EXPLICIT FORMULAS FOR THE JUMP OF Q-DEGREES

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Abstract

In the context of the axiom of projective determinacy, Q - degrees have been proposed as the appropriate generalisations of the hyperdegrees to all the odd levels of the projective hierarchy. In chapter one we briefly review the basics of Q - theory.

In the second chapter we characterise the Q - jump operation in terms of certain two - person games and derive an explicit formula for the Q - jump. This makes clear the similarities between the Q - degrees and the constructibility degrees, the Q - jump operation being a natural generalisation of the sharp operation.

In chapter three we mix our earlier results with some forcing techniques to get a new proof of the jump inversion theorem for Q - degrees. We also extend some results about minimal covers in hyperdegrees to the Q - degrees. Many of our methods are immediately applicable to the constuctible degrees and provide new proofs of old results.

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Chapter 1

Background and definitions

It is well known that by adopting the axiom of Projective Determinacy (PD), much of the classical structure theory of the first two levels of the projective hierarchy can be lifted, with a periodicity of order two, to the higher levels of the hierarchy. The hyperdegrees are just the Δ^1_1 - degrees and so hyperarithmetic theory should have some "good" generalisations at all the odd levels of the projective hierarchy.

In view of the periodicity mentioned above, it is at first surprising to find that some of the basic results about hyperdegrees are false when they are naively generalised to the Δ^1_{2n+1} - degrees. For example, Kleene's basis theorem (i.e., every nonempty Σ^1_1 set of reals contains a real which is hyperarithmetic in the complete Π^1_1 set of integers) is false when it is generalised to Σ^1_{2n+1} sets of reals and Δ^1_{2n+1} - degrees. A closer analysis leads to a new notion, that of "Q_{2n+1} - degree" as the appropriate generalisation of hyperdegree to the 2n+1 - level of the projective hierarchy.

"Q - theory" was originally developed by Kechris, Martin and

Solovay (Ke,Ma,So). In this chapter we shall review the basic ingredients of "Q - theory." Generally we shall follow the conventions of Moschovakis (Mo). An account of Q - theory is (Ke, Ma,So). Our basic theory will be ZF + DC, any other hypotheses will be explicitly stated.

§1 : Notation and terminology

$i, j, k, m, n, s, t.$	denote integers, i.e., elements of ω.
$\alpha, \beta, \gamma, \ldots x, y, z.$	denote reals, i.e., elements of ω^ω.
$x \leq_T y$	means that x is recursive in y.
$x \leq_{2n+1} y$	means $\{ (m,n) \mid x(m) = n \}$ is $\Delta^1_{2n+1}(y)$, i.e., $x \in \Delta^1_{2n+1}(y)$.

1.1 The Δ^1_{2n+1} - degrees

The relation " $\leq_{2n+1}$ " is transitive and we can use it to define an equivalence relation " $\equiv_{2n+1}$ " on the reals:

$$x \equiv_{2n+1} y \quad \text{iff} \quad x \leq_{2n+1} y \;\&\; y \leq_{2n+1} x \;.$$

The equivalence classes of the relation " $\equiv_{2n+1}$ " are called the $\underline{\Delta^1_{2n+1}}$ - degrees. Thus, the Δ^1_{2n+1} - degree of x is;

$$[x]_{2n+1} = \{ y \mid x \equiv_{2n+1} y \} .$$

The relation " $\leq_{2n+1}$ " on reals gives rise to a canonical partial ordering on the Δ^1_{2n+1} - degrees which we shall also denote by $\leq_{2n+1}$, i.e.,

$$[x]_{2n+1} \leq_{2n+1} [y]_{2n+1} \quad \text{iff} \quad x \leq_{2n+1} y.$$

In the case n = 0 we get the Δ^1_1 - degrees which are just the hyperdegrees. All the above may be relativised to define the $\Delta^1_{2n+1}(x)$ - degrees, for any real x.

1.2 Games and strategies

Given a set of reals A we define a game G_A for two players (I and II) by;

I :	m_0	m_2	m_4	....	where $m_i \in \omega$.
II:	m_1	m_3		....	Let $\alpha = (m_0, m_1, m_2, \ldots\ldots)$

Player I wins the game iff $\alpha \in A$.

i.e., the two players in turn construct a real "α." Player I wins G_A if $\alpha \in A$, otherwise player II wins the game. The set A is called the payoff set (for player I). Often we shall refer to G_A as "the game A."

A strategy (for either player) in G_A is a function $f:\omega^{<\omega} \to \omega$. Player I is said to follow the strategy f in a play of G_A if:

$$\begin{aligned} m_0 &= f(\langle\ \rangle),\\ m_2 &= f(\langle m_1\rangle),\\ m_4 &= f(\langle m_1,m_3\rangle),\\ &\cdots\cdots\cdots\\ m_{2i} &= f(\langle m_1,m_3,\ldots\ldots,m_{2i-1}\rangle). \end{aligned}$$

In the same way we can define what it means for player II to follow the strategy g.

A strategy f may be effectively coded as a real. We shall reserve the letters σ and τ to denote codes for strategies in various games. We shall often call σ and τ themselves strategies.

$\sigma*\beta$ will denote player I's play when he follows a strategy σ and player II plays β.

$\alpha*\tau$ will denote player II's play when he follows a strategy τ and player I plays α.

We shall also need to consider restrictions of strategies to the finite plays in a game. Given a strategy $f:\omega^{<\omega} \to \omega$ with code σ, we can effectively code $f\restriction\omega^n$ as a real which we shall denote by $\sigma\restriction n$. Thus, for example, if σ is a strategy for player I in some game then $\sigma\restriction n$ determines his first n+1 moves given by σ.

1.3 The game quantifier $\Game$ and the pointclasses $M_k = \omega.k - \Pi^1_1$

The game quantifier $\Game$ is the key, in the context of determinacy hypotheses, to lifting structure theory up the projective hierarchy.

For a pointset $P \subseteq \omega^\omega \times X$ we define $\Game\alpha P \subseteq X$ as follows;

$x \in \Game\alpha P$ iff Player I has a winning strategy for the game;

$$\begin{array}{llllll} \mathrm{I}: & m_0 & & m_2 & \ldots\ldots & \alpha = (m_0, m_1, m_2, \ldots.) \\ \mathrm{II}: & & m_1 & & m_3 \ \ldots\ldots & \mathrm{I} \text{ wins iff } P(\alpha, x) \end{array}$$

For a pointclass Γ, $\Game\Gamma$ denotes the pointclass that consists of all the pointsets of the form $\Game\alpha P$, for some $P \in \Gamma$. For example $\Game\Sigma^0_1 = \Pi^1_1$, $\Game\Pi^1_{2n+1} = \Sigma^1_{2n+2}$ and assuming $\underset{\sim}{\Delta}^1_{2n}$ - determinacy $\Game\Sigma^1_{2n} = \Pi^1_{2n+1}$. For more details about the game quantifier see (Mo).

<u>Definition</u>. (<u>Difference hierarchy</u>) Let ξ be a recursive ordinal. <u>$\xi - \Pi^1_1$</u> denotes the pointclass that consists of all pointsets of the following form: For some recursive sequence $(A_\eta)_{\eta<\xi}$ of Π^1_1 sets we have, letting $A_\xi = \phi$, for each real x;

$x \varepsilon A$ iff the least $\eta \leq \xi$ such that $x \notin A_\eta$ is even.

For convenience, let $M_k = \omega.k - \Pi^1_1$, for $k = 0, 1, 2, 3, \ldots\ldots$

The pointclasses M_k form a hierarchy above Π^1_1 and Σ^1_1 but well within Δ^1_2 i.e.,

$$\Sigma^1_1, \Pi^1_1 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq M_3 \subsetneqq \ldots\ldots\ldots\ldots\ldots M_k \subsetneqq \cdots \subsetneqq \Delta^1_2 .$$

In fact there is a Δ^1_2 set $G \subseteq \omega \times \omega \times X$ which ω - parametrises the M_k sets of X uniformly in k, i.e., the sets $\{ x \mid G(m,k,x) \}$, for $m = 0, 1, 2, \ldots.$, are precisely the M_k subsets of X.

Under the hypothesis of $\underset{\sim}{\Delta}^1_m$ - determinacy we can use the game quantifier to lift this hierarchy to the m^{th} level of the projective hierarchy. The pointclasses $\Game^{m-1}M_k$ form a hierarchy above Σ^1_m, Π^1_m but all well within Δ^1_{m+1}. Further there is a uniform parametrisation of $\Game^{m-1}M_k$ in Δ^1_{m+1} in the sense described above.

1.4 Norms and scales

Let Γ be a pointclass and A a pointset. A (regular) norm on A is an onto map $\theta : A \to \kappa$, for some ordinal κ. θ is called a Γ - norm if the two relations " $\leq^*_\Phi$ " and " $<^*_\Phi$ " defined below are in Γ.

$$x \leq^*_\Phi y \quad \text{iff} \quad x \in A \ \& \ (y \notin A \text{ or } \Phi(x) \leq \Phi(y)).$$

$$x <^*_\Phi y \quad \text{iff} \quad x \in A \ \& \ (y \notin A \text{ or } \Phi(x) < \Phi(y)).$$

Γ is said to be normed if every pointset in Γ has a Γ - norm.

A scale on A is a sequence (Φ_n) of norms on A such that; if (x_i) is a sequence of reals that satisfies,

i) $x_i \in A$ for each i and $x_i \to x$ as $x \to \infty$.

and

ii) for each n, for all large i, $\Phi_n(x_i) = \text{constant} = \lambda_n$.

then,

$x \in A$ and for each n, $\Phi_n(x) \leq \lambda_n$.

A scale (Φ_n) is called a Γ - scale if the relations "R" and "S" defined below are both in Γ.

$$R(n,x,y) \quad \text{iff} \quad x \leq^*_{\Phi_n} y. \qquad S(n,x,y) \quad \text{iff} \quad x <^*_{\Phi_n} y.$$

Γ is scaled if every pointset in Γ has a Γ - scale.

