q}(1+\log s)^{\beta+1} f^*(s) ds/s$$

$$= I_{p1;\alpha+1} f^* + J_{q1;\beta+1} f^* .$$

Thus, (11.7) is proved by the above equivalence and the definitions of the spaces involved (cf. (11.1) and (9.3)).

To prove (11.9), note that the (quasi)norm of a function f in $\mathscr{L}^{p1}(\log \mathscr{L})^\alpha \cap \mathscr{L}^{q1}(\log \mathscr{L})^\beta$ is (cf. (11.2))

$$\int_0^1 (1-\log t)^\beta \Big(\int_t^\infty s^{1/q} f^*(s) ds/s\Big) dt/t + \int_1^\infty (1+\log t)^\alpha \Big(\int_0^t s^{1/p} f^*(s) ds/s\Big) dt/t. \tag{11.11}$$

After interchanging the order of integration and noting that

$$\int_0^1 (1-\log t)^\beta dt/t = |\beta+1|^{-1}; \quad \int_1^\infty (1+\log t)^\alpha dt/t = |\alpha+1|^{-1}$$

the (quasi)norm (11.11) becomes

$$|\beta+1|^{-1} I_{q1;\beta+1} f^* + |\alpha+1|^{-1} J_{p1;\alpha+1} f^* + |\alpha+1|^{-1}\int_0^1 s^{1/p} f^*(s) ds/s + |\beta+1|^{-1}\int_1^\infty s^{1/q} f^*(s) ds/s. \tag{11.12}$$

The norm of f in $L^{p1}(\log L)^{\alpha+1} \cap L^{q1}(\log L)^{\beta+1}$ is (cf. (9.4))

$$I_{q1;\beta+1}f^* + J_{p1;\alpha+1}f^* . \tag{11.13}$$

The quasinorm (11.12) clearly dominates the quasinorm (11.13). To verify the reverse inequality, we apply (7.2) to $\int_0^1 s^{1/p} f^*(s)ds/s$ and we apply (8.1) to $\int_1^\infty s^{1/q} f^*(s)ds/s$. This proves (11.9).

The remainder of the theorem is proved similarly.

The next theorem is the auxiliary space analogue of Theorem 10.5. Notice that now the spaces <u>are</u> ordered along the "diagonals" $\alpha + 1/a$ = constant and/or $\gamma + 1/c$ = constant (cf. Remark 10.2.i).

<u>THEOREM 11.2</u>: Suppose $0 < p < q \leq \infty$, $0 < a,b,c,d \leq \infty$ and $-\infty < \alpha,\beta,\gamma,\delta < \infty$. Then

$$\mathcal{L}^{pa}(\log \mathcal{L})^\alpha + \mathcal{L}^{qc}(\log \mathcal{L})^\gamma \subseteq \mathcal{L}^{pb}(\log \mathcal{L})^\beta + \mathcal{L}^{qd}(\log \mathcal{L})^\delta, \tag{11.14}$$

and

$$\mathcal{L}^{pa}(\log \mathcal{L})^\alpha \cap \mathcal{L}^{qc}(\log \mathcal{L})^\gamma \subseteq \mathcal{L}^{pb}(\log \mathcal{L})^\beta \cap \mathcal{L}^{qd}(\log \mathcal{L})^\delta, \tag{11.15}$$

whenever one of the following conditions holds:

$$\begin{array}{ll} \text{(i)} & \alpha + 1/a > \beta + 1/b \;;\; \gamma + 1/c > \delta + 1/d, \\ \text{(ii)} & \alpha + 1/a > \beta + 1/b \;;\; \gamma + 1/c = \delta + 1/d, \quad c \leq d, \\ \text{(iii)} & \alpha + 1/a = \beta + 1/b, \; a \leq b \;;\; \gamma + 1/c > \delta + 1/d, \\ \text{(iv)} & \alpha + 1/a = \beta + 1/b, \; a \leq b \;;\; \gamma + 1/c = \delta + 1/d, \; c \leq d. \end{array} \tag{11.16}$$

<u>Proof</u>: Let us first prove (11.14). Then all the quantities $\alpha + 1/a$, $\beta + 1/b$, $\gamma + 1/c$ and $\delta + 1/d$ are positive. First, we assume

$a > b$ and $c > d$. This is a subcase of (11.16.i), so $\alpha + 1/b > \beta + 1/b$ and $\gamma + 1/c > \delta + 1/d$. We need to prove that (cf. (11.1))

$$I_{pb;\beta}(A_p f^*) + J_{qd;\delta}(B_q f^*) \leq k(I_{pa;\alpha}(A_p f^*) + J_{qc;\gamma}(B_q f^*)) \qquad (11.17)$$

Statements (10.10) and (10.14) are valid are valid. Hence,

$$I_{pb;\beta}(A_p f^*) \leq kI_{pa;\alpha}(A_p f^*) \qquad (11.18)$$

and

$$J_{qd;\delta}(B_q f^*) \leq kJ_{qc;\gamma}(B_q f^*). \qquad (11.19)$$

These inequalities (11.18) and (11.19) imply the required result (11.17).

We next assume that $a > b$ and $c \leq d$. This is a subcase of (11.16.i) or (11.16.ii), so $\alpha + 1/a > \beta + 1/b$ and $\gamma + 1/c \geq \delta + 1/d$. Since the spaces $\mathcal{L}^{pa}(\log \mathcal{L})^{\alpha} + \mathcal{L}^{qc}(\log \mathcal{L})^{\delta}$ decrease with increasing γ, we may assume $\gamma + 1/c = \delta + 1/d$, $c < d$. In this case the inequality (11.18) still holds. By (10.3) and (10.4) we have

$$J_{qd;\delta}(B_q f^*) \leq (J_{qc;\gamma}(B_q f^*))^{c/d}(J_{q\infty;\gamma+1/c}(B_q f^*))^{1-c/d} .$$

