

KAČ-MOODY ALGEBRAS and STRING THEORY

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To Lisa

As regards strings:

For a theory may be formed by the mind, and therefore be shaped by its structures. However, it frequently happens that the descriptive power of a good theory so transcends the meager understanding of the original conception, that it is very hard to believe that the theory only reflects our cognitive structures. It would then appear to be in much closer contact with nature than with our minds; our minds are continually agasped at its applicability.

—Albert Einstein

Abstract

The focus of this thesis is on (1) the role of Kač-Moody algebras in string theory and the development of techniques for systematically building string theory models based on higher level ($K \geq 2$) KM algebras and (2) fractional superstrings, a new class of solutions based on $SU(2)_K/U(1)$ conformal field theories. The content of this thesis is as follows.

In chapter two we review KM algebras and their role in string theory. In the next chapter, we present two results concerning the construction of modular invariant partition functions for conformal field theories built by tensoring together other conformal field theories. This is based upon our research in ref. [2]. First we show how the possible modular invariants for the tensor product theory are constrained if the allowed modular invariants of the individual conformal field theory factors have been classified. We illustrate the use of these constraints for theories of the type $SU(2)_{K_A} \otimes SU(2)_{K_B}$, finding all consistent theories for K_A and K_B odd. Second we show how known diagonal modular invariants can be used to construct inherently asymmetric invariants where the holomorphic and anti-holomorphic theories do not share the same chiral algebra. Explicit examples are given.

Next, in chapter four we investigate some issues relating to recently proposed fractional superstring theories with $D_{\text{critical}} < 10$. Using the factorization approach of Gepner and Qiu, we systematically rederive the partition functions of the $K = 4, 8,$ and 16 theories and examine their spacetime supersymmetry. Generalized GSO projection operators for the $K = 4$ model are found. Uniqueness of the twist field, $\phi_{K/4}^{K/4}$, as source of spacetime fermions, is demonstrated. Our research was originally presented in refs. [3, 4]

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Figures 3.1-7 are from ref. [5].

Figures A.1-3 are from ref. [6].

Kač-Moody Algebras and String Theory

Chapter 1: Introduction

1.1 Reasons for String Theory Research

Elementary particle physics has achieved phenomenal success in recent decades, resulting in the Standard Model (SM), $SU(3)_C \times SU(2)_L \times U(1)_Y$, and the verification, to high precision, of many of its predictions. However, there are still several shortcomings or unsatisfying aspects of the theory. Consider, for example, the following:^[7]

1. The SM is very complicated, requiring measurement of some 21 free parameters, such as the masses the quarks and leptons and the coupling constants. We should expect the true fundamental theory to have at most one free parameter.
2. The SM has a complicated gauge structure. A gauge group that is the direct product of three gauge groups with independent couplings does not seem fundamental.
3. There seems to be a naturalness problem concerning the scale at which the electroweak (EW) symmetry, $SU(2)_L \times U(1)_Y$, breaks to the electromagnetic $U(1)_{EM}$. Why is this scale of 100 GeV so much smaller than the Planck scale of 10^{19} GeV? Although this is “explained” by the scale of the Higgs mass, fine-tuning is required in renormalization theory to keep the Higgs mass on the order of the symmetry breaking scale. This seems to suggest the need for supersymmetry at a higher scale.
4. Fine-tuning is also required to solve the strong CP problem.
5. The SM provides no unification with gravity, *i.e.*, no means of forming a consistent theory of quantum gravity.
6. The cosmological constant resulting from EW symmetry breaking should be approximately 50 orders of magnitude higher than the experimental limit. Solving this problem from the SM viewpoint again requires a fine-tuned cancellation.

These shortcomings have motivated a search for phenomenologically viable Grand Unified Theories (GUT's) that would unify SM physics through a single force and even for a Theory of Everything (TOE) that could consistently combine the SM with gravity. In the last decade, this pursuit has resulted in an intensive study of string theory, which involves only one truly elementary “particle,” a (closed) string-like (rather than point-like) object with a length on the order of the Planck scale, $l_{\text{Pl}} = 10^{-33}$ cm. In this theory all particles ordinarily regarded as “elementary” are explained as vibrational or internal modes of this fundamental string. One of the advantages of string theory is that it removes the infinities resulting from high-energy interactions of point-like particles, without requiring renormalization techniques. The supersymmetric version of string theory contains no ultraviolet divergences.

String theory is the first theory to successfully combine the SM forces with gravity. Any string theory with $D > 2$ contains an infinite tower of vibrational/internal excitation modes. Included in the closed (super)string spectrum is a massless spin-2 (and spin-3/2) state which can be identified with the graviton (and its supersymmetric partner, the gravitino).

1.2 Status of String Theory Phenomenology

In a sense string theory has been too successful following the explosion of interest in the mid-80's. The (super)string theory is inherently a (10) 26 dimensional spacetime theory. Although in both cases there are only a very few solutions for the theories when all spacetime dimensions are uncompactified, for each dimension that is compactified, there arise many more possible solutions. With only four uncompactified spacetime dimensions, there is a plethora (on the order of several million) of distinct solutions to the superstring theory. Many different approaches to “compactification,” *e.g.*, bosonic lattices and orbifolds, free fermions, Calabi-Yau manifolds, and $N = 2$ minimal models, have been devised. (Often there is, however, much overlap and sometimes even complete equivalence between alternative methods of compactification.) Four-dimensional solutions can be classified into two broad categories: (1) those solutions involving an actual geometrical compactification from ten uncompact-

ified dimensions, and (2) those with internal degrees of freedom having no equivalent representation in terms of six well-defined compactified dimensions.

There is a potential problem with solutions in the first class: such models with $N = 1$ spacetime supersymmetry (SUSY) and/or chiral fermions cannot contain massless spacetime scalar fields in the adjoint or higher dimensional representations of the gauge group.^[1,8,9] This presents a possible difficulty for string theory, because typical GUT's depend upon scalars in these representations to break the gauge symmetry down to the SM. In the usual approach, spontaneous symmetry breaking is brought about by vacuum expectation values (VEV's) of these scalars. Thus, the gauge groups of these string models either must break to the standard model near the string (Planck) scale or a non-standard Higgs breaking is required. An example of the first method is symmetry breaking by Wilson lines in Calabi-Yau vacua.^[10] Flipped $SU(5)$ is the primary example of the second approach.^[11] However, standard GUT's such as $SU(5)$ or $SO(10)$ are excluded from this class of string theory models.

In the first class of models, the absence of spacetime scalars in higher representations results from the association of geometrical compactification with level-one KM algebras. In other words, the connection of these models to level-one KM algebras is basically a byproduct of the classical idea of "compactification." Because of this, basing a model on level-one KM algebras has been the standard approach to string theory phenomenology. Starting from either the ten dimensional type-II or heterotic superstrings, four-dimensional spacetime has most often been derived through "spontaneous compactification" of the extra six dimensions. In ten uncompactified dimensions the only modular invariant heterotic string models with spacetime SUSY and gauge symmetry are the level-one $E_8 \otimes E_8$ and level-one $SO(32)$ solutions. (In ten uncompactified dimensions, the type-II string has $N = 2$ SUSY, but no gauge group.) Compactification of the extra six dimensions on a Calabi-Yau manifold or symmetric orbifold, naturally keeps the KM algebra at level-one. The resulting gauge group g , is a subgroup of either $E_8 \otimes E_8$ or $SO(32)$, and the representations of the gauge group that appear are determined by the level of the algebra. Models using bosonic lattice compactification, or equivalently complex world sheet fermions,^[12-14] likewise have

level-one KM algebras, with the associated gauge group being a subgroup of either $SO(12) \otimes E_8 \otimes E_8$ or $SO(44)$.

Models can be based on higher-level KM algebras, if the demand for a classical interpretation of compactification is relaxed. Such models fall into the second general class of string solutions and can contain scalars in the adjoint or higher representations. These states can exist in the spectrum if their gauge group arises from a level- $K \geq 2$ KM algebra on the world sheet. Examples are given in [1], where the approach to such models is via *real* fermions.

The unitary representations of a level- K KM algebra in a string model are required to satisfy (see section 2.2.)

$$K \geq \sum_{i=1}^{\text{rank } \mathcal{L}} n_i m_i, \quad (1.2.1)$$

where n_i are the Dynkin labels of the highest weight representation of the associated Lie algebra, \mathcal{L} , and m_i are the related co-marks. Based on this unitarity constraint, at level-one only the singlet, spinor, conjugate spinor and vector representations of $SO(4n + 2)$ can appear. For $SU(N)$ level-one, only the $\binom{N}{0}$ (*i.e.*, the singlet), $\binom{N}{1}$, $\binom{N}{2}$, \dots , $\binom{N}{N-1}$ representations are allowed,¹ while for E_6 level-one the 1, 27 and $\overline{27}$ representations can be present.

Until geometric “compactification” from ten dimensions is sacrificed, higher-level models cannot be reached. However, when the basic strategy is generalized, level-one models become much less special. If one starts with all spatial dimensions initially compactified, and not well defined spatially, the occurrence of level-one KM algebras is not necessarily favored. That is, after “decompactification” of three spatial dimensions, a gauge group in the $(3 + 1)$ -dimensional space based on higher level algebras becomes possible (and not unlikely).

The difference between the two classes of “four-dimensional” string models relates to the question of how valid it is to think geometrically about physics at the Planck

¹For $SU(N)$ at level- K , the rule for determining all unitary representations is the following: Only those representations that correspond to a Young tableau with K or fewer columns are allowed. Henceforth a level- K KM algebra, $\tilde{\mathcal{L}}$, based on a Lie algebra, \mathcal{L} , will often be denoted by \mathcal{L}_K , with the exception of those Lie algebras that already carry a subscript denoting the rank, *e.g.*, E_6 .

scale. Are the lattice, free fermion, and Calabi-Yau approaches to compactification too classical for Planck scale physics? Going beyond the classical notion of spatial dimensions was one reason that Gepner considered $N = 2$ minimal models,^[15] (even though Calabi-Yau manifolds and $N = 2$ minimal models were eventually found to be equivalent^[15]).² Many considerations suggest investigating string models based on higher level K-M algebras, even though the degrees of freedom of the models generally cannot be expressed in terms of compactified spatial dimensions.

