SOME TOPICS IN DESCRIPTIVE SET THEORY AND ANALYSIS

Thesis by

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For my darling

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Abstract

Coanalytic subsets of some well known Polish spaces are investigated. A natural norm (rank function) on each subset is defined and studied by using well-founded trees and transfinite induction as the main tools. The norm provides a natural measure of the complexity of the elements in each subset. It also provides a "Rank Argument" of the non-Borelness of the subset.

The work is divided into four chapters. In Chapter 1 nowhere differentiable continuous functions and Besicovitch functions are studied. Chapter 2 deals with functions with everywhere divergent Fourier series, and everywhere divergent trigonometric series with coefficients that tend to zero. Compact Jordan sets (i.e., sets without cavities) and compact simply-connected sets in the plane are investigated in Chapter 3. Chapter 4 is a miscellany of results extending earlier work of M. Ajtai, A. Kechris and H. Woodin on differentiable functions and continuous functions with everywhere convergent Fourier series.

Contents

Chapter 0

In this chapter we shall review the Descriptive Set-Theoretic results that we need. We shall not give proofs but we will give precise references.

§1. Coanalytic subsets and coanalytic norms. A *Polish space* is a complete, separable metric space. From now on X shall always denote a Polish space. A set $A \subseteq X$ is a *Borel subset* of X if it belongs to the smallest σ-algebra of subsets of X which contains all the open subsets of X. Let Y be a Polish space and $f:Y \to X$ be a function. We say that f is a *Borel measurable function* if for each open set A in X the set $f^{-1}[A]$ is a Borel subset of Y. A set $A \subseteq X$ is an *analytic subset* of X if there exists a Polish space Y and a Borel subset of $X \times Y$ such that A is the projection of B onto X, i.e.,

$$A = \{x \in X: \exists y \in Y \ ((x,y) \in B)\}$$

A set $A \subseteq X$ is a *coanalytic subset* of X if its complement $X - A$ is an analytic subset of X.

A coanalytic subset A of X is said to be *complete* if for any Polish space Y and coanalytic subset B of Y there is a Borel function $f:Y \to X$ such that $y \in B \Leftrightarrow f(y) \in A$. As it is known the Polish space $\mathbb{R}$, of the real numbers has a subset which is coanalytic and not Borel it follows that no complete coanalytic subset of X can be Borel. A proof that demonstrates that $A \subseteq X$ is not Borel by showing that A is complete coanalytic is usually referred to as a "Completeness Argument."

A *norm* on a set $A \subseteq X$ is just a map $\varphi : A \to ORD$, where ORD is the class of all ordinals. (A norm is sometimes also referred to as a *rank function*.) The map φ induces a pre-well-ordering $\leq_\varphi$ on A which is defined by

$$x \leq_\varphi y \iff \varphi(x) \leq \varphi(y).$$

Two norms are said to be *equivalent* if they induce the same pre-well-ordering.

Let A be a coanalytic subset of X. A norm $\varphi : A \to ORD$ is said to be a *coanalytic norm* if there is an analytic subset B of X^2, and a coanalytic subset C of X^2 such that

$$y \in A \Rightarrow \forall x[\{x \in A \text{ and } \varphi(x) \leq \varphi(y)\} \iff (x,y) \in B \iff (x,y) \in C].$$

It is known that every coanalytic subset has a coanalytic norm defined on it. Moreover this coanalytic norm is always equivalent to one which takes values in ω_1 , and it is by no means unique. (See [41].)

If the set A arises from natural considerations in Real Analysis, Harmonic Analysis or Point-Set Topology (which we shall call Analysis for short) the question of finding a coanalytic norm on A, which naturally reflects in some sense the properties of the elements of A, is of interest. (For instance if the coanalytic norm is such that "simple" elements of A have small ranks, i.e., the map φ sends "simple" elements to small ordinals, then φ induces a natural measure of the complexity of the elements of A.) The norm φ on A enables us to view A as a natural ω_1-hierarchy. We shall refer to coanalytic norms that arise out of Analysis and that reflect the properties of the elements of the coanalytic set in question as "natural coanalytic norms."

The following proposition is basic to our study of natural coanalytic norms.

<u>Proposition</u> <u>1</u>. Suppose $A \subseteq X$ is a coanalytic subset of X and $\varphi:A \to \omega_1$ is a coanalytic norm on A. Then A is Borel $\Longleftrightarrow$ $\varphi[A]$ is countable.

<u>Proof</u>. See [41] p. 196 and p. 213. □

From Proposition 1 we immediately see that to show A is not Borel, it will suffice to show that $\varphi[A]$ is unbounded in ω_1. Such a proof of the non-Borelness of A is usually referred to as a *Rank Argument*. There is a slight extension of this *Rank Argument* which depends on the following proposition.

<u>Proposition</u> <u>2</u>. Suppose A is a coanalytic subset of X and $\varphi:A \to \omega_1$ is a norm on A such that

(i) there is an analytic subset B of X^2 such that $x,y \in A \Rightarrow [\varphi(x) < \varphi(y) \Longleftrightarrow (x,y) \in B]$, and

(ii) φ is unbounded in ω_1 on A.

Then A is not a Borel subset of X.

<u>Proof</u>. See [26]. □

Remark. Observe that if P and Q are coanalytic subsets of X, $Q \subseteq P$ and $\varphi:P \to \omega_1$ is a coanalytic norm then Q and $\varphi \upharpoonright Q$ satisfy condition (i) of Proposition 2.

We end this section with two results which will aid us in constructing coanalytic norms.

Proposition 3. Let X and Y be Polish spaces and $A \subseteq X$, $B \subseteq Y$ be coanalytic subsets. Let also $f:X \to Y$ be a Borel measurable function with $f^{-1}[B] = A$ and $\varphi:B \to \omega_1$ be a coanalytic norm. Then the map $\psi:A \to \omega_1$ defined by $\psi(x) = \varphi(f(x))$ is also a coanalytic norm.

Proof. See [26]. □

Proposition 4. Let A be a coanalytic subset of X, $\varphi:A \to \omega_1$ be a norm and $\leq_\varphi$ the associated pre-well-ordering. Then φ is a coanalytic norm iff the initial segments of $\leq_\varphi$ are uniformly Borel (i.e., the subsets $\{x: x \leq_\varphi y\}$ are uniformly Borel in y).

Proof. See [22]. □

§2. Well-Founded Trees and Their Ranks. Let A be any non-empty set. We define A^* to be the collection of all finite sequences from A (including the empty sequence $\emptyset$), i.e.,

$$A^* = \bigcup_{n\in\omega} A^n, \text{ where } A^0 = \{\emptyset\}.$$

A *tree* T on A is any subset T of A^* such that $\langle a_1,\ldots,a_n,a_{n+1}\rangle \in T \Rightarrow \langle a_1,\ldots,a_n\rangle \in T$, for all $a_1, \ldots, a_n, a_{n+1} \in A$.

The elements of T are called *nodes*. By definition $\emptyset$ is always a node of any non-empty tree. We call $\emptyset$ the *root* of such a tree. A subset $S \subseteq T$ which is also a tree on A is called a *subtree* of T.

Let $\underset{\sim}{u}$ be a finite sequence from A and T be a tree on A. We define $T_{\underset{\sim}{u}}$ by

$$T_{\underset{\sim}{u}} = \bigcup_{n \in \omega} \{\underset{\sim}{v} \in A^n : \underset{\sim}{u}\frown\underset{\sim}{v} \in T\}.$$

It is easy to verify that $T_{\underset{\sim}{u}}$ is a tree on A. If $\underset{\sim}{u}$ is not a node of T then $T_{\underset{\sim}{u}} = \emptyset$. When $\underset{\sim}{u}$ is node of T we shall refer to $T_{\underset{\sim}{u}}$ as the *tree at the node* $\underset{\sim}{u}$ in T. A tree T on A is said to be *well-founded* provided there is no sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ from A such that for each n, $\langle a_1, \ldots, a_n \rangle \in T$.

Let T be a well-founded tree on A. We define by induction a sequence of trees as follows:

Put $T^0 = T$, $T^{\alpha+1} = \bigcup_{n \in \omega} \{\underset{\sim}{v} \in A^n : \exists a \in A(\underset{\sim}{v}\frown\langle a \rangle \in T^\alpha)\}$, and

$$T^\lambda = \bigcap_{\alpha<\lambda} T^\alpha \quad \text{for } \lambda \text{ a limit ordinal.}$$

Observe that the sequence $\langle T^\alpha \rangle_{\alpha \in ORD}$ is strictly decreasing, so for sufficiently large α, $T^\alpha = \emptyset$. Note also that if T is non-empty, then the least α such that $T^\alpha = \emptyset$ must be a successor ordinal.

<u>Definition</u>. Let T be a non-empty well-founded tree. We define the *rank* r(T) of T by

$$r(T) = \text{least } \alpha \text{ such that } T^{\alpha+1} = \emptyset$$

If $\underset{\sim}{v}$ is a node of T we define the rank, $r(\underset{\sim}{v};T)$ of $\underset{\sim}{v}$ in T by $r(\underset{\sim}{v};T) = r(T_{\underset{\sim}{v}})$.

If T is the empty tree we adopt the convention that $r(T) = -1$. If T is not well-founded we let $r(T) = \infty$. It is easy to see that for any finite sequence $\underset{\sim}{u}$ from A that $(T^{\alpha})_{\underset{\sim}{u}} = (T_{\underset{\sim}{u}})^{\alpha}$ for each $\alpha \in ORD$. Using this it can be shown that

$$r(T) = \sup\{r(T_{\underset{\sim}{v}}) + 1 : \underset{\sim}{v} \in T, \underset{\sim}{v} \neq \emptyset\}$$

$$= \sup\{r(\underset{\sim}{v};T) + 1 : \underset{\sim}{v} \in T, \underset{\sim}{v} \neq \emptyset\}$$

It turns out that r is a coanalytic norm if we view the set of all well-founded trees on $\mathbb{N} = \{1,2,3,\ldots\}$ as a subset of a certain Polish space. Consider $\mathbb{N}^*$, the set of all finite sequences from $\mathbb{N}$. A tree T on $\mathbb{N}$ is a subset of $\mathbb{N}^*$ and so can be identified with its characteristic function $\chi_T : \mathbb{N}^* \to \{0,1\} = 2$. So a tree on $\mathbb{N}$ can be viewed as an element of the Polish space $2^{\mathbb{N}^*}$. Let WF be the set of all well-founded trees on $\mathbb{N}$ viewed as a subset of $2^{\mathbb{N}^*}$. Then we have the following result:

<u>Proposition</u> <u>5</u>. WF is a coanalytic subset of $2^{\mathbb{N}^*}$ and $r : WF \to \omega_1$ is a coanalytic norm.

<u>Proof</u>. See [26] □

The final result we need in this section is a classically known result which is a corollary of the recent and much more powerful Kunen-Martin Theorem. Let $\preccurlyeq$ be a binary relation on X. We say that $\preccurlyeq$ is a *strict relation* if $x \preccurlyeq y \Rightarrow \neg(y \preccurlyeq x)$. We say that $\preccurlyeq$ is *well-founded* if there is no sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ such that $x_{n+1} \preccurlyeq x_n$ for all $n \in \mathbb{N}$. Finally $\preccurlyeq$ is said to be analytic, if when viewed as a subset of X^2 it is analytic. Let $\preccurlyeq$ be well-founded strict relation on X. We associate with $\preccurlyeq$ a tree $T_{\preccurlyeq}$ defined by

$$\langle x_1,\ldots,x_n \rangle \in T_{\preccurlyeq} \Leftrightarrow x_n \preccurlyeq x_{n-1} \preccurlyeq \cdots \preccurlyeq x_1$$

We define the *length* of the relation $\preccurlyeq$ as the rank of the tree $T_{\preccurlyeq}$.

<u>Proposition 6</u>. Let $\preccurlyeq$ be a strict analytic well-founded relation on X. Then $\preccurlyeq$ has countable length.

<u>Proof</u>. See [41] p. 103. □

§3. <u>Tree Description and Cantor-Bendixson Analysis</u>. Let $A \subseteq X$ be a coanalytic subset of X. A very useful way of obtaining a coanalytic norm on A is to associate with each element, x of X, a tree (or a set of trees) such that the tree (resp. set of trees) associated with x is well-founded iff $x \in A$. This provides a way of associating an ordinal, namely the rank of the tree (resp. strict supremum of the ranks of the trees) to each element of A. This gives a norm on A and with a little bit of luck it is usually easy to check that this norm is a coanalytic norm. We shall refer to such a process of obtaining a coanalytic norm as a *Tree Description.*

Another process of obtaining a coanalytic norm on A is to associate with each element, x of X, a nested sequence of closed sets such that this sequence stabilizes at the empty set, $\emptyset$, iff $x \in A$. We illustrate this process by the following example and shall refer to it as a *Cantor-Bendixson Analysis*. Let $K = \langle K[0,1],\sigma\rangle$ be the Polish space of all non-empty compact subsets of [0,1], with σ being the Hausdorff metric. Let CS be the subset of K consisting of all countable sets in K. Then it is easy to show that CS is a coanalytic subset of K. Now for $A \in K$ define A' by

$$A' = \{x \in A\colon x \text{ is not an isolated point of } A\}.$$

Then A' is also in K so that $'$ can be viewed as a "derivative" operation on K. We define by induction a sequence $\langle A^{\alpha}\rangle_{\alpha\in ORD}$ as follows:

Put $A^0 = A$, $A^{\alpha+1} = (A^{\alpha})'$, and $A^{\lambda} = \bigcap_{\alpha<\lambda} A^{\lambda}$ for λ a limit ordinal.

Then it is easy to see that $\langle A^{\alpha}\rangle$ stabilizes at $\emptyset$ iff $A \in CS$. Moreover if we define $\rho(A)$ to be the least α such that $A^{\alpha+1} = \emptyset$, then $\rho\colon A \to \omega_1$ is a coanalytic norm.

§4: <u>Notations</u> <u>and</u> <u>Conventions</u>. Most of the notations we use will be standard and when we depart from standard practice we shall point this out. When the Polish space X is understood it is usual to talk about Borel (analytic , etc.) sets rather than Borel (analytic, etc.) subsets. The modern practice is to refer to analytic sets as $\underset{\sim}{\Sigma}^1_1$ sets and coanalytic sets as $\underset{\sim}{\Pi}^1_1$ sets. Borel sets are then $\underset{\sim}{\Delta}_1^1$ sets. The $\underset{\sim}{\Sigma}_2^1$ sets are defined as projections of $\underset{\sim}{\Pi}_1^1$ sets, and $\underset{\sim}{\Pi}_2^1$ sets are defined as compliments of $\underset{\sim}{\Sigma}_2^1$ sets. We shall have no need to go higher up this hierarchy so we shall bow to tradition and use analytic instead of $\underset{\sim}{\Sigma}_1^1$, etc.

We shall always take $\mathbb{N}$ to be the set of positive integers, i.e., $\mathbb{N} = \{1,2,3,\ldots\}$. When we need to consider the set of non-negative integers we shall use ω. Also if $\underset{\sim}{u} = \langle a_1,\ldots,a_m\rangle$ and $\underset{\sim}{v} = \langle b_1,\ldots,b_n\rangle$ we denote by $\underset{\sim}{u}{}^\frown\underset{\sim}{v}$, the concatenation $\langle a_1,\ldots,a_m,b_1,\ldots,b_n\rangle$. Finally in referring to results from different chapters we use Proposition m.n to mean Proposition n from Chapter m.

Chapter 1

Introduction. It was not always clear that there could exist a continuous function which was differentiable at no point. (By "differentiable" we of course mean having a finite derivative. Such functions are called nowhere differentiable continuous functions.) In fact in 1806 M. Ampère [02] even tried to show that no such function could exist and not too many mathematicians disagreed with him (but it must also be said that not many were convinced by his "proof"). Of the early attempts in constructing a nowhere differentiable continuous function mention must be made of B. Bolzano. In a manuscript dated around 1830 Bolzano constructed a continuous function on an interval and showed that it was not differentiable on a dense set of points. (It was later shown by R. Rychlik [43] that this function was in fact nowhere differentiable.)

Around 1873 K. Weierstrass constructed the first nowhere differentiable continuous function. This discovery was published by du Bois-Reymond [12] in 1874 and prior to this no such function was ever published. An example of a nowhere differentiable function published in C. Cellerier [09] was thought (see [53]) to have been discovered as early as 1850 by Cellerier but of this we are very much in doubt. Also a function considered by B. Riemann around 1860 and very often thought of as being nowhere differentiable turns out to be differentiable at certain points (see [15], [16] or [47]). So the honour of the discovery of the first nowhere differentiable continuous function goes to

Weierstrass.

Later many more examples of nowhere differentiable continuous functions were constructed and it became fashionable to ask that more stringent requirements be satisfied (for instance, instead of being nowhere differentiable, the function might be required to have no derivative, finite or infinite). In 1925 A. Besicovitch [07] constructed a continuous function with no one-sided derivative, finite or infinite. Such functions are called Besicovitch functions in honour of their discoverer. Functions which satisfy even more stringent requirements than the Besicovitch functions have been constructed by A. P. Morse [40].

The status of nowhere differentiable continuous functions took a different twist when in 1931 S. Mazurkiewicz [39] showed that the set of all such functions is a co-meager subset of the set of all continuous functions of period 1. (See also Banach [03].) (So that in the sense of Baire Category the functions which are not nowhere differentiable are exceptional.) This provided an abstract proof of the existence of nowhere differentiable continuous functions. A little later S. Saks [44] showed that the set of all Besicovitch functions was a meager subset of the set of all continuous functions. So we cannot get an abstract existence proof as before. However J. Malý [34] recently showed that the Besicovitch functions was co-meager in a certain restricted class of continuous functions, thus retrieving the situation.

Let C = $\langle C[0,1],d \rangle$ be the Polish space of all real valued continuous function on [0,1] with d being the supnorm metric given by

$$d(f,g) = \sup\{|f(x)-g(x)|: x \in [0,1]\}$$

Let ND be the set of all nowhere differentiable functions and BF be the set of all Besicovitch functions in C. (It is understood that at the endpoints, 0 and 1, one-sided derivatives are considered.) Then it is easy to show that ND and BF are coanalytic subsets. R. D. Mauldin [36] showed that ND is not a Borel subset (there is a small error in [36]; for the correction see [37]) and in a communication with Kechris (see [25]) indicated that he also had a proof that BF was also not Borel. Kechris later [25] showed that ND and BF are complete coanalytic subsets (and hence they can't be Borel).

In this chapter we shall investigate a "natural" rank function on ND, the definition of which is essentially due to Kechris and Woodin (see [23]). We give a Tree Description of ND to get an auxiliary rank function ρ. Our natural rank function r is then easily defined in terms of ρ. It turns out that r is a coanalytic norm on ND. The rank function r provides a natural measure of the complexity of the functions in ND. The rank of a function, r(f), measures in some sense how "close" the function f "came to being differentiable." We shall show that the functions with smallest rank, namely 1, are precisely the set BC, of the Banach functions in ND. (A Banach function is a function such that at each point at least one of the Dini derivatives are infinite. What Banach had essentially shown in his proof of the co-meagerness of ND was that BC was co-meager, hence the name.) The Banach functions are easily seen to be nowhere differentiable but as the rank r(f) increases it becomes more difficult to see that f is nowhere differentiable. So the functions become more complicated as the rank increase.

We next consider the ranks of certain natural examples in ND to see that our intuitive idea that natural examples should have small ranks is reasonable. We will also show for each ordinal $1 < \alpha < \alpha_1$, how to construct a function $f \in$ BF such that $r(f) = \alpha$. Since r is coanalytic norm on ND this will provide Rank Arguments of the non-Borelness of BF and ND. Finally we formulate a Cantor-Bendixson Analysis which gives rise to the same rank function r. This Analysis is much more complicated than the ones given in [01] and [26] but it has some interesting aspects. One of the interesting aspects of this Cantor-Bendixson Analysis is that it involves a "simultaneous induction," as opposed to a "parametric induction" which is for instance used in [26] to define the rank function on the set D, of everywhere differentiable functions in C.

§1. <u>Tree Description: The Rank Functions ρ and r</u>. In this section we study the set ND by associating with each element of ND a countable number of well-founded trees. But first we check that BF and ND are coanalytic subsets.

<u>Proposition 1</u>. ND and BF are coanalytic subsets of C.

<u>Proof</u>. It will suffice to show that C - ND and C - BF are analytic subsets. We have

$$C - ND = \{f \in C : \exists x \in [0,1] \text{ (f is differentiable at x)}\}$$

Now f is differentiable at x iff $\forall n\ \exists m$ such that

$$\left.\begin{array}{l} \forall h_1, h_2 \text{ with } 0 < |h_1|, |h_2| < 1/m \\ \text{and } x + h_1,\ x + h_2 \in [0,1] \\ \text{we have } \left| \dfrac{f(x+h_1)-f(x)}{h_1} - \dfrac{f(x+h_2)-f(x)}{h_2} \right| \le \dfrac{1}{n} \end{array}\right\} \quad (*)$$

Let $E(n,m) = \{(f,x) \in C \times [0,1] : (*) \text{ holds}\}$. Then it is easy to see that $E(n,m)$ is closed, and consequently $\bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} E(n,m)$ is Borel. C - ND is the projection of this Borel set onto C and so it is analytic.

$$C - BF = \{f \in C : \exists x \in [0,] \text{(f has a one-sided derivative (possibly infinite) at x)}\}$$

Now f has a one-sided derivative at x iff $\forall n\ \exists m$

$\forall \varepsilon \in \{-1,1\}$ such that $\forall h_1, h_2$ with $0 < h_1, h_2 < 1/m$

and $x + \varepsilon h_1,\ x + \varepsilon h_2 \in [0,1]$

$$\left|\frac{f(x+\varepsilon h_1)-f(x)}{\varepsilon h_1} - \frac{f(x+\varepsilon h_2)-f(x)}{\varepsilon h_2}\right| \leq \frac{1}{n},$$

<u>or</u> $\forall h$ with $0 < h < 1/m$ and $x + \varepsilon h \in [0,1]$

we have $\frac{f(x+\varepsilon h)-f(x)}{\varepsilon h} \geqslant n$,

<u>or</u> $\forall h$ with $0 < h < 1/m$ and $x + \varepsilon h \in [0,1]$

we have $\frac{f(x+\varepsilon h)-f(x)}{\varepsilon h} \leq -n$.

(**)

We now put $F(n,m) = \{(f,x) \in C \times [0,1] : (**) \text{ holds}\}$ and proceed as before to conclude that C-BF is analytic. □

With each $f \in C$ and each positive rational M (M should be thought of as being large) we will associate a tree T_f^M which reflects the properties of f. But first some notation. Let $\mathbb{Q}^+$ be the set of all positive rational numbers. Let also R[0,1] be the collection of all non-empty intervals open in [0,1], and Q[0,1] be the set of intervals in R[0,1] with rational endpoints. Observe that Q[0,1] is countable. For any interval $I \in R[0,1]$ with endpoints $a,b (a < b)$ and any $f \in C$ we define the *difference quotient* $\Delta_f(I)$, of f over I by $\Delta_f(I) = \frac{f(b)-f(a)}{b-a}$.

<u>Definition</u>. Let $f \in C$ and $M \in \mathbb{Q}^+$. We define the tree T_f^M on Q[0,1] as follows:

$$\langle I_1,\ldots,I_n\rangle \in T_f^M \Leftrightarrow I_1 = [0,1] \text{ and } \forall i = 2,\ldots,n \text{ we have}$$

(i) $I_i \in Q[0,1]$, $\bar{I}_i \subseteq I_{i-1}$, $|I_1| \leq 1/i$ and

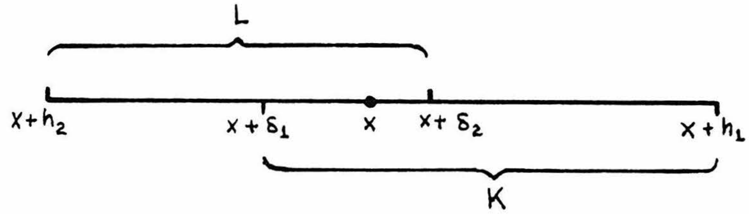
(ii) $\forall K,L \in R[0,1]$ with $I_n \subseteq c,L \subseteq_{i-1}$ we have

$$|\Delta_f(K)-\Delta_f(L)| \leq M/(i-1).$$

By $\bar{I}_i$ we mean the closure of I_i and $|I_i|$ is the length of I_i. We note in passing that the essential part of the definition is contained in condition (ii). Observe that if $M' \geqslant M$, then we immediately have from the definition that $T_f^M \subseteq T_f^{M'}$. The next result gives the fundamental relation between f and the associated trees T_f^M.

<u>Proposition 2</u>. Let $f \in C$. Then $f \in ND \Leftrightarrow \forall M \in \mathbb{Q}^+$ (T_f^M is well-founded).

<u>Proof</u>. "$\Rightarrow$": Suppose for some $M \in \mathbb{Q}^+$, T_f^M is not well-founded. Then there is a sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ such that $\langle I_1,\ldots,I_n\rangle \in T_f^M$ for each $n \in \mathbb{N}$. Let $\{x\} = \bigcap_{n\in\mathbb{N}} \bar{I}_n$. We shall show that f is differentiable at x. Fix m. Let h_1 , $h_2 \neq 0$ be such that $x + h_1$, $x + h_2 \in I_m$. Let also $K,L \in R[0,1]$ be such that $\{x\} \subseteq K$, $L \subseteq I_m$, endpoints $(K) = \{x+h_1,x+\delta_1\}$ and endpoints $(L) = \{x+h_2,x+\delta_2\}$ where δ_1 and δ_2 are chosen so that $|\delta_1| < |h_1| < |h_2|$ and $\frac{h_1-\delta_1}{h_1} = \frac{h_2-\delta_2}{h_2}$. Then



$$\left|\frac{f(x+h_1)-f(x)}{h_1} - \frac{f(x+h_2)-f(x)}{h_2}\right| \le \left|\frac{f(x+\delta_1)-f(x)}{h_1} - \frac{f(x+\delta_2)-f(x)}{h_2}\right|$$

$$+ \left|\frac{h_1-\delta_1}{h_1}\cdot\frac{f(x+h_1)-f(x+\delta_1)}{h_1-\delta_1} - \frac{h_2-\delta_2}{h_2}\cdot\frac{f(x+h_2)-f(x+\delta_2)}{h_2-\delta_2}\right|$$

$$\le \left|\frac{f(x+\delta_1)-f(x)}{h_1}\right| + \left|\frac{f(x+\delta_2)-f(x)}{h_2}\right| + \left|\frac{h_1-\delta_1}{h_1}\right| \cdot \left|\Delta_f(K)-\Delta_f(L)\right|.$$

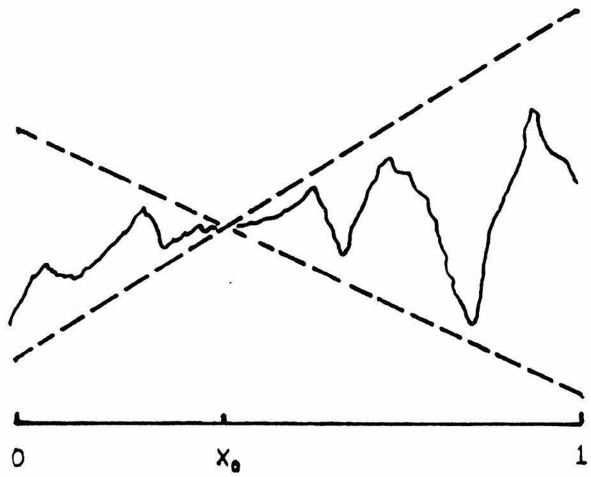
Now $|\Delta_f(K)-\Delta_f(L)| \le M/m$ (because $K \cap L$ contains an I_n for a large enough n) and by the continuity of f $\left|\frac{f(x+\delta_1)-f(x)}{h_1}\right|$, $\left|\frac{f(x+\delta_2)-f(x)}{h_2}\right| \to 0$ and $\left|\frac{h_1-\delta_1}{h_1}\right| \to 1$ as $\delta_1 \to 0$. So

$$\left|\frac{f(x+h_1)-f(x)}{h_1} - \frac{f(x+h_2)-f(x)}{h_2}\right| \le \frac{M}{m}. \qquad (*)$$

Since x is an interior point of I_m in $[0,1]$ and (*) is true for all m it follows that f is differentiable at x.

"$\Leftarrow$": Suppose now that $f \notin ND$. We shall show that for some $M \in \mathbb{Q}^+$ there is a sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ such that $\langle I_1,\dots,I_n\rangle \in T_f^M$

for each $n \in \mathbb{N}$. Choose $x_0 \in [0,1]$ such that f is differentiable at x_0. Then it is easy to see that there is a $c \in \mathbb{Q}^+$ such that f(x) lies between $c\cdot(x-x_0)$ and $-c\cdot(x-x_0)$ for each $x \in [0,1]$.



Take M = 2c. Since f is differentiable of x_0, $\forall n\ \exists m(n)$ such that $\forall h$ with $0 < |h| < 1/m(n)$ and $x_0 + h \in [0,1]$ we have $\left|\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0)\right| \leq \frac{M}{2n}$. We may assume without loss of generality that m(n) is strictly increasing. Let $p_n = \max\{0, x_0 - \frac{1}{2}m(n)\}$ and $q_n = \min\{1, x_0+1/2m(n)\}$. For $n \geqslant 2$ choose $I_n \in Q[0,1]$ such that $x_0 \in I_n \subseteq [p_n,q_n]$ and $\bar{I}_n \subseteq I_{n-1}$ (I_1 is of course [0,1]). Then $|I_n| \leq |q_n-p_n| = 1/m(n) \leq 1/n$ .So for each n , $\langle I_n,\ldots,I_n\rangle$ satisfies condition (i) of the tree T_f^M. We will show that it also satisfies condition (ii). Let $K,L \in R[0,1]$ be such that $x_0 \in K \cap L$ and endpoints (K) =0{a,b} and endpoints (L) = {c,d} . Then for all such $K,L \subseteq I_n$ ($n \geqslant 2$) we have

$$|\Delta_f(K)-\Delta_f(L)| = \left|\frac{f(b)-f(a)}{b-a} - \frac{f(d)-f(c)}{d-c}\right|$$

$$\leq \left|\frac{f(b)-f(a)}{b-a} - f'(x_0)\right| + \left|\frac{f(d)-f(c)}{d-c} - f'(x_0)\right|$$

$$\leq \left|\frac{f(b)-f(x_0)}{b-x_0} - f'(x_0)\right| + \left|\frac{f(a)-f(x_0)}{d-c} - f'(x_0)\right|$$

$$+ \left|\frac{f(b)-f(x_0)}{d-x_0} - f'(x_0)\right| + \left|\frac{f(c)-f(x_0)}{c-x_0} f'(x_0)\right|$$

$$\leq 4\cdot M/4n = M/n.$$

Also for all such $K,L \subseteq I$, we have $|\Delta_f(K)-\Delta_f(L)| \leq |\Delta_f(K)| + |\Delta_f(L) \leq c + c = M/1$. So $|\Delta_f(K)-\Delta_f(L)| \leq M/n$ for all $\{x_0\} \subseteq K,L \subseteq I_n$. Since $x_- \in I_n$, for all n it follows that condition (ii) for T_f^M is satisfied, and so $\langle I,\dots,I_n\rangle \in T_f^M$ for each $n \in \mathbb{N}$. So T_f^M is not well-founded. □

Let $f \in ND$ and $M \in \mathbb{Q}^+$. Since Q[0,1] is countable the rank of the tree T_f^M is countable. Also since $T_f^{M'} \supseteq T_f^M$ doe $M' \geqslant M$ we have $\sup\{r(T_f^M)+1 : M \in |^+\} = \sup\{r(T_f^N)+1 : N \in \mathbb{N}\} < \omega_1$.

<u>Definition</u>. We define the rank function $\rho : ND1 \to \omega_1$ by

$$\rho(f) = \sup\{r(T_f^M)+1 : M \in \mathbb{Q}^+\}.$$

<u>Proposition 3</u>. $\rho : ND \to \omega_1$ is a coanalytic norm.

<u>Proof</u>. Let $\beta : Q[0,1] \to \mathbb{N}$ be a Borel measurable bijection. Define a

map $\tau : C \to 2^{\mathbb{N}^*}$ by $\tau(f) = T_f$, where T_f is the tree (viewed as an element of $2^{\mathbb{N}^*}$) given by

$$T_f = \{\emptyset\} \cup \bigcup_{N \in \mathbb{N}} \{<N, \beta(I), \ldots, (I_n)> : <I_1, \ldots, I_n> \in T_f^N\} .$$

Then $f \in ND \iff T_f \in WF$. So ND $\tau^{-1}[WF]$. Moreover $\rho(f) = \sup\{r(T_f^N)+1 : N \in \mathbb{N}\} = r(T_f) = r(\tau(f))$. Also it is easy to see that τ is a Borel measurable function. Hence by Propositions 0.3 and 0.5 it follows that ρ is a coanalytic norm.

□

Our next aim is to show that $\rho(f)$ is always a limit ordinal. But first we need some definitions and two lemmas.

Definition. Let $I \in R[0,1]$ and T be a tree on R[0,1]. We define the subtree $T \restriction I$ of T by

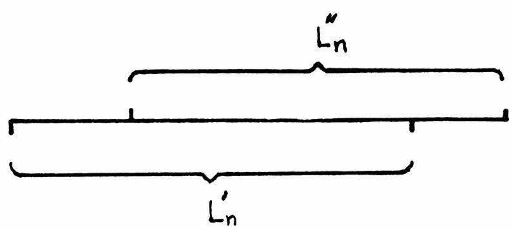
$$<I_1, I_2, \ldots, I_n> \in T \restriction I \iff <I_1, I_2, \ldots, I_n> \in T \text{ and } I_2 \subseteq I .$$

Let $f \in ND$ and $M \in \mathbb{Q}^+$. For each $x \in [0,1]$ we define

$$r(T_f^M; x) = \min\{r(T_f^M \restriction I) : x \in I \in R[0,1]\}$$

Lemma 4. If $r(T_f^M) \geq \omega \cdot \alpha$ then there is an $x \in [0,1]$ such that $r(T_f^{ML}; x) \geq \omega \cdot \alpha$.

Proof. We shall find a sequence $<L_n>_{n \in \mathbb{N}}$ of nested closed intervals with $1/n \leq |L_n| < 2/n$ such that $r(T_f^M \upharpoonright L_n) \geqslant \omega \cdot \alpha$ for each $n \in \mathbb{N}$. Taking $x \in \bigcap_{n \in \mathbb{N}} L_n$ we will then get $r(T_f^M;x) \geqslant \omega \cdot \alpha$. Take $L_1 = [0,1]$. We construct L_n by induction on n. Given L_n choose closed intervals L_n' and L_n'' such that $L_n' \cup L_n'' = L_n$; $|L_n'|$, $|L_n''| < 2/(n+1)$ and $|L_n' \cap L_n''| \geqslant 1/(n+1)$.