Therefore, to prove (11.19) (and hence, (11.17)) in this case it will suffice to show

$$J_{q\infty;\gamma+1/c}(B_q f^*) \leq kJ_{qc;\gamma}(B_q f^*). \qquad (11.20)$$

Since $\gamma + 1/c > 0$, we may replace $(1+\log t)^{\gamma+1/c}$ by $(\log t)^{\gamma+1/c}$ since the two function are asymptotic as $t \to \infty$. We write

$$(\log t)^{\gamma c+1} = (\gamma c+1)\int_1^t (\log u)^{\gamma c} du/u. \tag{11.21}$$

Then, for $1 \leq t < \infty$, we have

$$\begin{aligned}(\log t)^{\gamma c+1}\Big(\int_t^\infty s^{1/q} f^*(s)ds/s\Big)^c \\ &= (\gamma c+1)\int_1^t [\log^\gamma u\int_t^\infty s^{1/q} f^*(s)ds/s]^c du/u \\ &\leq (\gamma c+1)\int_1^t [\log^\gamma u\int_u^\infty s^{1/q} f^*(s)ds/s]^c du/u \;.\end{aligned}$$

Taking c^{th} roots and passing to the supremum (over $1 \leq t < \infty$) we obtain the required result (11.20) (cf. (9.2) and (4.2)).

Third, we assume $a \leq b$ and $c > d$. This is either a subcase of (11.16.i) or (11.16.iii) so that $\gamma + 1/c > \delta + 1/d$ and $\alpha + 1/a \geq \beta + 1/b$. We may assume $\alpha + 1/a = \beta + 1/b$ and $a < b$. In this case (11.19) still holds. To prove (11.18) (and hence, (11.17)), Lemma 10.3 shows it will suffice to show that

$$I_{p\infty;\alpha+1/a}(A_p f^*) \leq kI_{pa;\alpha}(A_p f^*). \tag{11.22}$$

As in the previous case, we write

$$(\log 1/t)^{\alpha a+1} = (\alpha a+1)\int_t^1 (\log u)^{\alpha a} du/u, \tag{11.23}$$

and proceed as before to prove the required result (11.22).

Finally, we assume $a \leq b$ and $c \leq d$. We may suppose $a < b$, $c < d$, $\gamma + 1/c = \delta + 1/d$ and $\alpha + 1/a = \beta + 1/b$. We then have (11.20) and (11.22). This completes the proof (via Lemma 10.3).

The inclusion (11.15) is proved in a similar fashion.

There is a similar theorem for the $\mathcal{M}$ spaces. Its proof is fashioned after the proof of Theorem 11.2 so we omit it.

THEOREM 11.3: Suppose $0 < p < q \leq \infty$, $0 < a,b,c,d \leq \infty$ and $-\infty < \alpha, \beta, \gamma, \delta < \infty$. Then

$$\mathcal{M}^{pa}(\log \mathcal{M})^{\alpha} + \mathcal{M}^{qc}(\log \mathcal{M})^{\gamma} \subseteq \mathcal{M}^{pb}(\log \mathcal{M})^{\beta} + \mathcal{M}^{qd}(\log \mathcal{M})^{\delta}, \tag{11.24}$$

and

$$\mathcal{M}^{pa}(\log \mathcal{M})^{\alpha} \cap \mathcal{M}^{qc}(\log \mathcal{M})^{\gamma} \subseteq \mathcal{M}^{pb}(\log \mathcal{M})^{\beta} \cap \mathcal{M}^{qd}(\log \mathcal{M})^{\delta}, \tag{11.25}$$

whenever one of the conditions (11.16) hold.

The next theorem gives some special inclusion relations for the spaces $\mathcal{L}$, $\mathcal{M}$ and L.

THEOREM 11.4: Suppose $0 < p < q \leq \infty$, $0 < a,b \leq \infty$ and $-\infty < \alpha, \beta < \infty$. If $\alpha + 1/a > 0$ and $\beta + 1/b > 0$, then

$$\mathcal{L}^{pa}(\log \mathcal{L})^{\alpha} + \mathcal{L}^{qb}(\log \mathcal{L})^{\beta} \subseteq L^{p1} + L^{q1} \tag{11.26}$$

and

$$\mathcal{M}^{pa}(\log \mathcal{M})^{\alpha} + \mathcal{M}^{qb}(\log \mathcal{M})^{\beta} \subseteq L^{p\infty} + L^{q\infty}. \tag{11.27}$$

If $\alpha + 1/a < 0$ and $\beta + 1/b < 0$, then

$$L^{p1} \cap L^{q1} \subseteq \mathcal{L}^{pa}(\log \mathcal{L})^{\alpha} \cap \mathcal{L}^{qb}(\log \mathcal{L})^{\beta} \tag{11.28}$$

and

$$L^{p\infty} \cap L^{q\infty} \subseteq \mathcal{M}^{pa}(\log \mathcal{M})^{\alpha} \cap \mathcal{M}^{qb}(\log \mathcal{M})^{\beta}. \tag{11.29}$$

Proof: We choose γ to satisfy

$$0 < \gamma + 1 < \min(\alpha + 1/a, \beta + 1/b).$$

Applying (11.14) and (11.7), we have

$$\mathcal{L}^{pa}(\log \mathcal{L})^{\alpha} + \mathcal{L}^{qb}(\log \mathcal{L})^{\beta} \subseteq \mathcal{L}^{p1}(\log \mathcal{L})^{\gamma} + \mathcal{L}^{q1}(\log \mathcal{L})^{\gamma}$$
$$= L^{p1}(\log L)^{\gamma+1} + L^{q1}(\log L)^{\gamma+1}.$$

But, since $\gamma + 1 > 0$, the last space is clearly contained in $L^{p1} + L^{q1}$. This proves (11.26). The inclusion (11.27) follows in the same way from (11.24) and (11.8).

The inclusions (11.28) and (11.29) follow similarly by choosing γ so that

$$\max(\alpha + 1/a, \beta + 1/b) < \gamma + 1 < 0$$

and applying (11.15) and (11.9) (to prove (11.28)) or applying (11.25) and (11.10) (to prove (11.29)).