Often we shall need a slightly stronger notion than that of scale, namely "very good scale." In this case instead of the above we require (Φ_n) to have the following properties;

i) for all x, z; $\Phi_n(x) \leq \Phi_n(z) \rightarrow \forall\, i \leq n\ \Phi_i(x) \leq \Phi_i(x)$.

ii) if for each i, $x_i \in A$ and for all large m, $\Phi_m(x_i)$ is constant, then $x = \lim x_i$ exists and the same conclusions as before hold.

§2 : Preliminary results

Assuming $\underset{\sim}{\Delta}^1_{2n}$ - determinacy an extensive theory of Π^1_{2n+1} and Σ^1_{2n+2} sets has been developed. Much of this theory is based on the three periodicity theorems:

First Periodicity Theorem. (Martin - Moschovakis; (Mo)) Assume $\underset{\sim}{\Delta}^1_{2n}$ - determinacy. Then Π^1_{2n+1} and Σ^1_{2n+2} are normed.

Second Periodicity Theorem. (Moschovakis; (Mo)) Assume $\underset{\sim}{\Delta}^1_{2n}$ - determinacy. Then Π^1_{2n+1} and Σ^1_{2n+2} are scaled.

Third Periodicity Theorem. (Moschovakis; (Mo)) Assume $\underset{\sim}{\Delta}^1_{2n}$ - determinacy. If player I has a winning strategy in a Σ^1_{2n} game then he has a winning strategy that is Δ^1_{2n+1}.

Moschovakis (Mo) has also shown that the particular properties claimed for the various pointclasses in the periodicity theorems, are propagated up the projective hierarchy by means of the game quantifier. Versions of these theorems apply to other pointclasses with suitable closure properties. An example of such a result is:

2.1 Theorem. (Steel; (St)) Assume $\bigcup_k \Game^{m-1}\underset{\sim}{M}_k$ - determinacy. Then; for each $k \geq 1$, every set in $\Game^m M_k$ admits a very good scale (ϕ_n) such that each ϕ_n is a $\Game^m M_{k+n+1}$ - norm (uniformly in k,n).

The proof of the third periodicity theorem then gives:

2.2 Theorem. (Moschovakis) Assume $\bigcup_k \Game^{m-1}\underset{\sim}{M}_k$ - determinacy. Suppose $A \in \Game^{m-1}M_k$ for some k. If player I has a winning strategy in the game A, then, player I has a winning strategy σ such that $\sigma\restriction n \in \Game^m M_{k+n+1}$, uniformly in n.

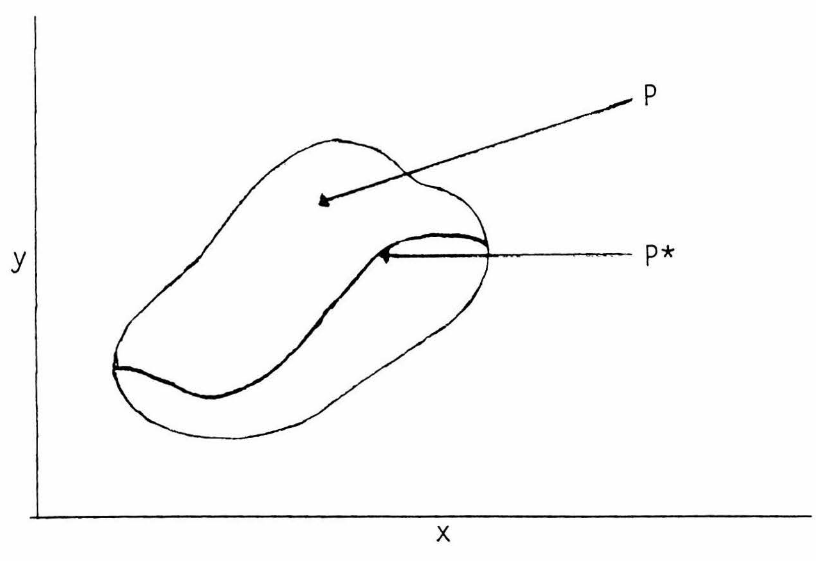
We shall be able to use the last two results working with $\underset{\sim}{\Delta}^1_{2n}$ - determinacy since;

<u>2.3 Theorem.</u> (Kechris - Woodin; (Ke,Wo)) For $n \geq 1$; $ZF + DC$ proves; $\underset{\sim}{\Delta}^1_{2n}$ - determinacy iff $\bigcup_k \Game^{2n-1} \underset{\sim}{M}_k$ - determinacy.

We shall need the following two corollaries to the periodicity theorems.

<u>2.4 The uniformisation theorem</u>

<u>Definition.</u> A pointclass Γ is <u>uniformised</u> if for every pointset $P(x,y)$ in Γ there is a pointset $P^*(x,y)$ also in Γ such that; $P^* \subseteq P$ and for each x, $\exists y P(x,y)$ iff $\exists ! y P^*(x,y)$. I.e.,



A scaled pointclass with suitable closure properties can easily be uniformised. The second periodicity theorem now gives:

2.4 Theorem. (Mo) Assume $\underset{\sim}{\Delta}^1_{2n}$ - determinacy. Then, Π^1_{2n+1} and Σ^1_{2n+2} are uniformised.

2.5 The bounded quantification theorem

Definition. Let Γ be a pointclass and A a pointset. A is called Γ - bounded if for every pointset $P(x,y)$ in Γ, the set $R(x)$ defined by; $R(x)$ iff $\exists y \varepsilon A\ P(x,y)$, is also in Γ.

A consequence of the first periodicity theorem is:

2.5 Theorem. (Mo) Assume $\underset{\sim}{\Delta}^1_{2n}$ - determinacy. Then, Δ^1_{2n+1} is Π^1_{2n+1} - bounded.

§3: The basics of Q - theory

In hyperarithmetic theory the jump of a real x is taken to be the Δ^1_1 - degree of the complete $\Pi^1_1(x)$ set of integers W^x_1. It is very tempting, in view of the periodicity phenomena present in the projective hierarchy, to take the Δ^1_{2n+1} - degrees together with the jump operation $x \to W^x_{2n+1}$ (for some complete $\Pi^1_{2n+1}(x)$ set of integers W^x_{2n+1}) and expect many of the results about hyperdegrees to generalise. This unfortunately does not happen and instead we need to look at "Q - degrees."

A good example of a result which fails to generalise in a naive way is Kleene's basis theorem.

<u>Definition.</u> A set of reals C is called a <u>basis</u> for the pointclass Γ if; every nonempty Γ set of reals contains some real in C.

<u>Kleene's basis theorem.</u> (Kleene; (Mo)) The reals Δ^1_1 in the complete Π^1_1 set of integers are a basis for Σ^1_1 .

However;

<u>Theorem.</u> (Martin - Solovay; (Ke,Ma,So)) Assume $\underset{\sim}{\Delta}^1_{2n}$ - determinacy. Then, the reals that are Δ^1_{2n+1} in the complete Π^1_{2n+1} set of integers are <u>not</u> a basis for Σ^1_{2n+1}.

By considering the new notion of Q - degree we shall see that Kleene's theorem may be generalised.

3.1 The largest thin Π^1_{2n+1} set of reals C_{2n+1} and the first nontrivial Π^1_{2n+1} - singleton y^0_{2n+1}

Under the hypothesis of $\underset{\sim}{\Delta}^1_{2n}$ - determinacy there is a largest thin (i.e., containing no perfect set) Π^1_{2n+1} set of reals. We denote this set by C_{2n+1}.

C_{2n+1} is closed under " $\equiv_{2n+1}$ " and so is a collection of Δ^1_{2n+1} - degrees. Further, the partial ordering " $\leq_{2n+1}$ " on the Δ^1_{2n+1} - degrees becomes a wellordering when it is restricted to the Δ^1_{2n+1} - degrees of C_{2n+1}.

__Definition__. A real x is called a __Π^1_{2n+1} - singleton__ if the set $\{ x \}$ is (as a subset of the reals) Π^1_{2n+1} .

The set of all Π^1_{2n+1} - singletons is clearly a subset of C_{2n+1} and so the Δ^1_{2n+1} - degrees of the Π^1_{2n+1} - singletons are well - ordered by " $\leq_{2n+1}$." A Π^1_{2n+1} - singleton which is also Δ^1_{2n+1} is called __trivial__, otherwise it is called __nontrivial__.

__Definition__. The __first nontrivial Π^1_{2n+1} - singleton, y^0_{2n+1}__, is (a representative of the Δ^1_{2n+1} - degree of) the first, with respect to the wellordering " $\leq_{2n+1}$ " on the Δ^1_{2n+1} - degrees of Π^1_{2n+1} - singletons, nontrivial Π^1_{2n+1} - singleton.

All the above may be relativised to define y^x_{2n+1} for any real x.

3.2 The set Q_{2n+1}

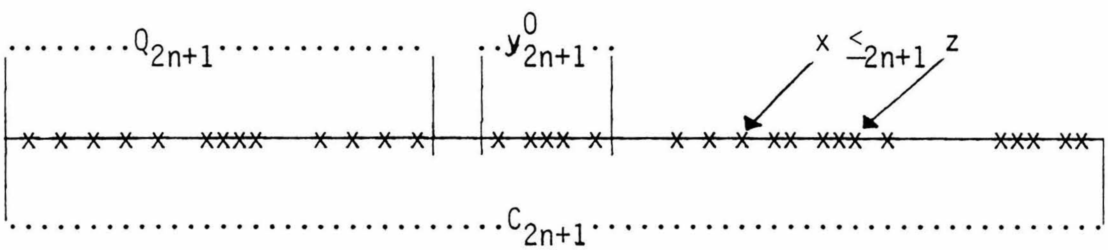
Contained in C_{2n+1} is another naturally defined set Q_{2n+1}. It has several (non - trivially) equivalent definitions, e.g.,

Definition A. Q_{2n+1} is the largest Π^1_{2n+1} - bounded set.

