THE RAMSEY PROPERTY AND DEGREES IN THE ANALYTICAL HIERARCHY

Thesis by

Ilias George Kastanas

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1981

(Submitted January 26, 1981)

To my parents

and my sister

ΑΡΜΟΝΙΗ ΑΦΑΝΗΣ ΦΑΝΕΡΗΣ ΚΡΕΙΤΤΩΝ

ΗΡΑΚΛΕΙΤΟΣ

Acknowledgment

I would like to extend my deepest thanks to my advisor, Prof. Alexander S. Kechris, for teaching me mathematical logic and for his unfailing and invaluable assistance and encouragement.

I have benefited from conversations with many logicians and from many inspiring seminar talks; in particular I would like to thank Leo Harrington, Tony Martin, Yiannis Moschovakis, Robert Solovay and John Steel.

The Caltech Mathematics Department provided continued support and a pleasant environment of the highest academic standard.

Abstract

In Chapter I we review some known results about the Ramsey theory for partitions of reals, and we present a certain two-person game such that if either player has a winning strategy then a homogeneous set for the partition can be constructed, and conversely. This gives alternative proofs of some of the known results. We then discuss possible uses of the game in obtaining effective versions and prove a theorem along these lines.

In Chapter II we study the structure of initial segments of the Δ^1_{2n+1}-degrees, assuming Projective Determinacy. We show that every finite distributive lattice is isomorphic to such an initial segment, and hence that the first-order theory of the ordering of Δ^1_{2n+1}-degrees is undecidable.

In Chapter III we extend Friedberg's Jump Inversion theorem to Q_{2n+1}-degrees, after noticing that it fails for Δ^1_{2n+1}-degrees. We assume again Projective Determinacy.

Table of contents

Chapter 0	Background and definitions	1
Chapter I	On the Ramsey property	4
Chapter II	Initial segments of Δ^1_{2n+1}-degrees	23
Chapter III	The Jump Inversion theorem for Q_{2n+1}-degrees	45

Chapter 0

BACKGROUND AND DEFINITIONS

Our basic theory is ZF + DC; other hypotheses are explicitly stated. We denote Projective Determinacy by PD.

For definitions of the recursive, arithmetical, analytical and projective sets in product spaces of ω, 2^ω and ω^ω and for their basic properties we refer to [15] and [17]. For set-theoretic background see [5]. Our terminology and notation is in general that of [15].

Definition 0.1 A pointclass Γ is reflecting if for any $A \in \Gamma$, $A \subset \omega$ and any $P \in \Gamma$, $P \subset \omega^\omega$ we have

$$P(A) \Rightarrow \exists X \ (\ X \subset A \ \text{ and } \ X \in \Delta \ \text{ and } \ P(X) \ )$$

where $\Delta = \Gamma \cap \check{\Gamma}$.

The pointclass Π^1_1 is not reflecting; Σ^1_2 is. Under PD all pointclasses Π^1_{2n+1} , Σ^1_{2n+2} $(n > 0)$ are reflecting. For an account see [9].

Definition 0.2 (PD) C_{2n+1} is the largest countable Π^1_{2n+1} set of reals, and C_{2n+2} is the largest countable Σ^1_{2n+2} set of reals.

We mention some of their properties: C_{2n+2} is the set of reals that are recursive in some element of C_{2n+1} . The set C_m is made up of Δ^1_m - degrees (a Δ^1_m - degree is a set of reals that is an equivalence class for the equivalence relation $\alpha \equiv_{\Delta_m} \beta \Leftrightarrow \alpha \in \Delta^1_m(\beta)$ and $\beta \in \Delta^1_m(\alpha)$). The Δ^1_m - degrees in the set C_m are well-ordered by $\alpha \leq_{\Delta_m} \beta \Leftrightarrow \alpha \in \Delta^1_m(\beta)$. For these and other results see [7].

Definition 0.3 Given $S \subset \omega^\omega$ let $H_{2n+1}(S) = \{ \alpha : \forall\beta \in S (\alpha \in \Delta^1_{2n+1}(\beta)) \}$; we call it the hull of S. If S is a nonempty Σ^1_{2n+1} set then $H_{2n+1}(S)$ is called a Σ^1_{2n+1} - hull. We let now Q_{2n+1} = the union of all Σ^1_{2n+1} - hulls .

For an account of Q-theory, due to Kechris and Martin-Solovay, see [7] and [10]. We mention some results, assuming PD : The set Q_{2n+1} is Π^1_{2n+1} . Every Σ^1_{2n+1} - hull is Π^1_{2n+1} - bounded (this means that if $R(\alpha, x)$ is Π^1_{2n+1} then so is $\exists\alpha \in H_{2n+1}(S)\, R(\alpha, x)$). The set Q_{2n+1} is the largest Σ^1_{2n+1} - hull, and the largest Π^1_{2n+1} - bounded set. Relativizing to an arbitrary real β we may define the set $Q_{2n+1}(\beta)$. We define also $\alpha \leq_{Q_{2n+1}} \beta \Leftrightarrow \alpha \in Q_{2n+1}(\beta)$, and $\alpha \equiv_{Q_{2n+1}} \beta \Leftrightarrow \alpha \in Q_{2n+1}(\beta)$ and $\beta \in Q_{2n+1}(\alpha)$. This is an equivalence relation, and the equivalence classes are called Q_{2n+1} - degrees. The set C_{2n+1} consists of such degrees. The set Q_{2n+1} is the largest initial segment of C_{2n+1} closed under $\leq_{\Delta_{2n+1}}$; it consists of the Δ^1_{2n+1} - degrees in C_{2n+1} up to and not including the degree of the first nontrivial (i.e. non - Δ^1_{2n+1}) Π^1_{2n+1} singleton y_0^{2n+1}. Relativizing to α we have y_α^{2n+1}. If $\alpha \leq_{Q_{2n+1}} \beta$ then y_α^{2n+1}

$\leq_{\Delta_{2n+1}} y_\beta^{2n+1}$, and y_α^{2n+1} plays the role of the <u>jump</u> for Q_{2n+1} - degrees. The set Q_{2n+1} is closed under the Δ^1_{2n+1} - jump.

To obtain an <u>ordinal assignment</u> for the Q_{2n+1} - degrees we proceed as follows.

<u>Definition 0.4</u> $\lambda_{2n+1} = \sup\{\xi : \xi$ is the length of a Σ^1_{2n+1} wellfounded relation on $\omega^\omega\} = \sup\{\xi : \xi$ is the length of a Δ^1_{2n+1} prewellordering of $\omega^\omega\}$. Relativizing to α we obtain $\lambda_{2n+1}(\alpha)$. Finally $k_{2n+1}(\alpha) = \sup\{\lambda_{2n+1}(\langle\alpha,\beta\rangle) : \lambda_{2n+1}(\langle\alpha,\beta\rangle) < \lambda_{2n+1}(y_\alpha^{2n+1})\}$.

Of course λ_{2n+1} is the ordinal assignment for the Δ^1_{2n+1} - degrees, e.g. the Spector Criterion holds: $\underset{\sim}{d} \leq_{\Delta_{2n+1}} \underset{\sim}{e} \Rightarrow [\underset{\sim}{d}' \leq_{\Delta_{2n+1}} \underset{\sim}{e} \Leftrightarrow \lambda_{2n+1}(d) < \lambda_{2n+1}(e)]$. Now we have $\lambda_{2n+1}(\alpha) < k_{2n+1}(\alpha) < \lambda_{2n+1}(y_\alpha^{2n+1})$, $k_{2n+1}(\alpha)$ is invariant under $\equiv_{Q_{2n+1}}$,

$\alpha \leq_{Q_{2n+1}} \beta \Rightarrow k_{2n+1}(\alpha) \leq k_{2n+1}(\beta)$, and the Spector Criterion is true for Q_{2n+1} - degrees : $\underset{\sim}{d} \leq_{Q_{2n+1}} \underset{\sim}{e} \Rightarrow [\underset{\sim}{d}' \leq_{Q_{2n+1}} \underset{\sim}{e} \Leftrightarrow k_{2n+1}(d) < k_{2n+1}(e)]$. Naturally $\underset{\sim}{d}'$ is the degree of y_d^{2n+1}.

The relation $k_{2n+1}(\alpha) \leq k_{2n+1}(\beta)$ is Σ^1_{2n+1}.

We also give a generalization of Reflection: If P is Π^1_{2n+1} then $\exists\alpha \in Q_{2n+1}\ P(\alpha) \Rightarrow \exists\alpha \in \Delta^1_{2n+1}\ P(\alpha)$.

Chapter I

ON THE RAMSEY PROPERTY

1. Background

The first two levels of the analytical hierarchy admit an extensive theory, which can be developed within the framework of classical mathematics. This is no longer true for higher levels; there exist models of ZFC where basic theorems of the above-mentioned theory, appropriately generalized, hold true and other models of ZFC where the same theorems fail (assuming models of ZFC exist at all - but that is an article of faith). For an account of these matters see [15].

Various new axioms have been employed to remedy this. The Axiom of Constructibility gives a complete but rather pathological picture, while the Axiom of Measurable Cardinals can only prove results one step up the hierarchy, and then the independence phenomena resume. By far the most lively and fruitful new axiom has been the Axiom of Determinacy, in its various forms, e.g. Projective Determinacy or even full AD (This needs a word of explanation: AD contradicts the Axiom of Choice. However it is quite likely that AD holds in the model $L[\omega^{\omega}]$, and most questions of descriptive set theory relativize to that model. Consult [15].). Under Determinacy the basic theory of the first two levels generalizes to all levels.

Some of the basic theorems have been called Regularity Theorems ([12]); they ascribe nice properties to sets. Typical nice properties are Lebesgue measurability, the property of Baire and the Perfect Set property.

Now there is a certain pattern in the proofs of these theorems. We discuss first the property of Baire ([6] or [15]). Define the following game (Banach-Mazur game on the integers) : Given any set $A \subset \omega^{\omega}$ player I plays a finite sequence of integers $s_0 \in \omega^{<\omega}$, then player II plays $s_1 \in \omega^{<\omega}$, then player I plays $s_2 \in \omega^{<\omega}$, and so on. A run of the game produces a real, $s_0{}^\frown s_1{}^\frown s_2{}^\frown \ldots$. If this real belongs to A, I wins. If it belongs to the complement of A, II wins. It is not hard to establish the Banach-Mazur Theorem: Player I has a winning strategy in the above game iff A is comeager in some non-empty open set, and player II has a winning strategy iff A is meager. It follows that if for every closed set C this game on $A \cap C$ is determined (i.e. if either player has a winning strategy) then A has the property of Baire. Hence for every interesting pointclass the Determinacy of all games in it implies that every set in the pointclass has the property of Baire.

As another illustration, we define a game on a set $A \subset 2^{\omega}$. Player I plays $s_0 \in 2^{<\omega}$, then II plays $n_0 \in 2 = \{0, 1\}$, then I plays $s_1 \in 2^{<\omega}$, then II plays $n_1 \in 2$, etc. Player I wins iff $s_0{}^\frown n_0{}^\frown s_1{}^\frown n_1{}^\frown \ldots \in A$. It is easy to show that I has a winning strategy iff A has a nonempty perfect subset, and II has a winning strategy iff A is countable. Again we have that if this game is determined for sets in a certain pointclass then the Perfect Set property holds, i.e. every set in the pointclass either is countable or it contains a nonempty perfect subset.

The pattern is obvious: devise a game on A such that I has a winning strategy iff property $\chi(A)$ holds, and II has a winning strategy iff $\psi(A)$ holds. Then Determinacy ensures that $\chi(A)$ or $\psi(A)$ holds.

We pose now the question: does the Ramsey property fit the above pattern?

First some pertinent definitions. Let $A \subset [\omega]^{\omega}$ = the set of infinite sets of integers. Then A has a <u>homogeneous set</u> H if, by definition, $H \in [\omega]^{\omega}$ and either every infinite subset of H belongs to A or every infinite subset of H belongs to the complement of A. A has the <u>Ramsey property</u> iff it has a homogeneous set.

Not every set has the Ramsey property, but it takes a blunt use of the Axiom of Choice to furnish a counterexample: Well-order $[\omega]^{\omega}$ by $\prec$ and define $\mathcal{J} \subset [\omega]^{\omega}$ by $S \in \mathcal{J} \Leftrightarrow \exists T\ (T \subset S$ and $T \prec S)$. Then $\mathcal{J}$ has no homogeneous set.