The central charge of the level- K algebra (which measures the contribution to the conformal anomaly of the world sheet theory) is

$$c_{\mathcal{L}} = \frac{K \dim \mathcal{L}}{K + \frac{1}{2} C_A}, \quad (1.2.2)$$

where C_A is the quadratic Casimir for the adjoint representation. For simply-laced groups the central charge equals the rank of the group at level one and monotonically approaches the dimension of the group as $K \rightarrow \infty$. This demonstrates that heterotic models constructed from free real world-sheet bosons, ∂X^i , compactified on a lattice (or equivalently free complex fermions) include only simply-laced level-one algebras.³ Hence, as stated above, both the $E_8 \otimes E_8$ and $SO(32)$ ten-dimensional models are level-one. With compactification that treats left-moving and right-moving modes of the string symmetrically the level remains one.

Construction of models with higher-level gauge groups requires asymmetry between the left- and right-moving fields on the world-sheet.^[1] Associated with this property of the fields there are asymmetric modular invariants. Systematically constructing asymmetric modular invariants has proven very difficult, except for the special case of models based on free bosons or fermions. However, even for asymmetric

²We suggest that this indicates more than just the mathematical equivalences of the approaches as demonstrated through Landau-Ginsburg potentials. It is another example that makes us question the meaningfulness of the concept of well-defined compactified spatial dimensions of Planck scale length.

³The rank of the group is at least equal to the number of bosons, since each bosonic operator ∂X^i generates a $U(1)$ KM algebra. Also note that we have specified simply-laced, because, as eqs. (2.2.3) and (2.2.9) together indicate, the central charge for non-simply-laced algebras is greater than the rank of the group even at level-one.

models, use of lattice bosons (or equivalently complex fermions) limits the possibilities to level one-models. The first and simplest alternative is to use real fermions instead.^[1] However, to date, no phenomenologically viable model has been found using this approach. A more general method for constructing (asymmetric) modular invariant tensor products of KM algebras (and of conformal field theories) has not been developed. Several years of research has shown that an enormous collection of consistent free fermion models exist, of which only a small percentage are actually left-right symmetric. Perhaps, a systematic approach to developing asymmetric modular invariants for tensor products of higher-level KM algebras could produce a new class of string models with viable phenomenology. Steps toward developing this approach is the focus of chapter 3 of this thesis.

1.2.a *Phenomenological Restrictions*

Recent results from LEP have resulted in tighter constraints for viable string models. Using renormalization group equations (RGE), the measured high precision values of the standard model coupling constants have been extrapolated from M_Z to near the Planck scale. It was found that the RGE for the minimal supersymmetric standard model with just two Higgs doublets predict a unification of the three coupling constants g_3 , g_2 and g_1 for $SU(3) \times SU(2)_L \times U(1)_Y$, respectively, at about 10^{16} GeV. For string theory this naively poses a problem since the string unification scale is generally required, at tree level, to be near the Planck scale (around 10^{18-19} GeV). Three classes of solutions have been proposed for resolving the potential inconsistency between these extrapolations and string theory.^[16]

The first proposal is to regard the unification of the couplings at 10^{16} GeV using the minimal SUSY standard model RGE as a coincidence, and to allow additional states between the electroweak scale and the string unification scale that raise the RGE unification scale. A second suggestion is that string threshold effects could significantly lower the string scale down to the minimal SUSY standard model RGE unification scale. The third possibility is that a grand unified gauge group results from a Kač-Moody algebra at level $K \geq 2$. As we have discussed, adjoint (and

higher) representations for Lorentz scalars become possible when the level of the KM algebra is greater than one. These adjoint scalars might allow $SU(5)$ or $SO(10)$ grand unification. Thus, the SUSY standard model couplings could unify at 10^{16} GeV and run upward from there with a common value to the string unification scale.

The last proposal appears most natural and appealing. The concept of a grand unified gauge group fits well with the idea of successive levels of increasing symmetry much better than does going directly from the symmetry of the standard model to the symmetry of the string. It seems far more natural for the strong force to merge with the electroweak significantly below the string scale, rather than where the gravitational coupling (and, additionally, all hidden sector gauge couplings) finally merge.

Thus, we will examine various aspects of higher-level Kač-Moody algebras in string models. In chapter 2 we review Kač-Moody algebras in greater depth and discuss their applications to string theory, including general properties of and restrictions on higher level models. In chapter 3 we develop tools for systematically constructing string models containing (asymmetric) higher-level KM algebras. Chapter 4 heads along a different direction as we investigate aspects of a potentially new class of string models with spacetime SUSY and critical dimensions below ten. These models seem to have a local world sheet symmetry that pairs the world sheet boson not with a fermion, but rather with a primary field of a higher-level $\frac{SU(2)}{U(1)}$ conformal field theory.

Chapter 2: Kač-Moody Algebras and String Theory

2.1 Review of Kač-Moody Algebras

At the heart of the gauge symmetries of string theory are not only Lie algebras, but the more complicated Kač-Moody (KM) algebras,^[17] for which the former are subalgebras. Because of the importance of KM algebras in string theory, we review them in this chapter before proceeding in the next chapter with our study of modular invariant partition functions for tensor products of KM algebras. Often in string theory the terms “affine algebra,” “affine Kač-Moody algebra,” and “Kač-Moody algebra” are used interchangeably. The imprecise use of these terms can be confusing, since there are actually three distinct classes of KM algebras, only one of which is “affine.”^[6] The basic step required to progress from Lie algebras⁴ to KM algebras is to relax the *finite*-dimension restriction on Lie algebras and consider *infinite*-dimensional generalizations. As we shall show, many of the features of semi-simple Lie algebras reappear in KM algebras. In fact, Lie algebras can be regarded as particular cases of KM algebras with the special property of being finite-dimensional.

Analogous to Lie algebras, KM algebras are defined by generalized Cartan matrices (or equivalently by Dynkin diagrams). We will discuss KM algebras in terms of these matrices. After lifting the finite-dimension restriction, examination of generalized Cartan matrices shows that KM algebras can be grouped into three distinct classes, called the “finite” (corresponding to standard Lie algebras), “affine,” and “indefinite” types. Within the affine class are two subclasses, denoted as “twisted” and “untwisted.”

Recall that the elements of an $l \times l$ -dimensional Cartan matrix, \mathbf{A} , for a Lie algebra, \mathcal{L} , of rank- l are defined by

$$A_{jk} = \frac{2\langle \alpha_j, \alpha_k \rangle}{\langle \alpha_j, \alpha_j \rangle}, \quad j, k \in I^{\mathcal{L}} \equiv \{1, 2, \dots, l\}, \quad (2.1.1)$$

where α_j is a simple root of the algebra. For Lie algebras, the inner product of two

⁴Specifically, compact simple or compact semi-simple Lie algebras, which is henceforth implied.

roots is defined by

$$\langle \alpha_j, \alpha_k \rangle \equiv \alpha_j \cdot \alpha_k = \sum_{m=1}^l (\alpha_j)_m (\alpha_k)_m. \quad (2.1.2)$$

(As we show, a more general definition applies for the inner product of roots in a KM algebra. See section 2.1.a.) Cartan matrices are defined by four properties:

- (a) $A_{jj} = 2$ for $j = 1, 2, \dots, l$;
- (b) $A_{jk} = 0, -1, -2,$ or -3 if $j \neq k$;
- (c) for $j \neq k$, $A_{jk} = 0$ if and only if (iff) $A_{kj} = 0$;
- (d) $\det \mathbf{A}$ and all proper principal minors of \mathbf{A} are positive.⁵

Classification of all $l \times l$ -dimensional matrices with these properties completely classifies Lie algebras of rank- l .

In the late 1960's Kač and Moody discovered that some of these properties could be relaxed to produce a new, enlarged set of algebras, with the primary difference being that the new algebras were infinite-dimensional. By infinite-dimensional, we mean that there is an infinite number of roots (equivalently an infinite number of generators) of the algebra. Their generalized⁶ Cartan matrix, \mathbf{A}^{KM} , is defined as $d_{\text{AKM}} \times d_{\text{AKM}}$ -dimensional with the properties that

- (a') $A_{jj} = 2$ for $j \in I^{\tilde{\mathcal{L}}} = \{0, 1, \dots, d_{\text{AKM}} - 1\}$;
- (b') for $j \neq k$ ($j, k \in I$), A_{jk} is a non-positive integer;
- (c') for $j \neq k$ ($j, k \in I$), $A_{jk} = 0$ iff $A_{kj} = 0$.

One modification is that property (d) has been lifted. No longer must the determinant or all proper minors of the matrix be positive. $\det \mathbf{A}^{\text{KM}} \leq 0$ is now allowed, with the rank- l of the matrix, $\tilde{\mathcal{L}}$, determined by the largest square submatrix of \mathbf{A}^{KM} with non-zero determinant. Thus, $l = d_{\text{AKM}}$ only when $\det \mathbf{A}^{\text{KM}} \neq 0$. Otherwise $l < d_{\text{AKM}}$. Second, non-diagonal elements $A_{jk} < -3$, for $j \neq k$, are permitted.

The basic ideas and terminology for roots and root subspaces for a complex KM

⁵A *principal minor* of \mathbf{A} is the determinant of a *principle submatrix* of \mathbf{A} , which is a submatrix consisting of elements A_{jk} in which j and k both vary over the same subset of indices. These quantities are *proper* if the subset of indices is a proper subset of the set of indices.

⁶There is additionally a slight change of notation. For Lie algebras, $I^{\tilde{\mathcal{L}}}$ should be altered to $I^{\mathcal{L}} = \{1, 2, \dots, l \equiv d_A\}$.

algebra, $\tilde{\mathcal{L}}$, are very similar to those for a semi-simple complex Lie algebra. The commutative subalgebra, \mathcal{H} , of $\tilde{\mathcal{L}}$ is referred to as the Cartan subalgebra (CSA) of $\tilde{\mathcal{L}}$, and the set of elements E^α of $\tilde{\mathcal{L}}$ possessing the property that

$$[\mathbf{h}, E^\alpha] = \langle \boldsymbol{\alpha}, \mathbf{h} \rangle E^\alpha ; \quad \text{for all } h \in \mathcal{H} , \quad (2.1.3)$$

form the root subspace $\tilde{\mathcal{L}}_\alpha$ corresponding to the root $\boldsymbol{\alpha}$. The set of roots $\boldsymbol{\alpha}_i$, for $i \in I^{\tilde{\mathcal{L}}}$, are the simple roots upon which the generalized Cartan matrix is based. A generic root, $\boldsymbol{\alpha}$, has the form

$$\boldsymbol{\alpha} = \sum_{i \in I} c_j^\alpha \boldsymbol{\alpha}_i, \quad (2.1.4)$$

where the set of c_j^α are either all non-negative integers or all non-positive integers.