Now observe that if $\langle I_1,\ldots,I_n,I_{n+1},\ldots,I_k\rangle \in T_f^M \upharpoonright L_n$ then $I_{n+1} \subseteq L_n'$ or $I_{n+1} \subseteq L_n''$. So $\langle I_1,I_{n+1},\ldots,I_k\rangle \in T_f^M \upharpoonright L_n'$ or $\langle I_1,I_{n+1},\ldots,I_k\rangle \in T_f^M \upharpoonright L_n''$. Suppose now that $r(T_f^M \epsilon L_n')$, $r(T_f^M \upharpoonright L_n'') \leq \beta < \omega \cdot \alpha$. Then

$$r(T_f^M \upharpoonright L_n) \leq \sup\{r(\underset{\sim}{u};T_f^M \upharpoonright L_n)+1 : \underset{\sim}{u} \in T_f^M \upharpoonright L_n \text{ and } |\underset{\sim}{u}|=n+1\} + n$$

$$\leq \max\{\sup\{r(\underset{\sim}{v};T_f^m \upharpoonright L_n')+1 : \underset{\sim}{v} \epsilon_f^M \upharpoonright L_n' \text{ and } |\underset{\sim}{v}|=2\} + n ,$$

$$\sup\{r(\underset{\sim}{v};T_f^M \restriction L_n'')+1 : \underset{\sim}{v}\in T_f^M \restriction L_n'' \text{ and } |\underset{\sim}{v}|=2h\} + n)$$

$$\le \max\{r(T_f^M \restriction L_n'), r(T_f^M \restriction L_n'')\} + n \le \beta + n < \omega\cdot\alpha$$

which contradicts the induction hypothesis. Thus

$$\max\{r(T_f^M \restriction L_n'),\ r(T_f^M \restriction L_n'')\} \geq \omega\cdot\alpha .$$

$$\text{Choose } L_{n+1} = \begin{cases} L_n' \text{ if } r(T_f^M \restriction L_n') \geq \omega\cdot\alpha \\ L_n'' \text{ if } r(T^M \restriction L_n') < \omega\cdot\alpha \text{ and } r(T_f^M \restriction L_n'') \geq \omega\cdot\alpha \end{cases}$$

Then L_{n+1} has the required properties and the induction step is complete. This completes the proof of the lemma. □

<u>Definition</u>. Let T be a tree on Q[0,1] and $k \in \mathbb{N}$. We define the subtree $[T]_k$ of T by

$$\langle I_1,I_2,\ldots,I_n\rangle \in [T]_k \iff \langle I_1,I_2,\ldots,I_n\rangle \in T \text{ and there exists}$$
$$J_2,\ \ldots,\ J_k \in Q[0,1] \text{ such that } \langle I_1,J_2,\ldots,J_k,I_2,\ldots,I_n\rangle \in T.$$

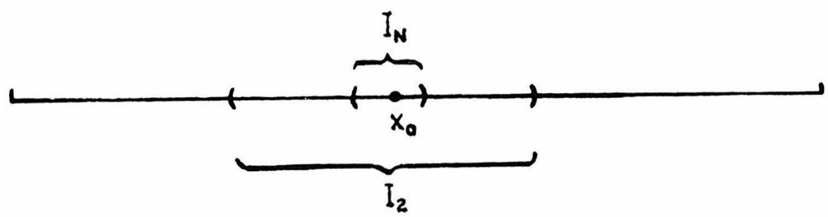
<u>Lemma 5</u>. Suppose $r(T_f^M \restriction I) \geq \omega\cdot\alpha$. Then for each $k \in \mathbb{N}$ $r([T_f^M \restriction I]_k) \geq \omega\cdot\alpha$.

<u>Proof</u>. Observe from the definition of the tree T_f^M that $\langle I_1,I_2,\ldots,I_n\rangle \in T_f^M$ implies $\langle I_1,I_{i_2},\ldots,I_{i_m}\rangle \in T_f^M$ for any subsequence $\langle i_2,\ldots,i_m\rangle$ of $\langle 2,\ldots,n\rangle$. Now $\sup\{r(\underset{\sim}{v}\ [T_f^M \in I]_k)\ :\ \underset{\sim}{v}\in[T_f^M \restriction I]_k$ and

$|\underset{\sim}{v}|=2\} + k \leq r(\langle I_1\rangle ; T_f^M \restriction I) \geqslant \omega\cdot\alpha$,since all the nodes in $T_f^M \restriction I$ lie below $\langle I_1\rangle$. So $\sup\{r(\underset{\sim}{v};[T_f^M \restriction I]_k) : \underset{\sim}{v} \in [T_f^M \restriction I]_k$ and $|\underset{\sim}{v}|=2\} \geqslant \omega\cdot\alpha$, otherwise the above inequality would not hold. Thus $r(\langle I_1\rangle;\{T_f^M \restriction I]_k \geqslant \omega\cdot\alpha$ and so $[T_f^M \restriction I]_k$ has rank $\geqslant \omega\cdot\alpha$. □

<u>Proposition 6</u>. For $f \in ND$, $\rho(f)$ is always a limit ordinal.

<u>Proof</u>. We shall show that $\rho(f) \geqslant \omega\cdot\alpha + 1$ implies $\rho(f) \geqslant \omega\cdot(\alpha+1)$. It will then follow that $\rho(f)$ must always be a limit ordinal. Suppose $\rho(f) \geqslant \omega\cdot\alpha + 1$. Then by definition $r(T_f^M) \geqslant \omega\cdot\alpha$ for some $M \in \mathbb{Q}^+$. By lemma 4 there exists an $x_0 \in [0,1]$ such that $r(T_f^M;x_0) \geqslant \omega\cdot\alpha$. Fix $N \in \mathbb{N}$. Choose $I_2 , \ldots, I_N \in Q[0,1]$ such that $|I_i| \leq 1/i$, $\bar{I}_i \subseteq I_{i-1}$ and $x_0 \in I_N$. (Here as always $I_1 = [0,1]$).



Now define the tree T_N by

$$\langle I_1,J_N,\ldots,J_n\rangle \in [T_f^M \restriction I_N]_N \Rightarrow$$

$$\langle I_1,I_2,\ldots,I_{N-1},J_N,\ldots,J_n\rangle \in T_N$$

and $<I_1,\ldots,I_i> \in T_N$ for each $i = 1, \ldots, N-1$.

It is easy to see that T_N is a subtree of $T_f^{M\cdot N}$. But

$$r(<I_1>;T_N) \geqslant r(<I_1> : [T_f^M \upharpoonright I_N]_N) + N - 2$$

$$\geqslant \omega\cdot\alpha + N - 2 \text{ by lemma 5.}$$

Hence $r(T_f^{M\cdot N}) \geqslant \omega\cdot\alpha + N - 2$. So

$$\begin{aligned}\rho(f) &= \sup\{r(T_f^{M\cdot N}) + 1 : N \in \mathbb{N}\} \\ &\geqslant \sup\{\omega\cdot\alpha + N - 1 : N \in \mathbb{N}\} = \omega\cdot(\alpha+1).\end{aligned}$$ □

<u>Definition</u>: For $f \in ND$ we define $r(f)$ to be the unique ordinal α such that $\rho(f) = \omega\cdot\alpha$.

It follows immediately that r is a coanalytic norm on ND. We will now characterize the functions for which r(f) is small. We make the following definition:

<u>Definition</u>: Let $f \in C$. We define the *amplitude* $A(f;x)$ of the difference quotient of f at x by

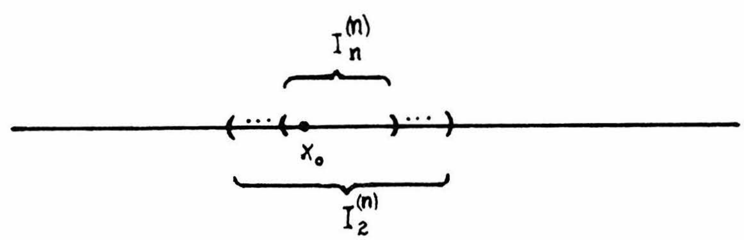
$$A(f;x) = \limsup_{h_1,h_2\to 0}\left|\frac{f(x+h_1)-f(x)}{h_1} - \frac{f(x+h_2)-f(x)}{h_2}\right|$$

It follows that $f \in ND$ iff $A(f;x) > 0$ for each $x \in [0,1]$. Moreover $A(f;x)$ is finite iff all the Dini derivatives at x are finite. So we can rewrite the set BC, of all the Banach functions as

$$BC = \{f \in C : A(f;x) = +\infty \text{ for each } x \in [0,1]\}$$

Proposition 7. $r(f) = 1 \Leftrightarrow f$ is a Banach function.

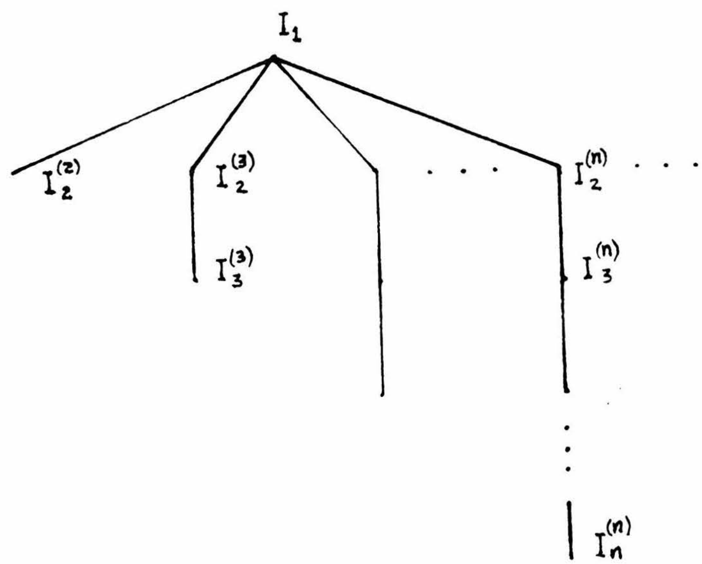
Proof. "$\Rightarrow$": Suppose f is not a Banach function. Then there is an $x_0 \in [0,1]$ such that $A(f;x_0)$ is finite. We can thus find an $M \in \mathbb{Q}^+$ such that for all $K,L \in R[0,1]$ with $x_0 \in K,L$ we have $|\Delta_f(K)-\Delta_f(L)| \leq M$. We shall show that $r(T_f^M) \geqslant \omega$, so that $r(f)$ cannot be 1. Fix n. Choose $I_2^{(n)} \in Q[0,1]$ such that $|I_2^{(n)}| \leq 1/n$ and $x_0 \in I_2^{(n)}$. By the continuity of f we can find $I_n^{(n)} \in Q[0,1]$ with $\bar{I}_n^{(n)} \subseteq I_2^{(n)}$ and $x_0 \in I_n^{(n)}$ such that $|\Delta_f(K)-\Delta_f(L)| \leq M/n$ for all $K,L \in R[0,1]$ with $I_n^{(n)} \subseteq K,L \subseteq I_2^{(n)}$. (It will suffice to choose $I_n^{(n)}$ with endpoints close enough to the endpoints of $I_2^{(n)}$.)



Now choose $I_i^{(n)} \in Q[0,1]$ such that $\bar{I}_i^{(n)} \subseteq I_{i-1}^{(n)}$ $(i=3,\ldots,n-1)$. Then it is easy to see that $\langle I_1, I_2^{(n)},\ldots,I_n^{(n)}\rangle \in T_f^M$. So $r(\langle I_1\rangle;T_f^M) \geqslant n$. But this is true for each $n \in \mathbb{N}$. Hence $r(\langle I_1\rangle;T_f^M) \geqslant \omega$. So $r(T_f^M) \geqslant \omega$ and we are done.

"$\Leftarrow$": Suppose now that $r(f) \neq 1$. Then $r(f) > 1$ and so

$\rho(f) \geqslant \omega \cdot 2$. Thus for some $M \in \mathbb{Q}^+$, $r(T_f^M) \geqslant \omega + 1$. So T_f^M must have a subtree as shown below:



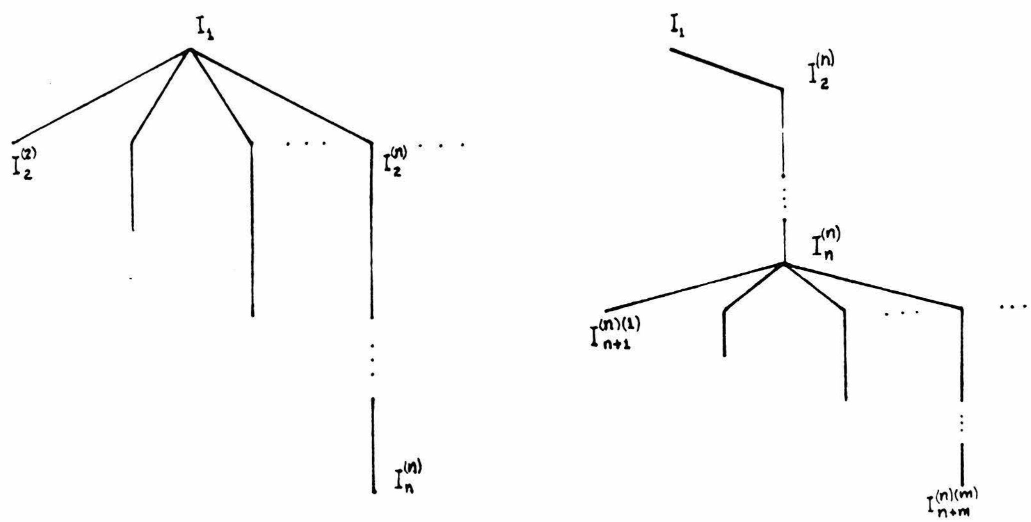
Let x_0 be a limit point of the set of all midpoints of the intervals $\{I_n^{(n)}\}_{n \geqslant 2}$. Then there are arbitrarily small intervals I in the tree T_f^M arbitrarily close to x_0. Let $h_1, h_2 \neq 0$ be such that $x_0 + h_1$, $x_0 + h_2 \in [0,1]$. Then as in the "$\Rightarrow$" direction of the proof of Proposition 2 we get that

$$\left| \frac{f(x_0+h_1)-f(x_0)}{h_1} - \frac{f(x_0+h_2)-f(x_0)}{h_2} \right| \leq M .$$

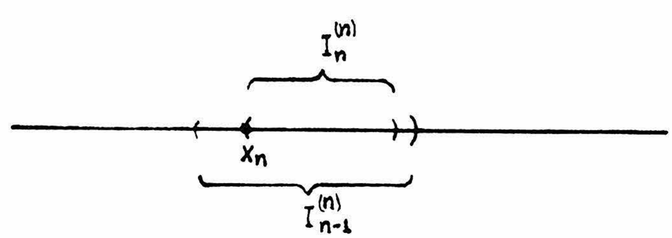
Since this is true for all h_1 , h_2 we get that $A(f;x_0) \leq M$. So f is not a Banach function. □

<u>Proposition 8</u>. Suppose there is a $c > 0$ such that $A(f;x) \geqslant c$ for each $x \in \{0,1\}$. Then $r(f) \leq 2$.

<u>Proof</u>. Suppose $r(f) > 2$. The $\rho(f) \geqslant \omega\cdot 3$. So there is an $M \in \mathbb{Q}^+$ such that $r(T_f^M) \geqslant \omega\cdot 2 + 1$. So T_f^M has a subtree as shown below, where each of the nodes $\langle I_1, I_2^{(n)}, \ldots, I_n^{(n)}\rangle$ is of rank at least ω in T_f^M.



Fix $n \in \mathbb{N}$ and consider the subtree of T_f^M through the node $\langle I_1, I_2^{(n)}, \ldots, I_n^{(n)}\rangle$. As in the "$\Leftarrow$" direction of the proof of Proposition 7 we see that there is a point $x_n \in \bar{I}_n^{(n)} \subseteq I_{n-1}^{(n)}$ such that $A(f;x_n) \leq M/(n-1)$.



But this is true for each $n \in \mathbb{N}$. So for large enough n we will have $A(f;x_n) < c$, which is a contradiction. Hence $r(f) \le 2$. □

Remark. It is easy to construct an example of a function f with $A(f;x_n) \to 0$ for some sequence $\{x_n\} \subseteq [0,1]$ but with $r(f) = 2$. (We will sketch this construction at the end of §3.) From this it follows that the converse of Proposition 8 is false.

§2. Some Natural Examples. Before we give our natural examples we will introduce a notation which will prove very handy. Let I be an interval with endpoints x and y (with $x \ne y$, and y not necessarily greater than x). For a continuous function f we define $\Delta_f(x,y) = \frac{f(y)-f(x)}{y-x}$. We now turn to our natural examples.

Weierstrass function: This function is defined by

$f(x) = \sum_{n=0}^{\infty} a^n \sin \pi(b^m x)$, where $0 < a < 1$ and b is an odd integer such that $ab > 1 + 3\pi/2$. In A. N. Singh [46] it is shown that for $k \in \mathbb{Z}$, $m \in N$

$$\Delta_f\left(x, x + \frac{2k}{b^{m+1}}\right) = -\pi \sum_{n=0}^{m-1} (ab)^n \sin \pi\, b^n\left(x + \frac{2k}{b^{m+1}}\,\theta\right)$$

$$-\pi(ab)^m \frac{\sin(k\pi/b)}{(k\pi/b)} \cdot \sin \pi(b^m x + k/b) \qquad (*)$$

for some θ with $|\theta| < 1$, by using the Mean Value Theorem. Now it is clear that the absolute value of the first m terms is less than $\sum_{n=0}^{m-1} \pi(ab)^n < \frac{\pi(ab)^m}{ab-1}$. Also it is easy to see that there are two integers k_1, k_2 with $|k_1|$, $|k_2| \leq 3b/4$ such that

$$\sin \pi(b^m x + k_1/b) \geq 1/\sqrt{2} \text{ and } \sin(b^m x + k_2/b) \leq -1/\sqrt{2} .$$

But for these k_i's (i = 1,2) we have

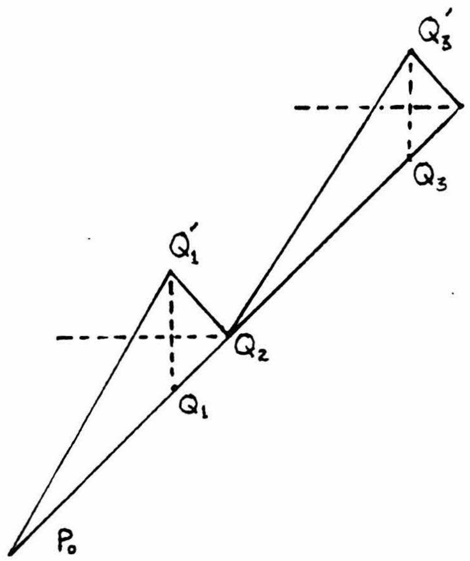
$$\left| \pi(ab)^m \cdot \frac{\sin(k_i\pi/b)}{(k_i\pi/b)} \cdot \sin \pi\left[b^m x + \frac{k_i}{b}\right] \right| \geq \frac{\pi(ab)^m}{3\pi/2}$$

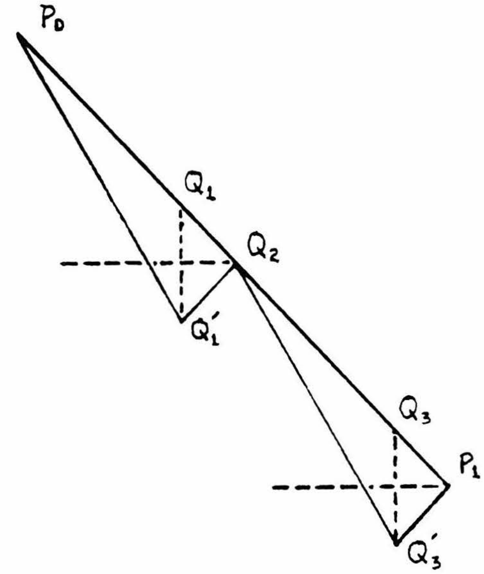
Hence the last term in (*) dominates and we have

$\limsup_{m\to\infty} \left| \Delta_f\left[x , x + \frac{2k_i}{b^{m+1}}\right] \right| = +\infty$, and so $A(f;x) = +\infty$ for each $x \in (0,1)$.

It is easy to compute directly as in [21] p. 405 that f has right derivative $+\infty$ at $x = 0$, and left derivative $-\infty$ at $x = 1$. Thus $A(f;x) = +\infty$ for each $x \in [0,1]$. From Proposition 7 it now follows that $r(f) = 1$.

Bolzano function: This function is constructed geometrically by iterating a basic operation $\mathfrak{J}_0$ on straight lines that are not parallel to one of the coordinate axes. Consider such a straight line with endpoints $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$. Let Q_1, Q_2 and Q_3 be the points which are 3/8, 1/2 and 7/8 the way from P_0 to P_1 on $[P_0,P_1]$ respectively.





Let Q_1' be the image of the Q_1 when it is reflected in the line $y = (y_0+y_1)/2$ and Q_3' be the image of the point Q_3 when it is reflected in the line $y = y_1$. $\mathcal{J}_0$ is defined to be the operation which takes the line segment P_0P_1 to the polygonal path $P_0Q_1'Q_2Q_3'P_1$. Now let $f_0(x) = x$ for $x \in [0,1]$. We define the function $f_1(x)$ by graph $(f_1) = \mathcal{J}_0$ (graph (f_0)). In general we define f_{n+1} by graph $(f_{n+1}) = \mathcal{J}_0$ (graph (f_n)). It is understood that the operation $\mathcal{J}_0$ is applied to each straight line segment of graph (f_n). This gives us a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ of continuous functions on [0,1]. It is easy to see that f_n converges to a continuous function f.

Rychlik [43] obtained a parametric representation of f by continuous functions φ and Φ, with φ strictly increasing as shown below

$$x = \varphi(\xi), \quad f(x) = \Phi(\xi) \qquad \xi \in [0,1].$$

Using this representation he was able to show that if

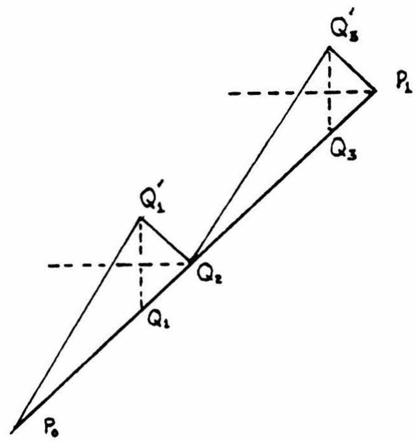
$$\xi = \frac{k_1}{4} + \frac{k_2}{4^2} + \frac{k_3}{4^3} \cdots + \frac{k_n}{4^n} + \cdots \quad , k_i \in \{0,1,2,3\}$$

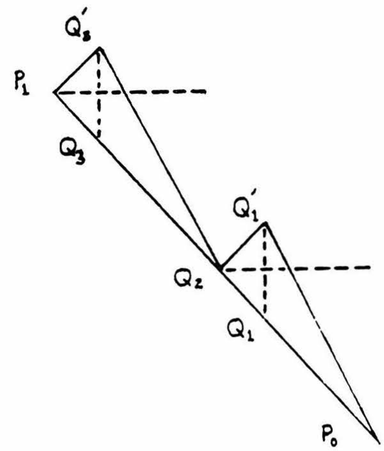
$$\xi_n' = \frac{k_1}{4} + \cdots + \frac{k_n}{4^n}, \quad \xi_n'' = \frac{k_1}{4} + \cdots + \frac{k_n+1}{4^n}$$

and we put $x_n' = \varphi(\xi_n')$, $x_n'' = \varphi(\xi_n'')$ then

$\frac{f(x_n'')-f(x_n')}{x_n''-x_n'} = \chi(k_1)\cdots\chi(k_n)$ and $x_n' \leq x = \varphi(\xi) \leq x_n''$ where χ is the function defined by $\chi(0) = \chi(2) = 5/3$, $\chi(1) = \chi(3) = -1$. From this it immediately follows that $A(f;x) \geq 1$ for each $x \in [0,1]$. Also an easy computation shows that $A(f;1) = 2$. So by Propositions 7 and 8 we get that $r(f) = 2$.

Kowalewski variant: G. Kowalewski [31] made a modification to the basic Bolzano operation $\mathfrak{I}_0$ to produce a variant of the Bolzano function which was in some sense more natural. We shall call the modified operation $\mathfrak{I}_1$. Consider a straight line P_0P_1 which is not parallel to a coordinate axis and with P_0 being lower than P_1 (i.e., if $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$ then $y_0 < y_1$). Let Q_1 , Q_2 , Q_3 be the points which are 3/8, 1/2 and 7/8 the way from P_0 to P_1 on $[P_0,P_1]$ respectively.





Again we let Q_1' be the image of Q_1 when it is reflected in the line $y = (y_0 + y_1)/2$ and Q_3' be the image of Q_3 when it is reflected in the line $y = y_1$. The operation $\mathfrak{I}_1$ is defined to be the one which takes the line segment P_0P_1 to the polygonal path $P_0Q_1'Q_2Q_3'P_1$. We proceed as before to obtain a continuous function f. Kowalewski [31] showed by geometrical means that f was nowhere differentiable. Singh [45] gave an analytic representation of this function as follows. A point $x \in (0,1)$ can be represented as

$$x = \frac{3^{a_1}}{8} k_1 + \frac{3^{a_2}}{8^2} k_2 + \cdots + \frac{3^{a_n}}{8^n} k_n + \cdots$$

if the a_n's and k_n's are suitably chosen with $k_n \in \{0,1,\ldots,7\}$. f(x) is then defined by

$$f(x) = \frac{3^{a_1}}{4} q_1 + \frac{3^{a_2}}{4^2} q_2 + \cdots + \frac{3^{a_n}}{4^n} q_n + \cdots$$

where the q_n's are chosen from $\{-2,-1,\ldots,5\}$ according to a prescribed rule. Using this representation Singh showed that if x and x′ are two points which first differ in their representation at the n-th place then $|x-x'| < (3/8)^{n-1}$. Moreover a point x″ can be found such that

$$|f(x'')-f(x)| \geqslant (3/4)^{n-2} \quad \text{and} \quad |x-x''| < |x-x'| .$$

Thus $\left|\frac{f(x'')-f(x)}{x''-x}\right| \geqslant 3\cdot 2^{n-3}$ and so $A(f;x) = +\infty$ for each $x \in (0,1)$. It is easy to see that at x = 0, f(x) has right derivative $+\infty$; and at x = 1, f(x) has left derivative $-\infty$. Thus $A(f;x) = +\infty$ for each $x \in [0,1]$ and so from Proposition 7 we have that r(f) = 1.

Takagi function: This function was defined by T. Takagi [52]. Let $t \in [0,1]$ be represented as

$$t = \frac{c_1}{2} + \frac{c_2}{2^2} + \frac{c_3}{2^3} + \cdots + \frac{c_n}{2^n} + \cdots , \quad c_i \in \{0,1\}$$

and put

$$\tau_n = \frac{c_n}{2^n} + \frac{c_{n+1}}{2^{n+1}} + \frac{c_{n+2}}{2^{n+2}} + \cdots$$

and

$$\tau_n' = \frac{(1-c_n)}{2^n} + \frac{(1-c_{n+1})}{2^n} + \frac{(1-c_{n+2})}{2^{n+2}} + \cdots = \frac{1}{2^{n-2}} - \tau_n.$$

The function f(t) is defined by

$$f(t) = \sum_{n=1}^{\infty} \gamma_n , \text{ where } \gamma_n = \begin{cases} \tau_n & \text{if } c_n = 0 \\ \tau_n' & \text{if } c_n = 1 \end{cases}$$

It is easy to see that $f(t) = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$, where

$$a_n = \begin{cases} \nu_n = \#\text{ of 1's among } c_1,\dots,c_n & \text{if } c_n = 0 \\ \pi_n = \#\text{ of 0's among } c_1,\dots,c_n & \text{if } c_n = 1 \end{cases}$$

Moreover it is clear that f is single-valued since the numbers of the form $t = m/2^n$ $(m,n \in \mathbb{N})$ (being the only t's with two representation) give rise to the same value f(t). We will prove that $A(f;t) \geqslant 1/2$ for each $t \in [0,1]$. Let us adopt the convention that the number t does not have $c_n = 1$ from some point onwards, unless $t = 1$ (in which case $c_n = 1$ for all $n \in \mathbb{N}$). Then each number has a unique representation. Now if $c_n = 0$ then $\Delta_f(t , t + 1/2^n) = \pi_n - \nu_n \, 2^{n+1}\tau_{n+1}$, and if $c_{n-1} = c_n = 0$ then $\Delta_f(t , t + 1/2^n) = \pi_n - \nu_n$. So if t is such that $c_n = 0$ and $c_{n+1} = 0$ then

$$\left| \Delta_f\left(t , t + \frac{1}{2^n}\right) - \Delta_f\left(t , t + \frac{1}{2^{n+1}}\right) \right| = 1 - 2^{n+1}\tau_{n+2}; \qquad (*)$$

and if t is such that $c_n = 0$ and $c_{n+1} = 1$ then

$$\left| \Delta_f\left(t , t + \frac{1}{2^n}\right) - \Delta_f\left(t , t + \frac{1}{2^{n+1}}\right) \right| = 2^{n+1}\tau_{n+2}. \qquad (**)$$

Now let $t \in [0,1]$. Then there are there possible cases:

<u>Case</u> <u>(i)</u>: $c_n = 0$ from some point onwards. So three is an n_0 such that $c_n = 0$ for all $n \geqslant n_0$. Thus $\tau_n = 0$ for all $n \geqslant n_0$. Thus from (*)

$$\left| \Delta_f\left(t,t + \frac{1}{2^n}\right) - \Delta_f\left(t,t + \frac{1}{2^{n+1}}\right) \right| = 1$$

for all $n \geqslant n_0$ and so $A(f;t) \geqslant 1$.

Case (ii): There are infinitely many 0's and infinitely many 1's in the unique expansion of t. So there is a sequence $\{n_k\}$ such that $c_{n_k} = 0$ and $c_{n_k+1} = 1$ for all $k \in \mathbb{N}$.

$$\left| \Delta_f\left[t, t + \frac{1}{2^{n_k}}\right] - \Delta_f\left[f, t + \frac{1}{2^{n_k+1}}\right] \right| = 2^{n+1}\tau_{n+2}$$

But $\left(t - 2^{-(n_k+1)}\right)$ has its n_k-th and $(n_k + 1)$-th term = 0 , so

$$\left| \Delta_f\left[t - \frac{1}{2^{n_k+1}}, t + \frac{1}{2^{n_k+1}}\right] - \Delta_f\left[t - \frac{1}{2^{n_k+1}}, t\right] \right| = 1 - 2^{n+1}\tau_{n+2}$$

using (*). Since at least one of $2^{n+1}\tau_{n+1}$, $(1-2^{n+1}\tau_{n+2})$ is greater than or equal to 1/2 we get that $(f;t) \geqslant 1/2$.

Case (iii): $t = 1$. In this case a direct computation shows that $f(1) = 0$ and $f(1-2^{-n}) = n/2^n$. This immediately shows that $A(f;1) = +\infty$.

So we have shown that $A(f;t) \geqslant 1/2$ for all $t \in [0,1]$. Thus $f \in ND$ and by Proposition 8, $r(f) \leq 2$.

Problem. What is the rank of the Takagi function?

Cellerier function: This function was given by Cellerier [09] and is defined by $f(x) = \sum_{n=1}^{\infty} a^{-n}\sin(\pi a^n x)$ where a is an even integer $\geqslant 1000$. Using the fact that a is an even integer we obtain as in

[21] p. 406

$$\Delta_f\left(x\ ,\ x + \frac{2}{a^m}\right) = \sum_{n=1}^{m-1} \cos(\pi a^n x) + \lambda\theta$$

where $|\theta| < 1$ and λ is a positive number depending only on a , which can be made arbitrarily small by taking a sufficiently large. Similarly we obtain

$$\Delta_f\left(x,x + \frac{1}{a^m}\right) = \sum_{n=1}^{m} \cos(\pi a^m x) - \frac{2}{\pi} \sin(\pi a^m xh) + \lambda'\theta'$$

where $|\theta'| < 1$ and λ' depends only on a and $\lambda' \to 0$ as $a \to \infty$. Thus

$$\Delta\left(x\ ,\ x + \frac{2}{a^{m+1}}\right)\ \Delta_f\left(x\ ,\ x + \frac{2}{a^m}\right) = \cos(\pi a^m x) + 2\lambda\eta$$

and $$\Delta_f\left(x\ ,\ x + \frac{2}{a^m}\right) - \Delta_f\left(x\ ,\ x + \frac{1}{a^m}\right) = \frac{2}{\pi} \sin(\pi a^m x) + 2\lambda'\eta'$$

where $|\eta|,|\eta'| < 1$. Using this and the fact that $a \geqslant 1000$ we get that

$$\left|\Delta_f\left(x\ ,\ x + \frac{2}{a^{m+1}}\right) - \Delta_f\left(x\ ,\ x + \frac{2}{a^m}\right)\right| \geqslant \frac{1}{2}$$

$$\text{or } \left|\Delta_f\left(x\ ,\ x + \frac{2}{\ }\right) - \Delta_f\left(x\ ,\ x + \frac{1}{a^m}\right)\right| \geqslant \frac{1}{2}\ .$$

Hence $A(f;x) \geqslant 1/2$ for each $x \in [0,1)$. A direct computation easily shows that $A(f;1) = +\infty$. Thus $f \in ND$ and by Proposition 8,

$r(f) \le 2$.

Problem: What is the rank of the Cellerier function?

Morse-Besicovitch functions: A. P. Morse [40] constructed functions on [0,1] which were such that

$$\limsup_{x \to z_-} \left| \frac{f(x)-f(z)}{x-z} \right| = +\infty \quad \text{for all } z \in [0,1]$$

and

$$\limsup_{x \to z_+} \left| \frac{f(x)-f(z)}{x-z} \right| = +\infty \quad \text{for all } z \in [0,1]$$

and were moreover Besicovitch functions. (See also [34].) Such functions are called Morse-Besicovitch functions for obvious reasons. Now from the definition of a Morse-Besicovitch function we immediately have $A(f;x) = +\infty$ for all $x \in [0,1]$. So it follows from Proposition 7 that $r(f) = 1$. We shall denote the class of all Morse-Besicovitch functions by MB.

Knopp functions: K. Knopp [27] gave a general method of constructing nowhere differentiable functions by using a sequence $\{u_n(x)\}$ of functions with certain properties. The sequence $u_n(x)$ is chosen so that $\sum_{n=0}^{\infty} \|u_n\|$ converges, therefore $\sum_{n=0}^{\infty} u_n(x)$ converges to a continuous function. For the details we also refer to [21] p. 407–409. Now the construction of Knopp guarantees that $A(f;x) = +\infty$ for each $x \in (0,1)$. So if a Knopp function f is in ND (it might not be in ND because it is possible that it has a finite one-sided derivative at $x = 0$ or $x = 1$) then we have $r(f) \leq 2$ by using Proposition 8. If in addition we also had that $A(f;0) = A(f;1) = +\infty$ then of course we would get $r(f) = 1$. Most of the nowhere differentiable functions which are expressed as an infinite series of functions can be obtained by the Knopp method. In particular so can the Weierstrass function.

