DETERMINATION OF THE MUTUAL INDUCTANCE OF END TURNS OF INDUCTION MOTORS

Thesis

by

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Contents.

Foreword.

The author wishes to thank the officials of the United States Electric Corporation for their cooperation in furnishing induction motors for test purposes, and the design data on these and other motors. Especially, thanks are due Mr. Barth of that company for his assistance in furnishing design data.

This work was carried out under the guidance and with the cooperation of Dr. Vaino Hoover, who is in a large measure responsible for the results obtained. His ideas and work have been drawn on a number of times in the following paper without special acknowledgment.

Finally, the assistance of Mr. Rubin Widess and Mr. William Pickles, who also worked on this problem, is gratefully acknowledged.

Determination of the Mutual Inductance of End Turns of Induction Motors

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Introduction.

The object in making the following investigation was to facilitate the obtaining of a more accurate determination of the reactance of the end turns of rotating alternating current machinery. Such formulae as are available at present are, as far as the author knows, based on incorrect assumptions, or fail to consider the various factors present. The accuracy of the results obtained when using such formulae is very small. Often the calculated reactance will not be within 400% of the correct value. Because of this, it was felt that a more accurate analysis of this problem would be of value, chiefly to the designing engineer,

In this report, the "end turns" of the machine windings are considered to be the part of the turns of the coils on the armature of the machine not lying in armature slots. These turns, the connecting conductors between the armature bars, lie against each other in a belt extending around the circumference of the armature.

In nearly all motor coils, the end turns consist of a short section extending straight out from the armature slot, a much longer section bent nearly parallel with the core, an abruptly curved end section, and two lengths similar to the first two. It is obvious that the accurate determination of the inductance of such coils must take into account the peculiar shape of the coils and the effect of the coils upon each other. It is this second effect which is usually neglected in determinations of the reactance. Actually this mutual inductance is of greater importance than the self inductance of the coils.

Analysis and Tests.

In this investigation, computations have been made of the self and mutual-inductance of the end turns of a 15 horse power 1000 r.p.m. induction motor having 48 armature slots. The theoretical value of the total induction of the coils has been checked by a series of tests in which the mutual inductance of the coils of this motor, and of two other motors, has been measured. The effects of the rotor, shaft, end bells, etc. upon the end turn inductance have been found.

Calculation of End Turn Inductance.

The end turns of the motor coils set up a flux in the air surrounding them. If the flux at the face of the iron is parallel to the face, the end-turn inductance will be the same as the inductance of two coils made up by cutting out the part of the coils imbedded in the armature and joining up the resultant triangular end portions into a roughly diamond-shaped coil. The mutual inductance of two such coils could be calculated for the various positions one could assume with respect to the other, i.e., for all positions which two motor coils could occupy relative to each other. However, considering that there would be a short straight section in each half of the coil where the motor coil extends out from the coil before the initial turn, the calculation of inductance of such coils is still rather involved. However, that little section could be straightened out, and if the total length of conductor were not shortened, there would be very little change in the mutual inductance of the coils. The three steps in the process outlined above are shown in Fig. 1. The resultant diamond-shaped coils can be handled nicely in mutual inductance calculations.

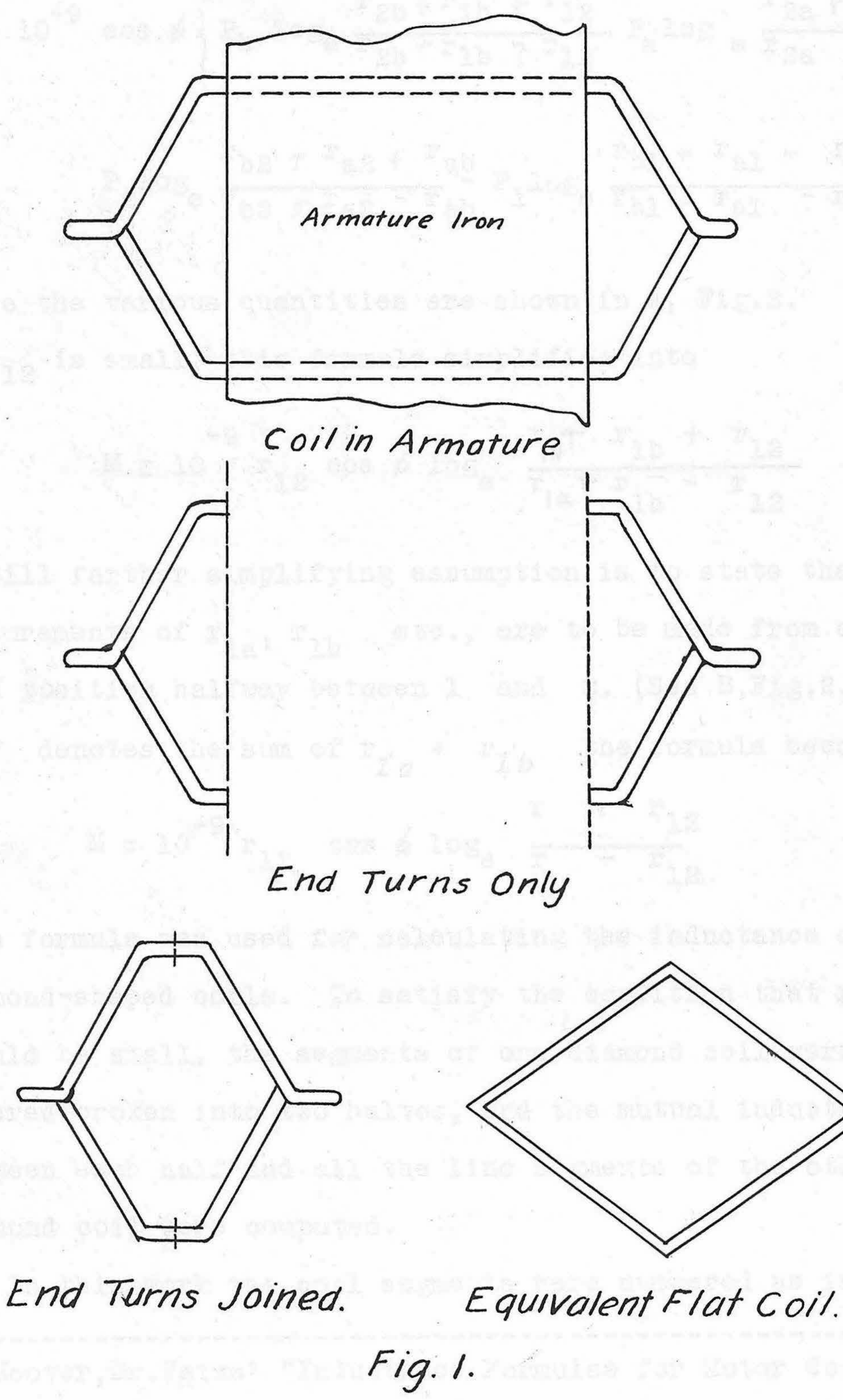
Armature Iron
Coil in Armature
End Turns Only
End Turns Joined.
Equivalent Flat Coil.

Fig. 1.

The mutual inductance between two intersecting segments is given by Dr. Vaino Hoover[1], as

$$M = 10^{-9} \cos \phi \left\{ P_b \log_e \frac{r_{2b} + r_{1b} + r_{12}}{r_{2b} + r_{1b} - r_{12}} - P_a \log_e \frac{r_{2a} + r_{1a} + r_{12}}{r_{2a} \; r_{1a} - r_{12}} \right.$$

$$\left. P_2 \log_e \frac{r_{b2} + r_{a2} + r_{ab}}{r_{b2} + r_{a2} - r_{ab}} - P_1 \log_e \frac{r_{b1} + r_{a1} + r_{ab}}{r_{b1} + r_{a1} - r_{ab}} \right\} \qquad (1)$$

where the various quantities are shown in A, Fig.2.

If r_{12} is small, this formula simplifies into

$$M = 10^{-9} \, r_{12} \cos \phi \log_e \frac{r_{1a} + r_{1b} + r_{12}}{r_{1a} + r_{1b} - r_{12}} \qquad (2)$$

A still farther simplifying assumption is to state that the measurements of r_{1a}, r_{1b} etc., are to be made from a and b to a position halfway between 1 and 2. (See B, Fig.2.)

If r denotes the sum of $r_{1'a} + r_{1'b}$ the formula becomes

$$M = 10^{-9} \, r_{12} \cos \phi \log_e \frac{r + r_{12}}{r - r_{12}} \qquad (3)$$

This formula was used for calculating the inductance of the diamond-shaped coils. To satisfy the condition that r_{12} should be small, the segments of one diamond coil were considered broken into two halves, and the mutual inductance between each half and all the line segments of the other diamond coil were computed.

In this work the coil segments were numbered as in Fig.3.

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1. Hoover, Dr. Vaino: "Inductance Formulae for Motor Coils".