A distinctive difference between Lie algebras and KM algebras is whereas the dimension, $n_{\mathcal{L}}$, of the CSA of Lie algebras is equal to the rank, l , of the Cartan matrix, this relation does not hold for KM algebras. Rather, for KM algebras^[6] the dimension of the generalized CSA is

$$n_{\tilde{\mathcal{L}}} = 2d_{A^{\text{KM}}} - l . \quad (2.1.5)$$

Only when $l = d_{A^{\text{KM}}}$ does $n_{\tilde{\mathcal{L}}} = l$.

For any KM algebra, the CSA \mathcal{H} can be divided into two parts, \mathcal{H}' and \mathcal{H}'' : \mathcal{H}' being a $d_{A^{\text{KM}}}$ -dimensional algebra with $\{H^i, i \in I^{\tilde{\mathcal{L}}}\}$ as its basis; and \mathcal{H}'' simply defined to be the $(d_{A^{\text{KM}}} - l)$ -dimensional complimentary subspace of \mathcal{H}' in \mathcal{H} . The H^i are the generators giving the first $d_{A^{\text{KM}}}$ components,

$$\alpha_j(H^i) \equiv \langle \boldsymbol{\alpha}_j, H^i \rangle , \quad (2.1.6)$$

of the simple roots, $\boldsymbol{\alpha}_j$. \mathcal{H}'' is non-trivial only when $\det \mathbf{A}^{\text{KM}} = 0$. Within \mathcal{H}' is a subset, \mathcal{C} , that forms the center of the KM algebra. The elements $\mathbf{h} \in \mathcal{C}$ commute with all the members of $\tilde{\mathcal{L}}$. That is, if $\mathbf{h} \in \mathcal{C}$ then

$$\langle \boldsymbol{\alpha}_j, \mathbf{h} \rangle = 0 \text{ for all } j \in I^{\tilde{\mathcal{L}}} . \quad (2.1.7)$$

That \mathcal{C} is $(d_{A^{\text{KM}}} - l)$ -dimensional is shown by elementary matrix theory. The proof is short and is as follows: Any element of $\mathbf{h} \in \mathcal{H}'$ has the form

$$\mathbf{h} = \sum_{j \in I} \mu_j H^j . \quad (2.1.8)$$

If \mathbf{h} is also in \mathcal{C} , then

$$\sum_{j \in I} \mu_j \boldsymbol{\alpha} \cdot H^j = 0 . \quad (2.1.9)$$

The \mathcal{H}' can always be rotated into the Chevalley basis^[6] where the set of eqs. (2.1.9) becomes the matrix eq.

$$\mathbf{A}^{\text{KM}} \boldsymbol{\mu} = 0 . \quad (2.1.10)$$

($\boldsymbol{\mu}$ is a column vector with entries μ_j .) Since \mathbf{A}^{KM} has rank l , elementary matrix theory shows that there are $d_{\text{A}^{\text{KM}}} - l$ linearly independent solutions to $\boldsymbol{\mu}$. The basis of the $d_{\text{A}^{\text{KM}}}$ -dimensional subspace $\mathcal{H}^- \equiv \mathcal{H} - \mathcal{C}$ (which includes \mathcal{H}'') can be formed from those elements $\mathbf{h}_+^k \in \mathcal{H}$, where $\langle \boldsymbol{\alpha}_j, \mathbf{h}_+^k \rangle = \delta_{jk}$ and $j, k \in I^{\tilde{\mathcal{L}}}$. Thus, no non-trivial element $\mathbf{h}'' = \sum_{k \in I} \lambda_k \mathbf{h}_+^k \in \mathcal{H}'' \in \mathcal{H}_+$ can be in the \mathcal{C} , since

$$\langle \boldsymbol{\alpha}_j, \mathbf{h}'' \rangle = \lambda_j . \quad (2.1.11)$$

2.1.a Categories of Kač-Moody Algebras

Matrices satisfying properties (a'-c') defining a generalized Cartan matrix can be divided into three categories, each corresponding to a unique class of KM algebras. The following three theorems define these classes

Theorem 2.1: A complex KM algebra, $\tilde{\mathcal{L}}$, is “finite” (equivalently, it is a Lie algebra), iff all the principle minors of the corresponding generalized Cartan matrix, \mathbf{A}^{KM} , are positive.

This constraint on the principle minors is equivalent to demanding that:

$$(F.1) \det \mathbf{A}^{\text{KM}} \neq 0;$$

$$(F.2) \text{ there exists a vector } \mathbf{u} > 0 \text{ of dim } d_{\text{A}^{\text{KM}}} \text{ such that } \mathbf{A}^{\text{KM}} \mathbf{u} > 0;^7 \text{ and}$$

$$(F.3) \mathbf{A}^{\text{KM}} \mathbf{v} \geq 0 \text{ implies } \mathbf{v} \geq 0.$$

Properties (F.1-3) imply that the associated algebra does not contain any *imaginary* roots, *i.e.*, roots α such that $\langle \alpha, \alpha \rangle \leq 0$,^[6] which corresponds to reimposing constraints (b) and (d). Hence, these Cartan matrices define finite Lie algebras.

⁷ $\mathbf{u} > 0$ is defined to mean $u_j > 0$ for all $j \in I$. Similar definitions apply when “<”, “≥”, or “≤” appear in vector relations.

Theorem 2.2: A complex KM algebra, $\tilde{\mathcal{L}}$, is “affine” iff its generalized Cartan matrix, \mathbf{A}^{KM} , satisfies $\det \mathbf{A}^{\text{KM}} = 0$ and all the proper minors of \mathbf{A}^{KM} are positive.

An equivalent definition of this class is to require that the matrix is such that:

$$(A.1) \det \mathbf{A}^{\text{KM}} = 0 \text{ but } l = d_{A^{\text{KM}}} - 1;$$

$$(A.2) \text{ there exists a vector } \mathbf{u} > 0 \text{ such that } \mathbf{A}^{\text{KM}} \mathbf{u} = 0; \text{ and}$$

$$(A.3) \mathbf{A}^{\text{KM}} \mathbf{v} \geq 0 \text{ implies } \mathbf{A}^{\text{KM}} \mathbf{v} = 0.$$

With these properties, this class of KM algebras must contain imaginary roots. The term “affine” is derived from the special characteristics of its generalized Weyl group.^[6] Each complex affine KM algebra, $\tilde{\mathcal{L}}^{\text{aff}}$, can be constructed from an associated complex simple Lie algebra, \mathcal{L} . The properties that $l = d_{A^{\text{KM}}} - 1$ along with $\det \mathbf{A}^{\text{KM}} = 0$, place a severe constraint on the one additional simple root $\equiv \alpha_0$. In terms of its l -dimension projection onto the Lie algebra subspace, which we denote by $\alpha_0^{\mathcal{L}}$, the constraint is

$$\alpha_0^{\mathcal{L}} = - \sum_{j=1}^{d_{A^{\text{KM}}}-1} \alpha_j \equiv -\psi, \quad (2.1.12)$$

where α_j are the simple roots of the Lie algebra, \mathcal{L} , and ψ is its highest root. The affine algebras are the class of KM algebras upon which the spacetime gauge groups of string theory are based; therefore, affine algebras are discussed in greater detail in the following (sub)sections.

The last class of KM algebras, called “indefinite” algebras is most simply defined by those generalized Cartan matrices that satisfy neither of the conditions of Theorems 2.1 or 2.2. Indefinite matrices have the following properties.

$$(I.1) \text{ there exists a } \mathbf{u} > 0 \text{ such that } \mathbf{A}^{\text{KM}} \mathbf{u} < 0; \text{ and}$$

$$(I.2) \mathbf{A}^{\text{KM}} \mathbf{v} \geq 0 \text{ and } \mathbf{v} \geq 0 \text{ imply that } \mathbf{v} = 0.$$

As (I.1-2) indicate, indefinite algebras also have imaginary roots.

For a specific $d_{A^{\text{KM}}}$ there are only a finite number of possible generalized Cartan matrices in the finite or affine classes. In the finite case, where $l = d_{A^{\text{KM}}}$, these matrices correspond to the standard simple Lie algebras, which are denoted by A_l , B_l , C_l , D_l , $E_{6,7,8}$, F_4 , and G_2 . In the affine case, where $l = d_{A^{\text{KM}}} - 1$, there is an untwisted generalization of each Cartan matrix associated with a Lie algebra of rank- l . Common notation for the affine algebras is to add a superscript of (1) to the

Lie algebra symbol. For example, the untwisted affine version of A_l is denoted by $A_l^{(1)}$. Additionally, for A_l , D_l , and E_6 there is a “twisted” generalization denoted by superscripts (2). There is also a second twisted affinization of D_4 , denoted by $D_4^{(3)}$. The twisted algebras result from either a \mathbb{Z}_2 or a \mathbb{Z}_3 rotation and projection of the roots of the untwisted affine algebra,^[6] and are non-simply-laced affinizations of simply-laced Lie algebras.

The third type of KM algebra, the “indefinite” class, is appropriately named because there is an infinite number of matrices that meet neither “finite” nor “affine” requirements for each value of d_{AKM} . All of these matrices correspond to non-isomorphic algebras not grounded in generalizations of the Lie algebras. Very few (if any) applications for indefinite KM algebras have been found. In particular, they appear to play no part in string theory.

To illustrate the differences of the three classes, consider the simplest non-trivial generalized Cartan matrix possible, the 2×2 -dimensional

$$\mathbf{A}^{\text{KM}} = \begin{pmatrix} 2 & -r \\ -s & 2 \end{pmatrix},$$

where r and s are positive integers. Now let us classify all of the possible KM algebras in each class associated with specific values for r and s .