We have seen several natural examples of functions in ND and in all cases we had $r(f) \leq 2$. This supports our intuitive idea that natural examples should have rank 1 or 2. Of course we cannot make this precise because the concept of "natural" is rather vague.

§3. <u>The Rank Function r is Unbounded in ω_1 on BF</u>. In this section our aim will be to show that for each $1 \leq \alpha < \omega_1$ there is an $f \in BF$ such that $r(f) = \alpha$. To this end we introduce the following definition:

<u>Definition</u>: Let $f \in ND$ and $M \in \mathbb{Q}^+$. For each $I \in Q[0,1]$ we define the subtree $T^M_{I,f}$ of T^M_f by $\langle I_1,\dots,I_n\rangle \in T^M_{I,f} \Leftrightarrow \langle I_1,\dots,I_n\rangle \in T^M_f$ and $\exists\ J_1, \dots, J_k \in Q\,[0,1]$ such that $\langle I_1,\dots,I_n,J_1,\dots,J_k\rangle \in T^M_f$ and $J_k \subseteq I$. $T^M_{I,f}$ is called the subtree of T^M_f *based* in I. For each $x \in [0,1]$ we also define

$$r(x;T^M_f) = \min\{r(T^M_{I,f}): x \in I \in Q[0,1]\}.$$

Note. $r(x;T_f^M)$ is to be distinguished from $r(T_f^M;x)$, which was defined in §1. $r(x;T_f^M)$ is a tool we'll need only in this section.

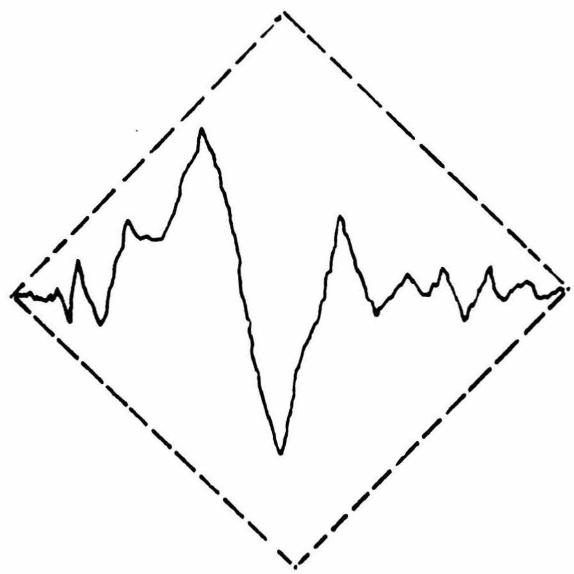
Lemma 9: Suppose $r(T_f^M) \geq \omega$. Then there is an $x \in [0,1]$ such that $r(x;T_f^M) = r(T_f^M)$.

Proof: The proof is very similar to Lemma 4. We shall find a nested sequence of closed intervals $\langle L_n \rangle_{n \in \mathbb{N}}$ with $1/n \leq |L_n| < 2/n$ such that $r(T^M_{L_n,f}) = r(T_f^M)$ for each $n \in \mathbb{N}$. Taking $x \in \bigcap_{n \in \mathbb{N}} L_n$ gives us the result. We take $L_1 = [0,1]$. Given L_n choose L_n', L_n'' as in Lemma 4. Now observe that if $\langle I_1,\ldots,I_{n+1} \rangle$ is a node of length $n + 1$ in $T^M_{L_n,f}$ then $I_{n+1} \subset L_n'$ or $I_{n+1} \subset L_n''$. Thus any node in $T^M_{L_n,f}$ of length $\geq n + 1$ lies in at least one of the trees $T^M_{L_n',f}$, $T^M_{L_n'',f}$. Moreover if $\underset{\sim}{v}$ is a node which is not in $T^M_{L_n'',f}$ then $\underset{\sim}{v}$ can only have nodes extending it which are of length at most n. So $r(\underset{\sim}{v};T^M_{L_n,f}) \leq n$. Since $r(T^M_{L_n,f}) = r(T_f^M) \geq \omega$ by the induction hypothesis, we see that $r(T_f^M) = \max\{r(T^M_{L_n',f}), r(T^M_{L_n'',f})\}$. Now let

$$L_{n+1} = \begin{cases} L'_n & \text{if } r(T^M_{L'_n,f}) = r(T^M_f) \\ L''_n & \text{if } r(T^M_{L'_n,f}) < r(T^M_{L''_n,f}) = r(T^M_f) \end{cases}$$

This completes the induction step and we are done. □

Consider the square, S with vertices at (0,0), (1/2,1/2), (1,0) and (1/2,-1/2). Let SF be the collection of all Besicovitch functions whose graph lie inside the square S. It is easy to see that S is non-empty



and by definition SF ⊆ BF. Let f ∈ C. By a *scaled copy* of f onto the interval [a,b] we mean the function g defined by $g(x) = f(\frac{x-a}{b-a})$, x ∈ [a,b]. Let I be a closed interval with rational endpoint and with $|I| > 0$. We define

R(I) = {J ⊆ I; J is a non-empty interval open in I}

Q(I) = {J ∈ R(I): J has rational endpoints}

Definition: Let $f \in C$ and $M \in \mathbb{Q}^+$. Then there is an obvious way to define the tree $T^M_{f\restriction I}$. We define $T^M_{f\restriction I}$ by

$\langle I_1,\dots,I_n\rangle \in T^M_{f\restriction I} \Leftrightarrow I_1 = I$ and $\forall i = 2, \dots, n$ we have

(i) $I_i \in Q(I)$, $\bar{I}_i \subseteq I_{i-1}$, $|I_i| \leq |I|/i$ and

(ii) $\forall K,L \in R(I)$ with $I_n \subseteq K,L \subseteq I_{i-1}$ we have $|\Delta_f(K)-\Delta_f(L)| \leq M/(i-1)$.

It follows immediately that if $g \in C$ and $g\restriction I$ is a scaled copy of $f \in C$ onto I then T^M_f and $T^M_{g\restriction I}$ are isomorphic.

Proposition 10: The rank function r is unbounded in ω_1 on SF.

Proof: It will suffice to show that for each $\alpha < \omega_1$ there is an $f \in SF$ such that $\rho(f) \geq \omega\cdot\alpha$. For $\alpha = 0,1$ or 2 there is nothing to prove because we know from Proposition 7 that $f \in SF \Rightarrow \rho(f) \geq \omega\cdot 2$. Suppose the result is true for α. We shall prove it for α. Let $K_n = [2^{-n},2^{-(n-1)}]$ $n \in \mathbb{N}$. Choose $f \in SF$ with $\rho(f) \geq \omega\cdot\alpha$. The basic idea is to put scaled copies of f/n onto K_n and so obtain a function $g \in C$. Since

$T^1_{g\restriction k_n} \cong T^1_{f/n} \equiv T^n_f$ we should get

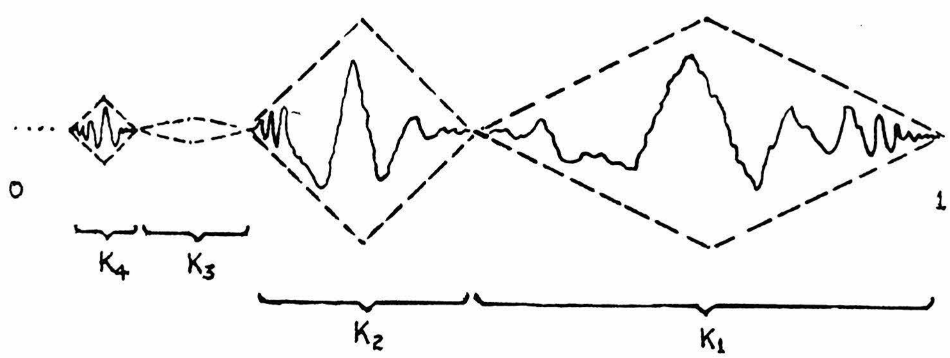
$$r(T^1_g) \geq \sup\{r(T_{g\restriction K_n}): n \in \mathbb{N}\}$$

$$= \sup\{r(T^n_f): n \in \mathbb{N} \geq \omega\cdot\alpha$$

So $\rho(g) > \omega\cdot\alpha$ and hence $\rho(g) \geq \omega\cdot(\alpha+1)$. But there are two problems here. First the g obtained will not be in SF (because g would be differentiable at x = 0). Moreover we might get problems with the inequality "$r(T^1_g) \geq \sup\{r(T_{g\restriction K_n}) : n \in \mathbb{N}\}$" if the rank of f is "concentrated" at its endpoints (i.e., if $r(x;T^M_f) < r(T^M_f)$ for all $x \in (0,1)$). So we need to modify our process accordingly.

Choose $f \in SF$ with $\rho(f) \geq \cdot\alpha$ such that for each $M \in \mathbb{Q}^+$ there is an $x_M \in (0/3,2/3)$ with $r(x_M;T^M_f) = r(T^M_f)$ (by virtue of Lemma 9). This latter condition can be easily obtained by replacing f by five scaled copies of f onto each fifth of the interval [0,1]. Define the function g by g(0) = 0, and

$$g\restriction K_n = \begin{cases} \text{scaled copy of } f \text{ onto } K_n & n \text{ even} \\ \text{scaled copy of } \frac{f}{2n} \text{ onto } K_n & n \text{ odd} \end{cases}$$



Then it is clear that $g \in C$, and by construction g has no unilateral derivative, finite or infinite at any point, except perhaps at x = 0. We show that g has no right derivative at x = 0.

Let $B = \sup\{|f(x)| : x \in [0,1]\} > 0$. Let x_0 be a point in [0,1] at which B is attained (i.e. $|f(x_0)| = B$) and let x_n be the image of x_0 in K_n when f is scaled onto Kn. Then $g(x_{2n}) = B/2^{-2n}$, since $|K_{2n}| = 2^{-2n}$. Now $x_{2n} \le 2\cdot 2^{-2n}$ because $x_{2n} \in K_{2n} = [2^{-2n}, 2^{-(2n-1)}]$. So $\left|\frac{g(x_{2n})-g(0)}{x_{2n}-0}\right| \geq \frac{B\cdot 2^{-2n}}{2\cdot 2^{-2n}} = \frac{B}{2}$ for all $n \in \mathbb{N}$.

But $\left|\frac{g(2^{-2n})-g(0)}{2^{-2n}-0}\right| = 0$ for all $n \in \mathbb{N}$. Hence g has no right derivative at x = 0. Now it is easy to see that the graph of g must lie in the square S, so $g \in SF$. We claim that $\rho(g) \geq \omega\cdot(\alpha+1)$.

Consider now the functions $g\restriction K_n$ for n odd. Let T_n be the tree given by $T_n = [T^1_{H_n}, g\restriction K_n]_3$ where H_n is the middle open third of K_n. Then each interval in the tree T_n is open in [0,1] and

$$r(T_n) + 3 \geq r(T^1_{g\restriction K_n}) .$$

Moreover $\quad T_n \cong [T^1_{(1/3,2/3)}, f/2n]_3 \equiv [T^{2n}_{(1/3,2/3)}, f]_3 .\quad$ So

$$\rho(f) = \sup\{r(T^{2n}_f) + 1 : n \in \mathbb{N}, n \text{ odd}\}$$

$$\leq \sup\{r([T^{2n}_{(1/3,2/3)}, f]_3) + 4 : n \in \mathbb{N}, n \text{ odd}\}$$

$$= \sup\{r(T_n) + 4 : n \in \mathbb{N}, n \text{ odd}\}$$

Since $\rho(f)$ is a limit ordinal it follows that

$$\sup\{r(T_n): n \in \mathbb{N}, n \text{ odd}\} \geqslant \rho(f) .$$

Now let T_n' be the tree defined by

$$<[0,1],I_2,\ldots,I_n> \in T_n' \iff <K_n,I_2,\ldots,I_k> \in T_n.$$

We claim that T_n' is a subtree of T_g^4. Because of the way T_n was chosen it will suffice to show that for any interval I_m in T_n' and for all $K,L \in R[0,1]$ with $I_m \subseteq K,L \subseteq [0,1]$ we have

$$|\Delta_g(K)-\Delta_g(L)| \leq 4.$$

Now if K contains an encpoint of some K_n then $|\Delta_g(K)| \leq 2$ by construction; and if $K \subseteq K_n$ for some n then $|\Delta_g(K)| = |\Delta_g(K) - \Delta_g(K_n| \leq 1$ (since T_n was a subtree of $T^1_{g \upharpoonright K_n}$). The same holds for L. So we always have

$$|\Delta_g(K) - \Delta_g(L)| \leq |\Delta_g(K)| + |\Delta_g(L)| = 4.$$

Thus T_n' is a subtree of T_g^4. Hence

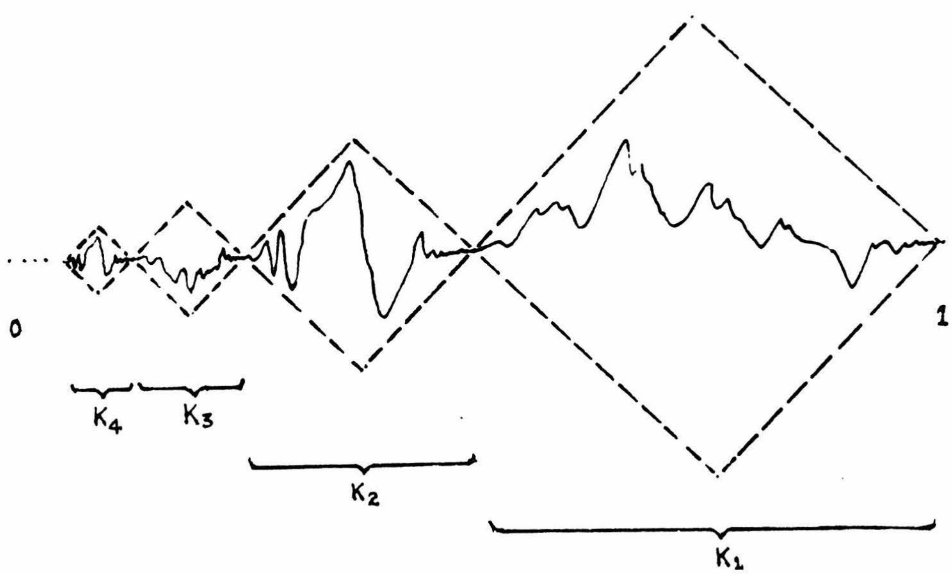
$$\begin{aligned} r(T_g^4) &\geqslant \sup\{r(T_n'): n \in \mathbb{N}, n \text{ odd}\} \\ &= \sup r(T_n): \\ &\geqslant \rho(f) \geqslant \omega\cdot\alpha \end{aligned}$$

So $\rho(g) = \sup\{r(T^M) + 1: M \in \mathbb{Q}^+\}\ \omega\cdot\alpha + 1$. Since $\rho(g)$ is a limit ordinal it follows that $\rho(g) \geqslant \omega\cdot(\alpha+1)$.

Suppose now that the result is true for all $\alpha < \lambda$ where λ is a limit ordinal. Let $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ be an increasing sequence of ordinals with $\lim \alpha_n = \lambda$. For each $n \in \mathbb{N}$ there is an $h_n \in SF$ with $p(h_n) \geq \omega \cdot (\alpha_n + 1)$ such that for each $M \in \mathbb{Q}^+$ there is an x_M^n in $(1/3, 2/3)$ with $r(x_M^n; T_{h_n}^M) = r(T_{h_n}^M)$. So for each $n \in \mathbb{N}$ there is an $M_n \in \mathbb{Q}^+$ also such that $r(T_{h_n}^{M_n}) \geq \omega \cdot \alpha_n$. Now let $f_n = h_n / (M_n + 1)$. Then $r(T_{f_n}) \geq r(T_{h_n}^{M_n}) \geq \omega \cdot \alpha_n$ and $r(x_{M_n}^n; T_{f_n}^{M_n}) = r(T_{f_n}^{M_n})$ for each $n \in \mathbb{N}$.

We proceed as before by defining g by $g(0) = 0$ and

$$g \restriction K_n = \begin{cases} \text{scaled copy of } f_2 \text{ onto } K_n & n \text{ even} \\ \text{scaled copy of } f_n \text{ onto } K_n & n \text{ odd} \end{cases}$$



As before we see that the scaled copy of f_2 in K_{2n} for $n \in \mathbb{N}$ ensures that g has no derivative at x = 0. Also the graph of g is clearly in the square, S. So $g \in SF$.

Finally by the same argument as before we get

$$\begin{aligned} r(T^4_g & \geq \sup\{r(T^1_{f_n}): n \in \mathbb{N}, n \text{ odd}\} \\ & \geq \sup\{\omega \cdot \alpha_n: n \in \mathbb{N}, n \text{ odd}\} \\ & = \omega \cdot \lambda \end{aligned}$$

Thus $\rho(g) \geq \omega \cdot (\lambda + 1) \geq \omega \cdot \lambda$ and we are done.

□

<u>Corollary 11</u>. ND and BF are not Borel subsets of C.

<u>Proof</u>. The result for ND follows immediately from Proposition 10 and Proposition 0.1. The result for BF follows from Proposition 10 and Proposition 0.2. □

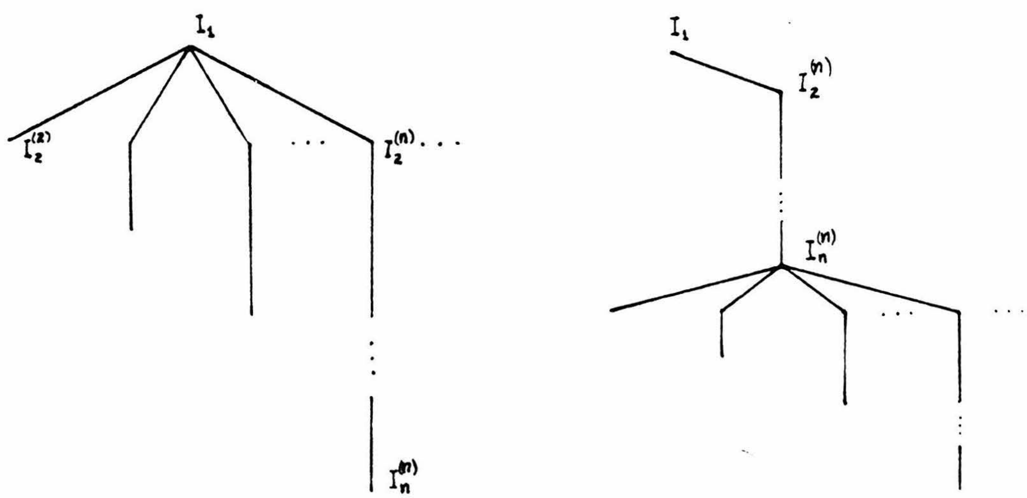
The next result shows that by refining the process given in proposition 10, we could find for each $1 \leq \alpha < \omega_1$ an $f \in BF$ such that $r(f) = \alpha$.

<u>Proposition 12</u>. For each $1 \leq \alpha < \omega_1$ there is an $f \in BF$ such that $r(f) = \alpha$.

<u>Proof</u>. For $\alpha = 1$ we simply take any $f \in BC \cap BF$, a

Morse-Besicovitch function would do nicely. We will show that for each $2 \leq \alpha < \omega_1$ there is an $f \in SF \subseteq BF$ such that $r(f) = \alpha$, by induction on α. For $\alpha = 1$ take $f \in SF$ with $A(f;x) = +\infty$ for each $x \in (0,1)$ (such an f can be easily constructed from a Morse-Besicovitch function by literally "squeezing" its graph into the square). Then by Proposition 8 it follows that $r(f) = 2$.

Now suppose the result is true for α, $\alpha \geqslant 2$. Let $f \in SF$ be such that $\rho(f) = \omega \cdot \alpha$ and for each $M \in \mathbb{Q}^+$ there is an $x_M \in (1/3,2/3)$ with $r(x_M;T_f^M) = r(T_f^M)$. Let g be constructed from f as in Proposition 10. We claim that $\rho(g) = \omega \cdot (\alpha+1)$. This will give us a function g with $r(g) = \alpha + 1$, so that the result will be true for $\alpha + 1$. Because of Proposition 10 we need only prove that $p(g) \leq \omega \cdot (\alpha+1)$. So T_g^M has a subtree as shown below with each of the nodes $\underset{\sim}{v} = \langle I_1, I_2^{(n)}, \ldots, I_n^{(n)} \rangle$ satisfying $r(\underset{\sim}{v}_n;T_f^M) \geqslant \omega \cdot \alpha$.



Fix $n \in \mathbb{N}$ and consider the subtree T_n through the node $\underset{\sim}{v}_n$ defined by

$$\underset{\sim}{u} \in T_n \Leftrightarrow \underset{\sim}{u} \in T_g^M, \text{ and } \underset{\sim}{u} \prec \underset{\sim}{v}_n \text{ or } \underset{\sim}{f}_n \prec \underset{\sim}{u}.$$

Since $r(\underset{\sim}{v}_n;T_n) \geq \omega\cdot\alpha$ it follows as in Lemma 4 that

$\exists x_n \in \bar{I}_n^{(n)} \subseteq I_{n-1}^{(n)}$ such that $r(T_n;x_n) \geq \omega\cdot\alpha$. But from the way g was constructed we know that $r(T_g^M;x) < \omega\cdot\alpha$ for each $x \neq 0$ (because in K_n we had $r(T^M_{g\restriction K_n}) < \omega\cdot\alpha$) . So x_n must be 0. Now as in Proposition 8 we see that $A(g{:}0) \leq M/(n-1)$. Since this is true for each n we get that $A(g;0) = 0$ which implies $g \notin$ ND, a contradiction. So we have $\rho(g) = \omega\cdot(\alpha+1)$.

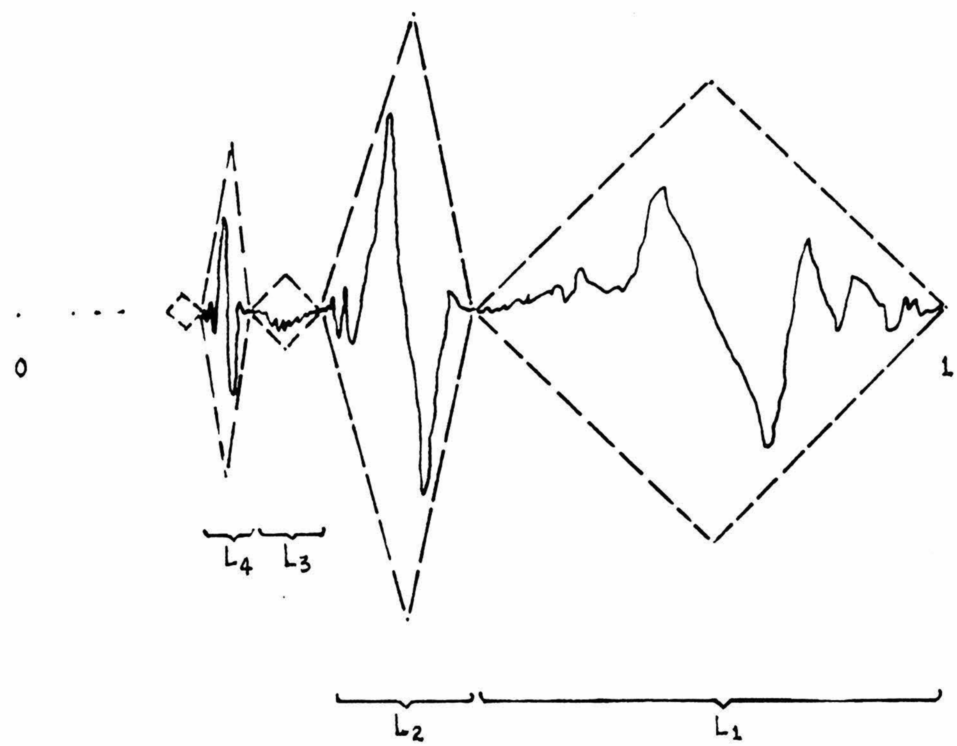
To complete the proof we need to show tht if the result is true for all $\alpha < \lambda$, λ a limit ordinal then it is also true for λ. In order to do this we must modify the construction given in Proposition 10 (because the g we ended up with there always satisfied $\rho(g) \geq \omega\cdot(\lambda+1)$) . Suppose the result is true for all $\alpha < \lambda$. Let $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence of ordinals with $\lim \alpha_n = \lambda$. Choose $f_n \in$ SF as before in Proposition 10 such that $r(f_n;0) = r(f_n;1) = 1$ for each $n \in \mathbb{N}$ also. Define g by $g(0) = 0$ and

$$g\restriction L_n = \begin{cases} \text{scaled copy of } 2nf \text{ onto } L_n & n \text{ even} \\ \text{scaled copy of } f_n \text{ onto } L_n & n \text{ odd} \end{cases}$$

where $L_n = [1/n+1,1/n]$ $n = 1,2,3,\ldots$. Then once again it is easy to see that $g \in$ SF. Also for each odd n, we can show exactly as in Proposition 10 that $r(T_g^{16n}) \geq \omega\cdot\alpha_n$. Thus

$$\rho(g) = \sup\{r(T_g^M) + 1 : M \in \mathbb{Q}^+\}$$

$\geqslant \sup\{r(T_g^{16n}) + 1 : n \in \mathbb{N},\ n \text{ odd}\} = \lambda.$

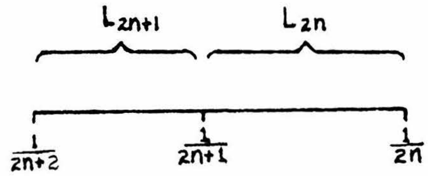


So we need to show that $\rho(g) \leq \lambda$, and the proof will be complete.

Fix $M \in \mathbb{Q}^+$ and consider the intervals L_n $(n \in \mathbb{N})$. Let $B = \sup\{|f_2(x)| : x \in [0,1]\} > 0$ and suppose B is attained at x_0 (i.e., $|f_2(x_0)| = B$). Let x_n be the image of x_0 in L_{2n} when $2(2n)f_2$ is scaled onto L_{2n}. Then $g(x_n) = \frac{4nB}{2n(2n+1)} = \frac{2B}{2n+1}$.

$$\text{So} \quad \left|\frac{g(x_{2n})-g(\frac{1}{2n+2})}{x_{2n}-\frac{1}{2n+2}}\right| \geqslant \frac{\frac{2B}{2n+1}}{\frac{1}{2n}-\frac{1}{2n+2}}$$

$$= \frac{n(n+1)B}{2n+1} > \frac{n}{2}B$$



Now let $n_0 = [M/B] + 1$. Then for all $n \geqslant n_0$, no subinterval J of L_{2n+1} is in the tree T_g^M (otherwise we would have $J \subseteq K$, $L \subseteq [0,1]$ but $|\Delta_g(K)-\Delta_g(L)| \leq nB/2 < M$ if we take K,L as $K = (\frac{1}{2n+2}, \frac{1}{2n+1})$, $L = (\frac{1}{2n+2}, x_{2n})$). So for each $x \in [0,1]$ we have

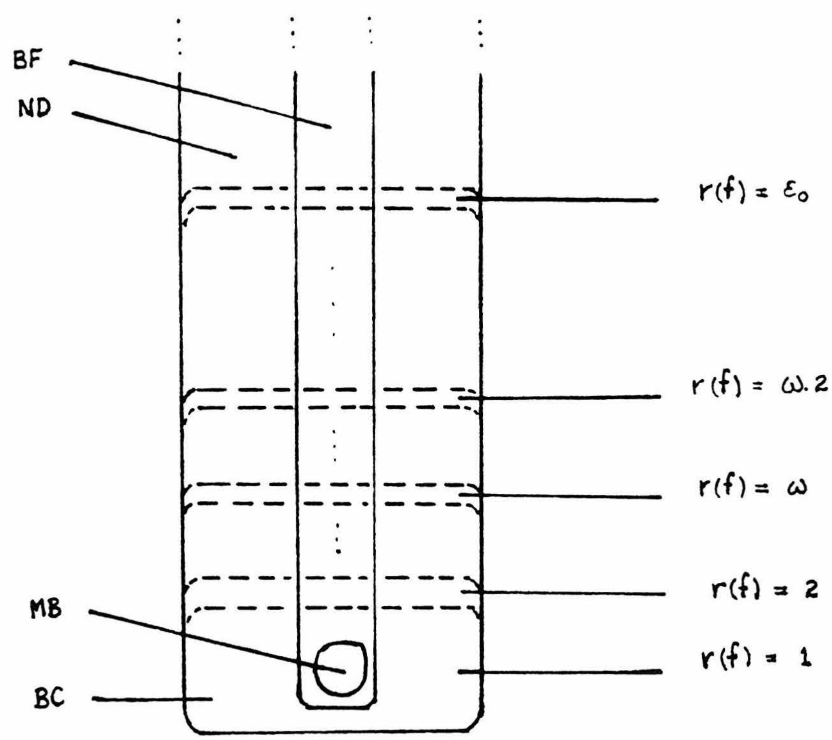
$$r(T_g^M;x \leq \sup\{r(T_{f_n}^M;x)\colon n \leq 2n_0\}$$
$$\leq \sup\{o(f_n)\colon n \leq 2n_0\} \leq \rho(f_{2n_0}).$$

So $I(T_g^M) < p(f_{2n_0}) + \omega < \omega\cdot\lambda$ (otherwise by Lemma 4 we could get an $x \in [0,1]$ with $r(T_g^M;x) \geqslant o(f_{2n_0})$). Thus $r(T_g^M) < \omega\cdot\lambda$ for each $M \in \mathbb{Q}^+$. So

$$\rho(g) = \sup\{r(T_g^M) + 1\colon M \in \mathbb{Q}^+\} \leqslant \omega.\lambda$$

since $\omega\cdot\lambda$ is a limit ordinal. This now shows that $\rho(g) = \omega\cdot\lambda$ and the proof is done. □

It follows immediately from the definition of r that $r(c \cdot f) = r(f)$ for each $c \neq 0$ and $f \in ND$. Proposition 11 thus shows that there are ω_1 many essentially distinct Besicovitch functions. Proposition 11 also shows that each level of the natural ω_1-hierarchy that r induces on ND and BF is nonempty. We draw a picture to exhibit this structure as shown below:



Before we conclude this section we make some remarks about the construction of a function with $r(f) = 2$, but such that $A(f;x_n) \to 0$ for some sequence $\{x_n\}$ in $[0,1]$ (that was promised in §1). Let K_n be as in the proof of Proposition 10. Choose a function $f \in SF$ with $A(f;x) = +\infty$ except at $0, 1/2$ and 1 where $A(f;x) \le 1$. Now define g by $g(0) = 0$ and

$$g \upharpoonright K_n = \begin{cases} \text{scaled copy of } 2^{n/2}f \text{ onto } K_n & n \text{ even} \\ \text{scaled copy of } 2^{-n/2}f \text{ onto } K_n & n \text{ odd} \end{cases}$$

Then it is easy to verify that $g \in BF$, but is not in SF. The reason is because $A(g;0) = +\infty$. Now from the construction of g we have $r(T_g^M) < \omega \cdot 2$ for each $M \in \mathbb{Q}^+$. Also $r(T_f^M;0) < \omega$, for each M (otherwise $A(g;0)$ would be finite). So $\forall M\ (r(T_g^M) \le \omega \cdot 2)$. So $\rho(g) \le \omega \cdot 2$ (otherwise there would be an M with $r(T_g^M) > \omega \cdot 2$). Since $r(f) = 2$, we see that $r(g)$ is exactly 2. But $A(g;x_n) \le 2^{-n/2}$ for all odd n, where x_n is the midpoint of K_n. So we are done.

§4: Cantor-Bendixson Analysis.

In this our last section of Chapter 1 we formulate an alternative description of the rank function r, by means of a Cantor-Bendixson Analysis.

Recall the definitions of R(I) and Q(I) from the beginning of §3. We will denote by $\bar{Q}(I)$ the set of all closed subintervals of I that have rational endpoints and length greater than 0. For each $f \in C$ and, $M \in \mathbb{Q}^+$ and $J \in \bar{Q}[0,1]$ we shall define a sequence $\langle P^{\alpha}_{M,f\restriction J}\rangle_{\alpha \in ORD}$ of closed sets, and a relation $Q(W,x,P^{\alpha}_{M,f\restriction J})$ which reflect the properties of f. M is to be thought of as being large and $Q(W,x,P^{\alpha}_{M,f\restriction J})$ is to be interpreted as the relation "W witnesses that $x \in P^{\alpha}_{M,f\restriction J}$ ". Here W will range over the closed subsets of J.

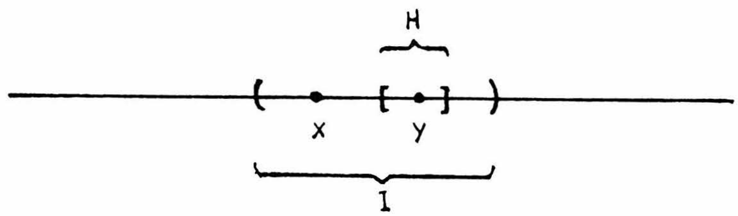
Definition. We define the set $P_{M,f\restriction J}$ by

$$x \in P_{M,f\restriction J} \iff \sup\{|\Delta_f(K)-\Delta_f(L)|: x \in K, L \in R(J)\} \leq M.$$

We define the sets $P^{\alpha}_{M,f\restriction J}$ and the relation $Q(W,x,P^{\alpha}_{M,f\restriction J})$ by induction as follows: Let

$$P^{1}_{M,f\restriction J} = P_{M,f\restriction J} \text{ and } Q(W,x,P^{1}_{M,f\restriction J}) \iff x \in W \cap P_{M,f\restriction J}$$

$$P^{\alpha+1}_{M,f\restriction J} = \{x \in J\text{: For all } I \in Q(J) \text{ with } x \in I\ \exists H \in \bar{Q}(I),$$
$$\exists y \in \text{int}_J(H),\ \exists V \subseteq P_{M,f\restriction J} \text{ s.t. } Q(V,y,P^{\alpha}_{M\cdot|I|,f\restriction H})\}$$



$$Q(W,x,P^{\alpha+1}_{M,f\restriction J}) \iff \text{For all } I \in Q(J) \text{ with } x \in I \ \exists H \in \bar{Q}(I), \exists y \in \text{int}_J(H),\ \exists V \subseteq W \cap P_{M,f\restriction J} \text{ such that } W(V,y,P^{\alpha}_{M\cdot|I|,f\restriction H}); \text{ and}$$

$$P^{\lambda}_{M,f\restriction J} = \bigcap_{\alpha<\lambda} P^{\alpha}_{M,f\restriction J}, \quad Q(W,x,P^{\lambda}_{M,f\restriction J}) \iff \forall\alpha < \lambda\ Q(W,x,P^{\alpha}_{M,f\restriction J})$$

for λ a limit ordinal.

