TOPICS IN DESCRIPTIVE SET THEORY RELATED TO EQUIVALENCE RELATIONS, COMPLEX BOREL AND ANALYTIC SETS

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Στη μητερα μου Αννα και στον πατερα μου Ευσταθιο

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Abstract

The purpose of this doctoral dissertation is first to show that certain kinds of invariants for measures, self-adjoint and unitary operators are as far from complete as possible and second to give new natural examples of complex Borel and analytic sets originating from Analysis and Geometry.

The dissertation is divided in two parts.

In the first part we prove that the measure equivalence relation and certain of its most characteristic subequivalence relations are generically S_∞-ergodic and unitary conjugacy of self-adjoint and unitary operators is generically turbulent.

In the second part we prove that for any $0 \leq \alpha < \infty$, the set of entire functions whose order is equal to α is $\mathbf{\Pi}^0_3$-complete and the set of all sequences of entire functions whose orders converge to α is $\mathbf{\Pi}^0_5$-complete. We also prove that given any line in the plane and any cardinal number $1 \leq n \leq \aleph_0$, the set of continuous paths in the plane tracing curves which admit at least n tangents parallel to the given line is $\mathbf{\Sigma}^1_1$-complete and the set of differentiable paths of class C^2 in the plane admitting a canonical parameter in $[0,1]$ and tracing curves which have at least n vertices is also $\mathbf{\Sigma}^1_1$-complete.

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Chapter I

A strong generic ergodicity property for measures, self-adjoint and unitary operators

Introduction

The results in this chapter are joint work of the author and A.S.Kechris.

One of the main trends of current research in Descriptive Set Theory is the study of natural equivalence relations arising in other branches of mathematics, in the sense of determining their relative complexity under the notion of Borel reducibility.

Definition: *Let X, X' be any Polish spaces and let E, E' be any equivalence relations on X, X' respectively. Then E is said to be* **Borel reducible** *to E' when there exists a Borel function $f : X \rightarrow X'$ with the property that $xEy \iff f(x)E'f(y)$, whenever x, y are in X.*

An important notion in the study of equivalence relations is the notion of generic S_∞-ergodicity, where S_∞ stands for the group of permutations of $\mathbf{N}$.

Definition: *Let X be any Polish space and let E be any equivalence relation on X. Then E is said to be* **generically S_∞-ergodic** *if every E-equivalence class is meager and for any Polish space Y and for any Baire measurable function $f : X \rightarrow Y^{\mathbf{N}}$ with the property that $xEy \Rightarrow \{f(x)(n) : n \in \mathbf{N}\} = \{f(y)(n) : n \in \mathbf{N}\}$, whenever x, y are in X, there exist an E-invariant co-meager subset A of X and a countable subset $\mathcal{C}$ of Y such that $x \in A \Rightarrow \{f(x)(n) : n \in \mathbf{N}\} = \mathcal{C}$, whenever $x \in X$. In particular, since by setting $u \cong_Y v \iff \{u(n) : n \in \mathbf{N}\} = \{v(n) : n \in \mathbf{N}\}$, whenever u, v are in $Y^{\mathbf{N}}$, we may canonically identify $Y^{\mathbf{N}}/\cong_Y$ with the set of all countable subsets of Y, generic S_∞-ergodicity implies that any E-invariants of elements of X, which are computed in a Baire measurable way and can be represented as countable subsets of a Polish space, must generically trivialize.*

The notion of generic S_∞-ergodicity for equivalence relations is related to the concept of generic turbulence for Polish group actions. The following definition is due to G. Hjorth.

Definition: *Let G be any Polish group acting continuously on a Polish space X and let $x \in X$. For any open neighborhood U of x in X and for any symmetric open neighborhood V of 1^G in G, the (U,V)-local orbit $O(x,U,V)$ of x in X is defined, as follows: $y \in O(x,U,V)$ if and only if there exist $g_0, g_1, ..., g_k$ in V such that if $x_0 = x$ and $x_{i+1} = g_i \cdot x_i$ for every index $i \leq k$, then all the x_i's are in U and $x_{k+1} = y$. The action of G on X is called* **turbulent** *at the point x, symbolically $x \in T_G^X$, if for any such U and V, there exists an open neighborhood U' of x in X such that $U' \subseteq U$ and $O(x,U,V)$ is dense in U'.*

The concept of turbulence is a property of the orbits of the action in the sense that if G is any Polish group acting continuously on a Polish space X and E_G^X stands for the corresponding orbit equivalence relation, then T_G^X is E_G^X-invariant, while the main result concerning the concept of turbulence is the following theorem of G. Hjorth [7].

Theorem: *Let G be any Polish group acting continuously on a Polish space X in such a way that the orbits of the action are meager and at least one orbit is dense. Then the following are equivalent:*
(i) The action of G on X is **generically turbulent**, *in the sense that T_G^X is co-meager in X.*
(ii) $(\exists x \in T_G^X)(\overline{G \cdot x} = X)$.

(iii) E_G^X *is generically* S_∞*-ergodic, in the sense that for any Polish space* Y *and for any Baire measurable function* $f : X \to Y^{\mathbf{N}}$ *with the property that* $xE_G^X y \Rightarrow \{f(x)(n) : n \in \mathbf{N}\} = \{f(y)(n) : n \in \mathbf{N}\}$*, whenever* x*,* y *are in* X*, there exist an* E_G^X*-invariant co-meager subset* A *of* X *and a countable subset* $\mathcal{C}$ *of* Y *such that* $x \in A \Rightarrow \{f(x)(n) : n \in \mathbf{N}\} = \mathcal{C}$*, whenever* $x \in X$*.*
(iv) The same as in (iii) but with "Baire measurable" replaced by "C-measurable" and "co-meager" replaced by "dense G_δ*."*
(v) For any Polish space Y *on which* S_∞ *acts in such a way that the action is Borel and for any Baire measurable function* $f : X \to Y$ *with the property that* $xE_G^X y \Rightarrow f(x)E_{S_\infty}^Y f(y)$*, whenever* x*,* y *are in* X*, there exists an* E_G^X*-invariant co-meager subset* A *of* X *for which* $f[A]$ *is contained in a single* $E_{S_\infty}^Y$*-equivalence class.*
(vi) The same as in (v) but with "Baire measurable" replaced by "C-measurable" and "co-meager" replaced by "dense G_δ*."*
(vii) For any relational language L*, consisting of countably many symbols, and for any Baire measurable function* $f : X \to X_L$ *with the property that* $xE_G^X y \Rightarrow f(x) \cong f(y)$*, whenever* x*,* y *are in* X*, there exists an* E_G^X*-invariant co-meager subset* A *of* X *for which all countable models in* $f[A]$ *are equivalent up to* $\cong$*, where* X_L *is the Polish space of all countable models for* L *whose universe is* $\mathbf{N}$*.*
(viii) The same as in (vii) but with "Baire measurable" replaced by "C-measurable" and "co-meager" replaced by "dense G_δ*."*

Remark: Part (v) of the above mentioned theorem of G. Hjorth explains the terminology S_∞-ergodic.

Our main purpose in this chapter is to show that any invariants for the measure equivalence relation and for certain of its most characteristic subequivalence relations and any unitary conjugacy invariants of self-adjoint and unitary operators, as well, which are computed in a Baire measurable way and can be represented as countable subsets of a Polish space or more generally as orbits of an S_∞-action or equivalent countable models up to isomorphism, must generically trivialize. In fact, we obtain the following results:

Theorem 1: *If X is any compact perfect Polish space and $P(X)$ stands for the Polish space of probability Borel measures on X, equipped with the* weak**-topology, while $\mu \sim \nu \iff (\mu \ll \nu \;\wedge\; \nu \ll \mu)$, whenever μ, ν are in $P(X)$, then $\sim$ is generically S_∞-ergodic. (The same is true if X is any compact smooth manifold of arbitrary dimension and we replace $\sim$ by $\sim_{C^r}$, where $\mu \sim_{C^r} \nu$ iff $\mu \sim \nu$ and both Radon-Nikodym derivatives $\frac{d\mu}{d\nu}$ and $\frac{d\nu}{d\mu}$ are differentiable functions of class C^r, whenever $r \in \mathbf{N} \cup \{\infty\}$.)*

Theorem 2: *Let $\mathbf{H}$ be any infinite-dimensional separable complex Hilbert space and let $U(\mathbf{H})$ stand for the Polish group of unitary operators on $\mathbf{H}$ and $S_1(\mathbf{H})$ stand for the Polish space of self-adjoint operators on $\mathbf{H}$ with norm at most one, both equipped with the strong topology. Then the conjugation action of $U(\mathbf{H})$ on both $U(\mathbf{H})$ and $S_1(\mathbf{H})$ is generically turbulent.*

Remark: The key tool for the proof of Theorems 1 and 2 is the above mentioned theorem of G. Hjorth.

1. Preliminaries

1.1. Measure and integration

The purpose of the present section is to present certain facts from the theory of measure and integration, which form part of the folklore of the subject, in order to facilitate the reader with the proof of Theorem 3.1 below. In fact, J.R. Choksi and M.G. Nadkarni proved the above mentioned result in [1] for the case of the unit circle, but their proof relied on ideas and results originating from Harmonic Analysis and the Theory of Martingales.

In what follows let X denote an arbitrary but fixed compact Polish space and let $C(X,\mathbf{R})$ stand for the Banach space of all continuous real-valued functions on X. Then, by virtue of the Riesz Representation Theorem and the Banach-Alaoglou Theorem, $\{\Lambda \in C(X,\mathbf{R})^* : \|\Lambda\| \leq 1 \ \wedge \ (1,\Lambda) = 1 \ \wedge \ (\forall f \in C(X,\mathbf{R}))(f \geq 0 \Rightarrow (f,\Lambda) \geq 0)\}$ equipped with the weak*-topology, can be viewed as the Polish space of probability Borel measures on X, which is usually denoted by $P(X)$, and the most central result concerning $P(X)$ is the so called **Portmanteau Theorem**:

The following are equivalent:
(i) $\mu_n \to \mu$ in $P(X)$ as $n \to \infty$;
(ii) for any $f \in C(X,\mathbf{R})$, $\int_X f d\mu_n \to \int_X f d\mu$ as $n \to \infty$;
(iii) for any open $O \subseteq X$, $\liminf\limits_{n\to\infty} \mu_n(O) \geq \mu(O)$;
(iv) for any closed $F \subseteq X$, $\limsup\limits_{n\to\infty} \mu_n(F) \leq \mu(F)$;
(v) for any Borel $B \subseteq X$ for which $\mu(\partial B) = 0$, $\lim\limits_{n\to\infty} \mu_n(B) = \mu(B)$.

In addition, when a countable dense subset $\{c_k : k \in \mathbf{N}\}$ is fixed, setting $\mathcal{B} = \{\cup_{i<n} B(c_{k_i}; 2^{-l_i}) : (k_0, ..., k_{n-1}), (l_0, ..., l_{n-1}) \in \mathbf{N}^n \text{ and } n \in \mathbf{N}\}$ and

$\mathcal{C} = \{\sum_{i=0}^{n} r_i \chi_{B_i} : (r_0, ..., r_n) \in (\mathbf{Q}_+^*)^{n+1}$ and $(B_0, ..., B_n) \in \mathcal{B}^{n+1}$ while $n \in \mathbf{N}\}$, where $\mathbf{Q}_+^*$ stands for the positive rationals, the following are true:

(a) For any non-empty finite subset M of $P(X)$, for any non-empty Borel subset A of X and for any $\epsilon > 0$, there exists $B \in \mathcal{B}$ such that $\mu(A \triangle B) < \epsilon$ for every $\mu \in M$.
(b) For any $\mu, \nu \in P(X)$, $\mu \perp \nu \iff (\forall (m,n) \in \mathbf{N}^2)(\exists B \in \mathcal{B})(\mu(B) < 2^{-n} \ \wedge \ \nu(B^c) < 2^{-(m+n)})$.
(c) For any $\mu \in P(X)$, $\mathcal{C}$ is dense in the **closed convex cone** $L_+^1(X,\mu) = \{f \in L^1(X,\mu) : f \geq 0, \mu - \text{a.e.}\}$ of $L^1(X,\mu)$.
(d) If $K(X)$ stands for the Polish space of all compact subsets of X, equipped with the Vietoris topology, then $P(X) \times K(X) \ni (\mu, K) \mapsto \mu(K) \in [0,1]$ is upper semi-continuous.
(e) If X is perfect, then there exists a continuous probability Borel measure on X which is fully supported.

1.2. Functional analysis and topological groups

As before, the purpose of the present section is to present certain facts from functional analysis and the theory of topological groups, which form part of the folklore of the subject, in order to facilitate the reader with the proof of Theorems 3.1 and 4.1 below.

If X, Y are any Polish spaces from which X is compact, then $C(X,Y)$, equipped with the topology of uniform convergence, constitutes a separable Frechet space and if d_Y is any complete compatible metric on Y, then

$$d_{C(X,Y)}(f,g) = \max_{x \in X} d_Y(f(x), g(x)) \ (f, g \in C(X,Y))$$

constitutes a complete compatible metric on $C(X,Y)$. In particular, $C(X,\mathbf{R})$ constitutes a separable Frechet space and the topology of uniform convergence constitutes a Polish group topology on $C(X,\mathbf{R}_+^*)$, where $\mathbf{R}_+^*$ stands for the positive reals.

If X constitutes a compact smooth manifold of arbitrary dimension and $r \in \mathbf{N} \cup \{\infty\}$, then the least topology on $C^r(X,\mathbf{R})$ for which the mappings $C^r(X,\mathbf{R}) \ni f \mapsto d^s f \in C(X, L^s(\mathbf{R}^{\dim(X)},\mathbf{R}))$, where $s \in \mathbf{N}$ and $0 \leq s \leq r$, are continuous (the $C(X, L^s(\mathbf{R}^{\dim(X)},\mathbf{R}))$'s being equipped with the topology of uniform convergence, while for any index s, $L^s(\mathbf{R}^{\dim(X)},\mathbf{R})$ denotes the linear space of s-linear forms on $\mathbf{R}^{\dim(X)}$ and $L^0(\mathbf{R}^{\dim(X)},\mathbf{R}) = \mathbf{R}$) is called the **Whitney topology** on $C^r(X,\mathbf{R})$ and $C^r(X,\mathbf{R})$ equipped with this topology constitutes a separable Frechet space, a complete compatible metric for which is given by the formula

$$d_{C^r(X,\mathbf{R})}(f,g) = \sum_{0 \leq s \leq r; s \in \mathbf{N}} 2^{-s} \frac{d_{C(X,L^s(\mathbf{R}^{\dim(X)},\mathbf{R}))}(d^s f, d^s g)}{1 + d_{C(X,L^s(\mathbf{R}^{\dim(X)},\mathbf{R}))}(d^s f, d^s g)}$$

$$(f, g \in C^r(X,\mathbf{R})) \ .$$

Proposition: *The Whitney topology constitutes a Polish group topology on* $C^r(X,\mathbf{R}_+^*)$.

Proof: In order to prove that the Whitney topology on $C^r(X,\mathbf{R}_+^*)$ constitutes a Polish group topology it is enough to prove that multiplication is separately continuous and to this end it is enough to prove the following claim:

If $h \in C^s(X,\mathbf{R})$, where $0 \leq s \leq r$ and $s \in \mathbf{N}$, then there exists a constant $C > 0$, which depends only on s and h, such that $\|d^s(fh) - d^s(gh)\|_\infty \leq C \cdot \|d^s f - d^s g\|_\infty$ $(f, g \in C^r(X, \mathbf{R}))$.

Setting $n = \dim(X)$, it is not difficult to prove by induction on s that if $(i_1, ..., i_s) \in \{1, ..., n\}^s$ and (U, ϕ) is any admissible chart on X, where $\phi(u) = (x_1(u), ..., x_n(u))$ $(u \in U)$, then for any $f \in C^r(X, \mathbf{R})$ and for any $p \in U$, $(d^s f)_p(\mathbf{e}_{i_1}, ..., \mathbf{e}_{i_s}) = \frac{\partial^s(f \circ \phi^{-1})}{\partial x_{i_1} ... \partial x_{i_s}}(\phi(p))$ ($\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ denoting the standard basis in $\mathbf{R}^n$). In addition, if $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{N}^n$, then we set $|\alpha| = \sum\limits_{i=1}^{n} \alpha_i$ and we denote by D^α the product of the commuting partial differential operators $\frac{\partial^\alpha}{\partial x_i^{\alpha_i}}$, $1 \leq i \leq n$ on $C^{|\alpha|}(\mathbf{R}^n, \mathbf{R})$, while if $\alpha, \beta \in \mathbf{N}^n$, then $\alpha \leq \beta \iff (\forall k \in \{1, ..., n\})(\alpha_k \leq \beta_k)$ and if $\alpha \leq \beta$, then we denote by C_β^α the product of $C_{\beta_k}^{\alpha_k}$, $1 \leq k \leq n$. So let s and h be as in the statement of the claim and let (U, ϕ) be any admissible chart on X, where $\phi(u) = (x_1(u), ..., x_n(u))$ $(u \in U)$, while $(i_1, ..., i_s) \in \{1, ..., n\}^s$ is arbitrary but fixed. Setting $\alpha_k = \text{card}(\{t \in \{1, ..., n\} : i_t = k\})$, $1 \leq k \leq n$ and $\alpha = (\alpha_1, ..., \alpha_n)$, for any $p \in U$,

$$(d^s(fh))_p(\mathbf{e}_{i_1}, ..., \mathbf{e}_{i_s}) - (d^s(gh))_p(\mathbf{e}_{i_1}, ..., \mathbf{e}_{i_s})$$
$$= \frac{\partial^s((fh) \circ \phi^{-1})}{\partial x_{i_1} ... \partial x_{i_s}}(\phi(p)) - \frac{\partial^s((gh) \circ \phi^{-1})}{\partial x_{i_1} ... \partial x_{i_s}}(\phi(p))$$
$$= D^\alpha((f \circ \phi^{-1})(h \circ \phi^{-1}))(\phi(p)) - D^\alpha((g \circ \phi^{-1})(h \circ \phi^{-1}))(\phi(p))$$
$$= \sum_{\alpha \leq \beta} C_\beta^\alpha \cdot D^\beta(h \circ \phi^{-1})(\phi(p)) \cdot (D^{\alpha-\beta}(f \circ \phi^{-1})(\phi(p)) - D^{\alpha-\beta}(g \circ \phi^{-1})(\phi(p))) ,$$

where for any $\beta \in \mathbf{N}^n$ such that $\beta \leq \alpha$, there exists $(j_1, ..., j_s) \in \{1, ..., n\}^s$ such that

$$D^{\alpha-\beta}(f \circ \phi^{-1})(\phi(p)) - D^{\alpha-\beta}(g \circ \phi^{-1})(\phi(p))$$
$$= \frac{\partial^s(f \circ \phi^{-1})}{\partial x_{j_1} ... \partial x_{j_s}}(\phi(p)) - \frac{\partial^s(g \circ \phi^{-1})}{\partial x_{j_1} ... \partial x_{j_s}}(\phi(p))$$
$$= (d^s f)_p(\mathbf{e}_{j_1}, ..., \mathbf{e}_{j_s}) - (d^s g)_p(\mathbf{e}_{j_1}, ..., \mathbf{e}_{j_s}) .$$

Therefore, the compactness of X is easily seen to imply that there exists a constant $C > 0$, which depends only on s and h such that $\|d^s(fh) - d^s(gh)\|_\infty \leq C \cdot \|d^s f - d^s g\|_\infty$, whenever $f, g \in C^r(X, \mathbf{R})$.
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Proposition: *If $\mu \in P(X)$, then the least topology τ_μ on $C^r(X, \mathbf{R}_+)$ which extends the Whitney topology on $C^r(X, \mathbf{R}_+)$ and for which the inclusion map $C^r(X, \mathbf{R}_+) \hookrightarrow L^1_+(X, \mu)$ is continuous, is Polish and*

$$\rho_\mu(f, g) = d_{C^r(X,\mathbf{R})}(f, g) + \int_X |f - g| d\mu \ (f, g \in C^r(X, \mathbf{R}_+))$$

constitutes a complete metric on $(C^r(X, \mathbf{R}_+), \tau_\mu)$.

Proof: If $\mathbf{A}$ and $\mathbf{B}$ are any countable bases for the topologies on $C^r(X, \mathbf{R})$ and $L^1(X, \mu)$ respectively, then it is not difficult to verify that $\mathbf{C} = \{A \cap B \cap C^r(X, \mathbf{R}_+) : A \in \mathbf{A},\ B \in \mathbf{B} \text{ and } A \cap B \cap C^r(X, \mathbf{R}_+) \neq \emptyset\}$ constitutes a countable base for τ_μ and τ_μ is easily seen to coincide with the topology induced by the metric ρ_μ on $C^r(X, \mathbf{R}_+)$, while if $(f_k)_{k \in \mathbf{N}}$ is any Cauchy sequence in $(C^r(X, \mathbf{R}_+), \rho_\mu)$ then $(f_k)_{k \in \mathbf{N}}$ constitutes a Cauchy sequence in both $C^r(X, \mathbf{R}_+)$ and $L^1_+(X, \mu)$; therefore, $(f_k)_{k \in \mathbf{N}}$ converges to some function f in $C^r(X, \mathbf{R}_+)$ and to some function g in $L^1_+(X, \mu)$ as well. But since convergence in $C^r(X, \mathbf{R})$ obviously implies uniform convergence on compacts and therefore pointwise convergence, while convergence in $L^1(X, \mu)$ implies convergence in measure and therefore the existence of a subsequence $(f_{k_l})_{l \in \mathbf{N}}$ of $(f_k)_{k \in \mathbf{N}}$ which converges to g almost everywhere with respect to μ, it follows that $f = \lim\limits_{k \to \infty} f_k = \lim\limits_{l \to \infty} f_{k_l} = g$ almost everywhere with respect to μ, which implies that $f = g$ in $(C^r(X, \mathbf{R}_+), \rho_\mu)$ and $\rho_\mu(f_k, f) = d_{C^r(X,\mathbf{R}_+)}(f_k, f) + \int_X |f_k - f| d\mu \to 0$ as $k \to \infty$.
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1.3. Operator theory

The purpose of the present section is to give a brief survey of some basics in Spectral Theory.

In what follows let $\mathbf{H}$ denote an arbitrary but fixed infinite-dimensional separable Hilbert space and let $L(\mathbf{H})$ stand for the space of all bounded linear operators on $\mathbf{H}$, equipped with the strong topology. A function E that assigns to every Borel subset B of a given Polish space X a projection $E(B)$ in $L(\mathbf{H})$ in such a way that $E(X)$ is the identity operator I on $\mathbf{H}$ and for any sequence $(B_n)_{n\in\mathbf{N}}$ of pairwise disjoint Borel subsets of X, $E(\cup_{n\in\mathbf{N}}B_n)x = \sum_{n=0}^{\infty} E(B_n)x$ $(x \in \mathbf{H})$, with respect to the strong topology, of course, is called a **spectral measure**. Spectral measures correspond to complex measures and give rise to spectral integrals in the following sense: If E is any function that assigns to every Borel subset B of a given Polish space X, a projection $E(B)$ in $L(\mathbf{H})$, then E constitutes a spectral measure, if and only if, $E(X) = I$ and for any $x, y \in \mathbf{H}$, $E_{x,y}(B) = (E(B)x, y)$ $(B \in \mathbf{B}(X))$ constitutes a complex Borel measure on X, and if E constitutes a spectral measure, then for any bounded Borel function $f : X \to \mathbf{C}$, there exists a unique operator $\int_X f dE$ in $L(\mathbf{H})$, which is usually refered to as the **spectral integral** of f with respect to E and is characterized by the property that $((\int_X f dE)x, y) = \int_X f dE_{x,y}$ for every $x, y \in \mathbf{H}$.

The most central result in Spectral Theory is the so called **Spectral Theorem** which states the following:

Spectral Theorem: *If T is any normal bounded linear operator on $\mathbf{H}$ and*

$\sigma(T)$ stands for the spectrum of T, then there exists a unique spectral measure $E^T : \mathbf{B}(\sigma(T)) \to L(\mathbf{H})$ for which T is the spectral integral of the identity function on $\sigma(T)$ with respect to E^T. In addition, if for any $x \in \mathbf{H}$, $\mathbf{H}_x^T$ stands for the closure of the linear subspace of $\mathbf{H}$ generated by the family $\{E^T(B)x : B \in \mathbf{B}(\sigma(T))\}$, then there exists a unique cardinal number $1 \leq \kappa(T) \leq \aleph_0$, which is usually refered to as the **spectral multiplicity** *of T, for which there exists a sequence $(u_i)_{i<\kappa(T)}$ of unit vectors in $\mathbf{H}$ such that the following conditions are satisfied:*
(i) The $\mathbf{H}_{u_i}^T$'s are pairwise orthogonal;
(ii) $\mathbf{H} = \oplus_{i<\kappa(T)}\mathbf{H}_{u_i}^T$;
(iii) $i < j < \kappa(T) \Rightarrow E_{u_j,u_j}^T \ll E_{u_i,u_i}^T$;
(iv) E_{u_0,u_0}^T constitutes a representative of the **maximal spectral type** *of T, in the sense that for any $x \in \mathbf{H}$, $E_{x,x}^T \ll E_{u_0,u_0}^T$;*
(v) For any $i < \kappa(T)$, if $U_i^T : L^2(\sigma(T), E_{u_i,u_i}^T) \to \mathbf{H}_{u_i}^T$ denotes the Hilbert space isomorphism defined by the relations $(U_i^T)\chi_B = E^T(B)u_i$ $(B \in \mathbf{B}(\sigma(T)))$, then for any $f \in L^2(\sigma(T), E_{u_i,u_i}^T)$ and for any $z \in \sigma(T)$, $((U_i^T)^{-1}T(U_i^T))f(z) = z \cdot f(z)$.