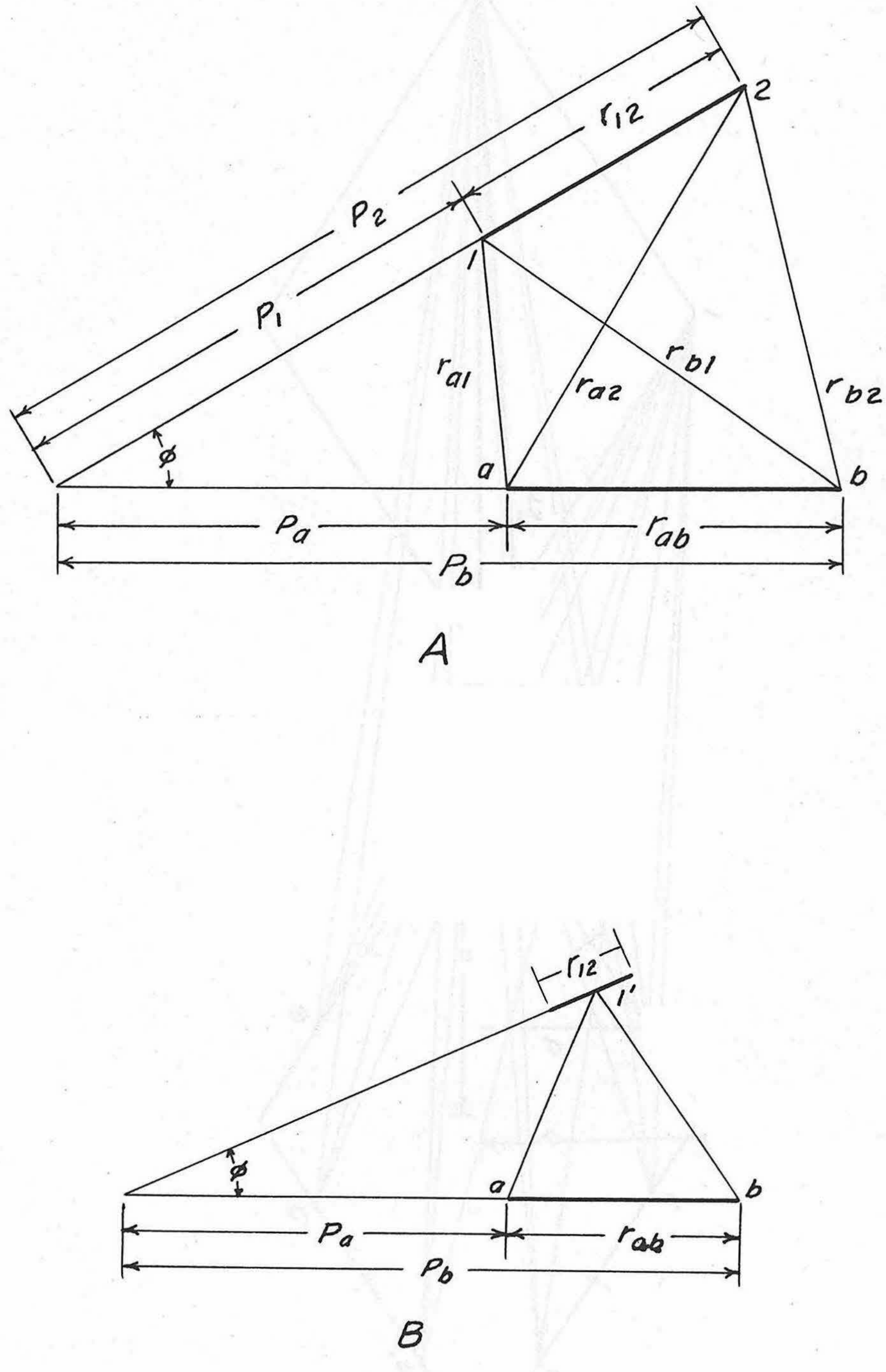


Fig. 2

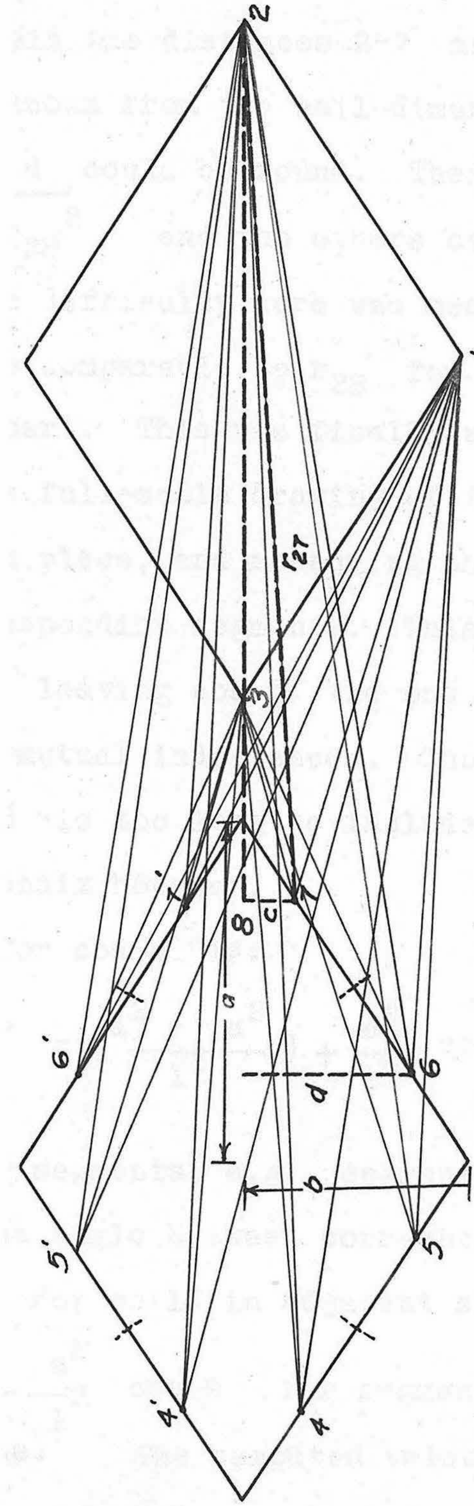


Fig. 3.

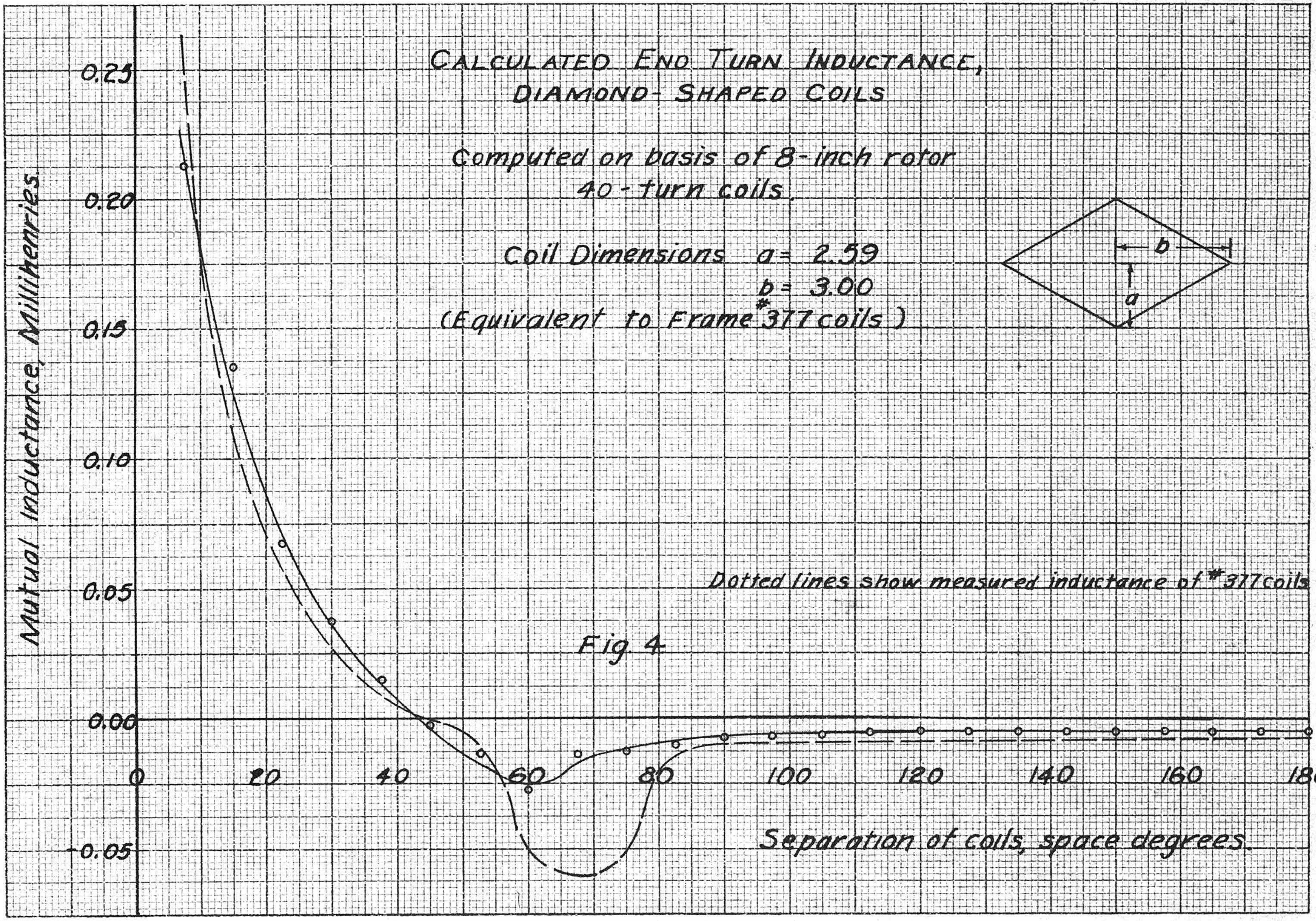
The first problem was to find the various lengths involved in the calculations. For example, to find the mutual inductance between line segment 7 and segment 2-1 it was necessary to obtain the distances 2-7 and 1-7. The lengths l and a were known from the coil dimensions, so the lengths c and d could be found. Then the distance 2-7 would be $\sqrt{c^2 + r_{28}^2}$ and the others could be found accordingly. The difficulty here was measuring the various projected lengths comparable to r_{28} for the two coils in all possible slots apart. This was finally accomplished by actually making a full-scale drawing of the motor armature, with the coils in place, and measuring the desired distance between the corresponding segments. This gave all the distances needed, leaving cos ϕ the one unknown in the equation for the mutual inductances. The development of the formula for cos ϕ is too long to include here; it is to be found in the Appendix, however.

The final value for cos ϕ is:

$$\cos \phi = \left(\frac{l^2 - a^2}{l^2}\right) + \frac{a^2}{l^2} \cos \theta \qquad (4)$$

for corresponding segments, e.g., segments 12 and 67 or 4'5'. θ is the angle between corresponding coil sides in the armature. For coils in adjacent slots, $\theta = \frac{1}{48} \times 360^\circ = 7.5^\circ$

$M = \left(\frac{l^2 - a^2}{l^2}\right) - \frac{a^2}{l^2} \cos \theta$ for segments on opposite sides of the center line. The computed values of the inductance of the diamond shaped coil are plotted in Fig.4.



CALCULATED END TURN INDUCTANCE,
DIAMOND-SHAPED COILS
Computed on basis of 8-inch rotor
40-turn coils.
Coil Dimensions a = 2.59
b = 3.00
(Equivalent to Frame #377 coils)
b
a
Dotted lines show measured inductance of #377 coils
Mutual Inductance, Millihenries
0.25
0.20
0.15
0.10
0.05
0.00
-0.05
0
20
40
60
80
100
120
140
160
180
Separation of coils, space degrees.

Fig. 4

Inductance Measurements.

To check the computed values of end turn inductance, a series of measurements of the mutual reactances of the end turns were carried out. The first consideration in this work was to obtain a reasonably simple,accurate method of measuring the mutual inductance of two coils. There are a number of bridges which are designed to do this, but the balance of such bridges usually takes quite some time, and the apparatus is rather unwieldy and delicate. It was felt that better results could be obtained in less time by a direct comparison method than by a bridge circuit. A circuit was eventually evolved that fitted the purpose very nicely. Measurements could be made rapidly, and to an accuracy of within one percent, which was well within the limits desired.

Test Procedure.

The complete diagram of the circuit used in these measurements is shown in Fig.5, and views of the equipment in Figs. 6 and 7.

Three-phase 110-volt alternating current is supplied to a "phase-shifting transformer", which consists of a trans- former wound like a three-phase induction regulator without

Wiring Diagram of Apparatus Used in Mutual Inductance Tests.