On the other hand, there are many positive results about the Ramsey property. We list some of them:

<u>Theorem 1.1</u> (Galvin-Prikry, [3]) Borel sets have the Ramsey property.

<u>Theorem 1.2</u> (Silver, [19]) Analytic sets have the Ramsey property.

<u>Theorem 1.3</u> (Silver, [19]) Assuming measurable cardinals exist, $\underset{\sim}{\Sigma}^1_2$ sets have the Ramsey property.

<u>Theorem 1.4</u> (Solovay, Harrington-Kechris, [4]) Assuming Projective Determinacy, projective sets have the Ramsey property.

<u>Theorem 1.5</u> (Prikry, [16]) Assuming $AD_{\mathbb{R}}$, all sets have the Ramsey property.

An easy consequence of unpublished results of Martin, Moschovakis, Solovay and Steel is

<u>Theorem 1.6</u> Assuming AD + $V=L[\omega^{\omega}]$, all sets have the Ramsey property.

Also, Solovay has proved some results about the complexity of homogeneous sets:

<u>Theorem 1.7</u> (Solovay, [20]) A Σ^0_1 set either has a hyperarithmetical homogeneous set in the Σ^0_1 side or else an arbitrary homogeneous set in the Π^0_1 side (the arbitrary set is actually recursive in Kleene's $\mathcal{O}$, by the Kleene Basis Theorem). A Δ^0_1 set has a hyperarithmetical homogeneous set.

<u>Theorem 1.8</u> (Solovay, [20]) A hyperarithmetical set has a homogeneous set in L_α, where α is the first recursively inaccessible ordinal.

Optimal bases for Π^1_1 sides of partitions are not known. Similarly for Σ^1_2.

We return now to our question: can we obtain the Ramsey property by an appropriate game, like the other Regularity properties?

A clue comes from Ellentuck's proof of Theorem 1.2 ([1]). He identifies sets having the "completely Ramsey" property with sets having the property of Baire in the Mathias topology. The definitions are as follows: If s is a finite set of integers and A an infinite one, with every member of s less than any member of A (denoted $s < A$), we call $\langle s,A\rangle$ a Mathias condition. A set $X \in [\omega]^{\omega}$ belongs to the Mathias neighborhood (s,A) iff $s \subset X \subset s \cup A$. Condition $\langle s,A\rangle$ extends $\langle t,B\rangle$ iff $t \subset s$ and $s-t \subset B$ and $A \subset B$; this is a partial ordering. The Mathias topology is strictly finer than the classical one on $[\omega]^{\omega}$. Finally, $P \subset [\omega]^{\omega}$ is completely Ramsey iff for every Mathias condition $\langle s,A\rangle$ there is an extension $\langle s,A'\rangle$ (i.e. $A' \subset A$) with $(s,A') \subset P$ or $(s,A') \subset [\omega]^{\omega}-P$. This is stronger than the Ramsey property, which says only that there exists an A with $(\emptyset,A) \subset P$ or $(\emptyset,A) \subset [\omega]^{\omega}-P$.

One may define a Banach-Mazur game on any p.o. set (the one we defined in page 5 was on $\omega^{<\omega}$). Player I plays some condition p_0, then II plays p_1 extending p_0, then I plays p_2 extending p_1, etc. If the sequence $p_0, p_1, \ldots$ determines a real in some pre-specified way (e.g. for Mathias conditions $s_0 \cup s_1 \cup s_2 \cup \ldots$) then we have a game on a set of reals, and in certain cases (e.g. if the p.o. set is countable) the Banach-Mazur theorem holds ([8]).

Prikry used the Banach-Mazur game with Mathias conditions to establish, from $AD_{\mathbb{R}}$, that all sets are Ramsey (Theorem 1.5). The Mathias topology does not have a countable basis, but by a result of Oxtoby the Banach-Mazur theorem holds if one assumes some form of the

Axiom of Choice - "there exists a wellordering of the reals" suffices for Oxtoby's proof. Using this and Ellentuck's results Prikry proves that

$$\forall P \subset [\omega]^{\omega} \Big(\text{(I has a winning strategy in the Banach-Mazur game)} \Leftrightarrow \exists\langle s,A\rangle \ \forall\langle t,B\rangle \leq \langle s,A\rangle \ \exists C \subseteq B \ [\ (t,C) \subset P\ ]$$

$$\text{and}$$

$$\text{(II has a winning strategy in the Banach-Mazur game)} \Leftrightarrow \forall\langle s,A\rangle \ \exists B \subset A \ [\ (s,B) \subset [\omega]^{\omega} - P\ ] \Big).$$

Prikry's actual statement is slightly weaker; the above version follows from his proof. Now he uses a metamathematical trick: the above sentence is Π^2_1 , and it has been proved in ZF + "there exists a wellordering of the reals" ; a well-known lemma says that it can be proved in ZF + $DC_{\mathbb{R}}$. Hence $AD_{\mathbb{R}}$ easily implies that every set has the Ramsey property.

We would like to find a direct proof, starting with a winning strategy and using it to construct the homogeneous set. This would follow the pattern described earlier; the proofs of the Regularity theorems are quite direct. Also, a direct method might be useful in proving effective versions of the Ramsey theorem, i.e. calculating the complexity of homogeneous sets.

We have not found such a direct proof using the above game. For one thing, it seems closely related to the completely Ramsey property, which is stronger than Ramsey. A different game, however, similar to the one used in [13], works fine for the Ramsey property, and the proof is constructive. We present the game and the theorem concerning it in the next section.

2. <u>The main theorem</u>

For $\varphi \subset [\omega]^{\omega}$ we define the game G_{φ} as follows:

I	II	
A_0		
	n_0, B_0	$n_0 \in A_0$, $B_0 \subset A_0$, $n_0 < B_0$
A_1		$A_1 \subset B_0$
	n_1, B_1	$n_1 \in A_1$, $B_1 \subset A_1$, $n_1 < B_1$

etc.

I wins iff $\{ n_0 , n_1 , \ldots \} \in \varphi$

Capital letters denote infinite sets of integers.

We have now the following theorem.

<u>Theorem 2.1</u> a) I has a winning strategy in G_{φ} iff there is a homogeneous set in φ (i.e. an infinite H such that every infinite subset of it belongs to φ).

b) II has a winning strategy in G_{φ} iff for every A there is a subset of it homogeneous in $[\omega]^{\omega}-\varphi$.

<u>Proof of a)</u> Let τ be a winning strategy for I. Since any run of the game where I follows τ produces a set in φ it is enough to find a particular run, producing H, such that for any $H' \subset H$ there is some run of the game using τ and producing H'. To ensure this we build $\{ a_0 , a_1 \ldots \} = H$ by choosing appropriate moves for II, using the following construction.

Suppose $\tau(\emptyset) = A_0$ (i.e. τ instructs I to play A_0 as his first move). Call any string A_0 , $\langle n_0, B_0 \rangle$, $\tau(A_0 , \langle n_0, B_0 \rangle)$, $\langle n_1, B_1 \rangle$, $\ldots$ ending with some $\tau(\ldots)$ a <u>partial run</u> of the game with I following τ. Define also $A|m = \{ n : n \in A \text{ and } n > m \}$.

Stage 0 Let $a_0 = \min A_0$

Stage 1 Index the substages by members of $\mathcal{P}(\emptyset)$ with a_0 adjoined:

Substage $\{a_0\}$ Let $\tau(A_0 , \langle a_0, A_0|a_0\rangle) = A_1$.

Let $a_1 = \min A_1$.

Stage 2 Index the substages by members of $\mathcal{P}(\{a_0\})$ with a_1 adjoined:

Substage $\{a_1\}$ Let $\tau(A_0 , \langle a_1, A_1|a_1\rangle) = A_2^1$

Substage $\{a_0, a_1\}$ Let $\tau(A_0 , \langle a_0, A_0|a_0\rangle , A_1 , \langle a_1, A_2^1\rangle)$ $= A_2^2 \equiv A_2$.

Let $a_2 = \min A_2$.

Stage 3 Index the substages by members of $\mathcal{P}(\{a_0, a_1\})$ with a_2 adjoined:

Substage $\{a_2\}$ $\tau(A_0, \langle a_2, A_2|a_2\rangle) = A_3^1$

Substage $\{a_1, a_2\}$ $\tau(A_0, \langle a_1, A_1|a_1\rangle, A_2^1, \langle a_2, A_3^1\rangle) = A_3^2$

Substage $\{a_0, a_2\}$ $\tau(A_0, \langle a_0, A_0|a_0\rangle, A_1, \langle a_2, A_3^2\rangle) = A_3^3$

Substage $\{a_0, a_1, a_2\}$ $\tau(A_0, \langle a_0, A_0|a_0\rangle, A_1, \langle a_1, A_2^1\rangle,$ $A_2^2, \langle a_2, A_3^3\rangle) = A_3^4 \equiv A_3$.

Let $a_3 = \min A_3$.

Before defining Stage k+1 note that $A_0 \supset A_1 \supset A_2^1 \supset A_2^2 \supset A_3^1 \supset$ $\supset A_3^2 \supset A_3^3 \supset A_3^4 \ldots$ Note also that the partial run of the game corresponding to, say, $\{x, y, z\}$ is a continuation of the partial run for $\{x, y\}$; all the partial runs follow τ. This is the state of affairs we want to preserve.

Now suppose Stage k has been completed, with $a_k = \min A_k$.

<u>Stage k+1</u> Consider $\mathcal{P}(\{a_0, a_1, \ldots a_{k-1}\})$ and adjoin a_k to each one of its members, obtaining the finite sets $s_1, s_2, \ldots s_m$ $(m = 2^k)$ which will index the substages. (Note: When we describe this whole construction on the binary tree a specific ordering will arise.)

<u>Substage s_1</u> Locate the partial run for $s_1-\{a_k\}$ in some previous stage, append $\langle a_k, A_k | a_k\rangle$ as a move for II and apply τ to obtain A^1_{k+1} .

<u>Substage s_2</u> Locate the partial run for $s_2-\{a_k\}$ in some previous stage, append $\langle a_k, A^1_{k+1}\rangle$ and apply τ to obtain A^2_{k+1} .

.

.

.

<u>Substage s_m</u> Locate the partial run for $s_m-\{a_k\}$ in some previous stage, append $\langle a_k, A^{m-1}_{k+1}\rangle$ and apply τ to obtain $A^m_{k+1} \equiv$ $\equiv A_{k+1}$.

Let $a_{k+1} = \min A_{k+1}$.

This completes the description of the construction.

Another way to present the construction is the binary tree diagram in Figure 1, page 13. I's moves are given by τ, II's moves are chosen as shown. Of course $a_i = \min A_i$. The set $H = \{a_0, a_1, \ldots\}$ is obtained from the run of the game developing on the leftmost branch of the tree, i.e. A_0, $\langle a_0, A_0 | a_0\rangle$, A_1, $\langle a_1, A^1_2\rangle$, A_2, $\langle a_2, A^3_3\rangle$, ...