As seen in Theorems 11.1 - 11.4, the quasinorms (11.1) - (11.4) are quite convenient for technical purposes. The next theorem gives an equivalent reformulation of these quasinorms. As we shall see in Chapter IV, these new expressions for the quasinorms on the auxiliary Lorentz-Zygmund spaces are the most natural for interpolation purposes.

THEOREM 11.5: Let $0 < p < q \leq \infty$, $0 < a,b < \infty$ and $-\infty < \alpha,\beta < \infty$. Then

$$\|f\|_{\mathcal{L}^{pa}(\log \mathcal{L})^{\alpha} + \mathcal{L}^{qb}(\log \mathcal{L})^{\beta}} \sim \|(A_p + B_q)f^*\|_{L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta}}\,; \qquad (11.30)$$

$$\|f\|_{\mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} \cap \mathscr{L}^{qb}(\log \mathscr{L})^{\beta}} \sim \|(A_p + B_q)f^*\|_{L^{pa}(\log L)^{\alpha} \cap L^{qb}(\log L)^{\beta}} ; \tag{11.31}$$

$$\|f\|_{\mathscr{M}^{pa}(\log \mathscr{M})^{\alpha} + \mathscr{M}^{qb}(\log \mathscr{M})^{\beta}} \sim \|(C_p + D_q)f^*\|_{L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta}} ; \tag{11.32}$$

$$\|f\|_{\mathscr{M}^{pa}(\log \mathscr{M})^{\alpha} \cap \mathscr{M}^{qb}(\log \mathscr{M})^{\beta}} \sim \|(C_p + D_q)f^*\|_{L^{pa}(\log L)^{\alpha} \cap L^{qb}(\log L)^{\beta}} . \tag{11.33}$$

Proof: In each of the statements above it is clear from the definitions of the spaces involved (cf. (9.3), (9.4), (11.1) - (11.4)) that the right-hand side dominates the left. In order to prove the reverse inequality, let us concentrate on (11.30). We have implicitly that $\alpha + 1/a > 0$ and $\beta + 1/b > 0$. The right-hand side of (11.30) is (cf. (9.3))

$$I_{pa;\alpha}(A_p + B_q)f^* + J_{qb;\beta}(A_p + B_q)f^*,$$

which is dominated by (cf. (2.1) and (11.1))

$$c\left\{\|f\|_{\mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} + \mathscr{L}^{qb}(\log \mathscr{L})^{\beta}} + I_{pa;\alpha}(B_q f^*) + J_{qb;\beta}(A_p f^*)\right\}. \tag{11.34}$$

We let $\lambda = 1/p - 1/q > 0$ and make the estimate (cf. (9.1) and (4.2)):

$$I_{pa;\alpha}(B_q f^*) \le \left(\int_0^1 [t^{\lambda}(1-\log t)^{\alpha}]^a dt/t\right)^{1/a} \int_1^{\infty} s^{1/q} f^*(s)ds/s$$
$$+ \left(\int_0^1 [t^{\lambda}(1-\log t)^{\alpha} \int_t^1 s^{1/q} f^*(s)ds/s]^a dt/t\right)^{1/a}.$$

Noting the first integral is finite and applying the Hardy inequality (2.9), we obtain

$$I_{pa;\alpha}(B_q f^*) \le c\left(J_{q1;0}f^* + I_{pa;\alpha}f^*\right). \tag{11.35}$$

Estimating $J_{qb;\beta}(A_p f^*)$ similarly (using the Hardy inequality (2.10)), we get

$$J_{qb;\beta}(A_p f^*) \le c\left(I_{p1;0}f^* + J_{qb;\beta}f^*\right) . \tag{11.36}$$

Since $A_p f^* + B_q f^*$ dominate f^* (cf. Lemma 4.1), the estimates (11.35) and (11.36) show that

$$I_{pa;\alpha}(B_q f^*) + J_{qb;\beta}(A_p f^*) \le c\Big\{\|f\|_{L^{p1} + L^{q1}}$$

$$+ \|f\|_{\mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} + \mathscr{L}^{qb}(\log \mathscr{L})^{\beta}}\Big\} .$$

Now, the inclusion (11.26) shows that

$$I_{pa;\alpha}(B_q f^*) + J_{qb;\beta}(A_p f^*) \le c\|f\|_{\mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} + \mathscr{L}^{qb}(\log \mathscr{L})^{\beta}} . \tag{11.37}$$

The estimates (11.34) and (11.37) establish the reverse inequality for (11.30). The remaining equivalences are proved similarly using the variants of Hardy's inequalities given in Theorem 2.5. The details are omitted.

12. The Embedding Theorem

The auxiliary Lorentz-Zygmund spaces introduced in Section 11 are related to the Lorentz-Zygmund spaces of Section 9 by the following basic embedding theorem.

THEOREM 12.1: Suppose $0 < p < q \leq \infty$, $1 \leq a \leq b \leq \infty$ and $-\infty < \alpha, \beta < \infty$. If $\alpha + 1/a > 0$ and $\beta + 1/b > 0$, then

$$\begin{aligned} &L^{pa}(\log L)^{\alpha+1} + L^{qb}(\log L)^{\beta+1} \subseteq \mathcal{L}^{pa}(\log \mathcal{L})^{\alpha} + \mathcal{L}^{qb}(\log \mathcal{L})^{\beta} \\ &\subseteq L^{pa}(\log L)^{\alpha+1/a} + L^{qb}(\log L)^{\beta+1/b} \subseteq \mathcal{M}^{pa}(\log \mathcal{M})^{\alpha} + \mathcal{M}^{qb}(\log \mathcal{M})^{\beta} \quad (12.1) \\ &\quad \subseteq L^{pa}(\log L)^{\alpha} + L^{qb}(\log L)^{\beta} . \end{aligned}$$

If $\alpha + 1/a < 0$ and $\beta + 1/b < 0$, then

$$\begin{aligned} &L^{pa}(\log L)^{\alpha+1} \cap L^{qb}(\log L)^{\beta+1} \subseteq \mathcal{L}^{pa}(\log \mathcal{L})^{\alpha} \cap \mathcal{L}^{qb}(\log \mathcal{L})^{\beta} \\ &\subseteq L^{pa}(\log L)^{\alpha+1/a} \cap L^{qb}(\log L)^{\beta+1/b} \subseteq \mathcal{M}^{pa}(\log \mathcal{M})^{\alpha} \cap \mathcal{M}^{qb}(\log \mathcal{M})^{\beta} \quad (12.2) \\ &\quad \subseteq L^{pa}(\log L)^{\alpha} \cap L^{qb}(\log L)^{\beta} . \end{aligned}$$

Proof: 1. The last inclusion in both (12.1) and (12.2) is a consequence of Lemma 4.1 and the definitions of the spaces involved ((9.3), (9.4), (11.3), (11.4)).

