Some Results on Projective Equivalence Relations

Thesis By

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Abstract

We construct a Π^1_1 equivalence relation E on ω^ω for which there is no largest E-thin, E-invariant Π^1_1 subset of ω^ω. Then we lift our result to the general case. Namely, we show that there is a Π^1_{2n+1} equivalence relation for which there is no largest E-thin, E-invariant Π^1_{2n+1} set under projective determinacy. This answers an open problem raised in Kechris [Ke2].

Our second result in this thesis is a representation for thin Π^1_3 equivalence relations on u_ω. Precisely, we show that for each thin Π^1_3 equivalence relation E on u_ω, there is a Δ^1_3 in the codes map $p\colon \omega^\omega \to u_\omega$ and a Π^1_3 in the codes equivalence relation e on u_ω such that for all real numbers x and y,

$$xEy \iff (p(x), p(y)) \in e.$$

This lifts Harrington's result about thin Π^1_1 equivalence relations to thin Π^1_3 equivalence relations.

Dedication

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I would like to begin by expressing my gratitude to my advisors, Professor A.S. Kechris and Professor G. Hjorth for helping me to choose the right problems and for their guidance and encouragement during my graduate years at Caltech.

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1. Introduction

As ZFC fails to resolve many important questions about general sets, and even large cardinals cannot determine the size of ω^ω, set theorists turn to consider the definable objects. It is descriptive set theorists' main interest to study various set theoretic properties of definable objects. ω^ω used to be our magic garden with various flowers like Π^1_1 sets, Σ^1_3 singletons and so on. Now, there is also strong interest in the quotient space ω^ω/E by definable equivalence relation E. Our interests are also restricted to this new playground in this thesis.

Let us have a basic picture of this classical garden of set theory at first; a more detailed and global description can be found from [Mo1] or [Ke6].

We will call $\mathcal{X}$ a product space, if $\mathcal{X}$ can be written as a product of ω, ω^ω, 2^ω. We call a product space $\mathcal{X}$ perfect if and only if it is perfect (i.e., no isolated point) under the product topology. In this thesis, we will always use $\mathcal{X}$, $\mathcal{Y}$ to represent product spaces; we also always use x, y, z and the corresponding letters with scripts to represent typical elements in product spaces.

Let $\{N_n\}_{n\in\omega}$ be the "natural" basis of a product space $\mathcal{X}$,

(1) we will call a set $A \subseteq \mathcal{X}$ Σ^0_1 (in some $x \in \omega^\omega$) or semirecursive (in some $x \in \omega^\omega$) set if there is a recursive (in x) function $\epsilon\colon \omega \to \omega$ such that $A = \bigcup_{n\in\omega} N_{\epsilon(n)}$.

(2) We will call an $A \subseteq \mathcal{X}$ Π^0_1 (in x) if and only if $\mathcal{X} \setminus A$ is Σ^0_1 (in x). Σ^0_1 sets are the effective version of the open sets, which are the open sets

that we have algorithms to determine the membership in them.

(3) We will call a set $A \subseteq \mathcal{X}$ Σ^1_1 if and only if there is a $B \subseteq \mathcal{X} \times \omega^\omega$ such that $A = \mathrm{proj}_{\omega^\omega}(B)$, namely, for all $x \in \mathcal{X}$, $x \in A$ if and only if $\exists y \in \omega^\omega$ such that $(x, y) \in B$. We also write $\mathrm{proj}_{\omega^\omega}(B)$ as $p(B)$ or even pB in this thesis.

(4) $A \subseteq \mathcal{X}$ is called a Π^1_1 (in x) set if and only if $\mathcal{X} \setminus A$ is a Σ^1_1 (in x) set.

(5) In general, we can define Σ^1_{k+1} (in x) sets as the projections along ω^ω of Π^1_k (in x) sets, and Π^1_{k+1} (in x) sets as the compliments of Σ^1_{k+1} (in x) sets.

(6) We call a set Δ^1_k (in x) if and only if it is both Σ^1_k (in x) and Π^1_k (in x).

(7) We call a set $A \subseteq \mathcal{X}$ $\boldsymbol{\Sigma}^1_k$ ($\boldsymbol{\Pi}^1_k$, $\boldsymbol{\Delta}^1_k$, $\boldsymbol{\Sigma}^0_k$, $\boldsymbol{\Pi}^0_k$, $\boldsymbol{\Delta}^0_k$) if and only if it is Σ^1_k (Π^1_k, Δ^1_k, Σ^0_k, Π^0_k, Δ^0_k respectively in some x).

(8) We call a function $f\colon \mathcal{X} \to \mathcal{Y}$ Σ^1_k (Π^1_k, Δ^1_k, Σ^0_k, Π^0_k, Δ^0_k, $\boldsymbol{\Sigma}^1_k$, $\boldsymbol{\Pi}^1_k$, $\boldsymbol{\Delta}^1_k$, $\boldsymbol{\Sigma}^0_k$, $\boldsymbol{\Pi}^0_k$, $\boldsymbol{\Delta}^0_k$) if and only if $G_f = \{(x, n) : f(x) \in N(\mathcal{Y}, n)\}$ is Σ^1_k (Π^1_k, Δ^1_k, Σ^0_k, Π^0_k, Δ^0_k, $\boldsymbol{\Sigma}^1_k$, $\boldsymbol{\Pi}^1_k$, $\boldsymbol{\Delta}^1_k$, $\boldsymbol{\Sigma}^0_k$, $\boldsymbol{\Pi}^0_k$, $\boldsymbol{\Delta}^0_k$ respectively), where $N(\mathcal{Y}, n)$ is a basis of $\mathcal{Y}$.

(9) We call an ordinal α Σ^1_k (Π^1_k, Δ^1_k, Σ^0_k, Π^0_k, Δ^0_k, Σ^1_k in x, Π^1_k in x, Δ^1_k in x, Σ^0_k in x, Π^0_k in x, Δ^0_k in x) if and only if there is a Σ^1_k (Π^1_k, Δ^1_k, Σ^0_k, Π^0_k, Δ^0_k, Σ^1_k in x, Π^1_k in x, Δ^1_k in x, Σ^0_k in x, Π^0_k in x, Δ^0_k in x respectively) well-ordering on a subset of ω which has the ordertype α. Each of these ordinals can be coded by a real number.

It is well known that a set $A \subseteq \mathcal{X}$ is $\boldsymbol{\Delta}_1^1$ if and only if it is a Borel set and a function is $\boldsymbol{\Delta}_1^1$ if and only if it is a Borel function.

We will call $A \subseteq \mathcal{X}$ perfect if and only if A is closed and has no isolated points. We will call $A \subseteq \mathcal{X}$ thick if it includes a perfect subset. Of course, we will call A thin if and only if A is not thick.

The continuum problem is the chief stimulus for studying perfect sets. The continuum hypothesis cannot be determined from ZFC. Even large cardinals fail to determine the size of the continuum. However, the effective version of the continuum hypothesis (i.e., the perfect set theorem) can be proved from suitable large cardinals. Speaking roughly, every "definable" uncountable set of real numbers is thick, hence, equinumerous with ω^ω.

Definable thin subsets of a product space $\mathcal{X}$ were extensively investigated by various researchers in descriptive set theory. Let us summarize some related results about thin sets below.

At first we know that for any perfect product space $\mathcal{X}$, there is no largest thin Σ_1^1 subset of $\mathcal{X}$. It suffices to prove this for ω^ω, since any perfect product space is homeomorphic to ω^ω through a Δ_1^1 bijection. Suppose we have a largest one among all of the thin Σ_1^1 subsets of ω^ω, it must be $B = \{x : x \text{ is } \Delta_1^1\}$ since a Σ_1^1 subset of ω^ω is thin if and only if it contains only Δ_1^1 reals by the effective perfect set theorem. However, B is not Σ_1^1 by Kleene's lower classification theorem on Δ_1^1.

However, for Π^1_1, we have a different story.

Theorem (Guaspari, Kechris, Sacks) 1.1. *For any given product space $\mathfrak{X}$, there is a largest thin Π^1_1 subset $C_1(\mathfrak{X}) \subseteq \mathfrak{X}$ that includes all thin Π^1_1 subsets of $\mathfrak{X}$.*

The largest thin Π^1_1 subset of ω^ω can be defined as

$$C_1(\omega^\omega) = \{x : \forall y(\omega_1^x \leq \omega_1^y \to x \in \Delta^1_1(y))\}$$
$$= \{x : x \in L_{\omega_1^x}\},$$

where

$$\omega_1^x = \text{the least ordinal which is not recursive in } x$$
$$= \text{the least ordinal which is not } \Delta^1_1 \text{ in } x.$$

A classical construction of the largest thin Π^1_1 set using Π^1_1 norms can be found in [Mo1] also.

Assuming that all $\mathbf{\Sigma}_1$ games on ω are determined, the projection of $C_1(\mathfrak{X} \times \omega^\omega)$ to $\mathfrak{X}$ gives us the largest Σ^1_2 subsets of $\mathfrak{X}$. So, we also have the largest Σ^1_2 subset of $\mathfrak{X}$. The largest thin Σ^1_2 subset of ω^ω is actually equal to the set of all constructible reals.

It is easy to see that there is no largest thin Π^1_2 subset of any perfect product space $\mathfrak{X}$. Otherwise, suppose A is the largest thin Π^1_2 subset of ω^ω, A cannot be $\mathfrak{X}$, so $B = \mathfrak{X} \setminus A \neq \emptyset$, B must contain a Δ^1_2 real x by the basis theorem for Δ^1_2 sets. Now, $A \cup \{x\}$ is larger than A but still thin and Π^1_2.

In general, we have the following

Theorem (Kechris and Moschovakis) 1.2.

(1) *Assume* $Det(\boldsymbol{\Sigma}^1_{2n})$. *For each perfect product space* $\mathfrak{X}$, *there is a largest thin* Π^1_{2n+1} *set* $C_{2n+1}(\mathfrak{X}) \subseteq \mathfrak{X}$ *of* $\mathfrak{X}$.

(2) *Assume* $Det(\boldsymbol{\Sigma}^1_{2n+1})$. *For each perfect product space* $\mathfrak{X}$, *there is a largest thin (or equivalently countable)* Σ^1_{2n+2} *set* $C_{2n+2}(\mathfrak{X}) \subseteq \mathfrak{X}$ Σ^1_{2n+2}.

(3) *Assume* $Det(\boldsymbol{\Sigma}^1_{2n})$. *For every perfect product space* $\mathfrak{X}$, *there is no largest thin* Σ^1_{2n+1} *subset of* $\mathfrak{X}$ *which contains every thin* Σ^1_{2n+1} *subset of* $\mathfrak{X}$.

(4) *Assume* $Det(\boldsymbol{\Delta}^1_{2n})$. *For all perfect product spaces, there is no largest thin* Π^1_{2n+2} *subset of* $\mathfrak{X}$.

We will investigate similar properties in the context of $\mathfrak{X}/E$ for some definable equivalence relation E in this thesis. We could easily lift all the definitions we mentioned before to the context of $\mathfrak{X}/E$ by considering $[A]_E = \{x : \exists a \in A(xEa)\}$ for any $A \subseteq \mathfrak{X}/E$. Equivalently, we can consider E-invariant sets but still work in $\mathfrak{X}$. We will follow the latter.

Fix E a definable equivalence relation on $\mathfrak{X}$ and Γ a lightface pointclass. In this thesis, we only care about the cases that Γ is Σ^1_k or Π^1_k for some $k \in \omega$. We will call a set A

(1) *E-invariant* if and only if for any $x, y \in \omega^\omega$, $x \in A \wedge xEy \implies y \in A$,

(2) *E-thick* if and only if there is a perfect subset $B \subseteq A$ such that for any $x, y \in B$, $x \neq y \implies x \not{E} y$,

(3) *E-thin* if and only if it is not E-thick,

(4) *largest E-thin, E-invariant Γ set* if and only if it is a E-thin, E-invariant Γ set that contains all of other E-thin, E-invariant Γ sets.

We will see when we have a largest E-thin, E-invariant Γ subset of a perfect product space for a Γ' equivalence relation E, where Γ, Γ' the above classes.

If Γ is Σ^1_{2n+1} or Π^1_{2n}, the solution is immediate. We have two recursive equivalence relations E_1 and E_2 such that there is a largest E_1-thin E_1-invariant Γ set but no largest E_2-thin E_2 invariant set. We can let E_1 be the largest equivalence relation on $\mathcal{X}$ (i.e., $\mathcal{X} \times \mathcal{X}$) and E_2 be the smallest equivalence relation on $\mathcal{X}$ (i.e., the identity relation $id(\mathcal{X})$).

Problem 1.3. *Assuming Projective Determinacy, for what Π^1_{2n+1} equivalence relations E on a perfect product space, is there a largest E-thin, E-invariant Σ^1_{2n+1} set or a largest E-thin E-invariant Π^1_{2n+2} set? It is obviously true for thin equivalence relations. But, is it true or false that for any thick Π^1_{2n+1} equivalence relation E, there is no largest E-thin E-invariant Σ^1_{2n+1} set or largest E-thin E-invariant Π^1_{2n+2} sets?*

If Γ is Π^1_{2n+1} or Σ^1_{2n+2}, the classical results about $E = id(\mathcal{X})$ tell us that there is a recursive equivalence relation E such that we have a largest E-thin E-invariant Π^1_{2n+1} set and a largest E-thin E-invariant Σ^1_{2n+2} set. But, how about the other Π^1_{2n+1} equivalence relations. We will deal with this problem in the following chapters.

In Chapter 1, we will prove that there is a Π^1_1 equivalence relation E on ω^ω, for which there is no largest E-thin, E-invariant Π^1_1 subset of ω^ω. Then, we prove a similar result in a more general context, we get a Π^1_{2n+1} equivalence relation E on ω^ω for which there is no largest E-thin, E-invariant Π^1_{2n+1} subset of ω^ω. In the last section, we lift Harrington's representation theorem for thin Π^1_1 equivalence relations to thin Π^1_3 equivalence relations assuming that $\forall x \in \omega^\omega (x^\sharp$ exists).

2. Largest E-thin, E-invariant Sets below Δ^1_3

In [Ke1], Kechris proved the following

Theorem (Kechris) 2.1. *Let E be a Π^1_1 equivalence relation on ω^ω, if $A \subseteq \omega^\omega$ is Π^1_1 and E-thin, then for each $x_0 \in A$, there is A_0 Δ^1_1 in an ordinal smaller than $\omega_1^{x_0}$ such that*

$$x_0 \in A_0 \subseteq [x_0]_E \cap A.$$

For a Π^1_1 E, if we let

$$A = \{x : \exists S(S \text{ is } \Delta^1_1 \text{ in an ordinal smaller than } \omega_1^x \text{ and } x \in S \subseteq [x]_E)\}.$$

A is clearly a Π^1_1 set which contains every E-thin Π^1_1 subset of ω^ω.

A is also E-thin, otherwise, we can play the ordinary forcing trick to blow up the continuum to get a contradiction. If A is E-thick, we can expand our universe $\mathbf{V}$ to some generic extension $\mathbf{V}[G]$ by some so that in $\mathbf{V}[G]$, we have $2^{\aleph_0} = \aleph_2$. Shoenfield absoluteness guarantees that A is still E-thick in $\mathbf{V}[G]$. So, there must be $\aleph_2$ many S's to witness the membership in A. But that is impossible because we only have $\aleph_1$ many such S's in $\mathbf{V}[G]$.

So we get a Π^1_1 set which is E-thin and contains every E-thin Π^1_1 subset of ω^ω as a subset. As every Σ^1_2 set can be decomposed as $\aleph_1$ union of Borel sets, we can adjust the definition a little bit to get a largest E-thin Σ^1_2 set.

Let

$$C_2(E) = \{x : \exists S(\ S \text{ is } \Delta^1_1 \text{ in a countable ordinal } \wedge x \in S \subseteq [x]_E)\}.$$

In [Ke1], Kechris observed that this $C_2[E]$ is the largest E-thin, E-invariant Σ^1_2 set for the Π^1_1 equivalence relation E. Hence we have

Theorem(Kechris) 2.2. *For any Π^1_1 equivalence relation E on a product space $\mathfrak{X}$, there is a largest E-thin, E-invariant Σ^1_2 set.*

If E is actually Δ^1_1, Kechris also noticed the following

Theorem(Kechris) 2.3. *For any Δ^1_1 equivalence relation E on a product space $\mathfrak{X}$, there is a largest E-thin, E-invariant Π^1_1 set.*

The largest E-thin, E-invariant set mentioned above can be defined as

$$C_1(E) = \bigcup\{C : C \text{ is an } E\text{-equivalence class}$$
$$\wedge \forall x \in C (C \text{ is } \Delta^1_1 \text{ in an ordinal smaller than } \omega^x_1)\}.$$

After these results, Kechris raised the following

Problem(Kechris). *Is it true or false that for any Π^1_1 equivalence relation E, there will be a largest E-thin, E-invariant Π^1_1 subset of ω^ω?*

We will answer this negatively. Namely, we will construct a Π^1_1 equivalence relation E for which there is no largest E-thin, E-invariant Π^1_1 subset of ω^ω.

The idea is the following. We list all possible candidates of the largest E-thin, E-invariant Π^1_1 sets, i.e., we list all Π^1_1 subsets of ω^ω as $\{A_n\}_{n\in\omega}$. We will construct our equivalence relation E step by step, the possibility of each A_n as the largest E-thin, E-invariant subset of ω^ω is destroyed at some step of our construction. If A_n is given attention at some stage of our construction, we will pick some Π^1_1 singleton $\{x\}$ which has some nice properties. If this x is not in A_n, we will let $\{x\}$ be an equivalence class of our E which is being constructed. Hence, A_n cannot be the largest E-thin, E-invariant Π^1_1 set since $A_n \cup \{x\}$ is obviously larger than A_n and still an E-thin, E-invariant Π^1_1 set. If this x is already in A_n, we will find some real y which is not in A_n and put (x, y) into the equivalence relation E. Hence, A_n cannot be E-invariant. To make the above idea work, we have to make our construction carefully, otherwise, we cannot guarantee that our equivalence relation E is Π^1_1. That is why we need the Kleene recursion theorem.

Let

$$\delta^1_2 = \sup\{\alpha : \alpha \text{ is a countable ordinal coded by a } \Delta^1_2 \text{ real}\}.$$

We will call an ordinal α stable if and only if for all Σ_1 formulas $\varphi(x_1, \cdots, x_n)$ in the Levy hierarchy of ZFC formulas, and for any $a_1, \cdots, a_n \in L_\alpha$,

$$\mathbf{L} \models \varphi(a_1, \cdots, a_n) \iff L_\alpha \models \varphi(a_1, \cdots, a_n).$$

We will call an ordinal α weakly stable if and only if for all Σ_1 formulas without parameters in the Levy hierarchy of ZFC formulas,

$$\mathbf{L} \models \varphi \iff L_\alpha \models \varphi.$$

It is a well-known fact that

$$\delta^1_2 = \text{ the least stable ordinal}$$

$$= \text{ the least weakly stable ordinal .}$$

We do not need this result in our proof in this chapter, if we just replace all of the appearances of δ^1_2 by σ_0, the least stable ordinal. But this fact suggests that it should be enough to use the Π^1_1 singletons as our building blocks.

We will need the following

Lemma 2.4. *For any given $\alpha < \delta^1_2$, any sentence ψ which is true in $\mathbf{L}$, there is a β such that*

(1) $\alpha \leq \beta < \delta^1_2$,

(2) $L_\beta \models \mathrm{ZFC}^* \wedge \mathbf{V} = \mathbf{L} \wedge \psi$,

(3) $L_\beta = \mathrm{SkolemHull}(L_\beta)$,

(4) $x = \mathrm{Th}(L_\beta)$,

(5) x *is a* Π^1_1 *singleton,*

where ZFC* *means a large enough finite fragment of ZFC. We also use* ZFL* *to denote* ZFC* + $(\mathbf{V} = \mathbf{L})$.

Remark. This lemma claims that we have unbounded many such β's. We only need the existence of one β to prove our main result in this chapter. But they are actually equivalent. We state the lemma in only an apparently stronger form.

Proof. Since α is smaller than δ^1_2 which is the least weakly stable cardinal, α cannot be a weakly stable cardinal. So, there must be a Σ_1 formula φ such that

$\mathbf{L} \models \varphi$ and $\forall\beta < \alpha(L_\beta \not\models \varphi)$. Let us fix such a φ in our proof. Consider the set

$$A = \{x : \exists\beta(L_\beta \models (\mathrm{ZFL}^* \wedge \psi \wedge \varphi) \text{ and } x \text{ codes } \mathrm{Th}(L_\beta)\}.$$

Claim. A is not empty.

Proof of the claim: As δ_2^1 is stable, for any finite many sentences $\varphi_1, \cdots, \varphi_n$ of the language of set theory, we can find a $\gamma < \delta_2^1$ such that $L_\gamma \models \varphi_1 \wedge \cdots \wedge \varphi_n$. Let β_0 be an ordinal such that $L_{\beta_0} \models \mathrm{ZFL}^* \wedge \varphi \wedge \psi$. Let $M = \mathrm{SkolemHull}(L_{\beta_0})$. As M is elementary equivalent to L_{β_0}, $M \models \mathrm{ZFL}^*$. If we put enough axioms of ZFL into ZFL*, we can guarantee that M is isomorphic to some L_β for some $\beta \leq \beta_0$ since M is clearly wellfounded. Let x code the theory of L_β. This x is clearly in A. $\square$(claim)

Claim. A is Π_1^1.