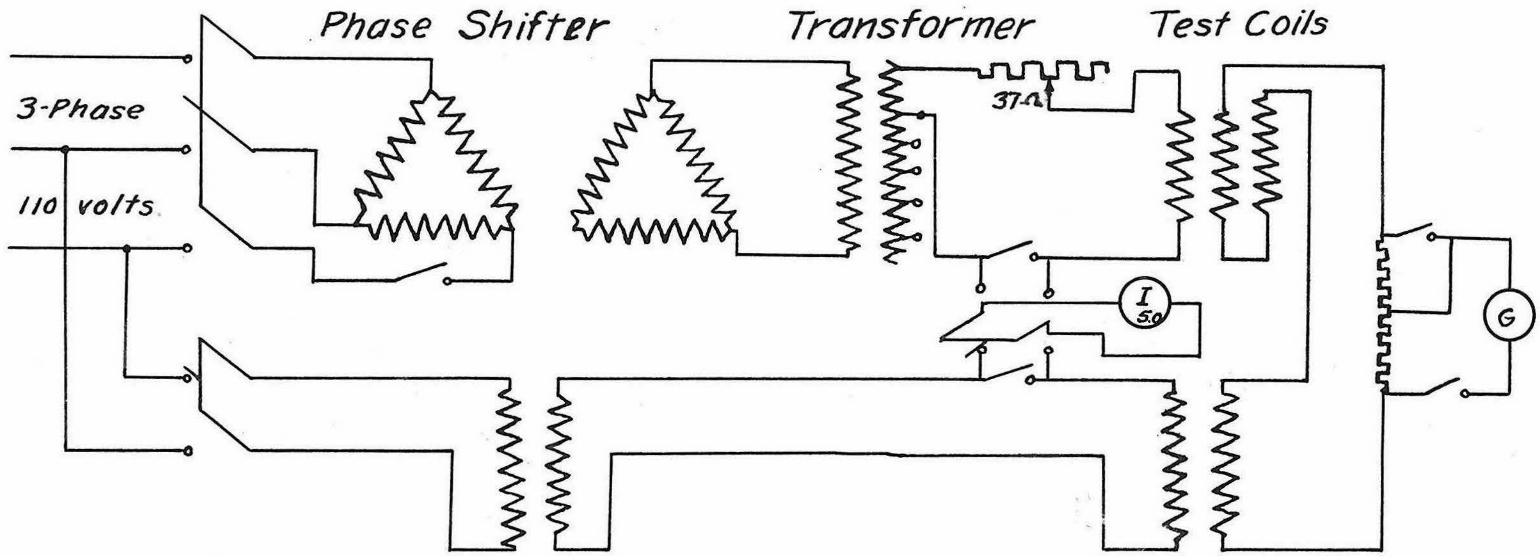


Fig. 5.

Fig. 6. General View of Test Setup.

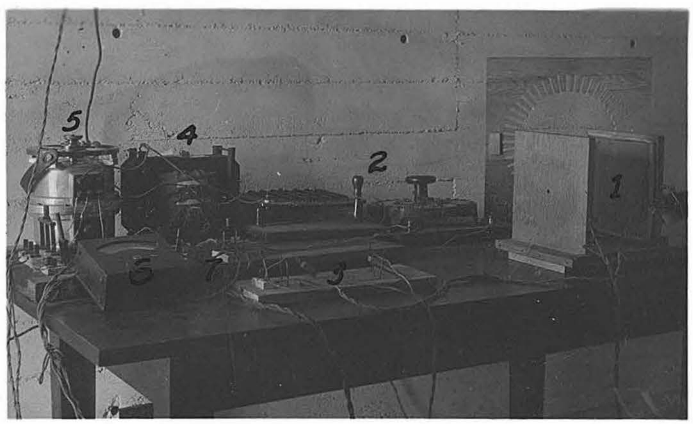


Fig 7. 1-Mutual Inductance, 2-Rheostats, 3-Voltage Divider, 4-Transformer, 5-Phase Shifter, 6-Ammeter, 7-Current-Measuring Switch.

connections between primary and secondary. As the secondary is rotated, the voltage of the delta-wound secondary is shifted in phase with respect to the supply voltage. Only one phase of the secondary is used, connected to a stepdown transformer having taps for 16.4, 19.3, 38.5, 57.8, and 96.2 volts. At present the 16.4 volt tap is used. Leads are taken from this tap to the primary coil of the two whose mutual inductance is to be determined, which is in series with a 30-ohm and a 5-ohm rheostat to adjust the primary current. On the other circuit, a single phase of the supply voltage is used to excite a single-phase induction regulator, which is used to vary the current in the primary of the standard mutual inductance. By means of a double pole double throw switch and two single pole short-circuiting switches, the current in the primary coil or the primary of the standard mutual inductance can be measured. The secondaries of the standard mutual inductance and the coils to be measured are connected in series opposition to a sensitive Leeds and Northrup vibration galvanometer. A resistance is connected across the galvanometer terminals with a tap so that for coarse adjustment only a part of the total voltage is impressed on the meter.

In operation, both primary coils are energized, and the currents I_1 and I_2 in the two circuits adjusted until the vibration galvanometer shows zero deflection. When this occurs, the voltage due to mutual induction on one side of the circuit is equal and opposite to that on the other side, or in the usual vectorial notation,

$$jwM_1 I_1 = jw M_2 I_2$$

or, $$M_1 I_1 = M_2 I_2 .$$

$$M_1 \neq \frac{I_2}{I_1} M_2 \qquad (5)$$

where M_1 is the mutual inductance to be measured, M_2 is the mutual inductance of the standard, and I_1 and I_2 are the currents in the respective branches. Since the last three quantities are known, this simple equation gives the value of the inductance measured.

It might be considered that the phase-shifting transformer was unnecessary. It was absolutely necessary to use it to obtain the balance under the usual test conditions. When measuring the mutual inductance of the end turns of two motor coils with the rotor in place, it was found that the effect of the rotor conductors on the secondary coil was to introduce a voltage out of phase with the component induced by the primary coil. Thus, the resultant voltage of induction would not be in phase with the voltage of induction from the standard, and the resultant of the two voltages in opposition would never be zero. By shifting the phase of the primary current, the voltage of induction in the secondary coil could be placed in exact phase opposition with that from the standard, and a perfect balance could thus be obtained.

No standard mutual inductance of the range desired was available, so one was made up. It consisted of two square coils, of 100 and 50 turns respectively, tapped for 40 and

80 % on both primary and secondary, supported upon two vertical boards, so that the axis of both coils coincided. The 100-turn coil was stationary; the 50-turn coil could be moved axially a distance of 40 centimeters. The distance of separation could be read on a scale attached to the base. This inductance proved ideal for the measurements taken, the mutual inductance ranging from 1.7 to 0.04 millihenries. The calibration curve is shown in Fig. 8; and the coils may be seen in Figs. 6 and 7.

To calibrate the coils, a ballistic galvanometer and standard Kelvin coil were used. For the wiring diagram, see Fig.9. The method of measurement consists of comparing the deflection of the ballistic galvanometer when current is suddenly reversed through the primary of the mutual inductance with the deflection when the Kelvin coil is dropped. A Kelvin coil is merely a coil of a definite number of turns suspended above the end of a bar magnet whose flux is known. As the coil falls over the magnet, it cuts a definite number of lines, and generates a voltage which may be computed. The voltage in the secondary of the mutual inductance when current is suddenly reversed in the primary depends upon the mutual inductance, i.e., the flux cutting the secondary. The formula for the mutual inductance is:

$$M = \frac{N_s \phi_s}{2 D_s} \times \frac{D_x}{I_x} \times 10^{-8} \qquad (6)$$

where M = mutual inductance between primary and secondary coils
N_s = Number of turns on Kelvin coil

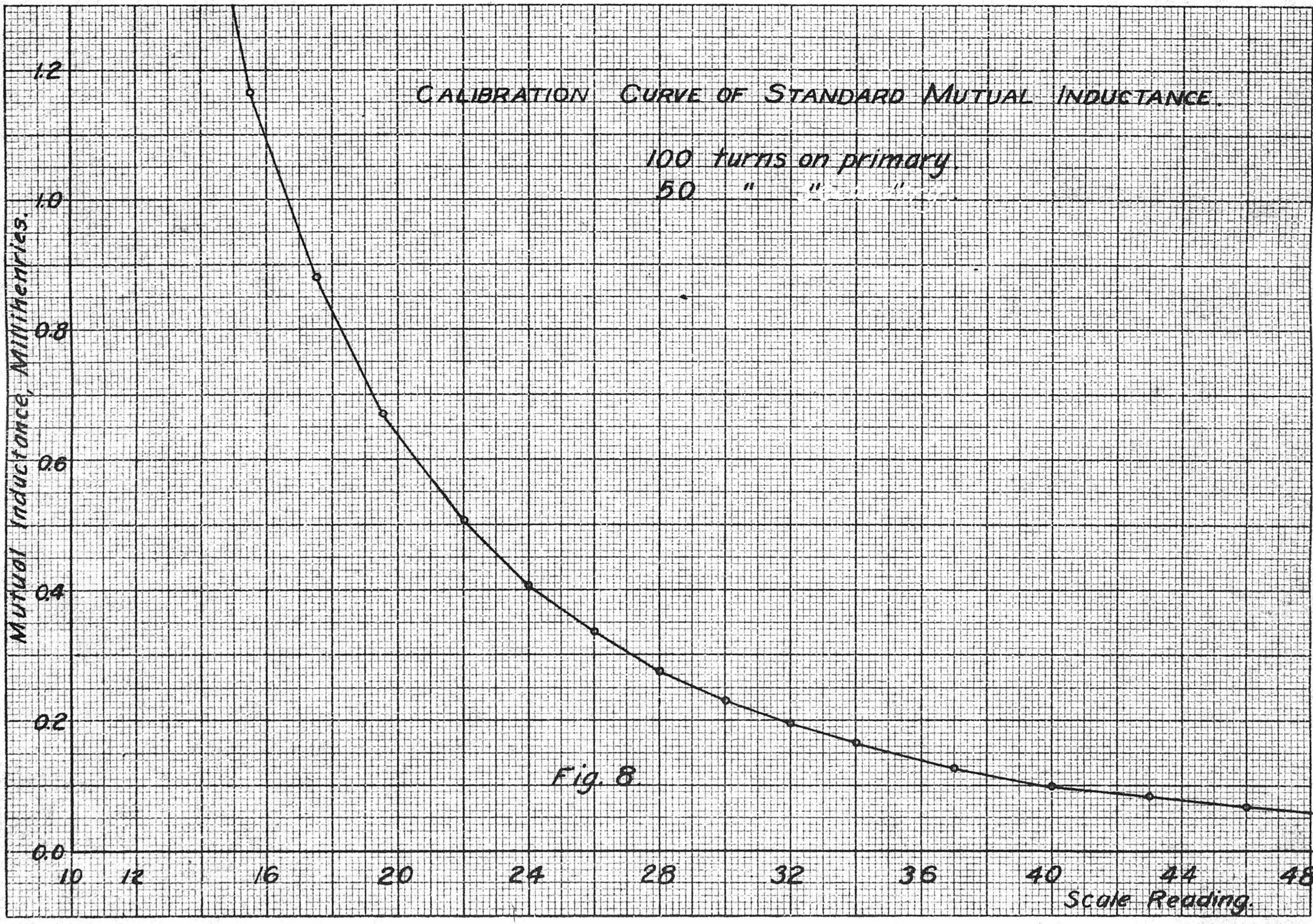

CALIBRATION CURVE OF STANDARD MUTUAL INDUCTANCE.
100 turns on primary.
50 "
Mutual Inductance, Millihenries.
0.0
0.2
0.4
0.6
0.8
1.0
1.2
10
12
16
20
24
28
32
36
40
44
48
Scale Reading.