Definition B. Q_{2n+1} is the largest Σ^1_{2n+1} - hull. (A set of reals P is a Σ^1_{2n+1} - hull if there is a nonempty Σ^1_{2n+1} set of reals B such that, for all reals x;
P(x) iff $\forall y \in B\ (x \leq_{2n+1} y)$.)

Q_{2n+1} , y^0_{2n+1}, C_{2n+1} and $\leq_{2n+1}$ are related as follows;

<u>3.1 Proposition</u>. (Ke,Ma,So) Assume $\underset{\sim}{\Delta}^1_{2n}$ - determinacy. Consider the prewellordering " $\leq_{2n+1}$ " on C_{2n+1}. Q_{2n+1} is a proper initial segment of C_{2n+1} and y^0_{2n+1} has minimal Δ^1_{2n+1} - degree in $C_{2n+1} - Q_{2n+1}$. i.e., we have the following picture of the Δ^1_{2n+1}-degrees of C_{2n+1};



Thus in a sense y^0_{2n+1} is the least real (with respect to " $\leq_{2n+1}$ ") which is "naturally" defined and not an element of Q_{2n+1}. We can of course relativise everything and define the notion of <u>Q_{2n+1} - degree</u> in the natural way: i.e.,

$$[x]_{Q_{2n+1}} = \{ z \mid x \in Q_{2n+1}(z) \ \& \ z \in Q_{2n+1}(x) \} .$$

In view of proposition 3.1 we take $x \mapsto y^x_{2n+1}$ as the <u>Q_{2n+1} - jump</u>. Now a version of Kleene's theorem holds:

<u>3.2 Theorem.</u> (Martin - Solovay; (Ke,Ma,So)) Assume $\underset{\sim}{\Delta}^1_{2n}$ - determinacy. The reals Δ^1_{2n+1} in y^0_{2n+1} are a basis for the Σ^1_{2n+1} sets of reals.

This theorem may be strengthened and it is in this form that we shall usually apply it:

<u>3.3 Theorem.</u> (Ke,Ma,So) Assume $\underset{\sim}{\Delta}^1_{2n}$ - determinacy. Suppose $x <_{2n+1} y^0_{2n+1}$; then every nonempty $\Sigma^1_{2n+1}(x)$ set of reals contains a real $z <_{2n+1} y^0_{2n+1}$.

In the case $n = 0$ definitions A and B give $Q_1 = \Delta^1_1$. For $n \geq 1$, Q_{2n+1} is substantially larger than Δ^1_{2n+1}. For instance $\Delta^1_{2n+1} \subseteq Q_{2n+1}$ and Q_{2n+1} is closed under the (Δ^1_{2n+1} - jump) operation $x \mapsto W^x_{2n+1}$.

<u>3.3 Q_{2n+1} and the pointclasses $\Game^m M_k$</u>

An explicit characterisation of Q_{2n+1} is;

<u>3.4 Theorem.</u> (Ke,Ma,So) Let $n \geq 1$ and assume $\underset{\sim}{\Delta}^1_{2n}$ - determinacy. Then, $Q_{2n+1} = \bigcup_k \mathfrak{D}^{2n} M_k \cap \omega^\omega$.

This characterisation is the starting point for the work of chapter two.

Chapter 2

A formula for the Q - jump

The characterisation of Q_{2n+1} given by theorem 3.4 of chapter one gives a natural way of defining a real "minimal" over Q_{2n+1}. Each $\Game^m M_k$ pointclass is ω-parametrised and there is a canonical sequence (U^m_k) of sets of integers such that, if A is a set of integers then;

$$A \in \Game^m M_k \quad \text{iff} \quad \exists n \forall t \; (t \in A \leftrightarrow \langle n,t \rangle \in U^m_k) .$$

We can code all the sets U^m_k as a real Y^0_{m+1}, i.e.,

Definition. $Y^0_{m+1} = \{ \langle k,t \rangle \mid t \in U^m_k \}$.

Definition

We can as usual relativise and in a similar way define Y^x_{m+1} for any real x. It is clear that $Y^x_{2n+1} \notin Q_{2n+1}(x)$ and in some sense Y^x_{2n+1} is the "least" real not in $Q_{2n+1}(x)$. The main result of this chapter is that Y^x_{2n+1} is a first nontrivial $\Pi^1_{2n+1}(x)$ - singleton and we may take $x \mapsto Y^x_{2n+1}$ as the Q_{2n+1} -

jump operation. This fact was conjectured in (Ke,Ma,So).

Martin (Ma) has shown, assuming sharps that;

i) $L[x] \cap \omega^{\omega} = \bigcup_{k} \ni M_k(x) \cap \omega^{\omega}$.

ii) $Y_2^x \equiv_T x^{\#}$.

This illustrates the connection between the constructible degrees and the Q_{2n+1} - degrees, i.e., the Q_{2n+1} degrees may be thought of as generalisations of the constructible degrees at the odd levels of the projective hierarchy with the Q_{2n+1} - jump operation corresponding to taking the sharp.

<u>Main theorem.</u> Assume $\underset{\sim}{\Delta}^1_{2n}$ - determinacy $(n \geq 1)$. Then, for each real x, Y^x_{2n+1} is a first nontrivial $\Pi^1_{2n+1}(x)$ - singleton.

We shall prove the theorem for the case $x = 0$. The relativised version may be proven in the same way. The result will follow as a corollary to a sequence of lemmas. The first two lemmas relate Y^0_{2n+1} to strategies in certain games and are reasonably straightfoward, the main argument is the proof of lemma 5.

In all of this chapter it is to be understood that $n \geq 1$ and we are working with the theory ZF + DC + $\underset{\sim}{\Delta}^1_{2n}$ - determinacy.

Lemma 1. Every game with $\partial^{2n-1}M_k$ payoff (for player I) has a winning strategy (for either player) which is recursive in Y^0_{2n+1}.

Proof. At this stage it is worth recalling theorem 2.3 of chapter one to see that every $\partial^{2n-1}\underset{\sim}{M}_k$ game is, in fact determined under the assumptions above.

If player I has a winning strategy in a $\partial^{2n-1}M_k$ game then by theorem 2.2 of chapter one he has a winning strategy that is recursive in Y^0_{2n+1}.

If player II has a winning strategy in a $\partial^{2n-1}M_k$ game then II has a $\partial^{2n-1}M_{k+1}$ payoff set and so as in the first case he has a winning strategy that is recursive in Y^0_{2n+1}.

Lemma 1 $\dashv$

Corollary 2. Y^0_{2n+1} is a Π^1_{2n+1} - singleton.

Proof. We can write;

$$U^{2n}_k = \{ t \mid \partial \alpha R^{2n-1}_k(t,\alpha) \},$$

where $R_k^{2n-1} \subseteq \omega \times \omega^\omega$ are $\Game^{2n-1}M_k$ sets such that;

$Y_{2n+1}^0 = \{ (k,t) \mid t \in U_k^{2n} \}$.

Further we have (by the uniform parametrisations of the $\Game^m M_k$ pointclasses) that the relations R_k^{2n-1} are Δ_{2n+1}^1 uniformly in k. Thus;

$$y \in \{ Y_{2n+1}^0 \} \quad \text{iff} \quad y \subseteq \omega \ \& \ (y)_0 = \phi \ \& \ \forall k \geq 1 \ ((y)_k = U_k^{2n}),$$

$$\text{iff} \quad y \subseteq \omega \ \& \ (y)_0 = \phi \ \& \ \forall k \geq 1 \ [$$
$$\forall t \ (t \in (y)_k \rightarrow \exists \sigma \leq_T y \ \forall \beta \ R_k^{2n-1}(t, \langle \sigma * \beta, \beta \rangle) \ ) \ \&$$
$$\forall t \ (t \notin (y)_k \rightarrow \exists \tau \leq_T y \ \forall \alpha \neg R_k^{2n-1}(t, \langle \alpha, \alpha * \tau \rangle) \ ) \ ].$$

The key point to notice in the above is that we can use y to bound the strategies in the $\Game^{2n-1}M_k$ games that are used to define the $(y)_k$'s.

Since the R_k^{2n-1}'s are Δ_{2n+1}^1 uniformly in k, inspection of the above fomula shows that Y_{2n+1}^0 is a Π_{2n+1}^1 - singleton.

Corollary 2 |

<u>Corollary 3.</u> $y_{2n+1}^0 \leq_{2n+1} Y_{2n+1}^0$.

<u>Proof.</u> Y_{2n+1}^0 is clearly a <u>nontrivial</u> Π_{2n+1}^1 - singleton and so

the corollary follows from the definition of y^0_{2n+1} as the first nontrivial Π^1_{2n+1} - singleton.

Corollary 3

Definition. A real y is called a Δ^1_{2n+1} - basis for the $\partial^{2n-1}M_k$ games if; for each integer k and each $\partial^{2n-1}M_k$ set of reals A, the game with payoff A (for player I) has a winning strategy (for either player) which is $\Delta^1_{2n+1}(y)$. We say that y is a recursive basis for the $\partial^{2n-1}M_k$ games if for each of these games there is a winning strategy recursive in y.

Definition

Thus, lemma 1 says that Y^0_{2n+1} is a recursive basis for the $\partial^{2n-1}M_k$ games. We complete the characterisation of Y^0_{2n+1} in terms of strategies in the $\partial^{2n-1}M_k$ games by showing that Y^0_{2n+1} is the "least" real that is a Δ^1_{2n+1} - basis for the $\partial^{2n-1}M_k$ games, i.e.,

Lemma 4. If y is a Δ^1_{2n+1} - basis for the $\partial^{2n-1}M_k$ games then $Y^0_{2n+1} \leq_{2n+1} y$.