When a complete orthonormal system $\{\mathbf{e}_n : n \in \mathbf{N}\}$ in $\mathbf{H}$ is fixed, to every normal bounded linear operator T on $\mathbf{H}$ a canonical representative of its maximal spectral type is assigned, as follows:

$$\mu_T(B) = \sum_{n=0}^{\infty} 2^{-(n+1)} E_{\mathbf{e}_n,\mathbf{e}_n}^T(B) \ (B \in \mathbf{B}(\sigma(T))) \ ,$$

while for any bounded Borel function $f : \sigma(T) \to \mathbf{C}$, the spectral integral of f with respect to E^T also constitutes a normal bounded linear operator on $\mathbf{H}$ and is usually denoted by $f(T)$. The importance of the canonical representative of the maximal spectral type of a normal bounded linear operator on $\mathbf{H}$ is demonstrated by the following proposition:

Proposition: *If T and $T_n, n \in \mathbf{N}$ are any normal bounded linear operators on $\mathbf{H}$ whose spectrum is contained in $K \in K(\mathbf{C}) \setminus \{\emptyset\}$ and $T_n \to T$ in $L(\mathbf{H})$ as $n \to \infty$, with respect to the strong topology, then $\mu_{T_n} \to \mu_T$ in $P(K)$ as $n \to \infty$.*

Proof: We will first prove the following claim:

For any continuous function $f : K \to \mathbf{C}$, $f(T_n) \to f(T)$ in $L(\mathbf{H})$ as $n \to \infty$, with respect to the strong topology.

Let u, v be any unit vectors on $\mathbf{H}$ and let $p(z) \in \mathbf{C}[z]$. Then, the proof of the Spectral Theorem shows that

$$((f(T_n) - f(T))u, v)$$

$$= \int_K (f-p) dE^{T_n}_{u,v} + ((p(T_n) - p(T))u, v) + \int_K (p-f) dE^{T}_{u,v}$$

so

$$|((f(T_n) - f(T))u, v)|$$

$$\leq \int_K |f-p||dE^{T_n}_{u,v}| + |((p(T_n) - p(T))u, v)| + \int_K |f-p||dE^{T}_{u,v}|$$

$$\leq 2 \cdot \|f-p\|_\infty \cdot \|u\| \cdot \|v\| + |((p(T_n) - p(T))u, v)|$$

for every $n \in \mathbf{N}$ and consequently

$$\|(f(T_n) - f(T))u\| = \sup_{\|v\|=1} |((f(T_n) - f(T))u, v)|$$

$$\leq 2 \cdot \|f-p\|_\infty \cdot \|u\| + \sup_{\|v\|=1} |((p(T_n) - p(T))u, v)|$$

$$= 2 \cdot \|f-p\|_\infty \cdot \|u\| + \|(p(T_n) - p(T))u\|$$

for every $n \in \mathbf{N}$. Hence, by virtue of the Stone-Weierstass Theorem, it is enough to prove that $\|(p(T_n) - p(T))u\| \to 0$ as $n \to \infty$. Indeed, if $p(z)$ has degree N and $p(z) = \sum_{k=0}^{N} \alpha_k z^k$, where the α_k's are in $\mathbf{C}$ and $\alpha_N \neq 0$, then the proof of the Spectral Theorem shows that

$$\|(p(T_n) - p(T))u\|$$

$$\leq \sum_{k=0}^{N} |\alpha_k| \cdot \|(T_n^k - T^k)u\|$$

$$\leq \sum_{k=0}^{N} |\alpha_k| \cdot k \cdot \left(\sup_{z\in K} |z|\right)^{k-1} \cdot \|(T_n - T)u\| \to 0$$

as $n \to \infty$.

Now to prove the proposition, by virtue of the Portmanteau Theorem, given any closed $F \subseteq K$, it is enough to show that $\limsup_{n\to\infty} \mu_{T_n}(F) \leq \mu_T(F)$ and since for any $n, N \in \mathbf{N}$, $\mu_{T_n}(F) \leq \sum_{k=0}^{N} 2^{-(k+1)} E^{T_n}_{\mathbf{e}_k,\mathbf{e}_k}(F) + \sum_{k>N} 2^{-(k+1)}$, it is enough to prove that for any $k \in \mathbf{N}$, $\limsup_{n\to\infty} E^{T_n}_{\mathbf{e}_k,\mathbf{e}_k}(F) \leq E^{T}_{\mathbf{e}_k,\mathbf{e}_k}(F)$, whenever $F \subseteq K$ is closed, or (equivalently) that $E^{T_n}_{\mathbf{e}_k,\mathbf{e}_k} \to E^{T}_{\mathbf{e}_k,\mathbf{e}_k}$ in $P(K)$ as $n \to \infty$; but this follows from the above mentioned claim, since for any $f \in C(K, \mathbf{C})$, $|\int_K f dE^{T_n}_{\mathbf{e}_k,\mathbf{e}_k} - \int_K f dE^{T}_{\mathbf{e}_k,\mathbf{e}_k}| = |((f(T_n) - f(T))\mathbf{e}_k, \mathbf{e}_k)| \leq \|(f(T_n) - f(T))\mathbf{e}_k\| \to 0$ as $n \to \infty$.
Οπερ εδει δειξαι

In the sequel we will focus on the Polish group $U(\mathbf{H})$ of all unitary operators on $\mathbf{H}$ and the Polish space $S_1(\mathbf{H})$ of all self-adjoint operators on $\mathbf{H}$ with norm at most one, considered in Theorem 5.1 below. $U(\mathbf{H})$ acts on both $U(\mathbf{H})$ and $S_1(\mathbf{H})$ by conjugation, the actions being continuous since they are separately continuous, and the most important Baire category results concerning these actions are summarized in the following theorem due to J.R. Choksi, M.G. Nadkarni [1], [2] and B. Simon [14].

Theorem: *The sets*

$$\mathcal{U}_1 = \{U \in U(\mathbf{H}) : \overline{U(\mathbf{H}) \cdot U} = U(\mathbf{H})\} \ ,$$

$$\mathcal{U}_2 = \{U \in U(\mathbf{H}) : \sigma(U) = \mathbf{T}\}$$

and

$$\mathcal{U}_3 = \{U \in U(\mathbf{H}) : \kappa(U) = 1\}$$

constitute conjugacy invariant dense G_δ*'s in* $U(\mathbf{H})$*, and the sets*

$$\Sigma_1 = \{S \in S_1(\mathbf{H}) : \overline{U(\mathbf{H}) \cdot S} = S_1(\mathbf{H})\} \ ,$$

$$\Sigma_2 = \{S \in S_1(\mathbf{H}) : \sigma(S) = [-1, 1]\}$$

and

$$\Sigma_3 = \{S \in S_1(\mathbf{H}) : \kappa(S) = 1\}$$

constitute conjugacy invariant dense G_δ*'s in* $S_1(\mathbf{H})$.

2. Generic S_∞-ergodicity for equivalence relations and the pseudo-Vaught transforms

Definition 2.1: *Let X be any Polish space and let E be any equivalence relation on X. Then E is said to be* **generically S_∞-ergodic** *if every E-equivalence class is meager and for any Polish space Y and for any Baire measurable function $f : X \to Y^{\mathbf{N}}$ with the property that $xEy \Rightarrow \{f(x)(n) : n \in \mathbf{N}\} = \{f(y)(n) : n \in \mathbf{N}\}$, whenever x, y are in X, there exist an E-invariant co-meager subset A of X and a countable subset $\mathcal{C}$ of Y such that $x \in A \Rightarrow \{f(x)(n) : n \in \mathbf{N}\} = \mathcal{C}$, whenever $x \in X$.*

Definition 2.2: *We shall say that an equivalence relation E on a given Polish space X* **admits an approximation by a Polish group action**, *when the following conditions are satisfied:*
(i) For any $x \in X$, there exists a Polish space Γ_x and a continuous mapping $\phi_x : \Gamma_x \to X$ such that $\phi_x[\Gamma_x] = [x]_E$.
(ii) There exists a Polish group G acting continuously on X with the property that for any $x \in X$, there exists an embedding $G \hookrightarrow \Gamma_x$ such that $\overline{G} = \Gamma_x$ and $\phi_x(g) = g \cdot x$, whenever $g \in G$.
(iii) For any $x \in X$ and for any $\gamma \in \Gamma_x$, there exists a homeomorphism $\psi_{x,\gamma} : \Gamma_x \to \Gamma_{\phi_x(\gamma)}$ with the property that $\phi_x(\delta) = \phi_{\phi_x(\gamma)}(\psi_{x,\gamma}(\delta))$, whenever $\delta \in \Gamma_x$.

Definition 2.3: *Let E be any equivalence relation on a given Polish space X and assume that it admits an approximation by a Polish group action. Then, keeping the same notations as in Definition 2.2, for any subset A of X its* **pseudo-Vaught transforms**, *A^* and $A^\triangle$, are defined as follows:*

$$A^* = \{x \in X : (\forall^* \gamma \in \Gamma_x)(\phi_x(\gamma) \in A)\} ,$$

and

$$A^{\triangle} = \{x \in X : (\exists^* \gamma \in \Gamma_x)(\phi_x(\gamma) \in A)\} \ .$$

The following proposition summarizes the basic properties of the pseudo-Vaught transforms:

Proposition 2.4: *Let E be any equivalence relation on a given Polish space X and assume that it admits an approximation by a Polish group action. Then, keeping the same notations as in Definition 2.2, we have the following:*

(a) The pseudo-Vaught transforms P^* and $P^{\triangle}$ of any subset P of X are E-invariant and

$$(P)_E \subseteq P^* \subseteq P^{\triangle} \subseteq [P]_E \ ,$$

where $(P)_E = \{x \in X : [x]_E \subseteq P\}$ and $[P]_E = \{x \in X : [x]_E \cap P \neq \emptyset\}$.
(b) For any $P \subseteq X$,

$$X \setminus P^{\triangle} = (X \setminus P)^*$$

and

$$X \setminus P^* = (X \setminus P)^{\triangle} \ .$$

(c) If P, Q are any subsets of X, then

$$P \subseteq Q \Rightarrow (P^{\triangle} \subseteq Q^{\triangle} \ \wedge \ P^* \subseteq Q^*) \ .$$

(d) If $P \subseteq X$ and $P_n \subseteq X$, whenever $n \in \mathbf{N}$, then

$$P = \cup_{n \in \mathbf{N}} P_n \Rightarrow P^{\triangle} = \cup_{n \in \mathbf{N}} P_n^{\triangle}$$

and

$$P = \cap_{n \in \mathbf{N}} P_n \Rightarrow P^* = \cap_{n \in \mathbf{N}} P_n^* \ .$$

(e) For any open $P \subseteq X$, P^* constitutes a G_δ.
(f) If $P \subseteq X$ is E_G^X-invariant and constitutes a G_δ, then P is contained in P^*. In particular, if $P \subseteq X$ is E_G^X-invariant and constitutes a dense G_δ, then P^* is E-invariant and constitutes a dense G_δ.

Proof: Parts $(b)-(d)$ are fairly straightforward and we will restrict ourselves in proving (a), (e) and (f).
(a) Since the fact that $(P)_E \subseteq P^* \subseteq P^\triangle \subseteq [P]_E$ is an immediate consequence of the definitions, we will restrict again ourselves in proving that both P^* and $P^\triangle$ are E-invariant. Indeed, if $x \in P^*$ and $y \in P^\triangle$, while $\gamma \in \Gamma_x$ and $\delta \in \Gamma_y$, then $\{\alpha \in \Gamma_x : \phi_x(\alpha) \in P\}$ is co-meager in Γ_x and $\{\beta \in \Gamma_y : \phi_y(\beta) \in P\}$ is non-meager in Γ_y, hence since the mappings $\psi_{x,\gamma} : \Gamma_x \to \Gamma_{\phi_x(\gamma)}$ and $\psi_{y,\delta} : \Gamma_y \to \Gamma_{\phi_y(\delta)}$ constitute homeomorphisms, while for any $\alpha \in \Gamma_x$ and for any $\beta \in \Gamma_y$, $\phi_x(\alpha) = \phi_{\phi_x(\gamma)}(\psi_{x,\gamma}(\alpha))$ and $\phi_y(\beta) = \phi_{\phi_y(\delta)}(\psi_{y,\delta}(\beta))$, it follows that $\psi_{x,\gamma}[\{\alpha \in \Gamma_x : \phi_x(\alpha) \in P\}] = \{\psi_{x,\gamma}(\alpha) : \alpha \in \Gamma_x \ \wedge \ \phi_x(\alpha) \in P\} = \{\psi_{x,\gamma}(\alpha) : \alpha \in \Gamma_x \ \wedge \ \phi_{\phi_x(\gamma)}(\psi_{x,\gamma}(\alpha)) \in P\} = \{\alpha' \in \Gamma_{\phi_x(\gamma)} : \phi_{\phi_x(\gamma)}(\alpha') \in P\}$ is co-meager in $\Gamma_{\phi_x(\gamma)}$ and consequently $\phi_x(\gamma) \in P^*$, while $\psi_{y,\delta}[\{\beta \in \Gamma_y : \phi_y(\beta) \in P\}] = \{\psi_{y,\delta}(\beta) : \beta \in \Gamma_y \ \wedge \ \phi_y(\beta) \in P\} = \{\psi_{y,\delta}(\beta) : \beta \in \Gamma_y \ \wedge \ \phi_{\phi_y(\delta)}(\psi_{y,\delta}(\beta)) \in P\} = \{\beta' \in \Gamma_{\phi_y(\delta)} : \phi_{\phi_y(\delta)}(\beta') \in P\}$ is non-meager in $\Gamma_{\phi_y(\delta)}$ and consequently $\phi_y(\delta) \in P^\triangle$.
(e) We choose at random a countable dense subset C of G and let d be any complete compatible metric on G. Given $x \in X$, since the mapping $\phi_x : \Gamma_x \to X$ is continuous, $\{\gamma \in \Gamma_x : \phi_x(\gamma) \in P\}$ is open in Γ_x and consequently it is co-meager in Γ_x iff it is dense in Γ_x or (equivalently) $(\forall n \in \mathbf{N})(\forall a \in C)(\exists b \in C)(d(a,b) < 2^{-n} \ \wedge \ b \cdot x \in P)$, which is easily seen to imply that P^* constitutes a G_δ.
(f) If $x \in P$, then our assumptions imply that G is contained in $\{\gamma \in \Gamma_x :$

$\phi_x(\gamma) \in P\}$ which constitutes a G_δ; therefore, $x \in P^*$.
Οπερ εδει δειξαι

The relation between the notion of generic S_∞-ergodicity and the notion of approximation by a Polish group action for equivalence relations is demonstrated by the following proposition:

Proposition 2.5: *Every equivalence relation all of whose equivalence classes are meager and which admits an approximation by a generically turbulent Polish group action is generically S_∞-ergodic.*

Proof: Let E be any equivalence relation on a given Polish space X and assume that all its equivalence classes are meager and that it admits an approximation by a generically turbulent Polish group action. Then, keeping the same notations as in Definition 2.2, the fact that the action of G on X is generically turbulent implies that if Y is any Polish space and $f : X \to Y^{\mathbf{N}}$ is any Baire measurable function with the property that $xEy \Rightarrow \{f(x)(n) : n \in \mathbf{N}\} = \{f(y)(n) : n \in \mathbf{N}\}$, whenever x, y are in X, there exists an E_G^X-invariant dense G_δ subset B of X and a countable subset $\mathcal{C}$ of Y such that $x \in B \Rightarrow \{f(x)(n) : n \in \mathbf{N}\} = \mathcal{C}$, whenever $x \in X$. Thus, setting $A = B^*$, Proposition 2.4 is easily seen to imply that A constitutes an E-invariant dense G_δ subset of X such that $x \in A \Rightarrow \{f(x)(n) : n \in \mathbf{N}\} = \mathcal{C}$, whenever $x \in X$.
Οπερ εδει δειξαι

3. Baire category in the space of probability Borel measures

Theorem 3.1: *If X is any compact perfect Polish space, then $P_c(X) = \{\mu \in P(X) : \mu \text{ is continuous}\}$, $P^*(X) = \{\mu \in P(X) : \text{supp}(\mu) = X\}$ and $\nu^\perp = \{\mu \in P(X) : \mu \perp \nu\}$ constitute $\sim$-invariant dense G_δ's in $P(X)$, while $\{\mu \in P(X) : \mu \ll \nu \text{ and } \frac{d\mu}{d\nu} \in C(X, \mathbf{R}^*_+)\}$ is dense and meager in $P(X)$, whenever $\nu \in P^*(X)$. In particular, if X constitutes a compact smooth manifold of arbitrary dimension, then $\{\mu \in P(X) : \mu \ll \nu \text{ and } \frac{d\mu}{d\nu} \in C^r(X, \mathbf{R}^*_+)\}$ is dense and meager in $P(X)$, whenever $\nu \in P^*(X)$ and $r \in \mathbf{N} \cup \{\infty\}$.*

Proof: We divide the argument in four steps:

a) $P^*(X)$ constitutes a $\sim$-invariant dense G_δ in $P(X)$:

If $\{O_n : n \in \mathbf{N}\}$ is any countable basis for the topology on X, then $P^*(X) = \cap_{n\in\mathbf{N}}(\{\mu \in P(X) : \mu(O_n) = 0\}^c)$ and by virtue of the Baire Category Theorem and the Portmanteau Theorem it is enough to prove that for any non-empty open $O \subseteq X$, $\text{int}(\{\mu \in P(X) : \mu(O) = 0\}) = \emptyset$. Towards a contradiction we assume the contrary and let $\lambda \in P(X)$, $\{f_0, ..., f_n\} \subseteq C(X, \mathbf{R})$ and $\epsilon > 0$ be such that $\{\mu \in P(X) : (\forall i \leq n)(|\int_X f_i d\mu - \int_X f_i d\lambda| < \epsilon)\} \subseteq \{\mu \in P(X) : \mu(O) = 0\}$, while $x \in O$ and $\eta > 0$ are such that $\frac{\eta}{1+\eta} \max\limits_{0\leq i\leq n}(|\int_X f_i d\lambda| + |f_i(x)|) < \epsilon$. Then $\kappa = \frac{\lambda+\eta\delta_x}{1+\eta} \in P(X)$ and $\kappa(O) = \frac{\eta}{1+\eta} > 0$, while for any $0 \leq i \leq n$, $\int_X f_i d\kappa - \int_X f_i d\lambda = \frac{1}{1+\eta}(\int_X f_i d\lambda + \eta f_i(x)) - \int_X f_i d\lambda = \frac{\eta}{1+\eta}(\int_X f_i d\lambda + f_i(x)) \Rightarrow |\int_X f_i d\kappa - \int_X f_i d\lambda| < \epsilon$, a contradiction.

b) For any $\nu \in P(X)$, $\nu^\perp$ constitutes a $\sim$-invariant dense G_δ in $P(X)$:

Since the set $\{x \in X : \nu(\{x\}) > 0\}$ is countable, if $\{O_n : n \in \mathbf{N}\}$ is any countable basis for the topology on X, an application of the Cantor-Bendixson Theorem shows that for any $n \in \mathbf{N}$, there exists $x_n \in O_n$ such that $\nu(\{x_n\}) = 0$. Hence, $D = \{x_n : n \in \mathbf{N}\}$ is countable dense in X and $\nu(D) = 0$, which implies that $\{\mu \in P(X) : \text{supp}(\mu)$ is finite and contained in $D\} \subseteq \nu^\perp$ and consequently $\nu^\perp$ is dense in $P(X)$. In addition, if $\{c_k : k \in \mathbf{N}\}$ is any countable dense subset of X and $\mathcal{B} = \{\cup_{i<n} B(c_{k_i}; 2^{-l_i}) : (k_0, ..., k_{n-1}), (l_0, ..., l_{n-1}) \in \mathbf{N}^n$ and $n \in \mathbf{N}\}$, then $\nu^\perp = \cap_{(m,n)\in\mathbf{N}^2} O_{mn}$, where for any $(m, n) \in \mathbf{N}^2$, $O_{mn} = \cup_{B\in\mathcal{B};\nu(B^c)<2^{-(m+n)}} \{\mu \in P(X) : \mu(B) < 2^{-n}\}$, and consequently given any $(m, n) \in \mathbf{N}^2$, we need only prove that O_{mn} is open in $P(X)$.

So let $\lambda \in O_{mn}$ and let $B \in \mathcal{B}$ be such that $\nu(B^c) < 2^{-(m+n)}$ and $\lambda(B) < 2^{-n}$. Since $\lambda(X) = 1$ and $\lambda(B) < 2^{-n} \leq 1$, it follows that $B^c \neq \emptyset$ and the regularity of ν implies that there exists an open subset O of X which contains B^c such that $0 \leq \nu(O) - \nu(B^c) < \frac{2^{-(m+n)} - \nu(B^c)}{2}$. Thus, in particular, $\nu(O) - \nu(B^c) < 2^{-(m+n)} - \nu(B^c) \Rightarrow \nu(O) < 2^{-(m+n)} \leq 1 \Rightarrow O^c \neq \emptyset$ and consequently B^c, O^c are disjoint non-empty closed subsets of X and an application of the Urysohn Lemma shows that there exists a continuous function $f : X \to [0, 1]$ such that $f = 1$ on O^c and $f = 0$ on B^c. It is enough to prove that $\{\mu \in P(X) : |\int_X f d\mu - \int_X f d\lambda| < \frac{2^{-n} - \lambda(B)}{2}\} \subseteq O_{mn}$. So let $\mu \in P(X)$ be such that $|\int_X f d\mu - \int_X f d\lambda| < \frac{2^{-n} - \lambda(B)}{2}$. By virtue of the regularity of μ, ν, there exist open subsets U, V of X containing O^c such that $0 \leq \mu(U) - \mu(O^c) < \frac{2^{-n} - \lambda(B)}{2}$ and $0 \leq \nu(V) - \nu(O^c) < \frac{2^{-(m+n)} - \nu(B^c)}{2}$; obviously, $W = U \cap V$ is an open subset of X containing O^c such that $0 \leq \mu(W) - \mu(O^c) < \frac{2^{-n} - \lambda(B)}{2}$ and $0 \leq \nu(W) - \nu(O^c) < \frac{2^{-(m+n)} - \nu(B^c)}{2}$. Since $\{B(c_k; 2^{-l}) : (k, l) \in \mathbf{N}^2\}$ constitutes a basis for the topology on X and $W \subseteq X$ is non-empty open, there exists a sequence $((k_i, l_i))_{i\in\mathbf{N}}$ of pairs of natural numbers such that $W = \cup_{i\in\mathbf{N}} B(c_{k_i}; 2^{-l_i})$ and

$W = \cup_{i\in\mathbf{N}} B(c_{k_i}; 2^{-l_i})$ and consequently there exists $j \in \mathbf{N}$ such that $0 \leq \nu(W) - \nu(C) < \frac{2^{-(m+n)}-\nu(B^c)}{2}$, where $C = \cup_{i<j} B(c_{k_i}; 2^{-l_i}) \in \mathcal{B}$. We remark that

$$C^c \subseteq (C^c \cap B) \cup B^c = (B \setminus C) \cup B^c$$

$$\subseteq (B \setminus O^c) \cup (O^c \setminus C) \cup B^c = (B \cap O) \cup (O^c \setminus C) \cup B^c$$

$$= (O \setminus B^c) \cup (O^c \setminus C) \cup B^c \subseteq (O \setminus B^c) \cup (W \setminus C) \cup B^c$$

and hence

$$\nu(C^c) \leq \nu(O \setminus B^c) + \nu(W \setminus C) + \nu(B^c)$$

$$= (\nu(O) - \nu(B^c)) + (\nu(W) - \nu(C)) + \nu(B^c)$$

$$< \frac{2^{-(m+n)} - \nu(B^c)}{2} + \frac{2^{-(m+n)} - \nu(B^c)}{2} + \nu(B^c) = 2^{-(m+n)} ,$$

while

$$\mu(C) \leq \mu(W) < \mu(O^c) + \frac{2^{-n} - \lambda(B)}{2}$$

$$= \int_{O^c} f d\mu + \frac{2^{-n} - \lambda(B)}{2} \leq \int_X f d\mu + \frac{2^{-n} - \lambda(B)}{2}$$

$$\leq \int_X f d\lambda + |\int_X f d\lambda - \int_X f d\mu| + \frac{2^{-n} - \lambda(B)}{2}$$

$$= \int_B f d\lambda + |\int_X f d\lambda - \int_X f d\mu| + \frac{2^{-n} - \lambda(B)}{2}$$

$$\leq \int_B 1 d\lambda + |\int_X f d\lambda - \int_X f d\mu| + \frac{2^{-n} - \lambda(B)}{2}$$

$$= \lambda(B) + |\int_X f d\lambda - \int_X f d\mu| + \frac{2^{-n} - \lambda(B)}{2}$$

$$< \lambda(B) + \frac{2^{-n} - \lambda(B)}{2} + \frac{2^{-n} - \lambda(B)}{2} = 2^{-n} .$$

We have thus proved that there exists $C \in \mathcal{B}$ such that $\nu(C^c) < 2^{-(m+n)}$ and $\mu(C) < 2^{-n}$, i.e., $\mu \in O_{mn}$.

c) $P_c(X)$ constitutes a $\sim$-invariant dense G_δ in $P(X)$:

We will first prove that $P_c(X)$ constitutes a G_δ. Since X is compact, both $P(X)$ and $K(X)$ are also compact and since $\text{proj}_{P(X)} : P(X) \times X \ni (\mu, x) \mapsto \mu \in P(X)$ is obviously continuous, while $X \ni x \mapsto \{x\} \in K(X)$ constitutes an embedding, it follows that the function $P(X) \times X \ni (\mu, x) \mapsto \mu(\{x\}) \in [0, 1]$ is upper semi-continuous and consequently its upper sections are closed and therefore compact, which implies that $P(X) \setminus P_c(X) = \{\mu \in P(X) : (\exists x \in X)(\mu(\{x\}) > 0)\} = \cup_{n \in \mathbf{N}} \text{proj}_{P(X)}[\{(\mu, x) \in P(X) \times X : \mu(\{x\}) \geq 2^{-n}\}]$ is K_σ in $P(X)$. The fact that $P_c(X)$ is dense in $P(X)$ will follow once we prove the fourth step of the argument, since $P_c(X) \cap P^*(X) \neq \emptyset$ and if $\nu \in P_c(X) \cap P^*(X)$, then $\{\mu \in P(X) : \mu \ll \nu \text{ and } \frac{d\mu}{d\nu} \in C(X, \mathbf{R}^*_+)\} \subseteq P_c(X) \cap P^*(X)$.

d) We divide the fourth step of the argument in two parts:

For any $\nu \in P^*(X)$, $\{\mu \in P(X) : \mu \ll \nu \text{ and } \frac{d\mu}{d\nu} \in C(X, \mathbf{R}^*_+)\}$ is dense and meager in $P(X)$.