Fig. 8.

ϕ = Number of lines in bar magnet cut by Kelvin coil

D_s = Deflection of ballistic galvanometer due to Kelvin coil

D_x = Deflection of ballistic galvanometer due to current reversal in primary of mutual inductance

I_x = Current in primary before reversal, amperes.

This formula is developed in Appendix B of this paper. The only factor that is not considered in the equation above is the resistance of the galvanometer circuit, which must be the same for both deflections.

So far, the apparatus would work just as well for the measurement of any mutual inductance in the working range. It now becomes necessary to obtain some method of insuring that only the mutual inductance of the end turns will be measured. After some study, it was decided to use the following procedure: Two flat coils (i.e.,coils that were not "pulled" to give the little kink in the end) were wound, each containing the same number of turns. With the exception of the absence of the kink on the end, the dimensions of these coils and usual form-wound motor coils were identical. These coils and others described in this paper are shown in Fig.10. The end bells and rotor of a motor were removed. There were no coils on the motor. A coil of the same number of turns as the two already made was wound on the motor. The wire was wound in the same slots that one coil would occupy, and in such a manner that the end connection was straight and as close as possible to the motor frame. Then one of the two prepared coils was placed in the same slots, and the coil sides of the two different coils brought as close together as possible.

Wiring Diagram for Calibration of Std. Mutual Inductance.

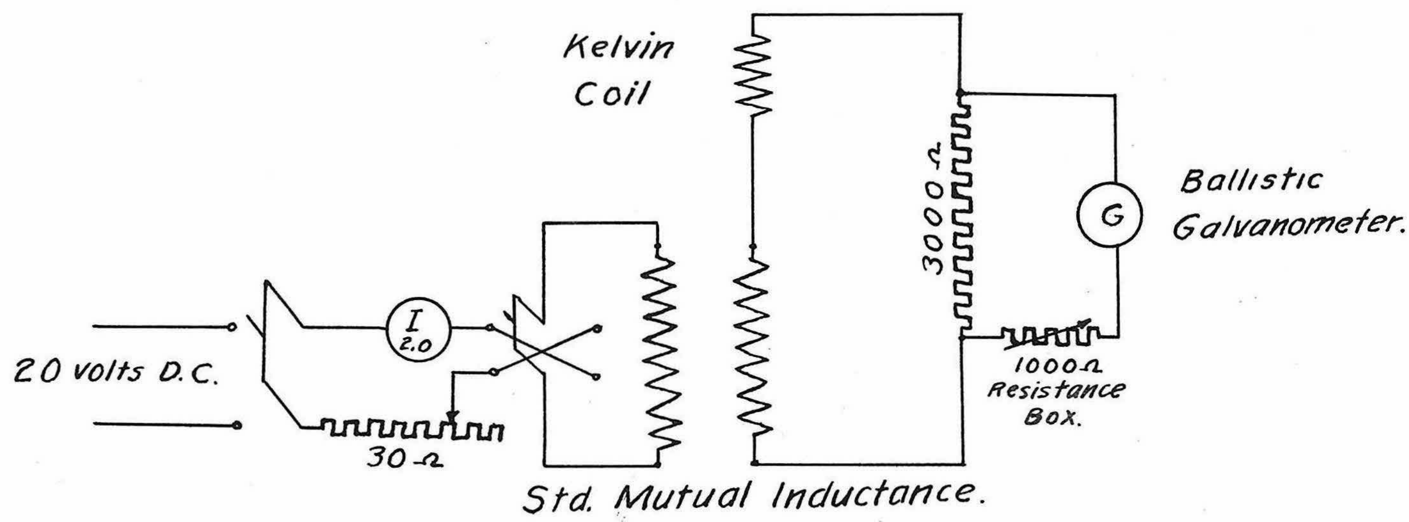


Fig. 9.

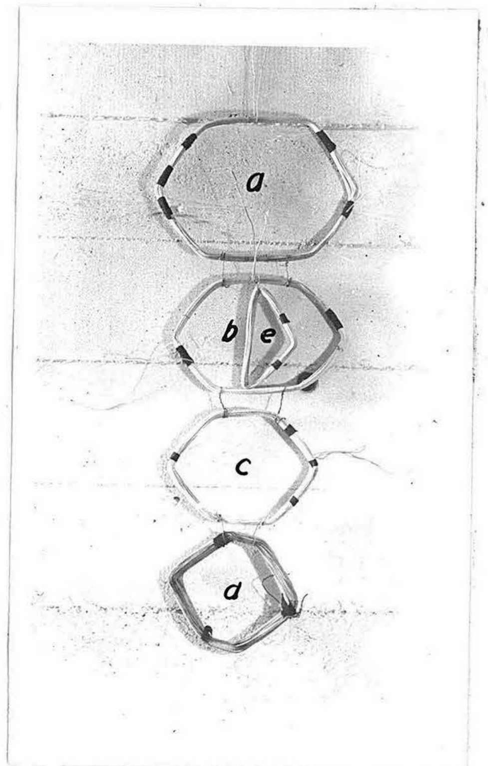


Fig. 10. Test Coils.

The two coils were then connected in series opposition, i.e., if the two leads were brought out to a source of voltage, current would flow around the formed coil in one direction, and around the smaller coil in the opposite direction. The second formed coil was then placed in whatever slots desired. This second formed coil was the primary, and the two other coils in series opposition the secondary of the system whose mutual inductance was to be measured. When alternating current was passed through the primary, a flux was set up both in the frame and in the end connections. The flux in the frame cut two coils whose conductors were in opposite directions, so there could be no resultant voltage of induction due to the flux in the frame. However, the flux from the end connections of the primary coil cut only the end connections of the formed coil, the other being wound as close to the frame as possible. Thus, the only voltage induced in the secondary coils will be that due to the mutual inductance of the end turns. Theoretically, perfect results should be obtained if the end connections of the inner secondary coil were coincident with the face of the frame, so that the end connections of the outer secondary coil would be the only conductors to cut the flux set up by the primary end turns. As will be seen later, a correction factor has to be applied because this could not be accomplished. Two views of the resultant coils actually used in the tests are shown in Figs. 11 and 12. It should be noted in both these views the care taken in holding the end turns of the inner secondary coil close to the face of the frame. It

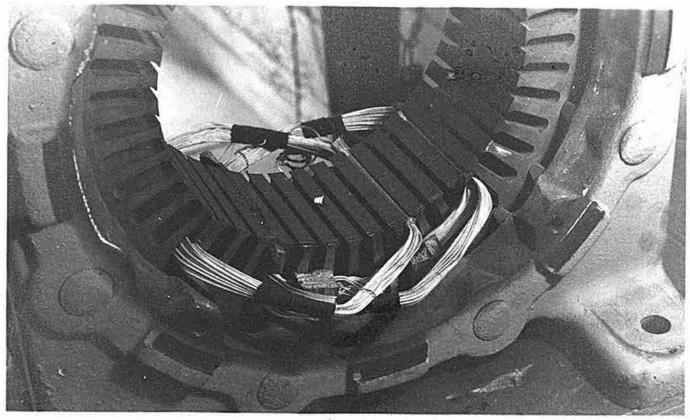


Fig. 11. Coils in Test Position.

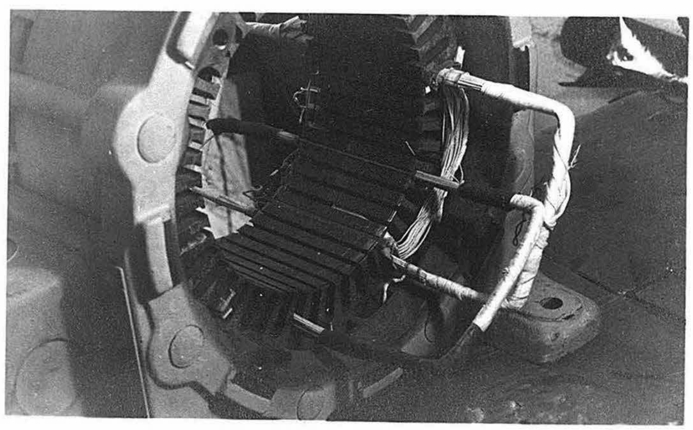


Fig. 12. Rectangular Coils in Place

was found that wooden wedges would best hold these inner end connections in position.

The set-up was then complete. An accurate method for measuring mutual inductance had been evolved, and an arrangement of coils had been discovered whereby the mutual inductance of the end turns of the motor coils could be measured with the coils in the motor. It was then possible to measure the inductance of the end turns of several motors, and to discover some factors affecting this inductance.

Test Data.

Four motors were available for test purposes. These were 5, 7.5, 10, and 15 horse power, 1000 r.p.m. 4-pole squirrel cage induction motors. There were 48 armature slots of the usual semi-inclosed type. The rotor diameter of all four was 8 inches. The axial length of iron in the core was 2, 2 1/2, 3 1/4, and 4 3/8 inches respectively, Otherwise, the motors were identical. In the actual testing, the 7.5 horse power motor was not used because of its similarity to the 5 h.p. motor. The motor coils had a span of 9 slots (1-10).