<u>Proof.</u> By the uniformity of the canonical parametrisations of the $\supset^m M_k$ sets there is a Δ^1_{2n+1} set $G(k,t,x)$ such that, for each k;

i) $G_k = \{ (t,x) \mid G(k,t,x) \}$ is a $\supset^{2n-1} M_k$ set.

ii) $U_k = \{ t \mid \supset x G_k(t,x) \}$ is the $\supset^{2n} M_k$ set that is used in the definition of Y^0_{2n+1}.

Now by definition of Y^0_{2n+1};

$(k,t) \varepsilon Y^0_{2n+1}$ iff $t \varepsilon U_k$,

iff $\supset x G_k(t,x)$,

iff $\exists \sigma \leq_{2n+1} y \; \forall \beta \; G(k,t,<\sigma * \beta, \beta>)$,

and this last expression is seen to be $\Pi^1_{2n+1}(y)$ by theorem 2.5. Also note;

$(k,t) \not\varepsilon Y^0_{2n+1}$ iff $t \not\varepsilon U_k$,

iff $\neg \supset x G_k(t,x)$,

iff Player II has a winning strategy in the game with payoff $\{ x \mid G_k(t,x) \}$ for player I. (This follows since the game is determined by theorem 2.3.)

Thus; $(k,t) \notin Y^0_{2n+1}$ iff $\exists \tau \leq_{2n+1} y \ \forall \alpha \neg G(k,t,\langle\alpha,\alpha * \sigma\rangle)$.

This last expression is also $\Pi^1_{2n+1}(y)$, hence, $Y^0_{2n+1} \leq_{2n+1} y$ as required.

Lemma 4 $\dashv$

By virtue of the preceding results the main theorem will follow from the next lemma.

Lemma 5. y^0_{2n+1} is a Δ^1_{2n+1} - basis for the $\Game^{2n-1}M_k$ games.

Proof. The argument is by contradiction. The proof is based on the argument used by Kechris and Woodin (Ke,Wo) to establish theorem 2.3 of chapter one (i.e., $\undertilde{\Delta}^1_{2n}$ - determinacy iff $\bigcup_k \Game^{2n-1}\undertilde{M}_k$ - determinacy), their argument in turn uses techniques of Martin (Ma) and ideas of Kechris and Solovay (Ke,So). The basic idea is to approximate $\Game^{2n-1}M_k$ games with Δ^1_{2n} games by using the Martin measure on the Turing degrees. (A set has measure one if it contains a cone of Turing degrees. The hypothesis of $\undertilde{\Delta}^1_{2n}$ - determinacy ensures that for each $\undertilde{\Sigma}^1_{2n}$ set of reals, either it or its complement has measure one.)

For notational convenience we shall take n = 2, k = 2 and assume towards a contradiction that $A \subseteq \omega^\omega \times \omega^\omega$ is a $\supset^3 M_2$ set which does not have a winning strategy $\leq_5 y_5^0$, i.e., assume;

(*) $\forall\ \sigma,\tau \leq_5 y_5^0\ \exists\ \alpha,\beta\ [\ A(\alpha,\alpha*\sigma)\ \&\ \neg A(\tau*\beta,\beta)\ ]$.

<u>Definition.</u> A countable set of reals M is called <u>x - good</u> if;

i) $x \in M$.

ii) $y,z \in M \rightarrow \langle y,z\rangle \in M$, i.e., M is closed under pairing.

iii) $(\ y \in M\ \&\ z \leq_T y\ ) \rightarrow z \in M$, i.e., M is downward closed under "$\leq_T$."

<u>Definition</u>

We shall regard countable sets of reals as being (via some coding) reals themselves. The relation (on "M" and "x") "M is x - good" is easily seen to be arithmetical.

For each real $z <_5 y_5^0$ construct a chain,

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots\cdots\cdots \subseteq M_k \subseteq \cdots\cdots$$

of countable sets of reals, as follows:

Let,

$S^0 = \{\ M \mid M \text{ is } z \text{ - good } \&\ \forall\ \sigma,\tau \leq_T z\ \exists\ \alpha,\beta \in M\ [\ A(\alpha,\alpha*\sigma)\ \&\ \neg A(\tau*\beta,\beta)\ ]\ \}$, then,

i) S^0 is nonempty (by (*)).

ii) S^0 is a $\Delta^1_5(z)$ set of reals.

Thus by the strengthened version of the Martin - Solovay basis theorem (theorem 3.3 of chapter one) there is some $M_0 <_5 y^0_5$ with $M_0 \varepsilon S^0$; fix such an M_0. Let;

$$S^1 = \{ M \mid M \text{ is } M_0 \text{ - good } \& \ \forall \sigma,\tau \leq_T M_0 \ \exists \alpha,\beta \ \varepsilon \ M \ [A(\alpha,\alpha*\sigma) \ \& \ \neg A(\tau*\beta,\beta)] \}.$$

As above, S^1 is a nonempty $\Delta^1_5(M_0)$ set of reals and so by the Martin - Solovay theorem, S^1 contains some element $M_1 <_5 y^0_5$. Also M_1 is M_0 - good and so in particular $M_0 \subseteq M_1$.

This construction can be continued, at each stage the strengthened Martin - Solovay basis theorem ensures that we can find $M_k <_5 y^0_5$. Let $M = \bigcup_k M_k$, then;

$$M \text{ is } z \text{ - good } \& \ \forall \sigma,\tau \ \varepsilon \ M \ \exists \alpha,\beta \ \varepsilon \ M \ [A(\alpha,\alpha*\sigma) \ \& \ \neg A(\tau*\beta,\beta)],$$

i.e., we have established the following;

$$(**) \quad \forall z <_5 y^0_5 \ \exists M \ [M \text{ is } z \text{ - good } \&$$
$$\forall \sigma,\tau \ \varepsilon \ M \ \exists \alpha,\beta \ \varepsilon \ M \ \{ A(\alpha,\alpha*\sigma) \ \& \ \neg A(\tau*\beta,\beta) \}].$$

Martin (Ma) gives a characterisation of the $\Game^m M_k$ sets. From this it follows that there is a formula θ of set theory such that, for every pair of uniform indiscernibles $u_1 < u_2$;

$$A(\alpha,\beta) \quad \text{iff} \quad \Game\gamma\ \Game\delta\ L[\alpha,\beta,\gamma,\delta] \models \theta(\alpha,\beta,\gamma,\delta,u_1,u_2).$$

<u>Notation.</u> In what follows c, d will be used to denote Turing degrees.

$\forall^* c\ R(c,\ldots\ldots)$	iff	$\exists c_0\ \forall c \geq_T c_0\ R(c,\ldots\ldots)$, i.e., on a cone of Turing degrees c, $R(c,\ldots\ldots)$ holds.
$\Game\gamma \leq_T c\ R(\gamma,\ldots\ldots)$	iff	$\exists\sigma \leq_T c\ \forall\beta \leq_T c\ R(<\sigma*\beta,\beta>,\ldots\ldots)$.
$\Game'\gamma\ R(\gamma,\ldots\ldots)$	iff	$\exists\sigma\ \forall\alpha\ R(<\alpha,\alpha*\sigma>,\ldots\ldots)$, i.e., player II has a winning strategy in the game with R as his payoff set.
$\Game'\gamma \leq_T c\ R(\gamma,\ldots)$	iff	$\exists\sigma \leq_T c\ \forall\alpha \leq_T c\ R(<\alpha,\alpha*\sigma>,\ldots\ldots)$.

Now we shall start to approximate the $\Game^3 M_2$ game A, the ultimate aim being to produce a "good" approximation that is Δ^1_4. Observe the following;

$$(***) \quad A(\alpha,\alpha^*\sigma) \quad \text{iff} \quad \exists\, \gamma \exists\, \delta \; L[\alpha,\alpha^*\sigma,\gamma,\delta] \models \theta(\alpha,\alpha^*\sigma,\gamma,\delta,u_1,u_2),$$

$$\text{iff} \quad \forall^* c \; \forall^* d \; \exists\, \gamma \leq_T c \; \exists \delta \leq_T d \; L[\alpha,\alpha^*\sigma,\gamma,\delta] \models \theta(\alpha,\alpha^*\sigma,\gamma,\delta,u_1,u_2).$$

Also, $\exists\, \delta\; L[\tau^*\beta,\beta,\gamma,\delta] \models \theta(\tau^*\beta,\beta,\gamma,\delta,u_1,u_2)$ defines a $\exists^2 \underset{\sim}{M}_2$ set (of γ's) and this set (being $\underset{\sim}{\Delta}^1_4$) is determined, hence;

$$\neg A(\tau^*\beta,\beta) \quad \text{iff} \quad \neg \exists \gamma \; [\; \exists \delta \; L[\tau^*\beta,\beta,\gamma,\delta] \models \theta(\tau^*\beta,\beta,\gamma,\delta,u_1,u_2) \;],$$

$$\text{iff} \quad \exists' \gamma \; \neg \exists \delta L[\tau^*\beta,\beta,\gamma,\delta] \models \theta(\tau^*\beta,\beta,\gamma,\delta,u_1,u_2).$$

Applying this argument once more and then arguing as in (***) we get;

$$(****) \quad \neg A(\tau^*\beta,\beta) \quad \text{iff} \quad \forall^* c \; \forall^* d \; \exists' \gamma \leq_T c \; \exists' \delta \leq_T d \; L[\tau^*\beta,\beta,\gamma,\delta] \models \neg\theta(\tau^*\beta,\beta,\gamma,\delta,u_1,u_2).$$

Now combine (**), (***) and (****) to get;

$$(+) \quad \forall z <_5 y^0_5 \; \exists M \; [\; M \text{ is } z\text{-good} \;\&\; \forall^* c \; \forall^* d \; \forall\, \sigma,\tau \in M \; \exists\, \alpha,\beta \in M$$
$$\{\; \exists\, \gamma \leq_T c \;\; \exists \delta \leq_T d \; L[\alpha,\alpha^*\sigma,\gamma,\delta] \models \theta(\alpha,\alpha^*\sigma,\gamma,\delta,u_1,u_2) \;\&$$
$$\exists' \gamma \leq_T c \; \exists' \delta \leq_T d \;\; L[\tau^*\beta,\beta,\gamma,\delta] \models \neg\theta(\tau^*\beta,\beta,\gamma,\delta,u_1,u_2) \;\} \;] .$$