By virtue of *b*), it is enough to prove that the set in question is dense. To this end given $\kappa \in P(X)$ such that $\text{supp}(\kappa) = \{x_0, ..., x_l\}$, $\{f_0, ..., f_n\} \subseteq C(X, \mathbf{R})$ and $\epsilon > 0$, we need only prove that there exists $\mu \in P(X)$ such that $\mu \ll \nu$ and $\frac{d\mu}{d\nu} \in C(X, \mathbf{R}^*_+)$, while for any $0 \leq i \leq n$, $|\int_X f_i d\mu - \int_X f_i d\kappa| < \epsilon$ or (equivalently) $|\int_X f_i d\mu - \sum_{k=0}^{l} \alpha_k f_i(x_k)| < \epsilon$, where $\alpha_k = \kappa(\{x_k\})$, $0 \leq k \leq l$. We set $M = \max_{0 \leq i \leq n} \|f_i\|_\infty$ and choose $\delta > 0$ such that the balls $B(x_k; \delta)$, $0 \leq k \leq l$ are pairwise disjoint and $\sup_{y \in B(x_k;\delta)} |f_i(y) - f_i(x_k)| < \frac{\epsilon}{3}$, for any pair of indices i, k. Given any $0 \leq k \leq l$, an application of the Urysohn Lemma for locally compact Hausdorff spaces shows that there exists a continuous function $\psi_k : X \to [0,1]$ such that $\psi_k = 1$ in $\overline{B}(x_k;\delta/2)$ and $\psi_k = 0$ out of $B(x_k;\delta)$, and if $\beta_k = \int_X \psi_k d\nu$, then, since $\psi_k = 1$ in $\overline{B}(x_k;\delta/2)$ and $\psi_k = 0$ out of $B(x_k;\delta)$, it follows that $\beta_k \geq \int_{B(x_k;\delta/2)} 1 d\nu = \nu(B(x_k;\delta/2)) > 0$ and $\beta_k = \int_{B(x_k;\delta)} \psi_k d\nu$. We set $\beta = \min_{0\leq k\leq l} \beta_k > 0$ and let $0 < \eta < 1$ be such that $\frac{\eta}{\eta+1} M(\beta^{-1}+\eta) < \frac{\epsilon}{3}$ and $\eta M < \frac{\epsilon}{3}$. Setting $h = \sum_{k=0}^{l} \alpha_k \beta_k^{-1} \psi_k + \eta$, it is not difficult to see that $\eta \leq h \leq \beta^{-1}+\eta$, $h : X \to [\eta, \beta^{-1}+\eta]$ is continuous and $\int_X h d\nu = 1+\eta$. Therefore, if $d\mu = \frac{h}{1+\eta} d\nu$, then $\mu \ll \nu$ and $\frac{d\mu}{d\nu} \in C(X, \mathbf{R}_+^*)$. Moreover, given any $0 \leq i \leq n$,

$$\int_X f_i d\mu - \sum_{k=0}^{l} \alpha_k f_i(x_k)$$

$$= \frac{1}{1+\eta} \int_X f_i h d\nu - \sum_{k=0}^{l} \alpha_k f_i(x_k)$$

$$= -\frac{\eta}{1+\eta} \int_X f_i h d\nu + \int_X f_i h d\nu - \sum_{k=0}^{l} \alpha_k f_i(x_k)$$

$$= -\frac{\eta}{1+\eta} \int_X f_i h d\nu$$

$$+ \sum_{k=0}^{l} \alpha_k \int_{B(x_k;\delta)} (f_i(y) - f_i(x_k)) \beta_k^{-1} \psi_k(y) d\nu(y) + \eta \int_X f_i d\nu \ ,$$

where

$$|\int_X f_i h d\nu| \leq \int_X |f_i| h d\nu \leq M(\beta^{-1}+\eta) \ ,$$

$$|\int_X f_i d\nu| \leq \int_X |f_i| d\nu \leq M$$

and given $0 \leq k \leq l$,

$$|\int_{B(x_k;\delta)} (f_i(y) - f_i(x_k)) \beta_k^{-1} \psi_k(y) d\nu(y)|$$

$$\leq \sup_{y\in B(x_k;\delta)} |f_i(y) - f_i(x_k)| \cdot \beta_k^{-1} \int_{B(x_k;\delta)} \psi_k(y) d\nu(y)$$

$$= \sup_{y \in B(x_k;\delta)} |f_i(y) - f_i(x_k)| < \frac{\epsilon}{3} \ ,$$

which implies that

$$|\int_X f_i d\mu - \sum_{k=0}^{l} \alpha_k f_i(x_k)| < \frac{\eta}{1+\eta} M(\beta^{-1} + \eta) + \frac{\epsilon}{3} \sum_{k=0}^{l} \alpha_k + \eta M < \epsilon \ .$$

If X constitutes a compact smooth manifold of arbitrary dimension, $\nu \in P^*(X)$ and $r \in \mathbf{N} \cup \{\infty\}$, $\{\mu \in P(X) : \mu \ll \nu$ and $\frac{d\mu}{d\nu} \in C^r(X, \mathbf{R}^*_+)\}$ is dense and meager in $P(X)$.

By virtue of $b)$, it is enough to prove that the set in question is dense. To this end given $\kappa \in P(X)$ such that $\text{supp}(\kappa) = \{x_0, ..., x_l\}$, $\{f_0, ..., f_n\} \subseteq C(X, \mathbf{R})$ and $\epsilon > 0$, we need only prove that there exists $\mu \in P(X)$ such that $\mu \ll \nu$ and $\frac{d\mu}{d\nu} \in C^r(X, \mathbf{R}^*_+)$, while for any $0 \leq i \leq n$, $| \int_X f_i d\mu - \int_X f_i d\kappa| < \epsilon$ or (equivalently) $| \int_X f_i d\mu - \sum_{k=0}^{l} \alpha_k f_i(x_k)| < \epsilon$, where $\alpha_k = \kappa(\{x_k\})$, $0 \leq k \leq l$. We set $M = \max_{0 \leq i \leq n} \|f_i\|_\infty$ and choose $\delta > 0$ and admissible charts $\phi_k : U_k \to B(\mathbf{0}; \delta)$, $0 \leq k \leq l$ on X such that the U_k's are pairwise disjoint and for any $0 \leq k \leq l$, $x_k = \phi_k^{-1}(\mathbf{0})$ and $\sup_{y \in U_k} |f_i(y) - f_i(x_k)| < \frac{\epsilon}{3}$, $0 \leq i \leq n$. Then there exists a C^∞-function $\phi : \mathbf{R}^{\dim(X)} \to [0, 1]$ such that $\phi = 1$ in $\overline{B}(\mathbf{0}; \delta/2)$ and $\phi = 0$ out of $B(\mathbf{0}; 2\delta/3)$ and for any $0 \leq k \leq l$, we set

$$\psi_k(x) = \begin{cases} \phi(\phi_k(x)) & , \text{ if } x \in U_k \\ 0 & , \text{ if } x \in X \setminus U_k \end{cases}$$

and thus obtain a C^∞-function $\psi_k : X \to [0, 1]$ with the property that $\psi_k = 1$ in $\phi_k^{-1}[\overline{B}(\mathbf{0}; \delta/2)]$ and $\psi_k = 0$ out of $U_k = \phi_k^{-1}[B(\mathbf{0}; \delta)]$. Given $0 \leq k \leq l$, if $\beta_k = \int_X \psi_k d\nu$, then since $\psi_k = 1$ in $\phi_k^{-1}[\overline{B}(\mathbf{0}; \delta/2)]$ and $\psi_k = 0$ out of $U_k = \phi_k^{-1}[B(\mathbf{0}; \delta)]$, it follows that $\beta_k \geq \int_{\phi_k^{-1}[B(0;\delta/2)]} 1 d\nu = \nu(\phi_k^{-1}[B(\mathbf{0}; \delta/2)]) > 0$

and $\beta_k = \int_{U_k} \psi_k d\nu$. We set $\beta = \min_{0\le k\le l} \beta_k > 0$ and let $0 < \eta < 1$ be such that $\frac{\eta}{\eta+1} M(\beta^{-1} + \eta) < \frac{\epsilon}{3}$ and $\eta M < \frac{\epsilon}{3}$. Setting $h = \sum_{k=0}^{l} \alpha_k \beta_k^{-1} \psi_k + \eta$, it is not difficult to see that $\eta \le h \le \beta^{-1} + \eta$, $h : X \to [\eta, \beta^{-1} + \eta]$ is differentiable of class C^r and $\int_X h d\nu = 1 + \eta$. Therefore, if $d\mu = \frac{h}{1+\eta} d\nu$, then $\mu \ll \nu$ and $\frac{d\mu}{d\nu} \in C^r(X, \mathbf{R}_+^*)$. Moreover, given any $0 \le i \le n$,

$$\int_X f_i d\mu - \sum_{k=0}^{l} \alpha_k f_i(x_k)$$

$$= \frac{1}{1+\eta} \int_X f_i h d\nu - \sum_{k=0}^{l} \alpha_k f_i(x_k)$$

$$= -\frac{\eta}{1+\eta} \int_X f_i h d\nu + \int_X f_i h d\nu - \sum_{k=0}^{l} \alpha_k f_i(x_k)$$

$$= -\frac{\eta}{1+\eta} \int_X f_i h d\nu$$

$$+ \sum_{k=0}^{l} \alpha_k \int_{U_k} (f_i(y) - f_i(x_k)) \beta_k^{-1} \psi_k(y) d\nu(y) + \eta \int_X f_i d\nu \ ,$$

where

$$|\int_X f_i h d\nu| \le \int_X |f_i| h d\nu \le M(\beta^{-1} + \eta) \ ,$$

$$|\int_X f_i d\nu| \le \int_X |f_i| d\nu \le M$$

and given $0 \le k \le l$,

$$|\int_{U_k} (f_i(y) - f_i(x_k)) \beta_k^{-1} \psi_k(y) d\nu(y)|$$

$$\le \sup_{y \in B(x_k;\delta)} |f_i(y) - f_i(x_k)| \cdot \beta_k^{-1} \int_{U_k} \psi_k(y) d\nu(y)$$

$$= \sup_{y \in U_k} |f_i(y) - f_i(x_k)| < \frac{\epsilon}{3} \ ,$$

which implies that

$$|\int_X f_i d\mu - \sum_{k=0}^{l} \alpha_k f_i(x_k)| < \frac{\eta}{1+\eta} M(\beta^{-1} + \eta) + \frac{\epsilon}{3} \sum_{k=0}^{l} \alpha_k + \eta M < \epsilon \ .$$

$O\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \delta\epsilon\iota\xi\alpha\iota$

4. Generic S_∞-ergodicity for measures

Theorem 4.1: *If X is any compact perfect Polish space and $P(X)$ stands for the Polish space of probability Borel measures on X, equipped with the* weak**-topology, while $\mu \sim \nu \iff (\mu \ll \nu \ \wedge \ \nu \ll \mu)$, whenever μ, ν are in $P(X)$, then $\sim$ is generically S_∞-ergodic. (The same is true if X is any compact smooth manifold of arbitrary dimension and we replace $\sim$ by $\sim_{C^r}$, where $\mu \sim_{C^r} \nu$ iff $\mu \sim \nu$ and both Radon-Nikodym derivatives $\frac{d\mu}{d\nu}$ and $\frac{d\nu}{d\mu}$ are differentiable functions of class C^r, whenever $r \in \mathbf{N} \cup \{\infty\}$.)*

Theorem 3.1 is easily seen to imply that every $\sim$-equivalence class is meager and consequently it is enough to prove that the equivalence relations considered in Theorem 4.1 admit approximations by generically turbulent Polish group actions. To this end we will first reveal the Polish group actions that approximate, in the sense of Definition 2.2, the equivalence relations considered in Theorem 4.1, and we will then prove that they are generically turbulent.

Proposition 4.2: *If X is any compact perfect Polish space, then we have the following:*
(i) For any $\mu \in P(X)$,

$$L^1_{++}(X,\mu) = \{f \in L^1(X,\mu) : f > 0, \ \mu\text{–a.e.}\}$$

constitutes a dense G_δ in $L^1_+(X,\mu)$, and the mapping

$$\Phi_\mu : L^1_+(X,\mu) \setminus \{0\} \to P(X)$$

defined by the relation

$$d(\Phi_\mu(f)) = (\int_X f d\mu)^{-1} f d\mu \ (f \in L^1_+(X,\mu) \setminus \{0\})$$

is continuous and satisfies the condition

$$\Phi_\mu[L^1_{++}(X,\mu)] = [\mu]_\sim \ .$$

(ii) $C(X, \mathbf{R}^*_+)$ *acts continuously on* $P(X)$ *via*

$$C(X, \mathbf{R}^*_+) \times P(X) \ni (f, \mu) \mapsto \Phi_\mu(f) \in P(X)$$

and for any $\mu \in P(X)$,

$$\overline{C(X, \mathbf{R}^*_+)} = L^1_{++}(X,\mu) \ .$$

(iii) For any $\mu \in P(X)$ *and for any* $f \in L^1_{++}(X,\mu)$, *the mapping*

$$\Psi_{\mu,f} : L^1_{++}(X,\mu) \ni g \mapsto g(\int_X f d\mu)/f \in L^1_{++}(X, \Phi_\mu(f))$$

costitutes a homeomorphism with the property that

$$\Phi_\mu(g) = \Phi_{\Phi_\mu(f)}(\Psi_{\mu,f}(g)) \ ,$$

whenever $g \in L^1_{++}(X,\mu)$.

Proof: *(i)* We divide the argument in two steps:

a) $L^1_{++}(X,\mu)$ constitutes a dense G_δ in $L^1_+(X,\mu)$:

If ϵ, δ are arbitrary positive rationals, then we set $H^{(\delta)}_\epsilon = \{f \in L^1_+(X,\mu) : \mu(\{x \in X : f(x) > \epsilon\}) > 1-\delta\}$ and let $H = \cap_{\delta \in \mathbf{Q}^*_+} \cup_{\epsilon \in \mathbf{Q}^*_+} H^{(\delta)}_\epsilon$. It is not difficult to verify that $H = L^1_{++}(X,\mu)$ and consequently we need only prove that for any $\delta \in \mathbf{Q}^*_+$, the set $H^{(\delta)} = \cup_{\epsilon \in \mathbf{Q}^*_+} H^{(\delta)}_\epsilon$ is open and dense in $L^1_+(X,\mu)$.

We shall first prove that $H^{(\delta)}$ is open in $L^1_+(X,\mu)$. So let $f \in H^{(\delta)}$ and let $\epsilon \in \mathbf{Q}^*_+$ be such that $\mu(\{x \in X : f(x) > \epsilon\}) > 1-\delta$. We set

$\eta = \mu(\{x \in X : f(x) > \epsilon\}) - (1-\delta) > 0$ and let g be any non-negative function in $L^1(X,\mu)$ such that $\int_X |f-g|d\mu < \frac{\epsilon\eta}{2}$. If $E = \{x \in X : |f(x) - g(x)| > \frac{\epsilon}{2}\}$, then obviously $\frac{\epsilon}{2}\mu(E) \le \int_E |f(x) - g(x)|d\mu(x) < \frac{\epsilon\eta}{2}$ and hence $\mu(\{x \in X : |f(x) - g(x)| > \frac{\epsilon}{2}\}) < \eta$. Thus, since $\{x \in X : g(x) \le \frac{\epsilon}{2}\} \cap \{x \in X : f(x) > \epsilon\} \subseteq \{x \in X : |f(x) - g(x)| > \frac{\epsilon}{2}\}$ and hence $\{x \in X : g(x) \le \frac{\epsilon}{2}\} \subseteq \{x \in X : f(x) \le \epsilon\} \cup \{x \in X : |f(x) - g(x)| > \frac{\epsilon}{2}\}$, it follows that $\mu(\{x \in X : g(x) \le \frac{\epsilon}{2}\}) \le \mu(\{x \in X : f(x) \le \epsilon\}) + \mu(\{x \in X : |f(x) - g(x)| > \frac{\epsilon}{2}\}) < 1 - \mu(\{x \in X : f(x) > \epsilon\}) + \eta = \delta$ and consequently $\mu(\{x \in X : g(x) > \frac{\epsilon}{2}\}) > 1-\delta$, i.e., $g \in H^{(\delta)}_{\epsilon/2}$. We have thus proved that $\{g \in L^1_+(X,\mu) : \int_X |f-g|d\mu < \frac{\epsilon\eta}{2}\} \subseteq H^{(\delta)}$ and consequently $H^{(\delta)}$ is open in $L^1_+(X,\mu)$.

What is left to show is that $H^{(\delta)}$ is dense in $L^1_+(X,\mu)$. So let f be any non-negative function in $L^1(X,\mu)$ and let $\epsilon \in \mathbf{Q}^*_+$. Then evidently $f + \frac{\epsilon}{2} \in L^1_{++}(X,\mu)$ and $f + \frac{\epsilon}{2} \in H^{(\delta)}$.

b) $\Phi_\mu : L^1_+(X,\mu) \setminus \{0\} \to P(X)$ is continuous and satisfies the condition $\Phi_\mu[L^1_{++}(X,\mu)] = [\mu]_\sim$:

If $f_n \to f$ in $L^1_+(X,\mu) \setminus \{0\}$ as $n \to \infty$, then evidently $|\int_X f_n d\mu - \int_X f d\mu| \le \int_X |f_n - f| d\mu \to 0$ as $n \to \infty$ and for any $g \in C(X,\mathbf{R})$, $|\int_X g f_n d\mu - \int_X g f d\mu| \le \|g\|_\infty \cdot \int_X |f_n - f| d\mu \to 0$ as $n \to \infty$. Therefore,

$$\int_X g d(\Phi_\mu(f_n)) = \int_X g f_n d\mu \cdot (\int_X f_n d\mu)^{-1} \to \int_X g f d\mu \cdot (\int_X f d\mu)^{-1} = \int_X g d(\Phi_\mu(f))$$

as $n \to \infty$, whenever $g \in C(X,\mathbf{R})$, and consequently $\Phi_\mu(f_n) \to \Phi_\mu(f)$ in $P(X)$ as $n \to \infty$, which implies that the mapping $\Phi_\mu : L^1_+(X,\mu) \setminus \{0\} \to P(X)$ is continuous. The fact that it satisfies the condition $\Phi_\mu[L^1_{++}(X,\mu)] = [\mu]_\sim$ follows immediately from the Radon-Nikodym Theorem.

(ii) It is straightforward to verify that $C(X,\mathbf{R}_+^*)\times P(X)\ni(f,\mu)\mapsto\Phi_\mu(f)\in P(X)$ constitutes an action whose continuity follows from part (i), while the density of $C(X,\mathbf{R}_+^*)$ in $L^1_{++}(X,\mu)$ will follow once we prove the following claim:

Let $\{c_k : k\in\mathbf{N}\}$ be any countable dense subset of X and given $(k_0,...,k_{n-1})\in\mathbf{N}^n$ and $r_i<s_i$ $(i<n)$ in $\mathbf{Q}_+^*$, where $n\in\mathbf{N}\setminus\{0\}$, let

$$h_{(k_0,...,k_{n-1};r_0,...,r_{n-1};s_0,...,s_{n-1})}:X\to[0,1]$$

be a continuous function satisfying the conditions

$$h_{(k_0,...,k_{n-1};r_0,...,r_{n-1};s_0,...,s_{n-1})}=1\text{ in }\cup_{i<n}\overline{B}(c_{k_i};r_i)$$

and

$$h_{(k_0,...,k_{n-1};r_0,...,r_{n-1};s_0,...,s_{n-1})}=0\text{ out of }\cup_{i<n}B(c_{k_i};s_i)$$

(whose existence is implied by Urysohn's Lemma for locally compact Hausdorff spaces). If $\mathcal{H}$ consists of all functions of the form:

$$\sum_{j=0}^{m-1}\alpha_j h_{(k_0^{(j)},...,k_{n_j-1}^{(j)};r_0^{(j)},...,r_{n_j-1}^{(j)};s_0^{(j)},...,s_{n_j-1}^{(j)})}\ +\ \alpha\ ,$$

where for any $0\le j<m$, $(k_0^{(j)},...,k_{n_j-1}^{(j)})\in\mathbf{N}^{n_j}$ and $r_i^{(j)}<s_i^{(j)}$ $(0\le i<n_j)$ are in $\mathbf{Q}_+^*$, while $n_j\in\mathbf{N}\setminus\{0\}$ and $\alpha,\alpha_j\in\mathbf{Q}_+^*$ (m being a positive integer), then $\mathcal{H}\cap C(X,\mathbf{R}_+^*)$ constitutes a countable set which is dense in $L^1_+(X,\mu)$.

Let $(\alpha_0,...,\alpha_{m-1})\in(\mathbf{Q}_+^*)^m$ and for any $0\le j<m$, let $(k_0^{(j)},...,k_{n_j-1}^{(j)})\in\mathbf{N}^{n_j}$ and $(s_0^{(j)},...,s_{n_j-1}^{(j)})\in(\mathbf{Q}_+^*)^{n_j}$, where $n_j\in\mathbf{N}\setminus\{0\}$, while $m\in\mathbf{N}\setminus\{0\}$. We set $\phi=\sum\limits_{j<m}\alpha_j\chi_{\cup_{i<n_j}B(c_{k_i^{(j)}};s_i^{(j)})}$ and let $\epsilon>0$ be arbitrary but fixed. It is enough to prove that there exists $h \in \mathcal{H}$ for which $\int_X |\phi - h| d\mu < \epsilon$. Since for any $0 \leq j \leq m$,

$$\cup_{n>0}(\cup_{i<n_j}\overline{B}(c_{k_i^{(j)}}; s_i^{(j)} - n^{-1})) = \cup_{i<n_j} B(c_{k_i^{(j)}}; s_i^{(j)}) ,$$

given $\delta > 0$, there exists an integer $n > 0$ such that

$$|\mu(\cup_{1 \leq \nu \leq n}(\cup_{i<n_j}\overline{B}(c_{k_i^{(j)}}; s_i^{(j)} - \nu^{-1}))) - \mu(\cup_{i<n_j} B(c_{k_i^{(j)}}; s_i^{(j)}))| < \delta$$

for every $0 \leq j < m$. We take $\delta = \frac{\epsilon}{2\sum\limits_{j<m} \alpha_j}$ and if k is the least positive integer for which $k^{-1} < \frac{\epsilon}{2}$, then setting

$$h = \sum_{j<m} \alpha_j h_{(k_0^{(j)},\ldots,k_{n_j-1}^{(j)}; s_0^{(j)}-n^{-1},\ldots,s_{n_j-1}^{(j)}-n^{-1}; s_0^{(j)},\ldots,s_{n_j-1}^{(j)})} \ + \ k^{-1} ,$$

it is not difficult to see that

$$\int_X |\phi - h| d\mu$$

$$\leq \sum_{j<m} \alpha_j \cdot \int_{\cup_{i<n_j} B(c_{k_i^{(j)}}; s_i^{(j)})} (1 -$$

$$h_{(k_0^{(j)},\ldots,k_{n_j-1}^{(j)}; s_0^{(j)}-n^{-1},\ldots,s_{n_j-1}^{(j)}-n^{-1}; s_0^{(j)},\ldots,s_{n_j-1}^{(j)})} \ ) \ d\mu \ + \ k^{-1}$$

$$\leq \sum_{j<m} \alpha_j \cdot \mu(\ (\cup_{i<n_j} B(c_{k_i^{(j)}}; s_i^{(j)})) \setminus$$

$$(\cup_{i<n_j}\overline{B}(c_{k_i^{(j)}}; s_i^{(j)} - n^{-1})) \ ) \ + \ k^{-1} \ < \ \epsilon \ .$$

(iii) It is not difficult to see that

$$\Theta : L^1(X, \mu) \ni g \mapsto g(\int_X f d\mu)/f \in L^1(X, \Phi_\mu(f))$$

constitutes an isometric isomorphism with inverse

$$\Theta^{-1} : L^1(X, \Phi_\mu(f)) \ni h \mapsto (\int_X f d\mu)^{-1} fh \in L^1(X, \mu)$$

mapping $L^1_+(X,\mu)$ onto $L^1_+(X,\Phi_\mu(f))$ and in particular $L^1_{++}(X,\mu)$ onto $L^1_{++}(X,\Phi_\mu(f))$, which implies that $\Psi_{\mu,f}$ constitutes a homeomorphism, while

$$\Phi_{\Phi_\mu(f)}(\Psi_{\mu,f}(g))(B)$$

$$= \int_B (g(\int_X f d\mu)/f) \cdot (\int_X (g(\int_X f d\mu)/f) d(\Phi_\mu(f)))^{-1} \cdot d(\Phi_\mu(f))$$

$$= \int_B (g(\int_X f d\mu)/f) \cdot (\int_X (g(\int_X f d\mu)/f) \cdot (\int_X f d\mu)^{-1} f \cdot d\mu \;)^{-1} \cdot (\int_X f d\mu)^{-1} f \cdot d\mu$$

$$= \int_B (\int_X g d\mu)^{-1} g \; d\mu = \Phi_\mu(g)(B)$$

for every Borel $B \subseteq X$ and consequently $\Phi_{\Phi_\mu(f)}(\Psi_{\mu,f}(g)) = \Phi_\mu(g)$, whenever $g \in L^1_{++}(X,\mu)$.