By $\text{int}_J(H)$ we mean the interior of H with respect to the topology of J. Observe that our definition is made by use of simultaneous induction on α, M and J. It is easy to see that $P^{\alpha}_{M,f\restriction J}$ is always a closed set. Moreover

$$\alpha \le \beta \Rightarrow P^{\beta}_{M,f\restriction J} \subseteq P^{\alpha}_{M,f\restriction J} \quad \text{and}$$

$$M \le M' \Rightarrow P^{\alpha}_{M,f\restriction J} \subseteq P^{\alpha}_{M',f\restriction J} . \qquad (*)$$

When J is the interval [0,1] we shall refer to $P^{\alpha}_{M,f\restriction J}$

simply as $P^{\alpha}_{M,f}$.

We will prove in Proposition 16 that $f \in ND \Leftrightarrow$ for each $M \in \mathbb{Q}^{+}$ $\forall \alpha < \omega_1$ such that $P^{\alpha}_{M,f} = \emptyset$. So for each $N \in \mathbb{N}$ $\exists \alpha_N < \omega_1$ such that $P^{\alpha}_{N,f} = \emptyset$ for all $\alpha \geq \alpha_N$. Thus by using (*) we see that if $\beta = \sup\{\alpha_N : N \in \mathbb{N}\}$ ω_1, we have that $P^{\alpha}_{M,f} =$ for all $\alpha \geq \beta$ and all $M \in \mathbb{Q}^{+}$. This allows us to make the following defintion:

<u>Definition</u>. We define a new rank function r_1 on ND by $r_1(f) =$ "least α such that$P^{\alpha}_{M,f} = \emptyset$ for all $M \in \mathbb{Q}^{+}$."
Our main goal in this section will be to show that r and r_1 define the same rank function.

<u>Definition</u>. Let $f \in ND$, $M \in \mathbb{Q}^{+}$ and $J \in \bar{Q}[0,1]$. For each subset W of J we define the subtree $W(T^{M}_{f\restriction J})$ of $T^{M}_{f\restriction J}$ by

$$\langle I_1,\ldots,I_n\rangle \in W(T^{M}_{f\restriction J}) \Leftrightarrow \langle I_1,\ldots,I_n\rangle \in T^{M}_{f\restriction J} \text{ and } W \cap I_n \neq \emptyset$$

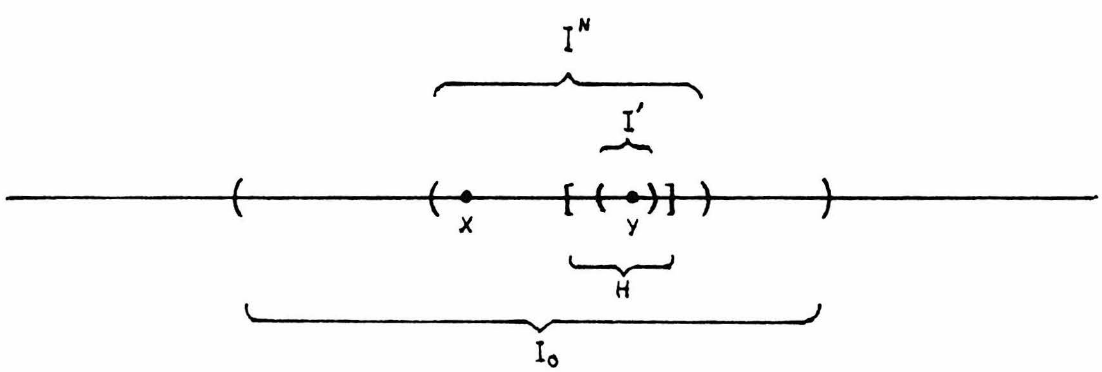
<u>Lemma 13</u>. If $x \in P^{\alpha}_{M,f\restriction J}$ and W satisfies $Q(W,x,P^{\alpha}_{M,f\restriction J})$ then $r(W(T^{M}_{f\restriction J});x) \geq \omega\cdot\alpha$.

<u>Proof</u>. We prove the result by simultaneous induction on α, M and J. For $\alpha = 1$ the result reduces to Proposition 7. Also the result follows trivially for limit ordinals

because of the way $P^{\lambda}_{M,f\restriction J}$ and $Q(W,x,P^{\lambda}_{M,f\restriction J})$ were defined.

So assume that the result is true for α, for all $M \epsilon \mathbb{Q}^{+}$ and all $j \epsilon \bar{Q}[0,1]$. We must show that it is true for $\alpha + 1$, for all $M \epsilon \mathbb{Q}^{+}$ and all $J \epsilon \bar{Q}[0,1]$. So fix M,J. Fix now $I_0 \epsilon Q(J)$ with $x \epsilon I_0$. It will suffice to show that $r(W(T^{M}_{f\restriction J})\restriction I_0) \geqslant \omega\cdot(\alpha+)$. We shall show that for each $N \epsilon \mathbb{N}$ the tree $W(T^{M}_{f\restriction J})\restriction I_0$ has a node of rank $\geqslant \omega\cdot\alpha + (N-1)$. The result follows immediately from this.

So we now fix $N \epsilon \mathbb{N}$. Choose $I^N \epsilon Q(J)$ such that $|I^N| \leq 1/N$ and $x \epsilon I^N \subseteq I_0$. From the assumptions on x we know that thee exists $H \epsilon \bar{Q}(I^N)$, $y \epsilon \mathrm{int}_J(H)$ and $V \subseteq W \cap P_{M,f\restriction J}$ such that $Q(V,y,P^{\alpha}_{M\cdot|I^N|,f\restriction H})$. Since $|I^N| \leq 1/N$ we also have $Q(V,y,P^{\alpha}_{M/N,f\restriction H})$ and by the induction hypothesis we thus get $r(V(T^{M/N}_{f1zkH});y) \geqslant \omega\cdot\alpha$. Now choose $I' \epsilon Q(H) \cap Q(J)$ with $y \epsilon I'$ and $|I'| < |J|/N$. Then Lemma 5 gives us $r([V(T^{M/N}_{f\restriction H})\restriction I']_N) \geqslant \omega\cdot\alpha$.



Choose $I_2, \ldots, I_N \in Q(H) \cap Q(J)$ such that $I' \subseteq I_N$ and $\bar{I}_i \subseteq I_{i-1}$, $|I_i| \leq |J|/i$ $(i = 2,\ldots,N)$. (Here I_1 is of course taken to be J.) Now define the tree T_N by $\langle I_1,\ldots,I_i\rangle \in T_N$ for each $i = 1, \ldots, N$ and

$$\langle N_H, J_2,\ldots,J_k\rangle \in [V(T^{M/N}_{f\restriction H})\restriction I']_N \Rightarrow$$
$$\langle I_1,\ldots,I_N,J_2,\ldots,J_k\rangle \in T_N.$$

It is then easy to see that

$$r(\langle J\rangle;T_N) \geqslant r(\langle H\rangle;[V(T^{M/N}_{f\restriction H})\restriction I']_N) + (N-1)$$
$$\geqslant \quad \omega\cdot\alpha + (N-1).$$

To complete the proof it will thus suffice to show that T_N is a subtree of $W(T^M_{f\restriction J})\restriction I_0$.

Let $\langle I_1,I_2,\ldots,I_n,\ldots,I_m\rangle$ be a node in T_N. We must show that $I_2 \subseteq I_0$, $W \cap I_m \neq \emptyset$, and for all $K,L \in R(J)$ with $I_m \subseteq K$, $L \subseteq I_n$ $|\Delta_f(K)-\Delta_f(L)| \leq M/n$. Now $I_2 \in Q(H)$ and $H \subseteq I_0$ so $I_2 \subseteq I_0$. Also

$$W \cap I_M \supseteq V \cap I_m \neq \emptyset$$

since $y \in V \cap I_m$ if $m \leq N$, and if $m > N$ then $V \cap I_M \neq \emptyset$ by the definition of $[V(T^{M/N}_{f\restriction H})\restriction I']_N$. Now if $n > N$ then $|\Delta_f(K)-\Delta_f(L)| \leq M/n$ from the definition of the tree $[V(T^{M/N}_{f\restriction H})\restriction I']_N$. Also if $2 \leq n \leq N$ then $|\Delta_f(K)-\Delta_f(L)| \leq M/N \leq M/n$ (since $|\Delta_f(K')-\Delta_f(L')| \leq M/N$ for all $K', L' \in R(H)$ with $I_m \subseteq K', L' \subseteq H$). Finally if $n = 1$ then $|\Delta_f(K) - \Delta_f(L)| \leq M$ because $P_{M,f\restriction J} \cap I_M \supseteq V \cap I_m \neq \emptyset$. So T_N is a subtree of $W(T^M_{f\restriction J})\restriction I_0$ and the proof is complete. □

Definition. We also define for each $k \in \mathbb{N}$ the subtree $[T]^k$ of T by

$\langle I_1, I_2, \ldots, I_n\rangle \in [T]^k \iff$ there exists $J_2, \ldots, J_{nk} \in Q(J)$ such that $\langle I_1, J_2, \ldots, J_{nk}\rangle \in (T)^k$ and $J_{i\cdot k} = I_{i+1}$ for all $i = 1, \ldots, n-1$.

We can show exactly as in Lemma 5 that the f llowing result is true.

Sub-lemma 14. Suppose T is a transitive sub-tree of $T^M_{f\restriction J}$, $I \in Q(J)$ and $r(T\restriction I) \geq \omega\cdot\alpha$. Then for each $k \in \mathbb{N}$ the subtree $[T\restriction I]^k$ is transitive and $r([T\restriction I]^k) \geq \omega\cdot\alpha$. □

Lemma 15. Let $f \in ND$ and suppose T is a transitive subtree of $T^M_{f\restriction J}$ with $r(T;x) \geq \omega\cdot\alpha$. Let also $W = \{z \in J: r(T;z) \geq \omega\}$. Then $Q(W,x,P_{M,f\restriction J})$. In particular $x \in P^{\alpha}_{M,f\restriction J}$.

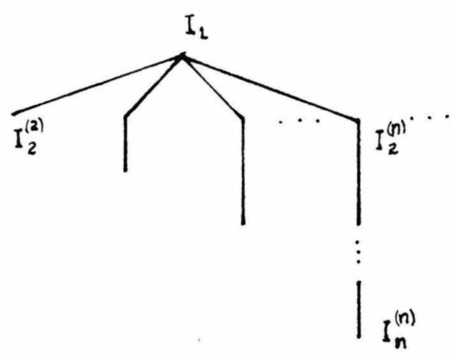
Proof. We prove the result by simultaneous induction on α, M and J. For $\alpha = 1$ the result reduces to Proposition 7, and for limit ordinals it follows routinely as in Lemma 13.

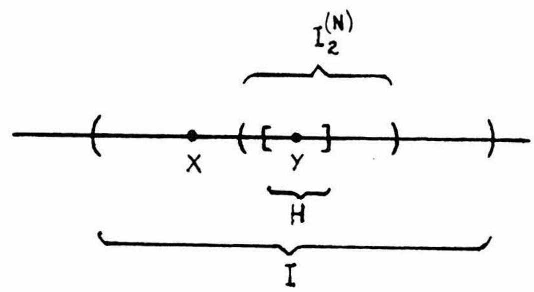
So assume that the result is true for α, for all $M \in \mathbb{Q}^+$ and all $J \in \bar{Q}[0,1]$. We have to show that it is true for $\alpha + 1$, for all $M \in \mathbb{Q}^+$ and all $J \in \bar{Q}[0,1]$. So fix J,M. It will suffice to show that

$$r(T;x) \geqslant \omega\cdot(\alpha+1) \Rightarrow Q(W,x,P^{\alpha+1}_{M,f\restriction J}).$$

Now fix $I \in Q(J)$ with $x \in I$. We must find $H \in \bar{Q}(I)$, $y \in \mathrm{int}_J(H)$ and $V \subseteq W \cap P_{M,f\restriction J}$ such that $Q(V,y,P^{\alpha}_{M\cdot|I|,f\restriction H})$.

Since $r(T;x) \geqslant \omega\cdot(\alpha+1)$ we have that $r(T\restriction I) \geqslant \omega\cdot(\alpha+1)$. So $T\restriction I$ has a subtree as shown below with each of the nodes $\langle I_1,I_2^{(n)},\ldots,I_n^{(n)}\rangle$ having rank $\geqslant\omega\cdot\alpha$ in $T\restriction I$.





Choose $N_1 \in \mathbb{N}$ so that $1/N_1 \leq |I|$. Choose also $N \geqslant N_1 + 1$ so that $1/N \leq I^{(N)}_{N_1}$ and put $H = \bar{I}^{(N)}_{N_1}$. Then $H \in \bar{Q}(I)$ since $\bar{I}^{(N)}_{N_1} \subseteq I_2^{(N)} \subseteq I$ by definition of the tree $T\restriction I$. Since $\langle I_1,I_2^{(N)},\ldots,I_N^{(N)}\rangle$ is a node of rank $\geqslant\omega\cdot\alpha$ in the tree $T\restriction I$, we have $r(T\restriction I_N^{(N)}) \geqslant \omega\cdot\alpha$. Now let T_N be the tree defined by

$$\langle H,J_2,\dots,J_n\rangle \in T_{N_1} \iff \langle J,J_2,\dots,J_n\rangle \in [T\restriction I_N^{(N)}]^N$$

T_N is essentially the same tree as $[T\restriction I_N^{(N)}]^N$ so by Sub-lemma 14, T_N is transitive. Also it is easy to see by a direct verification that T_N is a subtree of $T^{M/N_1}_{f\restriction H}$. Moreover $r(T_N) \geq \omega\cdot\alpha$.

So by lemma 4 we know there is a $y \in \bar{I}^{(N)}_{N_1} \subseteq I^{(N)}_N = \mathrm{int}_J(H)$ such that $r(T_N;y) \geq \omega\cdot\alpha$. Let $V = \{z \in H\colon r(T_N;z) \geq \omega\}$. Then $Q(V,y,P^{\alpha}_{M/H_1,f\restriction H})$ by the induction hypothesis. Since $1/N_1 \leq |I|$ we get that $Q(V,y,P^{\alpha}_{M\cdot|I|,f\restriction H})$. Moreover since T_N is essentially the tree $[T\restriction I_N^{(N)}]^N \subseteq T$ we have

$$r(T_N;z) \geq \omega \Rightarrow r(T;z) \geq \omega.$$

Thus $V \subseteq W$. Finally from the definition of W we know that $W \subseteq P_{M,f\restriction J}$ by Proposition 7. So $V \subseteq W \cap P_{M,f\restriction J}$ and we are done. □

<u>Proposition</u> <u>16</u>. Let $f \in C$. Then $f \in ND \iff$ for each $M \in \mathbb{Q}^+$, there exist $\alpha < \omega_1$ such that $P^{\alpha}_{M,f} = \emptyset$.

<u>Proof</u>: "⇒": Suppose $f \in ND$. Then for a fixed $M \in \mathbb{Q}^+$ we know that $r(T^M_f) < \omega\cdot r(f)$. So from Lemma 13, $P^{r(f)}_{M,f} = \emptyset$ (otherwise we would get $r(T^M_f;x) \geq \omega\cdot r(f)$ for some x, and this would give $r(T^M_f) \geq \omega\cdot r(f)$). So we are done.

"⇐": Suppose $f \notin ND$. Then there is an $x \in [0,1]$ such that f is differentiable at x. A direct verification shows that $Q(\{x\}, x, P^{\alpha}_{M,f})$ for all aga ϵ ORD. So $P^{\alpha}_{M,f}$ is never empty and we are done. □

<u>Proposition</u> <u>17</u>. For each $f \in ND$ $r_1(f) = r(f)$.

<u>Proof</u>. From lemma 13 we immediately have $r_1(f) \leq r(f)$. So we only need to show that $r(f) \leq r_1(f)$. Let $\alpha = r_1(f)$ and fix M. Then $P^{\alpha}_{M,f} = \emptyset$ and so by lemma 15 $r(T^m_f;x) < \omega\cdot\alpha$ for all $x \in [0,1]$. But this implies that $r(T^M_f) < \omega\cdot\alpha$. Since this is true for each M we have that $\rho(f) \leq \omega\cdot\alpha$. Thus $r(f) \leq \alpha_1 = r_1(f)$. □

<u>Final</u> <u>Remarks</u>. In this section we gave an alternative description of the rank function r, by using a Cantor-Bendixson Analysis. This Cantor-Bendixson Analysis was however very complicated. The fact that the trees T^M_f are transitive perhaps rules out the possibility of any simple Cantor-Bendixson Analysis (e.g., one which uses a derivative like operation) but this is not clear. So we pose the following problem.

<u>Problem</u>. Is there a simple Cantor-Bendixson Analysis which gives rise to the same rank function r?

Chapter 2

Introduction. In 1906 P. Fatou [14] asked whether a trigonometric series with coefficients tending to zero must converge on a set of positive measure. N. Lusin [33] answered this question in the negative by constructing such a series which was divergent a.e. (S. B. Stechkin [48] later showed this series was in fact everywhere divergent). Not much later H. Steinhaus [49] constructed such a series which was everywhere divergent. In the years that follow more examples of everywhere divergent trigonometric series with coefficients tending to zero were given, among them being the one by G. H. Hardy and J. E. Littlewood [18]. Steinhaus himself also produced another much simpler example and proved by elementary means that it was everywhere divergent (see [51]).

However none of these examples was a Fourier Series. So it was natural to ask whether there could be an everywhere divergent Fourier series (or at least a Fourier Series that is divergent a.e.) . Steinhaus [50] had shown there was an orthonormal series $\{\varphi_n\}$ and an integrable function f such that the orthonormal expansion of f with respect to $\{\varphi_n\}$ was everywhere divergent but this orthonormal sequence was of course artificial and so shed no light on the problem. Then Kolmogorov [28] showed in 1923 that there was an integrable function f whose Fourier series was divergent a.e. Using the same idea in 1926 Kolmogorov also showed (see [29]) that there was a function, f whose Fourier

Series was everywhere divergent. (In fact it turned out that the Fourier series of this f was unboundedly divergent everywhere.) Since then this has remained essentially the only way of constructing everywhere divergent Fourier Series.

It is very natural to ask whether the conjugate series of the Fourier series of Kolmogorov function is also a Fourier series. This turns out to be false but Y. M. Chen [10] showed that there was a function f such that the conjugate series of S(f) is a Fourier series and S(f) diverges unboundedly everywhere. It is also natural to ask whether there is a function whose Fourier series is such that lim $\sup|(S_n(f;x)| < \infty$ for each x. This was shown to be false because of Carlson's Theorem (see [30]). However J. Marcinkiewicz (see [35]) showed that there is an f such that lim $\sup|S_n(f;x)| < \infty$ a.e. and S(f) diverges a.e.

In another direction we can ask how fast can the coefficients of an everywhere divergent trigonometric series tend to zero. Stechkin [48] showed that there are everywhere divergent trigonometric series whose coefficients tend to zero as fast as permissible. More precisely if $r_n > 0$ for $n \geqslant 1$ and

$\sum_{n=1}^{\infty} (\min\{r_k : 1 \le k \le n\})^2 = +\infty$, then there exists $\{\varphi_n\} \subseteq \mathbb{R}$ such that

the series $\sum_{n=1}^{\infty} r_n \cos(nx-\varphi_n+\psi)$ diverges for all x and ψ.

Finally we can ask questions about the sequence of coefficients of an everywhere divergent trigonometric series with coefficients that tend to zero. How thin for example can this sequence be? (i.e., how large can the gaps of zeros between successive nonzero terms be?) We know that if the sequence is lacunary then trigonometric series will converge on a dense set (see [04], p. 186). A. S. Belov [06] however constructed examples which have large gaps of zeros between successive non-zero terms and which in a sense fall just short of being lacunary.

In this chapter we study the structure of the set of all everywhere divergent Fourier series, and the set of all everywhere divergent trigonometric series with coefficients which tend to zero. Let $\mathbb{T}$ be the unit circle and $C(\mathbb{T})^{\omega}$ be the Polish space of all sequences of continuous functions on $\mathbb{T}$. We will view $\mathbb{T}$ as the closed interval $[0,2\pi]$ with the points 0 and 2π identified. An element $\langle f_m \rangle$ of $C(\mathbb{T})^{\omega}$ will be denoted by $\underset{\sim}{f}$ when convenient. Let

$$DS = \{\underset{\sim}{f} \in C(\mathbb{T})^{\omega}\colon \underset{\sim}{f} \text{ is everywhere divergent}\}$$

$$DZ = \{\underset{\sim}{f} \in DS\colon \|f_{m+1}-f_m\| \to 0 \text{ as } m \to \infty\}$$

$$DT = \{\underset{\sim}{f} \in DZ\colon f_m \text{ is the m-th partial sum of a trigonometric series}\}, \text{ and}$$

$$DF = \{\underset{\sim}{f} \in DZ\colon f_m \text{ is the m-th partial sum of the Fourier series of some } f\}$$

Then DF and DT can be naturally identified with the set of everywhere divergent Fourier Series and the set of everywhere divergent trigonometric series with coefficients tending to zero.

The sets DF and DT may be viewed as subsets of the space (c_0) of all $\mathbb{Z}$-sequences which tend to zero. DF may also be viewed as a subset of the Polish space $L_1(\mathbb{T})$ of all Lebesgue integrable functions on $\mathbb{T}$. But no matter how we view DF and DT it is easy to see that they are coanalytic subsets. A. S. Kechris [25] showed that DF is a complete coanalytic subset and, by the general argument he presented there, the same result can be deduced for DT.

In this chapter we investigate a natural rank function r on DS. The analysis is very similar to that of Chapter 1 and although the definitions are sometimes slightly more complicated, the proofs are much simpler. The rank function r is a coanalytic norm and provides a natural measure of the complexity of the sequences in DS. $r(\underset{\sim}{f})$ measures in some sense the uniformity of the divergence of $\underset{\sim}{f}$. It turns out that sequences which are uniformly divergent have ranks 1 or 2 (c.f. [01] where it is shown that the functions of rank 1 are those with uniformly convergent Fourier Series). Also sequences which are unboundedly divergent have rank 1 (c.f. Proposition 1.7).

The rank function applies naturally to the set of everywhere divergent Fourier series and the set of everywhere divergent trigonometric series with coefficients tending to zero. We consider some natural examples of such series to confirm our intuitive idea that such examples should have small ranks (1 or 2).

We also show that for each $1 \leq \alpha < \omega_1$, there is a Fourier series with rank α. This will provide Rank Arguments of the non-Borelness of DF and DT.

§1. <u>Tree Description: The Rank Functions</u> ρ <u>and</u> <u>r</u>. As in Chapter 1 we study the set DS by associating with each $\underset{\sim}{f} \in$ DS a countable collection of well-founded trees. We first check that DS, DZ, DT and DF are coanalytic subsets.

<u>Proposition</u> <u>1</u>. DS, DZ, DT and DF are all coanalytic subsets.

<u>Proof</u>. Observe that $\underset{\sim}{f} \in C(\mathbb{T})^{\omega}$ - DS iff $\exists x\ \forall m \exists n$ such that

$$\forall m_1, m_2 \geqslant n \left(|f_{m_1}(x) - f_{m_2}(x)| \leq \tfrac{1}{m} \right). \qquad (*)$$

Let $E(m,n) = \{(f,x) \in C(\mathbb{T})^{\omega} \times \mathbb{T}: (*) \text{ holds}\}$. Then $E(m,n)$ is closed and so $\bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} E(m,n)$ is Borel. Hence $C(\mathbb{T})^{\omega}$ - DS is the projection of a Borel set onto $C(\mathbb{T})^{\omega}$ and so is analytic. Thus DS is a coanalytic subset of $C(\mathbb{T})^{\omega}$.

To see that DZ, DT and DF are also coanalytic observe that these sets are just subsets of DS which satisfy added Borel conditions. Thus DZ, DT and DF are intersections of DS with Borel sets and hence are coanalytic. If we view DT and DF as subsets of (c_0) then we also have that DT and DF are coanalytic subsets of (c_0). This is because the map $\{c_n\}_{n\in\mathbb{Z}} \mapsto \langle \sum_{k=-n}^{n} c_k e^{ikx} \rangle_{n\in\mathbb{N}}$ is a Borel measurable injection. So DT and DF are pre-images of coanalytic subsets under a Borel measurable map. Similarly DF viewed as a subset of $L_1(\mathbb{T})$ is also coanalytic because the map

$$f \mapsto \langle \sum_{k=-n}^{n} \hat{f}(k) e^{ikx} \rangle_{n\in\mathbb{N}}$$

is a Borel measurable injection. □

Let $Q(\mathbb{T})$ be the collection of all closed intervals which have length greater than zero and endpoints in $\pi\cdot\mathbb{Q}$. With each $\underset{\sim}{f} \in C(\mathbb{T})^\omega$ and $M \in \mathbb{Q}^+$ we shall associate a tree $T^M_{\underset{\sim}{f}}$ on $Q(\mathbb{T}) \times \mathbb{N}$.

<u>Definition</u>. We define the tree $T^M_{\underset{\sim}{f}}$ as follows

$$\langle (I_1,k_1), \ldots, (I_n,k_n)\rangle \in T^M_{\underset{\sim}{f}} \iff$$

(i) $I_1 = \mathbb{T}$, $k_1 = 1$ and for all $i = 2, \ldots, n$

the following conditions hold

(ii) $I_i \in Q(\mathbb{T})$, $|I_i| \le 2\pi/i$, $I_i \subseteq I_{i-1}$ and $k_i > k_{i-1}$,

(iii) for all $x \in I_n$ and all $m_1, m_2 \in [k_{i-1}, k_n]$ we have

$$|f_{m_1}(x) - f_{m_2}(x)| \leq M/(i-1).$$

It follows immediately from the definition that $T_{\underset{\sim}{f}}^M \subseteq T_{\underset{\sim}{f}}^{M'}$ if $M \leq M'$. Our next result tells us exactly when T_f^M is well founded for all $M \in \mathbb{Q}^+$.

<u>Proposition</u> <u>2</u>. $\underset{\sim}{f} \in DS \Leftrightarrow \forall M \in \mathbb{Q}^+ \; (T_{\underset{\sim}{f}}^M$ is well-founded).

<u>Proof</u>. "$\Rightarrow$": Suppose for some $M \in \mathbb{Q}^+$, $T_{\underset{\sim}{f}}^M$ is not well-founded. Then there is an infinite branch $\langle (I_n, k_n) \rangle_{n \in \mathbb{N}}$ in $T_{\underset{\sim}{f}}^M$. Let $\{x_0\} = \bigcap_{n \in \mathbb{N}} I_n$. We shall show that $\underset{\sim}{f}$ converges at x_0. Let $\varepsilon > 0$ be given. Choose i such that $M/i < \varepsilon$. Now let $m_1, m_2 \geq k_i$ be given. Since $\{k_n\}$ is strictly increasing there is a j such that $k_j \geq m_1, m_2$. So from the definition of the tree $T_{\underset{\sim}{f}}^M$ we have

$$\forall x \in I_J (|f_{m_1}(x) - f_{m_2}(x)| \leq M/i)$$

In particular since $x_0 \in I_J$ we get $|f_{m_1}(x_0) - f_{m_2}(x_0)| \leq M/i <$

ε, and so $\underset{\sim}{f}$ converges at x_0.

"$\Leftarrow$": Suppose $\underset{\sim}{f}$ converges at some x_0. Choose $M \in \mathbb{Q}^+$ such that $M \geqslant 2 \sup\{|f_m(x_0)|: m \in \mathbb{N}\} + 1$. Choose also a strictly increasing sequence $\langle k_n \rangle$ such that $k_1 = 1$, and

$$|f_{m_1}(x_0)-f_{m_2}(x_0)| < 1/3n \quad \text{for all } m_1,m_2 \geqslant k_n.$$

Let also $\langle I_n \rangle$ be a nested sequence of closed intervals in $\mathbb{T}$ with $x_0 \in I_n$ for all n, such that conditions (i) and (ii) of the definition of $T^M_{\underset{\sim}{f}}$ holds and

$$\forall x \in I_n \ \forall m \geqslant k_n(|f_m(x)-f_m(x)| \leq 1/3n).$$

(This last condition can be obtained because of the continuity of the functions f_m at x_0.) We shall show that $\langle (I_n,k_n) \rangle_{n \in \mathbb{N}}$ is an infinite branch in $T^M_{\underset{\sim}{f}}$. It will suffice to verify condition (iii) of the definition of the tree $T^M_{\underset{\sim}{f}}$. For all $x \in I_n$ and all $m_1,m_2 \in [1,k_n]$ we have

$$\begin{aligned} |f_{m_1}(x)-f_{m_2}(x) &\leq | f_{m_1}(x)-f_{m_1}(x_0)| + |f_{m_1}(x_0)-f_{m_2}(x_0)| \\ &\quad + |f_{m_2}(x_0)-f_{m_2}(x)| \\ &< \frac{1}{3n} + |f_{m_1}(x_0)| + |f_{m_2}(x_0)| + \frac{1}{3n} \\ &< 1/3n + (M-1) + 1/3n < M/1 \end{aligned}$$

Also for all $x \in I_n$ and all $m_1, m_2 \in [k_i, k_n]$, $i \geq 2$

$$|f_{m_1}(m)-f_{m_2}(x)| \leq |f_{m_1}(x)-f_{m_1}(x_0)| + |f_{m_1}(x_0)-f_{m_2}(x_0)| + |f_{m_2}(x_0)-f_{m_2}(x)|$$

$$< \frac{1}{3n} + \frac{1}{3i} + \frac{1}{3n} \leq \frac{1}{i} < \frac{M}{i}$$

So condition (iii) is satisfied and we are done. □

Let $\underset{\sim}{f} \in DS$ and $M \in \mathbb{Q}^+$. Since $Q(\mathbb{T}) \times \mathbb{N}$ is countable the rank of $T^M_{\underset{\sim}{f}}$ is countable. Also since $T^M_{\underset{\sim}{f}} \subseteq T^{M'}_{\underset{\sim}{f}}$ for $M \geq M'$ we have

$$\sup\{r(T^M_{\underset{\sim}{f}}) + 1 : M \in \mathbb{Q}^+\} = \sup\{r(T^N_{\underset{\sim}{f}}) + 1 : N \in \mathbb{N}\} < \omega_1 .$$

<u>Definition</u>. We define the rank function $\rho: ND \to \omega_1$ by

$$\rho(\underset{\sim}{f}) = \sup\{r(T^M_{\underset{\sim}{f}}) + 1 : M \in \mathbb{Q}^+\}.$$

<u>Proposition 3</u>. $\rho: ND \to \omega_1$ is a coanalytic norm.

<u>Proof</u>. The proof is identical to that of Proposition 1.3. □

Our next aim is to show that $\rho(\underset{\sim}{f})$ is always a limit ordinal.

We proceed exactly as in Chapter 1.

Definition. Let $I \in Q(\mathbb{T})$ and T be a tree on $Q(\mathbb{T}) \times \mathbb{N}$. We define the subtree $T \upharpoonright I$ of T by

$$\langle (I_1,k_1),(I_2,k_2),\dots,(I_n,k_n)\rangle \in \mathbb{T}\upharpoonright I \iff$$
$$\langle (I_1,k_1),(I_2,k_2),\dots,(I_n,k_n)\rangle \in T \text{ and } I_2 \subseteq I$$

Let $\underset{\sim}{f} \in DS$ and $M \in \mathbb{Q}^+$. For each $x \in \mathbb{T}$ we define

$$r(T^M_{\underset{\sim}{f}};x) = \min\{r(T^M_{\underset{\sim}{f}}\upharpoonright I): x \in \mathrm{int}(I),\ I \in Q(\mathbb{T})\}$$

Lemma 4. If $r(T^M_{\underset{\sim}{f}} \geq \omega\cdot\alpha$ then there is an $x \in \mathbb{T}$ such that $r(T^M_{\underset{\sim}{f}};x) \geq \omega\cdot\alpha$.

Proof. Same as that of Lemma 1.4. □

Definition. Let T be a tree on $Q(\mathbb{T}) \times \mathbb{N}$ and $p \in \mathbb{N}$. We define the subtree $[T]_p$ of T by

$\langle (I_1,k_1),\dots,(I_n,k_n)\rangle \in [T]_p \iff \langle (I_1,k_1),\dots,(I_n,k_n)\rangle \in T$ and there exist $(J_2,\ell_2),\dots,(J_p,\ell_p) \in Q(\mathbb{T}) \times \mathbb{N}$ such that $\langle (I_1,k_1),(J_2,\ell_2),\dots,(J_p,\ell_p),(I_2,k_2),\dots,(I_n,k_n)\rangle \in T$.

<u>Lemma</u> <u>5</u>. Suppose $r(T^M_{\underset{\sim}{f}}\upharpoonright I) \geqslant \omega\cdot\alpha$. Then for each $p \in \mathbb{N}$ $r([T^M_{\underset{\sim}{f}}\upharpoonright I]_p) \geqslant \omega\cdot\alpha$.

<u>Proof</u>. Same as that of Lemma 1.5. □

<u>Proposition</u> <u>6</u>. Let $\underset{\sim}{f} \in DS$. Then $\rho(\underset{\sim}{f})$ is a limit ordinal.

<u>Proof</u>. It will suffice to show that $\rho(\underset{\sim}{f}) \geqslant \omega\cdot\alpha + 1 \Rightarrow \rho(f) \geqslant \omega\cdot(\alpha+1)$. Suppose $\rho(\underset{\sim}{f}) \geqslant \omega\cdot\alpha + 1$. Then for some $M \in \mathbb{Q}^+$, $r(T^M_{\underset{\sim}{f}}) \geqslant \omega\cdot\alpha$. So by lemma 4 there is an $x_0 \in \mathbb{T}$ such that $r(T^M_{\underset{\sim}{f}};x_0) \geqslant \omega\cdot\alpha$.

Fix $N \in \mathbb{N}$. For $i = 1, \ldots, N$ choose $I_i \in Q(\mathbb{T})$ such that $I_i \subseteq I_{i-1}$, $|I_i| \leq 2\pi/i$, $x_0 \in I_N$ and $k_i = i$. (As usual $I_1 = \mathbb{T}$ and $k_1 = 1$.) Now define the tree T_N by

$$<(I_1,k_1),\ldots,(I_i,k_i)> \in T_N \qquad i = 1, \ldots, N, \text{ and}$$

$$<(I_1,k_1),(J_2,\ell_2),\ldots,(J_n,\ell_n)> \in [T^M_{\underset{\sim}{f}}\upharpoonright I_N]_N \Rightarrow$$

$$<(I_1,k_1),(I_2,k_2),\ldots,(I_N,k_N),(J_2,\ell_2),\ldots,(J_n,\ell_n)> \in T_N.$$

Then it is easy to see that T_N is a subtree of $T_{\underset{\sim}{f}}^{M\cdot N}$. But

$$r(\langle(I_1,k_1)\rangle;T_N) \geqslant r(\langle(I_1,k_1)\rangle;[T_{\underset{\sim}{f}}^{M} \restriction I_N]_N) + N - 2$$

$$\geqslant \omega\cdot\alpha + (N-2) \quad \text{by Lemma 5.}$$

Since this is true for each $N \in \mathbb{N}$ we have

$$\rho(\underset{\sim}{f}) = \sup\{r(T_{\underset{\sim}{f}}^{M\cdot N})+1 : N \in \mathbb{N}\}$$

$$\geqslant \sup\{r(T_N)+1 : N \in \mathbb{N}\}$$

$$\geqslant \sup\{\omega\cdot\alpha+(N-1) : N \in \mathbb{N}\} = \omega\cdot(\alpha+1).$$

This completes the proof. □

Definition. For $\underset{\sim}{f} \in DS$ we define $r(\underset{\sim}{f})$ to be the unique ordinal α such that $\rho(\underset{\sim}{f}) = \omega\cdot\alpha$.