Since M is countable and u_1 and u_2 are (uniform) indis - cernibles (+) implies;

$$(++) \quad \forall z <_5 y_5^0 \ \exists M \ [\ M \text{ is } z\text{ - good} \ \& \ \forall^* c \ \forall^* d \ \exists \ \xi_0, \xi_1, \xi_2 \ \forall \ \sigma, \tau \ \varepsilon \ M \ \exists \ \alpha, \beta \ \varepsilon \ M \ Q_\xi(\alpha, \beta, \sigma, \tau, c, d) \ ],$$

where,

$Q_\xi(\alpha,\beta,\sigma,\tau,c,d)$ is the following formula;

$$[\ \ni \gamma \leq_T c \ \ni \delta \leq_T d \ L_{\xi_0}[\alpha, \alpha^*\sigma, \gamma, \delta] \models \theta(\alpha, \alpha^*\sigma, \gamma, \delta, \xi_1, \xi_2) \ \&$$

$$\ni' \gamma \leq_T c \ \ \ni' \delta \leq_T d \ L_{\xi_0}[\tau^*\beta, \beta, \gamma, \delta] \models \neg\theta(\tau^*\beta, \beta, \gamma, \delta, \xi_1, \xi_2) \ ] \ .$$

<u>Note.</u> It is always to be understood that ξ_0, ξ_1, ξ_2, are countable ordinal variables coded as reals. We shall often write "ξ" for the triple (ξ_0,ξ_1,ξ_2). It is implicit in our notation that; $\xi_0 > \xi_2 > \xi_1$.

Let:

$$P(z,M) \equiv M \text{ is } z\text{ - good} \ \& \ \forall^* c \ \forall^* d \ \exists \xi \ \forall \ \sigma, \tau \ \varepsilon \ M \ \exists \ \alpha, \beta \ \varepsilon \ M \ Q_\xi(\alpha, \beta, \sigma, \tau, c, d),$$

i.e., $P(z,M)$ is [..............] in (++).

Now $P(z,M)$ is Σ^1_4 and we have (++) $\forall z <_5 y^0_5 \exists M\ P(z,M)$ thus, by the Martin - Solovay basis theorem the Π^1_4 set $\{ z \mid \forall M \neg P(z,M) \}$ must be empty, i.e.,

(+++) $\quad \forall z\ \exists M\ P(z,M)$.

By the (Σ^1_4) uniformisation theorem (see theorem 2.2 of chapter one) there is a total function $F: \omega^\omega \to \omega^\omega$ with Δ^1_4 graph such that if $M_x = F(x)$ then $P(x,M_x)$. Define;

$M(x) \equiv \{ M_y \mid y \leq_T x \ \&\ M_y \text{ is } x\text{ - good} \}$.

Note that $M(x)$ is a countable set and by (+++) we have;

(++++) $\quad \forall z\ \forall^* c\ \forall^* d\ \exists \xi\ \exists M \in M(z)\ \forall \sigma,\tau \in M\ \exists \alpha,\beta \in M\ Q_\xi(\alpha,\beta,\sigma,\tau,c,d)$.

Now we uniformise out "ξ." For each real z define the following function from the Turing degrees to (codes for) countable ordinals:

$f_z(c,d)$ = the least ξ such that; $\exists M \in M(z)\ \forall \sigma,\tau \in M\ \exists \alpha,\beta \in M\ Q_\xi(\alpha,\beta,\sigma,\tau,c,d)$ if such an M exists, undefined otherwise.

<u>Note:</u> $\forall^* c\ \forall^* d\ [\ f_z(c,d)$ is defined].

<u>The game G.</u>

Consider the following game G;

I	II	
$\langle\alpha, x_0\rangle$	$\langle x_1, \beta\rangle$	$x = \langle x_0, x_1\rangle$

I wins iff $\forall^* c\ \forall^* d\ \exists\gamma \leq_T c\ \exists\gamma \leq_T d$

$$L_{f_x(c,d)_0}[\alpha,\beta,\gamma,\delta] \models \theta(\alpha,\beta,\gamma,\delta,f_x(c,d)_1,f_x(c,d)_2).$$

This is a Δ^1_4 game and so by hypothesis it is determined. From this we shall get our desired contradiction. We first however need;

<u>Claim.</u> $\forall z\ \exists w \geq_T z\ \forall w' \geq_T w\ \forall^* c\ \forall^* d\ (\ f_{w'}(c,d) \geq f_w(c,d)\ )$.

<u>Proof.</u> We assume to the contrary and obtain a contradiction by constructing an infinite descending sequence of ordinals. In the proof and many times in the proof of the lemma, the as-

sumption "$\underset{\sim}{\Delta}^1_4$ - determinacy" enables us to commute "$\forall^*$" with other terms in a formula.

So assume the claim is false and pick z so that;

$$\forall w \geq_T z \, \exists w' \geq_T w \, \neg \forall^* c \; \forall^* d \, (f_{w'}(c,d) \geq f_w(c,d)).$$

If c_0 is such that, $\forall c \geq_T c_0 \; \forall^* d \, (f_w(c,d) \; \& \; f_{w'}(c,d)$ are defined), then the relation $R(c) \equiv c \geq_T c_0 \; \& \; \forall^* d \, (f_{w'}(c,d) \geq f_w(c,d))$ is $\underset{\sim}{\Delta}^1_4$, thus by $\underset{\sim}{\Delta}^1_4$-(Turing) determinacy we have:

$$\neg \forall^* c \; \forall^* d (f_{w'}(c,d) \geq f_w(c,d)) \rightarrow \forall^* c \; \neg \forall^* d \, (f_{w'}(c,d) \geq f_w(c,d)) .$$

This argument can be repeated and we get;

$$\neg \forall^* c \; \forall^* d (f_{w'}(c,d) \geq f_w(c,d)) \rightarrow \forall^* c \; \forall^* d (f_{w'}(c,d) < f_w(c,d)),$$

i.e., we have,

$$\forall w \geq_T z \, \exists w' \geq_T w \; \forall^* c \; \forall^* d \, (f_{w'}(c,d) < f_w(c,d)).$$

Now we can pick $z \leq_T z_0 \leq_T z_1 \leq_T z_2 \leq_T \cdots\cdots\cdots\cdots$ so that;

$\forall n \; \forall^* c \; \forall^* d \, (f_{z_{n+1}}(c,d) < f_{z_n}(c,d))$, and so;

$\forall^* c \; \forall^* d \; \forall n \, (f_{z_{n+1}}(c,d) < f_{z_n}(c,d))$ which is impossible.

Claim ┤

There are two cases to consider depending on which player wins the game G.

Case 1: Player I has a winning strategy in the game G.

Let $\tilde{\tau}$ be a winning strategy for player I in G. Use the claim to pick a real $w \geq_T \tilde{\tau}$ to satisfy;

$$(\#) \qquad \forall w' \geq_T w \ \forall^* c \ \forall^* d \ [\ f_{w'}(c,d) \geq f_w(c,d)\ ] \quad .$$

Let τ_0 be the strategy for player I in a game of the form;

$$\begin{array}{cc} \mathrm{I} & \mathrm{II} \\ \\ \alpha & \beta \end{array}$$

given by; $\quad \tau_0 * \beta = (\tilde{\tau} * \langle w,\beta\rangle\ )_0 \quad .$

Thus $\tau_0 \leq_T w$ and $\tilde{\tau} * \langle w,\beta\rangle = (\alpha, x_0)$ for some x_0 and $\alpha = (\tilde{\tau}*\langle w,\beta\rangle)_0$, i.e., if in the game G player II plays $\langle w,\beta\rangle$ then I answers (by playing his strategy $\tilde{\tau}$) with $\langle \tau_0 * \beta, x_0\rangle$, for some x_0.

From the definition of $f_w(c,d)$ we have;

$$\forall^* c \ \forall^* d \ \exists M \in M(w) \ \forall \sigma,\tau \in M \ \exists \alpha,\beta \in M \ Q_{f_w(c,d)}(\alpha,\beta,\sigma,\tau,c,d).$$

Now "$M(w)$" is a countable set and the relation

"$\forall\, \sigma,\tau \in M\ \exists\, \alpha,\beta \in M\ Q_{f_w(c,d)}(\alpha,\beta,\sigma,\tau,c,d)$" defines a $\underset{\sim}{\Delta}{}^1_2$ set of (c,d)'s, so by $\underset{\sim}{\Delta}{}^1_4$ - Turing determinacy we get;

$$(\#\#) \qquad \exists M \in M(w)\ \forall^* c\ \forall^* d\ \forall\, \sigma,\tau \in M\ \exists\, \alpha,\beta \in M\ Q_{f_w(c,d)}(\alpha,\beta,\sigma,\tau,c,d) \ .$$

Now fix $M_0 \in M(w)$ such that;

$$\forall^* c\ \forall^* d\ \forall\, \sigma,\tau \in M_0\ \exists\, \alpha,\beta \in M_0\ Q_{f_w(c,d)}(\alpha,\beta,\sigma,\tau,c,d) \ .$$

By definition of Q_ξ we have in particular;

$$\forall^* c\ \forall^* d\ \forall \tau \in M_0\ \exists \beta \in M_0\ \ni'\gamma \leq_T c\ \ \ni'\delta \leq_T d$$
$$L_{f_w(c,d)_0}[\tau^*\beta,\beta,\gamma,\delta] \models \neg\theta(\tau^*\beta,\beta,\gamma,\delta,f_w(c,d)_1,f_w(c,d)_2) \ .$$