2. To prove the first inclusion in (12.1) we note that the norm of a function f in $\mathcal{L}^{pa}(\log \mathcal{L})^{\alpha} + \mathcal{L}^{qb}(\log \mathcal{L})^{\beta}$ is (cf. (11.1), (9.1), (9.2))

$$\begin{aligned} &\left(\int_0^1 [(1-\log t)^{\alpha}\int_0^t s^{1/p} f^*(s)ds/s]^a dt/t\right)^{1/a} \\ &\quad + \left(\int_1^{\infty} [(1+\log t)^{\beta}\int_t^{\infty} s^{1/q} f^*(s)ds/s]^b dt/t\right)^{1/b}. \end{aligned} \qquad (12.3)$$

Since $\alpha + 1/a > 0$ and $\beta + 1/b > 0$, we may apply the Hardy inequalities (2.22) and (2.23) to show that (12.3) is dominated by

$$I_{pa;\alpha+1} f^* + J_{qb;\beta+1} f^*,$$

which is the norm of a function f in $L^{pa}(\log L)^{\alpha+1} + L^{qb}(\log L)^{\beta+1}$ (cf. (9.3)).

To prove the first inclusion in (12.2), it suffices show that (cf. (9.4), (11.4))

$$I_{qb;\beta}(B_q f^*) + J_{pa;\alpha}(A_p f^*) \le c(I_{qb;\beta+1} f^* + J_{pa;\alpha+1} f^*). \qquad (12.4)$$

Applying Minkowski's inequality, we obtain

$$I_{qb;\beta}(B_q f^*) \le \left(\int_0^1 [(1-\log t)^\beta \int_t^1 s^{1/q} f^*(s)ds/s]^b dt/t\right)^{1/b}$$
$$+ \left(\int_0^1 (1-\log t)^{\beta b} dt/t\right)^{1/b} \int_1^\infty s^{1/q} f^*(s)ds/s.$$

The integral $\int_0^1 (1-\log t)^{\beta b} dt/t$ is finite since $\beta + 1/b < 0$. We now apply the Hardy inequality (2.24) to show that

$$I_{qb;\beta}(B_q f^*) \le c(I_{qb;\beta+1} f^* + J_{q1;0} f^*). \qquad (12.5)$$

Estimating $J_{pa;\alpha}(A_p f^*)$ similarly (the Hardy inequality (2.25) now applies), we see that

$$J_{pa;\alpha}(A_p f^*) \le c(J_{pa;\alpha+1} f^* + I_{p1;0} f^*). \qquad (12.6)$$

Theorem 9.4 shows that

$$I_{p1;0} f^* + J_{q1;0} f^* \le c(I_{qb;\beta+1} f^* + J_{pa;\alpha+1} f^*). \qquad (12.7)$$

The estimates (12.5), (12.6) and (12.7) produce the required result (12.4).

3. To establish the second inclusion in (12.1), we shall show that

$$I_{pa;\alpha+1/a}f^* + J_{qb;\beta+1/b}f^* \leq c(I_{pa;\alpha}(A_p f^*) + J_{qb;\beta}(B_q f^*)). \qquad (12.8)$$

Suppose $0 < a < \infty$. The identity (11.23) and the remark (6.6) show that

$$I_{pa;\alpha+1/a}f^* \sim \left(\int_0^1 [t^{1/p}f^*(t)]^a \left(\int_t^1 (\log 1/u)^{\alpha a} du/u\right) dt/t\right)^{1/a} .$$

Next, interchange the order of integration to obtain

$$I_{pa;\alpha+1/a}f^* \sim \left(\int_0^1 (\log 1/u)^{\alpha a} \left(\int_0^u [t^{1/p}f^*(t)]^a dt/t\right) du/u\right)^{1/a} .$$

An application of the inequality (2.6) to the inner integral shows that

$$I_{pa;\alpha+1/a}f^* \leq cI_{pa;\alpha}(A_p f^*). \qquad (12.9)$$

When $a = \infty$, the inequality (12.9) still holds (cf. (4.6)). Using the identity (11.21) and proceeding as above (using (2.7) instead of (2.6)), we obtain

$$J_{qb;\beta+1/b}f^* \leq c(I_{pa;\alpha}(A_p f^*) + J_{qb;\beta}(B_q f^*)). \qquad (12.10)$$

The estimates (12.9) and (12.10) imply the desired result (12.8).

For the second inclusion in (12.2) we proceed as above, employing the identities

$$(1-\log t)^{\beta b+1} = |\beta b+1| \int_0^t (1-\log u)^{\beta b} du/u;$$

$$(1+\log t)^{\alpha a+1} = |\alpha a+1| \int_t^\infty (1+\log u)^{\alpha a} du/u,$$

which hold since $\alpha + 1/a < 0$ and $\beta + 1/b < 0$.