$O\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \delta\epsilon\iota\xi\alpha\iota$

Proposition 4.3: *If X is any compact smooth manifold of arbitrary dimension and $r \in \mathbf{N} \cup \{\infty\}$, then we have the following:*

(i) For any $\mu \in P(X)$, $C^r(X,\mathbf{R}^*_+)$ constitutes a dense G_δ in $C^r(X,\mathbf{R}_+)$ with respect to τ_μ, and the mapping

$$\Phi_\mu : C^r(X,\mathbf{R}_+) \setminus \{0\} \to P(X)$$

defined by the relation

$$d(\Phi_\mu(f)) = (\int_X f d\mu)^{-1} f d\mu \; (f \in C^r(X,\mathbf{R}_+) \setminus \{0\})$$

is continuous with respect to τ_μ and satisfies the condition

$$\Phi_\mu[C^r(X,\mathbf{R}^*_+)] = [\mu]_{\sim_{C^r}} \; .$$

(ii) $C^r(X,\mathbf{R}^*_+)$ acts continuously on $P(X)$ via

$$C^r(X,\mathbf{R}^*_+) \times P(X) \ni (f,\mu) \mapsto \Phi_\mu(f) \in P(X) \; .$$

(iii) For any $\mu \in P(X)$ *and for any* $f \in C^r(X, \mathbf{R}_+^*)$*, the mapping*

$$\Psi_{\mu,f} : (C^r(X, \mathbf{R}_+^*), \tau_\mu) \ni g \mapsto g(\int_X f d\mu)/f \in (C^r(X, \mathbf{R}_+^*), \tau_{\Phi_\mu(f)})$$

costitutes a homeomorphism with the property that

$$\Phi_\mu(g) = \Phi_{\Phi_\mu(f)}(\Psi_{\mu,f}(g)) \ ,$$

whenever $g \in C^r(X, \mathbf{R}_+^*)$.

Proof: (i) We divide the argument in two steps:

a) $C^r(X, \mathbf{R}_+^*)$ constitutes a dense G_δ in $C^r(X, \mathbf{R}_+)$ with respect to τ_μ:

Since $L^1_{++}(X, \mu)$ constitutes a G_δ in $L^1_+(X, \mu)$, say $L^1_{++}(X, \mu) = \cap_{i \in \mathbf{N}} U_i$, where the U_i's are open in $L^1_+(X, \mu)$, while $C^r(X, \mathbf{R}_+^*) = \cup_{\delta \in \mathbf{Q}_+^*} C^{(\delta)}$, where for any $\delta \in \mathbf{Q}_+^*$, the definition of the Whitney topology is easily seen to imply that $C^{(\delta)} = \{f \in C^r(X, \mathbf{R}_+) : \min_{x \in X} f(x) > \delta\}$ is open in $C^r(X, \mathbf{R}_+)$, it follows that $C^r(X, \mathbf{R}_+^*) = \cap_{i \in \mathbf{N}}(U_i \cap (\cup_{\delta \in \mathbf{Q}_+^*} C^{(\delta)}) \cap C^r(X, \mathbf{R}_+))$ constitutes a G_δ in $C^r(X, \mathbf{R}_+)$ with respect to τ_μ. Therefore, the claim will follow once we prove the density of $C^r(X, \mathbf{R}_+^*)$ in $C^r(X, \mathbf{R}_+)$ with respect to τ_μ:

So let $f \in C^r(X, \mathbf{R}_+)$ and let $\epsilon > 0$ be arbitrary but fixed. Then evidently $f + \frac{\epsilon}{2} \in C^r(X, \mathbf{R}_+^*)$ and for any $s \in \mathbf{N}$ for which $0 < s \leq r$, $d^s(f + \frac{\epsilon}{2}) = d^s f$, which implies that $\rho_\mu(f, f + \frac{\epsilon}{2}) < \epsilon$.

b) $\Phi_\mu : C^r(X, \mathbf{R}_+) \setminus \{0\} \rightarrow P(X)$ is continuous with respect to τ_μ and satisfies the condition $\Phi_\mu[C^r(X, \mathbf{R}_+^*)] = [\mu]_{\sim C^r}$:

The continuity of Φ_μ with respect to τ_μ follows from the definition of τ_μ and part (i) of Proposition 4.2, while the fact that it satisfies the condition $\Phi_\mu[C^r(X, \mathbf{R}^*_+)] = [\mu]_{\sim_{C^r}}$ follows from the Radon-Nikodym Theorem.

(ii) The fact that $C^r(X, \mathbf{R}^*_+) \times P(X) \ni (f, \mu) \mapsto \Phi_\mu(f) \in P(X)$ constitutes a continuous action follows from the definition of τ_μ and from part (ii) of Proposition 4.2.

(iii) It is not difficult to see that

$$\Theta : (C^r(X, \mathbf{R}_+), \tau_\mu) \ni g \mapsto g(\int_X f d\mu)/f \in (C^r(X, \mathbf{R}_+), \tau_{\Phi_\mu(f)})$$

constitutes a bijection with inverse

$$\Theta^{-1} : (C^r(X, \mathbf{R}_+), \tau_{\Phi_\mu(f)}) \ni h \mapsto (\int_X f d\mu)^{-1} f h \in (C^r(X, \mathbf{R}_+), \tau_\mu) \ ,$$

which maps $C^r(X, \mathbf{R}^*_+)$ onto itself and in order to prove that $\Psi_{\mu,f}$ constitutes a homeomorphism what we have to show is that Θ constitutes a homeomorphism. By symmetry, it is enough to show that Θ is continuous. Indeed, if $g_k \to g$ in $(C^r(X, \mathbf{R}_+), \tau_\mu)$, then $d_{C^r(X,\mathbf{R})}(g_k, g) \to 0$ and $\int_X |g_k - g| d\mu \to 0$ as $k \to \infty$, hence an application of the Radon-Nikodym Theorem shows that

$$\int_X |g_k(\int_X f d\mu)/f - g(\int_X f d\mu)/f| d(\Phi_\mu(f)) = \int_X |g_k - g| d\mu \to 0$$

as $k \to \infty$, while since given any natural number s and any function $w \in C^s(X, \mathbf{R})$, there exists a constant $C > 0$ depending only on s and w, for which $\|d^s(uw) - d^s(vw)\|_\infty \leq C \cdot \|d^s u - d^s v\|_\infty$, whenever $u, v \in C^s(X, \mathbf{R})$, it follows that $d_{C^r(X,\mathbf{R})}(g_k(\int_X f d\mu)/f, g(\int_X f d\mu)/f) \to 0$ as $k \to \infty$, which implies that $g_k \to g$ in $(C^r(X, \mathbf{R}_+), \tau_\mu)$. Finally, the fact that $\Phi_\mu(g) = \Phi_{\Phi_\mu(f)}(\Psi_{\mu,f}(g))$ for every $g \in C^r(X, \mathbf{R}^*_+)$ follows from part (iii) of Proposition 4.2.

$O\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \delta\epsilon\iota\xi\alpha\iota$

Lemma 4.4: *Let G be any Polish group acting continuously on a Polish space X and let $x \in X$. Suppose $G \cdot x$ is dense in X and there exists a fundamental system of open neighborhoods U of x in X with the property that for any $g \in G$ for which $g \cdot x \in U$, there exists $h \in G$ and a continuous path $[0,1] \ni t \mapsto h_t \in G$ such that $g \cdot x = h \cdot x$, $h_0 = 1^G$, $h_1 = h$ and $h_t \cdot x \in U$, whenever $t \in [0,1]$. Then the action of G on X is turbulent at the point x.*

Proof: Let V be any open neighborhood of x in X and let W be any symmetric open neighborhood of the identity in G. Then there exists an open neighborhood U of x in X which is contained in V and satisfies the condition stated in the formulation of the lemma. We need only prove that $O(x,U,W) = U \cap (G \cdot x)$. So let $g \in G$ be such that $g \cdot x \in U \cap (G \cdot x)$ and let $h \in G$ and $[0,1] \ni t \mapsto h_t \in G$ be as in the statement of the lemma. Then there exists a positive integer N such that for any s, t in $[0,1]$, $|s-t| \leq N^{-1} \Rightarrow h_s \cdot h_t^{-1} \in W$. Hence, setting $t_0 = 0$, $t_k = t_{k-1} + N^{-1}$ and $g_k = h_{t_k} \cdot h_{t_{k-1}}^{-1}$, whenever $1 \leq k \leq N$, it follows immediately that $g_k \in W$ and $g_k ... g_1 \cdot x = h_{t_k} \cdot x \in U$, whenever $1 \leq k \leq N$, while $g_N ... g_1 \cdot x = g \cdot x$. We have thus proved that $O(x,U,W) = U \cap (G \cdot x)$.
$O\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \delta\epsilon\iota\xi\alpha\iota$

Proposition 4.5: *If X is any compact perfect Polish space, then the action of $C(X, \mathbf{R}_+^*)$ on $P(X)$ described in Proposition 4.2 is turbulent at every $\mu \in P^*(X)$ and therefore generically turbulent. The same is true if X is any compact smooth manifold of arbitrary dimension and we replace $C(X, \mathbf{R}_+^*)$ by $C^r(X, \mathbf{R}_+^*)$ for every $r \in \mathbf{N} \cup \{\infty\}$.*

Proof: By virtue of Theorem 3.1, it is enough to verify that the system of open neighborhoods of $\mu \in P^*(X)$, which consists of the sets of the form

$$U_{\mu;f_0,\ldots,f_n;\epsilon} = \{\nu \in P(X) : (\forall i \leq n)(|\int\limits_X f_i d\nu - \int\limits_X f_i d\mu| < \epsilon)\} ,$$

where $\{f_0, ..., f_n\} \subseteq C(X, \mathbf{R})$ and $\epsilon > 0$, satisfy the condition stated in the formulation of Lemma 4.4. Indeed, if g is any function in the group considered, such that $g \cdot \mu \in U_{\mu;f_0,\ldots,f_n;\epsilon}$, then we need only take $h = (\int_X g d\mu)^{-1} \cdot g$ and $h_t = (1-t) + th$, whenever $t \in [0,1]$.
$O\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \delta\epsilon\iota\xi\alpha\iota$

Keeping the same notations as in Definition 2.2, in view of Propositions 4.2, 4.3 and 4.5, the following table indicates that the equivalence relations considered in Theorem 4.1 admit approximations by generically turbulent Polish group actions.

	X a compact perfect Polish space	X a compact smooth manifold and $r \in \mathbf{N} \cup \{\infty\}$
Γ_μ for $\mu \in P(X)$	$L^1_{++}(X, \mu)$	$(C^r(X, \mathbf{R}^*_+), \tau_\mu)$
G	$C(X, \mathbf{R}^*_+)$	$C^r(X, \mathbf{R}^*_+)$
ϕ_μ for $\mu \in P(X)$	$f \mapsto (\int_X f d\mu)^{-1} f d\mu$ whenever $f \in \Gamma_\mu$	$f \mapsto (\int_X f d\mu)^{-1} f d\mu$ whenever $f \in \Gamma_\mu$
$\psi_{\mu,f}$ for $f \in \Gamma_\mu$ and $\mu \in P(X)$	$g \mapsto g(\int_X f d\mu)/f$ whenever $g \in \Gamma_\mu$	$g \mapsto g(\int_X f d\mu)/f$ whenever $g \in \Gamma_\mu$

TABLE 4.1

Finally, we should mention that Theorem 1 goes through for any perfect Polish space X by considering a compactification of X:

Indeed, since X is homeomorphic to a G_δ subset of the Hilbert cube $[0,1]^{\mathbf{N}}$, the closure $\overline{X}$ of X in $[0,1]^{\mathbf{N}}$ obviously constitutes a compact perfect Polish space and if $(O_m)_{m\in\mathbf{N}}$ is any descending sequence of open subsets of $\overline{X}$ with the property that $X = \cap_{m\in\mathbf{N}} O_m$, then it is enough to notice that

$$P(X) = \cap_{(m,n)\in\mathbf{N}^2}\{\mu \in P(\overline{X}) : \mu(O_m) > 1 - 2^{-n}\}$$

constitutes a dense G_δ in $P(\overline{X})$, something that follows from the fact that the O_m's are dense in $\overline{X}$ and the functions $P(\overline{X}) \ni \mu \mapsto \mu(O_m) \in [0,1]$, $m \in \mathbf{N}$ are lower semi-continuous and consequently their lower sections are closed.

5. Generic turbulence for self-adjoint and unitary operators

Theorem 5.1: *Let* $\mathbf{H}$ *be any infinite-dimensional separable complex Hilbert space and let* $U(\mathbf{H})$ *stand for the Polish group of unitary operators on* $\mathbf{H}$*, while* $S_1(\mathbf{H})$ *stands for the Polish space of self-adjoint operators on* $\mathbf{H}$ *with norm at most one, both equipped with the strong topology. Then the conjugation action of* $U(\mathbf{H})$ *on both* $U(\mathbf{H})$ *and* $S_1(\mathbf{H})$ *is generically turbulent.*

Since the functions that assign to every operator in $U(\mathbf{H})$ a measure in $P(\mathbf{T})$ and to every operator in $S_1(\mathbf{H})$ a measure in $P([-1,1])$ as a canonical representative of its maximal spectral type are continuous, we will reduce the proof of Theorem 5.1 to Theorem 4.1. To this end we will first prove that there exist Borel inverses of the functions mentioned above that assign to every measure in $P^*(\mathbf{T}) \cap P_c(\mathbf{T})$ an operator in $U(\mathbf{H})$ and to every measure in $P^*([-1,1]) \cap P_c([-1,1])$ an operator in $S_1(\mathbf{H})$ and we will then prove that the Polish group actions considered in Theorem 5.1 are generically turbulent by proving that they satisfy the antecedents and part (iv) of the succedents of the theorem of G. Hjorth mentioned in the introduction.

Definition 5.2: *Given* $\mu \in P^*(\mathbf{T}) \cap P_c(\mathbf{T})$ *and* $\nu \in P^*([-1,1]) \cap P_c([-1,1])$*, let* f_μ *stand for the function* $e^{2\pi it} \mapsto \mu(\{e^{2\pi is} : 0 \leq s < t\}) \mathrm{mod} 1$ $(t \in [0,1))$*, which constitutes a homeomorphism of* $\mathbf{T}$ *onto* $\mathbf{R}/\mathbf{Z}$*, and let* g_ν *stand for the function* $x \mapsto \nu([-1,x])$ $(x \in [-1,1])$*, which constitutes a homeomorphism of* $[-1,1]$ *onto* $[0,1]$.

Lemma 5.3: *For any* $\mu \in P^*(\mathbf{T}) \cap P_c(\mathbf{T})$ *and for any* $\nu \in P^*([-1,1]) \cap P_c([-1,1])$*, the mapping* $\Phi_\mu : L^2([0,1), m_1) \ni f \mapsto f \circ f_\mu \in L^2(\mathbf{T}, \mu)$*, if we* *consider* $[0,1)$ *as a* **fundamental region** *of* $\mathbf{Z}$ *in* $\mathbf{R}$, *and* $\Psi_\nu : L^2([0,1],m_1) \ni g \mapsto g \circ g_\nu \in L^2([-1,1],\nu)$ *constitute Hilbert space isomorphisms.*

Proof: It is enough to prove that for any $f \in L^2([0,1),m_1)$ and for any $g \in L^2([0,1],m_1)$, $\int_0^1 |f(x)|^2 dx = \int_{\mathbf{T}} |f \circ f_\mu|^2 d\mu$ and $\int_0^1 |g(x)|^2 dx = \int_{-1}^1 |g \circ g_\nu|^2 d\nu$. Indeed, since the half-open intervals $[\alpha,\beta)$, where $0 \le \alpha < \beta \le 1$, and $(\gamma,\delta]$, where $-1 \le \gamma < \delta \le 1$, form semi-algebras which generate the Borel subsets of $[0,1)$ and $(-1,1]$ respectively, while setting $\phi : t \mapsto e^{2\pi i t}$ $(t \in [0,1))$, $(f_\mu \circ \phi)(\beta) - (f_\mu \circ \phi)(\alpha) = (\phi^{-1}\mu)([\alpha,\beta))$, whenever $0 \le \alpha < \beta \le 1$, an application of the Caratheodory Measure Extension Theorem shows that $\phi^{-1}\mu$ is the Borel measure that corresponds to the strictly increasing function $f_\mu \circ \phi$ and the restriction of ν on $(-1,1]$ is the Borel measure that corresponds to the strictly increasing function $g_\nu|(-1,1]$, which implies that for any $f \in L^2([0,1),m_1)$ and for any $g \in L^2([0,1],m_1)$,

$$\int_0^1 |f(x)|^2 dx = \int_0^1 |f(f_\mu(\phi(x)))|^2 (f_\mu \circ \phi)'(x) dx$$

$$= \int_0^1 |(f \circ f_\mu) \circ \phi|^2 d(\phi^{-1}\mu) = \int_{\mathbf{T}} |f \circ f_\mu|^2 d\mu$$

and

$$\int_0^1 |g(x)|^2 dx = \int_{(0,1]} |g(x)|^2 dx$$

$$= \int_{(-1,1]} |g(g_\nu(x))|^2 g_\nu'(x) dx = \int_{(-1,1]} |g \circ g_\nu|^2 d\nu$$

$$= \int_{-1}^1 |g \circ g_\nu|^2 d\nu \ .$$

$O\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \delta\epsilon\iota\xi\alpha\iota$

Lemma 5.4: *Given* $\kappa \in P^*(\mathbf{T}) \cap P_c(\mathbf{T})$ *and* $\lambda \in P^*([-1,1]) \cap P_c([-1,1])$, *if we consider* $[0,1)$ *as a fundamental region of* $\mathbf{Z}$ *in* $\mathbf{R}$ *the mapping* $U_\kappa f =$

$f_\kappa^{-1} \cdot f$ ($f \in L^2([0,1), m_1)$) *constitutes a unitary operator on* $L^2([0,1), m_1)$ *with spectral multiplicity one and with* $\mu_{U_\kappa} = \kappa$ *with respect to the standard basis in* $L^2([0,1), m_1)$ *which consists of the functions* $e_n : x \mapsto e^{2\pi i n x}$ ($x \in [0,1); n \in \mathbf{Z}$)*, while the mapping* $S_\lambda g = g_\lambda^{-1} \cdot g$ ($g \in L^2([0,1], m_1)$) *constitutes a self-adjoint operator on* $L^2([0,1], m_1)$ *with norm at most one, with spectral multiplicity one and with* $\mu_{S_\lambda} = \lambda$ *with respect to the standard basis in* $L^2([0,1], m_1)$ *which consists of the functions* $e_n : x \mapsto e^{2\pi i n x}$ ($x \in [0,1]; n \in \mathbf{Z}$).

Proof: For any $f \in L^2([0,1), m_1)$ and for any $g, h \in L^2([0,1], m_1)$,

$$(\|U_\kappa f\|_2)^2 = \int_0^1 |U_\kappa f|^2 dm_1 = \int_0^1 |f_\kappa^{-1} \cdot f|^2 dm_1$$

$$= \int_0^1 |f_\kappa^{-1}|^2 \cdot |f|^2 dm_1 = \int_0^1 |f|^2 dm_1 = (\|f\|_2)^2 ,$$

$$(\|S_\lambda g\|_2)^2 = \int_0^1 |g_\lambda^{-1} \cdot g|^2 dm_1 \le (\|g_\lambda^{-1}\|_\infty)^2 \cdot \int_0^1 |g|^2 dm_1 \le (\|g\|_2)^2$$

and

$$(S_\lambda g, h) = \int_0^1 g_\lambda^{-1} g \cdot \overline{h} \cdot dm_1 = \int_0^1 g \cdot \overline{g_\lambda^{-1} h} \cdot dm_1 = (g, S_\lambda h) ,$$

which implies that $U_\kappa \in U(L^2([0,1), m_1))$ and $S_\lambda \in S_1(L^2([0,1], m_1))$. Moreover, if $f \in L^2(\mathbf{T}, \kappa)$ and $g \in L^2([-1,1], \lambda)$, then

$$(\Phi_\kappa U_\kappa \Phi_\kappa^{-1} f)(\zeta) = (\Phi_\kappa U_\kappa (f \circ f_\kappa^{-1}))(\zeta)$$

$$= (\Phi_\kappa (f_\kappa^{-1} \cdot (f \circ f_\kappa^{-1})))(\zeta) = ((f_\kappa^{-1} \circ f_\kappa) \cdot ((f \circ f_\kappa^{-1}) \circ f_\kappa))(\zeta) = \zeta \cdot f(\zeta) ,$$

whenever $\zeta \in \mathbf{T}$, and

$$(\Psi_\lambda S_\lambda \Psi_\lambda^{-1} g)(x) = (\Psi_\lambda S_\lambda (g \circ g_\lambda^{-1}))(x)$$

$$= (\Psi_\lambda (g_\lambda^{-1} \cdot (g \circ g_\lambda^{-1})))(x) = ((g_\lambda^{-1} \circ g_\lambda) \cdot ((g \circ g_\lambda^{-1}) \circ g_\lambda))(x) = x \cdot g(x) ,$$

whenever $x \in [-1,1]$, which implies that $\Phi_\kappa U_\kappa \Phi_\kappa^{-1}$ has spectral multiplicity one and $E^{\Phi_\kappa U_\kappa \Phi_\kappa^{-1}}(B) f = \chi_B \cdot f$ ($B \in \mathbf{B}(\mathbf{T}); f \in L^2(\mathbf{T}, \kappa)$), while $\Psi_\lambda S_\lambda \Psi_\lambda^{-1}$

has spectral multiplicity one and $E^{\Psi_\lambda S_\lambda \Psi_\lambda^{-1}}(B)g = \chi_B \cdot g$ $(B \in \mathbf{B}([-1,1]); g \in L^2([-1,1],\lambda))$. Therefore, we deduce that U_κ has spectral multiplicity one and $E^{U_\kappa}(B) = \Phi_\kappa^{-1} E^{\Phi_\kappa U_\kappa \Phi_\kappa^{-1}}(B)\Phi_\kappa$ $(B \in \mathbf{B}(\mathbf{T}))$, while S_λ has spectral multiplicity one and $E^{S_\lambda}(B) = \Psi_\lambda^{-1} E^{\Psi_\lambda S_\lambda \Psi_\lambda^{-1}}(B)\Psi_\lambda$ $(B \in \mathbf{B}([-1,1]))$. Hence

$$E^{U_\kappa}_{e_n,e_n}(B) = (E^{U_\kappa}(B)e_n, e_n) = (E^{\Phi_\kappa U_\kappa \Phi_\kappa^{-1}}(B)\Phi_\kappa e_n, \Phi_\kappa e_n)$$

$$= \int_{\mathbf{T}} \chi_B \cdot (e_n \circ f_\kappa) \cdot \overline{(e_n \circ f_\kappa)}\, d\kappa = \int_B |e_n \circ f_\kappa|^2 d\kappa = \kappa(B) \ ,$$

whenever $B \in \mathbf{B}(\mathbf{T})$, and

$$E^{S_\lambda}_{e_n,e_n}(B) = (E^{S_\lambda}(B)e_n, e_n) = (E^{\Psi_\lambda S_\lambda \Psi_\lambda^{-1}}(B)\Psi_\lambda e_n, \Psi_\lambda e_n)$$

$$= \int_{-1}^{1} \chi_B \cdot (e_n \circ f_\lambda) \cdot \overline{(e_n \circ f_\lambda)}\, d\lambda = \int_B |e_n \circ f_\lambda|^2 d\lambda = \lambda(B) \ ,$$

whenever $B \in \mathbf{B}([-1,1])$, which implies that $E^{U_\kappa}_{e_n,e_n} = \kappa$ and $E^{S_\lambda}_{e_n,e_n} = \lambda$, whenever $n \in \mathbf{Z}$ and consequently $\mu_{U_\kappa} = \kappa$ and $\mu_{S_\lambda} = \lambda$.
Οπερ εδει δειξαι

Lemma 5.5: *The mappings* $F : P^*(\mathbf{T}) \cap P_c(\mathbf{T}) \ni \mu \mapsto f_\mu \in \mathrm{Hom}(\mathbf{T}, \mathbf{R}/\mathbf{Z})$ *and* $G : P^*([-1,1]) \cap P_c([-1,1]) \ni \nu \mapsto g_\nu \in \mathrm{Hom}([-1,1],[0,1])$ *constitute Borel injections.*

Proof: We will first prove that the mappings in question are injective: So let $\kappa, \lambda \in P^*(\mathbf{T}) \cap P_c(\mathbf{T})$ and $\mu, \nu \in P^*([-1,1]) \cap P_c([-1,1])$ be arbitrary but fixed and let $f_\kappa = f_\lambda$ and $g_\mu = g_\nu$. Then

$$\kappa(\{e^{2\pi i t} : \alpha \le t < \beta\}) = f_\kappa(e^{2\pi i\beta}) - f_\kappa(e^{2\pi i\alpha})$$

$$= f_\lambda(e^{2\pi i\beta}) - f_\lambda(e^{2\pi i\alpha}) = \lambda(\{e^{2\pi i t} : \alpha \le t < \beta\}) \ ,$$

whenever $0 \le \alpha < \beta \le 1$, and

$$\mu((\gamma,\delta]) = g_\mu(\delta) - g_\mu(\gamma) = g_\nu(\delta) - g_\nu(\gamma) = \nu((\gamma,\delta]) \ ,$$

whenever $-1 \leq \gamma < \delta \leq 1$, and since the sets $\{e^{2\pi it} : \alpha \leq t < \beta\}$, where $0 \leq \alpha < \beta \leq 1$, and the half-open intervals $(\gamma, \delta]$, where $-1 \leq \gamma < \delta \leq 1$, form semi-algebras which generate all the Borel subsets of $\mathbf{T}$ and $(-1, 1]$ respectively, an application of the Caratheodory Measure Extension Theorem shows that $\kappa = \lambda$ and the restrictions of μ, ν on $(-1, 1]$ coincide and consequently $\mu = \nu$.