Proof of claim: It suffices to show that for any real number x, $x \in A$ if and only if there is a β recursive in x such that x codes $\mathrm{Th}(L_\beta)$ and $\mathrm{ZFL}^* \subseteq \mathrm{Th}(L_\beta)$ and $\varphi, \psi \in \mathrm{Th}(L_\beta)$. Let x in A. From the definition of A, there is a β' such that such that x codes $\mathrm{Th}(L_{\beta'})$ and $\mathrm{ZFL}^* \subseteq \mathrm{Th}(L_{\beta'})$ and $\varphi, \psi \in \mathrm{Th}(L_{\beta'})$. Let M be the Skolem hull of L'_β. M must be isomorphic to some L_β for some $\beta \leq \beta'$ and $\mathrm{Th}(L_\beta) = \mathrm{Th}(L_{\beta'})$. As M is the Skolem hull of itself, M can be reconstructed effectively from its theory by a classical model theory construction. β, as the order type of Ord^M, is recursively reconstructible from $\mathrm{Th}(M) = \mathrm{Th}(L_\beta)$ as well. Hence, β is recursive in x. $\square$(claim)

Now, by the basis theorem for the Π_1^1 subsets of ω^ω, there is a x in A which is a Π^1_1 singleton. Let x code some $L_{\beta'}$, $\beta' < \delta^1_2$, since $L'_\beta \models \varphi$. We can play the same trick as before. Let M be the Skolem hull of $L_{\beta'}$. We know that M is isomorphic to some L_β for some $\beta \leq \beta' < \delta^1_2$. This β is what we want in the lemma. □(lemma)

Let $A \subseteq \omega \times \omega^\omega$ and $E \subseteq \omega \times \omega^\omega \times \omega^\omega$ be "good" universal Π^1_1 sets for which the *s-m-n* theorem applies. Let $A_k = \{x : (k, x) \in A\}$ and $E_k = \{(x, y) : (k, x, y) \in E\}$. Then $\{A_k\}_{k\in\omega}$ enumerate all Π^1_1 subsets of ω^ω in a Π^1_1 way and $\{E_k\}_{k\in\omega}$ enumerate all Π^1_1 subsets of $\omega^\omega \times \omega^\omega$ in a Π^1_1 way.

Lemma 2.5. *Let* $Thick(m, n) \iff A_m$ *is* E_n*-thick. Then* $Thick(m, n)$ *is* Σ^1_2.

This could be proved using Theorem 2.1. We give below a direct proof without using Theorem 2.1 because we think that the characterization of E-thickness that we obtain may be also interesting as well.

Proof. To make our notation simple, we just prove that it is Σ^1_2 to say that A is E-thick, for any given Π^1_1 $A \subseteq \omega^\omega$ and Π^1_1 $E \subseteq \omega^\omega \times \omega^\omega$. The proof for the general case is the same but notationally more complicated.

Fix a $T \subseteq (\omega \times \omega \times \omega)^{<\omega}$ to be a recursive pruned tree such that $\neg E = p[T]$ where $p[T]$ is the projection of $[T]$ to the second and third coordinates. Also fix a map π from ω^ω to

$$LO = \{x : x \text{ codes a linear ordering of a subset of } \omega\}$$

such that for all x, $x \in A \iff f(x) \in WO$, where

$$WO = \{x : x \text{ codes a well-ordering of a subset of } \omega\}.$$

Claim. the following are equivalent:

(1) A is E-thick.

(2) There is a continuous function $f\colon 2^\omega \to \omega^\omega$ such that

(i) for all x in 2^ω, $f(x) \in A$,

(ii) for all x, y in 2^ω, if $x \neq y$, then $f(x) \not\!E f(y)$.

(3) There is a countable ordinal α, a continuous function $f\colon 2^\omega \to \omega^\omega$ and a continuous function $g\colon \omega^\omega \times \omega^\omega \to \omega^\omega$ such that

(i) for all x, $\pi(f(x)) < \alpha$,

(ii) for all x, y in 2^ω, if $x \neq y$, then $f(x) \neq f(y)$,

(iii) for all x and y, if $x \neq y$, then $g(f(x), f(y))$ witnesses that $(f(x), f(y)) \notin E$.

(4) There is a real number r, a function $f_0\colon 2^{<\omega} \to 2^{<\omega}$ and a function $g_0\colon \omega^{<\omega} \times \omega^{<\omega} \to \omega^{<\omega}$ such that

(i) for all s and t in $2^{<\omega}$, if $s \subset t$, then $f_0(s) \subset f_0(t)$,

(ii) for any s and t in $2^{<\omega}$, if $len(s) = len(t)$, then $len(f_0(s)) = len(f_0(t))$, where $len(s)$ is the length of s,

(iii) for all s in $2^{<\omega}$, $f_0(s^\frown 0) \neq f_0(s^\frown 1)$,

(iv) for all (s_0, s_1) and (t_0, t_1) in $\omega^{<\omega} \times \omega^{<\omega}$, if $(s_0, s_1) \subseteq (t_0, t_1)$, then $g_0(s_0, s_1) \subseteq g_0(t_0, t_1)$,

(v) for all (s_0, s_1) in $\omega^{<\omega} \times \omega^{<\omega}$, if there is a $u \in \omega^{<\omega}$ such that $(u, s_0, s_1) \in T$, then $(g_0(s_0, s_1), s_0, s_1) \in T$,

(vi) for any s and t in $2^{<\omega}$, if $len(s) = len(t)$ and $s \neq t$, then there is a $u \in 2^{<\omega}$ such that $(u, f_0(s), f_0(t)) \in T$,

(vii) for all x in 2^ω, if $\forall n \in \omega \exists s \in 2^{<\omega}(x \restriction n = f_0(s))$, then x is recursive in r.

Proof of the above claim:

(4) $\Rightarrow$ (3): Assume that we have the f_0, g_0 and r satisfying all the requirements of (4). Let $\alpha = \omega_1^r$. Let $f = f_0^*$, namely, $f(x) = \cup_{n \in \omega} f_0(x \restriction n)$. This f is well-defined since f_0 is monotonic by (i). From (iii), f is one to one. f is also continuous from Theorem 2.6 in [K6]. From (v), we know that $\forall x \in 2^\omega(\pi(f(x)) < \alpha)$. We define g in a similar way, namely, $g(x, y) = g_0^*(x, y) = \cup_{n \in \omega} g_0(x \restriction n, y \restriction n)$. Then g is also continuous. Now, for any x and y in 2^ω, if $x \neq y$, there is a N such that for all $n \geq N$, $x \restriction n \neq y \restriction n$. Hence, $(g_0(f_0(x \restriction n), f_0(y \restriction n)), f_0(x \restriction n), f_0(y \restriction n)) \in T$ for all $n \geq N$ by (iv), (v) and (vi). As $[T]$ is closed, $(g(f(x), f(y)), f(x), f(y)) \in [T]$. Hence, $g(f(x), f(y))$ witnesses that $(f(x), f(y)) \notin E$ since $\neg E = p[T]$.

(3) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (1): Since 2^ω is compact, $f[2^\omega]$ is a compact subset of ω^ω. As ω^ω is a Hausdorf space, it must be a closed set. Hence, this perfect subset of A witnesses that A is E-thick.

(1) $\Rightarrow$ (4): Assume (1). The construction of f_0 is the routine Cantor scheme construction. (See [Ke6] for the Cantor scheme construction.) We can construct a Cantor scheme $(U_s)_{s \in 2^{<\omega}}$ (i.e., a family $(A_s)_{s \in 2^{<\omega}}$ of subsets of some Polish space $\mathfrak{X}$ such that (1) $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$, for all $s \in 2^{<\omega}$, and (2) $A_{s^\frown i} \subseteq A_s$, for all $s \in 2^{<\omega}$ and $i \in \{0, 1\}$.) such that

(1) U_s is of the form $N_t = \{x : x \text{ contains } t\}$ for some $t \in 2^{<\omega}$,

(2) if $len(s) = len(t)$, $U_s = N_{s'}$ and $U_t = N_{t'}$, then $len(s') = len(t')$,

(3) $diam(U_s) \leq 2^{-len(s)}$, i.e., if $U_s = N_{s'}$, then $len(s') \geq 2^{len(s)}$,

(4) $U_{s^\frown i} \subseteq U_s$, for any $s \in 2^{<\omega}$ and $i \in \{0, 1\}$,

(5) if $U_s = N_{s'}$ and $U_t = N_{t'}$, then there is some $u \in 2^{<\omega}$ such that $(u, s', t') \in T$.

This Cantor scheme can be constructed as usual by induction on the length of s since A is E-thick.

Let $f_0(s) = t$ if and only if $U_s = N_t$. Let $f(x) = f_0^*(x) = \cup_{n \in \omega} f_0(x \restriction n)$. f is a continuous injection from 2^ω into A. As $\pi[f[2^\omega]]$ is a $\mathbf{\Sigma}^1_1$ subset of WO, it is bounded below some countable ordinal α. Therefore, there is a $r \in \omega^\omega$ such that α is recursive in r. For any x such that $\forall n \in \omega \exists s \in 2^{<\omega}(x \restriction n = f_0(s))$, we know that $(\pi(f(x))$ codes a well-ordering with an order-type smaller than α, hence, recursive in r.

We can also construct the map g_0 by induction on the length of (s_0, s_1). We let $g_0(\emptyset) = \emptyset$. Suppose we have already defined g_0 for all (s_0, s_1) with

$len(s_0) = len(s_1) < n$, consider $t_0 = s_0{}^\frown i$ and $t_1 = s_1{}^\frown j$ for $i, j \in \omega$, if there is a $u \in 2^{<\omega}$ such that $(u, t_0, t_1) \in T$, we let $g_0(t_0, t_1)$ be some u such that $(u, t_0, t_1) \in T$. Otherwise, $g_0(t_0, t_1) = 0^n$. It is easy to check that g_0 works. □(claim)

As (4) is clearly Σ^1_2, it is Σ^1_2 to say that A is E-thick. So is $Thick(m, n)$. □

Let

$$\theta(m, n) \iff A_m \text{ is the largest } R_n\text{-thin } R_n\text{-invariant subset of } \omega^\omega.$$

Lemma 2.6. *$\theta(m, n)$ is absolute between* **L** *and* **V***.*

Proof. Let

$$\theta_0(m, n) \iff A_m \text{ is a } R_n\text{-thin } R_n\text{-invariant subset of } \omega^\omega .$$

$\theta_0(m, n)$ is Π^1_2 since $Thick(m, n)$ as specified as in the lemma before is Σ^1_2 and it is Π^1_2 to say that A_m is R_n invariant.

As $\theta(m, n) \iff \theta_0(m, n) \wedge \forall k(\theta_0(k, n) \implies A_k \subseteq A_m)$, $\theta(m, n)$ is absolute between **L** and **V** by the Shoenfield absoluteness theorem. □

Since $\theta(m, n)$ is absolute between **L** and **V**, we can assume that **V** = **L** without loss of generality, since if we proved that

$$\mathbf{L} \models \exists n(E_n \text{ is an equivalence relation } \wedge \forall m \neg\theta(m, n)),$$

we must have

$$\mathbf{V} \models \exists n(E_n \text{ is an equivalence relation } \wedge \forall m \neg\theta(m, n))$$

as well.

Let us define a set $A \subseteq \omega \times \omega^\omega \times \omega^\omega$ as following:

$(e, x, y) \in S \iff$

EITHER (1) $x = y$;

OR (2) $\exists\alpha\exists n \in \omega$ such that

(2.1) $L_\alpha = \text{SkolemHull}(L_\alpha)$,

(2.2) $L_\alpha \models \text{ZFL}^* \wedge \theta(n, e)$,

(2.3) x codes $\text{Th}(L_\alpha)$,

(2.4) x is a Π^1_1 singleton,

(2.5) $\forall m < n(L_\alpha \not\models \theta(m, e))$,

(2.6) $x \in A_n$,

(2.7) $L_\alpha \models$ "y is the $<_{L_\alpha}$-least element

subject to the following requirements:

(2.7.1) $y \in (L_\alpha \setminus A_n) \cap \omega^\omega$,

(2.7.2) y is not a Π^1_1 singleton,

(2.7.3) $\forall z(z$ is a Π^1_1 singleton $\rightarrow \neg(z, y) \in E_e)$;"

OR (3) $\exists\alpha\exists n \in \omega$ such that

(3.1) $L_\alpha = \text{SkolemHull}(L_\alpha)$,

(3.2) $L_\alpha \models \text{ZFL}^* \wedge \theta(n, e)$,

(3.3) y codes $\text{Th}(L_\alpha)$,

(3.4) y is a Π^1_1 singleton,

(3.5) $\forall m < n(L_\alpha \not\models \theta(m, e))$,

(3.6) $y \in A_n$,

(3.7) $L_\alpha \models$ "x is the $<_{L_\alpha}$-least element

subject to the following requirements:

(3.7.1) $x \in (L_\alpha \setminus A_n) \cap \omega^\omega$,

(3.7.2) x is not a Π^1_1 singleton,

(3.7.3) $\forall z(z$ is a Π^1_1 singleton $\rightarrow \neg(z, x) \in E_e)$."

Lemma 2.7. *S is a Π^1_1 subset of $\omega \times \omega^\omega \times \omega^\omega$.*

Proof. It suffices to show that (2) is a Π^1_1 formula. But

$$(2) \iff$$

$$\exists n[(2.4) \wedge (2.6) \wedge \exists\alpha((2.1) \wedge (2.2) \wedge (2.3) \wedge (2.5) \wedge (2.7)] \iff$$

$$\exists n[(2.4) \wedge (2.6) \wedge L_{\omega_1^{x,y}}(x, y) \models (\exists\alpha((2.1) \wedge (2.2) \wedge (2.3) \wedge (2.5) \wedge (2.7))].$$

The second equivalence comes from the fact that α can be recursively constructed from x and (2.1), (2.2), (2.3), (2.5) and (2.7) are Δ^1_1 formulas which are absolute between $L_{\omega_1^{x,y}}(x, y)$ and $\mathbf{L}$, and $L_{\omega_1^{x,y}}(x, y)$ correctly computes L_α for any $\alpha \in L_{\omega_1^{x,y}}(x, y)$.

Since $L_{\omega_1^{x,y}}(x, y) \models (\exists\alpha((2.1) \wedge (2.2) \wedge (2.3) \wedge (2.5) \wedge (2.7))$ is Π^1_1 by the Spector-Gandy theorem, it suffices to show that it is Π^1_1 to say that x is a Π^1_1

singleton. Let $B \subseteq \omega^\omega \times \omega$ be a universal Π^1_1 set. Let B^* be a Π^1_1 set which uniformizes B by the uniformization theorem of Π^1_1 sets. Namely, B^* is a Π^1_1 set such that

(1) for all $m \in \omega$, $\exists x \in \omega^\omega((x,m) \in B) \iff \exists x \in \omega^\omega((x,m) \in B^*)$

(2) for all $m \in \omega$, there is at most one x such that $(x,m) \in B^*$.

Then it clear that x is a Π^1_1 singleton if and only if $\exists m \in \omega((x,m) \in B^*)$. Hence, (2.4) is Π^1_1. $\square$

By the *s-m-n* theorem, there is a recursive function $f\colon \omega \longrightarrow \omega$ such that $\forall x \in \omega^\omega \forall y \in \omega^\omega((e,x,y) \in S \iff (f(e),x,y)) \in E$. Now, by the effective recursion theorem, there is a fixed point for f, i.e., there is a $e \in \omega$ such that for all real number x and y, $(e,x,y) \in S$ if and only if $(e,x,y) \in E$. From now on, let us fix this $e \in \omega$ in this chapter.

Lemma 2.8. *Suppose that $x \neq y$. If $(x,y) \in E_e$, then the one and only one element from the E_e-equivalent pair $\{x,y\}$ is a Π^1_1 singleton.*

Proof. It is clear from the construction.

Lemma 2.9. *E_e is an equivalence relation.*

Proof. (1) reflexity. Clear.

(2) symmetry. Clear.

(3) transitivity. Assume $(x,y) \in E_e$ and $(y,z) \in E_e$.

Case 1. $x = y$ or $y = z$, it is trivially true in this case.

Case 2. $x \neq y$ and $y \neq z$.

Subcase 2.1. y is a Π^1_1 singleton. In this case, neither x nor z can be a Π^1_1 singletons by the Lemma 2.8. Then,

$$x = z = <_L \text{ –the lease element subject to the same requirement.}$$

So, $(x, z) \in E_e$.

Subcase 2.2. y is not a Π^1_1 singleton. In this case, both x and z must be Π^1_1 singletons. Assume that $x \neq z$ towards a contradiction. Let

$$\alpha(x) = \text{ the unique } \alpha \text{ corresponding to } x \text{ in the construction,}$$

$$\alpha(z) = \text{ the unique } \alpha \text{ corresponding to } z \text{ in the construction.}$$

Without loss of generality, let us assume that $\alpha(x) < \alpha(z)$. Let n be the least number such that $L_{\alpha(z)} \models \theta(n, e)$. As $(y, z) \in E_e$, from the construction,

$L_{\alpha(z)} \models$ "y is the $<_{L_\alpha}$-least element subject to the following requirements:

$y \in (L_\alpha \setminus A_n) \cap \omega^\omega$,

y is not a Π^1_1 singleton,

$\forall z(z \text{ is a } \Pi^1_1 \text{ singleton } \rightarrow \neg(z, y) \in E_e)$."

But $x \in L_{\alpha(z)}$, since x is definable from $\alpha(x)$ in $L_{\alpha(z)}$ and $L_{\alpha(z)} =$ SkolemHull $(L_{\alpha(z)})$. $L_{\alpha(z)}$ would also think that x is a Π^1_1 singleton since $L_{\alpha(z)} \models$ ZFL*.

So, $L_{\alpha(z)} \models (x,y) \notin E_e$. But $L_{\alpha(z)} \models \text{ZFL}^*$, so we have $(x,y) \notin E_e$. Actually, if

$$L_\alpha \models KP + \text{ “every well-ordering is isomorphic to some ordinal in an order-preserving map,”}$$

L_α is absolute for all Π^1_1 formulas. We can always put enough axioms into ZFL* to guarantee this. But, $(x,y) \in E_e$, by our assumption. So, we have a contradiction. □

Lemma 2.10. *For any E_e-thin Π^1_1 set A,*

$$L \models \exists x(x \notin A \wedge\ x \text{ is not a } \Pi^1_1 \text{ singleton} \wedge \forall y(y \text{ is } \Pi^1_1\text{-singleton} \implies (x,y) \notin E_e)).$$

Proof. At first, we will show that A is thin. If A is thick, since every E_e-equivalence class has at most two elements, there are continuum many E_e inequivalent elements in A. But we know that A is E_e-thin, by Theorem 2.1, so, for every $x \in A$, we can find a A_0 which is Δ^1_1 in some ordinal smaller than ω_1^x such that $x \in A_0 \subseteq A$. We have only $\aleph_1$ many such A_0's. If we blow up the continuum to $\aleph_2$ by the product of $\aleph_2$ copies of Cohen forcing, we will get a contradiction.

$L_{\delta^1_2} \cap \omega^\omega$ is also a thin set by the same forcing trick. Now, by the Shoenfield absoluteness theorem, **L** think that both $\omega^\omega \cap L_{\delta^1_2}$ and A are thin. So is their union. Since $(L_{\delta^1_2} \cup A)$ is thin in $\mathbf{L}$, $\mathbf{L} \models \omega^\omega \cap (\mathbf{L} \setminus (L_{\delta^1_2} \cup A)) \neq \emptyset$. Let us pick any x in $\omega^\omega \cap (\mathbf{L} \setminus (L_{\delta^1_2} \cup A))$. This x is not a Π^1_1 singleton since all Π^1_1 singletons are in $L_{\delta^1_2}$. Hence this x is not in any L_α, for all α's which are coded by a Π^1_1 singleton in the sense of our construction. So, x is only E_e-equivalent to itself. So, this x witnesses the validity of the sentence of this lemma in $\mathbf{L}$. $\square$(lemma)

Theorem 2.11. *There is no largest E_e-thin E_e-invariant Π^1_1 subset of ω^ω.*

Proof. Assume that there is a largest E_e-thin E_e-invariant Π^1_1 subet of ω^ω. Let n be the least index for this set with the universal Π^1_1 set $A \subseteq \omega \times \omega^\omega$.

As A_n is E_e-thin,

$$\begin{aligned} L \models \exists x(x \notin A_n \wedge\ & x \text{ is not a } \Pi^1_1 \text{ singleton} \wedge \\ & \forall y(y \text{ is } \Pi^1_1\text{-singleton} \implies (x,y) \notin E_e) \\ & \forall m < n(\neg\theta(m,e)). \end{aligned}$$

By Lemma 2.4, we can always find x and α such that

(1) $\alpha < \delta^1_2$,

(2) $L_\alpha \models \text{ZFL}^* \wedge \theta(n,e) \wedge \forall m < n \neg\theta(m,e) \wedge \exists w(w \notin A_n \wedge$

w is not a Π^1_1-singleton $\wedge \forall y(y$ is a Π^1_1 singleton $\implies (w,y) \notin E_e))$,

(3) $L_\alpha = \text{SkolemHull}(L_\alpha)$,

(4) x codes $\text{Th}(L_\alpha)$ and x is a Π^1_1 singleton.