1. Finite (Lie) algebras: $\det \mathbf{A}^{\text{KM}} > 0$ so $rs < 4$. There are only three possibilities for non-equivalent algebras,
 - a. $r = 1, s = 1$ corresponding to A_2 ;
 - b. $r = 1, s = 2$ corresponding to B_2 ; and
 - c. $r = 1, s = 3$ corresponding to G_2 .
2. Affine algebras: $\det \mathbf{A}^{\text{KM}} = 0$ so $rs = 4$. There are only two inequivalent possibilities,
 - a. $r = 1, s = 4$ corresponding to $A_2^{(2)}$; and
 - b. $r = 2, s = 2$ corresponding to $A_1^{(1)}$.
3. Indefinite algebras: $\det \mathbf{A}^{\text{KM}} < 0$ so $rs > 4$. There is an infinite number of choices for r and s resulting in non-isomorphic algebras.

Since the finite class of KM algebras is simply composed of Lie algebras and the indefinite class, although the largest, seems to have little application to physics, we cease our study of them with this example of classification of 2×2 -dimensional generalized Cartan matrices. We now focus in greater detail on the affine algebras and their role in string theory.

2.1.b Affine Algebras

Having discussed the three classes of KM algebras, we focus here in detail on affine KM algebras. We generalized Cartan-Weyl basis. Recall from eq. (2.1.5) that the CSA, \mathcal{H} , of a KM algebra, $\tilde{\mathcal{L}}$, has dimension

$$n_{\tilde{\mathcal{L}}} = 2d_{AKM} - l ,$$

where d_{AKM} is the number of simple roots and l is the rank of the associated generalized Cartan matrix. \mathcal{H} can be divided into two parts, \mathcal{H}' of dimension d_{AKM} , and its complement \mathcal{H}'' of dimension $d_{AKM} - l$. Within \mathcal{H}' is the $(d_{AKM} - l)$ -dimensional center, \mathcal{C} , of $\tilde{\mathcal{L}}$. Applying this to the affine class of KM algebras, shows that:

- (1) \mathcal{H} has dimension $l + 2$.
- (2) \mathcal{H}' is $(l + 1)$ -dimensional with only one generator in the center.
- (3) \mathcal{H}'' is one-dimensional.

The l generators, denoted by H^p for $p \in I^{\mathcal{L}} = \{1, 2, \dots, l\}$, in \mathcal{H} but not in \mathcal{C} form the CSA of the Lie algebra $\mathcal{L} \subset \tilde{\mathcal{L}}$. Thus, affine CSA's contain two additional generators of \mathcal{H} not present in the Lie subalgebra. The single generator of the center is known as the level, K , of the algebra, and the generator of \mathcal{H}'' is called the scaling element, d . We can express generic roots of the KM algebra in the form⁸

$$\alpha_j = (\alpha_j^l, \alpha_j(K), \alpha_j(d)) ; \quad j \in I^{\tilde{\mathcal{L}}} , \quad (2.1.13)$$

where α_j^l forms the l -dimensional subvector that is associated solely with the Lie algebra \mathcal{L} . In this notation, the simple roots can be taken as

$$\alpha_p = (\alpha_p^l, 0, 0) \quad \text{for } p \in I^{\mathcal{L}} \quad \text{and} \quad \alpha_0 = (-\psi, 0, \alpha_0(d)) . \quad (2.1.14)$$

⁸ $\alpha_j(\mathbf{h}) \equiv \langle \alpha_j, \mathbf{h} \rangle$.

(Since K forms the center of the algebra, $\alpha_j(K) = 0$ by (2.1.3).)

Based on eq. (2.1.10), to a given affine Cartan matrix \mathbf{A}^{KM} is associated a single linearly independent d_{AKM} -dimensional vector $\boldsymbol{\mu} > 0$ such that $\mathbf{A}^{\text{KM}}\boldsymbol{\mu} = 0$. This vector is related to $\alpha_0(d)$ by

$$\boldsymbol{\delta} = \sum_{j=0}^l \mu_j \boldsymbol{\alpha}_j = (\boldsymbol{\delta}^l = 0, 0, \delta(d) = \alpha_0(d)) . \quad (2.1.15)$$

In other words, $\delta(H^p) = \delta(K) = 0$. d can be defined so that $\delta(d) \equiv \alpha_0(d) = 1$. $\boldsymbol{\delta}$ is an actual root of the theory, as are all integer multiples, $m\boldsymbol{\delta}$; $m \in \mathbb{Z}$. Thus, a general root has the form $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^l, 0, m)$. For $\boldsymbol{\alpha}^0 = 0$ we denote the associated operator by H_m^0 ; otherwise we denote the operator by E_m^α .

Consistency of the algebra^[6] forces $\boldsymbol{\delta}$ to be a “null root” (imaginary root as previously defined) with the property that

$$\langle \boldsymbol{\delta}, \boldsymbol{\delta} \rangle = \langle \boldsymbol{\delta}, \boldsymbol{\alpha}_j \rangle = 0 , \quad (2.1.16)$$

(this is more clearly seen in the Chevalley basis^[6]) and determines the generalization of (2.1.2) for two generic roots, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of an affine theory:

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = \boldsymbol{\alpha}^l \cdot \boldsymbol{\beta}^l + \alpha(K)\beta(d) + \alpha(d)\beta(K) . \quad (2.1.17)$$

Thus, only the Lie algebra components contribute to the inner product of any two simple roots,

$$\langle \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j \rangle = \boldsymbol{\alpha}_i^l \cdot \boldsymbol{\alpha}_j^l . \quad (2.1.18)$$

Using this generalized definition of an inner product, the Weyl reflection of a weight, $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^l, k, n)$, about a root, $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^l, 0, m)$, is

$$w_\alpha(\boldsymbol{\lambda}) = \boldsymbol{\lambda} - \boldsymbol{\alpha} \langle \boldsymbol{\lambda}, \boldsymbol{\alpha} \rangle \quad (2.1.19a)$$

$$= \left(\boldsymbol{\lambda}^l - 2[\boldsymbol{\lambda}^l \cdot \boldsymbol{\alpha}^l + km] \frac{\boldsymbol{\alpha}^l}{\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}}, k, n - 2[\boldsymbol{\lambda}^l \cdot \boldsymbol{\alpha}^l + km] \frac{m}{\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}} \right) \quad (2.1.19b)$$

This reflection can be split into two parts, a series of m translations by

$$t_{\alpha^l}(\boldsymbol{\lambda}) = \left(\boldsymbol{\lambda}^l + k \frac{\boldsymbol{\alpha}^l}{\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}}, k, n + \frac{1}{2k} \left\{ \boldsymbol{\lambda}^l \cdot \boldsymbol{\lambda}^l - (\boldsymbol{\lambda}^l + 2k \frac{\boldsymbol{\alpha}^l}{\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}})^2 \right\} \right) . \quad (2.1.19c)$$

followed by a Weyl reflection about α^l . The affine Weyl rotation is the product of these transformations,

$$w_\alpha(\boldsymbol{\lambda}) = w_{\alpha^l}(t_{\alpha^l}^m(\boldsymbol{\lambda})) . \quad (2.1.19d)$$

We conclude this general discussion of affine KM algebras with a listing of the algebra, itself. Adding the two additional generators, K and d , to the CSA of a Lie algebra, $\{H^p \equiv H_0^p; p \in I^\mathcal{L} = \{1, 2, \dots, l\}\}$, forms the affine CSA and enlarges the Lie algebra, \mathcal{L} ,⁹ from:

$$[H^p, H^q] = 0 \quad (2.1.20a)$$

$$[H^p, E^\alpha] = \alpha(H^i)E^\alpha \quad (2.1.20b)$$

$$[E^\alpha, E^\beta] = \begin{cases} \epsilon(\alpha, \beta)E^{\alpha+\beta} & \text{if } \alpha, \beta \text{ is a root} \\ \frac{2}{\alpha^2}\alpha \cdot \mathbf{H}, & \text{if } \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.1.20c)$$

(with $p, q \in I^\mathcal{L}$), to the full affine algebra

$$[H_m^i, H_n^j] = Km\delta^{ij}\delta_{m,-n} \quad (2.1.21a)$$

$$[H_m^p, E_n^\alpha] = \alpha(H^i), E_{m+n}^\alpha \quad (2.1.21b)$$

$$[E_m^\alpha, E_n^\beta] = \begin{cases} \epsilon(\alpha, \beta)E_{m+n}^{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root} \\ \frac{2}{\alpha^2}[\alpha \cdot \mathbf{H}_{m+n} + Km\delta_{m,-n}], & \text{if } \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.1.21c)$$

$$[K, T_n^a] = 0 \quad (2.1.21d)$$

$$[d, T_n^a] = nT_n^a \quad (2.1.21e)$$

⁹Eqs. (2.1.21) correspond to an *untwisted* affine KM algebra. The *twisted* algebras involve a $\mathbb{Z}_{q=2 \text{ or } 3}$ rotation by an outer automorphism of the untwisted KM algebra, which creates a \mathbb{Z}_q projection on the roots. The roots, $\alpha = (\alpha^\mathcal{L}, 0, m)$, may be classified by their eigenvalues, $\exp\{2\pi ip/q\}$ with $p \in \{0, 1, 2, \dots, q-1\}$, under this rotation. The related projection requires $m \pmod{q} = p$. The surviving $m = 0$ roots are isomorphic to the simple roots of a non-simply-laced Lie subalgebra, $\mathcal{L}^{(q)}$, of \mathcal{L} .

where $i, j \in I = \{0, 1, \dots, l\}$; $m, n \in \mathbb{Z}$, and T_n^a is any element of the algebra. The operators of the algebra have the hermiticity properties

$$H_m^{i\dagger} = H_{-m}^i, \quad E_m^{\alpha\dagger} = E_{-m}^{-\alpha}, \quad K^\dagger = K, \quad \text{and} \quad d^\dagger = d. \quad (2.1.22)$$