Since $\tau_0 \leq_T w$ and M_0 is "w - good" we may take "τ" to be τ_0 in the above and then argue as in (##) to deduce;

$$\exists \beta \in M_0\ \forall^* c\ \forall^* d\ \ \ni'\gamma \leq_T c\ \ \ni'\delta \leq_T d$$
$$L_{f_w(c,d)_0}[\tau_0{}^*\beta,\beta,\gamma,\delta] \models \neg\theta(\tau_0{}^*\beta,\beta,\gamma,\delta,f_w(c,d)_1,f_w(c,d)_2) \ .$$

Now choose $\beta_0 \in M_0$ to satisfy;

$$(\#\#\#) \qquad \forall^* c\ \forall^* d\ \ \ni'\gamma \leq_T c\ \ \ni'\delta \leq_T d$$
$$L_{f_w(c,d)_0}[\tau_0{}^*\beta_0,\beta_0,\gamma,\delta] \models \neg\theta(\tau_0{}^*\beta_0,\beta_0,\gamma,\delta,f_w(c,d)_1,f_w(c,d)_2) \ .$$

Put $\alpha_0 = \tau_0 * \beta_0$; then for some x_0:

	I	II	
	$\langle\alpha_0, x_0\rangle$	$\langle w, \beta_0\rangle$	$x = \langle x_0, w\rangle$

is a play of the game G in which I follows his winning strategy $\hat{\tau}$. Hence we must have;

$$\forall^* c \ \forall^* d \ \exists \gamma \leq_T c \ \exists \delta \leq_T d$$
$$L_{f_x(c,d)_0}[\alpha_0, \beta_0, \gamma, \delta] \models \theta(\alpha_0, \beta_0, \gamma, \delta, f_x(c,d)_1, f_x(c,d)_2) \ .$$

From this and (###) we shall finally get our desired contradiction by showing; $\forall^* c \ \forall^* d \ [\ f_w(c,d) = f_x(c,d)\ ]$.

<u>To show: $\forall^* c \ \forall^* d \ [\ f_w(c,d) \leq f_x(c,d)\ ]$.</u>

Since $w \leq_T x$ automatically by the claim and the definition of w we have $\forall^* c \ \forall^* d \ [\ f_w(c,d) \leq f_x(c,d)\ ]$.

<u>To show: $\forall^* c \ \forall^* d \ [\ f_x(c,d) \leq f_w(c,d)\ ]$.</u>

M_0 is w - good and τ_0, $\beta_0 \in M_0$ and so by the closure properties of "good" sets also α_0, x_0, $x = \langle x_0, w\rangle \in M_0$. Thus M_0 is also x - good. Since $M_0 \in M(w)$ we must have $M_0 = M_y$ for some

$y \leq_T w \leq_T x$ and so $M_0 \varepsilon M(x)$. Now by definition of M_0;

$$\forall^* c \ \forall^* d \ \exists M \varepsilon M(x) \ \forall \sigma,\tau \ \varepsilon M \ \exists \alpha,\beta \ \varepsilon M \ Q_{f_w(c,d)}(\alpha,\beta,\sigma,\tau,c,d),$$

and so by the definition of $f_x(c,d)$, $\forall^* c \ \forall^* d \ [\ f_x(c,d) \leq f_w(c,d)\ ]$. This completes the proof for case I.

Case II: Player II has a winning strategy in the game G.

In this case we can argue to a contradiction in a similar way to case I.

Lemma 5 ⊣

Lemma 5 as well as having the main theorem as a corollary may be viewed as a basis result. The proof shows that for each fixed $\Game^{2n-1} M_k$ game there is a winning strategy for one of the players which is $<_{2n+1} y^0_{2n+1}$. In fact this statement can be strengthened;

Theorem. Assume $\underset{\sim}{\Delta}^1_{2n}$ - determinacy $(n \geq 1)$. Then, for each fixed integer k there is a real $x <_{2n+1} y^0_{2n+1}$ which is a recursive basis for the $\mathfrak{D}^{2n-1}M_k$ games.

Proof. Let $A(t,x,y) \in \mathfrak{D}^{2n-1}M_k$ be ω - universal for the $\mathfrak{D}^{2n-1}M_k$ subsets of $\omega^\omega \times \omega^\omega$ and consider the following game G:

$$\begin{array}{ll} \mathrm{I} & \mathrm{II} \\ t & \\ & b_0 \\ a_0 & \\ & b_1 \\ a_1 & \\ & b_2 \\ \vdots & \vdots \end{array} \qquad \begin{aligned} \alpha &= (a_0,a_1,a_2,\ldots.) \\ \beta' &= (b_1,b_2,b_3,\ldots.) \\ \beta^* &= (b_0-1,b_1,b_2,..) \end{aligned}$$

II wins iff $[\ b_0 = 0\ \&\ \neg A(t,\alpha,\beta')\ ]$ or $[\ b_0 \neq 0\ \&\ A(t,\beta^*,\alpha)\ ]$.

Claim: Player II has a winning strategy in G.

Proof of claim. After player I has played t either,

(i) player I has a winning strategy in the game $A(t,\alpha,\beta)$

or

(ii) player II has a winning strategy in the game $A(t,\alpha,\beta)$.

In case (i) II can use I's winning strategy σ against α, i.e., $b_0 = \sigma(\langle\ \rangle)$, $b_{n+1} = \sigma(\langle a_0,a_1,\ldots..a_n\rangle)$, at the end of the play "[$b_0 \neq 0$ & $A(t,\beta^*,\alpha)$]" holds.

In case (ii) II can use player II's winning strategy σ for the game $A(t,x,y)$, i.e.,

$b_0 = 0, \quad b_{n+1} = \sigma(<a_0,\ldots\ldots a_n>)$,

at the end of the play "[$b_0 = 0$ & $A(t,\alpha,\beta')$]" holds.

Claim $\dashv$

G is a $\partial^{2n-1}M_{k+2}$ game and so II has a winning strategy $\sigma <_{2n+1} y^0_{2n+1}$ for G.

If $\sigma(<t>) = 0$ let σ_t be the strategy for player II in the game $A(t,\alpha,\beta)$ given by;

$\sigma_t(<a_0,a_1,\ldots\ldots>) = \sigma(<t,a_0,a_1,\ldots\ldots>)$,

then σ_t is a winning strategy for player II in this game.

If $\sigma(<t>) \neq 0$ then let σ_t be the winning strategy for player I in the game $A(t,\alpha,\beta)$ given by;

$\sigma_t(< >) = \sigma(<t>) - 1$

$\sigma_t(<a_0,a_1,\ldots\ldots>) = \sigma(<t,a_0,a_1,\ldots\ldots>)$.

Clearly $\sigma_t \leq_T \sigma$ and so taking $x = \sigma$ we have our theorem.

Theorem $\dashv$

Chapter 3

Some results about Q - degrees

In this chapter we shall use the explicit characterisation of the Q - degrees (see theorem 3.4 of chapter one) and the results of chapter two to prove some results about Q - degrees. Some of the methods will also be appropriate for the constructible degrees and will provide new proofs of known results.

§1: The jump inversion theorem for the Q - degrees; Cohen forcing in the Q - degrees

In this section techniques of "forcing in analysis" developed by Kechris (Ke) will be used together with the main result of chapter two to establish the jump inversion theorem for the Q - degrees. This result was originally proven by Kastanas (Ka). The new proof avoids the use of an ordinal assignment to the Q - degrees and is much closer to the proof of the jump inversion theorem for Turing degrees (Fr).

We shall first review the basic methods of forcing in the projective hierarchy.

Notation. In general we shall follow the conventions of the previous chapter.

$p, p_0, p_1, p_2, \ldots$ denote finite sequences of integers, i.e., elements of $\omega^{<\omega}$.

$p < \alpha$ iff $\forall i < \text{length}(p)\ [\ p(i) = \alpha(i)\ ]$.

$U_p = \{\ \alpha \varepsilon\ \omega^\omega \mid p < \alpha\ \}$, i.e., the U_p's are the basic open sets of the Baire space ω^ω.

$\bar{p} = p_0 {}^\frown p_1 {}^\frown p_2 \ \ldots\ = (p_0(0), \ldots, p_0(\text{length}(p_o)-1), p_1(0), \ldots)$.

Notation

Definition. Suppose A is a set of reals. We say that p forces A, and write $p \Vdash A$ (or $p \Vdash A(.)$), iff A is comeager on U_p.

Definition

Truth lemma. (see (Ke)) For all $A \subseteq \omega^\omega$ with the property of Baire; for a comeager set of reals x the following equivalence holds;

$$A(x) \quad \text{iff} \quad \exists p < x\ [\ p \Vdash A\ ].$$

Lemma

We can of course give a game theoretic characterisation of the forcing relation by considering the Banach - Mazur games, i.e.,

$p \Vdash A$ iff player II has a winning strategy in the following game:

$$\begin{array}{llllll} \mathrm{I}: & p_0 & & p_2 & \ldots\ldots & x = p^\frown p_0{}^\frown p_1{}^\frown \ldots \\ \mathrm{II}: & & p_1 & & p_3 \ \ldots\ldots & \mathrm{II} \text{ wins iff } A(x). \end{array}$$

The following result enables us to get a "good" estimate of the complexity of the forcing relation.

<u>The game formula.</u> (Ke) Let $P \subseteq \omega^\omega \times \omega^\omega$ then;

$\forall p_0 \exists p_1 \forall p_2 \exists p_3 \ldots [\forall a_0 \exists a_1 \forall a_2 \ldots P(\bar{p},\alpha)]$ iff
$\forall p_0 \forall a_0 \exists p_1 \exists a_1 \ldots P(\bar{p},\alpha)$,

provided the second game is determined. Here $\alpha = (a_0,a_1,\ldots)$.

Theorem |

The game formula says that in certain circumstances we can replace two applications of the game quantifier by just one application.