If $x \in A_n$, since

$$L_\alpha \models \exists w(w \notin A_n \wedge\ w \text{ is not a } \Pi^1_1\text{-singleton}$$

$$\wedge \forall w'(w' \text{ is a } \Pi^1_1 \text{ singleton} \implies (w, w') \notin E_e)),$$

let y be the $<_{L_\alpha}$-least real number in L_α. Then, $(x, y) \in E_e$ by the construction. This contradicts the assumption that A_n is E_e-invariant.

If $x \notin A_n$, then $A_n \cup \{x\}$ is still a E-thin, E-invariant Π^1_1 set but larger than A_n. This contradicts the assumption that A_n is the largest among all the E-thin, E-invariant Π^1_1 subsets of ω^ω.

Hence, for E_e, there is no largest E_e-thin E_e-invariant set. $\square$(theorem)

Remark. By theorem transfer theorem, we know that the same thing is true for all perfect product spaces $\mathfrak{X}$.

Next, let us consider Σ^1_1 equivalence relations. It is easier to construct a Σ^1_1 equivalence relation E for which there is no largest E-thin, E-invariant Π^1_1 set.

Let A be a Σ^1_1 but not Π^1_1 subset of ω. Let

$$(x, y) \in E \iff x = y \vee (x(0) = y(0) \in A).$$

E is clearly a Σ^1_1 equivalence relation. If there is a largest E-thin, E-invariant Π^1_1 set, say B, we can recover A from B in a Π^1_1 way.

Claim. $n \in A \iff \forall x(x(0) = n \implies x \in B)$.

Proof of the claim. For any $n \in A$, since $D_n = \{x : x(0) = n\}$ is E-thin, E-invariant Π^1_1 set, $D_n \subseteq B$. So, $x \in B$.

If $n \notin A$, we can clearly find a $x \notin B$ such that $x(0) = n$, since $D_n = \{x : x(0) = n\}$ is E-thick. $\square$

Hence, A would be Π^1_1, which contradicts the assumption on A.

From the argument above, we know that we can actually find a Σ^1_1 equivalence relation for which there is no E-thin Π^1_1 set which contains all recursive E-thin, E-invariant subsets of ω^ω.

Similar constructions would give us a Π^1_2 equivalence relation for which there is no largest E-thin, E-invariant Σ^1_2 subsets of ω^ω. Let $A \subseteq \omega$ be a Π^1_2 but non-Σ^1_2 set, For any x and y in ω^ω, let $(x, y) \in E$ if and only if either $x = y$ or $x(0) = y(0) \in A$. If there is a largest E-thin, E-invariant Σ^1_2 set B, A can be recovered from B in a Σ^1_1 way, namely, for all $n \in \omega$, $n \in A$ if and only if $\exists x(x(0) = n \wedge x \in B)$. Actually, there is no E-thin Σ^1_2 set which contains all recursive E-thin, E-invariant subsets of ω^ω.

The next problem is if there is a Σ^1_1 equivalence relation E on ω^ω for which there is no largest E-thin, E-invariant Σ^1_2 subset of ω^ω. Hjorth showed that the above is false under the assumption that there is $0^\sharp$.

Theorem(Hjorth) 2.12. *Assume that $0^\sharp$ exists. For any Σ^1_1 equivalence relation E, there is a largest E-thin, E-invariant Σ^1_2 subset C_E of ω^ω.*

To summarize, we have the following table:

Pointclass of E	Pointclass of the Largest Set			
	Σ_1^1	Π_1^1	Σ_2^1	Π_2^1
Δ_1^1	×	✓	✓	×
Σ_1^1	×	×	✓($0^\sharp$)	×
Π_1^1	×	×	✓	×
Δ_2^1	×	×	?	×
Σ_2^1	×	×	?	×
Π_2^1	×	×	×	×

Table on the existence of a largest E-thin, E-invariant set for a definable equivalence relation E below Δ_3^1.

The following problems are open:

Open Problems.

(1) *Is it true or false that for any Σ_2^1 equivalence relation E, there is a largest E-thin, E-invariant subset of ω^ω?*

(2) *Is it true or false that for any Δ_2^1 equivalence relation E, there is a largest E-thin, E-invariant subset of ω^ω?*

3. The General Case

In this chapter, we will solve the following problem negatively.

Problem. *Is it true that there is always a largest E-thin, E-invariant subset of ω^ω for any given Π^1_{2n+1} equivalence relation E?*

In Chapter 2, we solved the problem for Π^1_1 equivalence relations negatively. We constructed a Π^1_1 equivalence E for which there is no largest E-thin, E-invariant set. Using the work of Kechris and Martin in [KM1], it seems not difficult to lift our result to Π^1_3 equivalence relations. But the same argument cannot go any further without a generalization of the work in [KM1] to higher levels. It seems that Jackson has finished this generalization recently. But we can get around the difficulty by using Q-theory and the Martin-Solovay basis result for Σ^1_{2n+1} sets. We will answer the general problem negatively in this chapter.

Before we go any further, let us fix our notation first. Let $G^{\mathfrak{X}}$ be a good universal system for the Π^1_{2n+1} sets of the Polish space $\mathfrak{X}$. For simplicity, let $E = G^{\omega^\omega\times\omega^\omega} \subseteq \omega\times(\omega^\omega\times\omega^\omega)$, $A = G^{\omega^\omega} \subseteq \omega\times\omega^\omega$,$G = G^{\omega\times\omega\times\omega^\omega} \subseteq \omega\times(\omega\times\omega\times\omega^\omega)$. Let $\bar{G}$ uniformize G as a subset of $(\omega\times\omega\times\omega)\times\omega^\omega$, i.e.,

$$\forall m,n,k[\exists x(m,n,k,x)\in G \to \exists x(m,n,k,x)\in \bar{G}]$$

$$\wedge\forall m,n,k,x,y((m,n,k,x)\in\bar{G}\wedge(m,n,k,y)\in\bar{G}\to x=y).$$

We always use m,n,k,l,d,e for natural numbers and α,β,x,y,u,v for reals in this chapter. All of the other notations should be standard as in [Mo1].

We always assume $\mathbf{\Delta}^1_{2n}$ determinacy throughout this chapter.

Let A_l enumerate all the Π^1_{2n+1} sets. We will use a sequence $\{x_k\}_{k\in\omega}$ of Π^1_{2n+1} singletons to destroy the possibility of A_l being a largest E-thin, E-invariant set. For each l, if $x_l \in A_l$, we will introduce an element y such that yEx_l but $y \notin A_l$. This will make A_l not invariant. If $x_l \notin A_l$, we will ensure that x is the only element which is E-equivalent to x. That will make A_l not the largest, since $A_l \cup \{x\}$ will be invariant if A_l is. We have to work carefully to ensure that our equivalence relation E is Π^1_{2n+1}. We ensure this by using the recursion theorem and some Σ^1_{2n+1} elementary models generated by singletons.

Since our main building blocks for our Π^1_{2n+1} equivalence relation E are Π^1_{2n+1} singletons, let us review a theorem of Harrington about Π^1_{2n+1} singletons.

Theorem(Harrington) 3.1. *To each real α, we can associate a real y^α_{2n+1} such that:*

(1) *For each α, y^α_{2n+1} is a (representative of the $\Delta^1_{2n+1}(\alpha)$ degree of the) first non-trivial $\Pi^1_{2n+1}(\alpha)$ singleton. The formula*

$$H_{2n+1}(\alpha,\beta) \iff \beta = y^\alpha_{2n+1}$$

is Π^1_{2n+1}.

(2) *For each α, $\alpha \leq_T y^\alpha_{2n+1}$, and $\alpha \leq_T \beta \rightarrow y^\alpha_{2n+1} \leq_T y^\beta_{2n+1}$. In fact, these reductions are uniform. For instance, there is total recursive $p : \omega \rightarrow \omega$*

such that

$$\alpha = \{e\}^{\beta} \to y^{\alpha}_{2n+1} = \{p(e)\}^{y^{\beta}_{2n+1}}.$$

(3) *For all* α, β,

$$\alpha \leq^{Q}_{2n+1} \beta \to y^{\alpha}_{2n+1} \leq_T y^{\alpha}_{2n+1}.$$

(4) *Let*

$$\mathfrak{L}_{2n+1}(\alpha) = \{y_{A,\bar{\phi}} : A \in \Delta^1_{2n}, A \neq \emptyset, \bar{\phi} \text{ is an excellent}$$

$$\Delta^1_{2n+1}(\alpha) \text{ scale on } A\}.$$

Then $y^{\alpha}_{2n+1} \in \mathfrak{L}_{2n+1}(\alpha)$ *and every real in* $\mathfrak{L}_{2n+1}(\alpha)$ *is recursive in* y^{α}_{2n+1}. *In particular,* y^{α}_{2n+1} *is a recursive basis for* $\Sigma^1_{2n+1}(\alpha)$, *i.e., every non-empty* $\Sigma^1_{2n+1}(\alpha)$ *set contains a real recursive in* y^{α}_{2n+1}.

Proof. See [KMS] □

Definition. Let $U \subseteq \omega \times \omega^{\omega} \times \omega \times \omega)$ be a semi-recursive set universal for all semi-recursive sets of $\omega \times \omega$. Let $\bar{U}$ uniformize U as a subset of $(\omega \times \omega^{\omega} \times \omega) \times \omega$. We will call a real x good iff

(1) $x = (x^0, x^1)$,

(2) $x^0(i+1) = y^{x^0(i)}_{2n+1}, i.e., H_{2n+1}(x^0(i), x^0(i+1))$, where H_{2n+1} as in the above theorem of Harrington,

(3)

$$x^1(i,j,n) = \begin{cases} m & \text{if } \forall n \exists m \bar{U}(j, x^0(i), n, m) \text{ and } \bar{U}(j, x^0(i), n, m), \\ 0 & \text{if } \exists n \forall m \neg \bar{U}(j, x^0(i), n, m). \end{cases}$$

If x is good, then for any i,j, let $x_{i,j}$ be the real defined as following:

$$x_{i,j}(n) = m \quad \text{iff} \quad x^1(i,j,n) = m.$$

It is easy to see that $\{x_{i,j} : j \in \omega\} \subseteq \{y : y \leq_{\Delta^1_1} x^0(i)\}$. If x is good, let

$$M_x =_{d.f.} \{y : y \leq_T x^0(i) \text{ for some } i \in \omega\}$$

$$(= \{x_{i,j} : \text{ for some } i, j\}.)$$

It is Π^1_{2n+1} to say that x is good, namely, we have

Lemma 3.2. *There is a Π^1_{2n+1} formula $\phi(x)$ such that*

$$\phi(x) \iff \text{“}x \textit{ is good.”}$$

Proof. Clear. □

We have a natural well-ordering among reals of M_x for any good x. For any y and z in M_x, let (i_y, j_y) be the first pair of integers such that $y = x_{i_y,j_y}$ and (i_z, j_z) be the first pair of integers such that $y = x_{i_z,j_z}$ where $x_{i,j}$ is as given in the definition of being good. Let

$$y \preceq_{M_x} z \iff (i_y, j_y) \preceq_{\omega\times\omega} (i_z, j_z).$$

It is clear that this well-ordering is recursive in x.

Lemma 3.3. *If x is good, $\phi(x_0, \ldots, x_n)$ is projective, then there is an arithmetical formula $\psi(x, x_0, \ldots, x_n)$ such that*

$$M_x \models \phi(x_0, \ldots, x_n) \iff \psi(x, x_0, \ldots, x_n).$$

Proof. It is easy to see that we can always replace all quantifiers over reals by quantifiers over natural numbers because any element of M_x can be recursively recovered from x. □

Lemma 3.4. *If x is good, then $M_x \prec_{\Sigma^1_{2n+1}} \mathbf{V}$, i.e., for any Σ^1_{2n+1} formula $\phi(x_0, \ldots, x_n)$, any $y_0, \ldots, y_n \in M_x$*

$$M_x \models \phi(y_0, \ldots, y_n) \iff \phi(y_0, \ldots, y_n).$$

Proof. We use induction on the complexity of ϕ.

It is obviously true for arithmetical formulas.

Assume that it is true for all Σ^1_k (and hence Π^1_k) formulas with $k < 2n+1$. Let $\phi(x_0, \ldots, x_n) = \exists w \psi(w, x_0, \ldots, x_n)$ be Σ^1_{2n+1}. Let $y_0, \ldots, y_n \in M_x$, if $M_x \models \phi(y_0, \ldots, y_n)$, it is clear that $\phi(y_0, \ldots, y_n)$ holds. Now, assume that $\phi(y_0, \ldots, y_n)$ holds. As $y_0, \ldots, y_n$ is in M_x, there is an $i \in \omega$ such that $y_j \leq_T x(i)$ for $0 \leq j \leq n$. Let $D = \{w : \psi(w, y_0, \ldots, y_n)\}$, D is $\Pi^1_{2n}(x(i))$. By Martin-Solovay basis results, there is $w_0 \in D$ such that $w_0 \in \Pi^1_{2n+1}(x(i+1))$. By Harrington's theorem, $w_0 \leq_T x(i+2)$. Thus $w_0 \in M_x$. We get $M_x \models \phi(y_0, \ldots, y_n)$. □

Definition. Let $B \subseteq \omega \times \omega \times \omega \times \omega^\omega$ be the set defined as following:

$(d, e, l, x) \in B$ iff

(1) x is good,

(2) $M_x \models$ "$\forall k < l \quad A_k$ is not the largest E_e-thin E_e-invariant set,"

(3) $\forall k < l\ G_{d,e,k} \neq \emptyset$ and st.unif.$(G_{d,e,k}) \in M_x$ where st.unif. stands for standard uniformization, i.e., $\forall k < l, \exists y \in M_x((d,e,k,y) \in \bar{G})$.

Lemma 3.5. *B is Π^1_{2n+1}.*

Proof. (2) is Π^1_{2n+1} because of Lemma 3.4. □

Lemma 3.6. *There is a $d^* \in \omega$ such that*

$$(d^*, e, l, x) \in B \text{ if and only if} (d^*, e, l, x) \in G.$$

Proof. This is by the recursion theorem.

From now on, let us fix this d^*. We will construct our equivalence relation by the recursion theorem again.

Definition. Let $S \subseteq \omega \times \omega^\omega \times \omega^\omega$ be defined as following:

$(e, x, y) \in S$ iff

EITHER $x = y$,

OR for some l

(1) $x \in A_l$,

(2) $x =$ st.unif.$(B_{d^*,e,l})$, i.e., $(d^*, e, l, x) \in \bar{G}$,

(3) $y =$ the $\preceq_{M_x}$-least real in M_x subject to the following requirements:

(*i*) $\quad y \in (M_x \backslash A_l) \cap \omega^\omega$

(*ii*) $\quad \forall k < l\ y \not\mathrel{E}_e$ st.unif.$(G_{d^*,e,k})$.

OR for some l

(1) $y \in A_l$,

(2) $y = \text{st.unif.}(B_{d^*,e,l})$, i.e., $(d^*, e, l, y) \in \bar{G}$,

(3) $x =$ the $\preceq_{M_y}$-least real in M_y subject to the following requirements:

(i) $x \in (M_y \backslash A_l) \cap \omega^\omega$,

(ii) $\forall k < l, x \not\!\!E_e \text{st.unif.}(G_{d^*,e,k})$.

Lemma 3.7. *S is Π^1_{2n+1}.*

Proof. Lemma 3.4 implies that (3)'s are Π^1_{2n+1}. All others are clearly Π^1_{2n+1}.

□

By the recursion theorem again, we have the following

Lemma 3.8. *There is a natural number e^* such that*

$$(e^*, x, y) \in S \textit{ if and only if} (e^*, x, y) \in E.$$

Lemma 3.9. *E_{e^*} is an equivalence relation.*

Proof. Reflexity and symmetry are clear from our definition.

Let

$$r_k = \begin{cases} \text{st.unif.}(G_{d^*,e^*,k}), & \text{if } G_{d^*,e^*,k} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Actually, we will show that $G_{d^*,e^*,k} \neq \emptyset$ for all $k \in \omega$ later in this chapter.

We will next show that E_{e^*} is transitive. Assume $xE_{e^*}y$, and $yE_{e^*}z$.

Case 1: $x = y$ or $y = z$, it is obviously true.

Case 2: $x \neq y$ and $y \neq z$. From the construction, our E_{e^*} has the following properties:

(1) No E_{e^*}-equivalence class has more than two elements.

(2) If $xE_{e^*}y$, then either $x = y$ or exactly th of x and y must be a r_k for some $k \in \omega$.

Subcase 2.1: y is a Π^1_{2n+1} singleton. In this case, both x and z cannot be Π^1_{2n+1} singletons. So,

$$x = y = \text{ the } \leq_{M_y} \text{-least element subject to the same requirements.}$$

Subcase 2.2: y is not a Π^1_{2n+1} singleton. In this case, x and z are Π^1_{2n+1} singletons. Assume that $x \neq z$, towards a contradiction. Suppose l_x and l_y are the corresponding l in our construction of E_{e^*}, $l_x \neq l_y$ since x and y are uniquely determined by (d^*, e^*, l_x) and (d^*, e^*, l_y) respectively. Without loss of generality, we assume that $l_x < l_y$. From the construction, $x = \text{st.unif.}(B_{d^*,e^*,l_x})$ and $y = \text{st.unif.}(B_{d^*,e^*,l_y})$. As $l_x < l_y$, from the construction of B, we know that $\text{st.unif.}(B_{d^*,e^*,l_x}) \in M_y$. Now, we get a Π^1_{2n+1} singleton $x = r_{l_x} \in M_y$ such that $(z, x) \in E_{e^*}$, which contradicts with the assumption $yE_{e^*}z$ and $y \neq z$. □

Theorem 3.10. *Assuming $\boldsymbol{\Delta}^1_{2n}$ determinacy, there is no largest E_{e^*}-thin, E_{e^*}-invariant set for the Π^1_{2n+1} equivalence relation E_{e^*}.*

Proof. Let $A \subseteq \omega \times \omega^\omega$ universal for Π^1_{2n+1} sets as fixed at the beginning. By induction on l, we will show that for all l,

(1) $G_{d^*,e^*,l} \neq \emptyset$,

(2) for any good x, if $r_l \in M_x$, then

$$M_x \models \text{“}A_l \text{ is NOT the largest } E_{e^*}\text{-thin, } E_{e^*}\text{-invariant set,”}$$

(3) A_l is NOT the largest E_{e^*}-thin, E_{e^*}-invariant set.

Assume (1), (2) and (3) for all $k < l$. Now, we will show that they are still true when $k = l$.

(1)It is clear because any good x such that $x_i \in M_x$ for all $0 \leq i < l$ is an element of $G_{d^*,e^*,l}$, where $x_i = \text{st.unif.}(G_{d^*,e^*,i})$.

(2) Let x be a good real and $r_l \in M_x$. Let us work in M_x. If $r_l \notin A_l$ $(= A_l^{M_x})$, $[r_l]_{E_{e^*}} = \{r_l\}$. So, $A_l \cup \{r_l\}$ is E_{e^*}-thin, E_{e^*}-invariant but larger than A_l. Thus, we have

$$M_x \models \text{“}A_l \text{ is NOT the largest } E_{e^*}\text{-thin, } E_{e^*}\text{-invariant set.”}$$

If $r_l \in A_l$, as $\mathbf{V} \models \exists y(y \notin A_l \wedge y \not{E}_{e^*} r_0 \wedge \cdots \wedge y \not{E}_{e^*} r_{l-1})$ and $M_x \prec_{\Sigma^1_{2n+1}} \mathbf{V}$, there must be such y in M_x. Thus $M_x \models$ “A_l is not E_{e^*}-invariant.” Thus,

$$M_x \models \text{“}A_l \text{ is NOT the largest } E_{e^*}\text{-thin, } E_{e^*}\text{-invariant set.”}$$

(3) It is similar with (2) but simpler. $\square$

Let us give a summary of the relative results before we finish this chapter.

In [Ke1], Kechris proved the following

Theorem (Kechris) 3.11. *Assume $Det(\boldsymbol{\Delta}^1_2)$, let E be a Π^1_3 equivalence relation on ω^ω. If $A \subseteq \omega^\omega$ is Π^1_3 and E-thin, then for each $x_0 \in A$, there is an A_0 Δ^1_3 in an ordinal smaller than $\kappa_1^{x_0}$ such that*

$$x_0 \in A_0 \subseteq [x_0]_E \cap A.$$

For a Π^1_3 E, if we let

$$C_E = \{x : \exists S(S \text{ is } \Delta^1_3 \text{ in an ordinal smaller than } \kappa_1^x \text{ and } x \in S \subseteq [x]_E)\}.$$

A is an E-thin Π^1_3 set which contains every E-thin Π^1_3 subsets of ω^ω. So, there is a largest E-thin Π^1_3 subset of ω^ω.