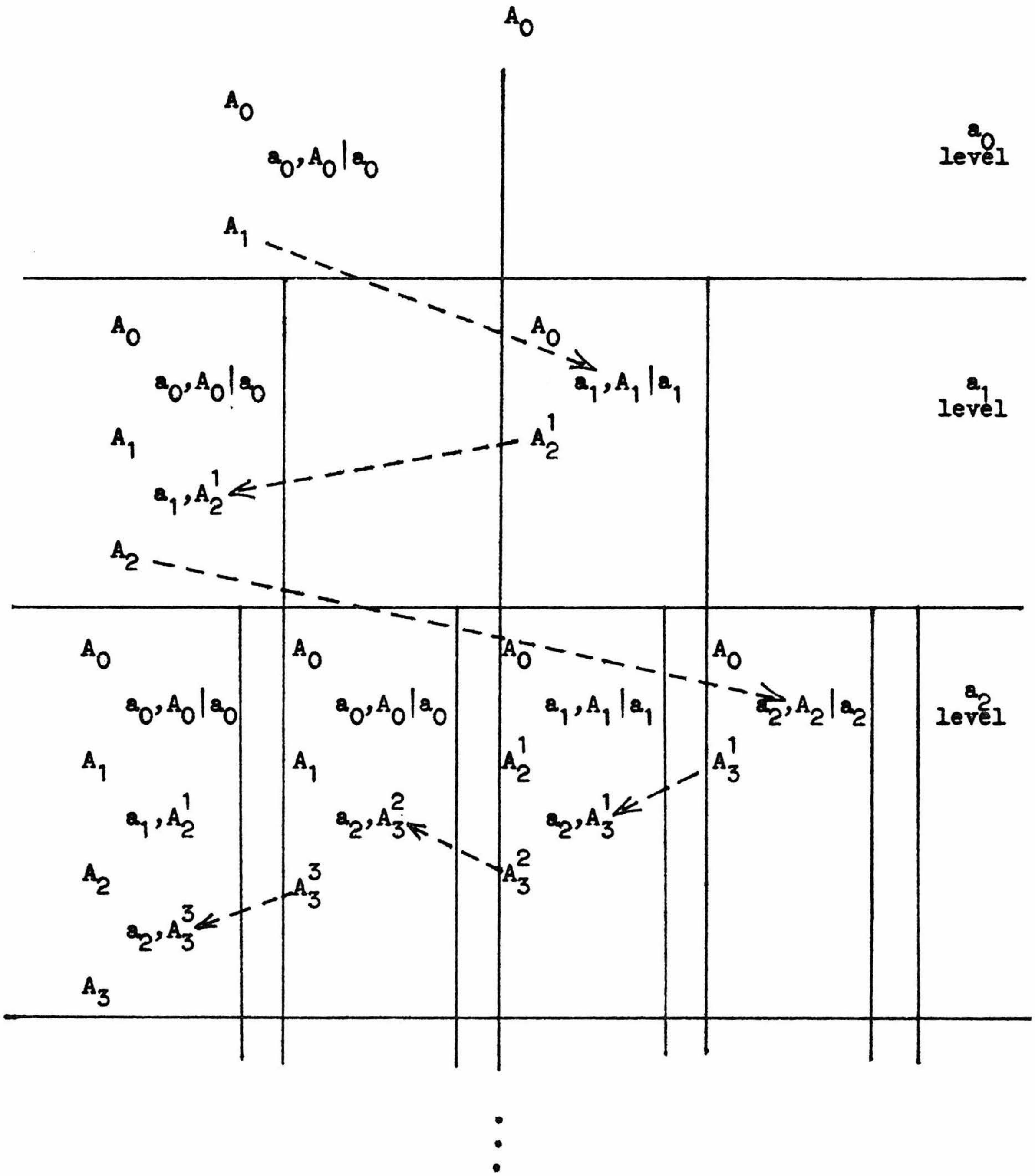

A_0
A_0
$a_0, A_0 | a_0$
A_1
a_0 level
A_0
$a_0, A_0 | a_0$
A_1
a_1, A_2^1
A_2
A_0
$a_1, A_1 | a_1$
A_2^1
a_1 level
A_0 A_0 A_0 A_0
$a_0, A_0 | a_0$ $a_0, A_0 | a_0$ $a_1, A_1 | a_1$ $a_2, A_2 | a_2$
a_2 level
A_1 A_1 A_2^1 A_3^1
a_1, A_2^1 a_2, A_3^2 a_2, A_3^1
A_2 A_3^3 A_3^2
a_2, A_3^3
A_3

Figure 1

We prove now that H is homogeneous in φ . The infinite subsets of H correspond to branches of the tree turning left infinitely often (at every splitting of the tree the right part is blank and the left contains a partial run of the game). For any infinite subset H' = $= \{ a_{i_1}, a_{i_2}, \dots \}$ we can find in stage i_1+1 (or using the branch in the tree) a partial run for $\{a_{i_1}\}$. Then we can find in stage i_2+1 (or again using the tree) a partial run for $\{a_{i_1}, a_{i_2}\}$ which extends the previous one, and so on. Hence there is a run of the game following τ and producing H'; therefore $H' \in \varphi$.

The converse of (a) is immediate: if there is a homogeneous set in φ then I plays it in his first move and ensures the win (e.g. he copies II's moves from then on).

Proof of b) Suppose II has a winning strategy σ . First we prove a lemma.

The σ_∞ lemma For every A_0 there exists an A, $A \subset A_0$, so that for every $m \in A$ there is an X and a Y, $Y \supset A|m$, with $\sigma(X) = \langle m, Y\rangle$. In fact for every partial run $C_1, \langle j_1, D_1\rangle, \dots, C_i, \langle j_i, D_i\rangle, A_0$ the same conclusion holds: there exists an A, $A \subset A_0$, so that for every $m \in A$ there is an X and a Y, $Y \supset A|m$, with $\sigma(C_1, \langle j_1, D_1\rangle, \dots, C_i, \langle j_i, D_i\rangle, X) = \langle m, Y\rangle$.

Proof of the lemma Let

$$\sigma(C_1, \langle j_1, D_1\rangle, \dots , C_i, \langle j_i, D_i\rangle, A_0) = \langle m_0, B_0\rangle$$

$$\sigma(C_1, \langle j_1, D_1\rangle, \dots , C_i, \langle j_i, D_i\rangle, B_0) = \langle m_1, B_1\rangle$$

$$\sigma(C_1, \langle j_1, D_1\rangle, \ldots, C_i, \langle j_i, D_i\rangle, B_1) = \langle m_2, B_2\rangle$$

.

.

.

and so on. Then $A = \{ m_0, m_1, m_2, \ldots \}$ has the property stated.

It is important to note that all properties of A in the lemma are inherited by any subset of A, i.e. the σ_∞ lemma holds for every $A' \subset A$. This is crucial for the construction that follows.

To obtain a homogeneous set we adapt the idea in the proof of (a): Use as induction hypothesis that when $\{ n_0, n_1, \ldots, n_k \}$ has been constructed every finite subset of it s arises from some partial run following σ. Then n_{k+1} must be chosen so that for every $s \cup \{ n_{k+1} \}$ there is a partial run following σ, in fact one that extends the partial run for s.

The construction below achieves this. For the sake of clarity we also give a binary tree version. Note that a partial run ending with a move of I is called a <u>position</u> for brevity.

<u>Stage 0</u> Suppose I's first move is A_0. Let $\sigma(A_0) = \langle n_0, C_0^0\rangle$. We provide for subsets of the eventual H that start with an integer other than n_0 :

<u>Substage {...}</u> Apply the σ_∞ lemma to the position C_0^0. Call the result B_0.

<u>Stage 1</u> $\sigma(A_0, \langle n_0, C_0^0\rangle; B_0) = \langle n_1, C_1^0\rangle$. We provide now for subsets starting with n_1 , and then for subsets starting with n_0 not followed by n_1 :

<u>Substage $\{n_1, \ldots\}$</u> Consider the position X^1, $\langle n_1, Y^1\rangle$, C_1^0 where X^1 and $\langle n_1, Y^1\rangle$ exist because of the σ_∞-lemma construction of B_0 ; it is easy to see that $C_1^0 \subset Y^1$. Apply the σ_∞ lemma to obtain C_1^1 .

<u>Substage $\{n_0, , \ldots\}$</u> Consider the position A_0, $\langle n_0, C_0^0\rangle$ C_1^1 . Apply the σ_∞ lemma to obtain C_1^2 , and rename it B_1.

<u>Stage 2</u> $\sigma(A_0, \langle n_0, C_0^0\rangle; B_0, \langle n_1, C_1^0\rangle; B_1) = \langle n_2, C_2^0\rangle$. We provide successively for subsets of the type $\{n_2, \ldots\}$, $\{n_1, n_2, \ldots\}$, $\{n_0, n_2, \ldots\}$ and $\{n_0, n_1, , \ldots\}$.

<u>Substage $\{n_2, \ldots\}$</u> Consider the position X^2, $\langle n_2, Y^2\rangle$, C_2^0 where we use again the σ_∞-lemma construction of B_0 . Apply the σ_∞ lemma to obtain C_2^1.

<u>Substage $\{n_1, n_2, \ldots\}$</u> Consider the position X^1, $\langle n_1, Y^1\rangle$, X^{12}, $\langle n_2, Y^{12}\rangle$, C_2^1 ; here X^1 and Y^1 were available already, while X^{12} and Y^{12} exist because of the σ_∞-lemma construction of C_1^1. Apply the σ_∞ lemma to obtain C_2^2 .

<u>Substage $\{n_0, n_2, \ldots\}$</u> Consider the position A_0, $\langle n_0, C_0^0\rangle$, X^{02}, $\langle n_2, Y^{02}\rangle$, C_2^2. As before we have used the σ_∞-lemma construction of B_1. Apply the σ_∞ lemma to obtain C_2^3 .

<u>Substage $\{n_0, n_1, , \ldots\}$</u> Consider the position A_0, $\langle n_0, C_0^0\rangle$, B_0, $\langle n_1, C_1^0\rangle$, C_2^3 . Apply the σ_∞ lemma to obtain C_2^4 , and rename it B_2 .

<u>Stage 3</u> $\sigma(A_0, \langle n_0, C_0^0\rangle; B_0, \langle n_1, C_1^0\rangle; B_1, \langle n_2, C_2^0\rangle; B_2) = \langle n_3, C_3^0\rangle$

... and so on.

The above exemplifies all the essential features of the construction, so that Stage k+1 should be clear. We omit its description, which would involve a mess of indices anyway.

The binary tree version of the above construction appears in Figure 2, page 18. Within each layer we proceed from right to left. At each splitting the box to the right corresponds to $n_i \notin$ the set, the box to the left corresponds to $n_i \in$ the set. All the right boxes are blank except for the last one on each layer. Downward arrows denote applications of the σ_∞ lemma. Player II's moves are dictated by σ, while I's moves are either copied in or they come from some application of the σ_∞ lemma (if they are X's). In fact one reads upwards until one meets a box with a downward arrow, i.e. an application of the σ_∞ lemma; one then uses it.

Clearly $C_0^0 \supset B_0 \supset C_1^0 \supset B_1 \supset C_2^0 \supset \ldots$ and within each partial run all sets behave, because of the properties ensured by the σ_∞ lemma.

The set $H = \{ n_0, n_1, n_2, \ldots\}$ arises from the run of the game developing on the leftmost branch of the tree, and σ has been followed in that run, so $H \in [\omega]^\omega - \varphi$. We prove now the homogeneity of H: If H' is an infinite subset of H then by following the corresponding branch in the tree we find coherent initial segments giving a run that follows σ and produces H'. For example if