It follows immediately that r is a coanalytic norm on DS. Moreover since DZ, DT and DF are intersections of DS with Borel sets, r is also a coanalytic norm on DZ, DT and DF. When we view the elements of DT as formal trigonometric series we shall also use r(S) to denote the rank of the series S. When we view the elements of DF as functions in $L_1(\mathbb{T})$ we shall use r(f) to denote the rank of the function f.

Our next goal is to characterize the elements of DS which have small rank. For this we introduce a notion of uniform divergence.

Definition. Let $\underset{\sim}{f} \in C(\mathbb{T})^{\omega}$. We say that $\underset{\sim}{f}$ is *strongly uniformly divergent* if

$$\exists \varepsilon_0 > 0\ \exists p_0 \forall n \forall x \exists m_1, m_2 \in [n, n+p_0](|f_{m_1}(x) - f_{m_2}(x)| \geq \varepsilon_0).$$

We say that $\underset{\sim}{f}$ is *unboundedly uniformly divergent* if

$$\forall B > 0 \forall n \exists p_0 \forall x \exists m_1, m_2 \in [n, n+p_0](|f_{m_1}(x) - f_{m_2}(x)| \geq B).$$

And we say that $\underset{\sim}{f}$ is *uniformly divergent* if

$$\exists \varepsilon_0 > 0\ \forall n \exists p_0 \forall x \exists m_1, m_2 \in [n, n+p_0](|f_{m_1}(x) - f_{m_2}(x)| \geq \varepsilon_0)$$

It follows immediately that

strong uniform divergence $\Rightarrow$ uniform divergence

unbounded uniform divergence $\Rightarrow$ uniform divergence.

To make the definitions clear we give the following examples.

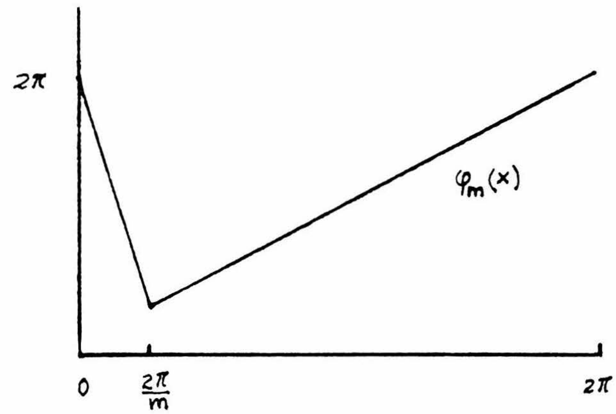
Example 1. $f_m(x) = (-1)^m \quad \forall m \in \mathbb{N}\ \forall x \in \mathbb{T}$.

Example 2. $f_m(x) = \log m \quad \forall m \in \mathbb{N} \quad \forall z \in \mathbb{T}$.

Example 3. $f_m(x) = \begin{cases} 1 & \text{if } m = 2^k \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$

Example 4. $$f_m(x) = \begin{cases} 0 & \forall x \in \mathbb{T}, \quad m \text{ odd} \\ \varphi_m(x) & \quad m \text{ even} \end{cases}$$

where $\varphi_m(x) = \begin{cases} x & x \in [2\pi/m, 2\pi] \\ 2\pi & x = 0 \\ \text{linear on } [0, 2\pi/m] \end{cases}$



Example 1 is strongly uniformly divergent but not unboundedly uniformly divergent. Example 2 is unbounded uniformly divergent but not strongly uniformly divergent. Example 3 is uniformly divergent but neither unboundedly uniformly divergent nor strongly uniformly divergent. Example 4 is everywhere divergent but not uniformly divergent.

<u>Proposition 7</u>. If $\underset{\sim}{f}$ is strongly uniformly divergent then $r(\underset{\sim}{f}) = 1$.

<u>Proof</u>. It will suffice to show that $r(T^M_{\underset{\sim}{f}})$ is finite for each $M \in \mathbb{Q}^+$. Since $\underset{\sim}{f}$ is strongly uniformly divergent we have

$$\exists \varepsilon_0 > 0\ \exists p_0 \forall n \forall x \exists_{m_1, m_2} \in [n, n+p_0]$$
$$\text{such that } |f_{m_1}(x) - f_{m_2}(x)| \geq \varepsilon_0 \qquad (*)$$

Let $i_0 = [M/\varepsilon_0] + 1$. We claim that any node in $T^M_{\underset{\sim}{f}}$ must have length at most $i_0 + p_0$. From this it will follow that $r(T^M_{\underset{\sim}{f}})$ is finite.

Suppose $\langle (I_1,k_1),\ldots,(I_q,k_q)\rangle$ is a node in $T^M_{\underset{\sim}{f}}$ of length greater than $i_0 + p_0$. Then $k_q \geqslant k_{i_0+p_0} \geqslant k_{i_0} + p_0$.

But from the definition of $T^M_{\underset{\sim}{f}}$ we have

$$\forall x \in I_n \ \forall m_1,m_2 \in [k_{i_0},k_q] \, |f_{m_1}(x)-f_{m_2}(x)| \le \frac{M}{i_0} < \varepsilon$$

which contradicts (*). Hence the results follow. □

Remark. The concept of strong uniform divergence is too restrictive to be of much use. In fact if for some $x_0 \in \mathbb{T}$ we have $f_{m+1}(x_0) - f_m(x_0) \to 0$ as $m \to \infty$, then it is easy to see that $\langle f_m \rangle$ cannot be strongly uniformly divergent. In particular the sequences in DZ cannot ever be strongly uniformly divergent. (Moreover example 3 shows that even the requirement $f_{m+1}(x_0) - f_m(x_0) \to 0$ for some x_0 , is not necessary in order not to have strong uniform divergence.) So we shall no longer concern ourselves with strong uniform divergence.

Definition. Let $\underset{\sim}{f} \in C(\mathbb{T})^\omega$ and $x_0 \in \mathbb{T}$. We define the *amplitude of divergence* of f at x_0 by

$$A(\underset{\sim}{f};x_0) = \limsup_{m_1,m_2\to\infty} |f_{m_1}(x_0)-f_{m_2}(x_0)|.$$

It is then clear that $\underset{\sim}{f}$ diverges at x_0 iff $A(\underset{\sim}{f};x_0) > 0$. We say that $\underset{\sim}{f}$ *diverges unboundedly* if for each $x \in \mathbb{T}$, $A(\underset{\sim}{f};x) = +\infty$.

Proposition 8. Let $\underset{\sim}{f} \in C(\mathbb{T})^{\omega}$. Then

(i) $\underset{\sim}{f}$ is unboundedly uniformly divergent $\Leftrightarrow$ $\underset{\sim}{f}$ is boundedly divergent.

(ii) $\underset{\sim}{f}$ is uniformly divergent $\Leftrightarrow$ there is a $c > 0$ such that $A(\underset{\sim}{f};x) \geqslant c$ for all $x \in \mathbb{T}$.

Proof. (i) The "$\Rightarrow$" direction is trivial so we shall only prove the "$\Leftarrow$" direction. So suppose $\underset{\sim}{f}$ is not unboundedly uniformly divergent. Then

$$\exists B_0 > 0 \exists n_0 \forall p \exists x \forall m_1,m_2 \in [n_0,n_0+p](|f_{m_1}(x)-f_{m_2}(x)|<B_0)$$

For each $p \in \mathbb{N}$ choose x_p such that

$$\forall m_1,m_2 \in [n_0,n_0+p] \text{ we have } |f_{m_1}(x_p)-f_{m_2}(x_p)| < B_0.$$

Now let x_0 be a limit point of the x_p's. We claim that $A(\underset{\sim}{f};x_0) < +\infty$. From this it follows that $\underset{\sim}{f}$ is not unboundedly divergent and this establishes the result. It will suffice to show that for $m_1,m_2 \geqslant n_0$

$$|f_{m_1}(x_0)-f_{m_2}(x_0)| \leq B_0.$$

So let $m_1,m_2 > n_0$ be given. Take $\delta > 0$. Choose p such that $p > m_1,m_2$ and

$$|f_{m_1}(x_p)-f_{m_1}(x_0)| < \delta, \ |f_{m_2}(x_p)-f_{m_2}(x_0)| < \delta$$

by using the continuity of f_{m_1} and f_{m_2} at x_0. Then

$$|f_{m_1}(x_0)-f_{m_2}(x_0)| \le |f_{m_1}(x_0)-f_{m_1}(x_p)| + |f_{m_1}(x_p)-f_{m_2}(x_p)| + |f_{m_2}(x_p)-f_{m_2}(x_0)| < \delta + B_0 + \delta = 2\delta + B_0.$$

Since this is true for all $\delta > 0$, we get that $|f_{m_1}(x_0)-f_{m_2}(x_0)| \le B_0$ and we are done.

(ii) The "$\Rightarrow$" direction is again trivial so we need only do the "$\Leftarrow$" direction. Suppose $\underset{\sim}{f}$ is not uniformly divergent. Then

$$\forall \varepsilon > 0 \exists n_0 \forall p \exists x \forall m_1 m_2 \in [n_0, n_0+p](|f_{m_1}(x)-f_{m_2}(x)| < \varepsilon).$$

Let $c > 0$ be given. We shall show that there is an x_0 such that $A(\underset{\sim}{f};x_0) < c$ and this will establish the result. Choose $\varepsilon = c/2$. Then $\exists n_0$ such that

$$\forall p \exists x_p \forall m_1 m_2 \in [n_0, n_0+p](|f_{m_1}(x_p)-f_{m_2}(x_p)| < c/2).$$

Let x_0 be a limit point of the x_p's. Then as in (i) we have for all $m_1, m_2 \geqslant n_0$ that $|f_{m_1}(x_0)-f_{m_2}(x_0)| \le c/2$. So $A(\underset{\sim}{f};x_0) \le c/2 < c$ and we are done. □

<u>Proposition 9</u>. Let $\underset{\sim}{f} \in DZ$. The following are equivalent:

(i) $\underset{\sim}{f}$ is unboundedly divergent

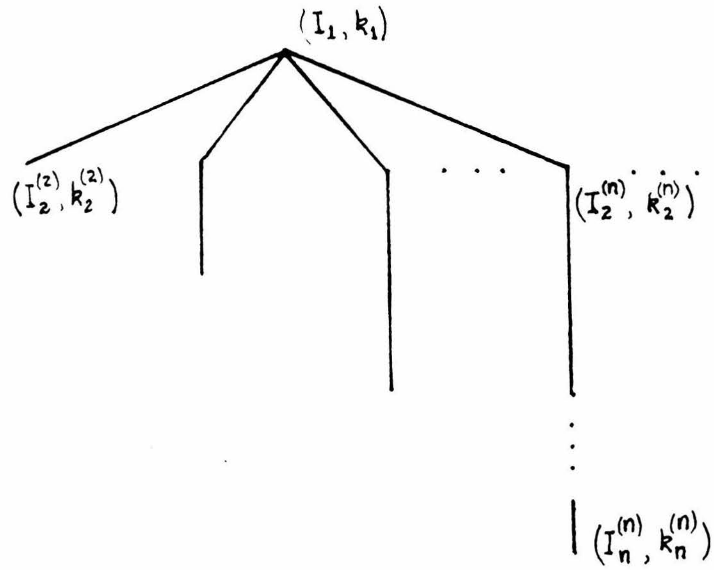
(ii) $r(\underset{\sim}{f}) = 1$

(iii) $\underset{\sim}{f}$ is unboundedly uniformly divergent.

Proof. (iii) ⇒ (i) is obvious, so it will suffice to show that (i) ⇒ (ii) and (ii) ⇒ (iii).

(i) ⇒ (ii): Suppose $r(\underset{\sim}{f}) > 1$. Then for some $M \in \mathbb{Q}^+$, $r(T^M_{\underset{\sim}{f}}) \geq \omega$.

So $T^M_{\underset{\sim}{f}}$ has a subtree as shown below.



Let x_n be the midpoint of $I_n^{(n)}$ and let x_0 be a limit point of the x_n's. We claim that $A(\underset{\sim}{f};x_0) < +\infty$. From this the result follows.

So we now prove the claim. Now from the definition of T^M_f we have

$$\forall x \in I_n^{(n)}\ \forall m_1,m_2 \in [1,k_n^{(n)}],\ |f_{m_1}(x)-f_{m_2}(x)| \leq M. \qquad (*)$$

Let m_1,m_2 be given. It will suffice to show that $|f_{m_1}(x_0)-f_{m_2}(x_0)| \leq M$.

Take $\delta > 0$ and choose, by the continuity of f_{m_1} and f_{m_2} at x_0, $n > m_1, m_2$ such that

$$|f_{m_1}(x_n)-f_{m_1}(x_0)| < \delta,\ |f_{m_2}(x_n)-f_{m_2}(x_0)| < \delta.$$

Then as in Proposition 8 by using (*) we get $|f_{m_1}(x_0)-f_{m_2}(x_0)| \le M + 2\delta$. Since this is true for all $\delta > 0$ we have that $|f_{m_1}(x_0)-f_{m_2}(x_0)| \le M$ and we are done.

(ii) ⇒ (iii): Suppose $\underset{\sim}{f}$ is not unboundedly uniformly divergent. Then

$$\forall x \in I_2 \forall m \in [1,q_N+N-1](|f_m(x)-f_m(x_{q_N})| \le M/3).$$

We claim that $<(I_1,k_1),(I_2,q_N),\ldots,(I_2,q_N+N-1)> \in T^M_{\underset{\sim}{f}}$. Indeed we have

$\forall x \in I_2\ \forall m_1,m_2 \in [1,q_N+N-1]$

$$|f_{m_1}(x)-f_{m_2}(x)| \le |f_{m_1}(x)-f_{m_1}(x_{q_N})| + |{}_{m_1}(x_{q_N})-f_{m_2}(x_{q_N})| + |f_{m_2}(x_{q_N})-f_{m_2}(x)| < M/3 + B_0 + M/3 \le M$$

Also for each $i = 2, \ldots, N-1$ we have $\forall x \in I_2$ and $\forall m_1,m_2 \in [q_N+i-2,q_N+N-1]$ that

$$|f_{m_1}(x)-f_{m_2}(x)| \le \sum_{k=0}^{n-2} \|f_{q_N+k+1}(x)-f_{q_N+k}(x)\|$$

$$\le \sum_{k=0}^{N-2} \|f_{q_N+k+1}-f_{q_N+k}(x)\|$$

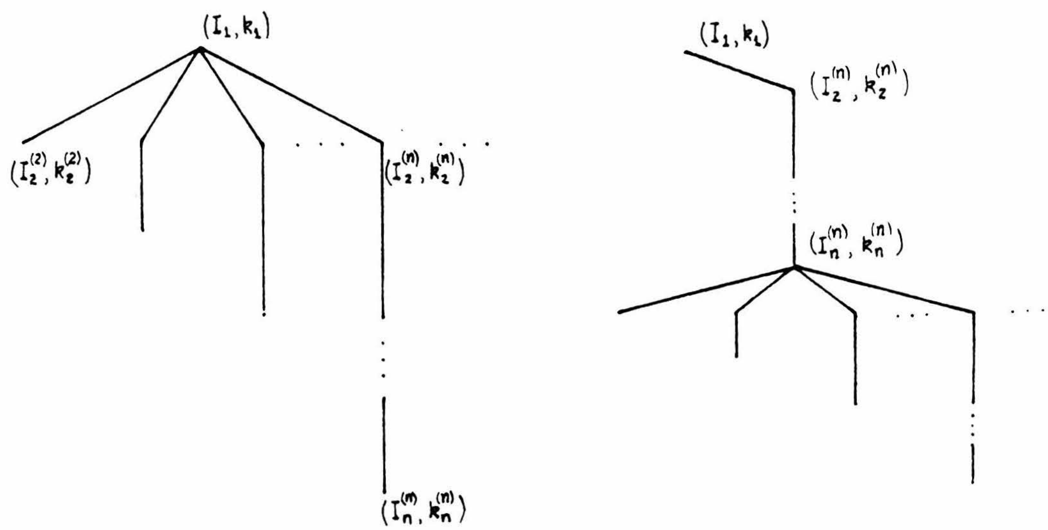
$$\le (N-1)\cdot\frac{M}{N^2} < \frac{M}{N} < \frac{M}{i}.$$

So our claim is verified and we are done. □

<u>Proposition</u> <u>10</u>. Let $\underset{\sim}{f} \in DS$ and suppose $\underset{\sim}{f}$ is uniformly divergent.

Then $r(\underset{\sim}{f}) \leq 2$.

<u>Proof</u>. In view of Proposition 8 (ii) it will suffice to prove that: "there is a $c > 0$ such that $A(\underset{\sim}{f};x) \geqslant c$ for each $x \in \mathbb{T} \Rightarrow r(f) \leq 2$." So assume its hypothesis. Now suppose $r(T^M_{\underset{\sim}{f}}) \geqslant \omega \cdot 2$. So $T^M_{\underset{\sim}{f}}$ has a subtree as shown below with each of the nodes $\langle I_1,k_1),(I_2^{(n)},k_2^{(n)}),\ldots,(I_n^{(n)},k_n^{(n)})\rangle$ being of rank at least ω.



Fix $n \in \mathbb{N}$ and consider the subtree through the node $\langle (I_1,k_1),\ldots,(I_n^{(n)},k_n^{(n)})\rangle$. As in the proof of "(i) $\Rightarrow$ (ii)" of Proposition 9 we see that there is a point $x_n \in I_n^{(n)}$ such that $A(\underset{\sim}{f};x_n) \leq M/n$. But this is true for each $n \in \mathbb{N}$. So for large enough n we get $A(\underset{\sim}{f};x_n) < c$ which is a contradiction. Hence $r(\underset{\sim}{f}) \leq 2$. $\square$

<u>Remark</u>. We know that example 4 is not uniformly divergent. It is

easy to see however that it has rank 2. In fact it is not difficult to construct an $\underset{\sim}{f} \in DF$ with $r(\underset{\sim}{f}) = 2$ and $A(\underset{\sim}{f};x_n) \to 0$ for some $\{x_n\} \subseteq \mathbb{T}$. (We sketch this construction at the end of §4). The converse of Proposition 10 is therefore false.

2. Some Natural Examples. In this section we investigate the ranks of some natural examples to once again verify our intuitive idea that they should be 1 or 2.

Kolmogorov functions: Kolmogorov [29] constructed a Lebesgue-integrable function whose Fourier series was everywhere divergent. It turned out that the Fourier series of this function was in fact unboundedly divergent everywhere. Such functions are now called Kolmogorov functions in honour of their discoverer. It follows immediately from the definition of a Kolmogorov function that it has rank 1 (because of Proposition 9). For more details on the Kolmogorov construction see [04] p. 455-464, or [30].

Lusin Series: This series was given by Lusin [33] and is defined as follows: Let

$$F(z) = H_0(z) + \sum_{p=1}^{\infty} \frac{1}{\sqrt{p}} H_p(z) \cdot z^{\lambda p}$$

$$\text{where } H_p(z) = \sum_{m=u}^{p} z^{m(p+1)} \theta_p(z \cdot e^{-2\pi i m/(p+1)})$$

$$\theta(z) = \sum_{n=0}^{p} z^n \text{ and } \lambda_p = \sum_{k=1}^{p} k^2.$$

Lusin [33] showed that $F(e^{ix})$ was everywhere divergent. Now put $S_1 = \text{Re } F(e^{ix})$ and $S_2 = \text{Im } F(e^{ix})$. Then it is easy to see that S_1 and S_2 are trigonometric series with coefficients tending to zero. Lusin [33] also showed that S_1 was divergent a.e.. Later Stechkin [48] showed that both S_1 and S_2 are everywhere unboundedly divergent. From Proposition 9 we thus get $r(S_1) = 1$ and $r(S_2) = 1$.

<u>Steinhaus Series</u>: Steinhaus [49] gave an example of an everywhere divergent trigonometric series with coefficients which tend to zero. The coefficients were defined by recurrence relations and were not capable of simple analytic expressions. We shall therefore refer to the simpler example Steinhaus later gave in [51] as the Steinhaus series. This series is defined by

$$S \sim \sum_{n=2}^{\infty} \frac{\cos n(x - \log \log n)}{\log n}$$

The proof that S is everywhere divergent is very elementary. Moreover it is easy to see from the proof given in [04] p. 76 that the amplitude of divergence of S at each x is at least 1/2. So by proposition 10 we have $r(S) \leq 2$. We shall see however that the Steinhaus series is a special Belov Series (which we will define in a moment). This will enable us to show that S is in fact unboundedly divergent everywhere. Thus $r(S) = 1$.

<u>Hardy-Littlewood</u> <u>Series</u>: These series are defined by

$$S_p \sim \sum_{n=1}^{\infty} n^{p-1/2} \cos(\alpha \log n + nx), \quad \tfrac{1}{2} > p \geqslant 0, \ \alpha > 0.$$

Hardy and Littlewood [18] had considered these series with the added restriction that $\alpha = 1/\log(a)$, where a is a positive integer with $a \not\equiv 1 \pmod 4$. It is clear that this restriction is purely artificial and we shall show that it can be removed. The Hardy-Littlewood series are also special Belov Series. We shall show that for $p > 0$, S_p is unboundedly divergent everywhere, and that for $p = 0$ the amplitude of divergence of S_p at each x is at least $1/3000 \sqrt{\alpha}$. So by Propositions 9 and 10 we have $r(S_p) = 1$ for $p > 0$, $r(S_p) \leq 2$ for $p = 0$.

<u>Problem</u>: What is the rank of S_p when $p = 0$?

<u>Herzog</u> <u>Series</u>: We shall consider a slight modification of the series given by F. Herzog [20]. Let

$$F(z) = \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \{z^{k_m} T_m(z) + z^{2k_m} T_m(z)\}$$

where $T_m(z) = 1 + z^m + z^{2m} + \cdots + z^{[m/12]m}$ and the exponents k_m form an increasing sequence of positive integers such that $z^{k_m} T_m(z)$ and $z^{2k_m} T_m(z)$ and $z^{k_n} T_n(z) + z^{2k_n} T_n(z)$ have no terms in common when $m \neq n$. Put $S_1 \sim \operatorname{Re} F(e^{ix})$. Herzog [20] showed that there is a universal constant $C > 0$ such that for each x there are infinitely many m's such that at least one of

$$e^{ik_m x} T_m(e^{ix}), \quad e^{i\cdot 2k_m x} T_m(e^{ix})$$

has a real part with absolute value greater than Cm. From this it immediately follows that S_1 is unboundedly divergent everywhere. Thus $r(S_1) = 1$.

Now let $G(z) = \sum_{n=1}^{\infty} \frac{(-1)^m}{m} \{z^{k_m} T_m(z) + z^{2k_m} T_m(z)\}$ and put $S_2 \sim \text{Re } G(e^{ix})$. Then it also follows that the amplitude of divergence of S_2 at each x is at least C. So $r(S_2) \leq 2$ by Proposition 10. But

$$G(e^{i\cdot 0}) \sim \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \{2[\tfrac{m}{12}] + 2\}$$

so S_2 diverges boundedly at $x = 0$. Hence by Proposition 9 we have $r(S_2) = 2$.

<u>Belov</u> <u>Series</u>: The following theorem was proved by Belov [06] and enabled him to produce numerous examples of everywhere divergent trigonometric series with coefficients that tend to zero.

<u>Theorem</u> <u>A</u> (Belov). Let $g(y)$ be a function defined on $[1,\infty)$ such that for some $a \geqslant 1$ it has a second derivative on $[a,\infty)$. Assume also that

(i) for $x \geqslant a$, $g''(y)$ is positive, non-increasing and tends to zero as $y \to \infty$,

(ii) $g'(y) \to \infty$ as $y \to \infty$, and

(iii) $\frac{1}{g''(y)} \cdot g''(y + \frac{\lambda}{\sqrt{g''(y)}}) \mapsto 1$ as $y \mapsto \infty$ for arbitrary λ.

Then for each N there exists $n_2 > n_1$ N with $n_2 - n_1 + 1 > 1/1000\sqrt{g''(N_2)}$ such that for all $n_1 \leq n \leq n_2$, all the values of $\cos g(n)$ have the same sign and $|\cos g(n)| > \cos(3\pi/8)$.

Corollary B. Let $\{r_n\}$ be a non-increasing sequence which tends to zero and put $S_g \sim \sum_{n=1}^{\infty} r_n \cos g(n)$. Then

(i) $\lim_{n \mapsto \infty} (r_n/\sqrt{g''n}) = +\infty$, implies S_g diverges unboundedly

(ii) there is a constant $C > 0$ such that $r_n \geqslant C\sqrt{g''(n)}$ for all n, implies S_g has amplitude of divergence at least $\frac{C \cdot \cos(3\pi/8)}{1000}$.

Corollary B follows easily from Theorem A. If we have a g that satisfies the hypotheses of Theorem A and we put $f(y) = g(y) + axy + bx + c$, $(a \neq 0)$ then Corollary B will give us an everywhere divergent trigonometric series, S_f with coefficients that tend to zero. (Here x should be viewed as a parameter rather than as a variable.) We shall call series that are produced this way Belov series. It follows immediately from corollary B that for any Belov series S, $r(S) \leq 2$.

We shall now show that the Steinhaus series is a Belov series and that it is unboundedly divergent. Let $g(y) = y\log\log y$. Put $f(y) = g(y) - x\cdot y$. Then g satisfies the hypotheses of Theorem A. Also $g''(y) = \quad \log y$. So if we put $r_n = 1/\log n$ then $r_n/\sqrt{g''(n)} \mapsto +\infty$ as $n \mapsto \infty$. Hence

$$S_f \sim \sum_{n=2}^{\infty} \frac{\cos(n\log\log n - nx)}{\log n} = \sum_{n=2}^{\infty} \frac{\cos n(x-\log\log n)}{\log n}$$

diverges unboundedly at each x. So the Steinhaus series has rank 1. Now we shall show that Hardy-Littlewood series are also Belov series. Let $g(y) = y \log y$ and put $f(y) = g(y) + x\cdot y$. Then g satisfies the hypotheses of Theorem A. Also $g''(y) = \alpha/y$. So if $r_n = n^{p-1/2}$, $\frac{1}{2} > p > 0$ then $\lim\limits_{n\mapsto\infty} (r_n/\sqrt{g''(n)}) = +\infty$; and if $r_n =$ $n^{-1/2}$ then $(r_n/\sqrt{g''(n)}) \geq 1/\sqrt{\alpha}$ for all n. Thus $S_p \sim \sum_{n=1}^{\infty} n^{p-1/2}$ $\cos(\alpha\log n + nx)$ diverges unboundedly when $\frac{1}{2} > p > 0$, and uniformly when $p = 0$.

3. <u>Cantor-Bendixson</u> <u>Analysis</u>. In this section we formulate an equivalent description of the rank function r on DZ by means of a Cantor-Bendixson Analysis. This will make it easy for us to show that r is unbounded in ω_1 on DF. Recall the definition $T^M_{\underset{\sim}{f}}$ from the beginning of §1. For each $N \in \mathbb{N}$ we define the tree $T^M_{\underset{\sim}{f},N}$ by

$T^M_{\underset{\sim}{f},N} = T^M_{\underset{\sim}{g}}$ where $\underset{\sim}{g}$ is the sequence defined by $g_m = f_{m+N}$. The Cantor-Bendixson Analysis is very similar to that in Chapter 1. We shall define a sequence $\langle P^\alpha_{M,\underset{\sim}{f},N})$ which reflect the properties of $\underset{\sim}{f}$. M is again to be thought of as being large and $Q(W, x, P^\alpha_{M,\underset{\sim}{f},N})$ is to be interpreted as W witnesses $x \in P^\alpha_{M,\underset{\sim}{f},N}$. W will range over the closed subsets of $\mathbb{T}$.

<u>Definition</u>. For each $M \in \mathbb{Q}^+$, $\underset{\sim}{f} \in DZ$, $N \in \mathbb{N}$ we define $P_{M,\underset{\sim}{f},N}$ by $x \in P_{m,\underset{\sim}{f},N} \iff \sup\{|f_{m_1}(x)-f_{m_2}(x)|: m_1,m_2 \geq N\} \leq M$. We define the sets $P^\alpha_{M,f,N}$ and the relation $Q(W, x, P^\alpha_{M,\underset{\sim}{f},N})$ by induction as follows: Let

$$P^1_{M,\underset{\sim}{f},N} = P_{M,\underset{\sim}{f},N} \text{ and } Q(W,x,P^1_{M,\underset{\sim}{f},N}) \iff x \in W \cap P_{M,\underset{\sim}{f},N}.$$

$P^{\alpha+1}_{m,\underset{\sim}{f},N} = \{x \epsilon P^\alpha_{M,\underset{\sim}{f},N}$: for all $I \epsilon Q(\mathbb{T})$ with $x \in int(I)$,

$\exists m \geq 1/|I|$, $\exists y \in I$, $\exists V \subseteq P_{M,\underset{\sim}{f},N}$ s.t. $Q(V, y, P^\alpha_{M\cdot|I|,\underset{\sim}{f},N+m})\}$

$Q(W, x, P^{\alpha+1}_{M,\underset{\sim}{f},N}) \iff$ for all $I \in Q(\mathbb{T})$ with $x \in int(I)$, $\exists m \geq 1/|I|$,

$\exists y \in I,\ \exists V \subseteq W \cap P_{M,\underset{\sim}{f},N}$ s.t. $Q(V\ ,\ y\ ,\ P^{\alpha}_{M,|I|,\underset{\sim}{f},N+m})$

and for λ a limit ordinal we let

$$P^{\lambda}_{M,\underset{\sim}{f},N} = \bigcap_{\alpha<\lambda} P^{\alpha}_{M,\underset{\sim}{f},N} \quad Q(W\ ,\ x\ ,\ P^{\lambda}_{M,\underset{\sim}{f},N}) \iff \forall \alpha < \lambda$$

$$Q(W,x,P^{\alpha}_{M,\underset{\sim}{f},N}).$$

Observe that our definition is made by use of simultaneous induction on α, M and N. It is easy to see that $P^{\alpha}_{M,\underset{\sim}{f},N}$ is a closed set. Also $\alpha \le \beta \Rightarrow P^{\beta}_{M,\underset{\sim}{f},N} \subseteq P^{\alpha}_{M,\underset{\sim}{f},N}$ and

$M \le M' \Rightarrow P^{\alpha}_{M,\underset{\sim}{f},N} \subseteq P^{\alpha}_{M',\underset{\sim}{f},N}$. When N = 1 we shall refer to $P^{\alpha}_{M,\underset{\sim}{f},N}$ simply as $P^{\alpha}_{M,\underset{\sim}{f}}$. We will show in Proposition 14 that $f \in$ DS $\iff$ for each $M \in \mathbb{Q}^{+}$, $\exists \alpha < \omega_1$ such that $P^{\alpha}_{M,\underset{\sim}{f}} = \emptyset$. As in chapter 1 this will allow us to make the following definition:

<u>Definition</u>. We define a new rank function r_1 on DS by

$$r_1(\underset{\sim}{f}) = \text{least } \alpha \text{ such that } P^{\alpha}_{M,\underset{\sim}{f}} = \emptyset \text{ for all } M \in \mathbb{Q}^{+}.$$

Our goal will then be to show that $r = r_1$ on DZ. It will turn out that r_1 is the same as r on DS except for an initial segment of length ω where r differs from r_1 by at most 1.

<u>Definition</u>. Let $f \in$ DS, $M \in \mathbb{Q}^{+}$ and $N \in \mathbb{N}$. For each subset W of $\mathbb{T}$

we define the subtree $W(T^M_{\underset{\sim}{f},N})$ of $T^M_{\underset{\sim}{f},N}$ by

$$\langle(I,k_1),\ldots,(I_n,k_n)\rangle \in W(T^M_{\underset{\sim}{f},N}) \Leftrightarrow \langle(I_1,k_1),\ldots,(I_n,k_n)\rangle \in T^M_{\underset{\sim}{f},N}$$

and $W \cap I_n \neq \emptyset$.

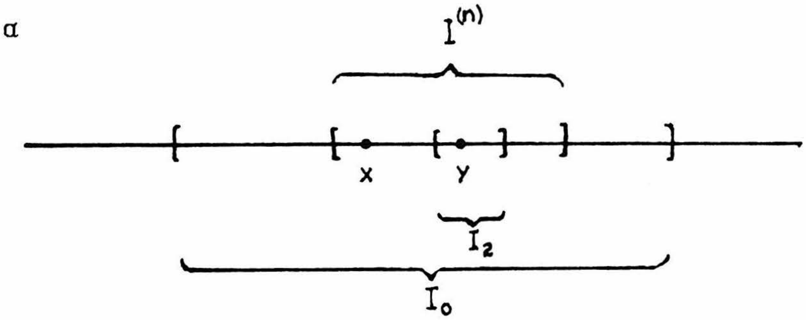
<u>Lemma 11</u>. If $x \in P^\alpha_{M,\underset{\sim}{f},N}$ and W satisfies $Q(W , x , P^\alpha_{M,\underset{\sim}{f},N})$ then

$r(W(T^{2m}_{\underset{\sim}{f},N}); x) \geq \omega\cdot\alpha$.

<u>Proof</u>. We prove the result by simultaneous induction on α, M and N. For $\alpha = 1$ the result follows from Proposition 9 and for limit ordinals the result follows trivially. Assume that the result is true for α, for all M and all N. We have to show that it is true for $\alpha + 1$, for all M and for all N. So fix M, N. Fix now $I_0 \in Q(\mathbb{T})$ with $x \in \text{int}(I_0)$. It will suffice to show that for each $n \in \mathbb{N}$ the tree $W(T^{2M}_{\underset{\sim}{f},N})\upharpoonright I_0$ has a node of rank at least $\omega\cdot\alpha + (n-1)$.

So fix $n \in \mathbb{N}$. Choose $I^{(n)} \in Q(\mathbb{T})$ such that $|I^{(n)}| \leq 1/n$, $I^{(n)} \subseteq I_0$ and $x \in \text{int}(I^{(n)})$. From the assumptions on x we know that there exists $m \geq 1/|I^{(n)}|$, $y \in I^{(n)}$ and $V \subseteq P_{M,\underset{\sim}{f},N}$ such that $Q(V , y , P^\alpha_{M\cdot|I^{(n)}|,\underset{\sim}{f},N+m})$. Since $|I^{(n)}| \leq 1/n$ we also have $Q(V , y , P^\alpha_{M/n,\underset{\sim}{f},N+m})$. By the induction hypothesis we thus get

$r(V(T^{2M/n}_{\underset{\sim}{f},N+m}):y) \geqslant \omega\cdot\alpha$



Now choose $I_2 \in Q(\mathbb{T})$ with $y \in \text{int}(I_2)$ and $I_2 \subseteq I^{(n)}$ such that $\sup\{|f_{m_1}(z)-f_{m_2}(z)|:N\le m_1,m_2\le N+m\} \le 2M/n$. (This is possible because $y \in P_{M/n,\underset{\sim}{f},N+m}$.) Then by lemma 5, $r([V(T^{2M/n}_{\underset{\sim}{f},N+m})\restriction I_2]_n) \geqslant \omega\cdot\alpha$.