Let

$$C_4(E) = \{x : \exists S(\ S \text{ is } \Delta^1_3 \text{ in a ordinal} < \kappa^1_3 \ \wedge x \in S \subseteq [x]_E)\}.$$

In [Ke1], Kechris observed that this $C_4[E]$ is the largest E-thin, E-invariant Σ^1_4 set for the Π^1_3 equivalence relation E. Hence we have the following

Theorem (Kechris) 3.12. *For any Π^1_3 equivalence relation E on a product space $\mathfrak{X}$, there is a largest E-thin, E-invariant Σ^1_4 set which contains every E-thin, E-invariant Σ^1_4 subset of $\mathfrak{X}$.*

If E is actually Δ^1_3, Kechris also noticed the following

Theorem(Kechris) 3.13. *For any Δ^1_3 equivalence relation E on a product space $\mathfrak{X}$, there is a largest E-thin, E-invariant Π^1_3 set.*

The largest E-thin, E-invariant set mentioned above can be defined as

$$C_3(E) = \bigcup\{C : C \text{ is an } E\text{-equivalence class} \\ \wedge \forall x \in C(C \text{ is } \Delta^1_3 \text{ in an ordinal smaller than } \kappa^x_1)\}.$$

A similar construction as in Chapter 2 would give us a Σ^1_3 equivalence relation E on ω^ω for which there is no largest E-thin, E-invariant Π^1_3 set. Also, we can construct a Π^1_4 equivalence relation E on ω^ω for which there is no largest E-thin, E-invariant Σ^1_4 set.

Hence, we have the following table:

Pointclass of E	Pointclass of the Largest Set			
	Σ^1_3	Π^1_3	Σ^1_4	Π^1_4
Δ^1_3	×	✓	✓	×
Σ^1_3	×	×	?	×
Π^1_3	×	×	✓	×
Δ^1_4	×	×	?	×
Σ^1_4	×	×	?	×
Π^1_4	×	×	×	×

Table on the existence of a largest E-thin, E-invariant set for a definable equivalence relation E below Δ^1_5.

The following problems are open:

Open Problems.

(1) *Is it true or false that for any Σ^1_4 equivalence relation E, there is a largest E-thin, E-invariant subset of ω^ω including all E-thin, E-invariant Σ^1_4*

subset of ω^ω?

(2) *Is it true or false that for any Δ^1_4 equivalence relation E, there is a largest E-thin, E-invariant subset of ω^ω including all E-thin, E-invariant Σ^1_4 subset of ω^ω?*

(3) *Is it true or false that for any Σ^1_3 equivalence relation E, there is a largest E-thin, E-invariant subset of ω^ω including all E-thin, E-invariant Σ^1_4 subset of ω^ω?*

In the most general case, we know less than in the case of the third and fourth levels. The following is a table which summarizes what we know right now.

Pointclass of E	Pointclass of the Largest Set			
	Σ^1_{2n+1}	Π^1_{2n+1}	Σ^1_{2n+2}	Π^1_{2n+2}
Δ^1_{2n+1}	×	?	?	×
Σ^1_{2n+1}	×	×	?	×
Π^1_{2n+1}	×	×	?	×
Δ^1_{2n+2}	×	×	?	×
Σ^1_{2n+2}	×	×	?	×
Π^1_{2n+2}	×	×	×	×

Table on the existence of a largest E-thin, E-invariant set for a definable equivalence relation E below Δ^1_{2n+3}.

The following problem is open:

Open Problems.

(1) *Is it true or false that for any Σ^1_{2n+2} equivalence relation E, there is a*

largest E-thin, E-invariant subset of ω^ω?

(2) *Is it true or false that for any Δ^1_{2n+2} equivalence relation E, there is a largest E-thin, E-invariant subset of ω^ω?*

(3) *Is it true or false that for any Σ^1_{2n+1} equivalence relation E, there is a largest E-thin, E-invariant subset of ω^ω?*

(4) *Is it true or false that for any Δ^1_{2n+1} equivalence relation E, there is a largest E-thin, E-invariant subset of ω^ω?*

(5) *Is it true or false that for any Π^1_{2n+1} equivalence relation E, there is a largest E-thin, E-invariant subset of ω^ω?*

4. Σ^1_3, Π^1_3 and Δ^1_3 in the Codes Subsets of u_ω

In this chapter, we will prove some technical lemmas on Σ^1_3, Π^1_3 and Δ^1_3 in the codes subsets of u_ω. On one hand results about the Σ^1_3 in the codes, Π^1_3 in the codes and Δ^1_3 in the codes sets are close relatives of results about the third level of the analytical hierarchy on real numbers. On the other hand, they can be considered as the Π^1_1, Σ^1_1 and Δ^1_1 subsets of $\langle u_\omega, <, \{u_n\}\rangle$ under the full determinacy. Even without AD, they often look like sets of the first level of the analytical hierarchy. All of the results in this chapter should be considered as folklore or direct generalizations of the classical results. However, we have to be careful when we generalize the classical results to this context. We collect here some results we will use later on to prove our theorem in Chapter 6 because it is not easy to find them in the literature. To fully investigate the analytical hierarchy on u_ω, the full determinacy is required to code all subsets of u_ω. We do not deal with it here. [Ke1] is a good reference for the related results, while [Ke2] is a good reference for the analytical hierarchy over $\aleph_1$.

We assume $\mathbf{\Delta}^1_2$ determinacy from now on. Harrington showed that $\mathbf{\Pi}^1_1$ determinacy implies that $\forall x \in \omega^\omega (x^\sharp$ exists), so, we have all the sharps for reals available. We adopt the standard coding system for the ordinals smaller than u_ω as in [KM1].

Let $w \in \mathrm{WO}_\omega$ if and only if $w = \langle n, x^\sharp\rangle$ for some $n \in \omega$, $x \in \omega^\omega$. For any

$w \in \mathrm{WO}_\omega$, if $w = \langle n, x^\sharp \rangle$, let

$$|w| = \tau_n^{L[x]}(u_1, \cdots, u_{k_n}),$$

where u_α is the α^{th} uniform indiscernible and τ_n is the n^{th} term in a recursive enumeration of all terms in the language of ZF+$\mathbf{V} = \mathbf{L}[\dot{x}]$ always taking ordinal values. From Solovay's theorem, $u_\omega = \{|w| : w \in \mathrm{WO}_\omega\}$.

For any w_1, w_2 in WO_ω, we say $w_1 \sim_\omega w_2$ if and only if $|w_1| = |w_2|$. It is clear that $\sim_\omega$ is an equivalence relation. We call $P(w, x)$ $\sim_\omega$*-invariant* on w if and only if for any w_1, w_2 in WO_ω, and x in ω^ω, if $w_1 \sim_\omega w_2$ and $P(w_1, x)$, then $P(w_2, x)$.

The following theorem is essential for getting results in the third level of the analytic hierarchy, and is the cornerstone of our results in this chapter too.

Theorem (Kechris and Martin) 4.1. *Assuming* $\mathbf{\Delta}^1_2$ *determinacy, if* $P(w, x)$ *is* Π^1_3 *and* $\sim_\omega$*-invariant on* w *and* $\exists w \in \mathrm{WO}_\omega P(w, x)$*, then* $\exists w \in \mathrm{WO}_\omega \cap \Delta^1_3(x) P(w, x)$.

Proof. See [KM1]. □

Corollary 4.2. *Assuming* $\mathbf{\Delta}^1_2$ *determinacy, if* $P(w, x)$ *is* $\sim_\omega$*-invariant on* w *and* Π^1_3*, and* $R(w, x) \iff \exists w \in \mathrm{WO}_\omega P(w, x)$*, then* $R(w, x)$ *is also* Π^1_3.

Let

$$\mathfrak{X}^{m,n,k} = u_\omega{}^m \times (\omega^\omega)^n \times \omega^k = \overbrace{u_\omega \times \cdots \times u_\omega}^{m \text{ times}} \times \overbrace{\omega^\omega \times \cdots \times \omega^\omega}^{n \text{ times}} \times \overbrace{\omega \times \cdots \times \omega}^{k \text{ times}}.$$

For $A \subseteq \mathfrak{X}^{m,n,k}$, let

$$A^* = \{(x_1, \cdots, x_m, y_1, \cdots, y_n, i_1, \cdots, i_k) : x_1, \cdots, x_n \in \mathrm{WO}_\omega; y_1, \cdots, y_n \in \omega^\omega;$$

$$i_1, \cdots, i_k \in \omega \text{ and } (|x_1|, \cdots, |x_m|, y_1, \cdots, y_n, i_1, \cdots, i_k) \in A\}$$

be the pullback of A under the coding map given by the uniform indiscernibles and Skolem terms at the beginning of this chapter.

For notational simplicity, we will use $\vec{x}$ for $x_1, \cdots, x_n$, $\vec{y}$ for $y_1, \cdots, y_m$, $\vec{i}$ for $i_1, \cdots, i_k$ and $\vec{\alpha}$ for $\alpha_1, \cdots, \alpha_m$. We use $\vec{x} \in \mathrm{WO}_\omega$ to express $x_1 \in \mathrm{WO}_\omega, \cdots,$ $x_m \in \mathrm{WO}_\omega$. We also use $\mathbf{x}$ and $\boldsymbol{\alpha}$ to represent $(\vec{x}, \vec{y}, \vec{i})$ and $(\vec{\alpha}, \vec{y}, \vec{i})$ respectively to simplify the notation further, if it is clear from the context. We write $|\vec{x}|$ to mean $(|x_1|, \cdots, |x_m|)$.

Definition. For $A \subseteq \mathfrak{X}^{m,n,k}$, we call A Σ^1_3 (Π^1_3, $\Sigma^1_3(x)$, $\Pi^1_3(x)$, Δ^1_3, $\Delta^1_3(x)$) in the codes if and only if $A^* \subseteq \mathfrak{X}^{0,m+n,k}$ is Σ^1_3 (Π^1_3, $\Sigma^1_3(x)$, $\Pi^1_3(x)$, Δ^1_3, $\Delta^1_3(x)$ respectively).

We will use $\Sigma^1_3(\mathfrak{X}^{m,n,k})$ ($\Pi^1_3(\mathfrak{X}^{m,n,k})$, $\Sigma^1_3(x, \mathfrak{X}^{m,n,k})$, $\Pi^1_3(x, \mathfrak{X}^{m,n,k})$, $\Delta^1_3(\mathfrak{X}^{m,n,k})$, $\Delta^1_3(x, \mathfrak{X}^{m,n,k})$ respectively) to represent the corresponding pointclasses.

It is easy to see the following

Lemma 4.3.

(1) *$\Sigma^1_3(\mathfrak{X}^{m,n,k})$, $\Pi^1_3(\mathfrak{X}^{m,n,k})$ and $\Delta^1_3(\mathfrak{X}^{m,n,k})$ are closed under $\wedge$, $\vee$, $\exists^{u_\omega}$, $\forall^{u_\omega}$. Moreover, $\Sigma^1_3(\mathfrak{X}^{m,n,k})$ is closed under $\exists^{\omega^\omega}$, $\Pi^1_3(\mathfrak{X}^{m,n,k})$ is closed*

under $\forall^{\omega^\omega}$ *and*

$\Delta^1_3(\mathfrak{X}^{m,n,k})$ *is closed under* $\neg$.

(2) $\Sigma^1_3(\mathfrak{X}^{m,n,k})$ *and* $\Pi^1_3(\mathfrak{X}^{m,n,k})$ *are* ω*-parametrized.*

Proof.

(1) By Lemma 4.1, $\Sigma^1_3(\mathfrak{X}^{m,n,k})$ is closed under $\forall^{u_\omega}$ and $\Pi^1_3(\mathfrak{X}^{m,n,k})$ is closed under $\exists^{u_\omega}$. All of the others are trivial.

(2) Let $U_0 \subseteq \omega \times \mathfrak{X}^{0,m+n,k}$ be a universal Σ^1_3 set for all Σ^1_3 subsets of $\mathfrak{X}^{0,m+n,k}$. Let

$$U_1 = \{(n, \vec{x}, \vec{y}, \vec{i}) : \vec{x} \in \mathrm{WO}_\omega\},$$

which is Δ^1_3. Let

$$U_2 = \{(n, \vec{x}, \vec{y}, \vec{i}) : (n, \vec{x}, \vec{y}, \vec{i}) \in U_1 \wedge \exists \vec{x}'(\vec{x}' \in \mathrm{WO}_\omega \wedge |\vec{x}'| = |\vec{x}| \wedge (n, \vec{x}, \vec{y}, \vec{i}) \in U_1)\}.$$

Let

$$W = \{(n, \vec{\alpha}, \vec{y}, \vec{i}) : \exists \vec{x} \in \mathrm{WO}_\omega(\vec{\alpha} = |\vec{x}|) \wedge (n, \vec{x}, \vec{y}, \vec{i}) \in U_2).$$

It is easy to check that W is $\Sigma^1_3(\mathfrak{X}^{m,n,k})$ and universal for all $\Sigma^1_3(\mathfrak{X}^{m,n,k})$ subsets of $\mathfrak{X}^{m,n,k}$.

It is similar to prove that $\Pi^1_3(\mathfrak{X}^{m,n,k})$ is ω-parametrized.

□

Remark. Similar facts hold for the pointclasses $\Sigma^1_3(x, \mathfrak{X}^{m,n,k})$, $\Pi^1_3(x, \mathfrak{X}^{m,n,k})$ and

$\Delta^1_3(x, \mathfrak{X}^{m,n,k})$ also.

Definition. Let $\alpha \in u_\omega$, we call $A \subseteq \mathfrak{X}^{m,n,k}$ Σ^1_3 in α in the codes (Π^1_3 in α in the codes respectively) if and only if $A^* \subseteq \mathfrak{X}^{0,m+n,k}$ is "uniformly" $\Sigma^1_3(x)$ ($\Pi^1_3(x)$ respectively) for all x such that $|x| = \alpha$, i.e., there is a Σ^1_3 (*pth* respectively) $B \subseteq \mathfrak{X}^{0,m+n+1,k}$ such that for all x coding α, for all $x_1, \cdots, x_m, y_1, \cdots, y_n$ in ω^ω and $i_1, \cdots, i_k$ in ω,

$$(x_1, \cdots, x_m, y_1, \cdots, y_n, i_1, \cdots, i_k) \in A^*$$
$$\iff (x_1, \cdots, x_m, y_1, \cdots, y_n, x, i_1, \cdots, i_k) \in B.$$

We will use $\Sigma^1_3(\alpha, \mathfrak{X}^{m,n,k})$ ($\Pi^1_3(\alpha, \mathfrak{X}^{m,n,k})$, $\Delta^1_3(\alpha, \mathfrak{X}^{m,n,k})$ respectively) to represent the corresponding pointclasses.

We will use ${}^*\Sigma^1_3(\mathfrak{X}^{m,n,k})$ (${}^*\Pi^1_3(\mathfrak{X}^{m,n,k})$, ${}^*\Delta^1_3(\mathfrak{X}^{m,n,k})$ respectively)to represent

$\bigcup_{\alpha \in \mathrm{WO}_\omega} \Sigma^1_3(\alpha, \mathfrak{X}^{m,n,k})$ ($\bigcup_{\alpha \in \mathrm{WO}_\omega} \Pi^1_3(\alpha, \mathfrak{X}^{m,n,k})$, $\bigcup_{\alpha \in \mathrm{WO}_\omega} \Delta^1_3(\alpha, \mathfrak{X}^{m,n,k})$ respectively).

It is easy to see that for any $A \subseteq \mathfrak{X}^{m,n,k}$, A is $\Sigma^1_3(\alpha)$($\Pi^1_3(\alpha)$ or $\Delta^1_3(\alpha)$) in the codes iff there is a Σ^1_3 (Π^1_3 or Δ^1_3 respectively) in the codes $B \subseteq u_\omega \times \mathfrak{X}^{m,n,k}$ such that $A = B_\alpha = \{(\vec{\alpha}, \vec{y}, \vec{i}) : (\alpha, \vec{\alpha}, \vec{y}, \vec{i}) \in B)\}$.

As usual, we have

Lemma 4.4.

(1) $\Sigma^1_3(\alpha, \mathfrak{X}^{m,n,k})$, $\Pi^1_3(\alpha, \mathfrak{X}^{m,n,k})$ *and* $\Delta^1_3(\alpha, \mathfrak{X}^{m,n,k})$ *are closed under* $\wedge$, $\vee$, $\exists^{u_\omega}$, $\forall^{u_\omega}$. *Moreover,* $\Sigma^1_3(\alpha, \mathfrak{X}^{m,n,k})$ *is closed under* $\exists^{\omega^\omega}$, $\Pi^1_3(\alpha, \mathfrak{X}^{m,n,k})$ *is closed under* $\forall^{\omega^\omega}$ *and* $\Delta^1_3(\alpha, \mathfrak{X}^{m,n,k})$ *is closed under* $\neg$.

(2) $\Sigma^1_3(\alpha, \mathfrak{X}^{m,n,k})$ *and* $\Pi^1_3(\alpha, \mathfrak{X}^{m,n,k})$ *are* ω*-parametrized.*

Remark. The relativized version holds as well.

Lemma 4.5. ${}^*\Sigma^1_3(\mathfrak{X}^{m,n,k})$ *and* ${}^*\Pi^1_3(\mathfrak{X}^{m,n,k})$ *are* u_ω*-parametrized, i.e., there are* Σ^1_3 *in the codes* $U \subseteq u_\omega \times \mathfrak{X}^{m,n,k}$ *and* Π^1_3 *in the codes* $V \subseteq u_\omega \times \mathfrak{X}^{m,n,k}$ *such that for any* $\Sigma^1_3(\alpha)$ *in the codes* $A \subseteq \mathfrak{X}^{m,n,k}$ *and* $\Pi^1_3(\alpha)$ *in the codes* $B \subseteq \mathfrak{X}^{m,n,k}$*, there are* $\beta_0, \beta_1 \in u_\omega$ *with*

$$A = U_{\beta_0} = \{(\vec{\alpha}, \vec{y}, \vec{i}) : (\beta_0, \vec{\alpha}, \vec{y}, \vec{i}) \in U\}$$

$$A = V_{\beta_1} = \{(\vec{\alpha}, \vec{y}, \vec{i}) : (\beta_1, \vec{\alpha}, \vec{y}, \vec{i}) \in V\}.$$

Proof. Let $f\colon u_\omega \times \omega \to u_\omega$ be a Δ^1_3 in the codes bijection between $u_\omega \times \omega$ and u_ω. Let $W \subseteq \omega \times \mathfrak{X}^{m+1,n,k}$ be a Σ^1_3 in the codes set universal for all Σ^1_3 in the codes sets of $u_\omega \times \mathfrak{X}^{m,n,k}$. Let $U \subseteq u_\omega \times \mathfrak{X}^{m,n,k}$ be defined as the following

$$(\alpha, \vec{\alpha}, \vec{y}, \vec{i}) \in U \iff ((f^{-1}(\alpha))_0, (f^{-1}(\alpha))_1, \vec{\alpha}, \vec{y}, \vec{i}) \in W.$$

It is easy to check that U works.

We can define V in a similar way. $\square$

Sometimes, we need a better parametrization for the *-pointclasses. We have the following refinement.

Lemma (The Good Parametrization Lemma) 4.6. *We can associate with each space* $\mathfrak{X}^{m,0,k}$ *a* Σ^1_3 *in the codes set* $G^{m,k} \subseteq u_\omega \times \mathfrak{X}^{m,0,k}$ *and a* Π^1_3 *in* *the codes set* $H^{m,k} \subseteq u_\omega \times \mathfrak{X}^{m,0,k}$ *such that*

(1) $G^{m,k}$ *is universal for all* ${}^*\Sigma^1_3(\mathfrak{X}^{m,0,k})$ *sets and* $H^{m,k}$ *is universal for all* ${}^*\Pi^1_3(\mathfrak{X}^{m,0,k})$ *sets,*

(2) *for* $P \subseteq \mathfrak{X}^{m,0,k}$,

$$P \in \Sigma^1_3(\mathfrak{X}^{m,0,k}) \iff \exists n \in \omega(P = G^{m,k}_n)$$

$$P \in \Pi^1_3(\mathfrak{X}^{m,0,k}) \iff \exists n \in \omega(P = H^{m,k}_n),$$

(3) *for each* m_1,m_2 k_1, k_2 *in* ω, *there are a* Δ^1_3 *in the codes (which means that the pullback of the graph is* Δ^1_3*)* $s^{m_1,m_2,k_1,k_2}_\Sigma : u_\omega \times \mathfrak{X}^{m_1,0,k_1} \to u_\omega$ *and* $s^{m_1,m_2,k_1,k_2}_\Pi : u_\omega \times \mathfrak{X}^{m_1,0,k_1} \to u_\omega$ *so that for each* $(\vec{\alpha}, \vec{i}) \in \mathfrak{X}^{m_1,0,k_1}$, $(\vec{\beta}, \vec{j}) \in \mathfrak{X}^{m_2,0,k_2}$ *and* $\varepsilon \in u_\omega$,

$$G^{m_1+m_2,k_1+k_2}(\varepsilon, \vec{\alpha}, \vec{\beta}, \vec{i}, \vec{j}) \iff G^{m_2,k_2}(s^{m_1,m_2,k_1,k_2}_\Sigma(\varepsilon, \vec{\alpha}, \vec{i}), \vec{\beta}, \vec{j}),$$

$$H^{m_1+m_2,k_1+k_2}(\varepsilon, \vec{\alpha}, \vec{\beta}, \vec{i}, \vec{j}) \iff H^{m_2,k_2}(s^{m_1,m_2,k_1,k_2}_\Pi(\varepsilon, \vec{\alpha}, \vec{i}), \vec{\beta}, \vec{j}),$$

and if $(\epsilon, \vec{\alpha}, \vec{i}) \in \omega^{m_1+k_1+1}$, *then* $s^{m_1,m_2,k_1,k_2}_\Pi(\epsilon, \vec{\alpha}, \vec{i}) \in \omega$.