If we prove that graph(F) and graph(G) constitute Borel subsets of the product spaces $(P^*(\mathbf{T}) \cap P_c(\mathbf{T})) \times \mathrm{Hom}(\mathbf{T}, \mathbf{R}/\mathbf{Z})$ and $(P^*([-1,1]) \cap P_c([-1,1])) \times \mathrm{Hom}([-1,1],[0,1])$ respectively, then since the mappings F and G are injective, an application of the Souslin Theorem will show that they are Borel, for

$$F^{-1}[U] = \mathrm{proj}_{P^*(\mathbf{T}) \cap P_c(\mathbf{T})}(\mathrm{graph}(F) \cap ((P^*(\mathbf{T}) \cap P_c(\mathbf{T})) \times U))$$

and

$$G^{-1}[V] = \mathrm{proj}_{P^*([-1,1]) \cap P_c([-1,1])}(\mathrm{graph}(G) \cap ((P^*([-1,1]) \cap P_c([-1,1])) \times V)) ,$$

whenever $U \subseteq \mathrm{Hom}(\mathbf{T}, \mathbf{R}/\mathbf{Z})$ and $V \subseteq \mathrm{Hom}([-1,1],[0,1])$ are non-empty open. So let $(\mu, f) \in (P^*(\mathbf{T}) \cap P_c(\mathbf{T})) \times \mathrm{Hom}(\mathbf{T}, \mathbf{R}/\mathbf{Z})$ and let $(\nu, g) \in (P^*([-1,1]) \cap P_c([-1,1])) \times \mathrm{Hom}([-1,1],[0,1])$ be arbitrary but fixed. Then

$$(\mu, f) \in \mathrm{graph}(F) \iff (\forall x \in \mathbf{Q} \cap [0,1))(\mu(\{e^{2\pi iy} : 0 \leq y < x\}) = f(e^{2\pi ix}))$$

and

$$(\nu, g) \in \mathrm{graph}(G) \iff (\forall t \in \mathbf{Q} \cap [-1,1])(\nu([-1,t]) = g(t)) ,$$

which implies that graph(F) and graph(G) are Borel in $(P^*(\mathbf{T}) \cap P_c(\mathbf{T})) \times \mathrm{Hom}(\mathbf{T}, \mathbf{R}/\mathbf{Z})$ and $(P^*([-1,1]) \cap P_c([-1,1])) \times \mathrm{Hom}([-1,1],[0,1])$ respectively, since for any $x \in [0,1)$ and for any $t \in [-1,1]$, the mappings

$$\mathrm{Hom}(\mathbf{T}, \mathbf{R}/\mathbf{Z}) \ni f \mapsto f(e^{2\pi ix}) \in \mathbf{R}/\mathbf{Z}$$

and

$$\mathrm{Hom}([-1,1],[0,1]) \ni g \mapsto g(t) \in [0,1]$$

are obviously continuous, while the Portmanteau Theorem is easily seen to imply that the mappimgs

$$P^*(\mathbf{T}) \cap P_c(\mathbf{T}) \ni \mu \mapsto \mu(\{e^{2\pi i y} : 0 \leq y < x\}) \in [0,1]$$

and

$$P^*([-1,1]) \cap P_c([-1,1]) \ni \nu \mapsto \nu([-1,t]) \in [0,1]$$

are also continuous.

Οπερ εδει δειξαι

Proposition 5.6: *(i) Viewing* $\mathbf{H}$ *as being* $L^2([0,1), m_1)$ *and considering* $[0,1)$ *as a fundamental region of* $\mathbf{Z}$ *in* $\mathbf{R}$*, the mapping*

$$P^*(\mathbf{T}) \cap P_c(\mathbf{T}) \ni \kappa \mapsto U_\kappa \in \mathcal{U}_2 \cap \mathcal{U}_3$$

is Borel and

$$\kappa \sim \kappa' \iff U_\kappa E_{U(\mathbf{H})}^{\mathcal{U}_2 \cap \mathcal{U}_3} U_{\kappa'} \ ,$$

whenever $\kappa, \kappa' \in P^*(\mathbf{T}) \cap P_c(\mathbf{T})$.

(ii) Viewing $\mathbf{H}$ *as being* $L^2([0,1], m_1)$*, the mapping*

$$P^*([-1,1]) \cap P_c([-1,1]) \ni \lambda \mapsto S_\lambda \in \Sigma_2 \cap \Sigma_3$$

is Borel and

$$\lambda \sim \lambda' \iff S_\lambda E_{U(\mathbf{H})}^{\Sigma_2 \cap \Sigma_3} S_{\lambda'} \ ,$$

whenever $\lambda, \lambda' \in P^*([-1,1]) \cap P_c([-1,1])$.

Proof: By virtue of Lemma 5.4, Lemma 5.5 and the Spectral Theorem, it is enough to prove that the mappings $\mathrm{Hom}(\mathbf{T}, \mathbf{R}/\mathbf{Z}) \ni f \mapsto U'_f \in U(\mathbf{H})$ and $\mathrm{Hom}([-1,1],[0,1]) \ni g \mapsto S'_g \in S_1(\mathbf{H})$ defined by the relations

$$U'_f u = f \cdot u \ (f \in \mathrm{Hom}(\mathbf{T}, \mathbf{R}/\mathbf{Z}); u \in L^2([0,1), m_1))$$

and

$$S'_g v = g \cdot v \ (g \in \mathrm{Hom}([-1,1],[0,1]); v \in L^2([0,1], m_1))$$

are continuous. Indeed, if $f_n \to f$ in $\mathrm{Hom}(\mathbf{T}, \mathbf{R}/\mathbf{Z})$ and $g_n \to g$ in $\mathrm{Hom}([-1,1],[0,1])$ as $n \to \infty$, then for any $u \in L^2([0,1), m_1)$ and for any $v \in L^2([0,1], m_1)$, $(\|(U'_{f_n} - U_f)u\|_2)^2 \leq (\|f_n - f\|_\infty)^2 \cdot (\|u\|_2)^2 \to 0$ and $(\|(S'_{g_n} - S_g)v\|_2)^2 \leq (\|g_n - g\|_\infty)^2 \cdot (\|v\|_2)^2 \to 0$ as $n \to \infty$.
Οπερ εδει δειξαι

Our next task is to show that the actions considered in Theorem 5.1 satisfy the antecedents of the theorem of G. Hjorth mentioned in the introduction.

Proposition 5.7: *(i) For any $S \in \Sigma_2$, $U(\mathbf{H}) \cdot S$ is dense in $S_1(\mathbf{H})$.*
(ii) For any $U \in \mathcal{U}_2$, $U(\mathbf{H}) \cdot U$ is dense in $U(\mathbf{H})$.

Proof: We will first prove the following claim:

Let $x_1, ..., x_n$ be n unit vectors in $\mathbf{H}$ such that $1 \leq i < j \leq n \Rightarrow |(x_i, x_j)| < \delta$, where $0 < \delta < 7^{-n}$. Then the unit vectors $e_1, ..., e_n$ obtained from $x_1, ..., x_n$ by the standard orthogonalization process satisfy the conditions $\|x_i - e_i\| < 7^i \delta$, $1 \leq i \leq n$.

By definition $e_m = \frac{y_m}{\|y_m\|}$ and $y_m = x_m - \sum_{k=1}^{m-1} (x_m, e_k) e_k$ for every $1 \leq m \leq n$, and we argue by induction on n. So let $n > 1$ and assume the claim for the natural number $n-1$. Then $\|x_i - e_i\| < 7^i\delta$, $1 \le i < n$ and what we need to show is that $\|x_n - e_n\| < 7^n\delta$. But setting $p(n) = (\sum_{k=1}^{n-1}(1+7^k)^2)^{1/2}$, $3p(n) < 7^n$ and $\|y_n - x_n\|^2 = \sum_{k=1}^{n-1} |(x_n, e_k)^2| \le \sum_{k=1}^{n-1} |(x_n, x_k) + (x_n, e_k - x_k)|^2 < \delta^2 p(n)^2$, which implies that $1 - \delta p(n) < \|y_n\| < 1 + \delta p(n)$ and $\|x_n - e_n\| = \|x_n - \frac{x_n + y_n - x_n}{\|y_n\|}\| \le \frac{|\|y_n\| - 1| + \|y_n - x_n\|}{\|y_n\|} < \frac{2\delta p(n)}{1 - \delta p(n)} \le 3\delta p(n) < 7^n\delta$.

Now let $S \in \Sigma_2$ and let $U \in \mathcal{U}_2$ be arbitrary but fixed. If $\{t_n : n \in \mathbf{N}\}$ is any countable dense subset of $[-1, 1]$ and T is the unique operator in $S_1(\mathbf{H})$ defined by the relations $T\mathbf{e}_n = t_n\mathbf{e}_n$ $(n \in \mathbf{N})$, then given $N \in \mathbf{N}$, we need only prove that there exist $V, W \in U(\mathbf{H})$ such that $VSV^{-1} \in \{S' \in S_1(\mathbf{H}) : (\forall n \le N)(\|(S' - T)\mathbf{e}_n\| < 2^{-N})\}$ and $WUW^{-1} \in \{U' \in U(\mathbf{H}) : (\forall n \le N)(\|(U' - \exp(i\pi T))\mathbf{e}_n\| < 2^{-N})\}$. Since $\sigma(S) = [-1, 1]$ and $\sigma(U) = \mathbf{T}$, the fact that the spectrum of a normal bounded linear operator on $\mathbf{H}$ coincides with its approximate point spectrum implies that there exist unit vectors $\mathbf{x}_0, ..., \mathbf{x}_N$ and $\mathbf{y}_0, ..., \mathbf{y}_N$ in $\mathbf{H}$ such that for any $0 \le n \le N$, both $\|S\mathbf{x}_n - t_n\mathbf{x}_n\|$ and $\|U\mathbf{y}_n - e^{i\pi t_n}\mathbf{y}_n\|$ are less than

$$\min\left\{\frac{\min\{|t_i - t_j| : 1 \le i < j \le n\}}{2^{N+3} \cdot 7^{N+1}}, \frac{\min\{|e^{i\pi t_i} - e^{i\pi t_j}| : 1 \le i < j \le n\}}{2^{N+3} \cdot 7^{N+1}}\right\} .$$

Hence, given $0 \le i < j \le N$,

$$(t_i - t_j)(\mathbf{x}_i, \mathbf{x}_j) = (t_i\mathbf{x}_i - S\mathbf{x}_i + S\mathbf{x}_i, \mathbf{x}_j) - (\mathbf{x}_i, t_j\mathbf{x}_j - S\mathbf{x}_j + S\mathbf{x}_j)$$

$$= (t_i\mathbf{x}_i - S\mathbf{x}_i, \mathbf{x}_j) + (\mathbf{x}_i, S^*\mathbf{x}_j) - (\mathbf{x}_i, t_j\mathbf{x}_j - S\mathbf{x}_j) - (\mathbf{x}_i, S\mathbf{x}_j) ,$$

so

$$|(\mathbf{x}_i, \mathbf{x}_j)| \le \frac{\|t_i\mathbf{x}_i - S\mathbf{x}_i\| + \|t_j\mathbf{x}_j - S\mathbf{x}_j\|}{|t_i - t_j|} < 2^{-(N+2)} \cdot 7^{-(N+1)} ,$$

while

$$(\mathbf{y}_i, \mathbf{y}_j) = e^{-i\pi t_i}(e^{i\pi t_i}\mathbf{y}_i - U\mathbf{y}_i, \mathbf{y}_j) + e^{-i\pi t_i}(U\mathbf{y}_i, \mathbf{y}_j)$$

and

$$U^{-1}(U\mathbf{y}_j - e^{i\pi t_j}\mathbf{y}_j) = \mathbf{y}_j - e^{i\pi t_j}U^{-1}\mathbf{y}_j$$

$$\Rightarrow U^{-1}\mathbf{y}_j = e^{-i\pi t_j}\mathbf{y}_j - e^{-i\pi t_j}U^{-1}(U\mathbf{y}_j - e^{i\pi t_j}\mathbf{y}_j)$$

$$\Rightarrow (U\mathbf{y}_i, \mathbf{y}_j) = (\mathbf{y}_i, U^{-1}\mathbf{y}_j) = e^{i\pi t_j}(\mathbf{y}_i, \mathbf{y}_j) - e^{i\pi t_j}(\mathbf{y}_i, U^{-1}(U\mathbf{y}_j - e^{i\pi t_j}\mathbf{y}_j)) \ ,$$

which implies that

$$(\mathbf{y}_i, \mathbf{y}_j) = e^{-i\pi t_i}(e^{i\pi t_i}\mathbf{y}_i - U\mathbf{y}_i, \mathbf{y}_j)$$

$$+ e^{-i\pi t_i}(e^{i\pi t_j}(\mathbf{y}_i, \mathbf{y}_j) - e^{i\pi t_j}(\mathbf{y}_i, U^{-1}(U\mathbf{y}_j - e^{i\pi t_j}\mathbf{y}_j)))$$

and consequently

$$|(\mathbf{y}_i, \mathbf{y}_j)| \leq \frac{\|e^{i\pi t_i}\mathbf{y}_i - U\mathbf{y}_i\| + \|e^{i\pi t_j}\mathbf{y}_j - U\mathbf{y}_j\|}{|e^{i\pi t_i} - e^{i\pi t_j}|} < 2^{-(N+2)} \cdot 7^{-(N+1)} \ .$$

Thus, an application of the claim proved above shows that the unit vectors $\mathbf{u}_0, ..., \mathbf{u}_N$, obtained from $\mathbf{x}_0, ..., \mathbf{x}_N$ by the standard orthogonalization process, and the unit vectors $\mathbf{v}_0, ..., \mathbf{v}_N$, obtained from $\mathbf{y}_0, ..., \mathbf{y}_N$ by the standard orthogonalization process, satisfy the conditions $\|\mathbf{x}_i - \mathbf{u}_i\| < 2^{-(N+2)}$, $0 \leq i \leq N$ and $\|\mathbf{y}_i - \mathbf{v}_i\| < 2^{-(N+2)}$, $0 \leq i \leq N$. Therefore, by extending $\{\mathbf{u}_0, ..., \mathbf{u}_N\}$ and $\{\mathbf{v}_0, ..., \mathbf{v}_N\}$ to two complete orthonormal systems $\{\mathbf{u}_n : n \in \mathbf{N}\}$ and $\{\mathbf{v}_n : n \in \mathbf{N}\}$ in $\mathbf{H}$ and setting V and W to be the unique elements of $U(\mathbf{H})$ defined by the relations $V\mathbf{u}_n = \mathbf{e}_n$ $(n \in \mathbf{N})$ and $W\mathbf{v}_n = \mathbf{e}_n$ $(n \in \mathbf{N})$, it follows that for any $0 \leq n \leq N$,

$$\|(VSV^{-1} - T)\mathbf{e}_n\| = \|SV^{-1}\mathbf{e}_n - t_nV^{-1}\mathbf{e}_n\|$$

$$= \|S\mathbf{u}_n - t_n\mathbf{u}_n\| = \|S(\mathbf{u}_n - \mathbf{x}_n) + (S\mathbf{x}_n - t_n\mathbf{x}_n) + t_n(\mathbf{x}_n - \mathbf{u}_n)\|$$

$$\leq 2\|\mathbf{u}_n - \mathbf{x}_n\| + \|S\mathbf{x}_n - t_n\mathbf{x}_n\| < 2 \cdot 2^{-(N+2)} \ + \ 2 \cdot 2^{-(N+3)} \cdot 7^{-(N+1)} < 2^{-N}$$

and

$$\|(WUW^{-1} - \exp(i\pi T))\mathbf{e}_n\| = \|UW^{-1}\mathbf{e}_n - e^{i\pi t_n}W^{-1}\mathbf{e}_n\|$$

$$= \|U\mathbf{v}_n - e^{i\pi t_n}\mathbf{v}_n\| = \|U(\mathbf{v}_n - \mathbf{y}_n) + (U\mathbf{y}_n - e^{i\pi t_n}\mathbf{y}_n) + e^{i\pi t_n}(\mathbf{y}_n - \mathbf{v}_n)\|$$

$$\leq 2\|\mathbf{v}_n - \mathbf{y}_n\| + \|U\mathbf{y}_n - e^{i\pi t_n}\mathbf{y}_n\| < 2\cdot 2^{-(N+2)} \; + \; 2\cdot 2^{-(N+3)}\cdot 7^{-(N+1)} < 2^{-N} \; .$$

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Proposition 5.8: *(i) For any* $\kappa \in P([-1,1])$, $\{S \in S_1(\mathbf{H}) : \mu_S \perp \kappa\}$ *constitutes a conjugacy invariant dense* G_δ *in* $S_1(\mathbf{H})$*, which implies that for any* $S \in S_1(\mathbf{H})$*,* $U(\mathbf{H}) \cdot S$ *is meager in* $S_1(\mathbf{H})$*.*
(ii) For any $\lambda \in P(\mathbf{T})$, $\{U \in U(\mathbf{H}) : \mu_U \perp \lambda\}$ *constitutes a conjugacy invariant dense* G_δ *in* $U(\mathbf{H})$*, which implies that for any* $U \in U(\mathbf{H})$*,* $U(\mathbf{H}) \cdot U$ *is meager in* $U(\mathbf{H})$*.*

Proof: (i) As there exists a countable dense subset $\{t_n : n \in \mathbf{N}\}$ of $[-1,1]$ such that $\kappa(\{t_n : n \in \mathbf{N}\}) = 0$, setting $T\mathbf{e}_n = t_n\mathbf{e}_n$ $(n \in \mathbf{N})$ we obtain a unique operator in $S_1(\mathbf{H})$ for which $\mu_T = \sum\limits_{n=0}^{\infty} 2^{-(n+1)}\delta_{t_n} \perp \kappa$ and since unitarily equivalent normal bounded linear operators on $\mathbf{H}$ have unitarily equivalent spectral measures, it follows that for any $R, S \in S_1(\mathbf{H})$, $RE_{U(\mathbf{H})}^{S_1(\mathbf{H})}S \Rightarrow \mu_R \sim \mu_S$, hence $U(\mathbf{H}) \cdot T \subseteq \{S \in S_1(\mathbf{H}) : \mu_S \perp \kappa\}$ and we need only appeal to the facts that $U(\mathbf{H}) \cdot T$ is dense in $S_1(\mathbf{H})$, $S_1(\mathbf{H}) \ni S \mapsto \mu_S \in P([-1,1])$ is continuous and $\{\mu \in P([-1,1]) : \mu \perp \kappa\}$ constitutes a G_δ in $P([-1,1])$.
(ii) As before, there exists a countable dense subset $\{t_n : n \in \mathbf{N}\}$ of $[-1,1]$ such that $\lambda(\{e^{i\pi t_n} : n \in \mathbf{N}\}) = 0$. Hence, setting $T\mathbf{e}_n = t_n\mathbf{e}_n$ $(n \in \mathbf{N})$ and $W = \exp(i\pi T)$, the fact that for any $B \in \mathbf{B}(\mathbf{T})$, $E^W(B) = E^T(\{t \in [-1,1] : e^{i\pi t} \in B\})$ implies that $\mu_W = \sum\limits_{n=0}^{\infty} 2^{-(n+1)}\delta_{e^{i\pi t_n}} \perp \lambda$ and since unitarily equivalent normal bounded linear operators on $\mathbf{H}$ have unitarily equivalent spectral measures, it follows that for any $U, V \in U(\mathbf{H})$, $UE_{U(\mathbf{H})}^{U(\mathbf{H})}V \Rightarrow \mu_U \sim \mu_V$, hence $U(\mathbf{H}) \cdot W \subseteq \{U \in U(\mathbf{H}) : \mu_U \perp \lambda\}$ and we need only appeal to the facts that $U(\mathbf{H}) \cdot W$ is dense in $U(\mathbf{H})$, $U(\mathbf{H}) \ni U \mapsto \mu_U \in P(\mathbf{T})$ is continuous and

$\{\mu \in P(\mathbf{T}) : \mu \perp \lambda\}$ constitutes a G_δ in $P(\mathbf{T})$.

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We are finally in position to prove Theorem 5.1.