2.2 Application to String Theory

Now we consider the specific role of affine KM algebras in string theory, where they provide the world sheet realization of spacetime gauge theories. Present in string models are sets of $(h, \bar{h}) = (1, 0)$ conformal fields, $J^a(z)$, called currents, which satisfy the OPE of a KM generator,

$$J^a(z)J^b(w) = \frac{\tilde{K}^{ab}}{(z-w)^2} + \frac{if^{abc}}{(z-w)}J^c + (\text{non-singular terms}). \quad (2.2.1)$$

f^{abc} are the structure constants for the Lie algebra, \mathcal{L} , with normalization

$$f^{abc}f^{dbc} = C_A\delta^{ad} = \psi^2\tilde{h}\delta^{ad}. \quad (2.2.2)$$

C_A is the quadratic Casimir of the adjoint representation of \mathcal{L} , \tilde{h} is the dual Coxeter number, and ψ denotes the highest root. For each simple factor of the algebra, a basis can be chosen such that $\tilde{K}^{ab} = \tilde{K}\delta^{ab}$. $K \equiv 2\tilde{K}/\psi^2$ is defined as the level of a simple factor of the KM algebra (as discussed in 2.1.b). Commonly in string theory the normalization of $\psi^2 = 2$ is used, which results in $K = \tilde{K} \in \mathbb{Z}$ and $\tilde{h} = C_A/2$. Also recall that

$$C_A = r_{\mathcal{L}}^{-1} \sum_{a=1}^{\dim \mathcal{L}} \alpha_a^2 \quad (2.2.3a)$$

$$= \frac{1}{r_{\mathcal{L}}} \left(n_L + \left(\frac{S}{L}\right)^2 n_S \right) \psi^2 \quad (2.2.3b)$$

$$\rightarrow \left(\frac{\dim \mathcal{L}}{r_{\mathcal{L}}} - 1 \right) \psi^2 \quad \text{for simply-laced algebras} \quad (2.2.3c)$$

where $r_{\mathcal{L}}$ is the rank of the algebra and α_a is a simple root. n_S and n_L are the number of short and long roots, respectively, and S and L are the lengths.

The presence of an underlying KM algebra is alternatively seen from the related commutation relations of the modes of the currents,¹⁰

$$J^a(z) = \sum_{n \in \mathbb{Z}} J_n^a z^{-n-1} \ , \ \text{where} \ J_n^a = \oint \frac{dz}{(2\pi iz)} z^{n+1} J^a(z) . \quad (2.2.4)$$

The commutation relations have the form

$$[J_m^a, J_n^b] = i f^{abc} J_{m+n}^c + \tilde{K} m \delta^{ab} \delta_{m,-n} \ , \quad (2.2.5)$$

where $m, n \in \mathbb{Z}$. As was discussed previously, these commutators define the untwisted affine KM algebra, $\tilde{\mathcal{L}}$, associated with a compact (semi)-simple Lie algebra, \mathcal{L} . The horizontal Lie subalgebra, \mathcal{L} , is formed from the algebra of the zero modes, J_0^a , for which the level does not appear. The full (infinite) set of J_n^a 's provides the affinization of the finite dimensional subalgebra of J_0^a 's.

In a heterotic string model these currents appear in the vertex operator for a spacetime gauge boson,^[1] *e.g.*,

$$V^a = \zeta_\mu \psi^\mu(\bar{z}) J^a(z) \exp\{ip \cdot X\} ; \quad p^\mu p_\mu = p^\mu \zeta_\mu = 0 . \quad (2.2.6)$$

X^μ is the spacetime string coordinate and ψ^μ is a left-moving Ramond-Neveu-Schwarz fermion. Thus, spacetime gauge fields imply the existence of a KM algebra on the world sheet. In other words, there is an extension to the standard Virasoro algebra that includes the affine KM currents. In OPE language, the extended Virasoro–KM algebra takes the form

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{(z-w)} \partial T(w) + \dots \quad (2.2.7a)$$

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)} + \dots \quad (2.2.7b)$$

$$J^a(z)J^b(w) = \frac{\tilde{K}^{ab}}{(z-w)^2} + \frac{i f^{abc}}{(z-w)} J^c + \dots \quad (2.2.7c)$$

¹⁰Any field, ϕ , with conformal dimension, h_ϕ , can be written in terms of the normal modes, ϕ_n , in a Laurent expansion,

$$\phi(z) = \sum_{n=-\infty}^{\infty} z^{-n-h_\phi} \phi_n \ , \ \text{where} \ \phi_n = \oint \frac{dz}{(2\pi iz)} z^{n+h_\phi} \phi(z) .$$

Equivalently, the algebra can be expressed in terms of commutation relations of normal modes:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \quad (2.2.8a)$$

$$[L_m, J_n^a] = -nJ_{m+n}^a \quad (2.2.8b)$$

$$[J_m^a, J_n^b] = if^{abc}J_{m+n}^c + \tilde{K}m\delta^{ab}\delta_{m,-n} . \quad (2.2.8c)$$

In eq. (2.2.7a) the contribution to the Virasoro central charge (conformal anomaly) from the KM algebra is

$$c_{\mathcal{L}} = \frac{\tilde{K} \dim \mathcal{L}}{\tilde{K} + C_A/2} = \frac{K \dim \mathcal{L}}{K + \tilde{h}} . \quad (2.2.9)$$

The energy-momentum tensor, itself, may be written in terms of the KM currents:^[18]

$$T(z) = \frac{1}{\beta} \sum_{a=1}^{\dim \mathcal{L}} : J^a(z) J^a(z) := \frac{1}{\beta} \left(\lim_{z \rightarrow w} \sum_{a=1}^{\dim \mathcal{L}} J^a(z) J^a(w) - \frac{\tilde{K} \dim \mathcal{L}}{(z-w)^2} \right) . \quad (2.2.10)$$

$\beta \equiv 2\tilde{K} + C_A = 2(K + \tilde{h})$ is a constant fixed either by the requirement that $T(z)$ satisfy (2.2.7a) or, equivalently, that the $J^a(z)$'s transform as dimension (1, 0) primary fields.

In terms of the mode expansion for $T(z)$, (2.2.10) translates into

$$L_n = \oint \frac{dz}{(2\pi iz)} z^{n+2} T(z) \quad (2.2.11a)$$

$$= \frac{1}{\beta} \sum_{m=-\infty}^{\infty} : J_{m+n}^a J_{-m}^a : . \quad (2.2.11b)$$

All states in the theory necessarily fall into representations of the Virasoro–KM algebra.^[1] Each representation (Verma module), $[\phi_{(r)}]$, is composed of a primary field $\phi_{(r)}$ (actually, a multiplet of fields $\phi_{(r)}^\lambda$), and all of its “descendent” fields. The descendent fields are the set of fields formed by acting on a primary field with all possible products of the raising operators L_{-m} and J_{-n}^a for $m, n \in \mathbb{Z}^+$,

$$\left\{ \prod_{i=1}^{\infty} (L_{-i})^{A_i} \prod_{a=1}^{\dim \mathcal{L}} (J_{-i}^a)^{B_i^a} | \phi_{(r)} \right\} , \quad (2.2.12)$$

where $A_i, B_i^a \in \{0, \mathbb{Z}^+\}$. $\phi_{(r)}$ transforms as a highest weight representation (r) of \mathcal{L} , as indicated by the leading term in the OPE of $\phi_{(r)}$ with the current $J^a(z)$,

$$J^a(z) \phi_{(r)}(w) = \frac{(T^a)_{(r)}^{(r')}}{(z-w)} \phi_{(r)} , \quad (2.2.13a)$$

$$= \frac{(t_{(r)}^a)^{\lambda'\lambda}}{(z-w)} \phi_{(r)}^\lambda . \quad (2.2.13b)$$

$t_{(r)}^a$ are representation matrices for \mathcal{L} in the representation (r) . These primary fields create states, called highest weight states, defined by

$$|\phi_{(r)}\rangle \equiv \phi_{(r)}(0)|\text{vacuum}\rangle , \quad (2.2.14)$$

that are representations of the zero-mode (Virasoro-Lie) algebra,¹¹

$$L_0|\phi_{(r)}\rangle = h_{(r)}|\phi_{(r)}\rangle , \quad \text{and} \quad J_0^a|\phi_{(r)}\rangle = (T^a)_{(r)}^{(r')}|\phi_{(r')}\rangle \quad \text{for } n = 0 , \quad (2.2.15a)$$

and

$$L_n|\phi_{(r)}\rangle = J_n^a|\phi_{(r)}\rangle = 0 \quad \text{for } n > 0 . \quad (2.2.15b)$$

From (2.2.15a), the general form for the the conformal dimension, $h_{(r)}$, of the primary field, $\phi_{(r)}$, is

$$h_{(r)} = \frac{C_{(r)}/2}{\tilde{K} + C_A/2} \quad (2.2.15c)$$

$$= \frac{C_{(r)}/\psi^2}{K + \tilde{h}} , \quad (2.2.15d)$$

where

$$C_{(r)} \equiv l_{(r)} \frac{\dim \mathcal{L}}{\dim (r)}$$

is the quadratic Casimir of the representation (r) , with $\text{tr } t_{(r)}^a t_{(r)}^b = l_{(r)} \delta^{ab}$. The dimensions the descendent fields are $h_{(r)} + \mathbb{Z}^+$. Specifically,

$$h = h_{(r)} + \sum_{i=1}^{\infty} \left(iA_i + \sum_{a=1}^{\dim \mathcal{L}} iB_i^a \right) \quad (2.2.16)$$

for the field

$$\prod_{i=1}^{\infty} (L_{-i})^{A_i} \prod_{a=1}^{\dim \mathcal{L}} (J_{-i}^a)^{B_i^a} |\phi_{(r)}\rangle . \quad (2.2.17)$$

An issue we wish to stress is that only states in representations (r) satisfying,

$$K \geq \sum_{i=1}^{r_{\mathcal{L}}} n_i m_i , \quad (2.2.18)$$

¹¹– L_0 can be identified with the scaling element, d , of the KM algebra.

may appear in sensible string models. n_i are the Dynkin labels of the highest weight of the representation (r) and m_i are the related co-marks¹² group associated with the Lie algebra, \mathcal{L} .^[5] Eq. (2.2.18) is the condition for unitarity of a representation. Within any KM algebra, the subset

$$\{J_1^{-\alpha}, J_{-1}^{\alpha}, K - \alpha \cdot \mathbf{H}\},$$

where α is a root in \mathcal{L} and \mathbf{H} is the vector of currents in the Cartan subalgebra, forms an $SU(2)$ subalgebra. If λ is the weight of a component, $\phi_{(r)}^{\lambda}$, of the multiplet $\phi_{(r)}$, then

$$\begin{aligned} 0 \leq \langle \phi_{(r)}^{\lambda} | J_1^{-\alpha} J_{-1}^{\alpha} | \phi_{(r)}^{\lambda} \rangle &= \langle \phi_{(r)}^{\lambda} | [J_1^{-\alpha}, J_{-1}^{\alpha}] | \phi_{(r)}^{\lambda} \rangle \\ &= \langle \phi_{(r)}^{\lambda} | (K - \alpha \cdot \mathbf{H}) | \phi_{(r)}^{\lambda} \rangle \\ &= (K - \alpha \cdot \boldsymbol{\lambda}) \langle \phi_{(r)}^{\lambda} | \phi_{(r)}^{\lambda} \rangle . \end{aligned} \quad (2.2.19)$$

Hence, $(K - \alpha \cdot \boldsymbol{\lambda})$ must be positive for all roots α and all weights $\boldsymbol{\lambda}$ in the representation (r). Thus,

$$K \geq \boldsymbol{\psi} \cdot \mathbf{A} \quad (2.2.20a)$$

$$= \sum_{i=1}^{r_{\mathcal{L}}} n_i m_i , \quad (2.2.20b)$$

where $\boldsymbol{\psi}$ is the highest root of \mathcal{L} and \mathbf{A} is the highest weight in the (r) representation. This is the first major constraint placed on highest weight states of Lie algebras and the associated primary fields that can appear in consistent string models.