Proof. We will prove the Σ^1_3 part of this lemma, i.e., we will construct the $G^{m,k}$ only. The Π^1_3 part can be proved similarly.

Let $h: u_\omega \times u_\omega \to u_\omega$ be a Δ^1_3 in the codes bijection between $u_\omega \times u_\omega$ and u_ω such that

(1) $h[\omega \times \omega] = \omega$

(2) there are Δ^1_3 $h_1: u_\omega \to u_\omega$ and $h_2: u_\omega \to \omega$ decoding h, i.e., if $h(\alpha, n) = \beta$, then $h_1(\beta) = \alpha$ and $h_2(\beta) = n$.

Let $U \subseteq \omega \times u_\omega \times \mathfrak{X}^{m,0,k}$ parametrize $\Sigma^1_3(u_\omega \times \mathfrak{X}^{m,0,k})$. Let

$$G^*(\alpha, \vec{a}, \vec{i}) = U(h_2(\alpha), h_1(\alpha), \vec{a}, \vec{i}).$$

It is easy to check that G^* satisfies (1) and (2). So, we can always assume that we have a parameterization system satisfying (1) and (2).

For $\mathfrak{X}^{m,0,k}$, fix $\pi_{m,k} \colon u_\omega \times \mathfrak{X}^{m,0,k} \to u_\omega$ a recursive in the codes bijection such that $\pi_{m,k}[\mathfrak{X}^{0,0,m+k+1}] \subseteq \omega$ and let $V \subseteq u_\omega \times u_\omega \times u_\omega$ be Σ^1_3 in the codes and universal for the ${}^*\Sigma^1_3(u_\omega \times u_\omega)$ subsets of $u_\omega \times u_\omega$ so that (1) and (2) hold. Define $G^{m,k} \subseteq u_\omega \times \mathfrak{X}^{m,0,k}$ by

$$G^{m,k}(\varepsilon, \vec{a}, \vec{i}) \iff V((\varepsilon)_0, (\varepsilon)_1, \pi_{m,k}((\varepsilon)_2, \vec{a}, \vec{i})).$$

It is clear that $G^{m,k}$ is Σ^1_3 in the codes. $\square$

Claim. $G^{m,k}$ is universal for the ${}^*\Sigma^1_3(\mathfrak{X}^{m,0,k})$ sets and satisfies (2).

Proof of the claim. Suppose $Q \subseteq \mathfrak{X}^{m,0,k}$ is ${}^*\Sigma^1_3(\mathfrak{X}^{m,0,k})$, let

$$Q'(\alpha, \beta) \iff Q(p_{m,k}(\pi^{-1}_{m,k}(\beta))),$$

where $p_{m,k} \colon u_\omega \times \mathfrak{X}^{m,0,k} \to \mathfrak{X}^{m,0,k}$ is the projection map on $\mathfrak{X}^{m,0,k}$. Now, Q' is ${}^*\Sigma^1_3(u_\omega \times u_\omega)$. For any α, taking $\beta = \pi_{m,k}(\alpha, \vec{a}, \vec{i})$, we have

$$\begin{aligned} Q(\vec{a}, \vec{i}) &\iff Q'(\alpha, \pi_{m,k}(\alpha, \vec{a}, \vec{i})) \\ &\iff V(\varepsilon, \alpha, \pi_{m,k}(\alpha, \vec{a}, \vec{i})) \\ &\iff G^{m,k}(\langle \varepsilon, \alpha, \alpha \rangle, \vec{a}, \vec{i}). \end{aligned}$$

If Q is Σ^1_3 in the codes, Q' is also Σ^1_3 in the codes, by our selection of V, we can choose $\varepsilon \in \omega$ above. If we choose $\alpha \in \omega$ too, we get $\varepsilon^* = \langle \varepsilon, \alpha, \alpha \rangle \in \omega$ and

$$Q(\vec{\alpha}, \vec{\imath}) \iff G^{m,k}(\varepsilon^*, \vec{\alpha}, \vec{\imath}).$$

□

Claim. $\{G^{m,k}\}_{m,k\in\omega}$ satisfies (3).

Proof of the claim. Fix $\mathfrak{X}^{m_1,0,k_1} = \{(\vec{\alpha}, \vec{\imath}) : (\vec{\alpha}, \vec{\imath}) \in \mathfrak{X}^{m_1,0,k_1}\}$ and $\mathfrak{X}^{m_2,0,k_2} = \{(\vec{\beta}, \vec{\jmath}) : (\vec{\beta}, \vec{\jmath}) \in \mathfrak{X}^{m_2,0,k_2}\}$. We will try to construct a Δ^1_3 in the codes $s_\Sigma^{m_1,m_2,k_1,k_2} \colon u_\omega \times \mathfrak{X}^{m,0,k} \to u_\omega$ so that

$$G^{m_1+m_2,k_1+k_2}(\varepsilon, \vec{\alpha}, \vec{\beta}, \vec{\imath}, \vec{\jmath}) \iff G^{m_2,k_2}(s_\Sigma^{m_1,m_2,k_1,k_2}(\varepsilon, \vec{\alpha}, \vec{\imath}), \vec{\beta}, \vec{\jmath}).$$

Put

$$P(\alpha, \beta) \iff$$

(*) $$G((\alpha)_0, f_0(p_{m_1,k_1}(\pi^{-1}_{m_1,k_1}((\alpha)_1)), f_0(p_{m_2,k_2}(\pi^{-1}_{m_1,k_2}((\alpha)_2)),$$

$$f_1(p_{m_1,k_1}(\pi^{-1}_{m_1,k_1}((\alpha)_1)), f_1(p_{m_2,k_2}(\pi^{-1}_{m_1,k_2}((\alpha)_2))),$$

where $f_0 \colon \mathfrak{X}^{m,0,k} \to \mathfrak{X}^{m,0,0}$ is defined as $f_0(\vec{\alpha}, \vec{\imath}) = \vec{\alpha}$, and $f_1 \colon \mathfrak{X}^{m,0,k} \to \mathfrak{X}^{0,0,k}$ is defined as $f_1(\vec{\alpha}, \vec{\imath}) = \vec{\imath}$. Now, P is Σ^1_3 in the codes, so there must be a $\varepsilon^* \in \omega$ such that

(**) $$P(\alpha, \beta) \iff V(\varepsilon^*, \alpha, \beta).$$

For arbitrary ε, $(\vec{\alpha}, \vec{\imath}) \in \mathfrak{X}^{m_1,0,k_1}$,$(\vec{\beta}, \vec{\jmath}) \in \mathfrak{X}^{m_2,0,k_2}$, let $\alpha = \langle \varepsilon, \pi_{m_1,k_1}(\varepsilon, \vec{\alpha}, \vec{\imath}) \rangle$, $\beta = \pi_{m_2,k_2}(\varepsilon, \vec{\beta}, \vec{\jmath})$, substituting in (*) and (**), we get

$$G^{m_1+m_2,k_1+k_2}(\varepsilon, \vec{\alpha}, \vec{\beta}, \vec{\imath}, \vec{\jmath}) \iff P(\langle \varepsilon, \pi_{m_1,k_1}(\varepsilon, \vec{\alpha}, \vec{\imath}) \rangle, \pi_{m_2,k_2}(\varepsilon, \vec{\beta}, \vec{\jmath}))$$

$$\iff V(\varepsilon^*, \langle \varepsilon, \pi_{m_1,k_1}(\varepsilon, \vec{\alpha}, \vec{\imath}) \rangle, \pi_{m_2,k_2}(\varepsilon, \vec{\beta}, \vec{\jmath})),$$

then by the definition of G^{m_2,k_2}, we have that

$$G^{m_1+m_2,k_1+k_2}(\varepsilon,\vec{\alpha},\vec{\beta},\vec{i},\vec{j}) \iff G^{m_2,k_2}(\langle\varepsilon^*,\langle\varepsilon,\pi_{m_1,k_1}(\varepsilon,\vec{\alpha},\vec{i})\rangle,\varepsilon\rangle,\vec{\beta},\vec{j}).$$

Let $s_\Sigma^{m_1,m_2,k_1,k_2}(\varepsilon,\vec{\alpha},\vec{i}) = \langle\varepsilon^*,\langle\varepsilon,\pi_{m_1,k_1}(\varepsilon,\vec{\alpha},\vec{i})\rangle,\varepsilon\rangle$, from the argument above, $s_\Sigma^{m_1,m_2,k_1,k_2}$ is just what we want. □

The following effective recursion theorem is the main tool we used in the next section to construct the Π^1_3 in the codes equivalence relation e on u_ω.

Lemma (Effective Recursion Theorem) 4.7. *Let $\{G^{m,k}\}_{m,k\in\omega}$ and $\{H^{m,k}\}_{m,k\in\omega}$ be the good parametrization system given by the last lemma.*

(1) *For any $^*\Sigma^1_3$ (or $^*\Pi^1_3$) in the codes subset A of $u_\omega \times \mathfrak{X}^{m,0,k}$, there is an $\varepsilon \in u_\omega$ such that*

$$A(\varepsilon,\vec{\alpha},\vec{i}) \iff G^{m,k}(\varepsilon,\vec{\alpha},\vec{i}),$$

(or

$$A(\varepsilon,\vec{\alpha},\vec{i}) \iff H^{m,k}(\varepsilon,\vec{\alpha},\vec{i}),$$

respectively).

(2) *For any Σ^1_3 (or Π^1_3) in the codes subset A of $\omega \times \mathfrak{X}^{m,0,k}$, there is an $\varepsilon \in \omega$ such that*

$$A(\varepsilon,\vec{\alpha},\vec{i}) \iff G^{m,k}(\varepsilon,\vec{\alpha},\vec{i}),$$

(or

$$A(\varepsilon,\vec{\alpha},\vec{i}) \iff H^{m,k}(\varepsilon,\vec{\alpha},\vec{i}),$$

respectively).

Proof. This can be proved by the classical diagonal method. We will prove the Σ^1_3 part only because the proof of the other part is similar.

(1) Let $A \subseteq u_\omega \times \mathfrak{X}^{m,0,k}$ is ${}^*\Sigma^1_3$ in the codes. We define for $(\vec{\alpha},\vec{i}) \in \mathfrak{X}^{m,0,k}$ and $\eta \in u_\omega$,

$$B(\eta,\vec{\alpha},\vec{i}) \iff A(s^{1,0,m,k}(\eta,\eta),\vec{\alpha},\vec{i}).$$

As A is ${}^*\Sigma^1_3$ in the codes, so is B, hence, for some $\varepsilon_B \in u_\omega$, one has

$$B(\varepsilon,\vec{\alpha},\vec{i}) \iff G^{m+1,k}(\varepsilon_B,\varepsilon,\vec{\alpha},\vec{i}).$$

Let $\varepsilon = s_\Sigma^{1,0,m,k}(\varepsilon_B,\varepsilon_B)$, we claim ε works, as

$$\begin{aligned} A(\varepsilon,\vec{\alpha},\vec{i}) &\iff A(s_\Sigma^{1,0,m,k}(\varepsilon_B,\varepsilon_B),\vec{\alpha},\vec{i}) && \text{by the definition of } \varepsilon \\ &\iff B(\varepsilon_B,\vec{\alpha},\vec{i}) && \text{by the definition of } B \\ &\iff G^{m+1,k}(\varepsilon_B,\varepsilon_B,\vec{\alpha},\vec{i}) && \text{by the definition of } s_\Sigma^{1,0,m,k} \\ &\iff G^{m,k}(\varepsilon,\vec{\alpha},\vec{i}) && \text{by the definition of } \varepsilon. \end{aligned}$$

(2) Let $A \subseteq \omega \times \mathfrak{X}^{m,0,k}$ is Σ^1_3 in the codes. We define for $(\vec{\alpha},\vec{i}) \in \mathfrak{X}^{m,0,k}$ and $n \in \omega$,

$$B(n,\vec{\alpha},\vec{i}) \iff A(s^{0,1,m,k}(n,n),\vec{\alpha},\vec{i}).$$

As A is Σ^1_3 in the codes, so is B, hence, for some $\varepsilon_B \in u_\omega$, one has

$$B(n,\vec{\alpha},\vec{i}) \iff G^{m,k+1}(\varepsilon_B,\vec{\alpha},n,\vec{i}).$$

Let $\varepsilon = s_\Sigma^{0,1,m,k}(\varepsilon_B,\varepsilon_B)$, from the definition of $s_\Sigma^{0,1,m,k}$, $\varepsilon \in \omega$. It is similar to show that ε works.

□

Lemma (Spector-Gandy Theorem for Π^1_3, Becker and Kechris) 4.8.

There is a tree T_2 satisfying the following requirements:

(1) $T_2 \subseteq (\omega \times u_\omega)^{<\omega}$ *is* Δ^1_3 *in the codes,*

(2) $p[T_2]$ *is the complete* Π^1_2 *set of* ω^ω.

(3) *For any* $A \subseteq \omega^\omega$, A *is* Π^1_3 *if and only if there is a* Σ_1 *formula* $\varphi_A(x)$ *such that*

$$\forall x \in \omega^\omega (x \in A \iff L_{\kappa_3^x}[T_2, x] \models \varphi_A).$$

Proof. Though the proof in [BK1] works for different trees, by checking the proof in detail, we can adopt the proof there to get this lemma. □

From now on, let us fix such a Martin-Solovay tree T_2.

Corollary (Spector-Gandy Type Theorem for $\Pi^1_3(\mathfrak{X}^{m,0,k})$) 4.9. *For any* $A \subseteq \mathfrak{X}^{m,0,k}$,

(1) A *is* Π^1_3 *in the codes if and only if there is a* Σ_1 *formula* φ_A *such that*

$$\forall (\vec{\alpha}, \vec{i}) \in \mathfrak{X}^{m,0,k}((\vec{\alpha}, \vec{i}) \in A \iff L_{\kappa_3}[T_2] \models \varphi_A(\vec{\alpha}, \vec{i})).$$

(2) A *is* Π^1_3 *in the codes if and only if there is a* Σ_1 *formula* φ_A *such that*

$$\forall (\vec{\alpha}, \vec{i}) \in \mathfrak{X}^{m,0,k}((\vec{\alpha}, \vec{i}) \in A \iff L_{\kappa_3}[T_2] \models \varphi_A(\vec{\alpha}, \vec{i})).$$

(3) A *is* Δ^1_3 *in the codes if and only if there is a* Δ_1 *formula* φ_A *such that*

$$\forall (\vec{\alpha}, \vec{i}) \in \mathfrak{X}^{m,0,k}((\vec{\alpha}, \vec{i}) \in A \iff L_{\kappa_3}[T_2] \models \varphi_A(\vec{\alpha}, \vec{i})).$$

From this corollary, it is easy to see that every Δ^1_3 in the codes subset of some u_ω is in $L_{\kappa_3}[T_2]$. The similar result is not true for Σ^1_3 and Π^1_3 subsets. But Professor Hjorth has the following

Lemma (Hjorth). *All Π^1_3 in the codes subsets of some $\alpha < u_\omega$ are in $L_{\kappa_3}[T_2]$.*

Now we are aiming towards proving the norm property and some corollaries from it. The following theorem of Solovay provides the norms for the pointclasses $\Pi^1_3(\mathfrak{X}^{m,n,k})$ and $\Pi^1_3(\alpha, \mathfrak{X}^{m,n,k})$.

Lemma (Solovay) 4.10. *Assuming $\boldsymbol{\Delta}^1_2$ determinacy, let $\sim$ be a Σ^1_3 equivalence relation on a space $(\omega^\omega)^l$, let $P \subseteq (\omega^\omega)^l$ be $\sim$-invariant (i.e., $x \in P \wedge x \sim y \rightarrow y \in P$), then there is a $\sim$-invariant (i.e., for any x and y in P, if $x \sim y$, then $\varphi(x) = \varphi(y)$) norm Π^1_3: $P \rightarrow Ord$.*

Proof. See [Ke2]. □

Corollary 4.11. *$\Pi^1_3(\alpha, \mathfrak{X}^{m,n,k})$ is normed, i.e., for any $\Pi^1_3(\alpha)$ in the codes $A \subseteq \mathfrak{X}^{m,n,k}$, there is a $\delta \in Ord$ and a $\varphi\colon A \rightarrow \delta$ such that the relations $\mathbf{x} \leq^*_\varphi \mathbf{x}^*$ and $\mathbf{x} <^*_\varphi \mathbf{x}^*$ are $\Pi^1_3(\alpha)$ in the codes, where*

$$\mathbf{x} \leq^*_\varphi \mathbf{x}^* \iff \mathbf{x} \in A \wedge (\mathbf{x}^* \notin A \vee \varphi(\mathbf{x}) \leq \varphi(\mathbf{x}^*)),$$

$$\mathbf{x} <^*_\varphi \mathbf{x}^* \iff \mathbf{x} \in A \wedge (\mathbf{x}^* \notin A \vee \varphi(\mathbf{x}) < \varphi(\mathbf{x}^*)).$$

Corollary (Kreisel Uniformization Theorem) 4.12. *If $A \subseteq \mathfrak{X}^{m,n,k} \times \omega$ is $\Pi^1_3(\alpha)$ in the codes, then there is a $B \subseteq \mathfrak{X}^{m,n,k} \times \omega$ which is $\Pi^1_3(\alpha)$ in the* *codes such that*

(1) $\forall(\vec{\alpha},\vec{x},\vec{i}) \in \mathfrak{X}^{m,n,k}(\exists n \in \omega(\vec{\alpha},\vec{x},\vec{i},n) \in A \to \exists! n(\vec{\alpha},\vec{x},\vec{i},n) \in B)$,

(2) $B \subseteq A$.

Corollary (Kuratowski Reduction Theorem) 4.13. *Let A and B be two $\Pi^1_3(\alpha)$ in the codes subsets of $\mathfrak{X}^{m,n,k}$, then there are two $\Delta^1_3(\alpha)$ in the codes A' and B' with $A' \subseteq A$, $B' \subseteq B$, $A \cup B = A' \cup B'$ and $A' \cap B' = \emptyset$.*

Corollary (Seperation Theorem) 4.14. *Let A and B be two $\Sigma^1_3(\alpha)$ in the codes sets of $\mathfrak{X}^{m,n,k}$, $A \cap B = \emptyset$. Then there is a $\Delta^1_3(\alpha)$ in the codes C which separates A from B, i.e., such that $A \subseteq C$ and $C \cap B = \emptyset$.*

With the help of these results, we can code the ${}^*\Delta^1_3$ subsets of $\mathfrak{X}^{m,0,k}$.

Lemma (Coding the ${}^*\Delta^1_3(\mathfrak{X}^{m,0,k})$ Sets) 4.15. *There is a pair (D,W) such that*

(1) *$D \subseteq u_\omega$ is Π^1_3 in the codes and $W \subseteq u_\omega \times \mathfrak{X}^{m,0,k}$ is Δ^1_3 in the codes in $D \times \mathfrak{X}^{m,0,k}$, i.e., there are Σ^1_3 in the codes $W^\Sigma \subseteq u_\omega \times \mathfrak{X}^{m,0,k}$ and Π^1_3 in the codes $W^\Pi \subseteq u_\omega \times \mathfrak{X}^{m,0,k}$ such that $W = (D \times \mathfrak{X}^{m,0,k}) \cap W^\Sigma = (D \times \mathfrak{X}^{m,0,k}) \cap W^\Pi$,*

(2) $\{W_\alpha : \alpha \in D\} = {}^*\Delta^1_3(\mathfrak{X}^{m,0,k})$,

(3) *if $A \subseteq u_\omega$ is ${}^*\Pi^1_3$ in the codes and $B \subseteq A \times \mathfrak{X}^{m,0,k}$ is ${}^*\Delta^1_3$ in the codes in $A \times \mathfrak{X}^{m,0,k}$, then there must be a Δ^1_3 in the codes function $f : u_\omega \to u_\omega$ such that for all $\alpha \in A$, $f(\alpha) \in D$ and $W_{f(\alpha)} = B_\alpha$.*

Proof. Let

$$V^0 = \{(\varepsilon, \vec{\alpha}, \vec{i}) : ((\varepsilon)_0, \vec{\alpha}, \vec{i}) \in H^{m,k}\},$$

$$V^1 = \{(\varepsilon, \vec{\alpha}, \vec{i}) : ((\varepsilon)_1, \vec{\alpha}, \vec{i}) \in H^{m,k}\}.$$

By the Kuratowski reduction theorem, let (U^0, U^1) reduce (V^0, V^1), i.e., U^0 and U^1 are Π^1_3 in the codes, $U^0 \subseteq V^0$, $U^1 \subseteq V^1$, $U^1 \cap U^1 = \emptyset$ and $U^0 \cup U^1 = V^0 \cup V^1$. Let

$$D = \{\varepsilon \in u_\omega : \forall (\vec{\alpha}, \vec{i}) \in \mathfrak{X}^{m,0,k}((\varepsilon, \vec{\alpha}, \vec{i}) \in U^0 \vee (\varepsilon, \vec{\alpha}, \vec{i}) \in U^1)\}.$$

$D \subseteq u_\omega$ is Π^1_3 in the codes. Let

$$W = \{(\varepsilon, \vec{\alpha}, \vec{i}) : \varepsilon \in D \wedge (\varepsilon, \vec{\alpha}, \vec{i}) \in U^0\}.$$

Clearly, W is Δ^1_3 in the codes in $D \times \mathfrak{X}^{m,0,k}$, as

$$\varepsilon \in D \wedge (\varepsilon, \vec{\alpha}, \vec{i}) \notin W \iff \varepsilon \in D \wedge (\varepsilon, \vec{\alpha}, \vec{i}) \in U^1,$$

by the definition of D.