4. To prove the third inclusion in (12.1), it is required to show

$$I_{pa;\alpha}(C_p f^*) + J_{qb;\beta}(D_q f^*) \le c(I_{pa;\alpha+1/a} f^* + J_{qb;\beta+1/b} f^*). \qquad (12.11)$$

We assume that $1 \le a,b < \infty$. The inequality (2.2) shows that

$$I_{pa;\alpha}(C_p f^*) \le c\Big(\int_0^1 (1-\log t)^{\alpha a} \int_0^t [s^{1/p} f^*(s)]^a ds/s \; dt/t\Big)^{1/a} .$$

We may now apply the Hardy inequality (2.22) to prove that

$$I_{pa;\alpha}(C_p f^*) \le c I_{pa;\alpha+1/a} f^* . \qquad (12.12)$$

To estimate $J_{qb;\beta}(D_q f^*)$, apply (2.3) to obtain

$$J_{qb;\beta}(D_q f^*) \le c\Big(\int_1^\infty (1+\log t)^{\beta b} \int_{t/2}^\infty [s^{1/q} f^*(s)]^b ds/s \; dt/t\Big)^{1/b}.$$

Utilizing the Hardy inequality (2.23) and changing variables ($t/2 \to t$), we see that

$$J_{qb;\beta}(D_q f^*) \le c\Big(\int_{1/2}^\infty [t^{1/q}(1+|\log t|)^{\beta+1/b} f^*(t)]^b dt/t\Big)^{1/b}. \qquad (12.13)$$

The estimates (12.12) and (12.13) combine to produce the required result (12.11). (The cases $a = \infty$, $b = \infty$ are easy and thus omitted).

To prove the third inclusion in (12.2) we proceed as above to obtain

$$\|f\|_{\mathcal{M}^{pa}(\log \mathcal{M})^{\alpha} \cap \mathcal{M}^{qb}(\log \mathcal{M})^{\beta}}$$

$$\leq c(\|f\|_{L^{pa}(\log L)^{\alpha+1/a} \cap L^{qb}(\log L)^{\beta+1/b}} + \|f\|_{L^{p1} + L^{q1}}).$$

An appeal to Theorem 9.4 completes the proof.

CHAPTER IV

THE INTERPOLATION THEORY

13. Operators of Weak Type (p,q;r,s).

Suppose $0 < p < r \leq \infty$ and $0 < q,s \leq \infty$ with $q \neq s$. Let

$$1/\eta = 1/p - 1/r, \quad 1/\varepsilon = 1/q - 1/s, \quad m = \eta/\varepsilon . \tag{13.1}$$

The quantity m represents the slope of the line segment σ joining the points $(1/p, 1/q)$ and $(1/r, 1/s)$. For each $f \in L^{p1} + L^{r1}$, let

$$\begin{aligned}(W(\sigma)f^*)(t) &= t^{-1/q}\int_0^{t^m} u^{1/p} f^*(u)du/u \\ &\quad + t^{-1/s}\int_{t^m}^{\infty} u^{1/r} f^*(u)du/u, \qquad 0 < t < \infty .\end{aligned} \tag{13.2}$$

<u>DEFINITION 13.1</u>: Let T be a quasilinear operator mapping measurable functions on $(\mathfrak{X},\mu)$ into measurable functions on $(\mathcal{Y},\nu)$. Suppose $0 < p < r \leq \infty$ and $0 < q,s \leq \infty$ with $q \neq s$. The operator T is of <u>weak type (p,q;r,s)</u> if T is defined on $L^{p1} + L^{r1}$ and the inequality

$$(Tf)^*(t) \leq c(W(\sigma)f^*)(t), \quad 0 < t < \infty, \tag{13.3}$$

is satisfied for all $f \in L^{p1} + L^{r1}$, where c is a constant independent of f.

The next theorem reformulates the notion of weak type (p,q;r,s) in terms of the averaging operators A_p, B_p, C_p and D_p of Section 4.

We first need a lemma.

LEMMA 13.2: Suppose $0 < p < r \leq \infty$ and $f \in L^{p1} + L^{r1}$. Then the function

$$g(t) = t^{1/p}([A_p + B_r]f^*)(t)$$

increases on $(0,\infty)$, and the function

$$h(t) = t^{1/r}([A_p + B_r]f^*)(t)$$

decreases on $(0,\infty)$.

Proof: The definitions (4.1), (4.2) and (13.1) show that

$$g(t) = \int_0^t u^{1/p} f^*(u)du/u + t^{1/\eta}\int_t^\infty u^{1/r} f^*(u)du/u.$$

Let $0 < s < t < \infty$. Then

$$g(s) - g(t) = -\int_s^t u^{1/p} f^*(u)du/u + s^{1/\eta}\int_s^t u^{1/r} f^*(u)du/u$$

$$+ (s^{1/\eta} - t^{1/\eta})\int_t^\infty u^{1/r} f^*(u)du/u .$$

The last term is less than or equal to zero since $\eta > 0$ (cf. (13.1)). On the other hand, we have

$$s^{1/\eta}\int_s^t u^{1/r} f^*(u)du/u \leq \int_s^t u^{1/\eta} u^{1/r} f^*(u)du/u$$

$$= \int_s^t u^{1/p} f^*(u)du/u .$$

Thus, $g(s) \leq g(t)$ as required. The proof that h is decreasing is similar.

THEOREM 13.3: Suppose $0 < p < r \leq \infty$ and $0 < q,s \leq \infty$ with $q \neq s$. Then a quasilinear operator T is of weak type $(p,q;r,s)$ if and only if the following inequality holds:

(a) for $q < s$,

$$t^{\varepsilon/q}([C_q + D_s](Tf)^*)(t^\varepsilon) \leq ct^{\eta/p}([A_p + B_r]f^*)(t^\eta), \quad 0 < t < \infty; \tag{13.4}$$

(b) for $q > s$,

$$t^{\varepsilon/q}([C_s + D_q](Tf)^*)(t^\varepsilon) \leq ct^{\eta/p}([A_p + B_r]f^*)(t^\eta), \quad 0 < t < \infty; \tag{13.5}$$

where c is a constant independent of f.

Proof (a): Since $q < s$, we have that $m > 0$ (cf. (13.1)). From Definition 13.1, T is of weak type $(p,q;r,s)$ if and only if

$$(Tf)^*(t) \leq ct^\zeta([A_p + B_r]f^*)(t^m), \qquad 0 < t < \infty, \tag{13.6}$$

where

$$\zeta = -1/q + m/p = -1/s + m/r. \tag{13.7}$$

It is required to show that (13.4) and (13.6) are equivalent.

Suppose (13.4) holds. By Lemma 4.1, the left-hand side of (13.4) is greater than or equal to $2t^{\varepsilon/q}(Tf)^*(t^\varepsilon)$. The resulting inequality, after the change of variable $t^\varepsilon \to t$, reduces to (13.6).