We shall work with the Q_{2n+1} - degrees, so for the rest of this section assume $\underset{\sim}{\Delta}^1_{2n}$ - determinacy.

Fix a Δ^1_{2n+1} set $G(k,t,x,y)$ which uniformly parametrises the $\Game^{2n-1}M_k$ subsets of $\omega^\omega \times \omega^\omega$, i.e.,

i) $G_k = \{ (t,x,y) \mid G(k,t,x,y) \} \varepsilon \Game^{2n-1}M_k$, for $k = 1, 2, \ldots\ldots$

ii) The $\Game^{2n-1}M_k$ subsets of $\omega^\omega \times \omega^\omega$ are precisely the sets $G_{k,t} = \{ (x,y) \mid G(k,t,x,y) \}$.

iii) $\langle k,t \rangle \ \varepsilon \ Y^x_{2n+1}$ iff $\Game yG(k,t,x,y)$.

For each pair of integers k, t let;

$A_{k,t} = \{ x \mid \Game yG(k,t,x,y) \}$

so that each $A_{k,t}$ is $\Game^{2n}M_k$ and so (by the game formula) has the property of Baire. Let;

$S_{k,t} = \{ x \ \varepsilon \ \omega^\omega \mid A_{k,t}(x) \quad \text{iff} \quad \exists p < x \ p \Vdash A_{k,t} \}$.

$T_{k,t} = \{ x \ \varepsilon \ \omega^\omega \mid \neg A_{k,t}(x) \quad \text{iff} \quad \exists p < x \ p \Vdash \neg A_{k,t} \}$.

so that by the truth lemma each of the $S_{k,t}$'s and $T_{k,t}$'s are comeager, thus;

$S = (\bigcap_{k,t} S_{k,t}) \cap (\bigcap_{k,t} T_{k,t})$ is comeager.

Definition. x forces its Q_{2n+1} - jump if $x \in S$.

Definition

Proposition 1. Suppose that x forces its Q_{2n+1} - jump, then;
$y^x_{2n+1} \equiv_{2n+1} \langle y^0_{2n+1}, x \rangle$.

Proof.

i) Clearly $\langle y^0_{2n+1}, x \rangle \leq_{2n+1} y^x_{2n+1}$.

ii) $\langle k,t \rangle \in Y^x_{2n+1}$ iff $\ni y G(k,t,x,y)$,

iff $\exists p < x \; p \Vdash \ni y G_{k,t}(.,y)$ [since x forces its jump and is in particular a member of $S_{k,t}$],

iff $\exists p < x \; \forall p_0 \exists p_1 \ldots . [\exists y_0 \; \forall y_1 \ldots$ $G(k,t,p\,\bar{p},y)$],

iff $\exists p < x \; \forall p_0 \; \exists y_0 \; \exists p_1 \; \forall y_1 \ldots . \; G(k,t,p^\frown\bar{p},y)$ [by the game formula],

iff $\exists p < x \; \exists \sigma \leq_T Y^0_{2n+1} \; \forall \alpha \; G(k,t,p^\frown\langle\alpha*\sigma\rangle_0, \langle\alpha*\sigma\rangle_1)$ [by lemma 1 of chapter two].

This last formula is easily seen to be $\Pi^1_{2n+1}(Y^0_{2n+1},x)$. Also;

$$<k,t> \notin Y^x_{2n+1} \quad \text{iff} \quad \neg\exists\, y G(k,t,x,y)\,,$$

$$\text{iff} \quad \exists p < x \;\; p \;\|\!\!- \neg\exists y G_{k,t}(.,y)\,,$$

$$\text{iff} \quad \exists p < x \;\exists \sigma \leq_T Y^0_{2n+1} \;\forall \beta$$
$$\neg G(k,t,p^\frown<\sigma*\beta>_0,<\sigma*\beta>_1)\,,$$

which is also $\Pi^1_{2n+1}(Y^0_{2n+1},x)$ hence, $Y^x_{2n+1} \leq_{2n+1} <Y^0_{2n+1},x>$ as required.

Proposition $\dashv$

It immediately follows that:

Corollary 1. $\{ x \mid <y^0_{2n+1},x> \equiv_{2n+1} y^x_{2n+1} \}$ is comeager.

Corollary $\dashv$

By using the explicit formula for the Q - jump the above corollary has been established without using the ordinal assignment to the Q - degrees.

Corollary 2. $S = \{ x \mid x$ forces its Q_{2n+1} - jump $\} \in \Pi^1_{2n+1}(Y^0_{2n+1})$.

Proof. By proposition 1 and the results of chapter two when $x \in S$ we can use $<Y^0_{2n+1},x>$ to bound strategies in $\exists^{2n-1}M_k(x)$

games, thus;

$$S(x) \quad \text{iff} \quad \forall k,t \{ [\exists yG(k,t,x,y) \ \& \ \exists p < x \ p \Vdash \exists yG(k,t,\cdot,y)] \text{ or } [\neg\exists yG(k,t,x,y) \ \& \ \exists p < x \ p \Vdash \neg\exists yG(k,t,\cdot,y)] \},$$

$$\text{iff} \quad \forall k,t \{ [\exists \sigma \leq_{2n+1} \langle x, Y^0_{2n+1} \rangle \ \forall \beta \ G(k,t,x,\sigma * \beta) \ \& \ \exists p < x \ p \Vdash \exists yG(k,t,\cdot,y)] \text{ or } [\exists \sigma \leq_{2n+1} \langle x, Y^0_{2n+1} \rangle \ \forall \alpha \ \neg G(k,t,x,\alpha * \sigma) \ \& \ \exists p < x \ p \Vdash \neg \exists yG(k,t,\cdot,y)] \}.$$

Now the relations " $\exists p < x \ p \Vdash \exists yG(k,t,\cdot,y)$ " and " $\exists p < x$ $p \Vdash \neg\exists yG(k,t,\cdot,y)$ " are $\Delta^1_{2n+1}(x, Y^0_{2n+1})$ as before, thus S is a $\Pi^1_{2n+1}(Y^0_{2n+1})$ set.

Corollary ⊣

Jump inversion theorem for the Q_{2n+1} - degrees. (Kastanas; (Ka))
Suppose $y^0_{2n+1} \leq_{2n+1} z$; then for some real x, $z \equiv_{2n+1} y^x_{2n+1}$.

Proof. The set of reals S that force their Q_{2n+1} - jump is $\Pi^1_{2n+1}(Y^0_{2n+1})$ and comeager, thus in the Banach - Mazur game with payoff S (for player II) there is a winning strategy for player II. By the third periodicity theorem and the game formula player II has a winning strategy $\sigma \in \Delta^1_{2n+1}(Y^0_{2n+1})$.

Now let $z \geq_{2n+1} y^0_{2n+1}$ and consider the following play in a Banach - Mazur game:

$$\begin{array}{llllll} \text{I} : & a_0 & & a_1 & & \ldots\ldots\ldots \\ \text{II}: & & p_0 & & p_1 & \ldots\ldots\ldots \end{array}$$

where II plays according to the strategy σ and I plays $z = (a_0, a_1, a_2, \ldots.)$

By definition of σ the real $x = \langle a_0\rangle^\frown p_0{}^\frown\langle a_1\rangle^\frown p_1{}^\frown \ldots\ldots\ldots$ forces its Q_{2n+1} - jump and so $y^x_{2n+1} \equiv_{2n+1} \langle x, y^0_{2n+1}\rangle$. Also,

$$y^x_{2n+1} \equiv_{2n+1} \langle x, y^0_{2n+1}\rangle \leq_{2n+1} \langle z, \sigma, y^0_{2n+1}\rangle \equiv_{2n+1} \langle z, y^0_{2n+1}\rangle \equiv_{2n+1} z$$

and

$$z \leq_{2n+1} \langle x, \sigma\rangle \leq_{2n+1} \langle x, y^0_{2n+1}\rangle \equiv_{2n+1} y^x_{2n+1} .$$

I.e., $z \equiv_{2n+1} y^x_{2n+1}$.

Theorem ⊣

§2: Cones of minimal covers in the Q - degrees

In this section we shall extend some results of Simpson (Si), concerning cones of minimal covers in the hyper - and constructible degrees, to the Q - degrees.

Definition. A Q_{2n+1} - degree x is said to be a minimal cover if there is some Q_{2n+1} - degree $y <_{Q_{2n+1}} x$ such that;

$$\forall z \, [\, (y \leq_{Q_{2n+1}} z \leq_{Q_{2n+1}} x) \rightarrow (z \equiv_{Q_{2n+1}} y \text{ or } z \equiv_{Q_{2n+1}} x) \,],$$

i.e., there is no Q_{2n+1} - degree strictly between y and x.

A cone of minimal covers in the Q_{2n+1} - degrees is a set of minimal covers of the form $\{ x \mid x_0 \leq_{Q_{2n+1}} x \}$. x_0 is called a base for the cone.

In the same way we can define minimal covers etc. for other notions of degree.

Definition

Simpson (Si) has shown:

1) Assume V = L. Then there is no cone of minimal covers in the hyperdegrees.

2) Assume $0^{\#}$ exists (i.e., assume Σ^1_1 - determinacy). Then there is a cone of minimal covers in the hyperdegrees.

Using a result of Jensen (Je) 2) is easily generalised to the constructible degrees granting Σ^1_2 - determinacy. The methods of chapter two give an alternative proof of this result:

<u>Theorem.</u> Assume $\underset{\sim}{\Delta}^1_2$ - determinacy. Then, there is a cone of minimal covers in the constructible degrees. Further, Y^0_3 is a base for this cone.

<u>Proof.</u> As usual by Sacks' forcing (Sa);

$\forall x \exists y$ ($[y]_L$ is minimal over $[x]_L$),

also let $M \subseteq \omega^\omega \times \omega^\omega$ be defined by;

$M(x,y)$ iff $[y]_L$ is minimal over $[x]_L$,

iff $L[x,y] \models$ ($[y]_L$ is minimal over $[x]_L$),

so that M is a $\Game M_1$ set.