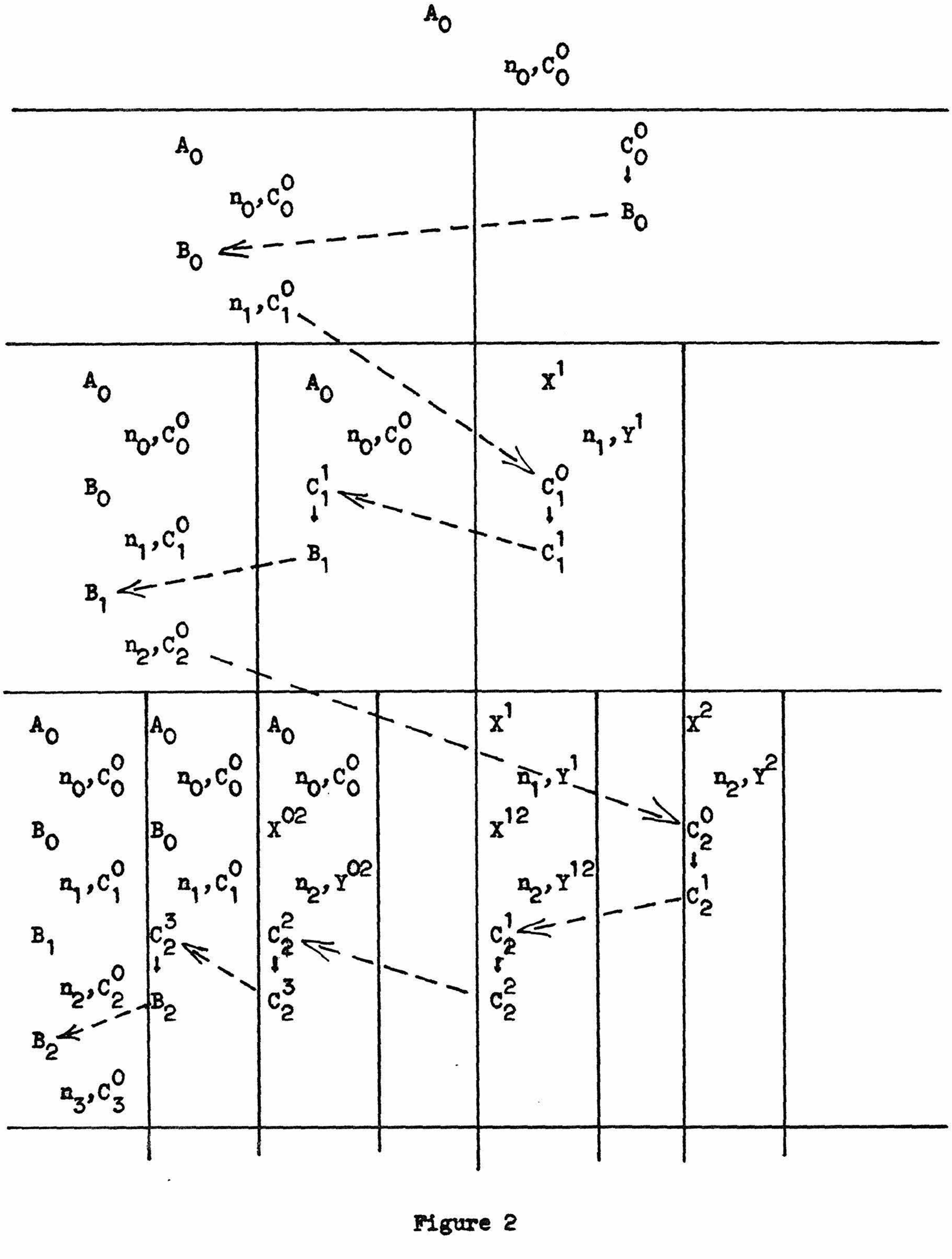

A0
n0,C0
C0
B0
B0
n1,C1
A0
A0
X1
n0,C0
n0,C0
n1,Y1
B0
C1
C1
n1,C1
B1
C1
B1
n2,C2
A0
A0
A0
X1
X2
n0,C0
n0,C0
n0,C0
n1,Y1
n2,Y2
B0
B0
X02
X12
C2
n1,C1
n1,C1
n2,Y02
n2,Y12
C2
B1
C2
C2
C2
n2,C2
B2
C2
C2
B2
n3,C3

Figure 2

$H' = \{ n_1, n_3, \dots \}$ then the run is X^1, $\langle n_1, Y^1 \rangle$, X^{13}, $\langle n_3, Y^{13} \rangle, \dots$ or if $H' = \{ n_0, n_2, n_3, \dots \}$ then the run is A_0, $\langle n_0, C_0^0 \rangle$, X^{02}, $\langle n_2, Y^{02} \rangle$, X^{023}, $\langle n_3, Y^{023} \rangle$, ... Since the run follows σ we have that $H' \in [\omega]^\omega - \varphi$.

The converse of (b) is again immediate: II plays the homogeneous set.

This concludes the proof of Theorem 2.1.

Remark The theorem holds also for G_φ played in $\langle s, A \rangle$ (instead of $\langle \emptyset, \omega \rangle$). This means that I's first move is some $A_0 \subset A$, II's first move is $\langle n_0, B_0 \rangle$ with $n_0 \in A_0$, $B_0 \subset A_0$ and $n_0 < B_0$, and so on; I wins iff $s \cup \{ n_0, n_1, \dots \} \in \varphi$. Then the theorem says that I has a winning strategy iff there is a homogeneous set in φ that lies in (s,A), and II has a winning strategy iff for every A' subset of A there is a homogeneous set in $[\omega]^\omega - \varphi$ that lies in (s,A').

3. Consequences and effectivization

Using Theorem 2.1 we can give alternative proofs of some of the results mentioned in Section 1. We have immediately a proof of the Galvin-Prikry theorem (Theorem 1.1): If φ is Borel then G_φ is determined by Martin's theorem ([14]); hence φ has a homogeneous set.

Likewise Prikry's theorem (Theorem 1.5) is a direct corollary of our result. It is an open problem whether $AD_{\mathbb{R}}$ may be replaced by AD in that theorem.

We turn now to effective results, motivated by Solovay's theorem (Theorem 1.7). Can we calculate the complexity of some homogeneous set if we know the complexity of the partition?

Theorem 3.1 (Kechris) (PD) A Π^1_{2n+1} partition has a Δ^1_{2n+1} homogeneous set in the Π^1_{2n+1} side, or some homogeneous set in the Σ^1_{2n+1} side. A Δ^1_{2n+1} partition has a Δ^1_{2n+1} homogeneous set. $(n \geq 1)$

Kechris' proof of the above result (unpublished) uses other ideas. With our methods we have only obtained the following partial result.

First, for each $\varphi \subset [\omega]^\omega$ define the game G^*_φ :

I	II	
A_0		s_0 is a finite subset of A_0 , B_0 is
	s_0, B_0	a subset of A_0 , $s_0 < B_0$. A_1 is a
A_1		subset of B_0. s_1 is a finite subset
	s_1, B_1	of A_1 , $B_1 \subset A_1$, $s_1 < B_1$ etc.

I wins iff $s_0 \cup s_1 \cup \ldots \in \varphi$.

We also assume, as part of the definition, that A_i, $B_i \in \Delta^1_{2n+1}$ and $\varphi \in \Pi^1_{2n+1}$.

We have then

<u>Theorem 3.2</u> (PD) I has a winning strategy in G^*_φ iff there is a Δ^1_{2n+1} homogeneous set in φ $(n \geq 1)$.

Before proving the theorem we discuss the ideas involved. We want to use Moschovakis' Third Periodicity Theorem ([15]) to obtain a definable winning strategy for I, and then use it in the manner of Section 2 to construct a definable homogeneous set. Now immediate application of Moschovakis' theorem is not useful because the payoff set is too complicated; however we can use Kechris' Asymmetric Game Formula ([8]) to reduce this complexity. The formula does not seem to apply to G_φ ; this is why we work with G^*_φ . There is still a problem with the Δ^1_{2n+1} character of the moves; one needs some way to describe them, e.g. the complete Π^1_{2n+1} set of integers W. This means that W will enter as a parameter; we take care of this by a reflection argument.

<u>Proof of the theorem</u> It is clear that if a Δ^1_{2n+1} homogeneous set exists then I plays it in his first move and wins the game.

For the converse, assume I has a winning strategy. We have then

$$\exists A_0 \ \forall s_0, B_0 \ \exists A_1 \ \ldots \ \forall \gamma \ S(\gamma, \sigma) \qquad \sigma = \cup s_i$$

for some $S \in \Sigma^1_{2n}$. By the Asymmetric Game Formula ([8],Appendix) we have that the above statement is equivalent to

$$\exists A_0 \ \forall s_0, B_0, \gamma(0) \ \exists A_1 \ \forall s_1, B_1, \gamma(1) \ \ldots \ S(\gamma, \sigma) \ .$$

Now we apply the Third Periodicity Theorem: since the description of permissible moves is recursive in W, the complete Π^1_{2n+1} set of integers, we have that there exists a winning strategy for I, τ, that is $\Delta^1_{2n+1}(W)$. It is easy to see that τ may be used to win G_φ , too. Applying the procedure of the proof of Theorem 2.1(a) we obtain A, a homogeneous set for the Π^1_{2n+1} side, with $A \in \Delta^1_{2n+1}(W)$. But the property of being homogeneous for a Π^1_{2n+1} set,

$$\forall B \; [\; B \subset H \Rightarrow \varphi(B) \;] \; ,$$

is itself Π^1_{2n+1} . Hence by Reflection (see Chapter 0) there exists a homogeneous set that is Δ^1_{2n+1}.

This concludes the proof of Theorem 3.2. To prove Theorem 3.1 by these methods it must be shown that if II has a winning strategy then there is some homogeneous set in the Σ^1_{2n+1} side of the partition.

Chapter II

INITIAL SEGMENTS OF Δ^1_{2n+1}-DEGREES

The purpose of this chapter is to prove a result about the structure of initial segments of the Δ^1_{2n+1}-degrees, partially ordered by $\leq_{\Delta_{2n+1}}$. (For definitions see Chapter 0, page 2)

Theorem (PD) Any finite distributive lattice is isomorphic to an initial segment of the Δ^1_{2n+1}-degrees.

Corollary (PD) The first-order theory of the Δ^1_{2n+1}-degrees with $\leq_{\Delta_{2n+1}}$ is undecidable.

These results have been proved in [22] for the case $n = 0$. We prove them for $n \geq 1$ below. For notational simplicity we work with $2n+1 = 3$ throughout.

1. Preliminaries

The following lemma gives useful information about $\leq_{\Delta_3}$.

Lemma 1.1 (PD) There is a fixed sequence $\{F_i\}$ of Δ^1_3 functions such that if $\lambda^\beta_3 = \lambda^0_3$ then $\alpha \leq_{\Delta_3} \beta \Leftrightarrow F_i(\beta) = \alpha$, for some i.

Proof [11].

This is a convenient characterization. To use it we must be able to find β's with the stated property, and this is what the next lemma furnishes. The definition of b-conditions and their ordering is in Section 3. The meaning of "for all sufficiently generic" (abbreviated f.a.s.g.) with respect to a partial ordering can be found in [8]; roughly, property A holds f.a.s.g. β iff for every condition p_0 there is a condition p_1 extending p_0 so that for every p_2 extending p_1 ... A holds for the real determined by the sequence $p_0, p_1, \ldots$

Lemma 1.2 (PD) For all sufficiently generic β (with respect to b-conditions), $\lambda_3^{\beta} = \lambda_3^{0}$.

Proof (Sketch) In [11] this lemma is shown for Δ_3^1 perfect trees, a particular case of b-conditions. However, beyond some general facts what is really used is the ability to carry out a fusion (or: splitting) argument. We show how to do this for b-conditions in Section 3, in the proof of Lemma 3.12. Hence the proof in [11] works in our more general setting.

To handle Δ_3^1 functions we need

Lemma 1.3 (PD) i) A total Δ_3^1 function is continuous on a comeager Δ_3^1 set. ii) A comeager Δ_3^1 set (in $(\omega^{\omega})^n$) contains a b-condition of the form $[T_1] \times [T_2] \times \ldots \times [T_n]$, where the T_i are Δ_3^1 perfect trees.

($[T]$ is the set of branches of the tree T)

Proof (Sketch) Again the proof of Lemma 1.7 in [11] suffices. For (ii) we perform a simple fusion argument, as in the proof of Lemma 3.12 .

Finally, we state the Δ-Selection Principle, the means of showing that various objects constructed are actually Δ^1_3 .

<u>The Δ-Selection Principle</u> (PD) If $\forall\alpha\, \exists n\ P(\alpha, n)$, with $P \in \Pi^1_3$, then there is a Δ^1_3 function f such that $P(\alpha, f(\alpha))$ holds.

<u>Proof</u> [15].

Let us also mention that Lemma 1.1 obviously holds for functions F of n variables, i.e. $\alpha \leq_{\Delta_3} \{ \beta_1, \beta_2, \dots \beta_n \}$ iff $F_i(\beta_1, \beta_2, \dots \beta_n) = \alpha$ for some F_i in a fixed countable sequence. In fact we may collect all such F's in a single countable sequence, thus providing for any n. Future uses of Lemma 1.1 tacitly assume this trivial extension.

2. Illustrative special cases

We consider the problem of finding initial segments isomorphic to diamond (i.e. $\mathcal{P}(2)$) and to the three-lattice (i.e. the linear ordering of three elements). This will illustrate the method and motivate some of the considerations in Section 3.

We use T's to denote Δ^1_3 perfect trees.

A) Proof for diamond

We use pairs (T_1, T_2) as conditions, a special case of the b-conditions of Section 3. Any condition determines the set of (α, β) such that $\alpha \in [T_1]$, $\beta \in [T_2]$. We order them naturally by inclusion.

We want an (α, β) such that $0, \alpha, \beta, \alpha \vee \beta$ realize diamond. (We abuse notation and confuse a real and its degree when convenient.) This will be the case if we take (α, β) sufficiently generic with respect to the notion of forcing (i.e. p.o. set) just described; we proceed to prove this.