Conversely, suppose (13.6) holds. Let $t > 0$. If $0 < u \leq t^\varepsilon$, then (13.6) and (13.7) show that

$$u^{1/q}(Tf)^*(u) \leq cu^{m/p}([A_p + B_r]f^*)(u^m).$$

Lemma 13.2 and the fact $m > 0$ show that the right-hand side is largest when $u = t^{\varepsilon}$. Taking the supremum over $0 < u \leq t^{\varepsilon}$, we obtain

$$t^{\varepsilon/q}(C_q(Tf)^*)(t^{\varepsilon}) \leq ct^{\eta/p}([A_p + B_r]f^*)(t^{\eta}). \tag{13.8}$$

Now fix u so that $t^{\varepsilon} \leq u < \infty$. Then (13.6) and (13.7) show that

$$u^{1/s}(Tf)^*(u) \leq cu^{m/r}([A_p + B_r]f^*)(u^m).$$

Lemma 13.2 and $m > 0$ imply that the right-hand side is largest when $u = t^{\varepsilon}$. Passing to the supremum over $t^{\varepsilon} \leq u < \infty$, we have

$$t^{\varepsilon/s}(D_s(Tf)^*)(t^{\varepsilon}) \leq ct^{\eta/r}([A_p + B_r]f^*)(t^{\eta}). \tag{13.9}$$

Now the identities

$$t^{\varepsilon/s} = t^{\varepsilon/q} \cdot t^{-1} \;;\; t^{\eta/r} = t^{\eta/p} \cdot t^{-1} \tag{13.10}$$

show that

$$t^{\varepsilon/q}(D_s(Tf)^*)(t^{\varepsilon}) \leq ct^{\eta/p}([A_p + B_r]f^*)(t^{\eta}). \tag{13.11}$$

The estimates (13.8) and (13.11) imply the required inequality (13.4).

The proof of part (b) proceeds similarly; only now $m < 0$, and so Lemma 13.2 implies that $u^{m/p}([A_p + B_r]f^*)(u^m)$ decreases while $u^{m/r}([A_p + B_r]f^*)(u^m)$ increases. This accounts for the interchange of q and s. The proof is now complete.

The operator $W(\sigma)$ defined by (13.2) is closely related to Calderón's $S(\sigma)$ operator introduced in [9, p. 288]. In fact, $W(\sigma) = S(\sigma)$ except in the important case $r = \infty$ where

$$(W(\sigma)f^*)(t) = t^{-1/q}\int_0^{t^m} u^{1/p}f^*(u)du/u + t^{-1/s}\int_{t^m}^{\infty} f^*(u)du/u,$$

$$(S(\sigma)f^*)(t) = t^{-1/q}\int_0^{t^m} u^{1/p}f^*(u)du/u.$$

Thus, in all cases we have

$$S(\sigma) \leq W(\sigma) \tag{13.12}$$

The following theorem is due to Calderón [9, Theorem 8].

THEOREM 13.4: (Calderón): Suppose $0 < p < r \leq \infty$ and $0 < q,s \leq \infty$ with $q \neq s$. Let T be a quasilinear operator of weak types (p,q) and (r,s). Then

$$(Tf)^*(t) \leq cS(\sigma)f^*(t), \quad 0 < t < \infty, \tag{13.13}$$

for all $f \in L^{p1} + L^{r1}$ if $r < \infty$, or for all $f \in L^{p1} + L^{\infty}$ if $r = \infty$, where c is a constant independent of f.

The next theorem shows when an operator of weak type $(p,q;r,s)$ is simultaneously of weak types (p,q) and (r,s).

THEOREM 13.5: Suppose $0 < p < r \leq \infty$, $0 < q,s \leq \infty$, $q \neq s$, and let T be a quasilinear operator.

(a) If T is of weak types (p,q) and (r,s), then T is of weak type $(p,q;r,s)$.

(b) If T is of weak type $(p,q;r,s)$, then T is of weak type (p,q). If, in addition, $r < \infty$, then T is also of weak type (r,s).

Proof: Part (a) is a direct consequence of (13.13) and (13.12).

To prove part (b) we are first required to show that

$$T : L^{p1} \to L^{q\infty}, \tag{13.14}$$

i.e., T is of weak type (p,q) (cf.(1.5.i)). Assume that $q < s$. Then the inequalities (13.8) and (13.9) remain valid. Suppose $f \in L^{p1}$. Let $t \to \infty$ in (13.8) to obtain

$$\|Tf\|_{q\infty} \leq c(\|f\|_{p1} + \overline{\lim_{t\to\infty}}\, t\int_{t^\eta}^{\infty} u^{1/r} f^*(u)du/u). \tag{13.15}$$

Using (13.1) and the fact $f \in L^{p1}$, we have

$$\overline{\lim_{t\to\infty}}\, t\int_{t^\eta}^{\infty} u^{1/r} f^*(u)du/u \leq \overline{\lim_{t\to\infty}} \int_{t^\eta}^{\infty} u^{1/p} f^*(u)du/u = 0 .$$

The required result (13.14) now follows from (13.15).

If, in addition, $r < \infty$, let $f \in L^{r1}$. Let $t \to 0$ in (13.9) to obtain

$$\|Tf\|_{s\infty} \leq c\left(\|f\|_{r1} + \overline{\lim_{t\to 0}}\, t^{-1}\int_0^{t^\eta} u^{1/p} f^*(u)du/u\right) .$$

The definition (13.1) and the fact $f \in L^{r1}$ show that

$$\overline{\lim_{t\to 0}}\, t^{-1}\int_0^{t^\eta} u^{1/p} f^*(u)du/u \leq \overline{\lim_{t\to 0}} \int_0^{t^\eta} u^{1/r} f^*(u)du/u = 0.$$

These estimates show that $T : L^{r1} \to L^{s\infty}$, i.e., T is weak type (r,s).

The case $q > s$ is treated similarly so we omit the details. This completes the proof.