Since $\mathcal{U}_2 \cap \mathcal{U}_3$ constitutes a conjugacy invariant dense G_δ in $U(\mathbf{H})$ and $\Sigma_2 \cap \Sigma_3$ constitutes a conjugacy invariant dense G_δ in $S_1(\mathbf{H})$, while the mappings

$$\Phi : \mathcal{U}_2 \cap \mathcal{U}_3 \ni U \mapsto \mu_U \in P^*(\mathbf{T})$$

and

$$\Phi_1 : \Sigma_2 \cap \Sigma_3 \ni S \mapsto \mu_S \in P^*([-1,1])$$

are continuous and the mappings

$$\Psi : P^*(\mathbf{T}) \cap P_c(\mathbf{T}) \ni \kappa \mapsto U_\kappa \in \mathcal{U}_2 \cap \mathcal{U}_3$$

and

$$\Psi_1 : P^*([-1,1]) \cap P_c([-1,1]) \ni \lambda \mapsto S_\lambda \in \Sigma_2 \cap \Sigma_3$$

are Borel, Lemma 5.4 and the proof of Theorem 4.1 show that if Y is any Polish space and $f : U(\mathbf{H}) \to Y^{\mathbf{N}}$, $g : S_1(\mathbf{H}) \to Y^{\mathbf{N}}$ are any C-measurable functions with the property that

$$UE_{U(\mathbf{H})}^{U(\mathbf{H})}V \Rightarrow \{f(U)(n) : n \in \mathbf{N}\} = \{f(V)(n) : n \in \mathbf{N}\} ,$$

whenever U, V are in $U(\mathbf{H})$, and

$$SE_{U(\mathbf{H})}^{S_1(\mathbf{H})}T \Rightarrow \{g(S)(n) : n \in \mathbf{N}\} = \{g(T)(n) : n \in \mathbf{N}\} ,$$

whenever S, T are in $S_1(\mathbf{H})$, there exist $\sim$-invariant dense G_δ subsets B of $P^*(\mathbf{T}) \cap P_c(\mathbf{T})$ and B_1 of $P^*([-1,1]) \cap P_c([-1,1])$ and countable subsets $\mathcal{C}$ and $\mathcal{C}_1$ of Y such that

$$\kappa \in B \Rightarrow \{f(\Phi(\kappa))(n) : n \in \mathbf{N}\} = \mathcal{C} ,$$

whenever $\kappa \in P^*(\mathbf{T}) \cap P_c(\mathbf{T})$, and

$$\lambda \in B_1 \Rightarrow \{f(\Phi_1(\lambda))(n) : n \in \mathbf{N}\} = \mathcal{C}_1 \ ,$$

whenever $\lambda \in P^*([-1,1]) \cap P_c([-1,1])$. Thus, setting $A = \Phi^{-1}[B]$ and $A_1 = \Phi_1^{-1}[B_1]$, we obtain unitary conjugacy invariant G_δ subsets of $\mathcal{U}_2 \cap \mathcal{U}_3$ and $\Sigma_2 \cap \Sigma_3$ respectively such that $\Psi[B] \subseteq A$ and $\Psi_1[B_1] \subseteq A_1$, while the facts that $\mathcal{U}_2 \subseteq \mathcal{U}_1$ and $\Sigma_2 \subseteq \Sigma_1$ show that A and A_1 are also dense in $U(\mathbf{H})$ and $S_1(\mathbf{H})$ respectively. Indeed, the implications

$$UE_{U(\mathbf{H})}^{U(\mathbf{H})}V \in A \Rightarrow UE_{U(\mathbf{H})}^{\mathcal{U}_2\cap\mathcal{U}_3}V \in A \Rightarrow \mu_U \sim \mu_V \in B \Rightarrow \mu_U \in B \Rightarrow U \in A$$

and

$$SE_{U(\mathbf{H})}^{S_1(\mathbf{H})}T \in A_1 \Rightarrow SE_{U(\mathbf{H})}^{\Sigma_2\cap\Sigma_3}T \in A_1 \Rightarrow \mu_S \sim \mu_T \in B_1 \Rightarrow \mu_S \in B_1 \Rightarrow S \in A_1$$

show that A and A_1 are unitary conjugacy invariant, while the facts that $\Phi \circ \Psi = \mathrm{id}$ and $\Phi_1 \circ \Psi_1 = \mathrm{id}$ show that $\Psi[B] \subseteq A$ and $\Psi_1[B_1] \subseteq A_1$. Therefore, by virtue of part *(iv)* of the theorem of G. Hjorth mentioned in the introduction, we need only prove that A and A_1 are contained in the saturation of $\Psi[B]$ and $\Psi_1[B_1]$ respectively according to unitary conjugacy: Indeed, if $U \in A \subseteq \mathcal{U}_2 \cap \mathcal{U}_3$ and $S \in A_1 \subseteq \Sigma_2 \cap \Sigma_3$, then $\mu_U = \Phi(U) \in B \subseteq P^*(\mathbf{T}) \cap P_c(\mathbf{T})$ and $\mu_S = \Phi_1(S) \in B_1 \subseteq P^*([-1,1]) \cap P_c([-1,1])$, hence $\mu_{\Psi(\mu_U)} = \mu_U$ and $\mu_{\Psi_1(\mu_S)} = \mu_S$ and the Spectral Theorem implies that $UE_{U(\mathbf{H})}^{\mathcal{U}_2\cap\mathcal{U}_3}\Psi(\mu_U) \in \Psi[B]$ and $SE_{U(\mathbf{H})}^{\Sigma_2\cap\Sigma_3}\Psi_1(\mu_S) \in \Psi_1[B_1]$.
$O\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \delta\epsilon\iota\xi\alpha\iota$

Finally, we should mention that a new proof of Theorem 1 was given by S. Solecki [15] and a new proof of the part of Theorem 2 concerning the equivalence relation induced by the action of the group of unitary operators on itself by conjugation was given by G. Hjorth [8].

References

[1] J.R. CHOKSI and M.G. NADKARNI, Baire category in spaces of measures, unitary operators and transformations, *Invariant Subspaces and Allied Topics*, edited by H. Helson and S. Yadov, Narosa Publ. Co., New Delhi, 1990, 147-163.

[2] J.R. CHOKSI and M.G. NADKARNI, *Genericity of certain classes of unitary and self-adjoint operators*, preprint.

[3] N. DUNFORD and J.T. SCHWARTZ, *Linear Operators, Part II: Spectral Theory*, Wiley Interscience Publishers, 1963.

[4] P.R. HALMOS, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, Second Edition, AMS Chelsea Publishing, 1957.

[5] L.A. HARRINGTON, A.S. KECHRIS and A. LOUVEAU, *A Glimm-Effros Dichotomy for Borel Equivalence Relations*, Journal of the AMS, Volume 3, Number 4, October 1990, 903-928.

[6] M.W. HIRSCH, *Differential Topology*, Springer-Verlag, 1997.

[7] G. HJORTH, *Classification and orbit equivalence relations*, preprint.

[8] G. HJORTH, *Non-smooth infinite dimensional group representations*, preprint.

[9] A.S. KECHRIS, *New directions in descriptive set theory*, preprint, 1998.

[10] A.S. KECHRIS, *Classical Descriptive Set Theory*, Springer-Verlag, 1995.

[11] M.G. NADKARNI, *Spectral Theory of Dynamical Systems*, Birkhauser, 1998.

[12] H.L. ROYDEN, *Real Analysis*, Third Edition, MacMillan Publishing Company, New York, 1988.

[13] W. RUDIN, *Functional Analysis*, Second Edition, McGraw-Hill International Editions, 1991.

[14] B. SIMON, *Operators with singular continuous spectrum: I. General operators*, Annals of Mathematics, 141(1995), 131-145.

[15] S. SOLECKI, *Polish group topologies*, preprint.

Chapter II

New natural examples of complex Borel and analytic sets

Introduction

Descriptive Set Theory is the study of definable sets in Polish (i.e., separable completely metrizable) spaces and one of the main trends of current research in the field is the classification of natural sets arising in other branches of mathematics, in the sense of computing their exact complexity (see, for example, the Introduction and Sections 23, 27, 33 and 37 of [5]).

Our main purpose in this chapter is to give new natural examples of complex Borel and analytic sets originating from Analysis and Geometry. In fact, we obtain the following results:

Theorem 1: *The set of Dirichlet series whose abscissa of absolute convergence is equal to* $-\infty$ *is* $\mathbf{\Pi}^0_3$*-complete.*

Theorem 2: *Given any non-negative real number* α*, the set of entire functions whose order is equal to* α *is* $\mathbf{\Pi}^0_3$*-complete and the set of all sequences of entire functions whose orders converge to* α *is* $\mathbf{\Pi}^0_5$*-complete.*

Theorem 3: *Given any line in the plane and any cardinal number* $1 \leq n \leq \aleph_0$*, the set of continuous paths in the plane tracing curves which admit at least* n *tangents parallel to the given line is* $\mathbf{\Sigma}^1_1$*-complete.*

Theorem 4: *Given any positive integer* N *and any cardinal number* $1 \leq n \leq \aleph_0$*, if* $-\infty < \alpha < \beta < +\infty$*, then the set of all functions in* $C([\alpha, \beta]^N, \mathbf{R})$ *whose graph in* $\mathbf{R}^{N+1}$ *admits at least* n *tangent* N*-dimensional hyperplanes parallel to* $\mathbf{R}^N$ *is* $\mathbf{\Sigma}^1_1$*-complete.*

Theorem 5: *For any cardinal number* $1 \leq n \leq \aleph_0$*, the set of differen-*

tiable paths of class C^2 in the plane admitting a canonical parameter in $[0,1]$ *and tracing curves which have at least* n *vertices is* $\mathbf{\Sigma}^1_1$*-complete, while the set of differentiable paths of class* C^3 *in the plane admitting a canonical parameter in* $[0,1]$ *and tracing curves which have at least* n *vertices is* $\mathbf{\Sigma}^0_2$ *if* $n < \aleph_0$ *and* $\mathbf{\Pi}^0_3$ *if* $n = \aleph_0$.

At the end of section 23 of [5], A.S. Kechris states: *In conclusion, we would like to mention that we do not know of any interesting "natural" examples of Borel sets in analysis or topology which are in one of the classes* $\mathbf{\Sigma}^0_\xi$ *or* $\mathbf{\Pi}^0_\xi$ *for* $\xi \geq 5$*, but not in a class with lower index.* Thus, Theorem 2 provides for the first time natural examples of complex Borel sets in Analysis or Topology that live in the fifth level of the Borel hierarchy. In addition, Theorem 5 was motivated by and should be contrasted with a generalization of the Four Vertex Theorem (see, for example, [11] on page 48 or [7] on pages 28-30) proved in [2]: *Every simple closed differentiable curve of class* C^3 *has at least four vertices.* Geometric properties that give rise to analytic sets which are not Borel were also given by O. Nikodym and W. Sierpinski (see, for example, [8], [9] and page 216 of [5]). Finally, to the best of our knowledge natural examples of complex Borel sets originating from Number Theory were given for the first time by H. Ki in his thesis [6]. As ill-founded trees on $\mathbf{N}$ form perhaps the archetypical $\mathbf{\Sigma}^1_1$-complete set (see, for example, [5] on page 209), the main tool for the proof of Theorems 3, 4 and 5 is the result proved in section 3, while on that what concerns the proof of Theorems 1 and 2 we should mention that $[.,.]$ stands for any standard coding of pairs of natural numbers by natural numbers and $(.)_0$, $(.)_1$ stand for the associated decoding functions, in the sense that $[(n)_0, (n)_1] = n$ for every $n \in \mathbf{N}$.

1. Complex Borel sets associated with Dirichlet series

By a Dirichlet series we mean a series of the form $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$, where s is a real number and $(a_n)_{n\in\mathbf{N}\setminus\{0\}}$ a sequence of complex numbers, and, by identifying $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ with $(a_n)_{n\in\mathbf{N}\setminus\{0\}}$, we may view $\mathbf{C}^{\mathbf{N}\setminus\{0\}}$ as the family of all Dirichlet series, while for any $a = (a_n)_{n\in\mathbf{N}\setminus\{0\}} \in \mathbf{C}^{\mathbf{N}\setminus\{0\}}$, $\sigma_a = \inf\left\{s \in \mathbf{R} : \sum_{n=1}^{\infty} \frac{|a_n|}{n^s} < +\infty\right\}$ is called the abscissa of absolute convergence of $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ (see, for example, [1] on page 225). It is not difficult to prove that the set of Dirichlet series whose abscissa of absolute convergence is less than $+\infty$ is $\mathbf{\Sigma}_2^0$-complete and we confine ourselves in proving the following result:

Theorem 1.1: *The set of Dirichlet series whose abscissa of absolute convergence is equal to $-\infty$ is $\mathbf{\Pi}_3^0$-complete.*

Proof: If $a \in \mathbf{C}^{\mathbf{N}\setminus\{0\}}$, then

$$\sigma_a = -\infty \iff \forall \nu \geq 1 \exists N \geq 1 \forall n \geq 1 (\sum_{k=1}^{n} |a_k| k^{\nu} \leq N) ,$$

which implies that the set $\{a \in \mathbf{C}^{\mathbf{N}\setminus\{0\}} : \sigma_a = -\infty\}$ is $\mathbf{\Pi}_3^0$, since the mapping $\mathbf{C}^{\mathbf{N}\setminus\{0\}} \ni a \mapsto a_n \in \mathbf{C}$ is obviously continuous, whenever $n \in \mathbf{N} \setminus \{0\}$.

To prove that $\{a \in \mathbf{C}^{\mathbf{N}\setminus\{0\}} : \sigma_a = -\infty\}$ is $\mathbf{\Pi}_3^0$-hard, it is enough to show that a set which is known to be $\mathbf{\Pi}_3^0$-hard is Wadge reducible to $\{a \in \mathbf{C}^{\mathbf{N}\setminus\{0\}} : \sigma_a = -\infty\}$ (see, for example, [5] on pages 156 and 169). Since $P_3 = \{x \in 2^{(\mathbf{N}\setminus\{0\})\times(\mathbf{N}\setminus\{0\})} : \forall m \forall^{\infty} n (x(m,n) = 0)\}$ is $\mathbf{\Pi}_3^0$-complete (see, for example, [5] on page 179), what we have to show is that $\{a \in \mathbf{C}^{\mathbf{N}\setminus\{0\}} : \sigma_a = -\infty\} \leq_W P_3$. So let

$$a_n^x = \begin{cases} 0 & , \text{ if } x((n)_0, (n)_1) = 0 \\ ((n)_0^2 \cdot n^{(n)_0})^{-1} & , \text{ if } x((n)_0, (n)_1) = 1 \end{cases} ,$$

whenever $x \in 2^{(\mathbf{N}\setminus\{0\})\times(\mathbf{N}\setminus\{0\})}$ and $n \in \mathbf{N} \setminus \{0\}$. Given $x, y \in 2^{(\mathbf{N}\setminus\{0\})\times(\mathbf{N}\setminus\{0\})}$ and $n \in \mathbf{N} \setminus \{0\}$, if $x(i,j) = y(i,j)$ for $1 \leq i,j \leq n$, then for any $1 \leq k \leq n$, $a_k^x = a_k^y$ and consequently $2^{(\mathbf{N}\setminus\{0\})\times(\mathbf{N}\setminus\{0\})} \ni x \mapsto (a_n^x)_{n\in\mathbf{N}\setminus\{0\}} \in \mathbf{C}^{\mathbf{N}\setminus\{0\}}$ is continuous. What is left to show is that for any $x \in 2^{(\mathbf{N}\setminus\{0\})\times(\mathbf{N}\setminus\{0\})}$, $x \in P_3 \iff (a_n^x)_{n\in\mathbf{N}\setminus\{0\}} \in \{a \in \mathbf{C}^{\mathbf{N}\setminus\{0\}} : \sigma_a = -\infty\}$.

If $x \notin P_3$ and $m \in \mathbf{N} \setminus \{0\}$ is such that $\{n \in \mathbf{N} \setminus \{0\} : x(m,n) = 1\}$ is infinite, then

$$\sum_{k=1}^{\infty} |a_k^x| k^m = \sum_{x((k)_0,(k)_1)=1} \frac{1}{(k)_0^2 \cdot k^{(k)_0}} \cdot k^m \geq \sum_{x(m,n)=1} \frac{1}{m^2 \cdot [m,n]^m} \cdot [m,n]^m$$

$$= \sum_{x(m,n)=1} \frac{1}{m^2} = +\infty$$

and hence $(a_n^x)_{n\in\mathbf{N}\setminus\{0\}} \notin \{a \in \mathbf{C}^{\mathbf{N}\setminus\{0\}} : \sigma_a = -\infty\}$.

If $x \in P_3$ and $\nu \in \mathbf{N} \setminus \{0\}$, then

$$\sum_{k=1}^{\infty} |a_k^x| k^\nu = \sum_{x((k)_0,(k)_1)=1} \frac{1}{(k)_0^2 \cdot k^{(k)_0}} \cdot k^\nu = \sum_{m=1}^{\infty} \sum_{x(m,n)=1} \frac{1}{m^2 \cdot [m,n]^m} \cdot [m,n]^\nu$$

$$= \sum_{m=1}^{\nu+1} \sum_{x(m,n)=1} \frac{1}{m^2 \cdot [m,n]^m} \cdot [m,n]^\nu + \sum_{m=\nu+2}^{\infty} \sum_{x(m,n)=1} \frac{1}{m^2} \cdot \frac{1}{[m,n]^{m-\nu}}$$

$$\leq \sum_{m=1}^{\nu+1} \sum_{x(m,n)=1} \frac{1}{m^2 \cdot [m,n]^m} \cdot [m,n]^\nu + \sum_{m=\nu+2}^{\infty} \sum_{x(m,n)=1} \frac{1}{m^2} \cdot \frac{1}{[m,n]^2}$$

$$\leq \sum_{m=1}^{\nu+1} \sum_{x(m,n)=1} \frac{1}{m^2 \cdot [m,n]^m} \cdot [m,n]^\nu + \sum_{m=\nu+2}^{\infty} \frac{1}{m^2} \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty$$

and hence $(a_n^x)_{n\in\mathbf{N}\setminus\{0\}} \in \{a \in \mathbf{C}^{\mathbf{N}\setminus\{0\}} : \sigma_a = -\infty\}$.

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2. Complex Borel sets associated with entire functions

Let $H(\mathbf{C})$ stand, as usual, for the Polish space of entire functions, equipped with the topology of "almost uniform convergence," namely the topology of uniform convergence on compacts, and for any $f \in H(\mathbf{C})$, let $M(r; f) = \max_{0\leq\theta<2\pi} |f(re^{i\theta})|$, whenever $r > 0$; then $\rho(f) = \limsup_{r\to+\infty} \frac{\log\log M(r;f)}{\log r}$ is called the order of the entire function f (see, for example, [4] on page 182) and if $c_n = \frac{f^{(n)}(0)}{n!}$, whenever $n \in \mathbf{N}$, and hence $f(z) = \sum_{n=0}^{\infty} c_n z^n$ $(z \in \mathbf{C})$, then $\rho(f) = \limsup_{n\to\infty} \frac{n\cdot\log n}{\log\frac{1}{|c_n|}}$ (see, for example, [4] on page 186).

Theorem 2.1: *For any* $0 \leq \alpha < \infty$, $\{f \in H(\mathbf{C}) : \rho(f) = \alpha\}$ *is* $\mathbf{\Pi}_3^0$-*complete and* $\{(f_k)_{k\in\mathbf{N}} \in H(\mathbf{C})^{\mathbf{N}} : \lim_{k\to\infty} \rho(f_k) = \alpha\}$ *is* $\mathbf{\Pi}_5^0$-*complete.*

Proof: If $f \in H(\mathbf{C})$ and $(f_k)_{k\in\mathbf{N}} \in H(\mathbf{C})^{\mathbf{N}}$, while $c_n = \frac{f^{(n)}(0)}{n!}$ and $c_{k,n} = \frac{f_k^{(n)}(0)}{n!}$, whenever $k, n \in \mathbf{N}$, then

$$\rho(f) = \alpha \iff \forall i \exists j \geq i \forall k \geq j \left(\frac{j\cdot\log j}{\log\frac{1}{|c_j|}} \geq \alpha - 2^{-i} \ \wedge\ \frac{k\cdot\log k}{\log\frac{1}{|c_k|}} \leq \alpha + 2^{-i} \right)$$

and

$$\lim_{k\to\infty} \rho(f_k) = \alpha \iff \forall i \exists j \forall k \geq j \forall l \exists m \geq l \forall n \geq m$$

$$\left(\frac{m\cdot\log m}{\log\frac{1}{|c_{k,m}|}} \geq \alpha - 2^{-i} \ \wedge\ \frac{n\cdot\log n}{\log\frac{1}{|c_{k,n}|}} \leq \alpha + 2^{-i} + 2^{-l} \right),$$

which implies that the sets $A_\alpha = \{f \in H(\mathbf{C}) : \rho(f) = \alpha\}$, $B_\alpha = \{(f_k)_{k\in\mathbf{N}} \in H(\mathbf{C})^{\mathbf{N}} : \lim_{k\to\infty} \rho(f_k) = \alpha\}$ are $\mathbf{\Pi}_3^0$ and $\mathbf{\Pi}_5^0$ respectively, since the mapping $H(\mathbf{C})^{\mathbf{N}} \ni (f_k)_{k\in\mathbf{N}} \mapsto f_k \in H(\mathbf{C})$ is obviously continuous, whenever $k \in \mathbf{N}$, and so is the mapping $H(\mathbf{C}) \ni f \mapsto f^{(\nu)}(0) \in \mathbf{C}$ (see, for example, [3] on page 192), whenever $\nu \in \mathbf{N}$.

To prove that A_α is $\mathbf{\Pi}^0_3$-hard, it is enough to show that a set which is known to be $\mathbf{\Pi}^0_3$-hard is Wadge reducible to A_α (see, for example, [5] on pages 156 and 169), and as $P_3 = \{x \in 2^{\mathbf{N}\times\mathbf{N}} : \forall m \forall^\infty n (x(m,n) = 0)\}$ is $\mathbf{\Pi}^0_3$-complete (see, for example, [5] on page 179), we will show that $A_\alpha \leq_W P_3$. So let

$$c^x_n = \begin{cases} \frac{1}{n^{n\cdot\phi_x(n)^{-1}}} & \text{, if } n \in \mathbf{N}\setminus\{0\} \\ 0 & \text{, if } n = 0 \end{cases},$$

where

$$\phi_x([m,n]) = \begin{cases} \alpha + 2^{-m} & \text{, if } x(m,n) = 1 \\ \alpha + 3^{-[m,n]} & \text{, if } x(m,n) = 0 \end{cases},$$

whenever $x \in 2^{\mathbf{N}\times\mathbf{N}}$ and $m, n \in \mathbf{N}$. Given any $x \in \mathbf{N}$, it is not difficult to prove that $|c^x_n|^{\frac{1}{n}} \to 0$ as $n \to \infty$; therefore, by setting $f_x(z) = \sum\limits_{n=0}^{\infty} c^x_n z^n$ ($z \in \mathbf{C}$) we obtain an entire function (see, for example, [3] on page 118) and what we want to show is that the mapping $2^{\mathbf{N}\times\mathbf{N}} \ni x \mapsto f_x \in H(\mathbf{C})$ is continuous. Indeed, we need only remark that for any integer N, if $x, y \in 2^{\mathbf{N}\times\mathbf{N}}$ and $x((n)_0,(n)_1) = y((n)_0,(n)_1)$, whenever $0 \leq n < N$, then $c^x_n = c^y_n$, whenever $0 \leq n < N$, and since for any $u \in 2^{\mathbf{N}\times\mathbf{N}}$ and for any integer $n \geq N$, $|c^u_n| \leq \frac{1}{n^{\frac{n}{\alpha+1}}}$, it follows that

$$\max_{|z|\leq R} |f_x(z) - f_y(z)| \leq 2 \cdot \sum_{n\geq N} \frac{R^n}{n^{\frac{n}{\alpha+1}}}$$

where $\sum\limits_{n\geq N} \frac{R^n}{n^{\frac{n}{\alpha+1}}} \to 0$ as $n \to \infty$, whenever $R > 0$. What is left to show is that for any $x \in 2^{\mathbf{N}\times\mathbf{N}}$, $x \in P_3 \iff \rho(f_x) = \alpha$.

If $x \notin P_3$, then there exists $m \in \mathbf{N}$ such that $\exists^\infty n(x(m,n) = 0)$, which implies that

$$\exists^\infty n \left(\frac{n \cdot \log n}{\log \frac{1}{|c^x_n|}} = \alpha + 2^{-m} \right)$$

and consequently $\rho(f_x) = \alpha + 2^{-m}$.

If $x \in P_3$, then for any $m \in \mathbf{N}$, there exists $n_m \in \mathbf{N}$ such that for any integer $n \geq n_m$, $x(m,n) = 0$ and hence

$$\frac{[m,n] \cdot \log[m,n]}{\log \frac{1}{|c^x_{[m,n]}|}} = \alpha + 3^{-[m,n]} < \alpha + 2^{-m} \ ;$$

therefore, given $M \in \mathbf{N}$,

$$\alpha < \frac{[m,n] \cdot \log[m,n]}{\log \frac{1}{|c^x_{[m,n]}|}} \leq \alpha + 2^{-M}$$

for every natural numbers m, n apart from the values $0 \leq n < n_m$, which proves that $\rho(f_x) = \alpha$.

To prove that B_α is $\mathbf{\Pi}^0_5$-hard, as before, it is enough to show that a set which is known to be $\mathbf{\Pi}^0_5$-hard is Wadge reducible to B_α. Since $S_4 = \{x \in 2^{\mathbf{N}\times\mathbf{N}} : \forall^\infty m \forall^\infty n(x(m,n) = 0)\}$ is $\mathbf{\Sigma}^0_4$-complete (see, for example, [5] on page 181), it is not difficult to prove that so is $S_4^* = \{x \in 2^{\mathbf{N}} : \forall^\infty m \forall^\infty n(x([m,n]) = 0)\}$. Indeed, we need only remark that the mapping $2^{\mathbf{N}} \ni x \mapsto (x([m,n]))_{(m,n)\in\mathbf{N}\times\mathbf{N}} \in 2^{\mathbf{N}\times\mathbf{N}}$ constitutes a homeomorphism whose inverse is $2^{\mathbf{N}\times\mathbf{N}} \ni x \mapsto (x((n)_0,(n)_1))_{n\in\mathbf{N}} \in 2^{\mathbf{N}}$. Therefore, $P_5^* = \{x \in 2^{\mathbf{N}\times\mathbf{N}} : \forall l(x_l \in S_4^*)\}$ is $\mathbf{\Pi}^0_5$-complete (see, for example, [5] on page 180), and what we have to show is that $P_5^* = \{x \in 2^{\mathbf{N}\times\mathbf{N}} : \forall l \forall^\infty m \forall^\infty n(x(l,[m,n]) = 0)\}$ is Wadge reducible to B_α. So let

$$c^x_{[l,m],n} = \begin{cases} \frac{1}{n^{n\cdot\phi_x(l,[m,n])^{-1}}} & , \text{if } n \in \mathbf{N}\setminus\{0\} \\ 0 & , \text{if } n = 0 \end{cases} ,$$

where

$$\phi_x(l,[m,n]) = \begin{cases} \alpha + 2^{-l} + n^{-1} & , \text{if } x(l,[m,n]) = 1 \text{ and } n \in \mathbf{N}\setminus\{0\} \\ \alpha + n^{-1} & , \text{if } x(l,[m,n]) = 0 \text{ and } n \in \mathbf{N}\setminus\{0\} \end{cases} ,$$

whenever $x \in 2^{\mathbf{N}\times\mathbf{N}}$ and $l, m \in \mathbf{N}$. Given any natural numbers l, m, it is not difficult to prove that for any $x \in 2^{\mathbf{N}\times\mathbf{N}}$, $|c^x_{[l,m],n}|^{\frac{1}{n}} \to 0$ as $n \to \infty$; therefore, by setting $f^x_{[l,m]}(z) = \sum\limits_{n=0}^{\infty} c^x_{[l,m],n} z^n$ $(z \in \mathbf{C})$, we obtain an entire function (see, for example, [3] on page 118) and what we want to show is that the mapping $2^{\mathbf{N}\times\mathbf{N}} \ni x \mapsto f^x_{[l,m]} \in H(\mathbf{C})$ is continuous.