It is well known that α (and β, of course) is generic with respect to Δ^1_3 perfect forcing, and consequently ([11]) is of minimal Δ^1_3 degree. That is, $x \leq \alpha$ implies $x \equiv \alpha$ or $x \equiv 0$ (we suppress the subscript Δ^1_3 from $\leq$ and $\equiv$). The proof is as follows: By Lemmas 1.2 and 1.1, $x \leq \alpha$ iff $F(\alpha) = x$. Now use Lemma 1.3 to claim that F is continuous on a comeager Δ^1_3 set, which contains a [T]. Find $T' \subset T$ so that F is either constant or one-to-one on [T′] (this well-known fact is proved in [11]). Since any [T] contains a Δ^1_3 real we have that $x \equiv 0$ or $x \equiv \alpha$.

It remains to show that $x < \alpha \vee \beta \Rightarrow x \leq \alpha$ or $x \leq \beta$. The argument will be as in the last paragraph, but instead of the "constant or one-to-one" property (which is not true any more) we use the following lemma.

Lemma 2.1 For every (T_1, T_2) there is a (T_1', T_2') contained in it such that

either F is constant on $\{\alpha\} \times [T_2']$, for all $\alpha \in [T_1']$

or F is constant on $[T_1'] \times \{\beta\}$, for all $\beta \in [T_2']$

or F is one-to-one (and continuous) on $[T_1'] \times [T_2']$.

Proof Using the by now familiar lemmas, F is continuous on a comeager Δ_3^1 set, which contains $[T] \times [T']$, for some $(T, T') \leq (T_1, T_2)$. This shows that without loss of generality we may assume F to be continuous to begin with.

Suppose the first two alternatives in the lemma fail, i.e. $\forall\, (T_1', T_2') \leq (T_1, T_2)$ the following hold:

$\exists \alpha \in [T_1']\ \exists \beta_1, \beta_2 \in [T_2']\quad F(\alpha, \beta_1) \neq F(\alpha, \beta_2)$ and

$\exists \beta \in [T_2']\ \exists \alpha_1, \alpha_2 \in [T_1']\quad F(\alpha_1, \beta) \neq F(\alpha_2, \beta)$.

These are used repeatedly to build a (T, T') on which F is one-to-one.

First, find α, γ_1 and γ_2 such that $F(\alpha, \gamma_1) \neq F(\alpha, \gamma_2)$. By continuity there exist initial segments of these reals s, t_1, t_2 such that for any α' starting with s, γ_1' starting with t_1 and γ_2' starting with t_2 $F(\alpha', \gamma_1')$ belongs to a neighborhood N_1, $F(\alpha', \gamma_2')$ belongs to a neighborhood N_2 and $N_1 \cap N_2 = \emptyset$. See Figure 3 in page 28.

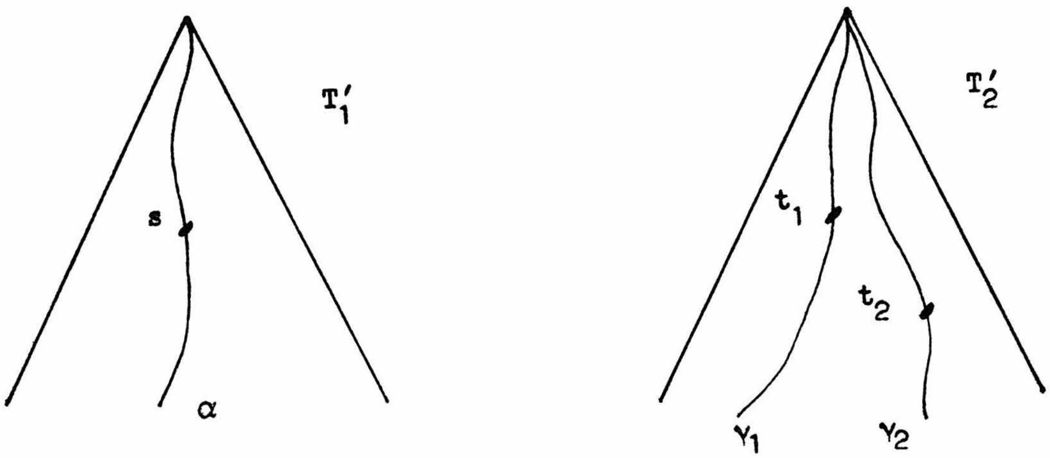


Figure 3

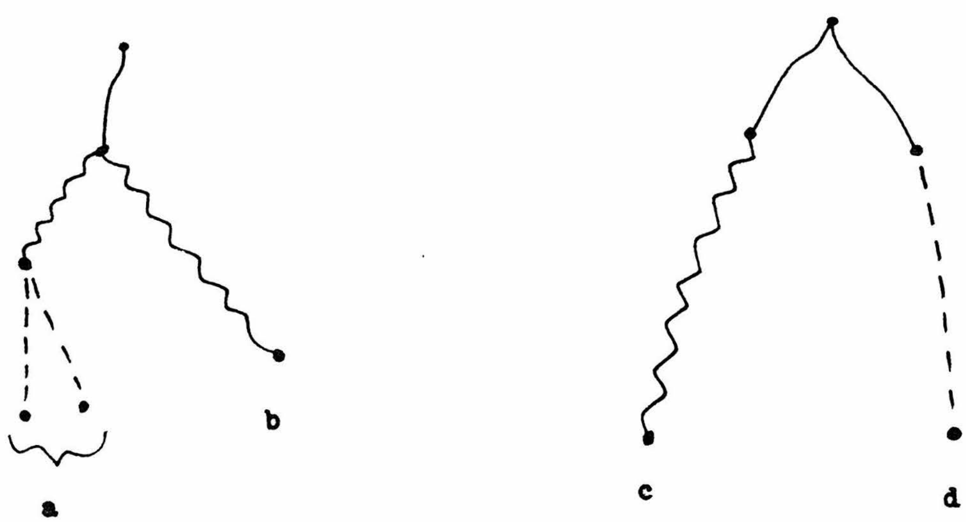


Figure 4

Now consider $(T_1')_s$, the perfect subtree extending s, $(T_2')_{t_1}$ and $(T_2')_{t_2}$. The above "separating" argument can be repeated. Repeat it twice, according to Figure 4 in page 28. Note that we have not picked a yet.

The inequality $F(\alpha_1, \beta_1) \neq F(\alpha_2, \beta_2)$ is satisfied if β_1 extends c and β_2 extends d (or the reverse), because of the solid line "separation". If they both extend c we still have the inequality if α_1 extends a (for either choice) and α_2 extends b (or the reverse), because of the wavy line "separation". To cover the remaining case we employ a "transfer": Consider some real α extending b and some real β extending d. Then $F(\alpha, \beta)$ will be outside at least one of the two neighborhoods produced by the broken line "separation". Ensure this by initial segments as before, (extending b and d in general) and keep the appropriate a. This is shown in Figure 5, page 30.

So we have the above inequality as long as (α_1, β_1) and (α_2, β_2) are not of the same type, where the types are ac, ad, bc, and bd. Now we iterate: the next step will produce incompatible extensions below each one of a,b,c,d with the same property for F. It is convenient first to perform extensions within T_1 and splittings within T_2 , using transfers to avoid more than one splitting in T_1, and then, after taking care of all cases, to reverse the procedure. See Figure 6, page 30, for the first part. The second part will involve extensions only for c_1, c_2, d_1, d_2 and splittings under

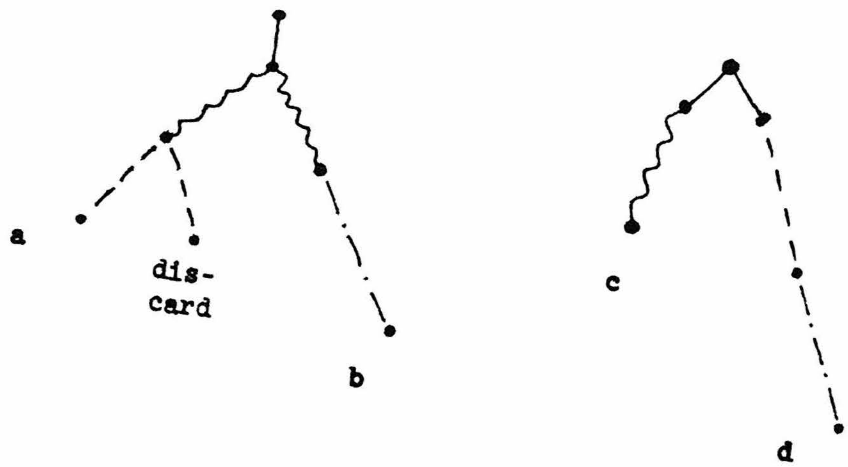


Figure 5

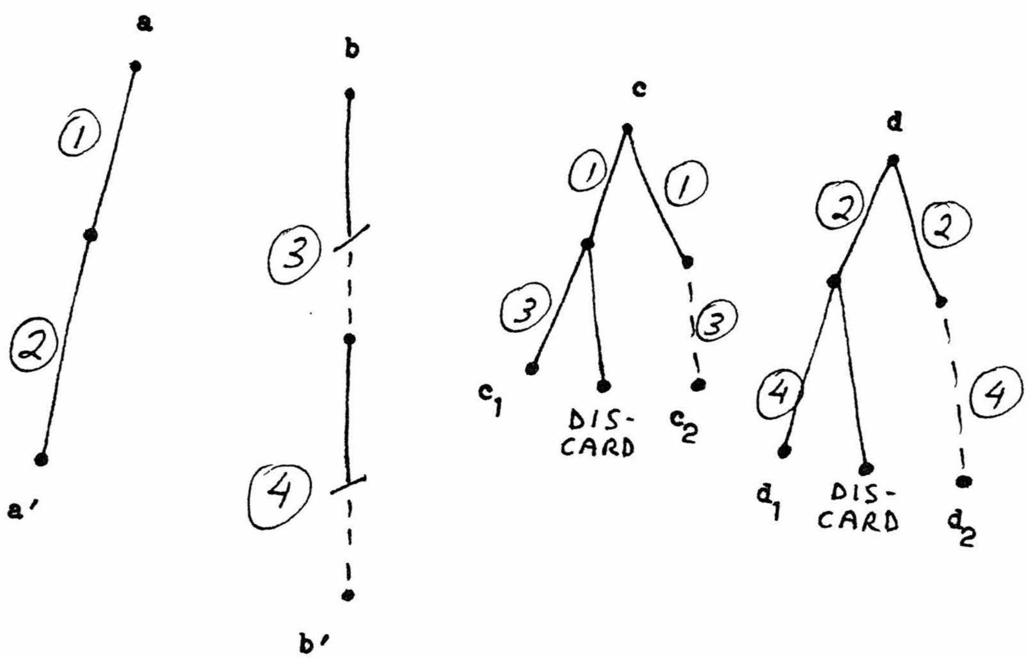


Figure 6

a' and b'. We have then a_1, a_2, b_1, b_2 and c_1, c_2, d_1, d_2 (not the same as the previous ones) and we are ready for the next step.

Continuing this procedure we obtain two perfect trees T and T′, and F is clearly one-to-one on $[T] \times [T']$ (if (α, β) and (α_1, β_1) differ then this happens at some finite stage; hence $F(\alpha, \beta) \neq$ $\neq F(\alpha_1, \beta_1)$). Using Δ^1_3-Selection we easily see that both trees are Δ^1_3. This concludes the proof of the lemma.

To finish the proof for diamond suppose $x \leq \alpha \vee \beta$. By genericity and the lemmas in Section 1 this is equivalent to $F(\alpha, \beta) = x$, for some F in the countable sequence. Apply now Lemma 2.1: if F is one-to-one on some condition we have $x \equiv \alpha \vee \beta$, if it is constant on some coordinate we have $x \leq \alpha$ or $x \leq \beta$, since [T] contains Δ^1_3 reals.

Finally, α and β cannot be of the same degree by the genericity of (α, β) and the fact that only countably many reals occupy a single degree.

The proof for diamond is now complete.

Remark Using $(T_0, \ldots, T_{n-1})$ we obtain an initial segment isomorphic to $\mathcal{P}(n)$.

B) Proof for the three-lattice

We want to find α, β so that $0 < \alpha < \alpha \vee \beta$ is an initial segment. Of course we must use different conditions.