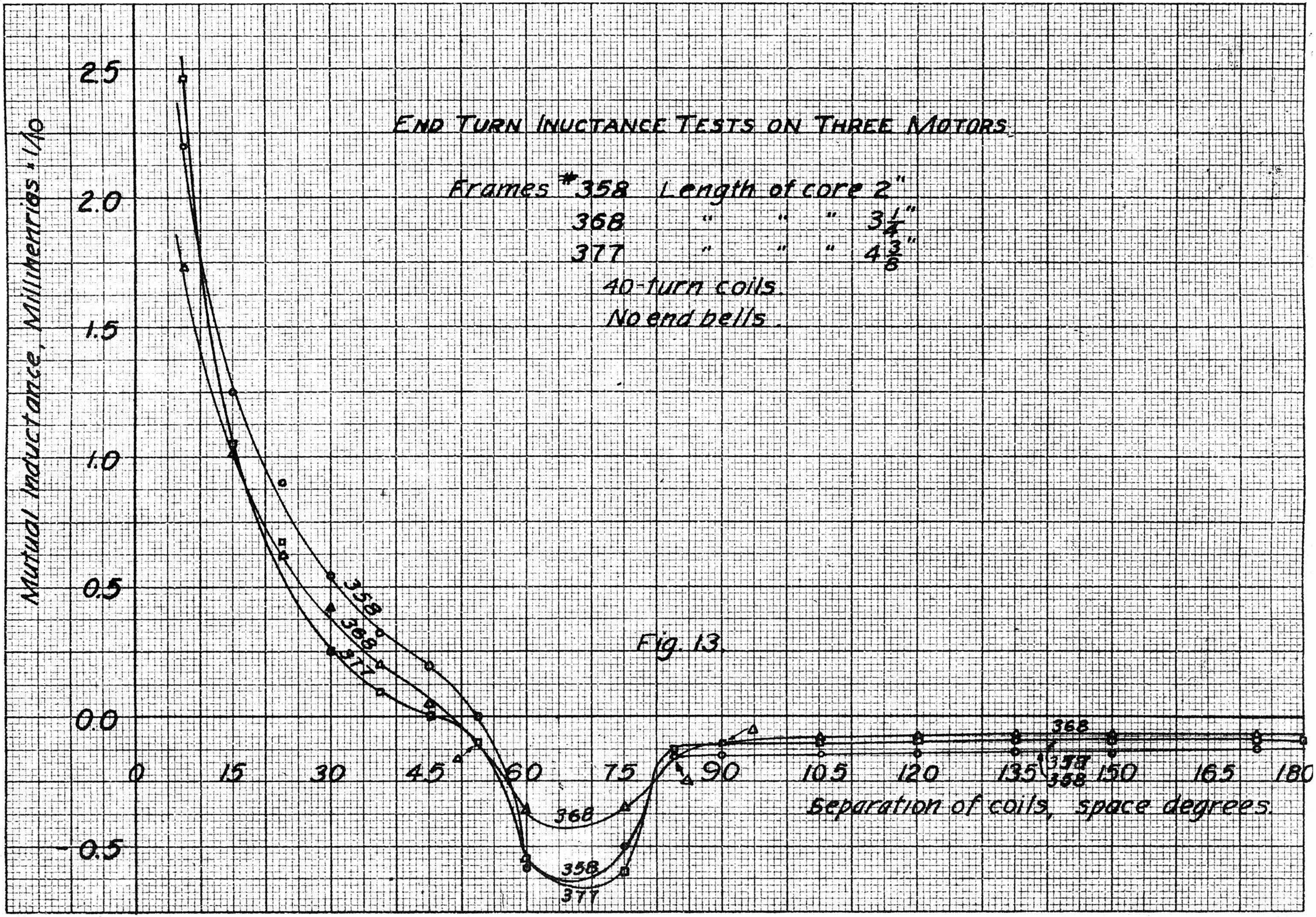
Inductance measurements were first made on all three machines using the standard sized coil for each, and arranging the coils in the way already described. These measurements were made with the primary in nearly all the slots up to half way around the motor armature. The rotor was in place for each measurement, but both end bells had been removed. The resultant mutual inductances have been plotted in Fig. 13. As would be expected, as the two coils were moved farther and farther apart, and turned each time through an angle of $\frac{1}{48} \times 360^\circ$, or 7.5°, the mutual inductance decreased rapidly, and at some point became zero. Beyond that point, the second coil would be reversed with respect to its former position, and the induced voltage was in the opposite direction to its former value. To avoid the difficulty of remembering that the induced voltage goes through a 180° phase change at that point, the mutual inductance beyond that point has been plotted as negative. It should be remembered that while the inductance will be called "negative" after reaching the zero point, there is actually no such thing as a negative inductance, and that the curves have been plotted that way only to show phase reversal.

The shape of all these curves is the same: the mutual inductance is, as would be expected, a maximum when the coils are in adjacent slots, and decreases to zero at around 60° around the circumference of the armature. It then rises to a negative maximum at about 75° and abruptly falls off to a very small value at 90°, which value is nearly constant for the remainder of the circumference. The similarity between Fig. 4 and Fig. 13 should be noted.

Effect of End Bells.

The effect of end bells upon the mutual inductance was next tested.

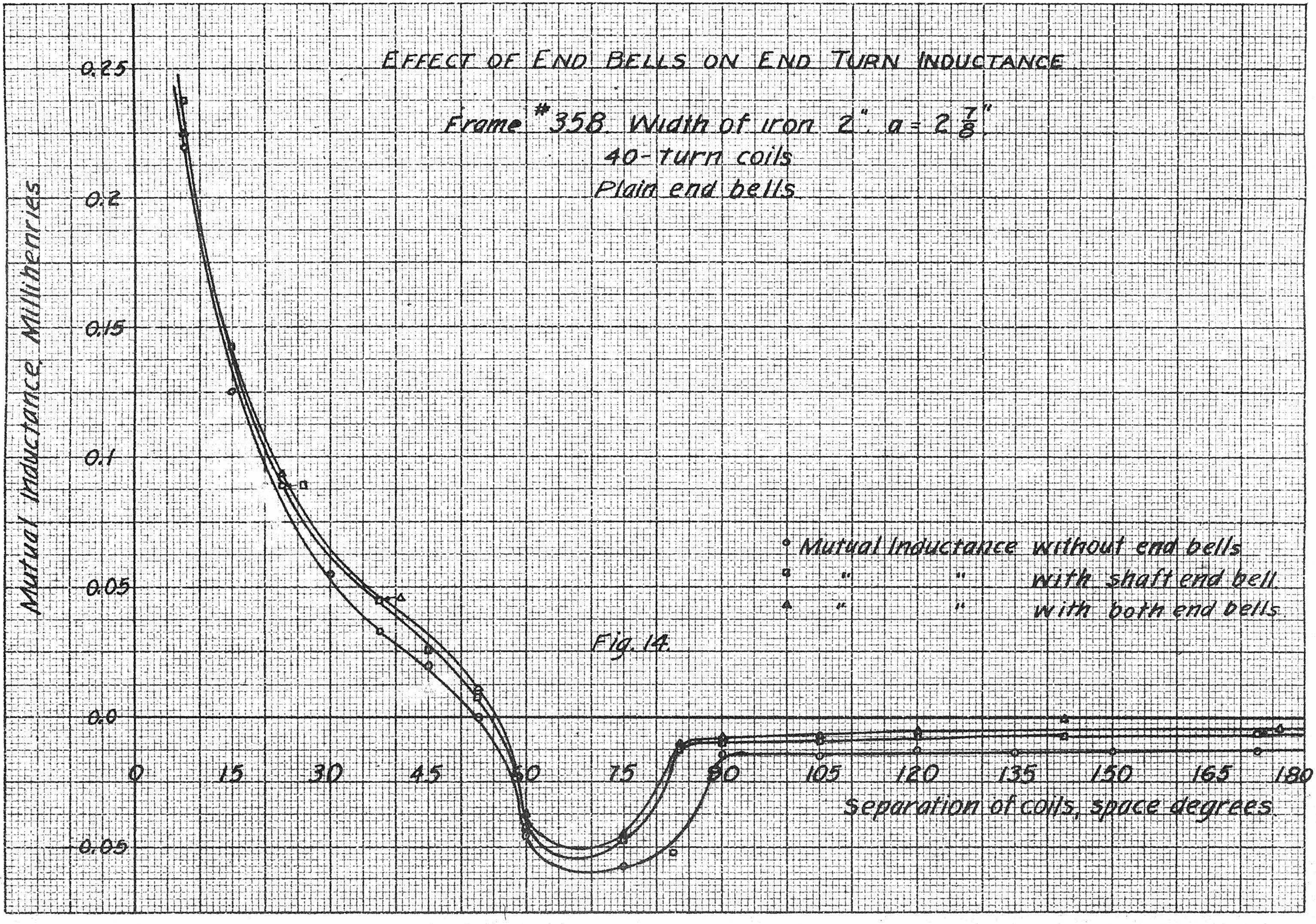
Ordinarily, it would be expected that the introduction of iron into the magnetic circuit would decrease the reluctance of the circuit, increasing the lines of flux threading the coils, and thus increasing the mutual inductance of the two coils. As a general conclusion, this is found to be true.

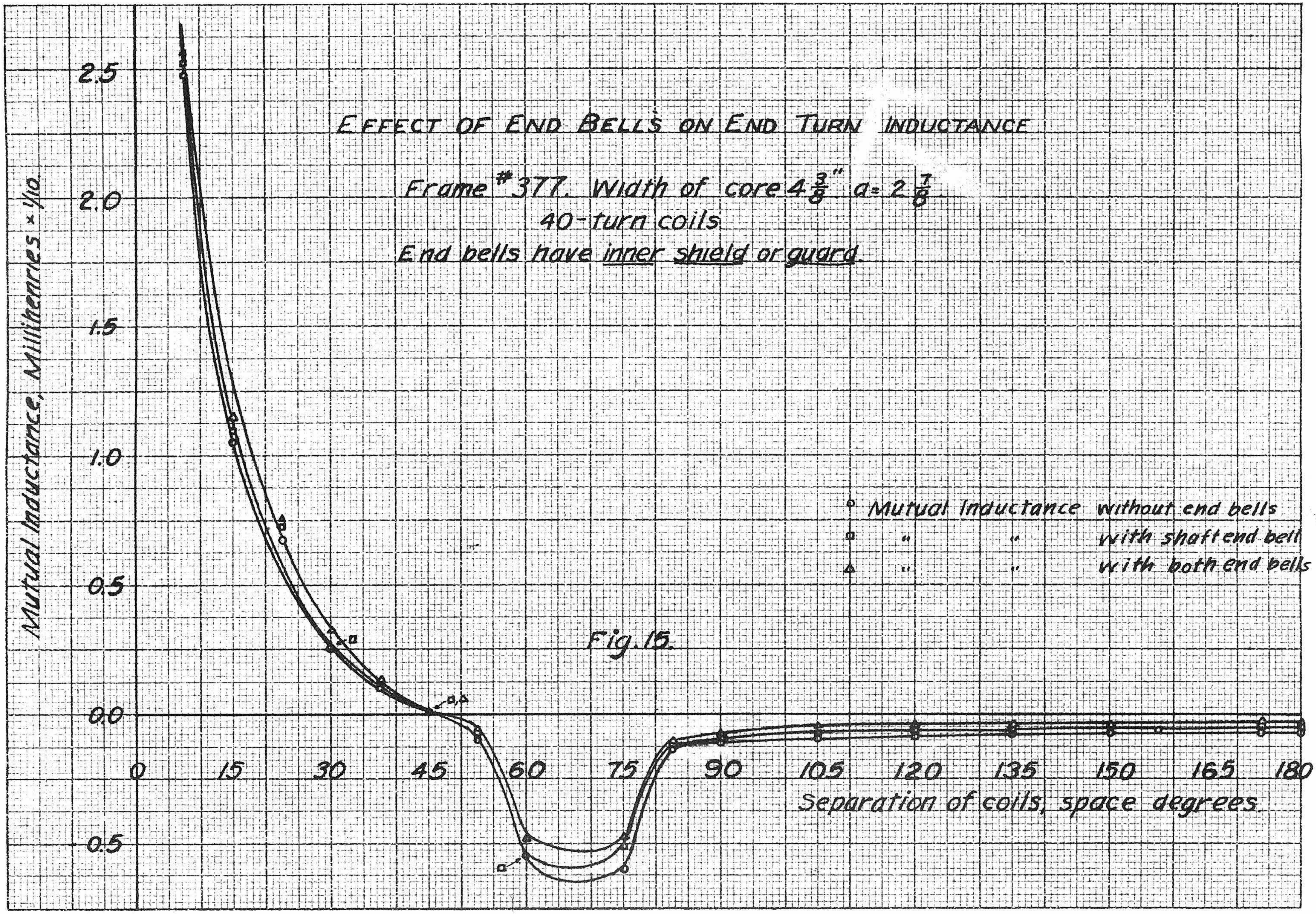

END TURN INUCTANCE TESTS ON THREE MOTORS.
Frames #358 Length of core 2"
368 " " " 3 1/4"
377 " " " 4 3/8"
40-turn coils.
No end bells.
Fig. 13.
Mutual Inductance, Millihenries ÷ 1/10
Separation of coils, space degrees.
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
0
15
30
45
60
75
90
105
120
135
150
165
180
358
368
377

Careful tests have been made with neither end bell on, with one end bell in position, and with both in place. The results of these runs are shown in Figs. 14 and 15. The first tests were made on the 5 h.p. motor. The end bells of this motor were of the ordinary type, consisting of a curved shell protecting the end turns and four radial arms supporting the bearing. The first test was of the inductance with the end bell on the shaft side in position. As was expected, the mutual inductance of the coils when close together was increased somewhat. When the coils were more than 90° apart, the mutual inductance fell off to approximately the same value as with no end bell. This would be expected, because the reluctance of the path when the coils are far apart is so great that the introduction of a little iron in the path does not appreciably change the flux.