14. Proof of the Main Results

We may now prove Theorems A,B and C which were stated in the Introduction.

Proof of THEOREM A: Recall that $0 < p < r \leq \infty$, $0 < q,s \leq \infty$ with $q \neq s$ and T is of weak type $(p,q;r,s)$. For $0 < \theta < 1$, u and v are defined by

$$\frac{1}{u} = \frac{\theta}{p} + \frac{1-\theta}{r} \ ; \ \frac{1}{v} = \frac{\theta}{q} + \frac{1-\theta}{s} \ . \tag{14.1}$$

Define η and ε by

$$\frac{1}{\eta} = \frac{1}{p} - \frac{1}{r} \ ; \ \frac{1}{\varepsilon} = \frac{1}{q} - \frac{1}{s} \ . \tag{14.2}$$

First suppose that $q < s$. Then, (13.4) shows that

$$t^{\varepsilon/q}([C_q + D_s](Tf)^*)(t^\varepsilon) \leq ct^{\eta/p}([A_p + B_r]f^*)(t^\eta) \ . \tag{14.3}$$

Applying the functional

$$\psi \to \left(\int_0^\infty [t^{\theta-1}(1+|\log t|)^\alpha \psi(t)]^a dt/t\right)^{1/a}$$

to both sides of (14.3), we obtain, by (14.1), (14.2) and a change of variable in both sides,

$$\begin{aligned} &\left(\int_0^\infty [t^{1/v}(1+|\log t|)^\alpha([C_q + D_s](Tf)^*(t)]^a dt/t\right)^{1/a} \\ &\quad \leq c\left(\int_0^\infty [t^{1/u}(1+|\log t|)^\alpha([A_p + B_r]f^*)(t)]^a dt/t\right)^{1/a} . \end{aligned} \tag{14.4}$$

The definition (14.1) shows that $p < u < r$ and $q < v < s$. Thus, Theorem 6.1 shows that in the expression (14.4), the operators

A_p, B_r, C_q and D_s are all equivalent to the identity. Hence, (14.4) is equivalent to the inequality

$$\|Tf\|_{va;\alpha} \leq c\|f\|_{ua;\alpha} .$$

This is precisely the desired conclusion

$$T : L^{ua}(\log L)^{\alpha} \to L^{va}(\log L)^{\alpha} .$$

The case $q > s$ is proved similarly utilizing (13.5). This completes the proof of Theorem A.

The following theorem gives an interpolation result for the auxiliary Lorentz-Zygmund spaces of Section 11.

THEOREM 14.1: Suppose $0 < p < r \leq \infty$ and $0 < q,s \leq \infty$ with $q \neq s$. Let T be a quasilinear operator of weak type $(p,q;r,s)$.

(i) If $\alpha + 1/a > 0$ and $\beta + 1/b > 0$, then

$$T : \mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} + \mathscr{L}^{rb}(\log \mathscr{L})^{\beta} \to \mathscr{M}^{qa}(\log \mathscr{M})^{\alpha} + \mathscr{M}^{sb}(\log \mathscr{M})^{\beta}. \qquad (14.5)$$

(ii) If $\alpha + 1/a < 0$ and $\beta + 1/b < 0$, then

$$T : \mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} \cap \mathscr{L}^{rb}(\log \mathscr{L})^{\beta} \to \mathscr{M}^{qa}(\log \mathscr{M})^{\alpha} \cap \mathscr{M}^{sb}(\log \mathscr{M})^{\beta}. \qquad (14.6)$$

Proof: Let η and ε be defined by (14.2). Let us first assume that $q < s$. The inequality (13.4) then holds:

$$t^{\varepsilon/q}([C_q + D_s](Tf)^*)(t^{\varepsilon}) \leq ct^{\eta/p}([A_p + B_r]f^*)(t^{\eta}). \qquad (14.7)$$

First, suppose $\alpha + 1/a > 0$ and $\beta + 1/b > 0$. Applying the functional

$$\psi \to \left(\int_0^1 [(1-\log t)^{\alpha}\psi(t)]^a dt/t\right)$$

to both sides of (14.7), we obtain, after a change of variable in each side,

$$I_{qa;\alpha}([C_q + D_s](Tf)^*) \le cI_{pa;\alpha}([A_p + B_r]f^*). \qquad (14.8)$$

Multiplying both sides of (14.7) by t^{-1}, the identities (13.10) show

$$t^{\varepsilon/s}([C_q + D_s](Tf)^*)(t^{\varepsilon}) \le ct^{\eta/r}([A_p + B_r]f^*)(t^{\eta}). \qquad (14.9)$$

Applying the functional

$$\psi \to \left(\int_1^{\infty} [(1+\log t)^{\beta}\psi(t)]^b dt/t\right)^{1/b}$$

to both sides of (14.9), we have, after a change of variable in each side,

$$J_{sb;\beta}([C_q + D_s](Tf)^*) \le cJ_{rb;\beta}([A_p + B_r]f^*). \qquad (14.10)$$

The sum of the inequalities (14.8) and (14.10), in conjunction with Theorem 11.5, shows that

$$\|Tf\|_{\mathcal{M}^{qa}(\log \mathcal{M})^{\alpha} + \mathcal{M}^{sb}(\log \mathcal{M})^{\beta}} \le c\|f\|_{\mathcal{L}^{pa}(\log \mathcal{L})^{\alpha} + \mathcal{L}^{rb}(\log \mathcal{L})^{\beta}}$$

This is precisely the assertion (14.5).

When $\alpha + 1/a$ and $\beta + 1/b$ are negative, the statement (14.6) is proved similarly by first applying the functional

$$\psi \to \left(\int_0^1 [(1-\log t)^{\beta}\psi(t)]^b dt/t\right)^{1/b}$$

to both sides of (14.9); then, we apply the functional

$$\psi \to \left(\int_1^\infty [(1+\log t)^{\alpha}\psi(t)]^a dt/t\right)^{1/a}$$

to both sides of (14.7). The sum of the resulting inequalities, together with Theorem 11.5, produces the desired result (14.6).

The case $q > s$ is proved similarly using the fundamental inequality (13.5). The proof is now complete.