Let T_n be the tree defined by

$$<(\mathbb{T},1),(I_2,m+2),\ldots,(I_2,m+i)> \in T_n \qquad i = 2, \ldots, n$$

$$\text{and } <(\mathbb{T},1),(J_2,\ell_2),\ldots,(J_p,\ell_p)> \in [[V(T^{2M/n}_{\underset{\sim}{f},N+m})\restriction I_2]_n$$

$$\Rightarrow <(\mathbb{T},1),(I_2,m+2),\ldots,(I_2,m+n),(J_2,\ell_2+m),\ldots,(J_p,\ell_p+m)> \in T_n.$$

Then it is easy to see that

$$r(<(\mathbb{T},1)>;T_n) \geqslant r(<(\mathbb{T},1)>;\ [V(T^{2M/n}_{\underset{\sim}{f},N+m})\restriction I_2]_n) + (n-1)$$

$$\geqslant \omega\cdot\alpha + (n-1).$$

So to complete the proof it will suffice to show that T_n is a subtree of $W(T^{2M}_{\underset{\sim}{f},N})\restriction I_0$. Let $<(J_1,k_1),\ldots,(J_n,k_n),\ldots,(J_p,k_p)>$ be a node in T_n. We must show that $J_2 \subseteq I_0$, $W \cap J_p \neq \emptyset$ and $\forall z \in J_p\ \forall m_1,m_2 \in [k_i,k_p]$ we have

$$|f_{N+m_1}(z)-f_{N+m_2}(z)| \leq 2M/i \qquad (*)$$

Now from the definition of T_n, J_2 must be I_2, so $J_2 = I_2 \subseteq I^{(n)} \subseteq I_0$. Also $W \cap J_p \supseteq V \cap I_p \neq \emptyset$ since $y \in V \cap I_m$ if $p \leq n$, and if $p > n$ then $V \cap I_m \neq \emptyset$ by the definition of $[V(T^{2M/n}_{\underset{\sim}{f},N+m})\restriction I_2]_n$.

Now if $i > n$ then (*) follows from the definition of the tree $[V(T^{2M/n}_{\underset{\sim}{f},N+m})]_n$. Also if $i \leq n$ then (*) follows because of the way we chose I_2. So T_n is indeed a subtree of $W(T^{2M}_{\underset{\sim}{f},N})\restriction I_0$ and we are done. □

Recall the definition of a *transitive* tree and the subtree $[T]^k$ of T from §4 Chapter 1. We have exactly as in Sub-lemma 1.14 the following result.

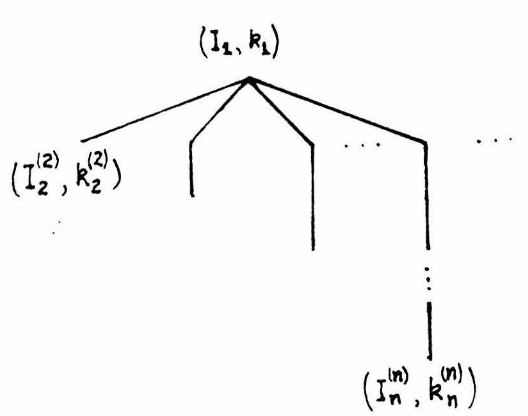
<u>Sub-lemma</u> <u>12</u>. Suppose T is a transitive subtree of $T^M_{\underset{\sim}{f},N}$, $I \in \mathbb{Q}(\mathbb{T})$ and $r(T\restriction I) \geqslant \omega\cdot\alpha$. Then for each $k \in \mathbb{N}$ the subtree $[T\restriction I]^k$ is transitive and $r([T\restriction I]^k) \geqslant \omega\cdot\alpha$. □

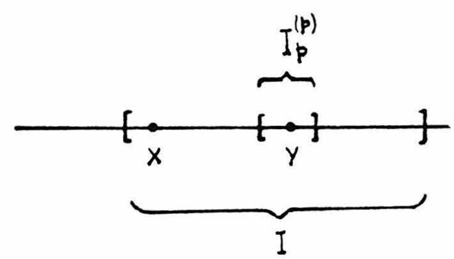
<u>Lemma</u> <u>13</u>. Let $f \in DZ$ and suppose T is a transitive subtree of $T^M_{\underset{\sim}{f},N}$ with $r(T;x) \geqslant \omega\cdot\alpha$. Let $W = \{z\in\mathbb{T} : r(T\,;\,z)\geqslant\omega\}$. Then $Q(W,x,P^{\alpha}_{M,\underset{\sim}{f},N})$. In particular $x \in P^{\alpha}_{M,\underset{\sim}{f},N}$.

<u>Proof</u>. The result is proved by simultaneous induction on α, M and N. For $\alpha = 1$ the result follows from Proposition 9, and for limit ordinals it is trivial as in Lemma 11. Assume that the result is true for α, all M and all N. We have to show that it is true for $\alpha + 1$, all M and all N. So fix M and N. It will suffice to show that

$$r(T;x) \geqslant \omega\cdot(\alpha+1) \Rightarrow Q(W\ ,\ x\ ,\ P^{\alpha+1}_{M,\underset{\sim}{f},N}).$$

Now fix $I \in Q(\mathbb{T})$ with $x \in \text{int}(I)$. We must find $m \geqslant 1/|I|$, $y \in I$ and $V \subseteq W \cap P_{M,\underset{\sim}{f},N}$ such that $Q(V\ ,\ y\ ,\ P^{\alpha}_{M\cdot|I|,\underset{\sim}{f},N+m})$. Since $r(T;x) \geqslant \omega\cdot(\alpha+1)$ we have that $r(T\restriction I) \geqslant \omega\cdot(\alpha+1)$. So $T\restriction I$ has a subtree as shown below with each of the nodes $\langle(I_1,k_1),\ldots,(I_n^{(n)},k_n^{(n)})\rangle$ having rank at least $\omega\cdot\alpha$ in $T\restriction I$.





Choose p such that $p \geqslant 1/|I|$ and let $m = k_p^{(p)}$. Then $m \geqslant p \geqslant 1/|I|$. Now let $\underset{\sim}{v}_n = \langle(I_1,k_1),\ldots,(I_n^{(n)},k_n^{(n)})\rangle$. Then the tree $T_{\underset{\sim}{v}_p}$ at the node $\underset{\sim}{v}_p$ is transitive and $r(T_{\underset{\sim}{v}_p}) \geqslant \omega\cdot\alpha$. So by

Sublemma 12, $r([T_{\underset{\sim}{v}_p}]^p) \geq \omega \cdot \alpha$. Let T_p be the tree defined by

$$<(\mathbb{T},1),(J_2,\ell_2+1-m),\ldots,(J_n,\ell_n+1-m)> \in T_p$$
$$\Leftrightarrow <(J_1,\ell_1),\ldots,(J_n,\ell_n)> \in [T_{v_p}]^p$$

Then T_p is transitive and it is easy to see that T_p is a subtree of $T^{M/p}_{\underset{\sim}{f},N+m}$. Moreover T_p has rank at least $\omega \cdot \alpha$. So by lemma 4 there is a $y \in I_p^{(p)} \subset I$ such that $r(T_p;y) \geq \omega \cdot \alpha$. Let $V = \{z : r(T_p;z) \geq \omega\}$. Then by the induction hypothesis $Q(V , y , P^{\alpha}_{M/p,\underset{\sim}{f},N+m})$. Since $1/p < 1/|I|$ we thus have $Q(V , y , P^{\alpha}_{M\cdot|I|,\underset{\sim}{f},N+m})$. We have now found our m, y and V. It

remains to show that $V \subseteq W \cap P_{M,f,N}$. Now from the definition of T_p we have that $r(T_p;z) \geq \omega$ implies $r(T;z) \geq \omega$. So $V \subseteq W$. Also from the definition of W we have that $W \subseteq P_{M,\underset{\sim}{f},N}$. So $V \subseteq W \cap P_{M,\underset{\sim}{f},N}$ and we are done. □

<u>Proposition</u> <u>14</u>. Let $\underset{\sim}{f} \in C(\mathbb{T})^{\omega}$. Then $\underset{\sim}{f} \in DS \Leftrightarrow$ for each $M \in \mathbb{Q}^+$, $\exists \alpha < \omega_1$ such that $P^{\alpha}_{M,\underset{\sim}{f}} = \emptyset$.

<u>Proof</u>. "$\Rightarrow$": Suppose f $\in$ DS. Then for a fixed $M \in \mathbb{Q}^+$ we know that $r(T^M_{\underset{\sim}{f}}) < \omega \cdot r(\underset{\sim}{f})$. So by lemma 11 we must have $P^{\alpha}_{M,\underset{\sim}{f}} = \emptyset$, where $\alpha = r(\underset{\sim}{f})$ (otherwise we would get an $x \in P^{\alpha}_{M,\underset{\sim}{f}}$ and this

would make $r(T^M_f) \geq \omega \cdot r(f)$) .

"$\Leftarrow$": Suppose $\underset{\sim}{f} \in DS$. Then there is an $x \in \mathbb{T}$ such that $\underset{\sim}{f}$ converges at x. A direct verification shows that $Q(\{x\}, x, P^{\alpha}_{M,\underset{\sim}{f}})$ holds for all $\alpha \in ORD$. So $P^{\alpha}_{M,f}$ is never empty and we are done.

□

Proposition 15. For each $\underset{\sim}{f} \in DZ$, $r(\underset{\sim}{f}) = r_1(\underset{\sim}{f})$.

Proof. Use Lemmas 11 and 13. □

Remark. By modifying the proof of Proposition 13 we can show that $r(T^M_{\underset{\sim}{f}}) \geq \omega\cdot(\alpha+1)$ implies $P^{\alpha}_{M,\underset{\sim}{f}} \neq \emptyset$, for all $\underset{\sim}{f} \in DS$. This shows that r and r_1 can differ by at most 1 if $r(\underset{\sim}{f}) < \omega$, and that $r(\underset{\sim}{f}) = r_1(\underset{\sim}{f})$ for all $\underset{\sim}{f} \in DS$ with $r(f) \geq \omega$.

4. The rank function, r is unbounded in ω_1 on DF.

The main aim of this section is to prove the following result.

Proposition 16. For each $1 \leq \alpha < \omega_1$ there is an $f \in L_1(\mathbb{T})$ such that $r(f) = \alpha$.

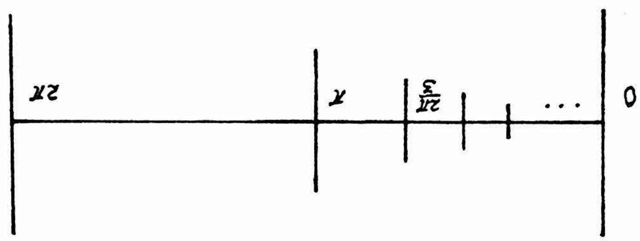
Proof. The proof is by induction on α. We shall show that for each α there is an $f_\alpha \in L_1(\mathbb{T})$ given by $f_\alpha = \lambda_\alpha \cdot f + \psi_\alpha$, such that $r(f_\alpha) = \alpha$. Here f is a Kolmogorov function and λ_α and ψ_α are chosen inductively. λ_α will be a $C^{(3)}$-function which is bounded by 1 and which is zero only on a prescribed closed countable set Ω_α.

We shall also need some other auxiliary functions besides the Kolmogorov function. Let ψ be a continuous function whose Fourier series has partial sums that are bounded in absolute value by 1, and converges everywhere except at x = 0. Let ϕ be a continuous function whose Fourier series converges everywhere except at x = 0 where it diverges unboundedly. For the existence of such functions see [04] p. 127-128.

Now for $\alpha = 1$ take $\Omega_1 = \emptyset$ and $\psi_0 \equiv 0$ to get $r(f_1) = 1$. For $\alpha = 2$ we take $\Omega_2 = \{0\}$ and $\psi_2 = \psi$ to get a function f_2 whose Fourier series diverges unboundedlly everywhere except at x = 0 where it diverges boundedly. From propositions 9 and 10 we get $r(f_2) = 2$. For $\alpha = 3$, we take $\Omega_3 = \{2\pi/n : n \geq 2\} \cup \Omega_2$ and

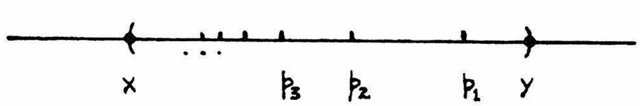
$$\psi_3 = \frac{1}{2}\psi_2 + \sum_{n=2}^{\infty} 4^{-n}\, \psi\left(x - \frac{2\pi}{n}\right).$$

In this case we see that $P^3_{M,f_3} = \emptyset$ for all $M \in \mathbb{Q}^+$ but $0 \in P^2_{1,f_3}$. Thus $r(f_3) = 3$.



The general situation splits into three cases:

<u>Case</u> <u>(i)</u>: $\alpha = \beta + 2$ for some ordinal β. In this case we take $\Omega_\alpha = \Omega_{\beta+1} \cup \Omega$ where Ω is the set which consists of a sequence of points from each of the intervals complementing $\Omega_{\beta+1}$ that converge to the left endpoint monotonically



$$p_i \to x, \quad (x,y) \subseteq \mathbb{T} - \Omega_{\beta+1}, \quad x,y \in \Omega_{\beta+1}.$$

Enumerate Ω as a countable sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ and put $\psi_\alpha = \frac{1}{2} \psi_{\beta+1} + \sum_{n=1}^{\infty} r^{-n} \psi(x-x_n)$. In this case we get that $0 \in P^{\beta+1}_{1,f_\alpha}$ but $P^{\alpha}_{M,f_\alpha} = \emptyset$ for all $M \in \mathbb{Q}^+$ so $r(f_\alpha) = \alpha$.

Case (ii): $\alpha = \lambda + 1$ where λ is a limit ordinal. In this case we choose an increasing sequence of successor ordinals α_n with $\lim \alpha_n = \lambda$. Let Ω_n' be a scaled copy of Ω_n onto $[\frac{2\pi}{n+1}, \frac{2\pi}{n}]$ and f'_{α_n} the function obtained by using Ω_n' instead of Ω_n. Then $\frac{2\pi}{n+1} \in P^{\alpha_n-1}_{1,f'_{\alpha_n}}$ but $P^{\alpha_n}_{M,f'_{\alpha_n}} = \emptyset$ for all $M \in \mathbb{Q}^+$. Let $\Omega_n' = \{x_m^{(n)} : m \in \mathbb{N}\}$. Put $\Omega_\alpha = \bigcup_{n\in\mathbb{N}} \Omega_n'$ and $\psi_\alpha = \sum_{m=1}^{\infty} 4^{-n} \psi_{\alpha_n}(x) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 4^{-n-m} \psi(x-x_m^{(n)})$. Then $0 \in P^{\alpha_n-1}_{1,f_\alpha}$ for all n, so $0 \in P^{\lambda}_{1,f_\alpha}$. However $P^{\lambda+1}_{M,f_\alpha} = \emptyset$ for all $M \in \mathbb{Q}^+$ because 0 is the only possible element of P^{λ}_{M,f_α} for any $M \in \mathbb{Q}$, and $P^{\lambda+1}_{M,f_\alpha} \neq \emptyset \Rightarrow P^{\lambda}_{M,f_\alpha}$ must be infinite. So we get that $r(f_\alpha) = \alpha$.

Case (iii): $\alpha = \lambda$ a limit ordinal. In this case we repeat the construction given in case (ii) except that we put

$$\psi_\alpha = \frac{1}{2}\,\phi(x) + \sum_{m=1}^{\infty} 4^{-n}\,\psi\alpha_n(x) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} 4^{-n-m}\,\psi(x-x_m^{(n)})$$

Ω_α remains the same as in case (ii). Because of the term $\frac{1}{2}$ $\emptyset(x)$, f_α now diverges unboundedly at $x = 0$. So $0 \notin P^\lambda_{M,f_\alpha}$ for any $M \in \mathbb{Q}^+$. Since 0 was the only possible element in P^λ_{M,f_α} it follows that $P_{M,f_\alpha} = \emptyset$ for all $M \in \mathbb{Q}^+$. So $r(f_\alpha) \le \lambda$. But we know that $\frac{2\pi}{n+1} \in P^{\alpha_{n-1}}_{M_n,f_\alpha}$ for large enough M_n's. So $r(f_\alpha) \geq \lambda$ and hence $r(f_\alpha) = \lambda = \alpha$. So we are done.

□

We are now in a position to give the construction we promised at the end of section 1. We wanted to construct a function f such that $r(f) = 2$ but whose Fourier series is not uniformly divergent. We proceed exactly as in case (iii).

Take f_0 to be a Kolmogorov function and $\Omega = \{2\pi/n : n \geq 2\}$. Choose a $\lambda \in C^{(3)}$ with $|\lambda| \le 1$ and $\lambda = 0$ only on Ω. Now define f by

$$f(x) = \lambda \cdot f_0(x) + \phi(x) + \sum_{n=2}^{\infty} 4^{-n}\psi(x-\tfrac{2\pi}{n})$$

where ϕ and ψ are as above. Then f has the required properties because $A(S(f);\frac{2\pi}{n}) = 4^{-n}$ for each $n \geq 2$.

Corollary 17. DS, DZ, DT and DF are all coanalytic but not Borel sets.

Proof. The rank function r is a coanalytic norm on each of these sets and is unbounded in ω_1. So by Proposition 0.1, they can't be Borel. □

Final Remarks. We have seen how to use the Cantor-Bendixson Analysis in order to show that r is unbounded in ω_1 on DF. Once again it is very plausible that there is no simple Cantor-Bendixson Analysis but this is not very clear. So we pose the following problem.

Problem. Is there a simple Cantor-Bendixson Analysis which gives rise to the same rank function r?

Chapter 3

Introduction. Let $K = \langle K(\mathbb{R}^2), d\rangle$ be the Polish space of all compact, non-empty subsets of the plane, $\mathbb{R}^2$ with d being the Hausdorff metric given by

$$d(A,B) = \sup\{dist(x,B)\ ,\ dist(A,y) : x \in A, y \in B\}$$

In this chapter we shall be interested in certain subsets of K. To define these subsets we need to introduce some definitions.

Let $A \in K$. By a *path* in A we mean a continuous function $\gamma:[a,b] \to A$ where a and b are real numbers with $a < b$. The path γ is said to be a arc if γ is an injective function. γ is said to be a *Jordan loop* if $\gamma:[a,b) \to A$ is injective and $\gamma(a) = \gamma(b)$. So Jordan arcs are homeomorphic to [0,1] and Jordan loops are homeomorphic to the unit circle $\mathbb{T}$. Let be a Jordan loop in A. By the Jordan curve Theorem we know that γ divides the plane into two open components, exactly one of which is bounded. We call the bounded component the *inside* of γ, written $ins(\gamma)$. A is said to have a *cavity* if there is a Jordan loop γ in A such that $ins(\gamma) \not\subseteq A$. A set without any cavities is called a *Jordan set.*

We now define the following subsets of K. Let JS be the collection of all Jordan sets in K, PC be the collection of all path-connected sets in K, and SC be the collection of all simply-connected sets in K. M. Ajtai (see [24])(see also H. Becker[05]) showed that PC is a $\underset{\sim}{\Pi}^1_2$ but not $\underset{\sim}{\Pi}^1_1$ subset of K. The question was raised as to whether PC is $\underset{\sim}{\Delta}^1_2$. This is still open. H. Becker [05] also showed that JS is $\underset{\sim}{\Pi}^1_1$ complete and that SC is not a $\underset{\sim}{\Sigma}^1_1$ subset of K. The question raised by Becker as to whether SC is $\underset{\sim}{\Pi}^1_1$ is also open (but see the final remark at the end of this chapter). We mention in passing that the collection of all connected sets in K is a closed subset of K (see [42] p.8).

<u>Remarks</u>. All of the Descriptive Set Theory we have used so far can be classified as the Classical Theory. There is also what is known as the Effective Theory. In the Effective Theory recursive functions and relations are defined on Polish spaces and the effective analogues of Borel, $\underset{\sim}{\Pi}^1_1$, and $\underset{\sim}{\Sigma}^1_1$ subsets, etc. are obtained (see [41]). The analogues are designated as Π^1_1, Σ^1_1 subsets, etc. (Σ^1_1 is referred to as "light-face sigma and one" and $\underset{\sim}{\Sigma}^1_1$ is referred to as "bold-face sigma one one," etc.) Ajtai actually showed that PC was Π^1_2 , and Becker showed that JS was Π^1_2. These results are slightly stronger than what was stated above but the proofs are the same. We shall say no more about this because we are mainly interested in the classical theory.

In this chapter we shall define and study a natural norm on JS. This norm is obtained by associating, with each set in JS, a collection of well-founded trees as in chapters 1 and 2. The well-founded trees are defined on sets with cardinality the same as $\mathbb{R}$ (unlike the case in chapters 1 and 2 where the trees were defined on countable sets). The norm provides a natural measure of the complexity of the sets in JS. It measures in some sense how "close the set came" to possessing a cavity. The sets of least possible rank are the convex sets, and if a set is locally path connected then it lies in the first or second level. We also give some natural examples of sets with small ranks. Finally we show that this norm is unbounded in ω_1 on SC. This provides a Rank Argument of the fact that JS is coanalytic but not Borel. It also shows that SC cannot be an analytic subset of K.

§1. Tree Description. In this section we study the sets in JS by associating with each $A \in JS$ a countable collection of well-founded trees. But first we check that JS is a coanalytic subset.

Proposition 1. JS is a coanalytic subset of K.

Proof. We shall show that K - JS is an analytic subset of K. Observe that $A \in K - JS \iff$ there is a continuous function $\gamma:[0,1] \to \mathbb{R}^2$ such that

$$\gamma \text{ is a Jordan loop in } A \text{ and } \mathrm{ins}(\gamma) \subseteq A \qquad (*)$$

Let $E = \{(A,\gamma)\in K\times C_2 : (*) \text{ holds}\}$. (Here C_2 is the Polish space of all continuous functions from [0,1] to $\mathbb{R}^2$.) Then it is easy to see that E is a Borel set. So K - JS is the projection of a Borel set onto K and hence is analytic. Thus JS is coanalytic. □

With each $A \in K$ and each $M \in \mathbb{Q}^+$ we shall associate a tree,

T_A^M which reflects in some way the properties of A. Here M should be thought of as being large. The trees T_A^M will be defined on the set of all Jordan arcs in A. This set has the same cardinality as $\mathbb{R}$. Before we give the definition of T_A^M we need some notation.

For a Jordan arc $\gamma:[a,b] \to \mathbb{R}^2$ we define $|\gamma|$ to be the distance between the endpoints of γ. We define the closure $\bar{\gamma}$ of γ to be the closed path, $\bar{\gamma}:[a,b+1] \to \mathbb{R}^2$ given by

$$\bar{\gamma}(t) = \begin{cases} \gamma(t) & t \in [a,b] \\ \gamma(a) & t = b+1 \\ \gamma \text{ linear on } [b,b+1] & \end{cases}$$

Let $\gamma:[a,b] \to \mathbb{R}^2$ be a closed path. Then $\mathbb{R}^2 - \gamma[a,b]$ is a union of open connected sets with exactly one element of the union being unbounded. We define the inside, $\text{ins}(\gamma)$ of γ by

$$\text{ins}(\gamma) = \text{union of all unbounded components of } \mathbb{R}^2 - \gamma[a,b]$$

Finally let $\gamma_1:[a,b] \to \mathbb{R}^2$ and $\gamma_2:[c,d] \to \mathbb{R}^2$ be paths. We say that $\gamma_1 \subseteq \gamma_2$ if $\gamma_1[a,b] \subseteq \gamma_2[c,d]$. We also define $\|\gamma_2-\gamma_1\|$, for $\gamma_1 \subseteq \gamma_2$ by

$$\|\gamma_2-\gamma_1\| = \sup\{\text{dist}(x,y) : x,y \in \{\gamma_1(a),\gamma_1(b)\} \cup (\gamma_2[c,d]-\gamma_1[a,b])\}$$

This definition is "very close to", but not the same as the Hausdorff metric distance between the sets $\gamma_2[c,d]$ and $\gamma_1[a,b]$.

Definition. We define the tree T_A^M as follows:

$\langle\gamma_1,\ldots,\gamma_n\rangle \in T_A^M \Leftrightarrow$ (i) $\gamma_1, \ldots, \gamma_n$ are Jordan arcs in A

(ii) $\exists r \in \mathbb{R}^2$ such that $B(r,1/M) \subseteq (\mathbb{R}^2 - A) \cap \mathrm{ins}(\bar{\gamma}_i)$ for all $i = 1, \ldots, n$, and

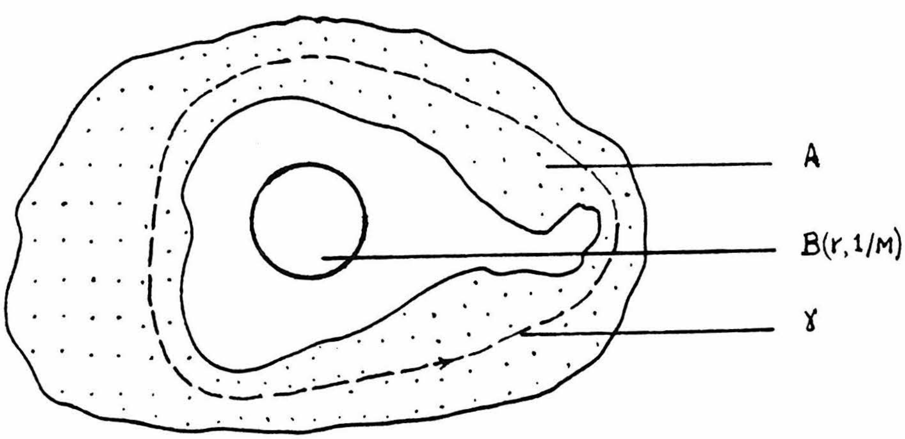
(iii) $\gamma_i \subseteq \gamma_{i+1}$, $\|\gamma_{i+1}-\gamma_i\| \leq M/i$ for all $i = 1, \ldots, n-1$.

Here $B(r,1/M)$ is the open disk with radius $1/M$ and center r.

Our next result gives the basic relation between the set A and the associated trees T_A^M.

Proposition 2. $A \in JS \Leftrightarrow \forall M \in \mathbb{Q}^+(T_A^M$ is well-founded).

Proof. "$\Leftarrow$": Suppose $A \in JS$. Then there exist $M \in \mathbb{Q}^+$, $r \in \mathbb{R}$ and a Jordan loop $\gamma:[a,b] \to A$ in A such that $B(r,1/M) \subseteq (\mathbb{R}^2-A) \cap \mathrm{ins}(\gamma)$.



We claim that T_A^M has an infinite branch. The result will follow from this. To prove the claim we shall construct an increasing sequence $\langle b_i\rangle_{i\in\mathbb{N}}$ such that for all i, $b_i < b$ and for all $t \in [b_i,b]$, $|\gamma(b)-\gamma(t)| \leq M/i$.

Since γ is continuous at b, $\exists b_1 \in (a,b)$ such that for all $t \in [b_1,b]$, $|\gamma(b)-\gamma(t)| \le M$. And given b_i we can choose $b_{i+1} \in (b_i,b)$ such that for all $t \in [b_{i+1},b]$, $|\gamma(b)-\gamma(t)| \le M/(i+1)$. So by induction we obtain our sequence $\langle b_i \rangle_{i\in\mathbb{N}}$. Now let $\gamma_i = \gamma\restriction[a,b_i]$. Then it is easy to see that γ_i is a Jordan arc in A and for each n, $\langle \gamma_1,\dots,\gamma_n \rangle \in T_A^M$. So $\langle \gamma_i \rangle_{i\in\mathbb{N}}$ is an infinite branch in T_A^M and the claim is proved.

"$\Rightarrow$": Now suppose that for some $M \in \mathbb{Q}^+$, T_A^M is not well-founded. Then there is an infinite branch $\langle \gamma_i \rangle_{i\in\mathbb{N}}$ in T_A^M. Without loss of generality we may assume that the γ_i's are such that $\gamma_i:[a_i,b_i] \to A$, $\gamma_i = \gamma_{i+1}\restriction[a_i,b_i]$, and the a_i's decrease to a and the b_i's increase to b (a and b being finite real numbers). From the definition of the tree T_A^M it follows that $\lim\limits_{i\to\infty} \gamma_i(a_i)$ and $\lim\limits_{i\to\infty} \gamma_i(b_i)$ both exist and

$$\lim_{i\to\infty} \gamma_i(a_i) = \lim_{i\to\infty} \gamma_i(b_i)$$

Moreover since A is compact this limit is in A.

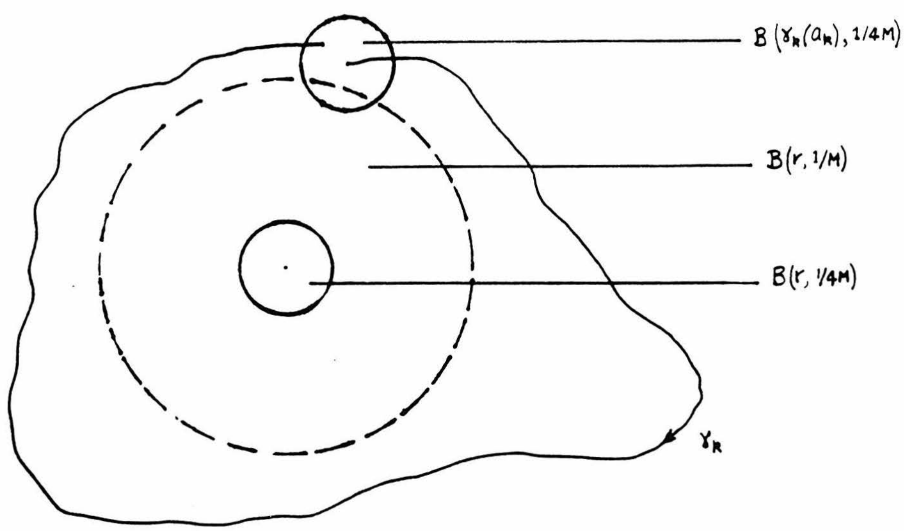
Let $\gamma:[a,b] \to A$ be defined by $\gamma\restriction[a_i,b_i] = \gamma_i$ and $\gamma(a) = \gamma(b) = \lim\limits_{i\to\infty} \gamma_i(a_i)$. Then it is clear that γ is continuous. Moreover γ is one to one on (a,b) since the γ_i's are one to one on $[a_i,b_i]$. Thus γ can intersect itself at most once.

We claim that there is an $r \in \mathbb{R}^2$ such that $B(r,1/4M) \subseteq (\mathbb{R}^2-A) \cap$

ins(γ). Indeed choose k such that M/k < 1/4M. Choose $r \in \mathbb{R}^2$ such that $B(r,1/M) \subseteq (\mathbb{R}^2-A) \cap \text{ins}(\bar{\gamma}_k)$. From the definition of the tree T_A^M we know that for each $n \geq k$, $\gamma_n[a_n,b_n] - \gamma_k[a_k,b_k]$ lies in a disk of diameter 1/4M which contains the point $\gamma_k(a_k)$. Thus for all $n \geq k$

$$\gamma_n[a_n,b_n] - \gamma_k[a_k,b_k] \subseteq B(\gamma_k(a_k),\tfrac{1}{4M}).$$

So the disk B(r,1/4M) lies in the inside of each of the paths $\bar{\gamma}_n$, $n \geq k$ and consequently in the inside of γ.



Now if γ is a Jordan loop then $A \notin JS$ and we are done. So suppose γ is not a Jordan loop. Then $\exists c \in (a,b)$ such that $\gamma(a) = \gamma(b) = \gamma(c)$. Let γ' be the path defined by

$$\gamma' = \begin{cases} \gamma\restriction[a,c] & \text{if } B(r,1/4M) \subseteq \text{ins}(\gamma\restriction[a,c]) \\ \gamma\restriction[c,b] & \text{if } B(r,1/4M) \not\subseteq \text{ins}(\gamma\restriction[a,c]) \\ & \text{and } B(r,1/4M) \subseteq \text{ins}(\gamma\restriction[c,b]) \end{cases}$$

Then γ' is a well-defined Jordan loop and this shows that $A \notin JS$. □

Definition. We define for each $A \in JS$, the rank function $\rho(A)$ by

$$\rho(A) = \sup\{r(T_A^M)+1 : M \in \mathbb{Q}^+\}$$

We emphasize here that we are using convention that the empty tree has rank -1.

Proposition 3: For each $A \in JS$, $\rho(A) < \omega_1$.

Proof. Let $A \in JS$. Fix $M \in \mathbb{Q}^+$ and consider the tree T_A^M. It is easy to see that the relation $\lessdot$, defined by

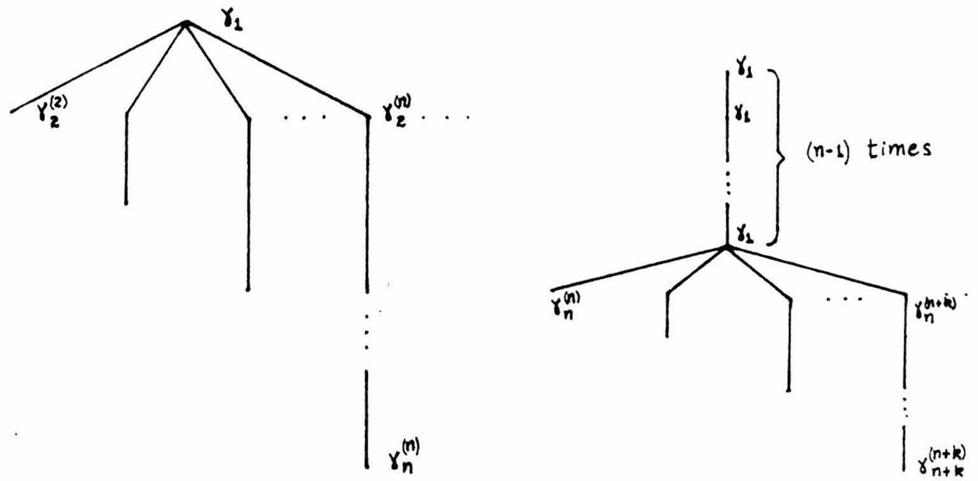
$$\underset{\sim}{u} \lessdot \underset{\sim}{v} \iff \underset{\sim}{u},\underset{\sim}{v} \in T_A^M \text{ and } \underset{\sim}{v} \text{ extends } \underset{\sim}{u}$$

is a strict analytic relation. Since $A \in JS$, T_A^M is well-founded and so $\lessdot$ is also well-founded. It therefore follows from Proposition 0.6 that $\lessdot$ has countable length. But the length of $\lessdot$ is just the rank of T_A^M, so $r(T_A^M) < \omega_1$. Thus

$$\begin{aligned}\rho(a) &= \sup\{r(T_A^M)+1 : M \in \mathbb{Q}^+\} \\ &= \sup\{r(T_A^N)+1 : N \in \mathbb{N}\} < \omega_1.\end{aligned}$$ □

Proposition 4. Let $A \in JS$. Then $\rho(A)$ is either a limit ordinal or the immediate successor of a limit ordinal.