One consequence of this, as we mentioned in chapter 1, is that string models based on level-1 KM algebras cannot have spacetime scalars in the adjoint representation. Naively, there would appear a way of escaping this. Since the KM currents transform in the adjoint representation we might use them to form spacetime scalars. Unfortunately, this cannot be done, at least for models with *chiral* fermions.^[19,20,1] The basic argument is as follows: The vertex operator, V_{scalar}^a for a spacetime scalar in a level-1 adjoint representation would necessarily have the form

$$V_{\text{scalar}}^a = O(\bar{z}) J^a(z) , \quad (2.2.21)$$

¹²See Figure A.1 of Appendix A for listings of the co-marks for each of the compact simple Lie algebras.

where J^a is one of the KM currents. Masslessness of the state requires that the anti-holomorphic operator, $O(\bar{z})$, have conformal dimension $\bar{h}_O = \frac{1}{2}$ and behave both in its OPE's and under GSO projections like an additional RNS fermion. Hence, the spacetime spinor degrees of freedom would fall into representations of the five-dimensional Lorentz group, $SO(4,1)$. Decomposition into $SO(3,1)$ spinors always gives non-chiral pairs. Thus, adjoint scalars and chiral fermions are mutually exclusive. Further, $N = 1$ SUSY, at least for models based on free field construction, also disallows these adjoint scalars.^[1,21] We have assumed in these arguments that the currents are not primary fields of the full Virasoro-KM algebra; comparison of eq. (2.2.7c) with (2.2.13a) proves this is, indeed, a valid assumption.

A second constraint on states is a bit more trivial. Since the gauge groups come from the bosonic sector of the heterotic string, the total contribution to the conformal anomaly from the gauge groups cannot exceed 22, *i.e.*,

$$c_{\text{KM}} = \sum_i \frac{K_i \dim \mathcal{L}_i}{K_i + \tilde{h}_i} \leq 22, \quad (2.2.22)$$

where the sum is over the different factors in the algebra and every $U(1)_K$ contributes 1 to the sum. This condition gives an upper bound to the levels for a GUT.^[8,9] For example, if the gauge group is $SO(10)$, the maximum level is seven, for E_6 it is four, while it is 55 for $SU(5)$.

In terms of the massless representations of the Lie algebras that appear, there is one additional constraint that is stronger than either of the first two. This constraint is on the conformal dimension of a primary field, (2.2.15c). Since the intercept for the bosonic sector of a heterotic string is one, a potentially massless state in an (r) representation cannot have $h_{(r)}$ greater than one. That is,

$$h_{(r)} = \frac{C_{(r)}/\psi^2}{K + \tilde{h}} \leq 1. \quad (2.2.23)$$

Chapter 3: Modular Invariant Partition Functions

3.1 Review of Characters, Partition Functions, and Modular Invariance

Recently, studies of classical string solutions have provided impetus for further research into two-dimensional conformal field theories. In particular, considerable effort has been spent in classifying modular invariant partition functions (MIPF's) of these theories. In any string model, there is corresponding to each (chiral) Verma module representation, $[\phi(z)]$, of the Virasoro algebra (or an extension of it such as a super-Virasoro or Virasoro-KM algebra) a character (a.k.a. partition function), $\chi_{[\phi]}$. The character is a trace over the Verma module on a cylinder,

$$\chi_{[\phi]} = \text{Tr}_{[\phi]} q^{L_{0\text{cyl}}} = q^{-c/24} \text{Tr}_{[\phi]} q^{L_{0\text{plane}}} , \quad (3.1.1)$$

where $q = \exp(2\pi i\tau)$, $\tau = \tau_1 + i\tau_2$, with $\tau_1, \tau_2 \in \mathbb{R}$, and the trace containing the conformal anomaly factor is defined on the complex plane.¹³ If we expand this character in terms of powers of q ,

$$\chi_{[\phi]} = q^{-c/24} \sum_{i=0}^{\infty} n_i q^{h_{\phi}+i} , \quad (3.1.2)$$

the integer coefficient, n_i counts how many (descendent) fields the Verma module contains at the i^{th} energy level. The one-loop partition function of the string model can be expressed in terms of bi-linears of the characters of the Verma modules,

$$Z(\tau, \bar{\tau}) = \sum_{a,b} N_{ab} \chi_a(\tau) \bar{\chi}_b(\bar{\tau}) \quad (3.1.3)$$

¹³The factor of $q^{-c/24}$ results from the stress-energy tensor, $T(z)$ not transforming homogeneously under a conformal transformation, but picking up a quantity equal to $c/12$ times the Schwarzian, $S(z, w)$. That is, under $w \rightarrow z = e^w$,

$$T_{\text{cyl}}(w) = \left(\frac{\partial z}{\partial w} \right)^2 T(z) + \frac{c}{12} S(z(w), w) ,$$

where $S(z(w), w) \equiv \frac{\partial z \partial^3 z - (3/2)(\partial^2 z)^2}{(\partial z)^2} = -\frac{1}{2}$. Thus, the L_0 defined on the cylinder is not equivalent to the L_0 defined on the complex plane, rather $L_{0\text{cyl}} = L_{0\text{plane}} - c/24$. If the only purpose for partition functions were to count the number of states at each level, the anomaly term could be effectively discarded. However, this term is very important with regard to modular invariance.

$$= \sum_{a,b} N_{ab} \text{Tre}^{2\pi i \tau_1 P} e^{-2\pi \tau_2 H} . \quad (3.1.4)$$

$H = L_0 + \bar{L}_0$ and $P = L_0 - \bar{L}_0$ are the Hamiltonian and momentum operators of the theory¹⁴ and $N_{ab} \in \mathbb{Z}$ corresponds to the number of times that the primary field associated with $\chi_a \bar{\chi}_b$ appears in the theory.

The term “modular invariant partition function” is understood, as above, to generally mean the MIPF for a genus-1 world sheet (a torus). In conformal field theory, a torus is characterized by a single complex parameter, the τ of the above equations. Geometrically, τ may be defined by making the following identifications on the complex plane:^[5]

$$z \approx z + n + m\tau , \quad n, m \in \mathbb{Z} , \quad \tau \in \mathbb{C} . \quad (3.1.5)$$

The more general definition

$$z \approx z + n\lambda_1 + m\lambda_2 , \quad n, m \in \mathbb{Z} , \quad \lambda_1, \lambda_2 \in \mathbb{C} , \quad (3.1.6)$$

leads to conformally equivalent tori under rescaling and rotation of λ_1 and λ_2 by the conformal transformation $z \rightarrow \alpha z$. Hence, only their ratio, $\tau \equiv \frac{\lambda_1}{\lambda_2}$, is a conformal invariant. Therefore λ_1 is set to one. Also, the freedom to interchange λ_1 and λ_2 allows us to impose $\text{Im} \tau > 0$. Thus, tori are characterized by complex τ in the upper-half plane. (See Figure 3.1.)

This is not the whole story though. It is not quite true that τ is a conformal invariant that cannot be changed by rescalings and diffeomorphisms. There are global diffeomorphisms, not smoothly connected to the identity, that leave the torus invariant, but change the parameter τ . They correspond to cutting the torus along either cycle a or b , twisting one of the ends by a multiple of 2π , and then gluing the ends back together. (See Figure 3.2.) Such operations are known as Dehn twists and

¹⁴Thus, from the statistical mechanics perspective, τ_2 can be viewed as either a Euclidean time that propagates the fields over the one-loop world sheet cylinder or as the inverse of a temperature. Analogously, P can be interpreted as a momentum operator that twists an end of the world sheet cylinder by τ_1 before both ends meet to form a torus.

generate all global diffeomorphisms of the torus. A Dehn twist around the a cycle, transforms τ into $\tau + 1$. (The related transformations of λ_1 and λ_2 are $\lambda_1 \rightarrow \lambda_1$ and $\lambda_2 \rightarrow \lambda_1 + \lambda_2$.) This transformation is commonly denoted as “ T ”,

$$T: \quad \tau \rightarrow \tau + 1 . \quad (3.1.7a)$$

The twist around the b cycle corresponds (after rotation and rescaling to bring λ_1 to one) to $\tau \rightarrow \frac{\tau}{\tau+1}$ and can be expressed in terms of T and another transformation, “ S ”, defined by

$$S: \quad \tau \rightarrow \frac{1}{\tau} . \quad (3.1.7b)$$

Specifically, $TST : \tau \rightarrow \frac{\tau}{\tau+1}$.