Indeed, we need only remark that for any integer $N \geq 2^l$, if $x, y \in 2^{\mathbf{N}\times\mathbf{N}}$ and $x(l,[m,n]) = y(l,[m,n])$, whenever $0 \leq n < N$, then since for any $u \in 2^{\mathbf{N}\times\mathbf{N}}$ and for any integer $n > N$, $|c^u_{[l,m],n}| \leq \frac{1}{n^{\frac{n}{\alpha+2^{-l}+1}}}$, it follows that

$$\max_{|z|\leq R} |f^x_{[l,m]}(z) - f^y_{[l,m]}(z)| \leq 2 \cdot \sum_{n>N} \frac{R^n}{n^{\frac{n}{\alpha+2^{-l}+1}}} ,$$

where $\sum\limits_{n>N} \frac{R^n}{n^{\frac{n}{\alpha+2^{-l}+1}}} \to 0$ as $N \to \infty$, whenever $R > 0$. Therefore, the definition of the product topology is easily seen to imply that the mapping $2^{\mathbf{N}\times\mathbf{N}} \ni x \mapsto (f^x_k)_{k\in\mathbf{N}} \in H(\mathbf{C})^{\mathbf{N}}$ is continuous and what is left to show is that for any $x \in 2^{\mathbf{N}\times\mathbf{N}}$, $x \in P^*_5 \iff (f^x_k)_{k\in\mathbf{N}} \in B_\alpha$, i.e., $x \in P^*_5 \iff \lim\limits_{k\to\infty} \rho(f^x_k) = \alpha$.

If $x \notin P^*_5$, then there exists $l \in \mathbf{N}$ and natural numbers $m_0 < m_1 < \ldots$ such that for any index i, $\exists^\infty n(x(l,[m,n]) = 1)$, which implies that

$$\exists^\infty n \left(\frac{n \cdot \log n}{\log \frac{1}{|c^x_{[l,m_i],n}|}} = \alpha + 2^{-l} + n^{-1} \right)$$

and consequently

$$\rho(f^x_{[l,m_i]}) = \limsup_{n\to\infty} \frac{n \cdot \log n}{\log \frac{1}{|c^x_{[l,m_i],n}|}} \geq \alpha + 2^{-l} ,$$

whenever $i \in \mathbf{N}$, which implies in its turn that the sequence $(\rho(f^x_k))_{k\in\mathbf{N}}$ does not converge to α.

So let $x \in P^*_5$ and let $l \in \mathbf{N}$. Then there exists $m_l \in \mathbf{N}$ such that for any integer $m \geq m_l$, $\forall^\infty n(x(l,[m,n]) = 0)$, which implies that

$$\forall^\infty n \left(\frac{n \cdot \log n}{\log \frac{1}{|c^x_{[l,m],n}|}} = \alpha + n^{-1} \right)$$

and consequently

$$\rho(f^x_{[l,m]}) = \limsup_{n\to\infty} \frac{n \cdot \log n}{\log \frac{1}{|c^x_{[l,m],n}|}} = \lim_{n\to\infty} (\alpha + n^{-1}) = \alpha \ ,$$

while if $0 \leq m < m_l$ and $n \in \mathbf{N} \setminus \{0\}$, then

$$\frac{n \cdot \log n}{\log \frac{1}{|c^x_{[l,m],n}|}} = \begin{cases} \alpha + 2^{-l} + n^{-1} & , \text{if } x(l,[m,n]) = 1 \\ \\ \alpha + n^{-1} & , \text{if } x(l,[m,n]) = 0 \end{cases} ,$$

which is easily seen to imply that

$$\alpha \leq \rho(f^x_{[l,m]}) = \limsup_{n\to\infty} \frac{n \cdot \log n}{\log \frac{1}{|c^x_{[l,m],n}|}} \leq \alpha + 2^{-l} \ .$$

Therefore, given $N \in \mathbf{N}$, $\alpha \leq \rho(f^x_{[l,m]}) \leq \alpha + 2^{-N}$, for any natural numbers l, m, apart from the values $0 \leq m < m_l$, $0 \leq l < N$, which proves that $\lim\limits_{k\to\infty} \rho(f^x_k) = \alpha$.
Οπερ εδει δειξαι

Corollary 2.2: *The order of an entire function is a Baire class two function which is not Baire class one.*

Proof: Since for any particular $0 \leq \alpha < \infty$, $\rho^{-1}[\{\alpha\}] = \{f \in H(\mathbf{C}) : \rho(f) = \alpha\}$ is $\mathbf{\Pi}^0_3$-complete and therefore not $\mathbf{\Pi}^0_2$, ρ is not Baire class one, while since if $f \in H(\mathbf{C})$ and $c_n = \frac{f^{(n)}(0)}{n!}$ for every $n \in \mathbf{N}$, then

$$\rho(f) = \lim_{n\to\infty} \sup_{m\geq n} \frac{m \cdot \log m}{\log \frac{1}{|c_m|}} \ ,$$

in order to prove that ρ is Baire class two, it is enough to prove that for any $n \in \mathbf{N}$,

$$s_n : H(\mathbf{C}) \ni f \mapsto \sup_{m\geq n} \frac{m \cdot \log m}{\log \frac{m!}{|f^{(m)}(0)|}} \in [0,\infty]$$

is Baire class one.

But if $U \subset [0,\infty]$ is non-empty open and $\infty \notin U$, then given $f \in H(\mathbf{C})$, $s_n(f) \in U$ if and only if

$$\exists r,s \in \mathbf{Q}\left((r-s,r+s) \subseteq U \ \wedge\ \forall m \geq n\left(\frac{m \cdot \log m}{\log \frac{m!}{|f^{(m)}(0)|}} \leq r+s\right)\right.$$

$$\left.\wedge\ \exists m \geq n\left(\frac{m \cdot \log m}{\log \frac{m!}{|f^{(m)}(0)|}} \geq r-s\right)\right)$$

and consequently $\{f \in H(\mathbf{C}) : s_n(f) \in U\}$ is $\mathbf{\Sigma}^0_2$. If $\infty \in U$, then $U = V \cup (\alpha,\infty]$, where $V \subset [0,\alpha)$ is non-empty open and $0 < \alpha < \infty$, hence

$$s_n(f) \in U \iff \left(s_n(f) \in V\ \vee\ \exists m \geq n\left(\frac{m \cdot \log m}{\log \frac{m!}{|f^{(m)}(0)|}} > \alpha\right)\right) ,$$

which implies that $\{f \in H(\mathbf{C}) : s_n(f) \in U\}$ is $\mathbf{\Sigma}^0_2$ and consequently s_n is Baire class one.

Οπερ εδει δειξαι

It is not difficult to prove that $A_\infty = \{f \in H(\mathbf{C}) : \rho(f) = \infty\}$ is $\mathbf{\Pi}^0_2$-complete and a straightforward computation shows that $B_\infty = \{(f_k)_{k\in\mathbf{N}} \in H(\mathbf{C})^{\mathbf{N}} : \lim_{k\to\infty} \rho(f_k) = \infty\}$ is $\mathbf{\Pi}^0_4$.

Open Problem: Is B_∞ $\mathbf{\Pi}^0_4$-complete?

3. Trees and functions in L^1

Let $1 \leq n \leq \aleph_0$ and let $-\infty < \alpha < \beta < \infty$. For any $i < n$ we set $I^{(i)}_{n;\emptyset} = \left[\alpha + i\frac{\beta-\alpha}{n}, \alpha + (i+1)\frac{\beta-\alpha}{n}\right]$, if $n < \aleph_0$, and $I^{(i)}_{\aleph_0;\emptyset} = \left[\alpha + \sum_{j=1}^{i} \frac{\beta-\alpha}{2^j}, \alpha + \sum_{j=1}^{i+1} \frac{\beta-\alpha}{2^j}\right]$, and if $s \in \mathbf{N}^{<\mathbf{N}}$ is such that the $I^{(i)}_{n;s}$'s are already defined, then for any $i < n$, we define the $I^{(i)}_{n;s\frown k}$'s, as follows: If $I^{(i)}_{n;s} = [a, b]$, then for any $k \in \mathbf{N}$, $I^{(i)}_{n;s\frown k} = \left[a + \sum_{j=1}^{2k+1} \frac{b-a}{2^j}, a + \sum_{j=1}^{2k+2} \frac{b-a}{2^j}\right]$. So let T be any tree on $\mathbf{N}$ and let $\phi_{n;T} = \prod_{i<n}\left(1 - \sum_{s\in T\setminus\{\emptyset\}} 2^{-\mathrm{length}(s)}\chi_{I^{(i)}_{n;s}}\right)$ and $\kappa_{n;T}(x) = \int_\alpha^x \phi_{n;T}(t)dt$ $(x \in [\alpha, \beta])$.

Theorem 3.1: *Given* $1 \leq n \leq \aleph_0$ *and given any tree* T *on* $\mathbf{N}$,

$$\text{if } (\kappa_{n;T})'_-(\beta) \text{ exists, then } (\kappa_{n;T})'_-(\beta) \geq \frac{1}{6}\ ,$$

$$T \in IF \Rightarrow (\forall i < n)(\exists \alpha_i \in \mathrm{Int}(I^{(i)}_{n;\emptyset}))(\kappa'_{n;T}(\alpha_i) = 0)$$

and

$$T \in WF \Rightarrow (\forall x \in [\alpha, \beta))((\kappa_{n;T})'_+(x) > 0)\ ,$$

while both mappings

$$Tr \ni T \mapsto \phi_{n;T} \in L^1([\alpha, \beta])$$

and

$$Tr \ni T \mapsto \kappa_{n;T} \in C([\alpha, \beta], \mathbf{R})$$

are well-defined and continuous.

Proof: We fix a tree T on $\mathbf{N}$ and for convenience we set $\phi^{(i)}_{n;T} = 1 - \sum_{s\in T\setminus\{\emptyset\}} 2^{-\mathrm{length}(s)}\chi_{I^{(i)}_{n;s}}$, whenever $i < n$. We remark that if $i < n$, $j < n$

and $i \neq j$, then $\phi^{(i)}_{n;T} = 1$ on $I^{(j)}_{n;\emptyset}$, which implies that $\phi_{n;T} = \phi^{(i)}_{n;T}$ on $I^{(i)}_{n;\emptyset}$, while our construction implies that for any $x \in I^{(i)}_{n;\emptyset}$ and for any $s \in \mathbf{N}^{<\mathbf{N}}$, there exists at most one $k \in \mathbf{N}$ for which $x \in I^{(i)}_{n;s\frown k}$, which implies that $\phi^{(i)}_{n;T}(x) \geq 1 - \sum\limits_{k=1}^{\infty} 2^{-k} = 0$ and $\phi^{(i)}_{n;T}(x) = 0$ iff there exists $\alpha \in [T]$ such that $x \in \bigcap_{k\in\mathbf{N}} I^{(i)}_{n;\alpha|k}$. In addition, for any positive integer i,

$$\frac{\kappa_{n;T}(\beta) - \kappa_{n;T}\left(\alpha + \sum\limits_{j=1}^{2i} \frac{\beta-\alpha}{2^j}\right)}{\beta - \left(\alpha + \sum\limits_{j=1}^{2i} \frac{\beta-\alpha}{2^j}\right)}$$

$$= \frac{1}{\beta - \left(\alpha + \sum\limits_{j=1}^{2i} \frac{\beta-\alpha}{2^j}\right)} \int\limits_{\alpha + \sum\limits_{j=1}^{2i} \frac{\beta-\alpha}{2^j}}^{\beta} \phi_{n;T}(t)dt$$

$$\geq \frac{1}{\beta - \left(\alpha + \sum\limits_{j=1}^{2i} \frac{\beta-\alpha}{2^j}\right)} \sum_{k=1}^{\infty} \frac{\beta - \alpha}{2^{2i+2k+1}} = \frac{1}{6},$$

which implies that if $(\kappa_{n;T})'_{-}(\beta)$ exists, then $(\kappa_{n;T})'_{-}(\beta) \geq \frac{1}{6}$.

So let $T \in WF$ and let $i < n$, while $x \in I^{(i)}_{n;\emptyset}$. Then there exists $s \in T$ of maximum length such that $x \in I^{(i)}_{n;s}$ and maximality implies that $x \in I^{(i)}_{n;s} \setminus \bigcup_{\nu\in\mathbf{N};s\frown\nu\in T} I^{(i)}_{n;s\frown\nu}$. If x is either the left endpoint or lies in the interior of $I^{(i)}_{n;s}$, then there exists $\epsilon > 0$ such that for $x < y < x+\epsilon$, $\phi^{(i)}_{n;T}(y) = 1 - \sum\limits_{k=1}^{\text{length}(s)} 2^{-k}$, which implies that $(\kappa_{n;T})'_{+}(x) = 1 - \sum\limits_{k=1}^{\text{length}(s)} 2^{-k}$. So let x be the right endpoint of $I^{(i)}_{n;s}$. If $s \neq \emptyset$, then there exists $\epsilon > 0$ such that for $x < y < x + \epsilon$, $\phi^{(i)}_{n;T}(y) = 1 - \sum\limits_{k=1}^{\text{length}(s)-1} 2^{-k}$, which implies that $(\kappa_{n;T})'_{+}(x) = 1 - \sum\limits_{k=1}^{\text{length}(s)-1} 2^{-k}$, while if $s = \emptyset$ and $i+1 < n$, then there exists $\epsilon > 0$ such that for $x < y < x+\epsilon$, $\phi^{(i+1)}_{n;T}(y) = 1$, which implies that $(\kappa_{n;T})'_{+}(x) = 1$. We have thus proved that $T \in WF \Rightarrow (\forall x \in [\alpha, \beta))((\kappa_{n;T})'_{+}(x) > 0)$.

So let $T \in IF$ and let $\alpha \in [T]$. If $i < n$, and α_i is the unique point contained in $\bigcap_{k\in\mathbf{N}} I^{(i)}_{n;\alpha|k}$, then we claim that $\kappa'_{n;T}(\alpha_i) = 0$. Indeed, if $k \in$

$\mathbf{N}$ and x, y are in $I^{(i)}_{n;\alpha|k}$, then $|\phi_{n;T}(x)-\phi_{n;T}(y)| = |\phi^{(i)}_{n;T}(x)-\phi^{(i)}_{n;T}(y)| \leq \sum_{s\in T\setminus\{\emptyset\}} 2^{-\mathrm{length}(s)}|\chi_{I^{(i)}_{n;s}}(x)-\chi_{I^{(i)}_{n;s}}(y)| \leq 2\sum_{j>k} 2^{-j} = 2^{-k+1}$ and hence if $x \neq \alpha_i$ lies in the interior of $I^{(i)}_{n;\alpha|k}$, while I stands for the interval defined by x and α_i, then, as $\phi^{(i)}_{n;T}(\alpha_i) = 0 \Rightarrow \phi_{n;T}(\alpha_i) = 0$, we obtain that $|\kappa_{n;T}(x)-\kappa_{n;T}(\alpha_i)| = \int_I \phi_{n;T}(t)dt = \int_I(\phi_{n;T}(t)-\phi_{n;T}(\alpha_i))dt \leq 2^{-k+1}\cdot|x-\alpha_i| \Rightarrow \left|\frac{\kappa_{n;T}(x)-\kappa_{n;T}(\alpha_i)}{x-\alpha_i}\right| \leq 2^{-k+1}$, and the claim follows. We have thus proved that $T \in IF \Rightarrow (\forall i < n)(\exists\alpha_i \in \mathrm{Int}(I^{(i)}_{n;\emptyset}))(\kappa'_{n;T}(\alpha_i) = 0)$.

What is left to show is that the mapping $Tr \ni T \mapsto \phi_{n;T} \in L^1([\alpha,\beta])$ is continuous, as the continuity of the mapping $Tr \ni T \mapsto \kappa_{n;T} \in C([\alpha,\beta],\mathbf{R})$ will then follow. (Indeed, it is enough to notice that for any f, g in $L^1([\alpha,\beta])$ and for any $x \in [\alpha,\beta]$, $|\int_\alpha^x f(t)dt - \int_\alpha^x g(t)dt| \leq \int_\alpha^\beta |f(t)-g(t)|dt$.) Given $i < n$, $s \in \mathbf{N}^{<\mathbf{N}}$ and $k \in \mathbf{N}$, it is not difficult to see that $m(I^{(i)}_{\aleph_0;\emptyset}) = \frac{\beta-\alpha}{2^{i+1}}$ and $m(I^{(i)}_{n;s\frown k}) = \frac{m(I^{(i)}_{n;s})}{4^{k+1}}$, while given T, T' in Tr, $\int_\alpha^\beta |\phi_{n;T'}(x)-\phi_{n;T}(x)|dx = \sum_{i<n}\int_{I^{(i)}_{n;\emptyset}} |\phi_{n;T'}(x)-\phi_{n;T}(x)|dx = \sum_{i<n}\int_{I^{(i)}_{n;\emptyset}} |\phi^{(i)}_{n;T'}(x)-\phi^{(i)}_{n;T}(x)|dx$, where for any $i < n$ and for any $x \in I^{(i)}_{n;\emptyset}$, $\phi^{(i)}_{n;T'}(x)-\phi^{(i)}_{n;T}(x) = \sum_{s\in T\setminus\{\emptyset\}} 2^{-\mathrm{length}(s)}\chi_{I^{(i)}_{n;s}}(x) - \sum_{s\in T'\setminus\{\emptyset\}} 2^{-\mathrm{length}(s)}\chi_{I^{(i)}_{n;s}}(x)$, which implies that $|\phi^{(i)}_{n;T'}-\phi^{(i)}_{n;T}| \leq 1$ and $\phi^{(i)}_{n;T'}-\phi^{(i)}_{n;T}$ vanishes on $I^{(i)}_{n;s} \setminus \cup_{k\in\mathbf{N}} I^{(i)}_{n;s\frown k}$ for every $s \in \mathbf{N}^{<\mathbf{N}}$; therefore, for any $i < n$ and for any $s \in \mathbf{N}^{<\mathbf{N}}$, $\int_{I^{(i)}_{n;s}} |\phi^{(i)}_{n;T'}(x)-\phi^{(i)}_{n;T}(x)|dx = \sum_{k=0}^{\infty}\int_{I^{(i)}_{n;s\frown k}} |\phi^{(i)}_{n;T'}(x)-\phi^{(i)}_{n;T}(x)|dx$, where for any $k \in \mathbf{N}$, $\int_{I^{(i)}_{n;s\frown k}} |\phi^{(i)}_{n;T'}(x)-\phi^{(i)}_{n;T}(x)|dx \leq m(I^{(i)}_{n;s\frown k}) = \frac{m(I^{(i)}_{n;s})}{4^{k+1}}$.

So let $T \in Tr$ be arbitrary but fixed and given $N \in \mathbf{N}\setminus\{0\}$, let $V_{T;N} = \{T' \in Tr : (\forall s \in \{0,1,...,N-1\}^{<N+1})(s \in T' \iff s \in T)\}$. It is not difficult to see that the $V_{T;N}$'s form a fundamental system of open neighborhoods of T in Tr. So let $N \in \mathbf{N}\setminus\{0\}$ be arbitrary but fixed and let $T' \in V_{T;N}$. Then, for any $i < n$ and for any $s \in \mathbf{N}^{<\mathbf{N}}$, $\int_{I^{(i)}_{n;s}} |\phi^{(i)}_{n;T'}(x)-\phi^{(i)}_{n;T}(x)|dx \leq$

$\sum_{k=0}^{N-1} \int_{I^{(i)}_{n;s}} |\phi^{(i)}_{n;T'}(x)-\phi^{(i)}_{n;T}(x)|dx + \sum_{k=N}^{\infty} \frac{m(I^{(i)}_{n;s})}{4^{k+1}} = \sum_{k=0}^{N-1} \int_{I^{(i)}_{n;s}} |\phi^{(i)}_{n;T'}(x)-\phi^{(i)}_{n;T}(x)|dx + m(I^{(i)}_{n;s})\frac{1}{3\cdot 4^N}$ and hence we obtain that

$$\int_{I^{(i)}_{n;\emptyset}} |\phi^{(i)}_{n;T'}(x)-\phi^{(i)}_{n;T}(x)|dx$$

$$\leq \sum_{k_0=0}^{N-1}\left(\sum_{k_1=0}^{N-1}\left(\cdots\left(\sum_{k_{N-1}=0}^{N-1}\int_{I^{(i)}_{n;(k_0,k_1,\ldots,k_{N-1})}} |\phi^{(i)}_{n;T'}(x)-\phi^{(i)}_{n;T}(x)|dx \; + \right.\right.\right.$$

$$\left.\left.\left. m(I^{(i)}_{n;\emptyset})\frac{1}{4^{k_0+1}}\cdots\frac{1}{4^{k_{N-2}+1}}\frac{1}{3\cdot 4^N}\right)\cdots\right) + m(I^{(i)}_{n;\emptyset})\frac{1}{4^{k_0+1}}\frac{1}{3\cdot 4^N}\right) + m(I^{(i)}_{n;\emptyset})\frac{1}{3\cdot 4^N} \; .$$

So let $(k_0,k_1,\ldots,k_{N-1}) \in \{0,1,\ldots,N-1\}^N$ be arbitrary but fixed. Then, given $s \in \mathbf{N}^{<\mathbf{N}}$, $s \perp (k_0,k_1,\ldots,k_{N-1}) \Rightarrow (\forall x \in I^{(i)}_{n;(k_0,k_1,\ldots,k_{N-1})})(\chi_{I^{(i)}_{n;s}}(x) = 0)$ and $s \subseteq (k_0,k_1,\ldots,k_{N-1}) \Rightarrow (s \in T \iff s \in T')$. Therefore, for any $x \in I^{(i)}_{n;(k_0,k_1,\ldots,k_{N-1})}$, $-\frac{1}{2^N} = -\sum_{k>N} 2^{-k} \leq \phi^{(i)}_{n;T'}(x)-\phi^{(i)}_{n;T}(x) \leq \sum_{k>N} 2^{-k} = \frac{1}{2^N}$, which implies that $\int_{I^{(i)}_{n;(k_0,k_1,\ldots,k_{N-1})}} |\phi^{(i)}_{n;T'}(x)-\phi^{(i)}_{n;T}(x)|dx \leq \frac{1}{2^N} m(I^{(i)}_{n;\emptyset})\frac{1}{4^{k_0+1}}\cdots\frac{1}{4^{k_{N-1}+1}}$ and as $\sum_{k=0}^{N-1}\frac{1}{4^{k+1}} = \frac{1}{3}\left(1-\frac{1}{4^N}\right)$ we obtain that

$$\int_{I^{(i)}_{n;\emptyset}} |\phi^{(i)}_{n;T'}(x)-\phi^{(i)}_{n;T}(x)|dx$$

$$\leq \sum_{k_0=0}^{N-1}\left(\sum_{k_1=0}^{N-1}\left(\cdots\left(\sum_{k_{N-1}=0}^{N-1}\frac{1}{2^N}m(I^{(i)}_{n;\emptyset})\frac{1}{4^{k_0+1}}\cdots\frac{1}{4^{k_{N-1}+1}} \; + \right.\right.\right.$$

$$\left.\left.\left. m(I^{(i)}_{n;\emptyset})\frac{1}{4^{k_0+1}}\cdots\frac{1}{4^{k_{N-2}+1}}\frac{1}{3\cdot 4^N}\right)\cdots\right) + m(I^{(i)}_{n;\emptyset})\frac{1}{4^{k_0+1}}\frac{1}{3\cdot 4^N}\right) + m(I^{(i)}_{n;\emptyset})\frac{1}{3\cdot 4^N}$$

$$\leq \frac{1}{2^N}m(I^{(i)}_{n;\emptyset})\cdot\left[\sum_{k_0=0}^{N-1}\left(\sum_{k_1=0}^{N-1}\left(\cdots\left(\sum_{k_{N-1}=0}^{N-1}\frac{1}{4^{k_0+1}}\cdots\frac{1}{4^{k_{N-1}+1}} \; + \right.\right.\right.\right.$$

$$\left.\left.\left.\left.\frac{1}{4^{k_0+1}}\cdots\frac{1}{4^{k_{N-2}+1}}\right)\cdots\right) + \frac{1}{4^{k_0+1}}\right) + 1\right]$$

$$= \frac{1}{2^N}m(I^{(i)}_{n;\emptyset})\cdot\sum_{k=0}^{N}\left(\frac{1}{3}\left(1-\frac{1}{4^N}\right)\right)^k$$

$$< \frac{1}{2^N} m(I_{n;\emptyset}^{(i)}) \cdot \sum_{k=0}^{\infty} \left(\frac{1}{3} \left(1 - \frac{1}{4^N} \right) \right)^k \leq \frac{3}{2^{N-1}} m(I_{n;\emptyset}^{(i)})$$

for every $i < n$, which implies that $\int_\alpha^\beta |\phi_{n;T'}(x) - \phi_{n;T}(x)| dx \leq 3(\beta - \alpha) 2^{-N+1}$ for every $T' \in V_{T;N}$.