Proof of THEOREM B: Recall that $0 < p < r \leq \infty$ and $0 < q,s \leq \infty$ with $q \neq s$. T is a quasilinear operator of weak type (p,q;r,s). For $1 \leq a \leq b \leq \infty$ and $1 \leq c \leq d \leq \infty$, the desired results are:

(a) If $\alpha + 1/a = \beta + 1/b > 0$ and $\gamma + 1/c = \delta + 1/d > 0$, then

$$T : L^{pa}(\log L)^{\alpha+1} + L^{rc}(\log L)^{\gamma+1} \to L^{qb}(\log L)^{\beta} + L^{sd}(\log L)^{\delta} .$$

(b) If $\alpha + 1/a = \beta + 1/b < 0$ and $\gamma + 1/c = \delta + 1/d < 0$, then

$$T : L^{pa}(\log L)^{\alpha+1} \cap L^{rc}(\log L)^{\gamma+1} \to L^{qb}(\log L)^{\beta} \cap L^{sd}(\log L)^{\delta} .$$

If $\alpha + 1/a > 0$ and $\gamma + 1/c > 0$, Theorem 14.1.i shows that

$$T : \mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} + \mathscr{L}^{rc}(\log \mathscr{L})^{\gamma} \to \mathscr{M}^{qa}(\log \mathscr{M})^{\alpha} + \mathscr{M}^{sc}(\log \mathscr{M})^{\gamma}. \quad (14.11)$$

But, the embedding theorem of Section 12 (cf. (12.1)) shows that

$$L^{pa}(\log L)^{\alpha+1} + L^{rc}(\log L)^{\gamma+1} \subset \mathscr{L}^{pa}(\log \mathscr{L})^{\alpha} + \mathscr{L}^{rc}(\log \mathscr{L})^{\gamma}. \quad (14.12)$$

The hypotheses (11.16.iv) now hold. Hence, we may use (11.24) and the embedding theorem of Section 12 to show

$$\mathcal{M}^{qa}(\log \mathcal{M})^{\alpha} + \mathcal{M}^{sc}(\log \mathcal{M})^{\gamma} \subseteq \mathcal{M}^{qb}(\log \mathcal{M})^{\beta} + \mathcal{M}^{sd}(\log \mathcal{M})^{\delta}$$
$$\subseteq L^{qb}(\log L)^{\beta} + L^{sd}(\log L)^{\delta}. \tag{14.13}$$

The inclusions (14.12) and (14.13), together with the result (14.11), prove the assertion of part (a).

Part (b) of Theorem B is proved similarly using (14.6), (12.2) and (11.25).

REMARKS 14.2 (i): Using the full force of the inclusion relations given by Theorem 10.5, Theorem B remains valid for various other choices of the parameters a, b, c, d and α, β, γ, δ. For example, suppose $a > b \geq 1$ and $\alpha + 1/a > \beta + 1/b$. Then (10.6) gives

$$L^{pa}(\log L)^{\alpha+1} + L^{rc}(\log L)^{\gamma+1} \subseteq L^{pb}(\log L)^{\beta+1} + L^{rc}(\log L)^{\gamma+1}. \tag{14.14}$$

If $\beta + 1/b > 0$ and $1 \leq c \leq d \leq \infty$ with $\gamma + 1/c = \delta + 1/d > 0$, then Theorem B shows that

$$T : L^{pb}(\log L)^{\beta+1} + L^{rc}(\log L)^{\gamma+1} \to L^{qb}(\log L)^{\beta} + L^{sd}(\log L)^{\delta}.$$

This result and the inclusion (14.14) prove that part (a) of Theorem B holds when $a > b \geq 1$, $\alpha + 1/a > \beta + 1/b > 0$, $1 \leq c \leq d \leq \infty$ and $\gamma + 1/c = \delta + 1/d > 0$.

(ii). The proofs of Theorems B1 and B2 are now simple consequences of Theorem B and Theorem 9.5.

Proof of THEOREM C: Recall that $0 < p < r \leq \infty$, $0 < q,s \leq \infty$ with $q \neq s$ and T is a quasilinear operator of weak type $(p,q;r,s)$. We also have $1 \leq a \leq b \leq \infty$, $0 < c \leq \infty$, $-\infty < \alpha,\beta,\gamma < \infty$ and $0 < \theta < 1$. The parameters u and v are given by

$$\frac{1}{u} = \frac{\theta}{p} + \frac{1-\theta}{r} \; ; \; \frac{1}{v} = \frac{\theta}{q} + \frac{1-\theta}{s} \quad .$$

Theorem A implies

$$T : L^{uc}(\log L)^{\gamma} \to L^{vc}(\log L)^{\gamma} \, . \tag{14.15}$$

Applying Theorem B (with $c = d = 1$, $\gamma = \delta = 0$), we obtain

$$T : L^{pa}(\log L)^{\alpha+1} + L^{r1}(\log L) \to L^{qb}(\log L)^{\beta} + L^{s1} \, . \tag{14.16}$$

The results (14.15) and (14.16) show that

$$\begin{aligned} T : \left(L^{pa}(\log L)^{\alpha+1} + L^{r1}(\log L)\right) + L^{uc}(\log L)^{\gamma} \\ \to \left(L^{qb}(\log L)^{\beta} + L^{s1}\right) + L^{vc}(\log L)^{\gamma}. \end{aligned} \tag{14.17}$$

An examination of the norms of the spaces involved (cf. (6.1), (9.3)) shows that the spaces in (14.17) reduce to $L^{pa}(\log L)^{\alpha+1} + L^{uc}(\log L)^{\gamma}$ and $L^{qb}(\log L)^{\beta} + L^{vc}(\log L)^{\gamma}$, respectively. Hence, the statement (14.17) reduces to

$$T : L^{pa}(\log L)^{\alpha+1} + L^{uc}(\log L)^{\gamma} \to L^{qb}(\log L)^{\beta} + L^{vc}(\log L)^{\gamma},$$

which is precisely the desired conclusion of Theorem C.a.i. The remaining statements in Theorem C are proved similarly.

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