S and T are the generators of the symmetry group of $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$, called the modular group of the torus. General modular transformations take the form

$$PSL(2, \mathbb{Z}) : \tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d} , \quad (3.1.8)$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. (The \mathbb{Z}_2 projection equates (a, b, c, d) with $(-a, -b, -c, -d)$ since both correspond to the same transformation of τ .) Thus, the true moduli space of conformally inequivalent tori is the upper-half plane modded out by the modular group. This region is called the fundamental domain, \mathcal{F} , of τ . The range for the fundamental domain is normally chosen to be

$$\mathcal{F} = \left\{ -\frac{1}{2} \leq \text{Re}\tau \leq 0, |\tau|^2 \geq 1 \cup 0 < \text{Re}\tau < \frac{1}{2}, |\tau|^2 > 1 \right\} . \quad (3.1.9)$$

A value of τ outside of the fundamental domain corresponds to a torus that is conformally equivalent to another produced by a τ in the fundamental domain. Any value of τ in the complex plane outside of the fundamental domain can be transformed, by a specific element of $PSL(2, \mathbb{Z})$, to the inside.

For a consistent string model, physical quantities, such as amplitudes, must be invariant under transformations of τ that produce conformally equivalent tori. That is, physical quantities must be “modular invariant”. This implies the necessity of a modular invariant partition function, because the one-loop vacuum-to-vacuum

amplitude, A , of a theory is the integral of the partition function, $Z(\tau, \bar{\tau})$ over the fundamental domain, \mathcal{F} ,

$$A = \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\text{Im}\tau)^2} Z(\tau, \bar{\tau}) . \quad (3.1.10)$$

Thus, consistency of a string theory requires that the one-loop partition function be invariant under both S and T transformations. Please note that although invariance of the one-loop partition function under S and T is necessary for a consistent model, it is not sufficient.^[22,12-14] Multi-loop partition functions must also be invariant under generalized modular transformations. Multi-loop invariance holds if, in addition to invariance at one-loop, there is invariance under a symmetry that mixes the cycles of neighboring tori of Riemann surfaces of genus $g > 1$ world sheets. This mixing is generally referred to as a U transformation.

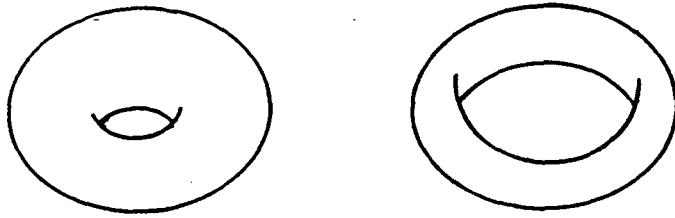


Fig. 3.1 Two conformally inequivalent tori

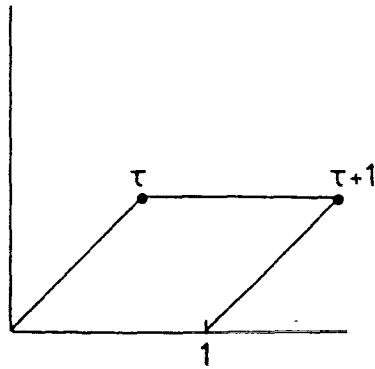


Fig. 3.2 Lattice representation of a two-dimensional torus defined by complex number τ

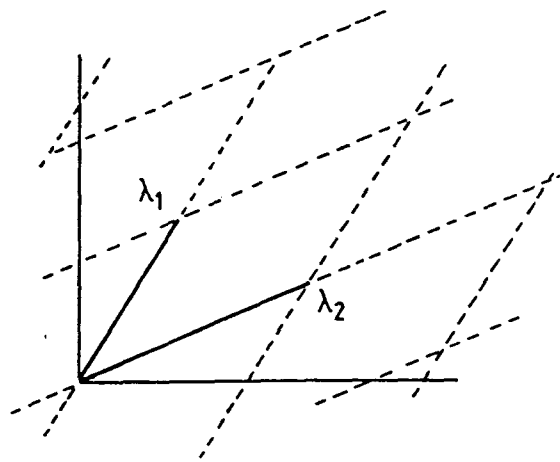


Fig. 3.3 Lattice representation of a two-dimensional torus defined by complex numbers λ_1 and λ_2

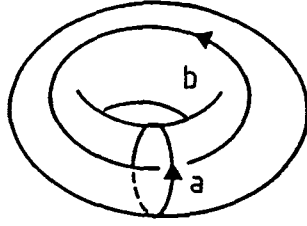


Fig. 3.4 The two independent cycles on the torus

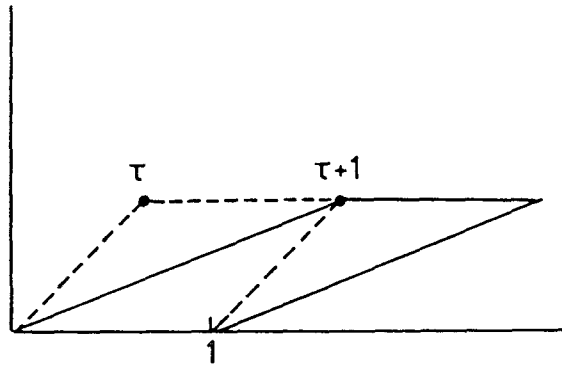


Fig. 3.5 Transformation of τ from Dehn twist around the a cycle

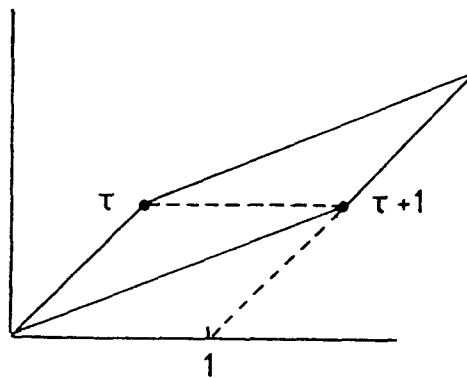


Fig. 3.6 Transformation of τ from Dehn twist around the b cycle

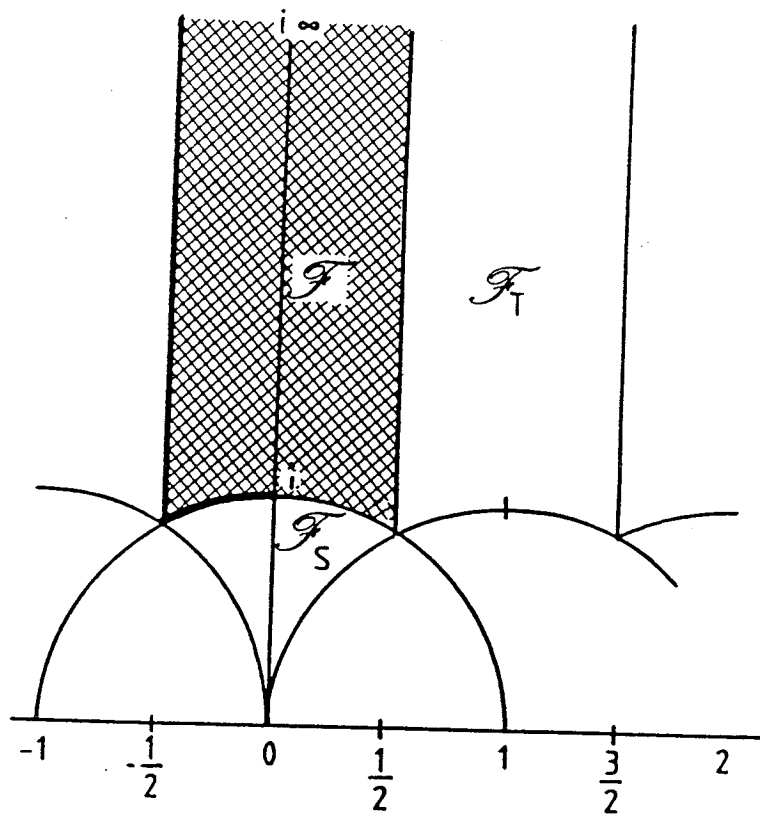


Fig. 3.7 Fundamental domain \mathcal{F} in moduli space
and its images under S and T

3.2 Complications for Models Based on General KM Algebras

Complete classification of modular invariant one-loop partition functions exists only for some of the simplest conformal field theories, in particular the minimal discrete series with $c < 1$, and the models based on $SU(2)_K$ Kač-Moody algebras.^[23] These MIPFs are formed from bilinears of characters, $\chi_l^{(K)}$, of $SU(2)_K$, which we label by twice the spin, $l = 2s$, of the corresponding $SU(2)$ representation ($l = 0$ to K).¹⁵ These MIPF's were constructed and found to be in one-to-one correspondence with the simply-laced Lie algebras:

$$\begin{aligned}
Z(A_{K+1}) &= \sum_{l=0}^K |\chi_l|^2; \quad K \geq 1 \\
Z(D_{\frac{K}{2}+2}) &= \begin{cases} \sum_{l_{EVEN}=0}^{\frac{K}{2}-2} |\chi_l + \chi_{K-l}|^2 + 2|\chi_{\frac{K}{2}}|^2; & K \in 4\mathbb{Z}^+ \\ \sum_{l_{EVEN}=1}^K |\chi_l|^2 + |\chi_{\frac{K}{2}}|^2 \\ \quad + \sum_{l_{ODD}=1}^{\frac{K}{2}-2} (\chi_l \chi_{K-l} + c.c.); & K \in 4\mathbb{Z}^+ + 2 \end{cases} \\
Z(E_6) &= |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2; \quad K = 10 \\
Z(E_7) &= |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 + |\chi_8|^2 \\
&\quad + [(\chi_2 + \chi_{14})\chi_8^* + c.c.]; \quad K = 16 \\
Z(E_8) &= |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2; \quad K = 27
\end{aligned} \tag{3.2.1}$$

The $D_{\frac{K}{2}+2}$ partition function is formed from twisting of the A_{K+1} partition function by the simple current $J_S = (0, K)$ for $K \in 4\mathbb{Z}^+$ or by the non-simple current $J_{NS} =$

¹⁵The values of l correspond to the dimensions of the highest weight representations (primary fields) meeting unitary conditions for an $SU(2)_K$ algebra. Throughout this chapter generic highest weight representations of an $SU(2)_K$ algebra are denoted by Φ_l or, where there will be no confusion, simply by l . When discussing holomorphic and anti-holomorphic primary fields, generically l will represent the former and \bar{l} the latter.