$O\pi\epsilon\rho \ \epsilon\delta\epsilon\iota \ \delta\epsilon\iota\xi\alpha\iota$

4. Analytic sets and tangents of continuous curves in the plane

A continuous path in the plane is a continuous mapping sending the interval $[0,1]$ into the Euclidean plane E^2. Therefore, we may view the Polish space $C([0,1],E^2)$ as the family of all continuous paths in the plane.

Theorem 4.1: *Given any line in the plane and any cardinal number $1 \leq n \leq \aleph_0$, the set of continuous paths in the plane tracing curves which admit at least n tangents parallel to the given line is $\mathbf{\Sigma}_1^1$-complete.*

Proof: Once $1 \leq n \leq \aleph_0$ is given, by choosing appropriately a coordinate system in the plane, it is enough to prove that the set of continuous paths in $\mathbf{R}^2$ tracing curves which admit at least n tangents parallel to the real line is $\mathbf{\Sigma}_1^1$-complete.

The fact that the set in question is $\mathbf{\Sigma}_1^1$-hard follows immediately from Theorem 3.1. Indeed, we need only consider for $\alpha = 0$ and $\beta = 1$ the mapping that assigns to every tree T on $\mathbf{N}$ the continuous path $t \mapsto (t, \kappa_{n;T}(t))$ ($t \in [0,1]$) in $\mathbf{R}^2$. Thus, what is left to show is that the set in question is actually $\mathbf{\Sigma}_1^1$, in case $n < \aleph_0$.

But this follows from the fact that given any $(x,y) \in C([0,1],\mathbf{R}^2)$, (x,y) traces a curve which has at least n tangents parallel to the real line iff there exists $(a_1,...,a_n,b_1,...,b_n) \in [0,1]^n \times \mathbf{R}^n$ with the properties

$$a_1 < ... < a_n \ ,$$

$$(\forall i \in \{1,...,n\})(b_i \neq 0) \text{ and}$$

$$(\forall i \in \{1,...,n\})(\forall \epsilon \in \mathbf{Q}_+^*)(\exists \delta \in \mathbf{Q}_+^*)(\forall r \in [0,1] \cap \mathbf{Q})$$

$$\left(0 < |r - a_i| < \delta \Rightarrow \left(\left| \frac{x(r) - x(a_i)}{r - a_i} - b_i \right| \leq \epsilon \wedge \left| \frac{y(r) - y(a_i)}{r - a_i} \right| \leq \epsilon \right)\right).$$

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Theorem 4.2: *Given any positive integer N and any cardinal number $1 \leq n \leq \aleph_0$, if $-\infty < \alpha < \beta < \infty$, then the set of all functions in $C([\alpha, \beta]^N, \mathbf{R})$ whose graph in $\mathbf{R}^{N+1}$ admits at least n tangent N-dimensional hyperplanes parallel to $\mathbf{R}^N$ is $\mathbf{\Sigma}_1^1$-complete.*

Proof: We will first prove that the set in question is $\mathbf{\Sigma}_1^1$-hard. To this end we consider the mapping that assigns to every tree T on $\mathbf{N}$ the continuous function $f_T : [\alpha, \beta]^N \ni (x_1, ..., x_N) \mapsto \kappa_{n;T}(x_1) + ... + \kappa_{n;T}(x_N) \in \mathbf{R}$. We remark that given $(a_1, ..., a_N) \in [\alpha, \beta]^N$, the graph of f_T in $\mathbf{R}^{N+1}$ admits a tangent N-dimensional hyperplane at the point $(a_1, ..., a_N, f_T(a_1, ..., a_N))$ iff f_T is differentiable at the point $(a_1, ..., a_N)$ or (equivalently) $\kappa_{n;T}$ is differentiable at the points $a_1, ..., a_N$, while the tangent N-dimensional hyperplane in question, if it exists, is perpendicular to the vector $(-\nabla f_T(a_1, ..., a_N), 1) = (-\kappa'_{n;T}(a_1), ..., -\kappa'_{n;T}(a_N), 1)$ and consequently it is parallel to $\mathbf{R}^N$ iff $\kappa'_{n;T}(a_1) = ... = \kappa'_{n;T}(a_N) = 0$. Therefore, an application of Theorem 3.1 shows that the set in question is $\mathbf{\Sigma}_1^1$-hard and what is left to show is that it is actually $\mathbf{\Sigma}_1^1$, in case $n < \aleph_0$.

But again this follows from the fact that given $f \in C([\alpha, \beta]^N, \mathbf{R})$, the graph of f in $\mathbf{R}^{N+1}$ admits at least n tangent N-dimensional hyperplanes parallel to $\mathbf{R}^N$ iff there exists $(\mathbf{a}^1, ..., \mathbf{a}^n) \in ([\alpha, \beta]^N)^n$ with the properties

$$1 \leq i < j \leq n \Rightarrow \mathbf{a}^i \neq \mathbf{a}^j \text{ and}$$

$$(\forall (i, \nu) \in \{1, ..., n\} \times \{1, ..., N\})(\forall \epsilon \in \mathbf{Q}_+^*)(\exists \delta \in \mathbf{Q}_+^*)(\forall r \in [\alpha, \beta] \cap \mathbf{Q})$$

$$\left(0 < |r - a^i_\nu| < \delta \Rightarrow \left|\frac{f(\mathbf{a}^i + (r - a^i_\nu)\mathbf{e}_\nu) - f(\mathbf{a}^i)}{r - a^i_\nu}\right| \le \epsilon\right) ,$$

where $\mathbf{e}_1, ..., \mathbf{e}_N$ denote the standard basis vectors in $\mathbf{R}^N$.

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5. Analytic sets and vertices of differentiable curves in the plane

By analogy to continuous paths in the plane we can define differentiable ones of any class. At this juncture we will restrict ourselves to the case when the differentiability class is C^2 or C^3 and identifying E^2 by $\mathbf{R}^2$, say by choosing a coordinate system, we view the Polish space

$$\mathcal{P}^2 = \left\{ (x,y) \in C^2([0,1],\mathbf{R}^2) : \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \right\}$$

as the family of all differentiable paths of class C^2 in the plane which admit a canonical parameter in $[0,1]$ and the Polish space

$$\mathcal{P}^3 = \left\{ (x,y) \in C^3([0,1],\mathbf{R}^2) : \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \right\}$$

as the family of all differentiable paths of class C^3 in the plane which admit a canonical parameter in $[0,1]$, since we are interested in the notion of curvature of a curve. Thus, if (x,y) is any path in $\mathcal{P}^2$ or $\mathcal{P}^3$, then the curvature κ of the curve traced by (x,y) is given by the formula $\kappa = \frac{dx}{ds} \cdot \frac{d^2y}{ds^2} - \frac{dy}{ds} \cdot \frac{d^2x}{ds^2}$ and depends at least continuously on the canonical parameter $s \in [0,1]$, while a point A on the curve traced by (x,y) is called a vertex if $\left(\frac{d\kappa}{ds}\right)_A = 0$, i.e., $\left(\frac{d\kappa}{ds}\right)_{s=a} = 0$, where a is the value of the canonical parameter for which $(x(a),y(a))$ constitutes the pair of Cartesian coordinates of the point A in the Euclidean plane E^2 (see, for example, [11] on page 26).

Theorem 5.1: *For any cardinal number $1 \leq n \leq \aleph_0$, the set of differentiable paths of class C^2 in the plane admitting a canonical parameter in $[0,1]$ and tracing curves which have at least n vertices is $\mathbf{\Sigma}^1_1$-complete, while the set of differentiable paths of class C^3 in the plane admitting a canonical* *parameter in* $[0,1]$ *and tracing curves which have at least* n *vertices is* $\mathbf{\Sigma}_2^0$ *if* $n < \aleph_0$ *and* $\mathbf{\Pi}_3^0$ *if* $n = \aleph_0$.

Proof: We will first prove that the set of differentiable paths of class C^2 in the plane admitting a canonical parameter in $[0,1]$ and tracing curves which have at least n vertices is $\mathbf{\Sigma}_1^1$-hard. To this end we consider for $\alpha = 0$ and $\beta = 1$ the mapping that assigns to every tree T on $\mathbf{N}$ the path in $\mathcal{P}^2$ defined, as follows: $x_{n;T}(s) = \int_0^s \cos(\psi_{n;T}(\xi))d\xi$ and $y_{n;T}(s) = \int_0^s \sin(\psi_{n;T}(\xi))d\xi$ for every $s \in [0,1]$, where $\psi_{n;T}(s) = \int_0^s \kappa_{n;T}(\xi)d\xi$ for every $s \in [0,1]$. It is not difficult to verify that the mapping $Tr \ni T \mapsto (x_{n;T}, y_{n;T}) \in \mathcal{P}^2$ is well-defined and given any $T \in Tr$, $T \in IF$ iff $(x_{n;T}, y_{n;T})$ traces a curve having at least n vertices; this follows from Theorem 3.1 and the fact that for any $T \in Tr$, the curvature of the curve traced by $(x_{n;T}, y_{n;T})$ is given by the function $\kappa_{n;T}$, as it follows from the proof of the theorem on the existence of a plane curve with given curvature (see, for example, [11] on page 27). What we need to show is that the mapping $Tr \ni T \mapsto (x_{n;T}, y_{n;T}) \in \mathcal{P}^2$ is continuous. By virtue of Theorem 3.1, if ϕ is either the identity, the sine or the cosine function, it is enough to show that the mappings $\Phi_1 : C([0,1],\mathbf{R}) \to C^1([0,1],\mathbf{R})$ and $\Phi_2 : C^1([0,1],\mathbf{R}) \to C^2([0,1],\mathbf{R})$, defined by the relations $\Phi_1(f)(x) = \int_0^x \phi(f(t))dt$ $(x \in [0,1]; f \in C([0,1],\mathbf{R}))$ and $\Phi_2(f)(x) = \int_0^x \phi(f(t))dt$ $(x \in [0,1]; f \in C^1([0,1],\mathbf{R}))$, are continuous.

The proof of the continuity of Φ_1 is left to the reader and since, for complete metric spaces, uniform convergence on compacts is equivalent to continuous convergence (see, for example, [10] on page 162), if $f_k \to f$ in $C^1([0,1],\mathbf{R})$ and $x_k \to x$ in $[0,1]$ as $k \to \infty$, it is enough to show that $\Phi_2(f_k)(x_k) \to \Phi_2(f)(x)$, $\Phi_2(f_k)'(x_k) \to \Phi_2(f)'(x)$ and $\Phi_2(f_k)''(x_k) \to \Phi_2(f)''(x)$ as $k \to \infty$. Indeed, the continuity of both ϕ and ϕ', the Lebesgue

Dominated Convergence Theorem (see, for example, Part One of [10]) and the fact that both $f_k(x_k) \to f(x)$ and $f'_k(x_k) \to f'(x)$ as $k \to \infty$ are easily seen to imply that $\Phi_2(f_k)(x_k) = \int_0^1 \phi(f_k(t))\chi_{[0,x_k]}(t)dt \to \int_0^1 \phi(f(t))\chi_{[0,x]}(t)dt = \Phi_2(f)(x)$, $\Phi_2(f_k)'(x_k) = \phi(f_k(x_k)) \to \phi(f(x)) = \Phi_2(f)'(x)$ and $\Phi_2(f_k)''(x_k) = \phi'(f_k(x_k)) \cdot f'_k(x_k) \to \phi'(f(x)) \cdot f'(x) = \Phi_2(f)''(x)$ as $k \to \infty$.

Our next step is to show that the set of differentiable paths of class C^2 in the plane admitting a canonical parameter in $[0,1]$ and tracing curves which have at least n vertices is $\mathbf{\Sigma}_1^1$, in case $n < \aleph_0$.

Indeed, we need only remark that given any $(x,y) \in \mathcal{P}^2$, the curve traced by (x,y) has at least n vertices iff there exists $(a_1, ..., a_n) \in [0,1]^n$ with the properties

$$a_1 < ... < a_n \text{ and}$$

$$(\forall i \in \{1,...,n\})(\forall \epsilon \in \mathbf{Q}_+^*)(\exists \delta \in \mathbf{Q}_+^*)(\forall r \in [0,1] \cap \mathbf{Q})$$
$$\left(0 < |r - a_i| < \delta \Rightarrow \left|\frac{x'(r)y''(r) - y'(r)x''(r) - x'(a_i)y''(a_i) + y'(a_i)x''(a_i)}{r - a_i}\right| \leq \epsilon\right).$$

Finally, we will prove that the set of differentiable paths of class C^3 in the plane admitting a canonical parameter in $[0,1]$ and tracing curves which have at least n vertices is $\mathbf{\Sigma}_2^0$ if $n < \aleph_0$ and $\mathbf{\Pi}_3^0$ if $n = \aleph_0$. To this end, given any positive integer N, it is enough to prove that the set $\mathcal{C}_N = \{(x,y) \in \mathcal{P}^3 : (\exists (a_1, ..., a_n) \in [0,1]^n)(1 \leq i < j \leq n \Rightarrow |a_i - a_j| \geq N^{-1} \wedge (\forall i \in \{1,...,n\})(x'(a_i)y'''(a_i) - y'(a_i)x'''(a_i) = 0))\}$ is closed, if $n < \aleph_0$. So let $(x_k, y_k) \to (x,y)$ in $\mathcal{P}^3$ as $k \to \infty$ and let $(x_k, y_k) \in \mathcal{C}_N$, whenever $k \in \mathbf{N}$. Then, for any $k \in \mathbf{N}$, there exists $(a_1^k, ..., a_n^k) \in [0,1]^n$ such that $1 \leq i < j \leq n \Rightarrow |a_i^k - a_j^k| \geq N^{-1}$ and $x'(a_i^k)y'''(a_i^k) - y'(a_i^k)x'''(a_i^k) = 0$, whenever $1 \leq i \leq n$. The compactness of $[0,1]^n$ implies that there exists a subsequence $((a_1^{k_j}, ..., a_n^{k_j}))_{j \in \mathbf{N}}$ of $((a_1^k, ..., a_n^k))_{k \in \mathbf{N}}$ which converges to some point $(a_1, ..., a_n)$ in $[0,1]^n$, and it is not difficult to prove that $1 \leq i < j \leq$

$n \Rightarrow |a_i - a_j| \geq N^{-1}$, while, as, for complete metric spaces, uniform convergence on compacts is equivalent to continuous convergence (see, for example, [10] on page 162), we deduce that $x'(a_i)y'''(a_i) - y'(a_i)x'''(a_i) = 0$ for every $1 \leq i \leq n$ and consequently $(x, y) \in \mathcal{C}_N$.

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Open Problem: Is the set of differentiable paths of class C^3 in the plane, which admit a canonical parameter in $[0, 1]$ and trace curves having infinitely many vertices, $\mathbf{\Pi}_3^0$-complete?

References

[1] T.M. APOSTOL, *Introduction to Analytic Number Theory*, Springer-Verlag, 1976.

[2] M. BARNER und F. FLOHR, *Der Vierscheitelsatz und seine Verallgemeinerungen*, Der Mathematikunterricht, Heft 4, 43-73, 1958.

[3] E. HILLE, *Analytic Function Theory*, Volume 1, Ginn and Company, 1959.

[4] E. HILLE, *Analytic Function Theory*, Volume 2, Ginn and Company, 1962.

[5] A.S. KECHRIS, *Classical Descriptive Set Theory*, Springer-Verlag, 1995.

[6] H. KI, *Topics in Descriptive Set Theory Related to Number Theory and Analysis*, Ph.D.Thesis, Caltech (1995).

[7] W. KLINGENBERG, *A Course in Differential Geometry*, Springer-Verlag, 1978.

[8] O. NIKODYM, *Sur les points rectilineairement accessibles des ensembles plans*, Fund.Math.VII (1925), p. 257-258.

[9] O. NIKODYM et W. SIERPINSKI, *Sur un ensemble ouvert, tel que la somme de toutes les droites qu' il contient est un ensemble non mesurable (B)*, Fund.Math.VII (1925), p. 259-262.

[10] H.L. ROYDEN, *Real Analysis*, Third Edition, MacMillan Publishing Company, 1988.

[11] J.J. STOKER, *Differential Geometry*, Wiley-Interscience, 1969.

Topics in descriptive set theory related to equivalence relations, complex Borel and analytic sets

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Summary

Two of the main trends of current research in Descriptive Set Theory are the study of natural equivalence relations arising in other branches of mathematics, in the sense of determining their relative complexity under the notion of Borel reducibility, and the classification of natural sets arising in other branches of mathematics, in the sense of computing their exact complexity.

Definition: *Let X, X' be any Polish spaces and let E, E' be any equivalence relations on X, X' respectively. Then E is said to be* **Borel reducible** *to E' when there exists a Borel function $f : X \to X'$ with the property that $xEy \iff f(x)E'f(y)$, whenever x, y are in X.*

An important notion in the study of equivalence relations is the notion of generic S_∞-ergodicity, where S_∞ stands for the group of permutations of $\mathbf{N}$.

Definition: *Let X be any Polish space and let E be any equivalence relation on X. Then E is said to be* **generically S_∞-ergodic** *if every E-equivalence class is meager and for any Polish space Y and for any Baire*

measurable function $f : X \to Y^{\mathbf{N}}$ *with the property that* $xEy \Rightarrow \{f(x)(n) : n \in \mathbf{N}\} = \{f(y)(n) : n \in \mathbf{N}\}$, *whenever* x, y *are in* X, *there exist an* E*-invariant co-meager subset* A *of* X *and a countable subset* C *of* Y *such that* $x \in A \Rightarrow \{f(x)(n) : n \in \mathbf{N}\} = C$, *whenever* $x \in X$. *In particular, since by setting* $u \cong_Y v \iff \{u(n) : n \in \mathbf{N}\} = \{v(n) : n \in \mathbf{N}\}$, *whenever* u, v *are in* $Y^{\mathbf{N}}$, *we may canonically identify* $Y^{\mathbf{N}}/\cong_Y$ *with the set of all countable subsets of* Y, *generic* S_∞*-ergodicity implies that any* E*-invariants of elements of* X, *which are computed in a Baire measurable way and can be represented as countable subsets of a Polish space, must generically trivialize.*

The notion of generic S_∞-ergodicity for equivalence relations is related to the concept of generic turbulence for Polish group actions. The following definition is due to G. Hjorth.

Definition: *Let* G *be any Polish group acting continuously on a Polish space* X *and let* $x \in X$. *For any open neighborhood* U *of* x *in* X *and for any symmetric open neighborhood* V *of* 1^G *in* G, *the* (U,V)*-local orbit* $O(x,U,V)$ *of* x *in* X *is defined, as follows:* $y \in O(x,U,V)$ *if and only if there exist* $g_0, g_1, ..., g_k$ *in* V *such that if* $x_0 = x$ *and* $x_{i+1} = g_i \cdot x_i$ *for every index* $i \leq k$, *then all the* x_i*'s are in* U *and* $x_{k+1} = y$. *The action of* G *on* X *is called* **turbulent** *at the point* x, *symbolically* $x \in T_G^X$, *if for any such* U *and* V, *there exists an open neighborhood* U' *of* x *in* X *such that* $U' \subseteq U$ *and* $O(x,U,V)$ *is dense in* U'.

The concept of turbulence is a property of the orbits of the action in the sense that if G is any Polish group acting continuously on a Polish space X and E_G^X stands for the corresponding orbit equivalence relation, then T_G^X is

E_G^X-invariant, while the main result concerning the concept of turbulence is the following theorem of G. Hjorth.

Theorem: *Let G be any Polish group acting continuously on a Polish space X in such a way that the orbits of the action are meager and at least one orbit is dense. Then the following are equivalent:*

(i) The action of G on X is **generically turbulent**, *in the sense that T_G^X is co-meager in X.*

(ii) $(\exists x \in T_G^X)(\overline{G \cdot x} = X)$.

(iii) E_G^X is generically S_∞-ergodic, in the sense that for any Polish space Y and for any Baire measurable function $f : X \to Y^{\mathbf{N}}$ with the property that $xE_G^X y \Rightarrow \{f(x)(n) : n \in \mathbf{N}\} = \{f(y)(n) : n \in \mathbf{N}\}$, whenever x, y are in X, there exist an E_G^X-invariant co-meager subset A of X and a countable subset $\mathcal{C}$ of Y such that $x \in A \Rightarrow \{f(x)(n) : n \in \mathbf{N}\} = \mathcal{C}$, whenever $x \in X$.

(iv) The same as in (iii) but with "Baire measurable" replaced by "C-measurable" and "co-meager" replaced by "dense G_δ."

(v) For any Polish space Y on which S_∞ acts in such a way that the action is Borel and for any Baire measurable function $f : X \to Y$ with the property that $xE_G^X y \Rightarrow f(x)E_{S_\infty}^Y f(y)$, whenever x, y are in X, there exists an E_G^X-invariant co-meager subset A of X for which $f[A]$ is contained in a single $E_{S_\infty}^Y$-equivalence class.

(vi) The same as in (v) but with "Baire measurable" replaced by "C-measurable" and "co-meager" replaced by "dense G_δ."

(vii) For any relational language L, consisting of countably many symbols, and for any Baire measurable function $f : X \to X_L$ with the property that $xE_G^X y \Rightarrow f(x) \cong f(y)$, whenever x, y are in X, there exists an E_G^X-invariant co-meager subset A of X for which all countable models in $f[A]$ are equivalent

$f[A]$ are equivalent up to $\cong$, where X_L is the Polish space of all countable models for L whose universe is $\mathbf{N}$.
(viii) The same as in (vii) but with "Baire measurable" replaced by "C-measurable" and "co-meager" replaced by "dense G_δ."

Remark: Part (v) of the above mentioned theorem of G. Hjorth explains the terminology S_∞-ergodic.

Our first purpose in this doctoral dissertation is to show that any invariants for the measure equivalence relation and for certain of its most characteristic subequivalence relations and any unitary conjugacy invariants of self-adjoint and unitary operators, as well, which are computed in a Baire measurable way and can be represented as countable subsets of a Polish space or more generally as orbits of an S_∞-action or equivalent countable models up to isomorphism, must generically trivialize. In fact, we obtain the following results:

Theorem 1: *If X is any compact perfect Polish space and $P(X)$ stands for the Polish space of probability Borel measures on X, equipped with the* weak**-topology, while $\mu \sim \nu \iff (\mu \ll \nu \ \wedge \ \nu \ll \mu)$, whenever μ, ν are in $P(X)$, then $\sim$ is generically S_∞-ergodic. (The same is true if X is any compact smooth manifold of arbitrary dimension and we replace $\sim$ by $\sim_{C^r}$, where $\mu \sim_{C^r} \nu$ iff $\mu \sim \nu$ and both Radon-Nikodym derivatives $\frac{d\mu}{d\nu}$ and $\frac{d\nu}{d\mu}$ are differentiable functions of class C^r, whenever $r \in \mathbf{N} \cup \{\infty\}$.)*

Theorem 2: *Let* $\mathbf{H}$ *be any infinite-dimensional separable complex Hilbert space and let $U(\mathbf{H})$ stand for the Polish group of unitary operators on* $\mathbf{H}$ *and*

$S_1(\mathbf{H})$ stand for the Polish space of self-adjoint operators on $\mathbf{H}$ with norm at most one, both equipped with the strong topology. Then the conjugation action of $U(\mathbf{H})$ on both $U(\mathbf{H})$ and $S_1(\mathbf{H})$ is generically turbulent.

Our second purpose in this doctoral dissertation is to give new natural examples of complex Borel and analytic sets originating from Analysis and Geometry. In fact, we obtain the following results:

Theorem 1: *The set of Dirichlet series whose abscissa of absolute convergence is equal to $-\infty$ is $\mathbf{\Pi}^0_3$-complete.*

Theorem 2: *Given any non-negative real number α, the set of entire functions whose order is equal to α is $\mathbf{\Pi}^0_3$-complete and the set of all sequences of entire functions whose orders converge to α is $\mathbf{\Pi}^0_5$-complete.*

Theorem 3: *Given any line in the plane and any cardinal number $1 \leq n \leq \aleph_0$, the set of continuous paths in the plane tracing curves which admit at least n tangents parallel to the given line is $\mathbf{\Sigma}^1_1$-complete.*

Theorem 4: *Given any positive integer N and any cardinal number $1 \leq n \leq \aleph_0$, if $-\infty < \alpha < \beta < +\infty$, then the set of all functions in $C([\alpha,\beta]^N, \mathbf{R})$ whose graph in $\mathbf{R}^{N+1}$ admits at least n tangent N-dimensional hyperplanes parallel to $\mathbf{R}^N$ is $\mathbf{\Sigma}^1_1$-complete.*

Theorem 5: *For any cardinal number $1 \leq n \leq \aleph_0$, the set of differentiable paths of class C^2 in the plane admitting a canonical parameter in $[0,1]$ and tracing curves which have at least n vertices is $\mathbf{\Sigma}^1_1$-complete, while the* *set of differentiable paths of class* C^3 *in the plane admitting a canonical parameter in* $[0,1]$ *and tracing curves which have at least* n *vertices is* $\mathbf{\Sigma}^0_2$ *if* $n < \aleph_0$ *and* $\mathbf{\Pi}^0_3$ *if* $n = \aleph_0$.