As (2) is a particular case of (3) when $A = D$ and $B = H^{m,k}$, we will show (3) directly here.

Let $A \subseteq u_\omega$ be ${}^*\Pi^1_3$ in the codes and $B \subseteq A \times \mathfrak{X}^{m,0,k}$ be ${}^*\Pi^1_3$ in the codes. Let

$$B^0 = \{\varepsilon, \vec{\alpha}, \vec{i}) \in u_\omega \times \mathfrak{X}^{m,0,k} : \varepsilon \in A \wedge (\varepsilon, \vec{\alpha}, \vec{i}) \in B\},$$

$$B^1 = \{\varepsilon, \vec{\alpha}, \vec{i}) \in u_\omega \times \mathfrak{X}^{m,0,k} : \varepsilon \in A \wedge (\varepsilon, \vec{\alpha}, \vec{i}) \notin B\},$$

are both ${}^*\Pi^1_3$ in the codes in $u_\omega \times \mathfrak{X}^{m,0,k}$. Hence, there are Δ^1_3 in the codes functions $f_0 \colon u_\omega \to u_\omega$ and $f_1 \colon u_\omega \to u_\omega$, such that for $i = 0$ or 1,

$$(*) \qquad (\varepsilon, \vec{\alpha}, \vec{i}) \in B^i \iff (f_i(\varepsilon), \vec{\alpha}, \vec{i}) \in H^{m,k}.$$

Set $f(\varepsilon) = \langle f_0(\varepsilon), f_1(\varepsilon)\rangle$. We check that this f works. Fix $\varepsilon \in A$. We have $B_\varepsilon = B^0_\varepsilon$ and $B^1_\varepsilon = \mathfrak{X}^{m,0,k} \backslash B_\varepsilon$. Now,

$$\begin{aligned} V^0_{f(\varepsilon)} &= H^{m,k}_{f_0(\varepsilon)} && \text{by the definition of } V^0 \\ &= B^0_\varepsilon && \text{by } (*), \end{aligned}$$

and

$$\begin{aligned} V^1_{f(\varepsilon)} &= H^{m,k}_{f_1(\varepsilon)} && \text{by the definition of } V^0 \\ &= B^1_\varepsilon && \text{by } (*), \end{aligned}$$

so that $V^0_{f(\varepsilon)} \cap V^1_{f(\varepsilon)} = \emptyset$ and $V^0_{f(\varepsilon)} \cup V^1_{f(\varepsilon)} = \mathfrak{X}^{m,0,k}$. But this implies that $U^0_{f(\varepsilon)} = V^0_{f(\varepsilon)}$ and $U^0_{f(\varepsilon)} = V^0_{f(\varepsilon)}$, so that $U^0_{f(\varepsilon)} \cup U^1_{f(\varepsilon)} = \mathfrak{X}^{m,0,k}$, i.e., $f(\varepsilon) \in D$, and moreover

$$W_{f(\varepsilon)} = U^0_{f(\varepsilon)} = V^0_{f(\varepsilon)} = H^{m,k}_{f_0(\varepsilon)} = B^0_\varepsilon$$

as desired. $\square$

We have coded the ordinals smaller than u_ω using sharps at the beginning of this section. But that is not enough. We need an induction all the way to κ_3 to get a representation theorem for the Π^1_3 thin equivalence relations. We have to find a method to code all of the ordinals smaller than κ_3. We will do this coding using ordinals smaller than u_ω in the following.

Let $D \subseteq u_\omega$ Π^1_3 in the codes and $W \subseteq D \times u_\omega \times u_\omega$ Δ^1_3 in the codes in $D \times u_\omega \times u_\omega$ as given by the lemma above, let W^Σ and W^Π as given in the lemma above also.

Definition of LO_{u_ω} and WO_{u_ω}.

$$\mathrm{LO}_{u_\omega} = \{\alpha : \alpha \in D \text{ and } W_\alpha \text{ is a linear ordering on } u_\omega\}$$

$$\mathrm{WO}_{u_\omega} = \{\alpha : \alpha \in D \text{ and } W_\alpha \text{ is a well-ordering on } u_\omega\}$$

As for any $\alpha \in D$, $\alpha \in \mathrm{LO}_{u_\omega}$ if and only if

$$\forall \beta_0 < u_\omega \forall \beta_1 < u_\omega (\beta_0 W^\Pi_\alpha \beta_1 \rightarrow \beta_0 W^\Sigma_\alpha \beta_0 \wedge \beta_1 W^\Sigma_\alpha \beta_1)$$

$$\wedge \forall \beta_0 < u_\omega \forall \beta_1 < u_\omega (\beta_0 W^\Pi_\alpha \beta_1 \wedge \beta_1 W^\Pi_\alpha \beta_0 \rightarrow \beta_0 = \beta_1)$$

$$\wedge \forall \beta_0 < u_\omega \forall \beta_1 < u_\omega \forall \beta_2 < u_\omega (\beta_0 W^\Pi_\alpha \beta_1 \wedge \beta_1 W^\Pi_\alpha \beta_2 \rightarrow \beta_0 W^\Sigma_\alpha \beta_2)$$

$$\wedge \forall \beta_0 < u_\omega \forall \beta_1 < u_\omega (\beta_0 W^\Pi_\alpha \beta_0 \wedge \beta_1 W^\Pi_\alpha \beta_1 \rightarrow \beta_0 W^\Sigma_\alpha \beta_1 \vee \beta_1 W^\Sigma_\alpha \beta_0),$$

by the Kechris-Martin theorem, it is Σ^1_3 in the codes in $D \times \mathfrak{X}^{m,0,k}$. On the other hand, for any $\alpha \in D$, $\alpha \in \mathrm{LO}_{u_\omega}$ if and only if

$$\forall \beta_0 < u_\omega \forall \beta_1 < u_\omega (\beta_0 W^\Sigma_\alpha \beta_1 \rightarrow \beta_0 W^\Pi_\alpha \beta_0 \wedge \beta_1 W^\Pi_\alpha \beta_1)$$

$$\wedge \forall \beta_0 < u_\omega \forall \beta_1 < u_\omega (\beta_0 W^\Sigma_\alpha \beta_1 \wedge \beta_1 W^\Sigma_\alpha \beta_0 \rightarrow \beta_0 = \beta_1)$$

$$\wedge \forall \beta_0 < u_\omega \forall \beta_1 < u_\omega \forall \beta_2 < u_\omega (\beta_0 W^\Sigma_\alpha \beta_1 \wedge \beta_1 W^\Sigma_\alpha \beta_2 \rightarrow \beta_0 W^\Pi_\alpha \beta_2)$$

$$\wedge \forall \beta_0 < u_\omega \forall \beta_1 < u_\omega (\beta_0 W^\Sigma_\alpha \beta_0 \wedge \beta_1 W^\Sigma_\alpha \beta_1 \rightarrow \beta_0 W^\Pi_\alpha \beta_1 \vee \beta_1 W^\Pi_\alpha \beta_0),$$

it is Π^1_3 in the codes in $D \times \mathfrak{X}^{m,0,k}$. Hence, it is Δ^1_3 in the codes in $D \times \mathfrak{X}^{m,0,k}$.

As

$$\alpha \in \mathrm{WO}_{u_\omega} \iff \alpha \in \mathrm{LO}_{u_\omega} \wedge \forall x \in \omega^\omega (\forall n (x(n) \text{ codes some } \beta(n) < u_\omega \\ \wedge \beta(n+1) W_\alpha \beta(n)) \rightarrow \exists n (\beta(n+1) = \beta(n))),$$

WO_{u_ω} is Π^1_3 in the codes.

Definition. For each $\alpha \in \mathrm{WO}_{u_\omega}$, let

$$\|\alpha\| = \text{the length of the wellordering coded by } \alpha.$$

Lemma 4.16. *For any* $\alpha \in \mathrm{WO}_{u_\omega}$,

$$(L_{\|\alpha\|}[T_2])^{L_{\kappa_3}[T_2]} = L_{\|\alpha\|}[T_2].$$

Proof. It suffices to show that $\|\alpha\|^{L_{\kappa_3}[T_2]} = \|\alpha\|$.

If $\alpha \in \mathrm{WO}_{u_\omega}$, let $R_\alpha(\xi, \eta)$ be the Δ^1_3 in the codes wellordering on u_ω coded by α. As R_α is Δ_1 in the model $L_{\kappa_3}[T_2]$ and $L_{\kappa_3}[T_2] \models \mathrm{KP}$, $R_\alpha \in L_{\kappa_3}[T_2]$. $L_{\kappa_3}[T_2] \models \mathrm{KP}$ also implies that there is a rank function for R_α in $L_{\kappa_3}[T_2]$, since R_α is a wellordering in $\mathbf{V}$. This can be proved by an induction along the wellordering R_α. So, it makes sense to talk about $\|\alpha\|^{L_{\kappa_3}[T_2]}$.

If $\|\alpha\|^{L_{\kappa_3}[T_2]} = \beta$, then there is an order preserving one-to-one onto map $h \colon u_\omega \rightarrow \beta_0$ in $L_{\kappa_3}[T_2]$. This h is also in $\mathbf{V}$. This h witnesses that $\|\alpha\| = \beta$. $\square$

Definition. For any α, β in u_ω, let

$$\alpha \preceq^* \beta \iff \alpha \in \mathrm{WO}_{u_\omega} \wedge (\beta \in \mathrm{WO}_{u_\omega} \rightarrow \|\alpha\| \leq \|\beta\|,$$

$$\alpha \prec^* \beta \iff \alpha \in \mathrm{WO}_{u_\omega} \wedge (\beta \in \mathrm{WO}_{u_\omega} \rightarrow \|\alpha\| \leq \|\beta\|.$$

Lemma 4.17. *Both $\preceq^*$ and $\prec^*$ are Π^1_3 in the codes.*

Proof. Let D,W, W^Π and W^Σ as in Lemma 4.15. Let $\varphi^\Sigma(\varepsilon, \xi, \eta)$ be a Π_1 formula such that for all ε, ξ and η in u_ω,

$$(\varepsilon, \xi, \eta) \in W^\Sigma \iff L_{\kappa_3}[T_2] \models \varphi^\Sigma(\varepsilon, \xi, \eta).$$

Let $\varphi^\Pi(\varepsilon, \xi, \eta)$ be a Σ_1 formula such that for all ε, ξ and η in u_ω,

$$(\varepsilon, \xi, \eta) \in W^\Pi \iff L_{\kappa_3}[T_2] \models \varphi^\Pi(\varepsilon, \xi, \eta).$$

It is easy to check that for all α and β in u_ω,

$$\begin{aligned}\alpha \preceq^* \beta \iff \alpha \in \mathrm{WO}_{u_\omega} & \\ \wedge\, L_{\kappa_3}[T_2] \models & \begin{cases} \exists y \exists f (y = u_\omega \wedge\ f \text{ is a function from } y \text{ to } y \\ \wedge\ f \text{ is one-to-one on the domain of } \alpha \\ \wedge\, \forall \xi, \eta \in y (\varphi^\Pi(\alpha, \xi, \eta) \wedge \neg\varphi^\Sigma(\alpha, \xi, \eta) \\ \qquad \rightarrow \varphi^\Sigma(f(\beta), \xi, f(\eta)) \wedge \neg\varphi^\Pi(f(\beta), \xi, f(\eta)))) \end{cases}\end{aligned}$$

$\alpha \preceq^* \beta$ is clearly Π^1_3 in the codes, as we can replace $y = u_\omega$ by $L_y[T_2] \models \mathrm{KP}$.

It is also easy to check that for all α and β in u_ω,

$$\alpha \prec^* \beta \iff \alpha \in \mathrm{WO}_{u_\omega}$$

$$\wedge L_{\kappa_3}[T_2] \models \begin{cases} \exists y \exists f \exists z \in y (y = u_\omega \wedge f \text{ is a function from } y \text{ to } z \\ \wedge \ f \text{ is one-to-one on the domain of } \alpha \\ \wedge \forall \xi, \eta \in y (\varphi^\Pi(\alpha, \xi, \eta) \wedge \neg\varphi^\Sigma(\alpha, \xi, \eta) \\ \qquad \rightarrow \varphi^\Sigma(f(\beta), \xi, f(\eta)) \wedge \neg\varphi^\Pi(f(\beta), \xi, f(\eta)))). \end{cases}$$

Hence, $\prec^*$ is also Π^1_3 in the codes, by the Spector-Gandy theorem. □

The following relation will be needed in the next section also.

Definition. We define

$$\text{Lim}(\sigma) \iff \sigma \in \text{WO}_{u_\omega} \wedge \ \sigma \text{ codes a limit ordinal}$$

$$\text{Succ}(\eta, \sigma) \iff \eta \in \text{WO}_{u_\omega} \wedge \sigma \in \text{WO}_{u_\omega} \wedge \ \sigma \text{ codes an ordinal}$$

$$\text{which is the successor of the ordinal coded by } \eta.$$

Lemma 4.18. *Both* Lim *and* Succ *are* Δ^1_3 *in the codes in* WO_{u_ω}.

Proof. Let

$$\text{Lim}^\Sigma(\sigma) \iff \forall \alpha \in u_\omega (\alpha \prec^* \sigma \rightarrow \exists \beta \in u_\omega (\alpha \prec^* \beta \prec^* \sigma)),$$

$$\text{Lim}^\Pi(\sigma) \iff \forall \alpha \in u_\omega (\sigma \not\preceq^* \alpha \rightarrow \exists \beta \in u_\omega (\alpha \prec^* \beta \prec^* \sigma)).$$

It clear that Lim^Σ and Lim^Π witness that Lim is Δ^1_3 in the codes in WO_{u_ω}.

For $\text{Succ}(\eta, \sigma)$, we can define

$$\text{Succ}^\Sigma(\eta, \sigma) \iff \forall \alpha \in u_\omega (\alpha \prec^* \sigma \rightarrow \eta \not\prec^* \alpha),$$

$$\text{Succ}^\Pi(\eta, \sigma) \iff \exists \alpha \in u_\omega (\eta \prec^* \alpha \prec^* \sigma).$$

Theses Succ^{Σ} and Succ^{Π} witness that Succ is Δ^1_3 in the codes in WO_{u_ω}. □

By the proceeding lemma, both $\mathrm{Lim}(\sigma)$ and $\mathrm{Succ}(\eta, \sigma)$ are Δ^1_3 in the codes. The same remark after the proceeding lemma applies to these relations too.

Now, let us proceed towards our last lemma in this chapter. We will show how to decompose Π^1_3 in the codes sets into the union of κ_3 many ${}^*\Delta^1_3$ sets in a uniform and effective way.

Lemma 4.19. *Let $G^{m,k} \subseteq u_\omega \times \mathfrak{X}^{m,0,k}$ and $H^{m,k} \subseteq u_\omega \times \mathfrak{X}^{m,0,k}$ as given in Lemma 4.6, then there are Δ^1_3 in the codes $f, g \colon u_\omega \times u_\omega \to u_\omega$ such that*

(1) *for any $\alpha, \varepsilon \in \mathrm{WO}_{u_\omega}$,*

$$G^{m,k}_{f(\alpha,\varepsilon)} = H^{m,k}_{g(\alpha,\varepsilon)},$$

(2) *for any $\varepsilon, \alpha_1, \alpha_2 \in \mathrm{WO}_{u_\omega}$,*

$$\|\alpha_1\| \leq \|\alpha_2\| \to G^{m,k}_{f(\alpha_1,\varepsilon)} \subseteq G^{m,k}_{f(\alpha_2,\varepsilon)} \wedge H^{m,k}_{f(\alpha_1,\varepsilon)} \subseteq H^{m,k}_{f(\alpha_2,\varepsilon)},$$

(3) *for any $\varepsilon \in u_\omega$,*

$$H^{m,k}_{\varepsilon} = \bigcup_{\alpha \in \mathrm{WO}_{u_\omega}} G^{m,k}_{f(\alpha,\varepsilon)} = \bigcup_{\alpha \in \mathrm{WO}_{u_\omega}} H^{m,k}_{g(\alpha,\varepsilon)}.$$

Proof. Let φ_H be the Σ_0 formula such that for all $\varepsilon \in u_\omega$ and $(\vec{\alpha}, \vec{i}) \in \mathfrak{X}^{m,0,k}$,

$$H(\varepsilon, \vec{\alpha}, \vec{i}) \iff L_{\kappa_3}[T_2] \models \exists x \varphi_H(x, \varepsilon, \vec{\alpha}, \vec{i}).$$

For any $\alpha, \varepsilon \in u_\omega$ and $(\vec{\alpha}, \vec{i}) \in \mathfrak{X}^{m,0,k}$, let

$$A(\alpha, \varepsilon, \vec{\alpha}, \vec{i}) \iff (\alpha \in \mathrm{WO}_{u_\omega} \to L_{\|\alpha\|}[T_2] \models \exists x \varphi_H(x, \varepsilon, \vec{\alpha}, \vec{i})).$$

$A(\alpha, \varepsilon, \vec{\alpha}, \vec{i})$ is Δ^1_3 in the codes, because

$$A(\alpha, \varepsilon, \vec{\alpha}, \vec{i}) \iff \alpha \in \mathrm{WO}_{u_\omega} \to (L_{\|\alpha\|}[T_2])^{L_{\kappa_3}[T_2]} \models \exists x \varphi_H(x, \varepsilon, \vec{\alpha}, \vec{i})),$$

$$\iff \alpha \in \mathrm{WO}_{u_\omega} \to L_{\kappa_3}[T_2] \models (L_{\|\alpha\|}[T_2] \models \exists x \varphi_H(x, \varepsilon, \vec{\alpha}, \vec{i})),$$

and $L_{\|\alpha\|}[T_2] \models \exists x \varphi_H(x, \varepsilon, \vec{\alpha}, \vec{i})$ can be expressed by a Π_1 formula. Let

$$A(\alpha, \varepsilon, \vec{\alpha}, \vec{i}) \iff G^{m+2,k}(\varepsilon_A, \alpha, \varepsilon, \vec{\alpha}, \vec{i}).$$

We have that

$$A(\alpha, \varepsilon, \vec{\alpha}, \vec{i}) \iff G^{m,k}(s^{2,0,m,k}_{\Sigma}(\varepsilon_A, \alpha, \varepsilon), \vec{\alpha}, \vec{i}).$$

Let $f(\alpha, \varepsilon) = s^{2,0,m,k}_{\Sigma}(\varepsilon_A, \alpha, \varepsilon)$.

Now, for any $\alpha, \varepsilon \in u_\omega$ and $(\vec{\alpha}, \vec{i}) \in \mathfrak{X}^{m,0,k}$, let

$$B(\alpha, \varepsilon, \vec{\alpha}, \vec{i}) \iff (\alpha \in \mathrm{WO}_{u_\omega} \wedge L_{\|\alpha\|}[T_2] \models \exists x \varphi_H(x, \varepsilon, \vec{\alpha}, \vec{i})).$$

Similarly, $B(\alpha, \varepsilon, \vec{\alpha}, \vec{i})$ is Π^1_3 in the codes, so, there is a $\varepsilon_B \in u_\omega$ such that

$$B(\alpha, \varepsilon, \vec{\alpha}, \vec{i}) \iff H^{m+2,k}(\varepsilon_A, \alpha, \varepsilon, \vec{\alpha}, \vec{i}).$$

We have that

$$B(\alpha, \varepsilon, \vec{\alpha}, \vec{i}) \iff H^{m,k}(s^{2,0,m,k}_{\Pi}(\varepsilon_A, \alpha, \varepsilon), \vec{\alpha}, \vec{i}).$$

Let $g(\alpha, \varepsilon) = s^{2,0,m,k}_{\Pi}(\varepsilon_A, \alpha, \varepsilon)$. It is easy to see that f and g work. $\square$

5. A Technical Lemma

In this chapter, we will prove Hjorth's lemma mentioned in Chapter 4. The following proof is suggested by him.

Lemma (Hjorth). *Assuming* $\mathbf{\Delta}^1_2$ *determinacy, for any* $x \in \omega^\omega$ *and* $n \in \omega$, *if* A *is a* $\Pi^1_3(x)$ *in the codes subset of* u_n, *then* $A \in L_{\kappa_3^x}[T_2, x]$.

Proof. We will show this by induction on $n \in \omega$. For each $x \in \omega^\omega$ and $n \in \omega$, let $P(n, x)$ stand for the proposition corresponding to the x and n, i.e., for any $x \in \omega^\omega$ and $n \in \omega$, if A is a $\Pi^1_3(x)$ in the codes subset of u_n, then $A \in L_{\kappa_3^x}[T_2, x]$.

Base Case. $n = 1$.

At first, we will show that for any $x \in \omega^\omega$, $\alpha < u_1 = \omega_1$ and $n \in \omega$, if A is a $\Pi^1_3(x)$ in the codes subset of α, then $A \in L_{\kappa_3^x}[T_2, x]$.