Suppose we attempt to use the same argument. Instead of Lemma 2.1 we now need a lemma that will say, roughly, "either F is constant on all $\{\alpha\} \times [T]$ or it is one-to-one". In a sense we have a weaker hypothesis from which to obtain one-to-one-ness; so we will allow more general conditions. The definition is: (α, β) belongs to the condition p iff $\alpha \in [T]$ and $\beta \in [T_\alpha]$, where T_α depends continuously on α. Of course this whole object is assumed to be in Δ^1_3 . The ordering is by inclusion.

We have now

Lemma 2.2 $\forall q \quad \exists p \leq q$ so that

either F is constant on every $\{\alpha\} \times [T_\alpha]$ of p

or F is one-to-one on p.

Proof If the first alternative fails we have that

$\forall p \leq q \quad \exists \beta \quad \exists \gamma_1, \gamma_2 \in [T_\beta] \quad F(\beta, \gamma_1) \neq F(\beta, \gamma_2)$.

Apply this to q and obtain β, γ_1 and γ_2. Then find a subtree which avoids β and apply the above again, obtaining β', γ_1' and γ_2' such that $F(\beta', \gamma_1') \neq F(\beta', \gamma_2')$. See Figure 7, page 33.

We may assume that $F(\beta', \gamma_1')$ and $F(\beta', \gamma_2')$ are both different from $F(\beta, \gamma_1)$ and $F(\beta, \gamma_2)$, because otherwise we apply the hypothesis once again and select one of the two values, whichever works.

Using continuity we ensure this state of affairs by initial segments. Now the result is iterated and we obtain a condition on which F is one-to-one.

Clearly the argument in (A) plus Lemma 2.2 finish the proof for the three-lattice.

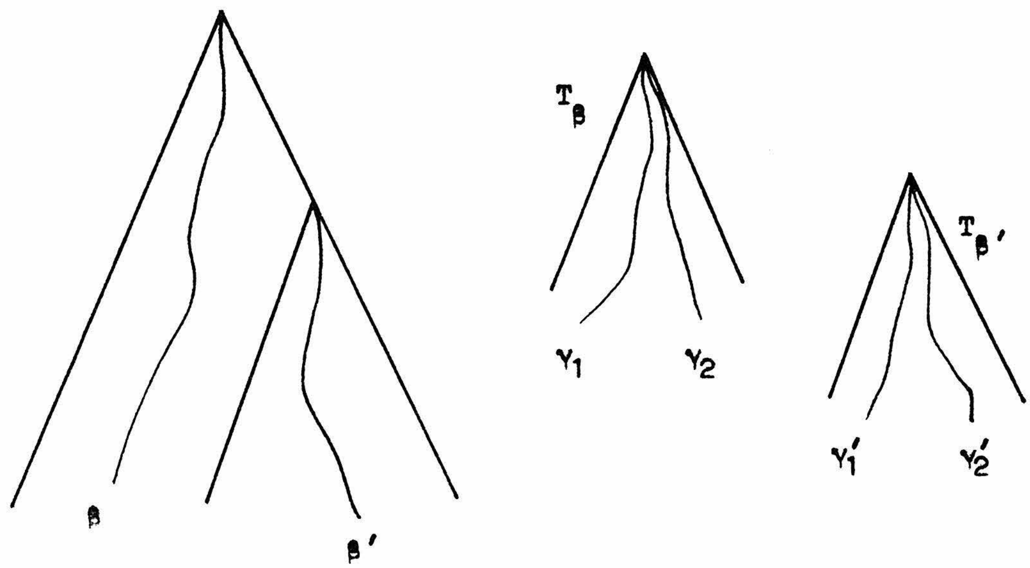


Figure 7

3. Proof of the theorem

A) Some preliminaries

The two cases discussed contain the germ of the general proof. To realize a sublattice of $\mathcal{P}(n)$, for each $a \leq b$ that holds in $\mathcal{P}(n)$ and does not hold in the sublattice we must "disperse" the $(T_0, T_1, \ldots, T_{n-1})$ condition in the appropriate coordinates, much as we did in proceeding from diamond to the three-lattice. This vague remark gives a clue for the general definition of conditions (the "$\equiv_a^b$-isomorphism" requirement below). Also, we must develop a general method for handling all the separation and transfer arguments in building conditions by fusion. The appropriate generalization of these arguments involves the notion of "a-splitting".

We begin by defining b-conditions and a-splittings and establishing their basic properties. For all this we owe an essential debt to [22].

B) b-conditions, a-splittings and their properties

Let $A \subset \omega$ be finite and L be a sublattice of $\mathcal{P}(A)$; this way we obtain all finite distributive lattices. The ordering is $\subset$, 0 is $\emptyset$ and 1 is A.

If $b \in L$ consider $(2^{\omega})^{b}$ and call its elements p, q, ... We explain notation by an example: if b is $\{0, 3, 4\}$ then p is $(\alpha_0, \alpha_3, \alpha_4)$ and p(3) is α_3, a binary real. For purposes of coding let p* be the real $(\alpha_0(0), \alpha_3(0), \alpha_4(0), \alpha_0(1), \alpha_3(1), \alpha_4(1), \ldots)$. If s is a binary string of length n then $[s]_b$ is the set of p's such that p*'s first n numbers are given by s. The $[s]_b$'s form a basis for the usual topology on $(2^{\omega})^b$.

For $a \leq b$ define the projection $\pi^b_a\colon (2^{\omega})^b \to (2^{\omega})^a$ by keeping only the reals with index in a. Let now $\equiv^b_a$ be the equivalence relation induced on $(2^{\omega})^b$, i.e. $p \equiv^b_a q$ iff p and q agree on reals with indices in a. Note that $[s^\frown 0]_b \cap [s^\frown 1]_b = \emptyset$, $\pi^b_a[s^\frown 0]_b$ and $\pi^b_a[s^\frown 1]_b$ are either equal or disjoint, and $\pi^b_a[s^\frown 0]_b \neq [s^\frown 0]_a$.

Definition 3.1 A b-isomorphism is a function $\phi\colon B \to C$, $B, C \subset (2^{\omega})^b$, such that

1) it is a homeomorphism (with respect to the induced topology on B and C)

2) it is an isomorphism (with respect to the relations $\equiv^b_a$ restricted on B, C for all $a \leq b$)

3) it is in Δ^1_3 .

Definition 3.2 A b-condition is a b-isomorphic image of $(2^{\omega})^b$.

We use P, Q, R with occasional embellishments to denote b-conditions. We order them by inclusion. Clearly they generalize the conditions used in 2A and 2B.

Lemma 3.1 The class of b-isomorphisms is closed under compositions, inverses and restrictions; therefore, if P, Q are b-conditions by virtue of the b-isomorphisms $\phi\colon (2^{\omega})^b \to P$, $\psi\colon (2^{\omega})^b \to Q$ then $\phi\psi$ is also a b-isomorphism, giving the b-condition ϕQ, and $\phi Q \subset P$.

Proof Obvious.

In what follows b is usually understood, so we omit it as a superscript if no confusion can arise.

Lemma 3.2 If Ψ is a b-isomorphism with domain P, A and B are subsets of P and $a \leq b$ then $\pi_a A$, $\pi_a B$ are equal (resp. disjoint) iff $\pi_a \Psi A$, $\pi_a \Psi B$ are equal (resp. disjoint).

Proof Obvious.

Lemma 3.3 Let P be a b-condition, $p_i \in P$ and $c_i \leq b$ for i = 1,2, ... n. Then there exists a $p \in P$ satisfying $p \equiv_{c_i} p_i$ for all i iff $p_i \equiv_{c_i \cap c_j} p_j$ for all i, j = 1,2, ... n.

Proof If $p_i \equiv_{c_i} p \equiv_{c_j} p_j$ then $p_i \equiv_{c_i \cap c_j} p_j$.

For the converse we handle first the case $P = (2^{\omega})^b$: Define p(x)(n) to be $p_i(x)(n)$ if $x \in c_i$, 0 if $x \in b - \bigcup c_i$. Now for arbitrary P, given by $\Phi: (2^{\omega})^b \to P$, find an x so that $x \equiv_{c_i} \Phi^{-1} p_i$ and apply Φ.

Lemma 3.4 Given P and $a \leq b$ there exist $Q_0, Q_1 \subset P$ so that

$$c \leq a \quad \Rightarrow \quad \pi_c Q_0 = \pi_c Q_1 = \pi_c P \qquad \text{and}$$
$$c \leq b \wedge c \not\leq a \quad \Rightarrow \quad \pi_c Q_0 \cap \pi_c Q_1 = \emptyset .$$

Proof Define $R_i = \{ p : p \in (2^{\omega})^b$ and $\forall x \in b-a \;\; p(x)(0) = i \}$. If $\Phi: (2^{\omega})^b \to P$ then ΦR_0, ΦR_1 work.

Lemma 3.5 For any Q_0, Q_1 the set $I = \{ c : c \leq b$ and $\pi_c Q_0 = \pi_c Q_1 \}$ is an ideal in L.

Proof Closure under $\leq$ is immediate. For $\cup$ we prove that given $c, d \in I$ $\pi_{c \cup d} Q_0 \subset \pi_{c \cup d} Q_1$; then $\supset$ follows by symmetry. So let $q_0 \in Q_0$; we will find $q_1 \in Q_1$ such that $q_0 \equiv_{c \cup d} q_1$. This is done as follows: Since $c, d \in I$ we can find $q_{10}, q_{11} \in Q_1$ so that $q_0 \equiv_c q_{10}$ and $q_0 \equiv_d q_{11}$. Applying Lemma 3.3 to $(2^{\omega})^b$ we see that $q_{10} \equiv_{c \cap d} q_{11}$. Applying Lemma 3.3 to Q_1 we obtain a q_1 such that $q_1 \equiv_c q_{10}$ and $q_1 \equiv_{\bar{d}} q_{11}$. Therefore $q_1 \equiv_{c \cup d} q_0$.

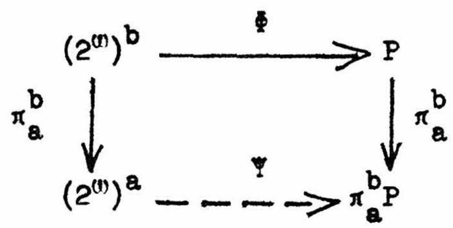
<u>Lemma 3.6</u> Given Q_1, Q_2, Q_3 define $c_{ij} = \bigcup \{ c : c \leq b$ and $\pi_c Q_i = \pi_c Q_j \}$. Then the intersection of any two of c_{12}, c_{13}, c_{23} is contained in the third.

<u>Proof</u> Obvious.

The next lemma helps in visualizing the structure of b-conditions corresponding to complicated lattices by reducing it to simpler cases.

<u>Lemma 3.7</u> If P is a b-condition and $a \leq b$ then $\pi_a^b P$ is an a-condition, and $\pi_c^a(\pi_a^b P) = \pi_c^b P$.

<u>Proof</u>



Lemma 3.8 will be useful in "thinning down" conditions.

<u>Lemma 3.8</u> If $a \leq b$, P is a b-condition, Q is an a-condition and $Q \subset \pi_a^b P$ then

i) $P \cap (\pi_a^b)^{-1} Q$ is a b-condition

ii) $\pi_c^b(P \cap (\pi_a^b)^{-1} Q) = \pi_c^b P \cap \pi_c^b (\pi_a^b)^{-1} Q$.

<u>Proof</u> (i) First for $P = (2^\omega)^b$: If $\Psi : (2^\omega)^a \to Q$ then define $\Phi : (2^\omega)^b \to (2^\omega)^a$ by $\Phi(p(x))$ being $p(x)$ if $x \in b-a$, and $\Psi \pi_a^b(p(x))$ if $x \in a$. This shows that $(\pi_a^b)^{-1} Q$ is a b-condition.

Now for arbitrary P, given by $\Phi : (2^\omega)^b \to P$: Construct

$\Psi : (2^{\omega})^a \to \pi^b_a P$ as in the proof of Lemma 3.7, so that $\Psi \pi^b_a = \pi^b_a \Phi$. Then $\Psi^{-1}Q$ is an a-condition; by the case $(2^{\omega})^b$ above, we have that $(\pi^b_a)^{-1}\Psi^{-1}Q$ is a b-condition. Now apply Φ.