It was rather surprising to find that the addition of the other end bell did not materially increase the mutual inductance. This was checked in both runs. The important factor in both tests proved to be the presence of the end bell on the shaft end.

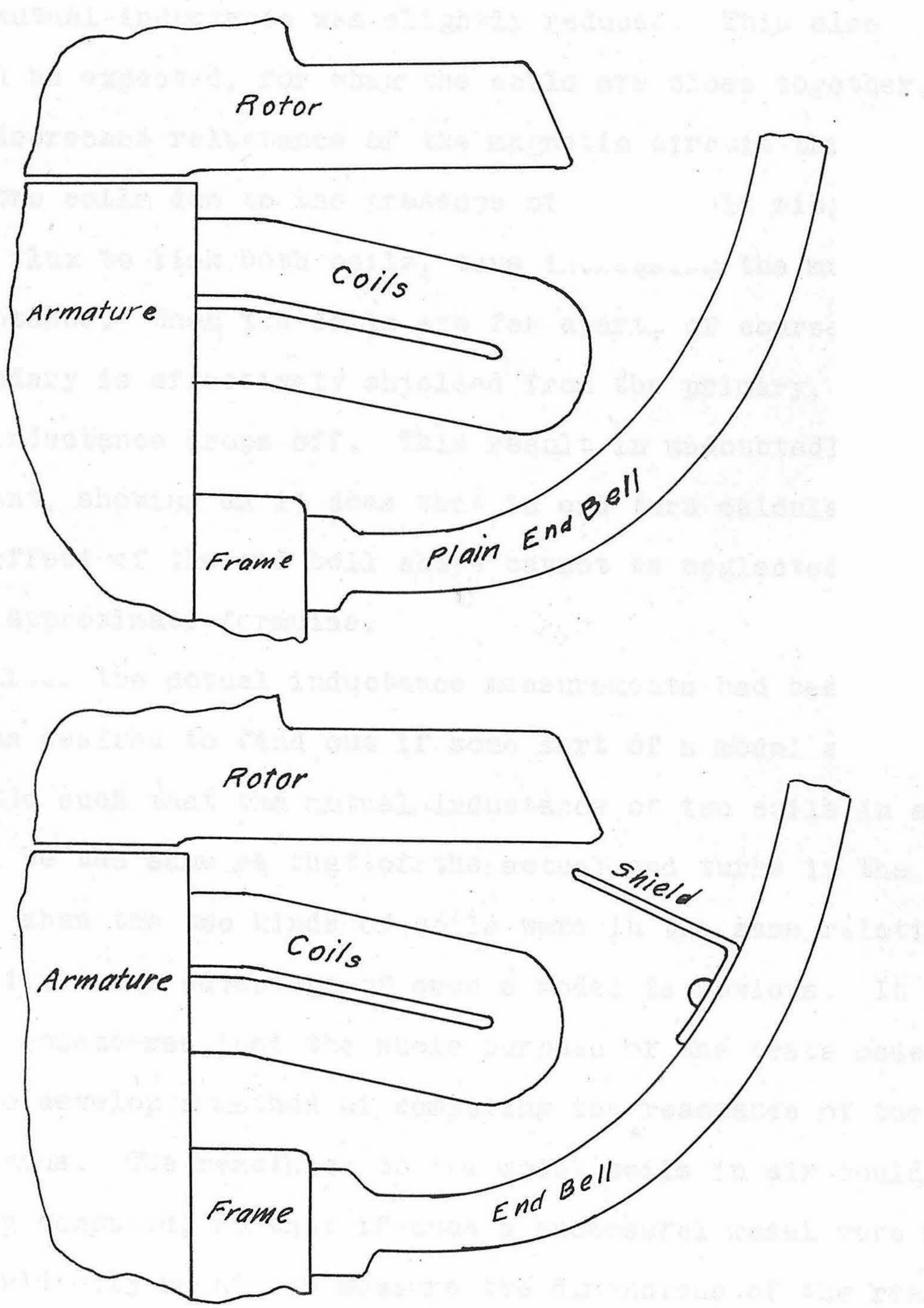
The end bells of the 10 h.p. motor were of slightly different construction from those of the 5 h.p. motor in that there was an extra guard ring on the exterior of the bell. This shield of pressed metal extended back from the end bell over the end connections (for comparison of the two bell shapes, see Fig. 16). It was found that when


EFFECT OF END BELLS ON END TURN INDUCTANCE
Frame #358. Width of iron 2". a = 2 7/8"
40-turn coils
Plain end bells
Mutual Inductance without end bells
" " with shaft end bell
" " with both end bells
Mutual Inductance, Millihenries
0.25
0.2
0.15
0.1
0.05
0.0
0.05
0
15
30
45
60
75
90
105
120
135
150
165
180
Separation of coils, space degrees
Fig. 14.



EFFECT OF END BELLS ON END TURN INDUCTANCE
Frame #377. Width of core 4 3/8" a = 2 7/8
40-turn coils
End bells have inner shield or guard.
Mutual Inductance without end bells
" " with shaft end bell
" " with both end bells
Mutual Inductance, Millihenries × 1/10
2.5
2.0
1.5
1.0
0.5
0.0
0.5
0
15
30
45
60
75
90
105
120
135
150
165
180
Separation of coils, space degrees.

Fig. 15.



End Bell Shapes
Fig. 16.

the end bell on the shaft end was fitted into place, the mutual induction of the end turns was increased when the coils were close together. When they were far apart, the mutual inductance was slightly reduced. This also would be expected, for when the coils are close together, the decreased reluctance of the magnetic circuit through the two coils due to the presence of the shield ring allows more flux to link both coils, thus increasing the mutual inductance. When the coils are far apart, of course, the secondary is effectively shielded from the primary, and the inductance drops off. This result in undoubtedly important, showing as it does that in end turn calculations the effect of the end bell shape cannot be neglected in even approximate formulae.

After the actual inductance measurements had been made, it was desired to find out if some sort of a model could be made such that the mutual inductance of two coils in air would be the same as that of the actual end turns in the motor when the two kinds of coils were in the same relative position. The advantage of such a model is obvious. It is to be remembered that the whole purpose of the tests made was to develop a method of computing the reactance of the end turns. The reactance of the model coils in air could be easily computed, so that if once a successful model were made, it would only remain to measure the dimensions of the real motor coils, translate these dimensions into those of a theoretically equivalent model, and compute the resulting inductance of the model.

This work has been carried out. It was stated under test procedure that the thickness of the end connections of the inner secondary coil should be zero for absolutely accurate results, as otherwise these end connections would cut some of the flux set up by the primary coil end connections, and thereby lower the resultant reactance voltage. Since such a condition was obviously impossible, in this part of the work special coils were made up. The reduction in the measured reactance due to the thickness of the inner secondary coil is directly proportional to the "end turn area" of this coil compared with the end turn area of the outer secondary coil. The end turn area is the projected area normal to the axis of the coils included between the end connections and the motor frame. Obviously, the inductance of the coils depends directly upon the area of the coils, so the reason for the statement made above is apparent. For very accurate measurements, it is desirable to have this ratio of areas as small as possible. For actual measurement of end turn reactance, this is impossible, for the end turn area of the outer secondary coil is fixed by the design of the motor coils, while the end turn area of the inner secondary coil is determined by the physical dimensions of the wires and the ability of the tester to force them as close to the motor frame as possible. However, in the determination of the dimensions of a model, the actual end turn reactance was not of importance, the vital part being the ability to reproduce the effects of whatever coils were used in the armature by coils in a model. For this reason, coils were wound for use in the motor having a very large end turn area compared with that of the inner secondary coil. These coils were of the simplest form possible, to facilitate possible later computations. They were merely rectangular coils, projecting from the frame about 4 1/2 inches. Since the average projection of the inner coil from the frame was approximately 3/16 inch, the end turn area was only 0.0417. The coils had the same span as the motor coils -- 1 and 10. The end turn inductance of these coils were measured just as in the former tests. This is plotted in Fig.17, curve 1.