To make the notations simple, we will drop the parameter x in the proof below, i.e., we will show that for any $\alpha < u_1 = \omega_1$ and $n \in \omega$, if A is a Π^1_3 in the codes subset of α, then $A \in L_{\kappa_3}[T_2]$. The relative version can be proved in a similar way.

Subcase 1. $\alpha < \omega$.

It is obviously true in this case.

Subase 2. $\alpha = \omega$.

This case is well-known. A proof about this case can be found from [KMS].

Subcase 3. $\alpha < \omega_1 = u_1$.

Let $M = L_{\kappa_3}[T_2]$, let

$$P = \{p : p \text{ is a function from some finite subset of } \omega \text{ to } \alpha\}.$$

We define a partial order on P as $p \leq q \iff p \supseteq q$. We call it $Coll(\omega, \alpha)$. It is clear that $Coll(\omega, \alpha) \in M$. Let $G \subseteq Coll(\omega, \alpha)$ be a M-generic subset of P. Then $f_G = \cup\{p : p \in G\}$ gives an onto map from ω to α.

It is easy to see that

$$M[G] = L_{\kappa_3}[T_2][G] = L_{\kappa_3}[T_2, G] = L_{\kappa_3}[T_2, f_G].$$

Let $\bar{A} = f_G^{-1}[A]$. Let w be a real coding f_G. Then, $\bar{A}$ is $\Sigma^1_3(w)$. Using and relativizing the argument in Subcase 2, we know that $\bar{A} \in M[G]$ and it is Δ_1 definable over $M[G]$ using the parameter G.

Let $\dot{A}$ be the forcing name such that $\dot{A}[G] = A$ in all $M[G]$. From the forcing theorem, $A = \{x : \ \Vdash \check{x} \in \dot{A}\}$. As the forcing relation is Δ_1, A is a Δ_1 definable subset of $\alpha \in L_{\kappa_3}[T_2]$ using the parameter $\dot{A} \in L_{\kappa_3}[T_2]$. Hence, $A \in L_{\kappa_3}[T_2]$.

Now, it is time to prove $P(1, x)$. We will show $P(1, 0)$ only, the relative case being similar.

Let A is a Δ^1_3 in the codes subset of u_1. Since A is Σ_1 over $L_{\kappa_3}[T_2]$, there is a Δ_0 formula $\psi(x, y)$ such that for all $\beta < \alpha$,

$$\beta \in A \iff L_{\kappa_3}[T_2] \models \exists y \psi(\beta, y).$$

For each $\beta \in A$, let

$$\rho(\beta) = \text{ the least } \gamma \text{ such that } L_\gamma[T_2] \models \exists y \psi(\beta, y).$$

It suffices to show that $\sup_{\beta \in A} \rho(\beta) < \kappa_3$, since if this is true, then

$$\beta \in A \iff L_{\sup_{\beta \in A} \rho(\beta)}[T_2] \models \exists y \psi(\beta, y).$$

This is a Δ_1 definition, so $A \in L_{\kappa_3}[T_2]$.

Now, we assume that $\sup_{\beta \in A} \rho(\beta) = \kappa_3$, towards a contradiction.

For any $\delta < \omega_1$, let $w(\delta)$ be a real in $L_{\kappa_3}[T_2]$ which codes δ. As $A \cap \delta \subseteq \delta < \omega_1$ and $A \cap \delta$ is $\Pi^1_3(w(\delta))$ in the codes, $A \cap \delta \in L_{\kappa_3^x}[T_2, w(\delta)] = L_{\kappa_3}[T_2]$ by Subcase 3. As

$$L_{\kappa_3}[T_2] \models \forall \beta \in A \cap \delta \exists \gamma (L_\gamma \models \exists y \psi(\beta, y)).$$

and

$$L_{\kappa_3}[T_2] \models \Delta_1 \text{ collection axiom},$$

$\rho[A \cap \delta]$ is bounded below κ_3. Let $\rho_\delta = \sup \rho[A \cap \delta]$. Since $\sup_{\beta \in A} \rho(\beta) = \kappa_3$, then $(\rho_\delta)_{\delta \in \omega_1}$ is a non-decreasing sequence of ordinals cofinal in κ_3. So, $cf(\kappa_3) = cf(\omega_1) = \aleph_1$, this contradicts the fact that $cf(\kappa_3) = \omega$.

Inductive Step. Assume $P(n, x)$, we will show $P(n+1, x)$. At first, we will show the following

Claim. For all $x \in \omega^\omega$ and $\alpha < u_{n+1}$, every $\Pi^1_3(x)$ in the codes subset of α is in $L_{\kappa_3}[T_2]$.

Proof of the claim. We will show this by contradiction. Assume that there is a $x \in \omega^\omega$, an $\alpha < u_{n+1}$ and a $\Pi^1_3(x)$ in the codes $A \subseteq \alpha$ such that $A \notin L_{\kappa_3}[T_2]$, towards a contradiction.

Let α be coded by some $y \in \mathrm{WO}_{u_\omega}$, i.e., $y = \langle k, z^\sharp \rangle$ such that

$$\alpha = \tau_k^{\mathbf{L}[z]}(u_1, \cdots, u_{l(k)}),$$

where $(\tau_n)_{n\in\omega}$ lists all the Skolem terms taking ordinal values in $\mathbf{L}[z]$ and $l(k)$ is the number of variable of τ_n.

Let $(\delta_i)_{i\in\omega}$ enumerate the first ω many $\mathbf{L}[z]$ indiscernibles after u_n in an increasing order. This sequence is definable from u_n over $\mathbf{L}(z^\sharp)$, hence in $\mathbf{L}[y]$.

Let

$$A = \{(\xi_1, \cdots, \xi_j, i, p) : i, j, p \in \omega, l(i) = j + p$$
$$\xi_1 \cdots, \xi_j \in u_n \text{ and } \tau_i^{\mathbf{L}[z]}(\xi_1, \cdots, \xi_j, \delta_1, \cdots, \delta_p) \in A\}.$$

Let $f \colon u_n^{<\omega} \to u_n$ be a Δ_1 $\mathbf{L}$-definable from u_n bijection, let π^Σ and π^Π be the Σ_1 and Π_1 formulas which define f over $\mathbf{L}$, i.e., for all $g \in \mathbf{L}$,

$$g = f \iff \mathbf{L} \models \pi^\Sigma(g, u_n) \iff \mathbf{L} \models \pi^\Pi(g, u_n).$$

Let $(\tau_{i_j})_{j\in\omega}$ be the Skolem terms which define f in the following sense: for any $\xi_1,\cdots,\xi_j \in u_n$,

$$\eta = f(\xi_1, \cdots, \xi_j) \iff \eta = \tau_{i_j}^{\mathbf{L}[z]}(\xi_1, \cdots, \xi_j, u_1, \cdots, u_n).$$

The sequence $(\tau_{i_j})_{j\in\omega}$ is simple enough, say, $\Delta^1_3(z)$ in the codes.

Let $(\tau_{m_j})_{j\in\omega}$ be the Skolem terms which define f^{-1} in the following sense: for any $\xi \in u_n$ and $j < len(f^{-1}(\xi))$,

$$\eta = (f^{-1}(\xi))_j \iff \eta = \tau_{m_j}^{\mathbf{L}[z]}(\xi, u_1, \cdots, u_n).$$

The sequence $(\tau_{m_j})_{j\in\omega}$ is also simple enough, say, $\Delta^1_3(z)$ in the codes.

Let $\hat{A} = f[\bar{A}] \subseteq u_n$. Since

$$\beta \in \hat{A} \iff f^{-1}(\beta) \in \bar{A}$$

$$\iff \exists i, j, p \in \omega, \xi_1, \cdots, \xi_j \in u_n(l(i) = j + p$$

$$\wedge\, \tau_i^{\mathbf{L}[z]}(\xi_1, \cdots, \xi_j, \delta_1, \cdots, \delta_p) \in A \wedge \bigwedge_{1\leq l\leq j} \xi_l = \tau_{m_l}^{\mathbf{L}[z]}(\beta, u_1, \cdots, u_n))$$

$$\iff \exists i, j, p \in \omega, \xi_1, \cdots, \xi_j, \gamma \in u_\omega(l(i) = j + p \wedge \gamma \in A$$

$$\tau_i^{\mathbf{L}[z]}(\tau_{m_1}^{\mathbf{L}[z]}(\beta, u_1, \cdots, u_n), \cdots, \tau_{m_j}^{\mathbf{L}[z]}(\beta, u_1, \cdots, u_n), \delta_1, \cdots, \delta_p) = \gamma),$$

$\hat{A}$ is $\Pi^1_3(x, z)$.

Let $\theta(x, y) = \exists y\theta_0(x, y)$ be a Σ_1 formula which define A over $L_{\kappa_3^x}[T_2, x]$. Let

$$\hat{\theta}(\xi) \iff \exists w \exists g \exists \gamma \exists i, j, p \in u_\omega(\gamma \in A \wedge \theta_0(\gamma, w) \wedge \mathbf{L} \models \pi^\Sigma(g, u_n)$$

$$\gamma = \tau_i^{\mathbf{L}[z]}((f^{-1}(\xi))_1, \cdots, (f^{-1}(\xi))_j, \delta_1, \cdots, \delta_p)),$$

$\hat{\theta}(\xi)$ is $\Sigma_1(u_n, x, y)$.

For $\beta \in A$, let

$$\varphi(\beta) = \text{ the least } \beta > u_\omega \text{ such that } L_\beta[T_2, x] \models \theta(\beta).$$

For $\hat{\beta} \in A$, let

$$\hat{\varphi}(\hat{\beta}) = \text{ the least } \hat{\beta} > u_\omega \text{ such that } L_{\hat{\beta}}[T_2, x, y] \models \hat{\theta}(\hat{\beta}).$$

It is easy to see that for every $\beta \in A$, there is a $\hat{\beta} \in \hat{A}$ such that $\hat{\varphi}(\hat{\beta}) \geq \varphi(\beta)$.

Since $A \notin L_{\kappa_3^x}[T_2, x]$, φ is unbounded below κ_3^x, i.e.,

$$\sup_{\beta \in A} \varphi(\beta) = \kappa_3^x.$$

As $\hat{A} \subseteq u_n$, $\hat{A}$ is $\Pi_3^1(x, y)$, by the induction hypothesis,

$$\sup_{\hat{\beta} \in \hat{A}} \hat{\varphi}(\hat{\beta}) < \kappa_3^{x,y}.$$

But,

$$\sup_{\hat{\beta} \in \hat{A}} \hat{\varphi}(\hat{\beta}) \geq \sup_{\beta \in A} \varphi(\beta),$$

we have that

$$\kappa_3^x < \kappa_3^{x,y}.$$

By Lemma 14.4 in [KMS],

$$y_3^x \leq_3^Q \langle x, y \rangle,$$

where y_3^x is an element in the first nontrivial $\Delta_3^1(x)$ degree of all $\Pi_3^1(x)$ singletons under the $\Delta_3^1(x)$ reduction, and $\leq_3^Q$ is the Q-reduction defined in [KMS]. We do not have to care about the exact definition of Q-reductions here, because we have

$$y_3^x \leq_3 \langle x, y \rangle,$$

actually, since $Q(x, y)$ consists of only trivial $\Pi_3^1(x, y)$ singletons and y_3^x is a $\Pi_3^1(x, y)$ singleton.

So far, we have showed the following:

$$\text{for any } y \text{ coding } \alpha,\ y_3^x \leq_3 \langle x, y \rangle.$$

Let

$$B = \{y : y \in \mathrm{WO}_{u_\omega} \wedge \forall w \in \mathrm{WO}_{u_\omega}(w \sim_\omega y \rightarrow y_3^x \in \Delta_3^1(x, w))\},$$

where $x \sim_\omega y$ iff x and y code the same ordinal smaller than u_ω. B is $\Pi_3^1(x)$. We have showed that B is not empty. By the Kechris-Martin theorem, there is a $\Delta_3^1(x)$ real $y^* \in B$. As $y_3^x \in \Delta_3^1(x, y^*)$, $y_3^x \in \Delta_3^1(x)$, which is a contradiction. □(Claim)

Now, let us prove $P(n+1, x)$. We will actually prove $P(n+1, 0)$ below, the relative case can be proved in a similar way.

Let A is a Δ_3^1 in the codes subset of u_{n+1}. Since A is Σ_1 over $L_{\kappa_3}[T_2]$, there is a Δ_0 formula $\psi(x, y)$ such that for all $\beta < \alpha$,

$$\beta \in A \iff L_{\kappa_3}[T_2] \models \exists y \psi(\beta, y).$$

For each $\beta \in A$, let

$$\rho(\beta) = \text{ the least } \gamma \text{ such that } L_\gamma[T_2] \models \exists y \psi(\beta, y).$$

It suffices to show that $\sup_{\beta \in A} \rho(\beta) < \kappa_3$, since if this is true, then

$$\beta \in A \iff L_{\sup_{\beta \in A} \rho(\beta)}[T_2] \models \exists y \psi(\beta, y),$$

and this is a Δ_1 definition, so $A \in L_{\kappa_3}[T_2]$.

Now, we assume that $\sup_{\beta \in A} \rho(\beta) = \kappa_3$ towards a contradiction.

For any $\delta < u_{n+1}$, Let $w(\delta)$ be a real in $L_{\kappa_3}[T_2]$ which codes δ. As $A \cap \delta \subseteq \delta < u_{n+1}$ and $A \cap \delta$ is $\Pi^1_3(w(\delta))$ in the codes, $A \cap \delta \in \mathbf{L}_{\kappa_3^x}[T_2, w(\delta)] = L_{\kappa_3}[T_2]$ by Subcase 2. As

$$L_{\kappa_3}[T_2] \models \forall \beta \in A \cap \delta \exists \gamma (L_\gamma \models \exists y \psi(\beta, y)).$$

and

$$L_{\kappa_3}[T_2] \models \Delta_1 \text{ collection axiom},$$

$\rho[A \cap \delta]$ is bounded below κ_3. Let $\rho_\delta = \sup \rho[A \cap \delta]$. Since $\sup_{\beta \in A} \rho(\beta) = \kappa_3$, then $(\rho_\delta)_{\delta \in u_{n+1}}$ is a non-decreasing sequence of ordinals cofinal in κ_3. So, $cf(\kappa_3) = cf(u_{n+1}) \neq \omega$, this contradicts the fact that $cf(\kappa_3) = \omega$. □(The Inductive Step)

□(Lemma)

6. Representation of Thin Π^1_3 Equivalence Relations

We have standard thin Π^1_1 equivalence relations on reals, namely, any equivalence relations Δ^1_1 reducible to Π^1_1 equivalence relations on ω. Harrington showed that these are all the thin Π^1_1 equivalence relations on reals actually.

Theorem (Harrington) 6.1. *For any thin Π^1_1 equivalence relation on ω^ω, there is a Δ^1_1 function p from ω^ω to ω and an equivalence relation e on ω such that for any x, y in ω^ω,*

$$xEy \iff (p(x), p(y)) \in e.$$

Proof (Harrington). See to [Ha1]. $\square$

Harrington's idea is as follows:

Let E be a thin equivalence relation on ω^ω, let $\{X_i\}_{i\in\omega}$ be a Δ^1_1 enumeration of Δ^1_1 subsets of ω^ω such that

(1) $\forall x \in \omega^\omega \exists i \in \omega (x \in X_i)$,

(2) $\forall i \in \omega \forall x \in \omega^\omega \forall y \in \omega^\omega (x \in X_i \wedge y \in X_i \rightarrow (x, y) \in E)$.

He defined $p\colon \omega^\omega \rightarrow \omega$ as

$$p(x) = i \iff i \text{ is the least natural number such that } x \in X_i.$$

It is natural to think to define an equivalence relation e on ω by letting $(i, j) \in e$ if and only if all real numbers in $p^{-1}(i) \cup p^{-1}(j)$ are E equivalent. But, this does not work, since if there is some i such that $p^{-1}(i) = \emptyset$, e defined as before will become $\omega \times \omega$. Harrington built this e step-by-step using induction all the way to ω_1^{CK}. At each step, he put only at most one carefully selected pair and all pairs induced by this pair into e. More precisely, he built a sequence of Δ_1^1 equivalence relations $\{e^\sigma\}_{\sigma<\omega^{CK}}$. Let e be the union of these e^σ. He started from $e^0 = id(\omega \times \omega)$. For σ a limit ordinal, he simply took a union to define e^σ. For σ a successor of a non-limit ordinal, he put nothing new into e^σ. For σ a successor of a limit ordinal λ, he put the first pair $(i, j) \notin e$ such that for unbounded many $\eta < \lambda$, the Π_1^1 assertion "all reals in $Y_i^\eta \cup Y_j^\eta$ are E-equivalent" can be seen to be true in less than λ steps, where $Y_i = \{x : (p(x), i) \in e\}$.

Many similarities between Π_1^1 and Π_3^1 were found by Kechris, Martin, Moschovakis, Solovay and others (see [Ke5] for a summary). For example, we have the prewell-ordering property, scale property, the Martin-Solovay representation theorem, the Spector-Gandy theorem for the third level, which are counterparts of the corresponding results for the first level. It seems that the following is a good analog:

$$\frac{\Pi_1^1}{\langle \omega, T_0 \rangle} \approx \frac{\Pi_3^1}{\langle u_\omega, T_2 \rangle},$$

where T_2 is the Martin-Solovay tree and T_0 the recursive tree on ω whose branches produce the complete Π_1^0 set of ω^ω. However, there are many differences between them, for example, the natural generalization of the basis theorem fails in the context of Σ_3^1, $L[T_2] \prec_{\Sigma_4^1} \mathbf{V}$ fails while $(L[T_0] =)L \prec_{\Sigma_2^1} \mathbf{V}$ holds. These similarities and differences make it pretty interesting to consider what will be the counterpart of the Harrington representation theorem in the third level. To work in the third level of the analytical hierarchy, we need some determinacy. From now on, we will always assume $\mathbf{\Delta}^1_2$ determinacy. We can show the following theorem later in this thesis.

Theorem. *For any thin Π^1_3 equivalence relation on ω^ω, there is a Δ^1_3 in the codes function p from ω^ω to u_ω and a Π^1_3 in the codes equivalence relation e on u_ω such that for any x, y in ω^ω,*

$$xEy \iff (p(x))e(p(y)).$$

Hjorth lifted Harrington's proof of the Silver perfect set theorem to the third level in [Hj1]. He had

Lemma (Folklore) 6.2. *Let E be a thin Π^1_3 equivalence relation on ω^ω. Then for any $x \in \omega^\omega$, there are $n \in \omega$, $\alpha \in u_\omega$, $D \subseteq \omega^\omega$, a Σ^1_3 $M \subseteq \omega^\omega \times \omega^\omega$ and a Π^1_3 $N \subseteq \omega^\omega \times \omega^\omega$ with*

(1) $\exists y \in \omega^\omega(\tau_n^{L[y]}(u_1, \dots, u_{k(n)}) = \alpha)$,

(2) $\forall y \in \omega^\omega(\tau_n^{L[y]}(u_1, \dots, u_{k(n)}) = \alpha \to D = M_y = N_y)$,

(3) $x \in D$,

(4) $D \subseteq [x]_E$.

From this lemma, it is easy to show

Lemma 6.3. *If E is a thin Π^1_3 in the codes equivalence relation, then there* *is a sequence $\{X_\alpha\}_{\alpha<u_\omega}$ such that*

(1) *$X_\alpha \subseteq$ some E-equivalence class,*

(2) *the relation "$x \in X_\alpha$" is Δ^1_3 in the codes,*

(3) *for all reals x, there is a α in u_ω such that $x \in X_\alpha$.*

Proof. Let E be a thin Π^1_3 equivalence relation. From the above lemma, for any $x \in \omega^\omega$, there is $\alpha \in u_\omega$ and $C \subseteq \omega^\omega$ such that

(1) $x \in C \subseteq [x]_E$,

(2) C is uniformly $\Delta^1_3(\alpha)$, i.e., there are Σ^1_3 $C^\Sigma \subseteq \omega^\omega \times \omega$ and Π^1_3 $C^\Pi \subseteq \omega^\omega \times \omega^\omega$ such that

$$\forall n \in \omega \forall y \in \omega^\omega (\alpha = \text{ the ordinal coded by } (n, y^\sharp) \rightarrow C = C^\Sigma_{(n,y^\sharp)} = C^\Pi_{(n,y^\sharp)}).$$

Let fix $D \subseteq u_\omega$, W, W^Π, $W^\Sigma \subseteq u_\omega \times u_\omega$ as in Lemma 4.15. Then, for any $x \in \omega^\omega$, there is an $\alpha \in D$ such that $x \in W_\alpha \subseteq [x]_E$. Now, let

$$A = \{(x, \alpha) : \alpha \in D \text{ and } x \in W_\alpha \subseteq [x]_E\}.$$

A is Π^1_3 in the codes. Let the Π^1_3 in the codes set A_0 uniformize A. As A_0 is the graph of some function, A_0 is actually Δ^1_3 in the codes. Let

$$B = \{\alpha \in u_\omega : \exists x \in \omega^\omega((x, \alpha) \in A_0)\}.$$

B is Σ^1_3 in the codes and $B \subseteq D$. Let B_0 be a Δ^1_3 in the codes set separating B from $u_\omega \setminus D$, i.e., $B \subseteq B_0 \subseteq D$. By effective induction, there is a Δ^1_3 in the codes function $f : u_\omega \to u_\omega$ enumerating B_0. Let $X_\alpha = W_{f(\alpha)}$. It is easy to check that $\{X_\alpha\}_{\alpha<u_\omega}$ works. $\square$

There are two major difficulties in lifting Harrington's result. In Harrington's proof, he can code the whole process of induction by a real number since only recursive ordinals are needed in his proof. He can guarantee that e^σ is Δ^1_1 by showing that it is both Σ^1_1 and Π^1_1 using a real coding the induction process. It is a method for showing Δ^1_1 from the top. We have to prove that our construction is Δ^1_3 from the bottom up since we are doing induction all the way to κ_3 which is a much larger ordinal compared with ω_1^{CK}, and we cannot code our process by a real number. That is why we have to develop the effective theory of Σ^1_3, Π^1_3 and Δ^1_3 in Chapter 4. We can show that our process is Δ^1_3 by two effective inductions. The other difficulty comes from a very good property of ω. It is a small cardinal but has many combinatorial properties of large cardinals. Harrington used the obvious fact that all finite sets of recursive ordinals have upper bounds. He needed the upper bound to freeze the construction at some step for all smaller natural numbers, to guarantee that every pair of natural numbers can get attention at some step of his construction. To lift Harrington's result, we have to consider an infinite set of ordinals. The existence of the upper bound is not trivial this time. However, Hjorth's lemma helps us out.