Proof. It will suffice to show that if $\rho(A) \geqslant \omega\cdot\alpha + 2$ then $\rho(A) \geqslant \omega\cdot(\alpha+1)$. Suppose $\rho(A) \geqslant \omega\cdot\alpha + 2$. Then for some $M \in \mathbb{Q}^+$ we have $r(T_A^M) \geqslant \omega\cdot\alpha + 1$. So T_A^M has a subtree, as shown below on the left, with the nodes $\langle\gamma_1,\ldots,\gamma_n^{(n)}\rangle$ having ranks that are increasing and with limit $\omega\cdot\alpha$.



We claim that $r(T_A^{M\cdot n}) \geqslant \omega\cdot\alpha + n$. From this the result follows readily. So we need to prove our claim. Let T_n be the tree defined by

$$\langle\gamma_1,\ldots,\gamma_1\rangle (i \text{ times}) \in T_n \qquad i = 1, \ldots, n-1$$

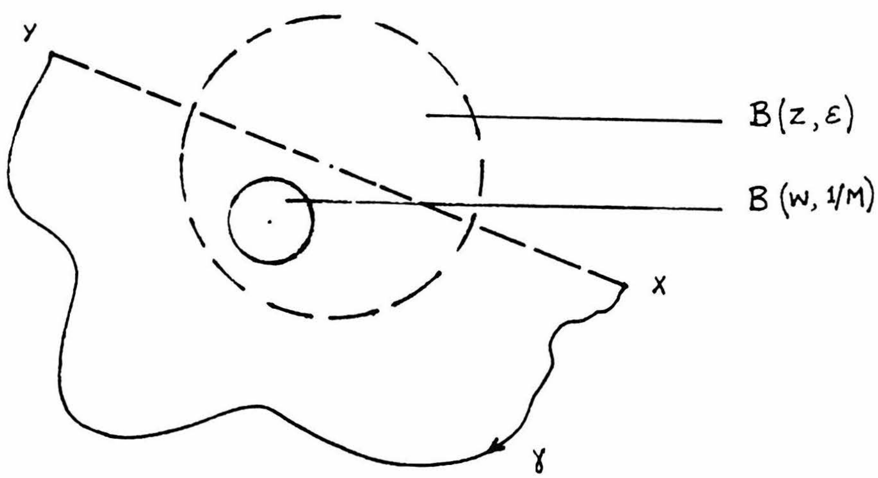
$$\langle\gamma_1,\ldots,\gamma_n,\ldots,\gamma_{n+k}\rangle \in T_A^M \Rightarrow \langle\underbrace{\gamma_1,\ldots,\gamma_1}_{(n-1)\text{ times}},\gamma_n,\ldots,\gamma_{n+k}\rangle \in T_n$$

Then it is easy to check that T_n is a subtree of $T_A^{M\cdot n}) \geqslant \omega\cdot\alpha + n$ and we are done. □

<u>Remark</u>. We use the convention that 0 is a limit ordinal. It is clear that $\rho(A)$ can never be 1 but we shall later see that $\rho(A)$ can be other immediate successors of limit ordinals.

<u>Proposition 5</u>. Let $A \in JS$. Then $\rho(A) = 0 \Leftrightarrow$ each path-component of A is convex.

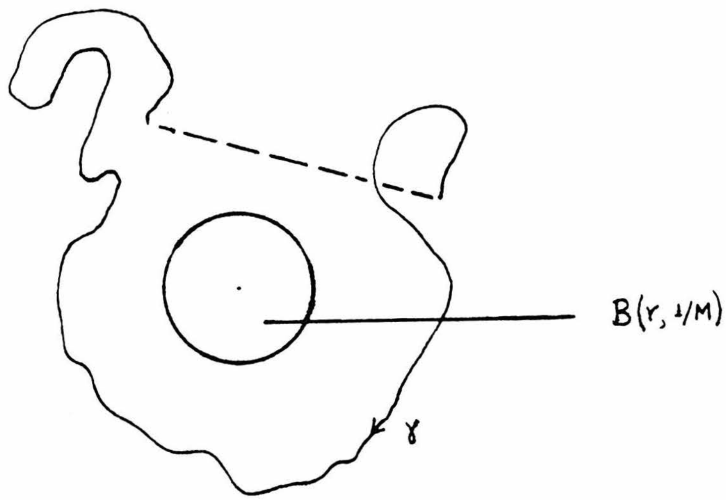
<u>Proof</u>. "$\Rightarrow$": Suppose P is a path-component of A which is not convex. Then there exist $x,y \in P$ such that the segment $[x,y] \not\subseteq P$. So there is a point z on $[x,y]$ such that $z \notin A$. Since A is closed, $(\mathbb{R}^2-A)$ is open and so there exists $\varepsilon > 0$ such that $B(z,\varepsilon) \cap A = \emptyset$.



Since P is path-connected there is a path in P connecting x and y, and by a standard result (see [13] p. 29) it follows that there is a Jordan arc γ in P connecting x and y. Choose $w \in B(z,\varepsilon/2)$ such that $B(w,\varepsilon/4) \subseteq ins(\bar{\gamma})$. Now let $M \in \mathbb{Q}^+$ be such that $M \geq \max\{|x-y|,4$ Then $|\gamma| \leq M/1$ and $B(w,1/M) \subseteq (\mathbb{R}^2-A) \cap ins(\bar{\gamma})$.

So $\langle\gamma\rangle \in T_A^M$ and hence $\rho(A) > 0$.

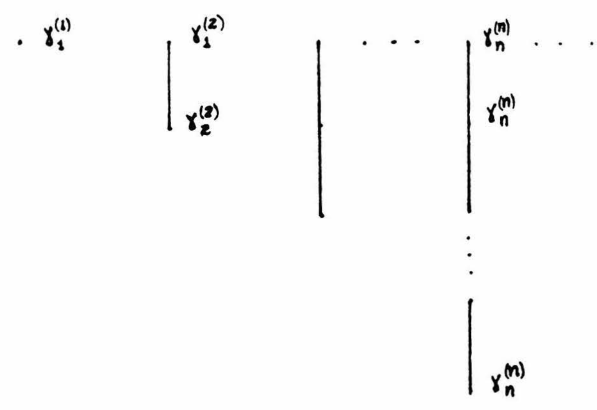
"$\Leftarrow$": Suppose $\rho(A) > 0$. Then there exist $M \in \mathbb{Q}^+$, $r \in \mathbb{R}^2$ and a Jordan arc γ in A such that $B(r,1/M) \subseteq (\mathbb{R}^2 - A) \cap \text{ins}(\bar{\gamma})$.



Let P be the path-component of A that contains γ. Then it is clear that P is not convex. So we are done. □

Proposition 6. Let $A \in JS$ and suppose that A is locally path-connected. Then $\rho(A) = 0$ or ω.

Proof. It will suffice to show that $\rho(A) \leq \omega$. Suppose $\rho(A) > \omega$. Then for some $M \in \mathbb{Q}^+$ we have $r(T_A^M) \geqslant \omega$. So T_A^M has a subtree as shown below.



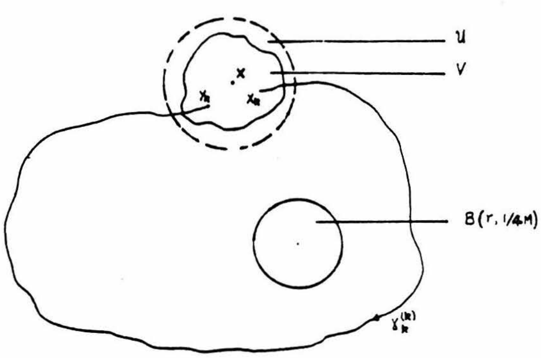
Here the $\gamma_j^{(i)}$'s are not necessarily distinct arcs. Let x_n be the initial point of $\gamma_n^{(n)}$ and y_n be the terminal point of $\gamma_n^{(n)}$. From the definition of the tree T_A^M we have that $|\gamma_n^{(n)}| \leq M/n$, so $|x_n - y_n| \leq M/n$. Let $r_n \in \mathbb{R}^2$ be such that for each n

$$B(r_n, 1/M) \subseteq (\mathbb{R}^2 - A) \cap \text{ins}(\bar{\gamma}_j^{(n)}) \quad \text{for } j = 1, \ldots, n.$$

Since $r_n \in \text{ins}(\bar{\gamma}_j^{(n)})$ it follows that $\{r_n\}_{n \in \mathbb{N}}$ is bounded because $\gamma_j^{(n)}$ is an arc in A. Let r be a limit point of $\{r_n\}_{n \in \mathbb{N}}$. By going to subsequences if necessary we may assume that $|r_n - r| < 1/2M$ for all $n \geqslant 2$. From this it immediately follows that

$$B(r, 1/2M) \subseteq (\mathbb{R}^2 - A) \cap \text{ins}(\bar{\gamma}_n^{(n)}) \quad \text{for all } n \geqslant 2.$$

Let x be a limit point of the sequence $\langle x_n \rangle_{n \in \mathbb{N}}$. Then $x \in A$ because A is closed. Let $U = B(x, 1/4M)$. Since A is locally path-connected there is an open set $V \subseteq U$ with $x \in V$ such that $V \cap A$ is path-connected. Since V is open and $x \in V$ we have for some large enough k that $x_k, y_k \in V$.



Since $V \cap A$ is path-connected there is a path in $V \cap A$ connecting x_k and y_k. This path together with $\gamma_k^{(k)}$ gives us a closed path γ in A with $B(r,1/2M) \subseteq (\mathbb{R}^2-A) \cap \mathrm{ins}(\gamma)$. Now we can extract from γ a Jordan loop γ' in A (as in [13] p. 29) such that $B(r,1/2M) \subseteq (\mathbb{R}^2-A) \cap \mathrm{ins}(\gamma')$. So $A \notin JS$, a contradiction. Hence the result follows.

□

Remarks. The converse of Proposition 6 is false. When we consider some natural examples in the next section we will see two that are not locally path-connected but which nonetheless have rank ω.

Definition. Let $A \subseteq \mathbb{R}^2$. We say that A has a *Hausdorff pseudo-cavity* if there exist $r \in \mathbb{R}^2$, $\varepsilon > 0$ and a sequence $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ of Jordan arcs in A such that for each $n \in \mathbb{N}$

(i) $B(r,\epsilon) \subseteq (\mathbb{R}^2-A) \cap \mathrm{ins}(\bar{\gamma}_n)$

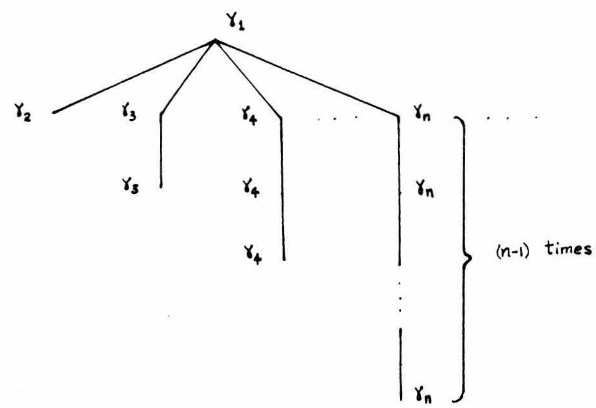
(ii) $\gamma_n \subseteq \gamma_{n+1}$, and

(iii) $|\gamma_n| \to 0$ as $n \to \infty$.

Proposition 7. Let $A \in JS$ and suppose that A has a Hausdorff pseudo-cavity. Then $\rho(A) \geqslant \omega\cdot 2$.

Proof. From the hypothesis we have that there exist $r \in \mathbb{R}^2$, $\varepsilon > 0$ and a sequence $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ of Jordan arcs in A such that $\gamma_n \subseteq \gamma_{n+1}$, $|\gamma| \to 0$ as $n \to \infty$ and $B(r,\epsilon) \subseteq (\mathbb{R}^2-A) \cap \mathrm{ins}(\bar{\gamma}_n)$. By considering a subsequence of $\langle\gamma_n\rangle_{n\in\mathbb{N}}$, if necessary we may assume that $|\gamma_n| \leq 1/n$ for all $n \in \mathbb{N}$.

Choose $M \in \mathbb{Q}^+$ such that $M \geqslant \max\{1/\varepsilon, \mathrm{diam}(A)\}$. Then $\|\gamma_n - \gamma_1\| \leq \mathrm{diam}(A) \leq M$ for all $n \in \mathbb{N}$. Now consider the tree, T shown below.



It is easy to see that T is a subtree of T_A^M. Moreover $r(T) = \omega + 1$. So $r(T_A^M) \geqslant \omega + 1$. Thus $\rho(A) \geqslant \omega + 2$ and hence by Proposition 4 we get that $\rho(A) \geqslant \omega\cdot 2$. □

<u>Remark</u>. The converse of Proposition 7 is also false. We shall give an example of a set in JS which has no Hausdorff pseudo-cavity but still has rank $\omega\cdot 2$ in the next section.

§2. <u>Some Natural Examples</u>. Before we give our natural examples we shall obtain two results which will aid us in calculating the ranks of some of the sets considered below.

<u>Lemma 8</u>. Suppose $A \in JS$ and $\rho(A) > \omega$. Then there is a disk $B(r,\delta)$ such that $\forall\varepsilon > 0$ there is a Jordan arc γ in A such that $|\gamma| < \varepsilon$ and $B(r,1/M) \subseteq (\mathbb{R}^2 - A) \cap \mathrm{ins}(\bar{\gamma})$.

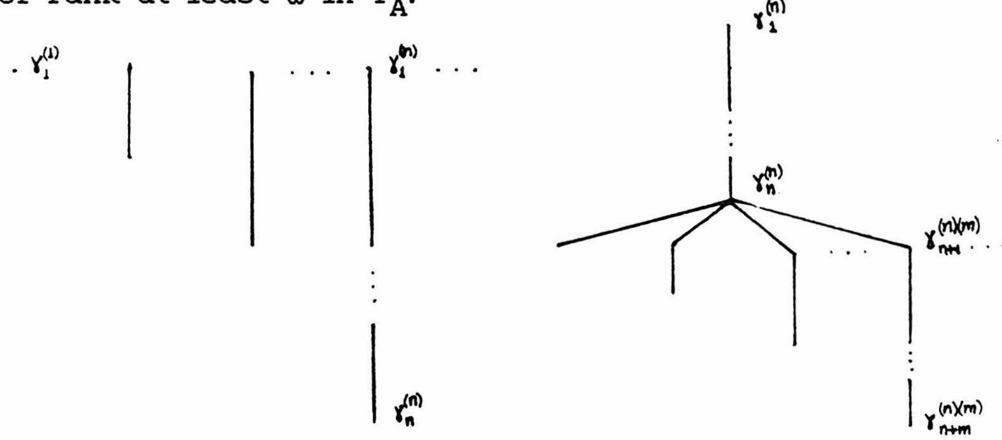
Proof. Suppose $\rho(A) > \omega$. Then for some $M \in \mathbb{Q}^+$, T_A^M has a subtree as shown in Proposition 6. Also as in Proposition 6 we can find an $r \in \mathbb{R}^2$ such that

$$B(r,1/2M) \subseteq (\mathbb{R}^2 - A) \cap \text{ins}(\bar{\gamma}_n^{(n)}) \quad \text{for all } n \geq 2.$$

Now take $\delta = 1/2M$. Since $|\gamma_n^{(n)}| \to 0$ as $n \to \infty$ the result follows immediately. □

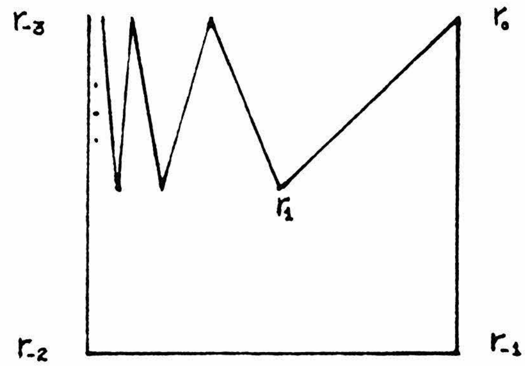
Lemma 9. Suppose $A \in JS$ and $\rho(A) > \omega \cdot 2$. Then for all $\varepsilon > 0$ there is a Jordan arc γ in A, such that $\forall \delta > 0$ there is a Jordan arc γ' in A with $\gamma \subseteq \gamma'$, $|\gamma'| < \delta$ and $\|\gamma - \gamma'\| < \varepsilon$.

Proof. Suppose $\rho(A) > \omega \cdot 2$. Then for some $M \in \mathbb{Q}^+$, T_A^M has a subtree as shown below where each of the nodes $\langle \gamma_1^{(n)}, \ldots, \gamma_n^{(n)} \rangle$ is of rank at least ω in T_A^M.



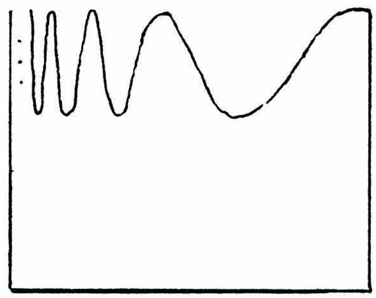
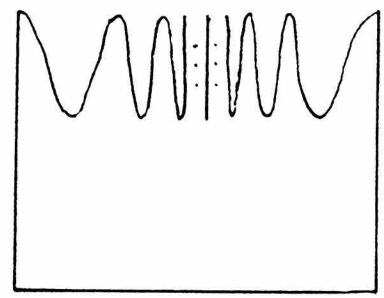
Now choose n such that $1/n < \varepsilon$ and let γ be $\gamma_n^{(n)}$. Choose m also so that $1/(n+m) < \delta$ and let γ' be $\gamma_{n+m}^{(n)(m)}$. Then from the properties of T_A^M it follows readily that γ and γ' fulfill the hypotheses imposed on them. □

<u>The Hausdorff saw</u>: Let $r_n(n=-3,-2,-1,...)$ be the following points in the plane $\mathbb{R}^2$: $r_{-3} = (0,1/2)$, $r_{-2} = (0,-3/2)$, $r_{-1} = (2,-3/2)$ and $r_n = (2\cdot 2^{-n},1/2\cdot(-1)^n)$ for $n \geqslant 0$. Let A be the union of the line segments $[r_n , r_{n+1}]$ $(n = -3,-2,-1,...)$. Then it is easy



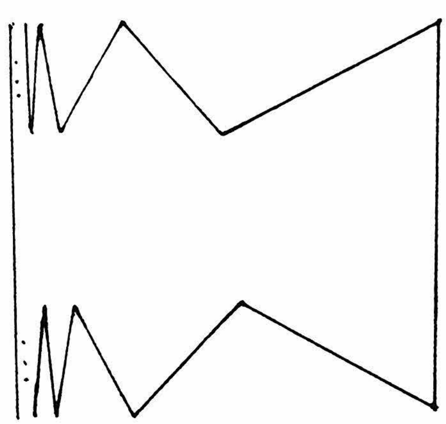
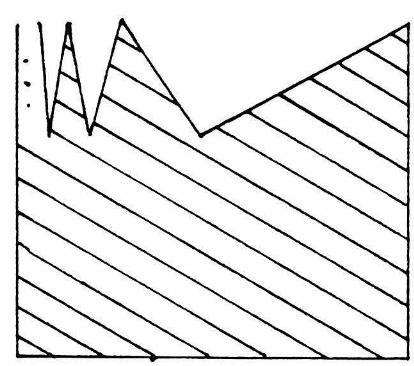
to see that A is an example of a compact simply-connected set which is not locally connected. The set A was first considered by Hausdorff (see [19] p. 180). Now it is clear that A has a Hausdorff pseudo-cavity so $\rho(A) \geqslant \omega\cdot 2$. And on the other hand we easily see that $\rho(A) \leq \omega\cdot 2$ because of lemma 9. Hence $\rho(A)$ is exactly $\omega\cdot 2$.

<u>Some other examples of rank exactly</u> $\omega\cdot 2$: Another commonly known example of a set of rank $\omega\cdot 2$ is the "sin(1/x)-circle" (also called the Warsaw circle)(see [42] p. 321). The sin(1/x)-circle consists of the graph of the function $f(x) = \sin(1/x)$ on the interval $(0,2/\pi]$ together with the line segments shown below on the left.

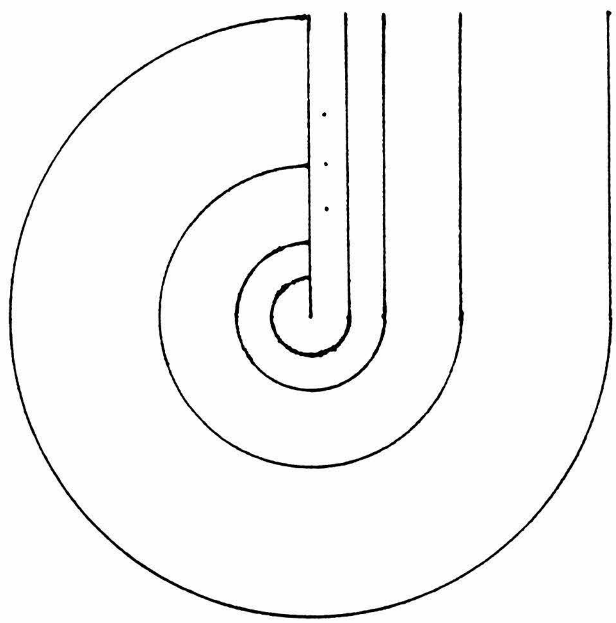
 

It is clear that the sin(1/x)-circle is homeomorphic to the Hausdorff saw. A set of rank $\omega \cdot 2$ which is not homeomorphic to the Hausdorff saw is shown above on the right. This set is a particular compactification of the real line $\mathbb{R}$, with an arc as the remainder (see [42] p. 321). The argument that this set has rank $\omega \cdot 2$ is identical to that for the case of the Hausdorff saw.

<u>Some sets of rank ω which are not locally path-connected</u>. The first example is essentially a two-sided Hausdorff saw and is shown below on the left. Another example is the Hausdorff saw together with its inside. (It is easy to show that the Hausdorff saw separates the plane into two open components exactly one of which is bounded. The bounded component is called the inside.)

The final example is much more complicated. Let A be the set consisting of the segements, $[r_n,s_n]$: where $r_n = (2^{-n+1},0)$, $s_n = (2^{-n+1},1)$ for $n \geq 1$ and $r_0 = (0,0)$ and $s_0 = (1,0)$; and the three-quarter circles of radii 2^{-n} ($n \in \mathbb{N}$) with centres at the origin and the missing quarters being in the first quadrant (see [32] p. 175). Then A



is a compact simply-connected set which is not locally path-connected. Now since none of these three sets is convex it follows from Proposition 5 that they must be of rank at least ω. But Lemma 8 also shows that these sets must be of rank at most ω. So they have rank exactly ω.

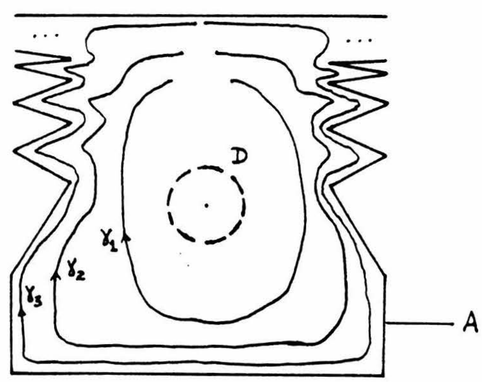
A set of rank $\omega + 1$: Consider the two-sided Hausdorff saw A. Let $\langle \gamma_n \rangle_{n \in \mathbb{N}}$ be a sequence of Jordan arcs in the inside of A which

(i) are pairwise disjoint and also disjoint from A

(ii) tend to A as $n \to \infty$, and

(iii) are such that $|\gamma_n| \to 0$ ad $n \to \infty$.

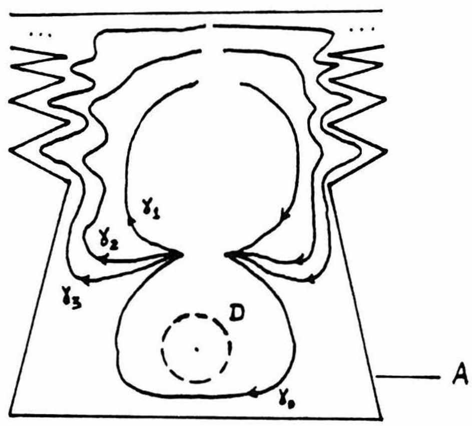
Let B be the set consisting of A and the γ_n's. (Here we follow the customary abuse of language and identify γ_n with its image.) Then B has rank $\omega + 1$. This example was essentially suggested to us by R. Edwards. Instead of the two-sided saw A, Edwards used a pseudo-arc (i.e., a set which separates the plane into two open components but which itself contains no Jordan arc).



First observe that for some large enough M the tree T_B^M is of rank ω. This is because there is a fixed disk D (shown in broken line above) such that $D \subseteq (\mathbb{R}^2 - B) \cap \text{ins}(\bar{\gamma}_n)$ for all n, and $|\gamma_n| \to 0$ as $n \to \infty$. Thus $\rho(B) \geq \omega + 1$.

Now from the definition of the tree T_B^M we know that the rank of T_B^M must be at most the supremum of the ranks of T_P^M where P is a path-component of B. So $r(T_B^M) \leq \omega$ for any $M \in \mathbb{Q}^+$, because each path-component P of B is either an arc or A, both of which have $r(T_P^M) < \omega$. Thus $\rho(B) \leq \omega + 1$. So we get that B has rank exactly $\omega + 1$.

<u>A set of rank $\omega\cdot 2$ which has no Hausdorff pseudo-cavity</u>: This is the last example we are going to give. Let B be the set constructed as in the previous example except that the arcs γ_n are all extensions of a Jordan arc γ_0 in the inside of A.



Then for some large enough $M \in \mathbb{Q}^+$ we see that T_B^M has rank at least $\omega + 1$. So $\rho(B) \geq \omega\cdot 2$ because of Proposition 4. Also from Lemma 9 it follows that B has rank at most $\omega\cdot 2$. Thus $\rho(B) = \omega\cdot 2$ and we are done.

<u>Remark</u>. Let A be a Hausdorff saw. We define the *amplitude* of A by

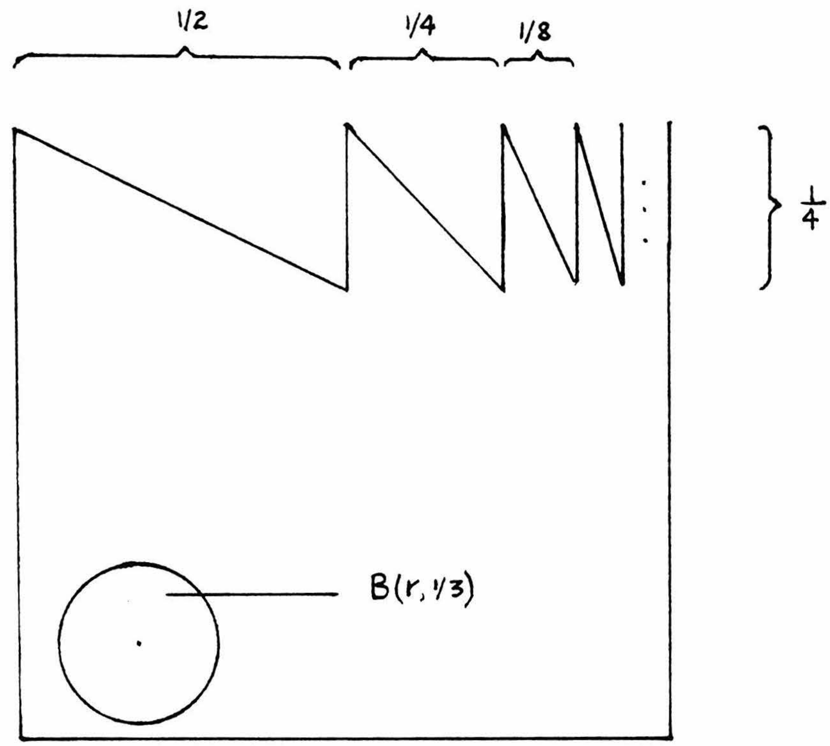
$$\text{amp}(A) = \liminf_{n\to\infty} \sup\{\|\gamma-\gamma'\|: \gamma' \subsetneq \gamma \text{ and } |\gamma|,|\gamma'| \leq 1/n \text{ where } \gamma \text{ and } \gamma' \text{ are Jordan arcs in } A\}$$

Then by considering Hausdorff saws with amplitudes decreasing to zero instead of Jordan arcs in the pen-ultimate example we can obtain a set of rank exactly $\omega\cdot 2 + 1$.

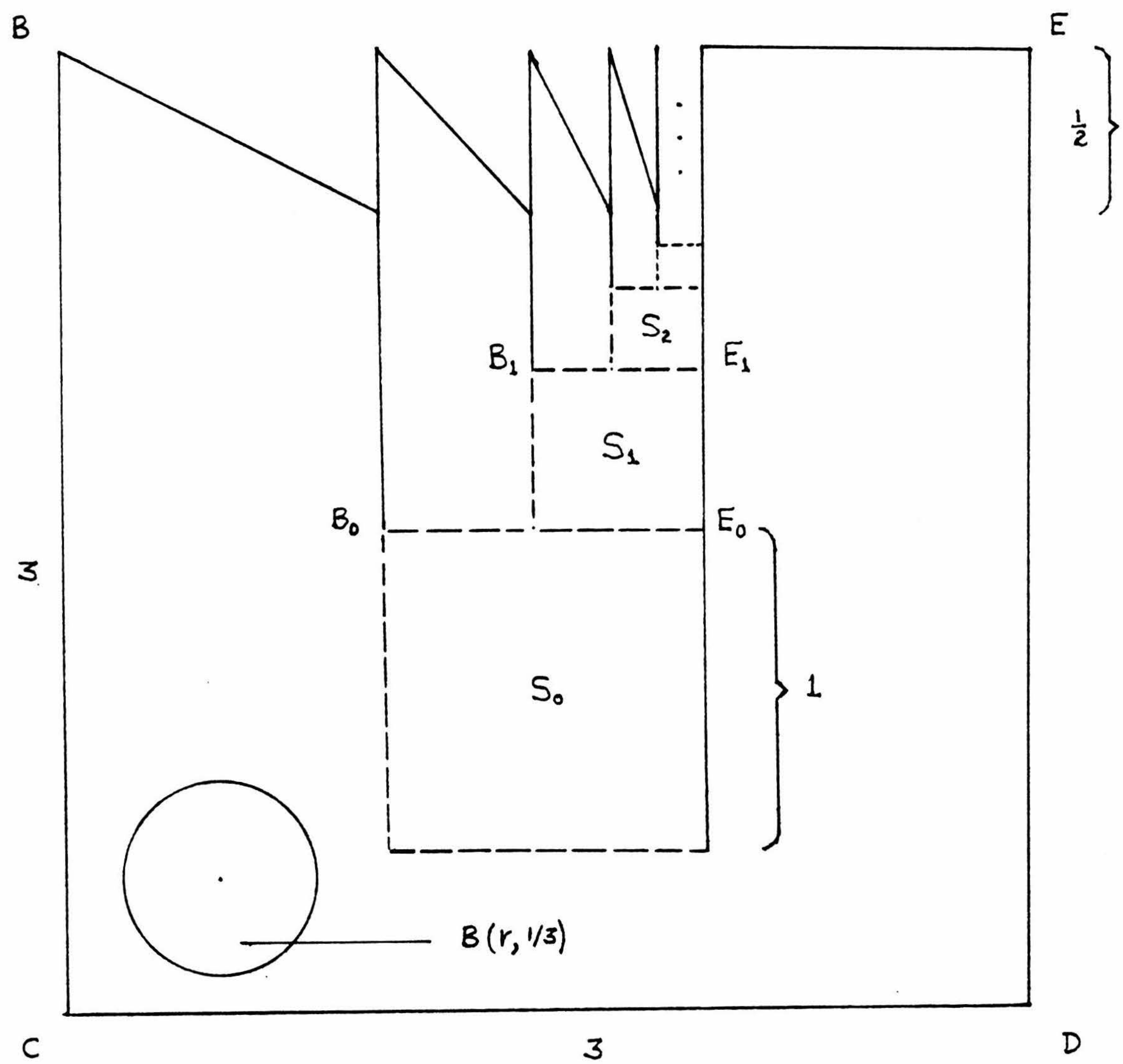
§3. <u>The Rank Function ρ is Unbounded in ω_1 on SC</u>. In this the last section of chapter 3 our main goal will be to prove the following result.

<u>Proposition 10</u>. The rank function ρ is unbounded in ω_1 on SC.

<u>Proof</u>. We will show by induction that for each $\alpha < \omega_1$ there is set $A_\alpha \in$ SC with $\rho(A_\alpha) \geqslant \omega \cdot \alpha$. For $\alpha = 0$, 1 and 2 the result is clear because of the natural examples presented in the previous section. We will first construct a set $A_3 \in$ SC such that $\rho(A_3) \geqslant \omega \cdot 3$. The induction steps for successor ordinals and for limit ordinals will essentially be the same as this construction. Let A_2 be the Hausdorff saw shown below.



Then a direct calculation shows easily that $r(T^3_{A_2}) \geqslant \omega + 1$. Let A_2' be the set obtained from A_2 by removing the line segments BC, CD and DE. Now let A_3 be the set obtained by inserting scaled copies of A_2' that fit exactly in the broken-line squares, S_n shown in the set below.



Observe that the squares S_n above have sides of length 2^{-n} $(n=0,1,2,\dots)$. Note also that the set A_3 is compact and simply-connected. We claim that $\rho(A_3) \geqslant \omega \cdot 3$. To prove this it will suffice to show that $r(T^3_{A_3}) \geqslant \omega \cdot 2 + 1$.

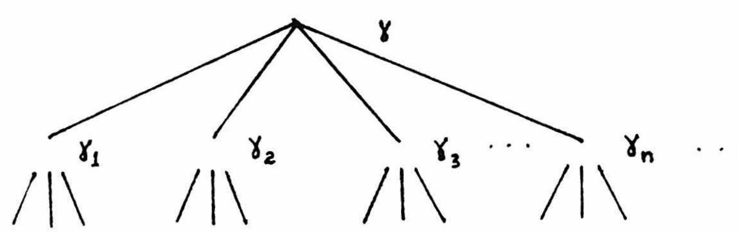
Let B_n and E_n be the top left and right vertices, respectively of the square S_n. Note that there is exactly one Jordan arc γ_n that connects B_n to E_n in A_3 (the broken lines are not parts of the set A_3). By considering the extensions of γ_n in the square S_n we see that

$$r(T^3_{A_3}) \geqslant r(T^{3\cdot 2^n}_{A_2}) .$$

This is because there is a scaled copy of A_2' in the square S_n and the set consisting of A_2' and the Jordan arc, γ_n is homeomorphic to the set A_2. Thus

$$r(T^3_{A_3}) \geqslant \sup\{r(T^{3\cdot 2^n}_{A_2}) : n \geqslant 0\} \geqslant \omega \cdot 2 .$$

Now each of the arc γ_n is an extension of the Jordan arc γ joining B and E. So we have a situation as shown below,



where the ranks of the nodes $\langle\gamma,\gamma_n\rangle$ in $T^2_{A_3}$ are unbounded in $\omega\cdot 2$. Thus $\langle\gamma\rangle$ has rank $\omega\cdot 2 + 1$ in $T^3_{A_3}$ and so $r(T^3_{A_3}) \geqslant \omega\cdot 2 + 1$. We thus have that $\rho(A_3) \geqslant \omega\cdot 3$.