Next, two coils having the same end turn area and end turn dimensions as the outer secondary coil, namely $4\frac{1}{2}$ x $5\frac{1}{2}$ inches, were wound up with the same number of turns. These coils were set up in a wooden frame constructed to the same dimensions as the motor armature. Thus, these end turn coils could be placed in the same position as those in the motor. The mutual inductance between these two coils was measured for all positions of one coil relative to the other. It is immediately seen that if the end turn inductance is independent of the motor iron, and if the inductance of the end turns on one side is independent of the coils on the other side (equivalent to saying that the flux from one set of end turns is screened from that on the other side), the end turn inductance of the coils in the motor should be just exactly

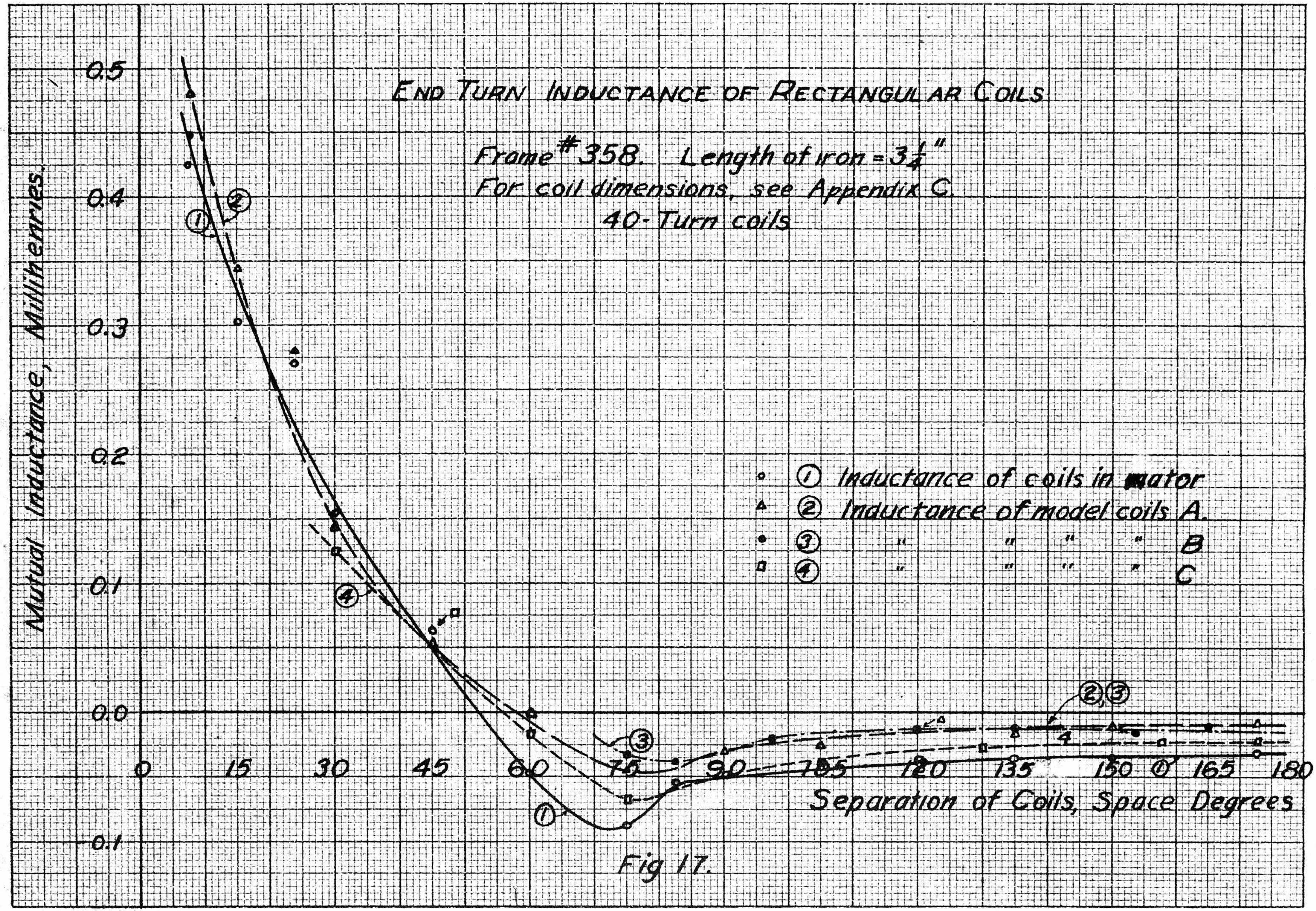

End Turn Inductance of Rectangular Coils
Frame # 358. Length of iron = 3 1/4"
For coil dimensions, see Appendix C.
40-Turn coils
① Inductance of coils in motor
② Inductance of model coils A.
③ " " " " B
④ " " " " C
Mutual Inductance, Millihenries.
0.5
0.4
0.3
0.2
0.1
0.0
0.1
0
15
30
45
60
75
90
105
120
135
150
165
180
Separation of Coils, Space Degrees

Fig 17.

twice that of these coils in the model. The results of this run are shown in Fig. 17, curve 2. It is apparent that when the coils are close together, the above premise is almost exactly true, while when coils are separated beyond 75°, it no longer holds.

The next assumption that was made was that the screening effect of the motor was not effective when the coils were somewhat widely separated. To check up on this, two more end turn coils similar to the first two were wound. Each two coils were separated by wooden spacers so that they were as far apart as the end turns of the motor coils (in other words, if the part of the motor coils in the armature were eliminated and wood introduced, similar coils would be produced). One of these composite coils is shown in b, Fig.18, below one of the motor coils, showing the similarity. The mutual inductance of the two composite coils was measured for separations greater than 75°. The plotted results are also shown in Fig.17, curve 3. This run is almost exactly identical with that of the former model run, and still is not close to the actual inductance.

There was still another possibility. If the face of the iron in the motor were identical with a flux line, there would be no difference in the field from either end turn coil if they were butted together. This follows from the fact that the flux from either coil would be normal to the coil face at the inner edge, and there would be no change in the field at the boundary when the two coils were put together. This

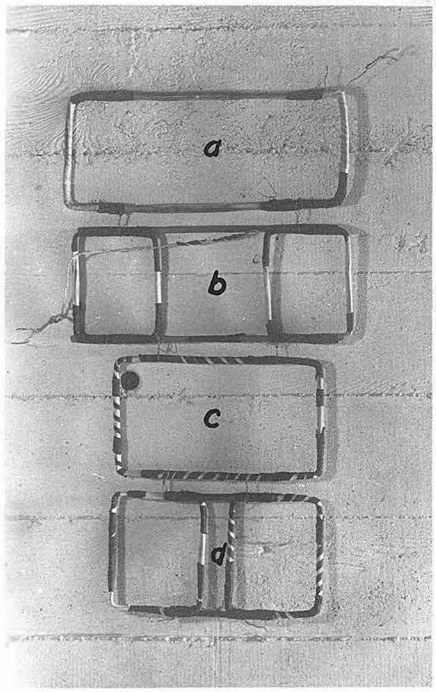


Fig. 18. Rectangular Test Coils.

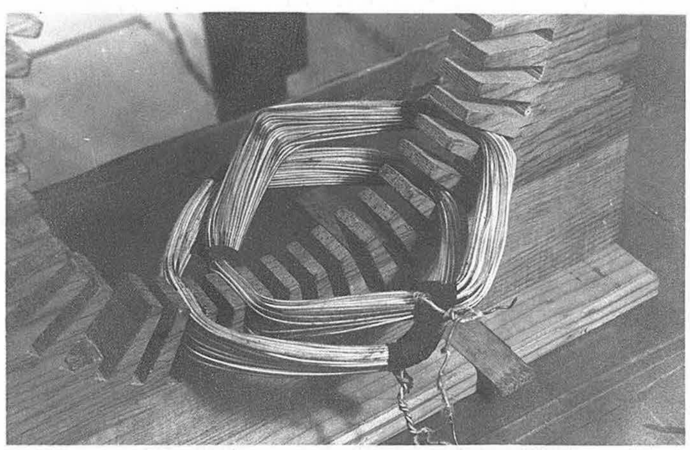


Fig. 19. Coils in Wooden Model.

assumption of the flux lines being coincident with the face of the frame is reasonable, for it is known that the flux inside the iron flows radially, and so must be nearly radial at the boundary between iron and air. This was considered in making up the next coils. Two coils with the same end turn dimensions and end turn area as the motor coils were made up. For a primary, a coil equivalent to butting two end turn coils together was made. Instead of a length of 4 1/2 inches, it had a length of 9 inches. The first two coils were attached side by side, with a separation equivalent to the mean distance of the inner secondary motor coil from the iron, namely 3/16 inch. The mutual inductance between the single coil primary and the two-coil secondary was measured in the wooden model as before. These coils are shown in c and d, Fig.18. The The mutual inductance is plotted as curve 4, Fig.17. This mutual inductance approaches most closely of all the methods tried the actual mutual inductance of the motor coils, although it is still not exactly equivalent.

To find out how accurate it was to make the assumption that the end turn reactance was equivalent to that of coils made by butting end turns from the opposite ends of the motor coils together, two coils were made of such dimensions that they were equivalent to the butting together of the end turns of the motor coils in the 15 h.p. motor. The end turn inductance of these coils were measured as before. A view of one of these coils is seen at the bottom of Fig.10. Fig.19 shows the two coils in test position in the wooden model. The results of these measurements are plotted in Fig. 20. For comparative purposes, the actual end turn inductance of the motor coils from which these small coils were made up has been also plotted on the same sheet.

As was found with the rectangular end turns, the butting-together effect is allowable only for separations over 50°.

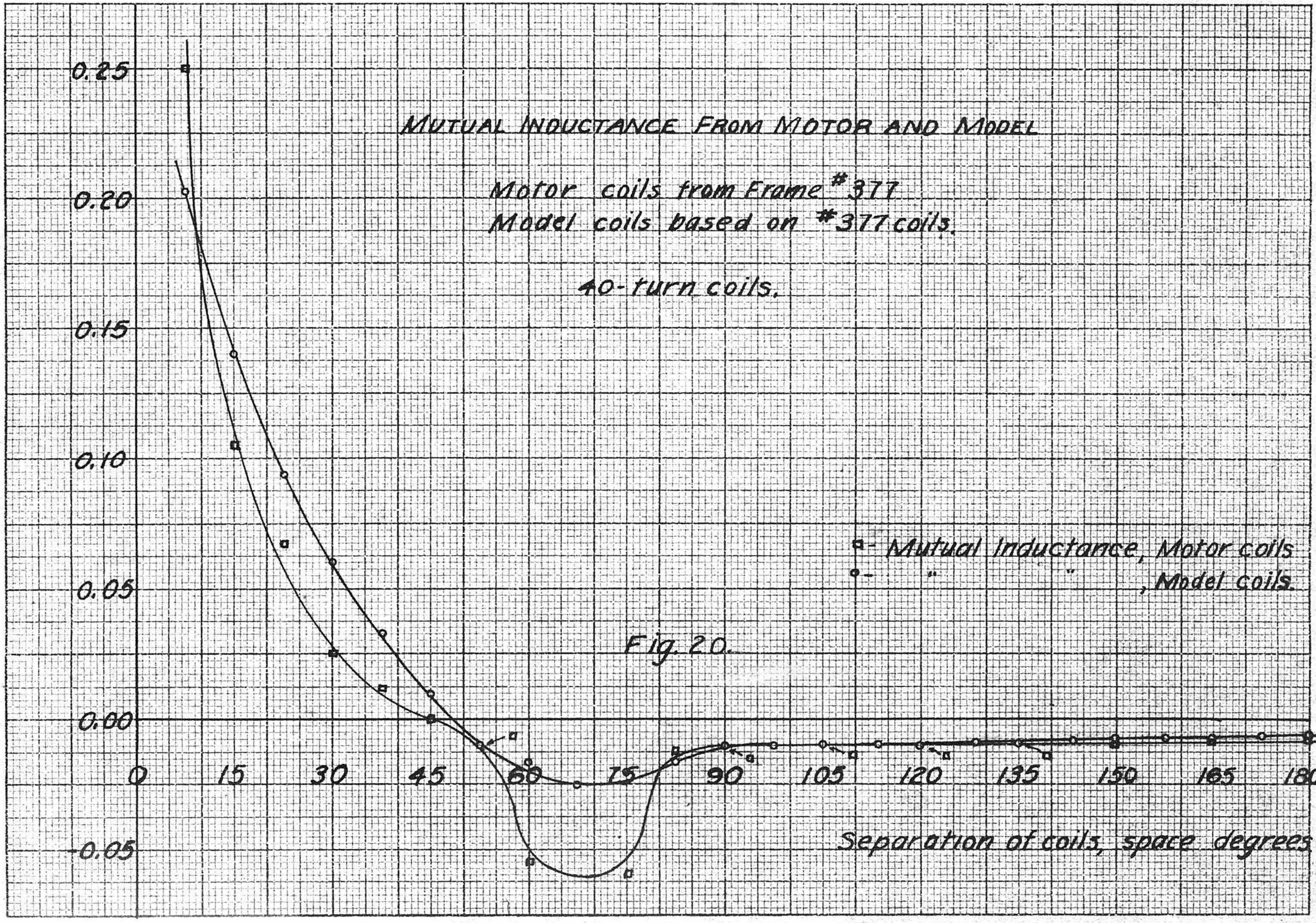

MUTUAL INDUCTANCE FROM MOTOR AND MODEL
Motor coils from Frame #377
Model coils based on #377 coils.
40-turn coils.
□- Mutual Inductance, Motor coils
○- " " , Model coils.
0.25
0.20
0.15
0.10
0.05
0.00
-0.05
0
15
30
45
60
75
90
105
120
135
150
165
180
Separation of coils, space degrees

Fig. 20.