For any $x \in \omega^\omega$, let $p(x)$ be the least $\alpha < u_\omega$ such that $x \in X_\alpha$. The graph of this p is Δ^1_3 in the codes.

We will build our equivalence relation e on u_ω by induction along ordinals up to κ_3, where

$$\kappa_3 = \text{the least ordinal } \kappa > u_\omega \text{ such that } L_\kappa[T_2] \models \text{KP}$$

$$= \sup\{\lambda : \lambda \text{ is the length of a } \Delta^1_3(\alpha) \text{ well-ordering}$$

$$\text{of subsets of } u_\omega \text{ for some } \alpha \in u_\omega\}.$$

Let us give an informal description of our construction before we go into the tedious details.

We will build a sequence $\{e_\sigma\}_{\sigma<\kappa_3}$ of ${}^*\Delta^1_3$ in the codes equivalence relations on u_ω. For each $\alpha \in u_\omega$, we let

$$Y^\sigma_\alpha = \{x : p(x) e_\sigma \alpha\}.$$

We will guarantee that all reals in Y^σ_α are E-equivalent and

$$\sigma < \sigma' \implies e_\sigma \subseteq e_{\sigma'}$$

from our construction.

(1) Let e_0 be the equality relation on u_ω.

(2) For σ a limit ordinal, let $e_\sigma = \bigcup_{\sigma'<\sigma} e_{\sigma'}$.

(3) For σ a non-limit ordinal, let $e_{\sigma+1} = e_\sigma$.

(4) For σ a limit ordinal, let us define $e_{\sigma+1}$ as follows: See if there are ordinals α and β smaller than u_ω such that $(\alpha, \beta) \notin e_\sigma$ and for unboundedly many $\eta < \sigma$ the Π^1_3 assertion:

(*) $$\text{all reals in } Y^\eta_\alpha \cup Y^\eta_\beta \text{ are } E\text{-equivalent}$$

can be seen to be true in $< \sigma$ steps. In our formal construction, we will work carefully to guarantee that Y_α^σ is $\Delta_3^1(\alpha, \sigma)$ in the codes so that the assertion $(*)$ above is really Π_3^1 in the codes. If there are no such pair of ordinals, let $e_{\sigma+1} = e_\sigma$. Otherwise, let (α, β) be the first such pair of ordinals under the natural well-ordering of $u_\omega \times u_\omega$, and let $e_{\sigma+1}$ be the smallest equivalence relation on u_ω such that $e_{\sigma+1} \supseteq e_\sigma$ and $(\alpha, \beta) \in e_{\sigma+1}$.

Then let $e = \bigcup_{\sigma < \kappa_3} e_\sigma$.

Now, let us go to the formal details to guarantee e is Π_3^1 in the codes. Basically, we need two effective inductions to guarantee that Y_α^σ is $\Delta_3^1(\alpha, \sigma)$ in the codes, one for $\Pi_3^1(\alpha, \sigma)$ and another for $\Pi_3^1(\alpha, \sigma)$.

Let $H \subseteq u_\omega \times u_\omega \times u_\omega \times u_\omega$ be a good universal Π_3^1 in the codes set for the ${}^*\Pi_3^1$ subsets of $u_\omega \times u_\omega \times u_\omega$, $G \subseteq u_\omega \times u_\omega \times u_\omega \times u_\omega \times u_\omega$ a universal Σ_3^1 in the codes set for the ${}^*\Sigma_3^1$ in the codes subsets of $u_\omega \times u_\omega \times u_\omega \times u_\omega$, i.e., $H = H^{3,0}$ and $G = G^{4,0}$. These G and H are trying to witness that Y_α^σ is $\Delta_3^1(\alpha, \sigma)$ in the codes.

Let

$$P(d, m, \alpha, \sigma, \beta) \iff \begin{cases} \forall x, y((G(d, m, p(x), \sigma, \alpha) \vee G(d, m, p(x), \sigma, \beta)) \\ \wedge(G(d, m, p(y), \sigma, \alpha) \vee G(d, m, p(y), \sigma, \beta)) \\ \rightarrow xEy). \end{cases}$$

After we "diagonalize" d and m, i.e., let $d = d^*$ and $m = m^*$, where d^* and m^* are the fixed points to be determined by the recursion theorem later in this chapter, $P(d, m, \alpha, \sigma, \beta)$ will mean that for all x, y in $Y_\alpha^\sigma \cup Y_\beta^\sigma$, $(x, y) \in E$, where $Y_\alpha^\sigma = \{x : (p(x), \alpha) \in e^\sigma\}$ and e^σ is the equivalence relation constructed up to the σ-step. This $P(d, m, \alpha, \sigma, \beta)$ is clearly Π_3^1 in the codes. So, there is a $l \in \omega$ such that

$$P(d, m, \alpha, \sigma, \beta) \iff H^{3,2}(l, \sigma, \alpha, \beta, d, m).$$

Hence, there is some Δ_3^1 in the codes function $s_\Pi^{2,1,2,0} : \omega \times \omega^\omega \times \omega^\omega \times \omega \times \omega \to u_\omega$ such that for all σ, α, β, d and m,

$$H^{3,2}(l, \sigma, \alpha, \beta, d, m) \iff H^{1,0}(s_\Pi^{2,1,2,0}(l, \alpha, \beta, d, m), \sigma).$$

The superscripts are pretty annoying, so we introduce some new notation to simplify them. Let

$$U(\epsilon, \alpha) \iff G^{1,0}(\epsilon, \alpha),$$

$$V(\epsilon, \alpha) \iff H^{1,0}(\epsilon, \alpha),$$

$$h(l, d, m, \alpha, \beta) = s_\pi^{2,1,2,0}(l, \alpha, \beta, d, m).$$

So,

$$P(d, m, \alpha, \sigma, \beta) \iff V(h(l, d, m, \alpha, \beta), \sigma).$$

Let g_U and f_V be given by the last lemma in Chapter 3.

Let

$$R^\Sigma(d, m, \alpha, \sigma, \beta) \iff \begin{cases} \operatorname{Lim}(\sigma) \wedge \neg H(m, \alpha, \sigma, \beta) \wedge \forall \sigma_0 \in u_\omega(\sigma_0 \prec^* \sigma \\ \quad \to \exists \sigma_1, \sigma_2 \in u_\omega(\sigma_1 \not\preceq^* \sigma_0 \wedge \sigma \not\preceq^* \sigma_1 \wedge \sigma \not\preceq^* \sigma_2 \\ \quad\quad \wedge U(g_U(\sigma_2, h(l, d, m, \alpha, \beta)), \sigma_1)). \end{cases}$$

After the diagonalization, R^{Σ} is a Σ^1_3 formula claiming that α is not equivalent to β up to the σ-th stage of our construction, before which there are unboundedly many stages at which the P can be witnessed to be true.

Let

$$R^{\Pi}(d,m,\alpha,\sigma,\beta) \iff \begin{cases} Lim(\sigma) \wedge \neg G(d,m,\alpha,\sigma,\beta) \wedge \forall \sigma_0 \in u_\omega(\sigma \not\preceq^* \sigma_0 \\ \qquad \rightarrow \exists \sigma_1, \sigma_2 \in u_\omega(\sigma_0 \prec^* \sigma_1 \prec^* \sigma \wedge \sigma_2 \prec^* \sigma \\ \qquad\qquad \wedge V(f_V(\sigma_2, h(l,d,m,\alpha,\beta)),\sigma_1)). \end{cases}$$

R^{Π} expresses the same fact as R^{Σ} after the diagonalization, i.e., for any α, $\beta \in u_\omega$ and $\sigma \in \mathrm{WO}_{u_\omega}$,

$$R^{\Sigma}(d^*,m^*,\alpha,\sigma,\beta) \iff R^{\Pi}(d^*,m^*,\alpha,\sigma,\beta),$$

where d^* and m^* are the fixed points to be determined by the recursion theorem later in this chapter. However, R^{Π} is a Π^1_3 formula now.

Let $B \subseteq \omega \times \omega \times u_\omega \times u_\omega \times u_\omega$ be defined as the follows:

for any d, m, α, σ, β in u_ω, $(d,m,\alpha,\sigma,\beta) \in B$ if and only if

EITHER $\alpha = \beta$,

OR $\sigma \in \mathrm{LO}_{u_\omega} \wedge \mathrm{Lim}(\sigma) \wedge \exists \eta(\sigma \not\preceq \eta \wedge G(d,m,\alpha,\sigma,\beta))$,

OR $\sigma \in \mathrm{LO}_{u_\omega} \wedge \exists \eta(\neg \mathrm{Lim}(\eta) \wedge \eta \in \mathrm{LO}_{u_\omega} \wedge \mathrm{Succ}(\eta,\sigma) \wedge G(d,m,\alpha,\eta,\beta))$,

OR $\sigma \in \mathrm{LO}_{u_\omega} \wedge \exists \eta(\mathrm{Lim}(\eta) \wedge \mathrm{Succ}(\eta,\sigma)$

$\wedge \exists n \in \omega \exists s \in {u_\omega}^{<n}(\wedge \alpha = s(0) \wedge \beta = s(n-1)$

$$\forall k < n-1(G(d,m,s(k),\eta,s(k+1))$$

$$\vee R(d,m,s(k),\eta,s(k+1)))),$$

where

$$R(d,m,\alpha,\sigma,\beta) \iff \begin{cases} R^{\Sigma}(d,m,\alpha,\sigma,\beta)\wedge \\ \forall\alpha'\beta'((\alpha',\beta') \prec_{u_\omega\times u_\omega} (\alpha,\beta)) \to \neg R^{\Pi}(d,m,\alpha,\sigma,\beta)). \end{cases}$$

From our construction above, B is Σ^1_3. Let d^* be the fixed point from the recursion theorem, i.e., for all m,α,σ,β in u_ω,

$$(d^*,m,\alpha,\sigma,\beta) \in B \iff G(d^*,m,l,\alpha,\sigma,\beta).$$

Let us fix this d^* from here on.

Let

$$Q(m,\alpha,\sigma,\beta) \iff \begin{cases} \forall x,y((G(d^*,m,p(x),\sigma,\alpha) \vee G(d^*,m,p(x),\sigma,\beta)) \\ \wedge(G(d^*,m,l,p(y),\sigma,\alpha) \vee G(d^*,m,l,p(y),\sigma,\beta)) \\ \to xEy). \end{cases}$$

Let $N \subseteq u_\omega \times u_\omega$ a good universal Π^1_3 set for all ${}^*\Pi^1_3$ subsets of u_ω and $M \subseteq u_\omega \times u_\omega$ a good universal Σ^1_3 set for all ${}^*\Sigma^1_3$ subsets of u_ω. Let f_N,g_M be the function given in the last lemma of Chapter 4.

By the *s-m-n* theorem, there must be a Δ^1_3 in the codes $\bar{h}\colon \omega^2 \times {u_\omega}^2$ such that for all m, α, σ, β,

$$Q(m,\alpha,\sigma,\beta) \iff N(\bar{h}(l',m,\alpha,\beta),\sigma),$$

where l' is a natural number which is the index of Q.

Let

$$S^{\Sigma}(m,\alpha,\sigma,\beta) \iff \begin{cases} \mathrm{Lim}(\sigma) \wedge \neg H(m,\alpha,\sigma,\beta) \wedge \forall\sigma_0 \in u_\omega(\sigma_0 \prec^* \sigma \\ \quad \to \exists\sigma_1,\sigma_2 \in u_\omega(\sigma_1 \not\preceq^* \sigma_0 \wedge \sigma \not\preceq^* \sigma_1 \wedge \sigma \not\preceq^* \sigma_2 \\ \quad\quad \wedge M(g_M(\sigma_2, \bar{h}(l',m,\alpha,\beta)),\sigma_1)). \end{cases}$$

S^{Σ} has similar meaning as R^{Σ}, the only difference is that at this stage, we have a fixed d^*.

Let

$$S^{\Pi}(m,\alpha,\sigma,\beta) \iff \begin{cases} \mathrm{Lim}(\sigma) \wedge \neg G(d^*,m,\alpha,\sigma,\beta) \wedge \forall\sigma_0 \in u_\omega(\sigma \not\preceq^* \sigma_0 \\ \quad \to \exists\sigma_1,\sigma_2 \in u_\omega(\sigma_0 \prec^* \sigma_1 \prec^* \sigma \wedge \sigma_2 \prec^* \sigma \\ \quad\quad \wedge N(f_N(\sigma_2, \bar{h}(l',m,\alpha,\beta)),\sigma_1)). \end{cases}$$

S^{Π} expresses the same fact as S^{Σ} but in a Π^1_3 form.

Let $A \subseteq u_\omega \times u_\omega \times u_\omega \times u_\omega$ be defined as the following: for any m, α, σ, β in u_ω, $(m,\alpha,\sigma,\beta) \in A$ if and only if

EITHER $\quad \alpha = \beta$,

OR $\quad \sigma \in \mathrm{LO}_{u_\omega} \wedge \mathrm{Lim}(\sigma) \wedge \exists\eta(\eta \prec^* \sigma \wedge H(m,\alpha,\sigma,\beta))$,

OR $\quad \sigma \in \mathrm{LO}_{u_\omega} \wedge \exists\eta(\neg\mathrm{Lim}(\eta) \wedge \eta \in \mathrm{LO}_{u_\omega} \wedge \mathrm{Succ}(\eta,\sigma) \wedge H(m,\alpha,\sigma,\beta))$,

OR $\quad \sigma \in \mathrm{LO}_{u_\omega} \wedge \exists\eta(\mathrm{Lim}(\eta) \wedge \mathrm{Succ}(\eta,\sigma)$

$\quad \wedge \exists n \in \omega \exists s \in {u_\omega}^{<n}(\wedge\alpha = s(0) \wedge \beta = s(n-1)$

$\quad\quad \forall k < n-1(H(m,s(k),\eta,s(k+1))$

$\quad\quad\quad \vee S(m,s(k),\eta,s(k+1))))$, where

$$S(m,\alpha,\sigma,\beta) \iff \begin{cases} S^{\Pi}(m,\alpha,\sigma,\beta)\wedge \\ \forall\alpha'\beta'((\alpha',\beta') \prec_{u_\omega\times u_\omega} (\alpha,\beta)) \to \neg S^{\Sigma}(m,\alpha,\sigma,\beta)). \end{cases}$$

It is easy to see that B is Π^1_3. Let m^* be the fixed point from the recursion theorem, i.e., for all α,σ,β in u_ω,

$$(m^*,\alpha,\sigma,\beta) \in B \iff H(m^*,\alpha,\sigma,\beta).$$

Let us fix this m^* from here. For any α,β in u_ω, let

$$\alpha e \beta \iff \exists\sigma \in \mathrm{WO}_{u_\omega} B(m^*,\alpha,\sigma,\beta),$$

this e is clearly a Π^1_3 in the codes equivalence relation on u_ω.

By the induction on $\|\sigma\|$, we have the following

Lemma 6.4. *For any α, β and σ in WO_{u_ω},*

$$G(d^*,m^*,\alpha,\sigma,\beta) \iff H(m^*,\alpha,\sigma,\beta).$$

This lemma guarantees that the e defined before this lemma is just the equivalence relation that we described informally at the beginning of this chapter.

Now, it is time to prove that for all x and y in ω^ω,

$$(x,y) \in E \iff (p(x),p(y)) \in e.$$

For any $\alpha \in u_\omega$, let

$$Y_\alpha = \{x : (p(x),\alpha) \in e\}.$$

It suffices to prove the following

Lemma 6.5. *For each α and β in ω^ω, if all reals in Y_α are E-equivalent to the reals in Y_β, then $(\alpha, \beta) \in e$.*

Proof. For this α and β, let

$$S = \{(\alpha', \beta') : (\alpha', \beta') \prec_{u_\omega \times u_\omega} (\alpha, \beta)\}.$$

This is a Π^1_3 subset of α, so, $A \in L_{\kappa_3}[T_2]$ by Hjorth's lemma in Chapter 3.

As

$$L_{\kappa_3}[T_2] \models \forall(\alpha', \beta') \in A \exists \sigma((\alpha', \beta') \in e^\sigma),$$

and

$$L_{\kappa_3}[T_2] \models \mathrm{KP},$$

there is a $B \in L_{\kappa_3}[T_2]$ such that

$$L_{\kappa_3}[T_2] \models \forall(\alpha', \beta') \in A \exists \sigma \in B((\alpha', \beta') \in e^\sigma).$$

Let $\sigma_0 = \sup\{\sigma : \sigma \in B\}$, $\sigma_0 < \kappa_3$ since $B \in L_{\kappa_3}[T_2]$. For this σ_0,

$$\forall \alpha' \beta'((\alpha', \beta') \prec_{u_\omega \times u_\omega} (\alpha, \beta) \wedge (\alpha', \beta') \in e \to (\alpha', \beta') \in e^\sigma_0).$$

Let us define an increasing sequence of ordinals smaller than κ_3 by letting

$$\sigma_{n+1} = \max(g_V(\sigma_n, h(l, d^*, \alpha, \beta)), f_U(\sigma_n, h(l, d^*, \alpha, \beta)),$$
$$g_N(\sigma_n, \bar{h}(l', m^*, \alpha, \beta)), f_M(\sigma_n, \bar{h}(l', m^*, \alpha, \beta))).$$

Since this sequence is $\Delta^1_3(\alpha, \beta)$ in the codes, its upper bound is also an ordinal $\Delta^1_3(\alpha, \beta)$ in the codes and hence smaller than κ_3. Let

$$\xi = \sup_{n \in \omega} \sigma_n.$$

If $(\alpha, \beta) \in e^{\xi}$, then $(\alpha, \beta) \in e$, we are done. So, we suppose that $(\alpha, \beta) \notin e^{\xi}$. In this case, from our construction, and because all reals in Y_α are E-equivalent to the reals in Y_β, $(\alpha, \beta) \in e^{\xi+1}$. Hence, $(\alpha, \beta) \in e$. □

Finally, we have our main theorem.

Theorem 6.6. *For any thin Π^1_3 equivalence relation on ω^ω, there is a Δ^1_3 in the codes function p from ω^ω to u_ω and a Π^1_3 in the codes equivalence relation e on u_ω such that for any x, y in ω^ω,*

$$xEy \iff (p(x), p(y)) \in e.$$

Proof. Let E be a Π^1_3 thin equivalence relation on u_ω, p and e as defined in this chapter.

From our construction, we know that Y_α must be a subset of some E equivalence class. We also know that Y_α is E-invariant from Lemma 6.5. So, Y_α is either an equivalence class or the empty set. But, it cannot be the empty set, otherwise, $(\alpha, \beta) \in e$ for all $\beta \in u_\omega$ from Lemma 6.5. So, for all $\beta \in u_\omega$, $Y_\alpha = Y_\beta$. Hence, $Y_\beta = \emptyset$ for all $\beta \in u_\omega$. This is impossible.

So, we have

$$(x, y) \in E \iff (p(x), p(y)) \in e.$$

□

It seems that Jackson lifted the Kechris-Martin theorem to higher levels. We hope his result could be used to lift Harrington's representation theorem further.

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SOME RESULTS ON PROJECTIVE EQUIVALENCE RELATIONS

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We construct a Π^1_1 equivalence relation E on ω^ω for which there is no largest E-thin, E-invariant Π^1_1 subset of ω^ω. Then we lift our result to the general case. Namely, we show that there is a Π^1_{2n+1} equivalence relation for which there is no largest E-thin, E-invariant Π^1_{2n+1} set under projective determinacy. This answers an open problem raised in Kechris [Ke2].

Our second result in this thesis is a representation for thin Π^1_3 equivalence relations on u_ω. Precisely, we show that for each thin Π^1_3 equivalence relation E on u_ω, there is a Δ^1_3 in the codes map $p\colon \omega^\omega \to u_\omega$ and a Π^1_3 in the codes equivalence relation e on u_ω such that for all real numbers x and y,

$$xEy \iff (p(x), p(y)) \in e.$$

This lifts Harrington's result about thin Π^1_1 equivalence relations to thin Π^1_3 equivalence relations.