(ii) The $\subset$ part is obvious. For $\supset$ suppose that $r = \pi^b_c p = \pi^b_c q'$ ($p \in P$, $\pi^b_a q' = q \in Q$). Since $Q \subset \pi^b_a P$ let $p' \in P$ be so that $\pi^b_a p' = q = \pi^b_a q'$. Then $p \equiv^b_c q' \equiv^b_a p'$, so $p \equiv^b_{c \cap a} p'$. By Lemma 3.3 there is a $p_0 \in P$ so that $p \equiv^b_c p_0 \equiv^b_a p'$, i.e. $\pi^b_c p_0 = r$ and $p_0 \in P \cap (\pi^b_a)^{-1}Q$.

We give now an important definition.

<u>Definition 3.3</u> Let $a \leq b$ and $\{P_i\}$, $i = 1,2,\ldots,r$, be b-conditions. Then the b-conditions $\{P^j_i\}$, $j = 0,1$ and $i = 1,2\ldots,r$, are an <u>a-splitting</u> of $\{P_i\}$ if $P^j_i \subset P_i$ and

1) $\pi_c P_i = \pi_c P_k \Rightarrow \pi_c P^j_i = \pi_c P^j_k$ for $c \leq a$

2) $\pi_c P^0_i \cap \pi_c P^1_i = \emptyset$ for $c \not\leq a$, $c \leq b$

3) $\pi_c P^0_i = \pi_c P^1_i$ for $c \leq a$.

<u>Lemma 3.9</u> Given an a-splitting as in Definition 3.3 adjoin P_{r+1}, a b-condition, to the $\{P_i\}$; then there exist P^0_{r+1}, P^1_{r+1} so that $\{P^j_i\}$, $i = 1,2,\ldots,r+1$, is still an a-splitting.

<u>Proof</u> If $r=0$ this is just Lemma 3.4. In general, define $c_{ik} = \bigcup\{ c : c \leq b$ and $\pi_c P_i = \pi_c P_k \}$, $c_i = c_{i,r+1}$, $c_0 = \bigcup c_m$ for $m \leq r$. Using Lemma 3.4 find b-conditions Q^0 and Q^1 so that $Q^0, Q^1 \subset P_{r+1}$, $c \leq a \cup c_0 \Rightarrow \pi_c Q^0 = \pi_c Q^1 = \pi_c P_{r+1}$, and $c \not\leq a \cup c_0 \Rightarrow \pi_c Q^0 \cap \pi_c Q^1 = \emptyset$. Now set $P^j_{r+1} = Q^j \cap \bigcap_{i \leq r} \pi^{-1}_{c_i} \pi_{c_i} P^j_i$.

<u>Claim</u> $\pi_{c_i} P^j_{r+1} = \pi_{c_i} P^j_i$, where $i = 1,2,\ldots,r$ and $j = 0,1$.

Granting the claim we have $\pi_{c_i} P_i^j \subset \pi_{c_i} (Q^j \cap \bigcap_{k<i} \pi_{c_k}^{-1} \pi_{c_k} P_k^j)$, hence applying Lemma 3.8 r times we see that P_{r+1}^j is a b-condition.

To establish the claim: $\subset$ is clear. For $\supset$, first we set $i = 1$ to simplify notation. Let p_1 be an arbitrary member of P_1^j. Define $p_2 \in P_2^j$, ..., $p_r \in P_r^j$ by induction: Suppose $p_1, \ldots, p_s$ $(1 \leq s < r)$ have already been defined so that $p_i \equiv_{c_{ik}} p_k$, for $i,k = 1,2,\ldots,s$. Since $\pi_{c_{s+1,i}} P_{s+1}^j = \pi_{c_{s+1,i}} P_i^j$, choose $q_i \in P_{s+1}^j$ so that $q_i \equiv_{c_{s+1,i}} p_i$. Then $q_i \equiv_{c_{s+1,i}} p_i \equiv_{c_{ik}} p_k \equiv_{c_{s+1,k}} q_k$. Using Lemma 3.6, $q_i \equiv_d q_k$ where $d = c_{s+1,i} \cap c_{s+1,k}$. Then by Lemma 3.3 we can find $p_{s+1} \in P_{s+1}^j$ so that $p_{s+1} \equiv_{c_{s+1,i}} q_i$, preserving the induction hypothesis. So we have now $p_1, \ldots, p_r$ in $P_1^j, \ldots, P_r^j$ respectively, such that $p_i \equiv_{c_{ik}} p_k$. Noting that each c_i is $\leq c_0$ we have $\pi_{c_i} Q^j = \pi_{c_i} P_{r+1} = \pi_{c_i} P_i$, and there exist $q_1, \ldots, q_r \in Q^j$ so that $q_i \equiv_{c_i} p_i$. As above we may use Lemmas 3.3 and 3.6 to obtain $q \in Q^j$ satisfying $q \equiv_{c_i} q_i$. But then $q \in \pi_{c_i}^{-1} \pi_{c_i} P_i^j$ for each i, so $q \in P_{r+1}^j$; also $\pi_{c_1} q = \pi_{c_1} p_1$. So the claim has been proved, and (1) of Definition 3.3 holds, with r replaced by r+1.

To verify (2), if $c \not\leq a$ then either $c \not\leq a \cup c_0$ (in which case $\pi_c P^0_{r+1} \cap \pi_c P^1_{r+1} \subset \pi_c Q^0 \cap \pi_c Q^1 = \emptyset$) or $c \cap c_i \not\leq a$ for some $i \in \{1,\ldots,r\}$ (in which case $\pi_{c \cap c_i} P^0_{r+1} \cap \pi_{c \cap c_i} P^1_{r+1} =$ $\pi_{c \cap c_i} P^0_i \cap \pi_{c \cap c_i} P^1_i = \emptyset$ and since $c \cap c_i \leq c$ we have that $\pi_c P^0_{r+1} \cap \pi_c P^1_{r+1} = \emptyset$).

Finally to verify (3) start with $p^0 \in P^0_{r+1}$. We can find $p^0_i \in P^0_i$ with $p^0_i \equiv_{c_i} p^0$ $(i = 1,\ldots,r)$; then we find $p^1_i \in P^1_i$ with $p^1_i \equiv_a p^0_i$; and then $p^1 \in P^1_{r+1}$ with $p^1 \equiv_{a \cap c_i} p^0$ $(i = 1,\ldots,r)$ i.e. $p^1 \equiv_{a \cap c_0} p^0$. By Lemma 3.3 choose $p \in P_{r+1}$ with $p \equiv_a p^0$ and $p \equiv_{c_0} p^1$. Since $\pi_{a \cup c_0} Q^1 = \pi_{a \cup c_0} P_{r+1}$ there is a $q \in Q^1$ with $q \equiv_{a \cup c_0} p$. Then $q \in P^1_{r+1}$ because $q \equiv_{c_0} p^1$ and $\pi_a q =$ $= \pi_a p^0$. So $\pi_a P^0_{r+1} \subset \pi_a P^1_{r+1}$. We can prove $\supset$ likewise.

<u>Lemma 3.10</u> Given an a-splitting as in Definition 3.3 suppose $Q^j_1 \subset P^j_1$ are b-conditions satisfying $\pi_a Q^0_1 = \pi_a Q^1_1$. Then there exist Q^j_i $(2 \leq i \leq r)$ so that $\{Q^j_i\}$, $i = 1,\ldots,r$ and $j = 0,1$, is an a-splitting of $\{P_i\}$.

<u>Proof</u> Define again $c_{ik} = \{ c : c \leq b$ and $\pi_c P_i = \pi_c P_k \}$ and let $c_i = c_{i1}$. Set $Q^j_i = P^j_i \cap \pi_{c_i}{}^{-1} \pi_{c_i} Q^j_1$ for $i = 2,\ldots,r$. Using Lemma 3.8 these are all b-conditions, and part (2) of Definition 3.3 holds trivially.

For (1) we must show $\pi_{c_{ik}} Q_i^j \subset \pi_{c_{ik}} Q_k^j$ for all i, k; however, Lemma 3.8 gives $\pi_c Q_k^j = \pi_c P_k^j \cap \pi_c \pi_{c_k}^{-1} \pi_{c_k} Q_1^j$ for all $c \leq b$, so that $\pi_{c_{ik}} Q_i^j \subset \pi_{c_{ik}} P_i^j = \pi_{c_{ik}} P_k^j$, and it suffices to show $\pi_{c_{ik}} Q_i^j \subset$ $\subset \pi_{c_{ik}} \pi_{c_k}^{-1} \pi_{c_k} Q_1^j$, i.e. given an arbitrary $q \in Q_i^j$ show that $q \equiv_{c_{ik}} q' \equiv_{c_k} q_1$ for some $q' \in (2^\omega)^b$ and $q_1 \in Q_1^j$. Using the definition of Q_i^j we can find $q_1 \in Q_1^j$ so that $q \equiv_{c_i} q_1$. Then by Lemma 3.6 $q \equiv_{c_{ik} \cap c_k} q_1$, and Lemma 3.3 gives q'.

For (3) it suffices, given $q \in Q_i^0$, to find $q_1 \in Q_1^1$ and $q' \in$ $\in (2^\omega)^b$ so that $q \equiv_a q' \equiv_{c_i} q_1$; this is done using Lemmas 3.3 and 3.6 .

<u>Lemma 3.11</u> Let Ω be a set of b-conditions that is open and dense, i.e. $\forall Q \, \exists R \in \Omega \; (R \subset Q)$ and $\forall Q \in \Omega \; \forall R \; (R \subset Q \Rightarrow R \in \Omega)$. Then given $\{P_i\}$ there is an a-splitting $\{P_i^j\} \subset \Omega$.

<u>Proof</u> By induction: Suppose Q_i^j $(i = 1,\ldots,r-1 \; , \; j = 0,1)$ have already been found (the case $r = 1$ is easy). Choose Q_r^0 and $Q_r^1 \subset P_r$ by Lemma 3.9; then $Q_r^{00} \subseteq Q_r^0$ with $Q_r^{00} \in \Omega$; $Q_r^{10} = Q_r^1 \cap \pi_a^{-1} \pi_a Q_r^{00}$; $P_r^1 \subset Q_r^{10}$ with $P_r^1 \in \Omega$; $P_r^0 = Q_r^{00} \cap$ $\cap \pi_a^{-1} \pi_a P_r^1$; and finally $P_i^j \subset Q_i^j$ by Lemma 3.10 .

C) The crucial lemma and the proof

The heart of the proof for diamond in 2A was Lemma 2.1; for the three-lattice in 2B, Lemma 2.2 . We present now a generalization of those lemmas that works for any finite distributive lattice L. Of course we use the apparatus of 3B.

Lemma 3.12 Given $b \in L$ and a b-condition P there exists a b-condition R, $R \subset P$, so that

either $\exists d < b$ so that F is constant on any $A \subset R$ with $\pi_d A$ = a singleton

or F is one-to-one on R (and continuous, of course).

Here F is a Δ^1_3 function.

Proof As in the proof of Lemma 2.1, we may assume without loss of generality that F is continuous.

Assume the first alternative fails, i.e. $\forall d < b \ \ \forall Q \subset P \ \ \exists A \subset Q$ with $\pi_d A$ a singleton and F is not constant on A. Considering two points in A that witness this and using the continuity of F it is easy to show that

(*) $\forall d < b \ \ \forall Q \subset P \ \ \exists R_1, R_2 \subset Q$ so that $\pi_d R_1 = \pi_d R_2$ and $F[R_1]$, $F[R_2]$ are contained in disjoint neighborhoods .

This property (*), which we express as "F separates R_1, R_2" , will now be iterated to produce a condition on which F will be one-to-one. This is a fusion argument, indexed by $2^{<\omega}$.