Conclusion

From the preceding study, several conclusions are obvious. In the first place, the present formulae for the calculation of end turn inductance are very much in error as they do not consider the presence of mutual inductance between end turns of different coils. This error amounts up, in test runs, to over 350%. Unless some large correction factor is added to the present formulae, they are comparatively useless and misleading.

The mutual inductance of the end turns can be computed fairly closely by a long but straightforward method, involving certain simplifying assumptions. The values obtained by this method check reasonably well with actual test figures from the same motor.

The mutual inductance decreases rapidly with the separation of the coils. When the coils are separated about 45°, the mutual inductance becomes zero. The inductance for separations greater than that may be considered negative in that the voltage induced in the coils will be opposite to its former value. A negative maximum is found at around 70° to 80° separation, after which the mutual inductance rapidly decreases to a low but practically constant negative value beyond 100°. Because of this negative maximum, calculation of the mutual inductance for coils separated by not more than 30° will give a fairly close approximation of the total mutual inductance of all coils, the values of positive inductance from 30° to 45° cancelling the negative value found farther out.

The mutual inductance may be either increased or decreased by the presence of end bells. Plain end bells with no shielding over the coils were found to increase the mutual inductance. However, if there are guards or shields over the tops of the coils, the mutual inductance is increased only until the shielding effect becomes of importance. This latter effect is first noticeable at about 60° separation. The relative length of core in the motor apparently has little if any effect upon the end turns inductance. This may be seen by referring to Fig.13.

Coils can be constructed from the dimensions of the actual motor coils that will give in air the same values of mutual induction found in the end turns of the motor coils when the machine is wound. Models such as these can be easily made and tested, and the values of mutual induction obtained much more easily than by computation by the present fairly exact means.

Finally, the testing of the end turn inductance of induction motors already wound can be quickly made by use of the circuit adopted for this work. The apparatus used in making the test is all standard, and the whole test run for a 48-slot armature should not take over an hour. Adjustments are simple and the accuracy is high.

In conclusion, the author hopes that some investigator will proceed from the end of this analysis. This paper has only touched on a few facts found in the study of one particular design of motor. There is much need for the development of a formula, theoretical or empirical, for the inductance of end turns, taking account of the mutual inductance of all coils with respect to each other. Since the end turn inductance is almost the sole factor in limiting initial current rush,etc., the development of such a formula should prove of much practical value.

Appendix A.

Derivation of cosϕ in mutual inductance calculation.

It was found that if each similar segment from the diamond coils in every slot about the armature circumference were moved parallel to itself until the outer ends of all the segments coincided, the segments formed elements on the surface of a cone of slant height <u>l</u> (the length of the segments) and radius of base <u>a</u> (half the coil span). Having this information, the angle ϕ between any two line segments was found as follows: Referring to Fig. 21, L_1 and L_2 are two segments. The angle between them is ϕ. The angle between the radii at the base is θ, the separation of the two coils in degrees. Evidently, from the figure

$$\sin \frac{\phi}{2} = \frac{a}{l} \sin \frac{\theta}{2} \tag{1}$$

$$\cos \phi = 1 - 2 \sin^2\left(\frac{\phi}{2}\right) = 1 - 2 \frac{a^2}{l^2} \sin^2\left(\frac{\theta}{2}\right)$$

But $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1 - \cos\theta)$

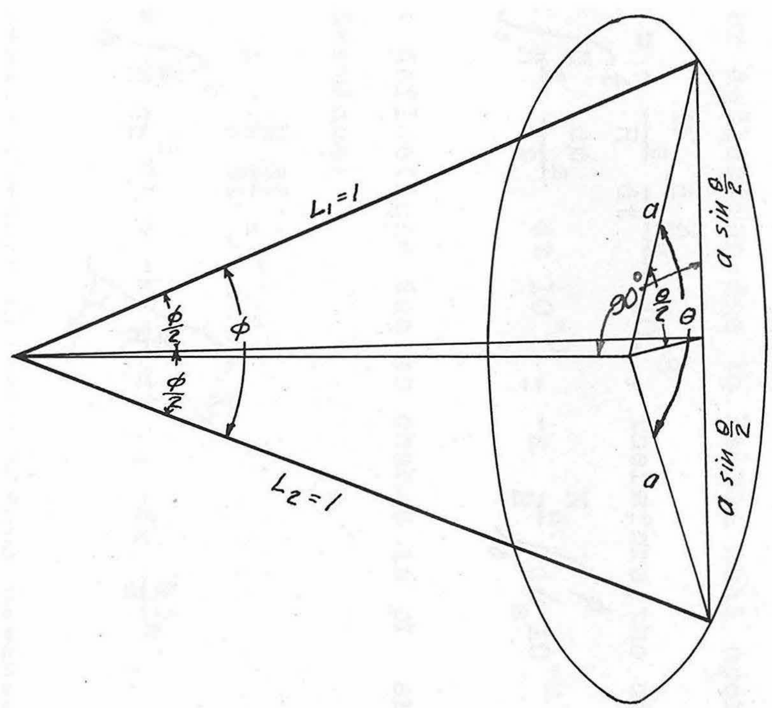
So, $\cos \phi = 1 - 2 \frac{a^2}{l^2} \cdot \frac{1}{2}(1 - \cos\theta)$

$$= 1 - \frac{a^2}{l^2}(1 - \cos\theta)$$

$$= \left(\frac{l^2 - a^2}{l^2}\right) + \frac{a^2}{l^2} \cos\theta \tag{2}$$

For segments in the opposite direction, it is merely necessary to replace θ by (θ 180°), which gives

$$\cos \phi = \left(\frac{l^2 - a^2}{l^2}\right) - \frac{a^2}{l^2} \cos\theta. \tag{3}$$



$L_1 = 1$
$L_2 = 1$
ϕ
$\phi/2$
$\phi/2$
a
a
$90°$
$\theta/2$
θ
$a \sin \frac{\theta}{2}$
$a \sin \frac{\theta}{2}$

Fig. 21.

Appendix B.

Derivation of Formula used in Standard Mutual Inductance Calibration.

The deflection of a ballistic galvanometer is directly proportional to the charge passed through it, i.e.,

$$D = k \int_0^t i \, dt$$

where the current is assumed zero at $t = 0$.

First consider deflection due to Kelvin coil operation:

$$i_1 = \frac{e}{R} = -\frac{N_s}{R}\frac{d\phi_s}{dt}10^{-8}, \text{ Therefore, the deflection is}$$

$$D_s = -k\int_0^t \frac{N_s}{R}\frac{d\phi_s}{dt}\,dt\,10^{-8} = -k\frac{N_s}{R}\int_0^{\phi} d\phi_s 10^{-8} = -k\frac{N_s\phi_s}{R}10^{-8} \quad (1)$$

Next consider deflections due to change in primary current in mutual inductance:

$$i_2 = \frac{e}{R} = -\frac{M}{R}\frac{di}{dt} =$$

$$D_x = -k\int_0^t \frac{M}{R}\frac{di}{dt}\,dt = -k\int_{-I_x}^{I_x}\frac{M}{R}\,di = -2k\frac{MI_x}{R}, \quad (2)$$

where I_x = current through primary before reversal.

From (2): $M = -\dfrac{R D_x}{2k I_x}$, and solving for k from (1):

$$k = -\frac{D_s R}{N_s \phi_s}10^{8}, \text{ so:}$$

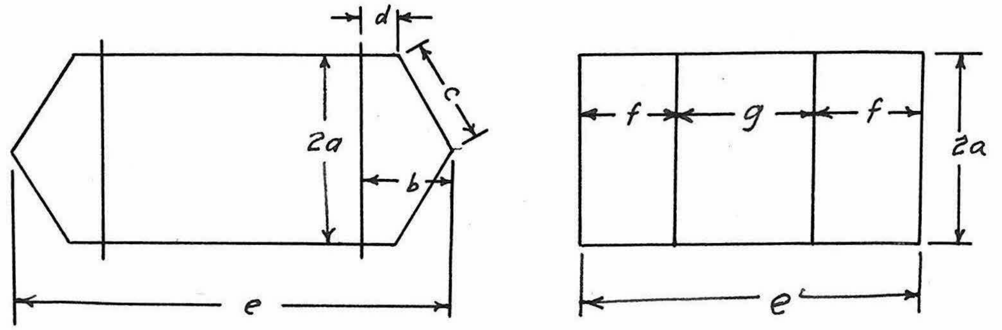
$$M = \frac{R D_x}{2 I_x}\frac{N_s \phi_s}{D_s R}10^{8}$$

$$= \frac{N_s \phi_s}{2 D_s}\frac{D_x}{I_x}10^{-8} \quad (3)$$

which is the desired formula.

Appendix C.

Coil Dimensions.



Coil	Frame	2a	b	c	d	e
Motor 1	377	5.25 in	2.25 in	3.00 in	0.5 in	9.50 in
" 2	368	" "	2.50 "	3.00 "	0.5 "	8.25 "
" 3	358	" "	2.50 "	3.00 "	0.5 "	7.00 "
Model Rect. 4	377	" "	f 4.50 "	g 4.375"	0.0 "	13.0 "
" " 5	Wood	" "	4.50 "	0.00 "	0.0 "	4.50 "
" " 6	Wood	" "	4.25 "	1.25 "	0.0 "	9.75 "
Model Hex. 7	Wood	" "	b 2.87 "	c 3.00 "	0.5 "	5.15 "

Notes:: 1 -- Fig. 10, number a

2 -- " ", number b

3 -- " ", number c

4 -- Fig. 18, number a

5 -- " ", number b

6 -- " ", number d

7 -- Fig. 10, number d