Let $A = \{ z \mid \exists x,y \leq_T z \,[\, z \equiv_T \langle x,y\rangle \,\&\, M(x,y) \,]\,\}$, then;

1) A is closed under "$\equiv_T$."

2) A is a $\Game M_1$ set.

3) A is unbounded in the Turing degrees (since $\forall x \exists y\, M(x,y)$).

Hence, by $\underset{\sim}{\Delta}^1_2$ - determinacy ($\equiv \bigcup_k \Game \underset{\sim}{M}_k$ - determinacy), A contains a cone of Turing degrees and since Y^0_3 is a recursive basis for the $\Game M_1$ games it is a base for this cone.

Now suppose that $z \geq_L Y_3^0$, then; $z \equiv_L \langle z, Y_3^0 \rangle \geq_T Y_3^0$ and also $\langle z, Y_3^0 \rangle \in A$. Hence, $\langle z, Y_3^0 \rangle$ is a minimal cover in the constructible degrees and thus so is z.

Theorem |

We now deal with minimal covers in the Q - degrees. Under the assumption of $\underset{\sim}{\Pi}^1_{2n+1}$ - determinacy, as well as there being a largest thin Π^1_{2n+1} set of reals there is also a largest countable Σ^1_{2n} set of reals which is denoted by C_{2n}. The reals in C_{2n} are in many ways "good" generalisations of the constructible reals to all the even levels of the projective hierarchy (see (Be) for more details). In particular;

1) $C_{2n} = \{x \mid \exists y \in C_{2n-1} \ (x \leq_T y) \}$.
2) $L[C_{2n}] \models ZF + DC + \underset{\sim}{\Delta}^1_{2n-1}$ - Determinacy.
3) $L[C_{2n}] \cap \omega^\omega = C_{2n}$.

It can now be seen that $\underset{\sim}{\Delta}^1_{2n+1}$ - determinacy is not enough to ensure that there is a cone of minimal covers in the Q_{2n+1} - degrees:

<u>Proposition.</u> Assume that $V = L[C_{2n+2}]$. Then there is no cone of minimal covers in the Q_{2n+1} - degrees.

Proof. The following argument is a generalisation of Simpson's (Si) proof that there is no cone of minimal covers in the hyperdegrees if $V = L$.

We will show that none of the reals in C_{2n+1} are minimal covers, the result will follow since C_{2n+1} is "unbounded" in $L[C_{2n+1}]$.

Suppose $x <_{Q_{2n+1}} y \in C_{2n+1}$. Then, $y \in C_{2n+1}(x)$ and $y \notin Q_{2n+1}(x)$. Now since y^x_{2n+1} is the first real above $Q_{2n+1}(x)$ in the canonical ("$\leq_{2n+1}$") prewellordering of $C_{2n+1}(x)$ we must have:

$$y^x_{2n+1} \leq_{2n+1} \langle x,y\rangle \equiv_{Q_{2n+1}} y, \quad \text{and} \quad x <_{Q_{2n+1}} y^x_{2n+1} \leq_{Q_{2n+1}} y.$$

Thus, if y is minimal over x then, $y^x_{2n+1} \equiv_{Q_{2n+1}} y$ and so y^x_{2n+1} would be minimal over x. This is clearly absurd. (If y^x_{2n+1} were minimal over x then by the Martin - Solovay basis theorem every nonempty $\Sigma^1_{2n+1}(x)$ set of reals would contain some real in $Q_{2n+1}(x)$. This is clearly not the case for the $\Sigma^1_{2n+1}(x)$ set $Q_{2n+1}(x)^c$.)

Proposition |

We also have the following result to complete the generalisation of Simpson's results:

Theorem. Assume $\Game^{2n}\underset{\sim}{M}_1$ - determinacy. Then there is a cone of minimal covers in the Q_{2n+1} - degrees.

Note: It has been conjectured (Ke,Ma,So) that $\Game^{2n}\underset{\sim}{M}_1$ - determinacy is equivalent to $\underset{\sim}{\Sigma}^1_{2n+1}$ - determinacy. Martin (Ma) and Harrington (Ha) have shown this to be the case for $n = 0$.

Proof. We shall do the case $n = 1$. The other cases are similar but they involve the use of more complicated ultrapowers than the one used below.

We define an inner model of ZFC which is a generalisation of L to the third level of the projective hierarchy as follows (see (Ke,Ma,So));

For each constructibility degree $d = [x]_L$ let $L[d] = L[x]$ and consider the ultrapower

$$M_3 = \prod_d \mathrm{HOD}^{L[d]}/\mu ,$$

where μ denotes the Martin measure on the constructibility degrees and $\mathrm{HOD}^{L[d]}$ is the inner model of all hereditarily ordinal definable within $L[d]$ sets.

The model M_3 has the following properties (see (Ke,Ma,So));

1) The set of reals of M_3 is Q_3.

2) For each real x, if $M_3[x]$ denotes the smallest inner

model of ZFC containing M_3 and x, then the reals of $M_3[x]$ are $Q_3(x)$. The definition of M_3 can of course be relativised; for any real x let

$$M_3(x) = \prod_d \mathrm{HOD}_x^{L[x,d]}/\mu \ .$$

Thus, $M_3(x)$ and $M_3[x]$ have the same reals (but it is not known if they are equal).

3) M_3 satisfies a "dual Schoenfield absoluteness theorem." I.e., for each Σ^1_3 formula $\Theta(x)$ there is a Π^1_3 formula $\Theta^*(x)$ which is effectively computable from Θ such that;

$$\Theta(x) \quad \text{iff} \quad M_3[x] \models \Theta^*(x),$$

and similarly interchanging the roles of Σ^1_3 and Π^1_3.

Fix a Σ^1_3 formula Θ such that for all reals x, y, z with $x,y \in Q_3(z)$ we have;

$$x \in Q_3(y) \quad \text{iff} \quad M_3(z)[x,y] = M_3(z) \models \Theta(x,y,z).$$

Now; $\quad y$ is minimal in the Q_3 - degrees over x

iff $\quad x \in Q_3(y)$ & $y \notin Q_3(x)$ & $\forall z \in Q_3(y)$ [($x \in Q_3(z)$) $\to$ ($z \in Q_3(x)$ or $y \in Q_3(z)$)].

iff $\quad M_3(\langle x,y\rangle) \models \Theta(x,y,\langle x,y\rangle)$ & $\neg\Theta(y,x,\langle x,y\rangle)$ & $\forall z$ [$\Theta(x,z,\langle x,y\rangle) \to$ { $\Theta(z,x,\langle x,y\rangle)$ or $\Theta(y,z,\langle x,y\rangle)$ }].

iff $M_3(\langle x,y\rangle) \models \Psi(x,y)$, for some formula Ψ of set theory.

iff $\forall^* d\ \{\ L[x,y,d] \models \text{"}HOD_{x,y} \models \Psi(x,y)\text{"}\ \}$.

This last expression is by results of Martin (Ma), $\Game^{2} M_1$. As in the case of the constructible degrees we have a cone of minimal covers in the Q_3 - degrees. Further, Y_4^0 is a base for this cone.

Theorem

References

(Be) H. Becker; Partially Playful Universes, in: Cabal Seminar 76 - 77, A. S. Kechris and Y. N. Moschovakis editors, Lecture Notes in Mathematics, No. 689, Springer, Berlin, 1978. pp. 55 - 90.

(Fr) R. Friedberg; A Criterion for the Completeness of Degrees of Unsolvability. J.Symb.Logic 22, 1957, pp. 159 - 160.

(Ha) L. Harrington; Analytic Determinacy and $0^{\#}$. J.Symb.Logic 43, 1978, pp. 685 - 693.

(Je) R. Jensen; Definable Sets of Minimal Degree, in: Mathematical Logic and Foundations of Set Theory, Proc. Internat. Colloq., Jerusalem 1968, Y. Bar-Hillel editor, North Holland Amsterdam, 1970, pp. 122 - 128.

(Ka) I. G. Kastanas; Ph.D. Thesis. California Institute of Technology, Pasadena, 1981.

(Ke) A. S. Kechris; Forcing in Analysis, in: Higher Set Theory, Proceedings, Oberwolfach, Germany, 1977, G. H. Muller and

D. S. Scott editors, Lecture Notes in Mathematics, No. 669, Springer, Berlin and New York, 1978, pp. 277 - 302.

(Ke,Ma,So) A. S. Kechris, D. A. Martin, R. M. Solovay; Introduction to Q - Theory, in: Cabal Seminar 79 - 81, A. S. Kechris, D. A. Martin and Y. N. Moschovakis editors, No. 839, Springer, Berlin, 1984, pp. 199 - 281.

(Ke,So) A. S. Kechris, R. M. Solovay; On the relative consistency strengths of Determinacy Hypotheses. To appear.

(Ke,Wo) A. S. Kechris, W. H. Woodin; The equivalence of Partition Properties and Determinacy. Proc. Natl. Acad. Sci. USA, Vol. 80, March 1983, pp. 1783 - 1786.

(Ma) D. A. Martin; The largest countable this, that and the other, in: Cabal Seminar 79 - 81, see (Ke,Ma,So),pp. 97 - 106.

(Mo) Y. N. Moschovakis; Descriptive Set Theory. North Holland Amsterdam, 1980.

(Sa) G. Sacks; Forcing with Perfect Closed Sets, in: Axiomatic

Set Theory, D. S. Scott editor, Proc. Symp. Pure Math., Vol. 13, Am. Math. Soc., Providence, R.I., 1967, pp. 331 - 355.

(Si) S. G. Simpson; Minimal Covers and Hyperdegrees. Trans. Am. Math. Soc., Vol. 209, 1975, pp. 45 - 64.

(St) J. R. Steel; Scales on Σ_1^1 Sets, in: Cabal Seminar 79 - 81, see (Ke,Ma,So), pp. 72 - 76.