Start by setting $Q_\emptyset = P$. Suppose Q_s has been defined for length(s) $\leq$ k so that

$$\pi_a Q_s = \pi_a Q_t \quad \Leftrightarrow \quad \pi_a[s]_b = \pi_a[t]_b \qquad \text{and}$$

$$\pi_a Q_s \cap \pi_a Q_t = \emptyset \quad \Leftrightarrow \quad \pi_a[s]_b \cap \pi_a[t]_b = \emptyset$$

for all s,t with length(s) = length(t) $\leq$ k and all $a \leq b$. This is our induction hypothesis. Consider $\{ [s\hat{}j]_b$: length(s) = k , $j = 0,1 \}$; it is an e-splitting of $\{ [s]_b$: length(s) = k } , for some unique $e \in L$. Using this e find an e-splitting of the collection $\{ Q_s$: length(s) = k } , namely $\{ Q_{s\hat{}j}$: length(s) = = k , $j = 0,1 \}$, so that F separates $Q_{s\hat{}0}$, $Q_{s\hat{}1}$. This is done by Lemma 3.11 (actually by a trivial extension); our property (*) guarantees density. It is easy to see that the induction hypothesis holds for these $Q_{s\hat{}j}$'s , so the process may continue.

Define now Φ: $(2^\omega)^b \to (2^\omega)^b$ by $\Phi(p) = \bigcap_k Q_{p|k}$, where $p|k$ codes the restriction of p to the first k arguments. We may arrange for the intersection to be a singleton by using at the n^{th} step conditions of diameter less than $1/2^n$. By the Δ^1_3 Selection Principle it is easy to see that Φ is Δ^1_3 . So Φ is a b-isomorphism and clearly F is one-to-one on R = range(Φ).

At long last we can complete the proof of our theorem.

<u>Proof of the theorem</u> Let L be a finite distributive lattice. Represent it as a sublattice of $\mathcal{P}(n)$, for minimal n. Form the corresponding b-conditions (for $b = n = \{0,1,\ldots,n-1\}$ and $a \in L$, $a \leq b$), consider them as a notion of forcing, and take an n-tuple that is sufficiently generic. Let us call it g. Suppose now

that $\alpha \leq g$ (here $\leq$ denotes Δ^1_3 reducibility and $\equiv$ denotes Δ^1_3 equivalence). By genericity and Lemmas 1.1 and 1.2 we have that $F_i(g) = \alpha$, for some F_i . Now by genericity and Lemma 3.12 either $\alpha \equiv g$ or $\alpha \leq \pi_d g$. So we perform a finite induction along the nodes of L and we see that $\pi^n_b g$, for $b \in L$, realizes distinct Δ^1_3-degrees forming an initial segment isomorphic to L.

Remark The same method works for sublattices of the lattice of all finite sets of integers.

Chapter III

THE JUMP INVERSION THEOREM FOR Q_{2n+1}-DEGREES

1. Background and definitions

One of the early results in the theory of Turing degrees (for basic information see [17]) was the following:

Friedberg Jump Inversion Theorem ([2]) If $\underset{\sim}{b} \geq \underset{\sim}{0}'$ then there exists an $\underset{\sim}{a}$ such that $\underset{\sim}{a}' = \underset{\sim}{a} \vee \underset{\sim}{0}' = \underset{\sim}{b}$.

Of course $\underset{\sim}{0}$ denotes the degree of the recursive sets, and $'$ denotes the Turing jump operation.

Next, the question was considered in the context of hyperdegrees. Let $\underset{\sim}{0}$ denote the hyperdegree of the hyperarithmetical sets and $'$ the hyperjump. Does the above theorem hold? The answer is yes ([21]):

Jump Inversion Theorem for Δ^1_1-degrees If $\underset{\sim}{b} \geq \underset{\sim}{0}'$ then there exists an $\underset{\sim}{a}$ such that $\underset{\sim}{a}' = \underset{\sim}{a} \vee \underset{\sim}{0}' = \underset{\sim}{b}$.

A natural question now is: does the inversion theorem hold for Δ^1_{2n+1}-degrees? (We are assuming PD, needless to say). By a well-known argument Determinacy implies that there exists some cone on which inversion holds (a cone , by definition, is $\{ \underset{\sim}{a} : \underset{\sim}{a} \geq \underset{\sim}{b} \}$, and $\underset{\sim}{b}$ is called the base of the cone). But what is the base of the cone? Is it again $\underset{\sim}{0}'$? (i.e. the Δ^1_{2n+1}-jump of the degree of Δ^1_{2n+1} sets). Surprisingly the answer is no:

<u>Theorem</u> (Kechris, unpublished) (PD) If $n \geq 1$, then no real in C_{2n+2} can be a base for a cone of inversion of the Δ^1_{2n+1}-jump. ("Cone of inversion" of course means that every member of the cone is the Δ^1_{2n+1}-jump of some Δ^1_{2n+1}-degree).

<u>Proof</u> For notational simplicity we let $2n+1 = 3$. If a member of C_4 were a base then it would be recursive in a member of C_3 , so without loss of generality assume a base b is in C_3. Consider the set $C = \{ \alpha : \exists \beta \in Q_3(\alpha) \ (\beta \in C_3 \text{ and } \alpha \leq_{\Delta_3} \beta) \}$. It is a subset of C_4 , and it is Π^1_3 , because the quantification is bounded. So it is countable, and hence a subset of C_3. Since $b \in C_3$ everything $\geq b$ in C_3 is the Δ^1_3 -jump of a member of C, thus a member of C_3. However the Δ^1_3-degrees in C_3 are wellordered with successor steps taken by the Δ^1_3-jump, so that a limit stage of this wellordering gives immediately a contradiction.

So the inversion theorem is a property of hyperdegrees that fails to generalize to Δ^1_{2n+1}-degrees, $n \geq 1$. Usually in such cases the validity of the property is restored if instead of Δ^1_{2n+1}-degrees we work with Q_{2n+1}-degrees. Indeed, it is the case that the jump inversion theorem holds for Q_{2n+1}-degrees, i.e. the base is again $\underset{\sim}{0}'$. Moreover we can establish that the Q_{2n+1}-jump is never one-to-one.

<u>Jump Inversion Theorem for Q_{2n+1}-degrees</u> (PD) If $\underset{\sim}{c}$ is a Q_{2n+1}-degree $\geq \underset{\sim}{0}'$ then there exist Q_{2n+1}-degrees $\underset{\sim}{a}$, $\underset{\sim}{b}$ such that $\underset{\sim}{a} \vee \underset{\sim}{b} = \underset{\sim}{a}' = \underset{\sim}{b}' = \underset{\sim}{c}$.

The rest of the chapter is devoted to the proof of this theorem.

2. <u>The proof</u>

For notational simplicity we work with $2n+1 = 3$. First we establish a lemma.

<u>Lemma 2.1</u> If $\underset{\sim}{0}' \not\leq \underset{\sim}{b}$ (i.e. $k_3^0 = k_3^b$) then $\underset{\sim}{b}' = \underset{\sim}{b} \vee \underset{\sim}{0}'$.

<u>Proof</u> By the Spector Criterion $\underset{\sim}{0}' \not\leq \underset{\sim}{b}$ iff $k_3^0 = k_3^b$. Now $k_3^0 < k_3^{\underset{\sim}{b} \vee \underset{\sim}{0}'}$, so again by the Spector Criterion $\underset{\sim}{b}' \leq \underset{\sim}{b} \vee \underset{\sim}{0}'$. The opposite inequality is obvious.

<u>Proof of the theorem</u> The set $\{ \alpha : k_3^\alpha = k_3^0$ and $\alpha \notin Q_3 \}$ is Σ_3^1 and comeager. In fact there is a sequence $D_0, D_1, \ldots$ of dense open sets, $\{D_i\} \in \Delta_3^1(y_0)$, such that $\bigcap D_i \subset \{ \alpha : k_3^\alpha = k_3^0$ and $\alpha \notin Q_3 \}$. We use these dense sets in the construction below.

We describe an inductive construction of reals a and b. Set

$a_{-1} = b_{-1} = \emptyset$

Inductive step: Suppose a_n, b_n have been constructed (they are finite sequences of integers). Consider the dense, open set D_{n+1} and extend a_n by a finite segment s, least in some fixed enumeration, so that the basic neighborhood defined by $a_n{}^\frown s$ is contained in D_{n+1}. Extend $b_n{}^\frown s$ by a finite segment t, least again, so that the basic neighborhood defined by $b_n{}^\frown s^\frown t$ is contained in D_{n+1}. Set now $a_{n+1} = a_n{}^\frown s^\frown t^\frown\{c(n)\}$, $b_{n+1} = b_n{}^\frown s^\frown t^\frown\{c(n)+1\}$.

This completes the inductive step.

Let now $a = \bigcup a_n$, $b = \bigcup b_n$. Since $a, b \in \bigcap D_i$ we have by Lemma 2.1 that $\tilde{a}' = \tilde{a} \vee \tilde{0}'$, $\tilde{b}' = \tilde{b} \vee \tilde{0}'$. Now $\tilde{a} \vee \tilde{0}' \geq \tilde{c}$, because using y_0 we may trace the construction of a and find all c(n)'s. Likewise $\tilde{b} \vee \tilde{0}' \geq \tilde{c}$. However $\tilde{a} \vee \tilde{0}' \leq \tilde{c}$, too, because $\tilde{0}' \leq \tilde{c}$ and the construction of a only needs y_0 and c. The same holds for b, and therefore we have $\tilde{a}' = \tilde{b}' = \tilde{a} \vee \tilde{0}' = \tilde{b} \vee \tilde{0}' = \tilde{c}$. Finally note that $\tilde{a} \vee \tilde{b} \geq \tilde{c}$, because if both a and b are available then considering the points where they differ c may be obtained. So we have $\tilde{a}' = \tilde{b}' = \tilde{a} \vee \tilde{b} = \tilde{c}$, and a, b cannot have the same degree.

Remark The reals a, b may also be chosen to be of minimal degree by using perfect trees in Q_3 instead of finite sequences.

References

1. E. Ellentuck, A new proof that analytic sets are Ramsey, J.Symb. Logic 39, (1974), pp.163-165.

2. R. Friedberg, A criterion for completeness of degrees of unsolvability, J.Symb.Logic 22, (1957), pp.159-160.

3. F. Galvin, K. Prikry, Borel sets and Ramsey's theorem, J.Symb.Logic 38, (1973), pp.193-198.

4. L. Harrington, A. Kechris, On the determinacy of games on ordinals, Ann.Math.Logic, (to appear).

5. T. Jech, Set Theory , Academic Press, 1978.

6. A. Kechris, M.I.T. notes, 1973

7. A. Kechris, The theory of countable analytical sets, Trans.Am.Math. Soc. 202, (1975), pp.259-297.

8. A. Kechris, Forcing in analysis, Higher Set Theory, Proceedings, Oberwolfach, Germany, 1977, G. H. Mueller and D. S. Scott, eds., Lecture Notes in Mathematics, Vol. 669, Springer-Verlag, 1978, pp.277-302.

9. A. Kechris, Spector second order classes and reflection, Generalized Recursion Theory II, J. E. Fenstad, R. O. Gandy, G. E. Sacks, eds., North Holland , 1978, pp. 147-183.

10. A. Kechris, Séminaire de Mathématiques 1978-9, Univ. de Paris VI, France.

11. A. Kechris, Forcing with Δ perfect sets and minimal Δ degrees, J.Symb.Logic, (to appear).

12. A. Kechris, An overview of descriptive set theory, Choquet Seminar Notes, (to appear)

13. R. Mansfield, A footnote to a theorem of Solovay, Logic Colloquium '77, A. Macintyre, L. Pacholski, J. Paris, eds.,North Holland, 1978, pp.195-198.

14. D. Martin, Borel determinacy, Ann. of Math., 102, (1975), pp.363-371.

15. Y. Moschovakis, Descriptive Set Theory, North Holland, 1980

16. K. Prikry, Determinateness and partitions, Proc.A.M.S. 54, (1976), pp.303-306.

17. H. Rogers, Theory of recursive functions and effective computability , McGraw-Hill, 1967.

18. G. Sacks, Forcing with perfect closed sets, Axiomatic Set Theory, D. Scott, ed.,Proceedings of Symposia in Pure Mathematics, Vol. 13, Am. Math. Society, Providence, R. I., 1967, pp.331-355.

19. J. Silver, Every analytic set is Ramsey, J.Symb.Logic 35, (1970), pp.60-64.

20. R. Solovay, Hyperarithmetically encodable sets, Trans.A.M.S. 239, (1978), pp.99-122.

21. S. Thomason, The forcing method and the upper semilattice of hyperdegrees, Trans.A.M.S. 129, (1967), pp.38-58.

22. S. Thomason, On initial segments of hyperdegrees, J.Symb.Logic 35, (1970), pp.189-197.