Now suppose that the result is true for α, where $\alpha \geqslant 3$. We construct $A_{\alpha+1}$ exactly as in the case of A_3. Just let A'_α be the set obtained from A_α by removing the segments BC, CD and DE. Then insert scaled copies of A'_α in the squares S_n in the same set used for A_3. An identical argument shows that $\rho(A_{\alpha+1}) \geqslant \omega\cdot(\alpha+1)$. Finally suppose that the result is true for all $\alpha < \lambda$, where λ is a limit ordinal. Let $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ be an increasing sequence of ordinals with $\lim \alpha_n = \lambda$. We proceed as before by inserting a scaled copy of A'_{α_n} in the square S_n. We obtain easily once again that $\rho(A_\lambda) \geqslant \omega\cdot\lambda$. This completes the proof. □

<u>Remarks</u>. We can show with a bit more effort that the sets A_α obtained above have rank exactly $\omega\cdot\alpha$, whenever α is a successor ordinal. A slight refinement of the process at the limit ordinal stages will also produce a set of rank exactly $\omega\cdot\lambda$. (The set A_λ , that we constructed above, turns to have rank exactly $\omega\cdot(\lambda+1)$.)

<u>Corollary 11</u>. JS is a coanalytic but not Borel subset of K. SC is

not an analytic subset of K.

Proof. Suppose JS is an analytic subset of K. Let $\prec$ be the relation defined by

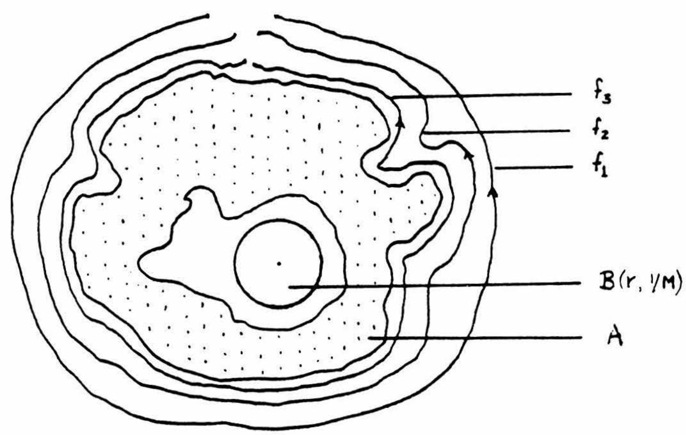
$$(A,M,\underset{\sim}{u}) \prec (B,N,\underset{\sim}{v}) \iff A = B,\ M = N \in \mathbb{Q}^+ \text{ and } \underset{\sim}{v} \text{ extends } \underset{\sim}{u} \text{ in } T_A^M.$$

Then it is easy to see that $\prec$ is a strict well-founded relation. So by Proposition 0.6, $\prec$ has countable length, α say. But this would mean that $\rho(A) \leq \alpha$ for any $A \in$ JS, which is a contradiction. So JS is not analytic. The same argument shows that SC is not analytic. □

We have seen that the rank function ρ, that we defined on JS, is a natural one and, although we haven't proved that ρ is a coanalytic norm, we were still able to get many of the analogous results in chapters 1 and 2. The question arises now as to whether ρ is a coanalytic norm. The difficulty here lies in the fact that the trees T_A^M (from which ρ is obtained) are "too big." It is very plausible however that ρ is a coanalytic norm so we make the following conjecture.

Conjecture. The rank function ρ is a coanalytic norm.

We will now describe briefly how to obtain a coanalytic norm ρ' on JS by making the trees T_A^M "smaller." ρ' is not very natural and not very easy to work with either. It will however be very easy to see that ρ' is a coanalytic norm. Fix, once and for all, a countable set $\mathcal{F}$, of Jordan arcs such that $\mathcal{F}$ is dense in the set of all Jordan arcs in the plane. For each $A \in K$, $M \in \mathbb{Q}^+$, we now define a tree S_A^M in a similar way to T_A^M. A typical node in S_A^M will be $<f_1,\ldots,f_n>$ as shown below.



The f_i's are chosen from $\mathcal{F}$ under conditions which will ensure that they converge to a closed curve in A. So the curve f_1 is chosen so that its distance from A is small and so that there is an open ball not in A which is in $\text{ins}(\bar{f}_1)$. f_2 is chosen closer to A and also close enough to f_1 so as to ensure that same open ball is contained in both $\text{ins}(\bar{f}_1)$ and $\text{ins}(\bar{f}_2)$.

The basic idea is to concoct S_A^M in such a way that if S_A^M has an infinite branch then $A \notin JS$. The converse result is always easy to obtain because $\mathcal{F}$ is dense in the set of Jordan arcs. The reason why the rank function ρ', so obtained, is a coanalytic norm is because $\mathcal{F}$ is countable. The proof is the same as that of Proposition 1.3.

Final Remarks. As was mentioned in the introduction a natural question is whether SC is coanalytic. Below is an idea of H. Becker of how one might go about proving this (if it is indeed true!). We have that $A \in SC \Leftrightarrow A \in JS$ and $\forall x,y \in A, \exists \gamma$ (γ is a path in A connecting x and y). So if the path γ can be chosen to be Borel in A it would follow that SC is coanalytic. (If the path γ can be chosen to be hyper-arithmetic in A (i.e., Δ^1_1 in A) then it will follow that SC is Π^1_1.) Now one way of showing that the path γ can be chosen in a Borel way is to use induction of the rank of A. The result is obvious for sets of rank 0 (because these are just the compact convex sets) but we have not been able to show that the induction step works. So the following question remains open.

Problem. Is the set SC coanalytic?

Chapter 4

Introduction. In this chapter we collect together a number of miscellaneous results about natural rank functions that are similar to those considered in the previous chapters.

Let $C = C[0,1]$ be the Polish space of all continuous real-valued functions on [0,1], and D be the subset of C consisting of all everywhere differentiable functions. (It is understood here that one sided derivatives are considered at the endpoints 0 and 1.) It was shown by S. Mazurkiewicz [38] that D is a complete coanalytic subset of C. Later A. S. Kechris and W. H. Woodin [26] defined a natural coanalytic norm on D, which we shall refer to as the *Kechris-Woodin rank function*. Also there is a process given by Denjoy [11] which recovers any function $f \in D$ from its derivative f' in α many steps, where α is a countable ordinal. This process provides another rank function on D, which is known as the *Denjoy rank function*. §1 is devoted to a comparison of the Kechris-Woodin and Denjoy rank functions. In §2 we show that there are functions of arbitrarily large Denjoy rank by using a simple construction. (This result was obtained before by A. Denjoy [11] but the constructions there were very complicated.) This provides another Rank Argument of the non-Borelness of D.

Let now $L_1(\mathbb{T})$ be the Polish space of all Lebesgue integrable functions on the unit circle $\mathbb{T}$, and CF be the set of functions in $L_1(\mathbb{T})$ with everywhere convergent Fourier series. Let also EC be the functions in CF that are continuous, i.e., EC = CF $\cap$ C($\mathbb{T}$). M. Ajtai and A. S. Kechris [01] showed that EC is a complete coanalytic subset of C($\mathbb{T}$). From this it follows easily that CF is a coanalytic but not Borel subset of $L_1(\mathbb{T})$. By specializing a construction of Z. Zalcwasser [54] (see also D. Gillespie and W. Hurewicz [17]), Ajtai and Kechris obtained a natural coanalytic norm on EC. This norm is referred to as the *Zalcwasser rank function*. In §3 we use the Zalcwasser rank to compare tests of everywhere convergence of Fourier series. It is well known that the Dini and Jordan tests are non-comparable but it is clear that in some sense the Dini test is much more powerful. We make this sense precise by associating with each test an ordinal, called its *strength*, by using the Zalcwasser rank function. Finally in §4 we show that the Zalcwasser rank function is unbounded in ω_1 on CF. This provides a Rank Argument of the non-Borelness of CF.

§1. <u>A Comparison of the Kechris-Woodin and Denjoy Rank Functions</u>. For the details of the constructions of these rank functions we refer to [26] and [08] p. 96.

Below we give a brief description. First is the Kechris-Woodin rank function. Let f $\in$ C and $\varepsilon \in \mathbb{Q}^+$. For a closed subset P of [0,1] we define $P'_{\varepsilon,f}$ by

$$P'_{\varepsilon,f} = \{x \in P: \forall \text{ open nghbd } U \text{ of } x\ \exists I, J \subseteq U$$

with I, J ϵ $\bar{Q}[0,1]$ and $I \cap J \cap P \neq \emptyset$

such that $|\Delta_f(I)-\Delta_f(J)| \geqslant \varepsilon\}$

The notation here is as in chapter 1 ($\bar{Q}[0,1]$ is the set of all closed subintervals of [0,1] with rational endpoints and length >0, etc.). It is easy to see that $P'_{\varepsilon,f}$ is closed. We may thus define by induction a sequence $\langle P^{\alpha}_{\varepsilon,f}\rangle_{\alpha\epsilon ORD}$ by putting

$$P^{0}_{\varepsilon,f} = [0,1] \qquad P^{\alpha+1}_{\varepsilon,f} = (P^{\alpha}_{\varepsilon,f})'_{\varepsilon,f}$$

$$P^{\lambda}_{\varepsilon,f} = \underset{\alpha<\lambda}{\S sS}\ P^{\alpha}_{\varepsilon,f} \quad \text{for } \lambda \text{ a limit ordinal.}$$

If f ϵ D then $P'_{\varepsilon,f}$ is nowhere dense in P, so $P^{\alpha}_{\varepsilon,f}$ is a strictly decreasing sequence of closed sets. Thus there exists a least countable ordinal $\alpha(\varepsilon,f)$ such that $P^{\alpha}_{\varepsilon,f} = \emptyset$ for all $\alpha \geqslant \alpha(\varepsilon,f)$. The Kechris-Woodin rank function is now defined by

$$|f|_{K-W} = \text{least } \alpha \text{ such that } P^{\alpha}_{\varepsilon,f} = \emptyset, \text{ for all } \varepsilon \in \mathbb{Q}^{+}$$
$$= \sup\{\alpha(\varepsilon,f) : \varepsilon \in \mathbb{Q}^{+}\}.$$

We now describe the Denjoy rank function. Let E be a closed subset of [0,1] and g be a measurable function on [0,1]. We define the set S 'E) by

S 'E) = {x ϵ E: for all intervals I with x ϵ I,

g is not integrable on $I \cap E\}$.

$S_g(E)$ is called the set of *non-summability* points of g with respect to E. For f ϵ C we define also the set $G_f(E)$ by

$G_f(E)$ = {x ϵ E: for all intervals I with x ϵ I,

$\Sigma_I |f(b_n)-f(a_n)|$ diverges}

where $\{(a_n,b_n)\}_{n\in\mathbb{N}}$ is the sequence of open intervals complimenting E in [0,1] and Σ_1 denotes that the summation is taken over only those intervals (a_n,b_n) that intersect I. $G_f(E)$ is called the set of *divergence points* of f with respect to E. It is easy to see that $S_g(E)$ and $G_f(E)$ are both closed subsets of E. Moreover if $f \in D$ then $S_{f'}(E)$ and $G_f(E)$ are both nowhere dense in E. (Here f′ denotes the derivative of E.) We can thus define a sequence of closed sets $\langle D_f^\alpha \rangle_{\alpha\in ORD}$ for a given $f \in D$ as follows:

$$\text{Put } D_f^0 = [0,1],\ D_f^{\alpha+1} = S_{f'}(D_f^\alpha) \cup G_f(D_f^\alpha)$$

$$\text{and } D_f^\lambda = \bigcup_{\alpha<\lambda} D_f^\alpha \text{ for } \lambda \text{ a limit ordinal.}$$

As before we see that there is a least countable ordinal, $\alpha(f)$ such that $D_f^\alpha = \emptyset$ for all $\alpha \geqslant \alpha(f)$. The Denjoy rank function is now defined by $|f|_{DJ} = \alpha(f)$. Our goal in this section will be to show that $|f|_{DJ} \leq |f|_{K-W}$. We first prove the following result.

Lemma 1. Let $f \in D$. Then $x \in D_f^\alpha \Rightarrow \forall \varepsilon \in \mathbb{Q}^+(x \in P_{\varepsilon,f}^\alpha)$.

Proof. We prove the result by induction on α. For $\alpha = 0$, there is nothing to prove because $D_f^0 = [0,1]$, and $P_{\varepsilon,f}^0 = [0,1]$ for all $\varepsilon \in \mathbb{Q}^+$. Also if the result for all $\alpha < \lambda$, where λ is a limit ordinal then it follows easily for λ. So we only need to deal with the successor ordinal case now.

Suppose the result is true for α. Let $x \in D_f^{\alpha+1}$. Fix $\varepsilon \in \mathbb{Q}^+$. We shall show that $x \in R_{\varepsilon,f}^{\alpha+1}$. To this end fix an open neighbourhood U of x. We need to show that $\exists I,J \subseteq U$ with $I,J \in$

$\bar{Q}[0,1]$ and $I \cap J \cap P^{\alpha}_{\varepsilon,f} \neq \emptyset$ such that $|\Delta_f(J)-\Delta_f(I)| \geq \varepsilon$. We have two cases: (i) $x \in S_{f'}(D^{\alpha}_f)$, and (ii) $x \in G_f(D^{\alpha}_f)$.

Case (i): Suppose $X \in S_{f'}(D^{\alpha}_f)$. Then for any interval I with $x \in I$, f' is not integrable over $I \cap D^{\alpha}_f$. In particular f' is unbounded on $I \cap D^{\alpha}_f$ whenever $x \in I$. Choose $I \subsetneq U$ with $I \in \bar{Q}[0,1]$. As f' is unbounded on $I \cap D^{\alpha}_f$ there is a $y \in I \cap D^{\alpha}_f$ such that $|f'(y)-\Delta_f(I)| \geq \varepsilon + 1$. Also we can choose $J \subsetneq U$ with $J \in \bar{Q}[0,1]$ and $y \in J$ such that $|\Delta_f(J)-f'(y)| \leq 1$ from the definition of the derivative f'. We thus get

$$|\Delta_f(J)-\Delta_f(I)| \geq \Delta_f|f'(y)-\Delta_f(I)| + -|\Delta_f(J)-f'(y)|$$
$$\geq \varepsilon + 1 - 1 = \varepsilon$$

By the induction hypothesis $D^{\alpha}_f \subsetneq P^{\alpha}_{\varepsilon,f}$, so $y \in I \cap J \cap P^{\alpha}_{\varepsilon,f}$ and hence $I \cap J \cap P^{\alpha}_{\varepsilon,f} \neq \emptyset$.

Case (ii): Suppose $x \in G_f(D^{\alpha}_f)$. Then for any interval I with $x \in I$, $\Sigma|f(b_n)-f(a_n)| = +\infty$. Here $\{a_n,b_n)\}$ are the intervals complementing D^{α}_f in [0,1] that intersect I. Choose $I \subsetneq U$ with $x \in I$ and $I \in \bar{Q}[0,1]$. As $\Sigma|f(b_n)-f(a_n)| = +\infty$ we must have

$$\limsup_{n\to\infty}\left|\frac{f(b_n)-f(a_n)}{b_n - a_n}\right| = \limsup_{n\to\infty}|\Delta_f([a_n,b_n])|$$
$$= +\infty.$$

So there is an $N \in \mathbb{N}$ such that $|\Delta_f([a_N,b_N])-\Delta_f(I)| \geq \varepsilon + 1$. By the continuity of f we can choose $J \subsetneq U$ with $J \in \bar{Q}[0,1]$ and $J \supseteq [a_N,b_N]$ such that $|\Delta_f([a,b])-\Delta_f(J)| \leq 1$. We thus get

$$|\Delta_f(J)-\Delta_f(I)| \geq |\Delta_f([a_N,b_N])-\Delta_f(I)|-|\Delta_f([a_N,b_N])-\Delta_f(J)|$$
$$\geq \varepsilon + 1 - 1 = \varepsilon$$

Now I must contain at least one of the endpoints a_N,b_N (otherwise $(a_N,b_N) \cap I = \emptyset$, a contradiction). So at least one of a_N,b_N is in $I \cap J \cap D_f^\alpha$. Since $D_f^\alpha \subseteq P_{\varepsilon,f}^\alpha$ by the induction hypothesis, we see that $I \cap J \cap P_{\varepsilon,f}^\alpha \neq \emptyset$. This completes the proof.

□

Proposition 2. For each $f \in D$, $|f|_{DJ} \leq |f|_{K-W}$.

Proof. Suppose $|f|_{K-W} = \alpha$. Then for all $\varepsilon \in \mathbb{Q}^+$, $P_{\varepsilon,f}^\alpha = \emptyset$, and consequently $D_f^\alpha = \emptyset$. Thus $|f|_{DJ} \leq \alpha = |f|_{K-W}$ and we are done. □

Remarks. It is easy to see that $|f|_{DJ} = 1$ iff f' is integrable and it was shown in [26] that $|f|_{K-W} = 1$ iff f' is continuous. Now consider function f given by

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

Then f' is bounded in [0,1] and so is integrable. Thus $|f|_{DJ} = 1$. But it was shown in [26] that $|f|_{K-W} = 2$, so the converse of Proposition 2 is false. In fact much more than this can be said. Let $BD_1 = \{f \in D: |f'(x)| \leq 1 \text{ for all } x \in [0,1]\}$. It was shown in [26] that the Kechris-Woodin rank function is unbounded in ω_1 on BD_1. But each function in BD_1 is integrable, so $|f|_{DJ} = 1$, for all $f \in BD_1$. We thus get that for each $\alpha \leq \omega_1$, there is an f_α such that $|f|_{DJ} = 1$ but $|f|_{K-W} \geq \alpha$.

§2. The Denjoy Rank Function is Unbounded in ω_1 on D. The main goal of this section is to prove the following result.

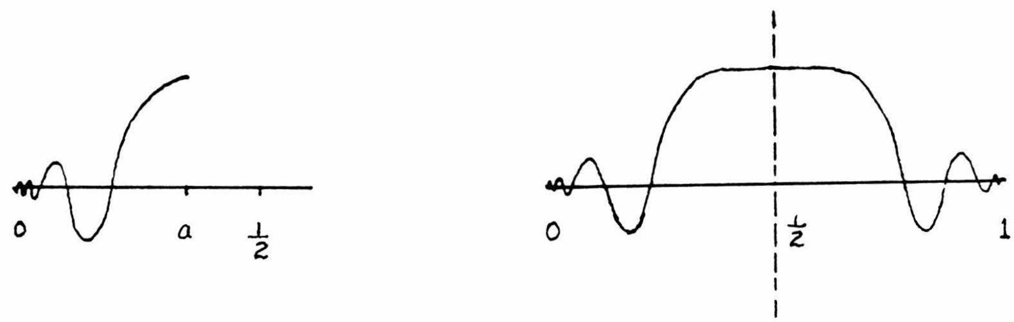
Proposition 3 (Denjoy). For each $\alpha < \omega_1$, there is an $f \in D$ such that $|f|_{DJ} \geq \alpha$.

Proof. We prove the result by induction on α. For $\alpha = 1$ there is nothing to prove. For $\alpha = 2$ we consider the function f_2 given by

$$f_2(x) = \begin{cases} x^2 \sin(x^{-2}) & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

It is a standard fact that f_2' is not integrable on any interval which contains the point $x = 0$. In fact we have $S_{f_2'}([0,1]) = \{0\}$ and $G_{f_2}[0,1] = \emptyset$. So $D^1_{f_2} = \{0\}$ and $D^2_{f_2} = \emptyset$. Thus $|f_2|_{DJ} = 2$.

We shall now construct a function f_3 such that $|f_3|_{DJ} = 3$. The same construction works for the successor ordinal of our induction, and with a slight and obvious modification for limit ordinal cases as well. Let $x = a$ be the first maximum of f_2 to the left of $x = 1/2$, and g be the function obtained by reflecting the graph of $f_2 \upharpoonright [0,a]$ in the line $x = 1/2$ and by joining the two maxima with a straight line.



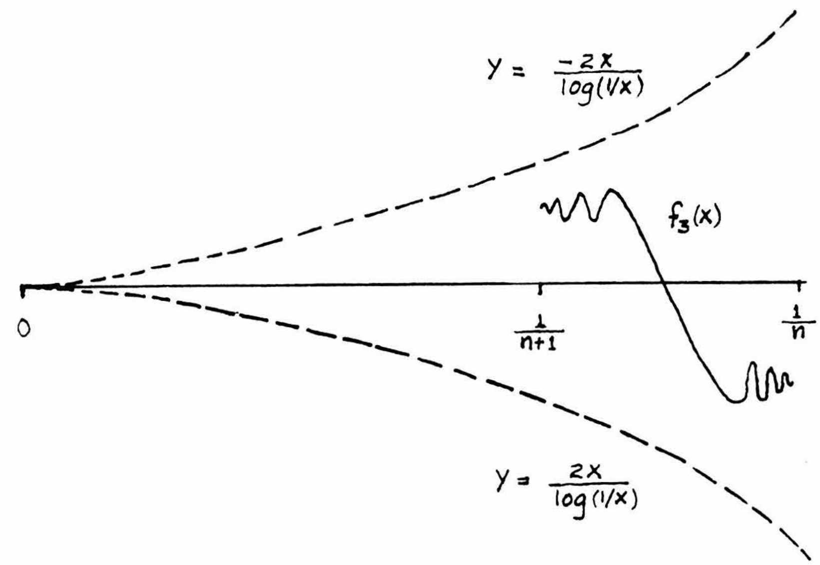
Now choose a differentiable function h on $[0,1]$ such that h' is continuous on $(0,1]$ and

$$h(\tfrac{1}{n}) = \frac{(-1)^n}{n \log n} \quad \text{for all } n \in \mathbb{N}, \text{ and}$$

$$\frac{2x}{\log(1/x)} \le h(x) \le \frac{-2x}{\log(1/x)} \quad \text{for } x \in \ 0,1).$$

Let g_n be a scaled copy of g that fits exactly in the interval $[1/(n+1),1/n]$. We define the function f_3 by

$$f_3(x) = \begin{cases} h(x) + g_n(x) & x \in [1/(n+1),1/n] \\ 0 & x = 0 \end{cases}$$



It is easy to see that $f_3 \in D$. It is also clear that $D^1_{f_3} = S_{f_3'}([0,1]) = \{0,1,1/2,1/3,...\}$. So $S_{f_3'}(D^1_{f_3}) = \emptyset$ because $D^1_{f_3}$ is countable. Also $G_{f_3}(D^1_{f_3}) = \{0\}$ by the choice of h. So $D^2_{f_3} = \{0\}$ and hence $D^3_{f_3} = \emptyset$. Thus $|f_3|_{DJ} = 3$. This completes the proof of the result.

□

<u>Remark</u> <u>1</u>. Using the process above we can actually check that at the successor ordinals we get a function $f_{\alpha+1}$ with $|f_{\alpha+1}|_{DJ} = \alpha + 1$. However at a limit ordinal λ we get a function f_λ with $|f_\lambda|_{DJ} = \lambda + 1$. The reason for this is because by its definition the Denjoy rank can never be a limit ordinal.

Remark 2. It was pointed out in [26] that although the Denjoy rank function is not a coanalytic norm, D and $|\cdot|_{DJ}$ satisfy condition (i) of Proposition 0.2. This together with Proposition 3 provides another Rank Argument of the non-Borelness of D.

§3. The Zalcwasser Rank Function and Tests for Everywhere Convergence of Fourier Series. We shall first give a brief description of the Zalcwasser rank function. For more details we refer to [01]. Let $\underset{\sim}{f} = \langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of functions in $C(\mathbb{T})$, and let P be a closed subset of $\mathbb{T}$. We define the *oscillation* $\omega_P(\underset{\sim}{f};x)$, of $\underset{\sim}{f}$ at x with respect to P by

$$\omega_P(x) = \inf_{\delta} \inf_{k} \sup_{m,n\geq k} \{|f_m(y)-f_n(y)| : |y-x|<\delta, y\in P\}.$$

Now define $P'_{\varepsilon,\underset{\sim}{f}}$ by $'_{\varepsilon,\underset{\sim}{f}} = \{x\in P : \omega_P(\underset{\sim}{f};x)\geq\varepsilon\}$. If $\langle f_n \rangle$ is an everywhere convergent sequence then it can be shown that $P'_{\varepsilon,\underset{\sim}{f}}$ is closed and nowhere dense in P. We can thus define a decreasing sequence $Z^{\alpha}_{\varepsilon,\underset{\sim}{f}}$ by induction as follows:

$$\text{Put } Z^{0}_{\varepsilon,\underset{\sim}{f}} = \mathbb{T}, \quad Z^{\alpha+1}_{\varepsilon,\underset{\sim}{f}} = (Z^{\alpha}_{\varepsilon,\underset{\sim}{f}})'_{\varepsilon,\underset{\sim}{f}}$$

and $Z^{\lambda}_{\varepsilon,\underset{\sim}{f}} = \bigcap_{\alpha<\lambda} Z^{\alpha}_{\varepsilon,\underset{\sim}{f}}$ for λ a limit ordinal.

As in the case of the Kechris-Woodin rank function we see that there is a least countable ordinal $\alpha(\varepsilon,\underset{\sim}{f})$ such that $Z^{\alpha}_{\varepsilon,\underset{\sim}{f}} = \emptyset$ for all $\alpha \geqslant \alpha(\varepsilon,\underset{\sim}{f})$. We then define the Zalcwasser rank function by

$$|\underset{\sim}{f}|_Z = \text{least } \alpha \text{ such that } Z^{\alpha}_{\varepsilon,\underset{\sim}{f}} = \emptyset \text{ for all } \varepsilon \in \mathbb{Q}^+$$

$$= \sup\{\alpha(\varepsilon,\underset{\sim}{f}) : \varepsilon \in \mathbb{Q}^+\}$$

The Zalcwasser rank function applies naturally to the set CF of all functions in $L_1(\mathbb{T})$ with everywhere convergent Fourier series by letting

$$|f|_Z = |\langle S_n(f)\rangle|_Z \text{ for } f \in CF$$

where $S_n(f)$ are the partial sums of the Fourier series of f.

Let now $\mathfrak{T}$ be a test for everywhere convergence of Fourier series. We define the strength, $S(\mathfrak{T})$ of $\mathfrak{T}$ by

$$S(\mathfrak{T}) = \sup\{|f|_Z+1 : \mathfrak{T} \text{ shows that the Fourier series of } f \text{ is everywhere convergent}\}$$

The aim of this section is to compute the strengths of the

Lipschitz, Jordan and Dini tests for everywhere convergence. For convenience we give the tests below.

Jordan Test: If f is of bounded variation on $\mathbb{T}$ then the Fourier series of f converges everywhere.

Dini Test: If at each point x of $\mathbb{T}$ the integral $\int_0^{\delta_x} \frac{f(x+t)+f(x-t)-2f(x)}{t}\,dt$ converges for some $\delta_x > 0$, then the Fourier series of f converges everywhere.

Lipschitz Test: If f is continuous on $\mathbb{T}$ and the modulus of continuity $\mu(\delta)$ satisfies $\mu(\delta)\log \delta \to 0$ as $\delta \to 0$ then the Fourier series of f converges uniformly on $\mathbb{T}$.

We will need the following definition and Theorem in calculating the strength of the Jordan test.

Definition: A sequence of functions $\langle f_n \rangle$, defined in a neighbourhood of x_0 and converging for $x = x_0$, is said to *converge continuously at* x_0 to a limit function f , defined in a neighbourhood of x_0, if $\forall \varepsilon > 0\ \exists\ \delta > 0$ and $N \in \mathbb{N}$ such that for all x with $|x-x_0| < \delta$ and all $n \geq N$ we have $|f_n(x)-f(x)| < \varepsilon$.

Theorem C. Let f be a function of bounded variation on $\mathbb{T}$. Then

$S_n(f)$ exhibits the Gibbs phenomenon at each essential (non-removable) point of discontinuity and converges continuously at each of the other points.

Proof. See [55] p. 61-62. □

Lemma 4. Let f be a function of bounded variation on $\mathbb{T}$. Then

$$|f|_Z = \begin{cases} 1 \text{ if f has no essential discontinuity} \\ 2 \text{ otherwise} \end{cases}$$

Proof. If f has no essential discontinuity then $S_n(f)$ is the same as the Fourier series of continuous function on $\mathbb{T}$ which is of bounded variation. So $S_n(f)$ converges uniformly in $\mathbb{T}$. But from [01] we know that $|f|_Z = 1$ iff $S_n(f)$ converges uniformly. So we are finished with the first part.

Now suppose f has an essential discontinuity. Then $|f|_Z \geqslant 2$. Let $\{x_n\}$ be the countably many points of essential discontinuity of f and $d_n = |\lim_{x \to x_n+} f(x) - \lim_{x \to x_n-} f(x)|$ be their respective jumps. Then by Theorem C we get that the oscillation of the sequence $S_n(f)$ in $\mathbb{T}$ is given by

$$\omega_{\mathbb{T}}(x) = \begin{cases} \ell d_n \text{ if } x = x_n \\ 0 \text{ otherwise,} \end{cases}$$

where ℓ is the Gibbs constant. Since f is of bounded variation there are finitely many x_n's , or there are infinitely many x_n's and $d_n \to 0$ as $n \to \infty$. So for a fixed $\varepsilon < 0$, $Z^1_{\varepsilon,f} = \{x : \omega_{\mathbb{T}}(x) \geqslant \varepsilon\}$ is a finite set. This shows that $Z^2_{\varepsilon,f} = \emptyset$ for each fixed $\varepsilon > 0$. Thus $|f|_Z \leq 2$ and we are done. □

<u>Proposition</u> <u>5</u>: Let $\mathfrak{J}_L$, $\mathfrak{J}_J$ and $\mathfrak{J}_D$ be the Lipschitz, Jordan and Dini tests respectively. Then $S(\mathfrak{J}_L) = 2$, $S(\mathfrak{J}_J) = 3$ and $S(\mathfrak{J}_D) = \omega_1$.

<u>Proof</u>. For $\mathfrak{J}_L$ there is nothing to prove because $\mathfrak{J}_L$ is a test for uniform convergence. The result for $\mathfrak{J}_J$ follows immediately from Lemma 4. So we need to show that $S(\mathfrak{J}_D) = \omega_1$. Consider the set $D(\mathbb{T})$ of all everywhere differentiable functions on $\mathbb{T}$. As indicated in [26] it can be shown that the Zalcwasser rank function is unbounded in ω_1 on $D(\mathbb{T})$. But for any $f \in D(\mathbb{T})$, the Dini test shows that $S_n(f)$ converges everywhere in $\mathbb{T}$. So we get that $S(\mathfrak{J}_D) = \omega_1$. □

§4. <u>The Zalcwasser Rank Function is Unbounded in ω_1 on CF</u>.

This final section of Chapter 4 is devoted to proving the following result.

<u>Proposition</u> <u>6</u>. For each $\alpha < \omega_1$, there is an $f \in CF$ such that $|f|_Z \geqslant \alpha$.

<u>Proof</u>. The proof is once again by induction on α. For $\alpha = 1$ there is nothing to prove. For $\alpha = 2$ we use a well known example of a function whose Fourier series converges everywhere but not

uniformly. Using this function we construct by a general process, a function of rank 3. The general case for successor ordinals and limit ordinals is essentially the same and is carried out as in Proposition 3.

Let $a \in \mathbb{N}$ and put $n_k = a^{k^2}$. Define f_a by

$$f_a(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \bar{Q}(x,n_k)$$

where $\bar{Q}(x,n)$ is the Fejér trigonometric polynomial given by

$$\bar{Q}(x,n) = \sum_{i=1}^{n} \frac{\sin(2n-i)x}{i} - \sum_{i=1}^{n} \frac{\sin(2n+i)x}{i}$$

It is easy to see that f_a is continuous and for sufficiently large a it is shown in [04] p. 125-127 that $S_n(f_a)$ converges everywhere, but not uniformly in any interval containing $x = 0$. From this it follows easily that $|f_a|_Z = 2$.

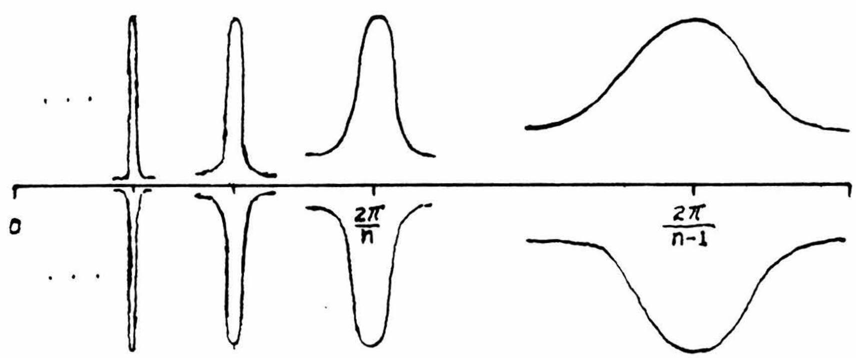
Now let $g_a = b \cdot f_a$ where $b = b(a)$ is chosen so that

$$\sup\{|S_k(g_a;x)| : x \in \mathbb{T}, k \in \mathbb{N}\} = 1$$

Observe that for a given open interval I with $0 \in I$, we can make the partial sums $S_k(g_a;x)$ arbitrarily small outside I by taking a large enough. Let $\langle I_n \rangle$ be a sequence of disjoint open intervals with $2\pi/n \in I_n$. Choose a_n large enough so that

$$|S_k(h_{a_n};x)| \le 2^{-n} \text{ for all } k \in \mathbb{N}, \text{ all } x \notin I_n$$

where the function h_{a_n} is defined by $h_{a_n}(x) = g_{a_n}(x-2\pi/n)$.



Now observe that the sum, $\sum_{n=1}^{\infty} S_k(h_{a_n};x)$ converges for each k, and that $f_3(x) = \lim_{k\to\infty} \sum_{n=1}^{\infty} S_k(h_{a_n};x)$ is a well defined function. It is easy to verify that $|f_3|_Z = 3$. This completes the proof. □

<u>Remark</u>. Observe that CF and the Zalcwasser rank function $|\cdot|_Z$, satisfy condition (i) of Proposition 0.2. This together with Proposition 6 gives a Rank Argument of the non-Borelness of CF.

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(All articles and Journals in Russian are transliterated using the British 2979 System. Everything else is as it is.)

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