$(1, K - 1)$ for $K \in 4\mathbb{Z}^+ + 2$.¹⁶

The exceptional invariants of E_6 and E_8 originate via conformal embeddings^[25] of $A_1 \subset C_2$ and $A_1 \subset G_2$ respectively. $Z(E_7)$ can be derived by the more involved process of first conformally embedding $\frac{SU(2)}{\mathbb{Z}_2} \otimes \frac{SU(3)}{\mathbb{Z}_3}$ in E_8 , and then gauging away the $\frac{SU(3)}{\mathbb{Z}_3}$ contribution. The reason for the correspondence between $SU(2)_K$ modular invariants and simply-laced Lie groups is not fully understood. General arguments have shown that for any simply-laced Lie group a modular invariant solution can be constructed for affine $SU(2)$ at a specific level.^[26] But we are not aware of a complete explanation as to why these are one-to-one. Expressing these partition functions in the general form

$$Z = \sum_{l, \bar{l}} N_{l, \bar{l}} \chi_l \bar{\chi}_{\bar{l}}, \quad (3.2.2)$$

it was realized that: (1) for each MIPF the values of $N_{l, \bar{l}}$ for $l = \bar{l}$ coincide with the exponents of the associated simply-laced Lie algebra. These exponents give the degree (minus one) of a system of independent generators of the ring of invariant polynomials in these algebras; (2) the level K at which a specific modular invariant exists obeys the rule $K + 2 = \kappa$, where κ is the Coxeter number of the Lie algebra.

Fields with both holomorphic and anti-holomorphic components are denoted by either $(\Phi_l, \bar{\Phi}_{\bar{l}})$ or (l, \bar{l}) . In either case the first element is holomorphic and the second is anti-holomorphic. The product of two such fields, resulting from tensoring $SU(2)_{K_A}$ and $SU(2)_{K_B}$ algebras, is denoted by $(l_A, \bar{l}_A; m_B, \bar{m}_B)$, where l_A and \bar{l}_A are holomorphic and antiholomorphic primary fields for the $SU(2)_{K_A}$ algebra. m_B and \bar{m}_B are to be interpreted similarly for the $SU(2)_{K_B}$.

¹⁶We follow the standard definition for simple currents.^[24] A simple current J is a primary field which when fused with any other primary field (including itself) Φ_l of the K-M algebra produces only one primary field as a product state:

$$J \otimes \Phi_l = \Phi_{l'}$$

A non-simple current J' , when fused with at least one other primary field (possibly itself), produces more than one primary field:

$$J' \otimes \Phi_l = \sum_{l'} \Phi_{l'}$$

Classification of MIPFs for tensor products of $SU(2)_{K_i}$ may shed more light on the underlying significance of this.

For tensor products of other theories no procedures have been developed that give all of the possible modular invariants, but a few simple algorithms exist for modifying a known modular invariant to produce another one, in particular the orbifold construction^[27] and the related operation of twisting by a simple current.^[28] In this chapter, we make some proposals aimed at the general problem of classifying all possible modular invariants for conformal field theories constructed by tensoring together models whose modular invariants are already known. By a tensor product of two theories, say A and B , we mean a theory whose chiral algebra includes the chiral algebras of both the A and B theories. As a consequence, the central charge of the combined theory will be the sum of those for the individual factors, the chiral blocks that make up amplitudes will be constructed from the products of the individual chiral blocks, and the characters will be products of the individual characters. Thus the partition function of the $A \otimes B$ theory is restricted to the form

$$Z^{AB} = \sum_{l,m,\bar{l},\bar{m}} N_{lm\bar{l}\bar{m}}^{AB} \chi_l^A \chi_m^B \bar{\chi}_{\bar{l}}^A \bar{\chi}_{\bar{m}}^B . \quad (3.2.3)$$

The approach taken here derives rules by iteration in the number of terms in the tensor products, *i.e.*, we consider the conditions placed on higher order tensor products by the requirements of modular invariance of lower order tensor products. We also discuss the degrees of freedom in MIPFs that remain after these conditions have been applied to the higher order terms. For an application of this process, we concentrate on the specific case of tensor products of two $SU(2)_{K_i}$ K-M algebras and their MIPFs. This is investigated for two reasons: (1) for insight into the density of MIPFs derived by simple currents compared to the total space of MIPFs;¹⁷ and (2) as a first step towards developing a systematic set of rules for constructing MIPFs

¹⁷Knowledge of the density of simple current MIPFs will play a significant role in understanding the total space of MIPFs. In the last few years A.N. Schellekens *et al.*^[28] have made significant progress towards complete classification of simple current modular invariants (SCMI's) for rational conformal field theories (RCFTs). These classifications appear amenable for generalization to SCMI's for tensor products of RCFTs. Therefore, understanding of the density of SCMI's

out of tensor products of characters for general K-M algebras and minimal models.^[29] The latter issue was first discussed in Ref. [1].

Completion of this set of rules generalizes the work in [12], [13] and [14], wherein the process for creating consistent (*i.e.*, modular invariant) models from tensor products of Ising models (the free fermion approach) is derived. These papers reveal how an infinite set of consistent free fermion models can be constructed, with the majority based on left-right (L-R) asymmetric modular invariants. Ref. [1] suggests that the majority of consistent models formed from tensor products of K-M algebras and minimal models may likewise be L-R asymmetric. As with the free fermion models, the L-R asymmetric cases may comprise the larger, and perhaps more interesting, class of models.

The combined tensor product theory is *not* restricted to be simply the product of the individual theories; the operators in the combined theory need not be diagonal (*i.e.*, left-right symmetric), and in general the fusion rules for the operator products will be modified. The latter point is the chief complication in the general problem. The allowed tensor product theories built from free bosons or fermions have been successfully categorized, because the possible fusion rules in these theories are almost trivial; likewise twisting a theory by a simple current gives unambiguously a new theory, because the new fusion rules are unambiguous. The difficulty of this procedure in general (compared to that for the free fermion or boson models) becomes clear from the transformation properties of the characters under S and T transformations, the generators of the modular group $PSL(2, \mathbb{Z})$.

For an Ising (free fermion) model, there are three non-zero characters. Each of these transforms under S or T into another one of the three characters (possibly times

compared to the total space of MIPFs is very constructive, for this will reveal the size of the space of solutions that cannot be found through Schellekens' approach.

a phase):

$$\begin{aligned}
T : \quad & \chi \begin{pmatrix} A \\ A \end{pmatrix} \rightarrow e^{i\pi/24} \chi \begin{pmatrix} A \\ P \end{pmatrix} \\
& \chi \begin{pmatrix} A \\ P \end{pmatrix} \rightarrow e^{-i\pi/24} \chi \begin{pmatrix} A \\ A \end{pmatrix} \\
& \chi \begin{pmatrix} P \\ A \end{pmatrix} \rightarrow e^{i\pi/12} \chi \begin{pmatrix} P \\ A \end{pmatrix}
\end{aligned} \tag{3.2.4}$$

$$\begin{aligned}
S : \quad & \chi \begin{pmatrix} A \\ A \end{pmatrix} \rightarrow \chi \begin{pmatrix} A \\ A \end{pmatrix} \\
& \chi \begin{pmatrix} P \\ A \end{pmatrix} \rightarrow \chi \begin{pmatrix} A \\ P \end{pmatrix} \\
& \chi \begin{pmatrix} A \\ P \end{pmatrix} \rightarrow \chi \begin{pmatrix} P \\ A \end{pmatrix}
\end{aligned}$$

where,

$$\chi \begin{pmatrix} A \\ A \end{pmatrix} = \chi_0 + \chi_{1/2}, \quad \chi \begin{pmatrix} A \\ P \end{pmatrix} = \chi_0 - \chi_{1/2}, \quad \text{and} \quad \chi \begin{pmatrix} P \\ A \end{pmatrix} = \sqrt{2} \chi_{1/16}, \tag{3.2.5}$$

Here $\chi_i, i = 0, 1/16, 1/2$ are the characters of the primary fields of conformal dimension $h = 0, 1/16, 1/2$ in the $c = 1/2$ critical Ising model. $P(A)$ denotes (anti-)periodic boundary conditions around one of the two non-contractible loops of the world sheet torus.^[26] In this case, the S and T transformations act on the characters in the manner of generic simple currents denoted by J_S or J_T , respectively, twisting the corresponding primary states Φ_i . In other words, the outcome of the transformation or fusion is, respectively, a single character or primary field:

$$\begin{aligned}
S : \quad & \chi_i \rightarrow \chi_j; & T : \quad & \chi_i \rightarrow \chi_k \\
J_S \otimes \Phi_i = & \Phi_j; & J_T \otimes \Phi_i = & \Phi_k .
\end{aligned}$$

However, in the generic case for a K-M algebra or minimal model, S transforms a character χ_l in the manner of a non-simple current, J_{NS} , acting on a generic primary field Φ_l . The outcome of the transformation or fusion is in general is not a single

term, but a sum of terms:

$$\begin{aligned} S : \chi_l &\Rightarrow \sum_{l'} S(l, l') \chi_{l'} \\ J_{NS} \otimes \Phi_l &= \sum_{l'} N_{J_{NS}l}^{l'} \Phi_{l'} \end{aligned} \quad (3.2.6)$$

(As shown by E. Verlinde^[30], the (positive integer) coefficients $N_{J_{NS}l}^{l'}$ are related to the matrix elements of the S transformation matrix:

$$N_{J_{NS}l}^{l'} = \sum_n \frac{S(J_{NS}, n) S(l, n) S(l', n)}{S(0, n)}, \quad (3.2.7)$$

where 0 denotes the vacuum state or identity field.) The complicated transformations of tensor products of generic K-M characters χ_l make complete classification of associated MIPFs much more difficult than in the free fermion or boson case.

In section 3.3 we consider the extent to which the integer coefficients $N_{lm\bar{l}\bar{m}}^{AB}$ in the partition function of the tensor product theory are constrained once we know all of the allowed possibilities for the corresponding coefficients $N_{l\bar{l}}^A$ and $N_{m\bar{m}}^B$ in the factor theories. In section 3.4 we investigate the more general problem of combining theories whose holomorphic and anti-holomorphic degrees of freedom need not possess the same chiral algebras. That is, we consider partition functions of the form, $Z^{AB} = \sum_{l, \bar{m}} N_{l\bar{m}} \chi_l^A \bar{\chi}_{\bar{m}}^B$. In the following sections we are interested ultimately in classifying consistent conformal field theories, not just modular invariant combinations of characters. Accordingly, we invoke consistency conditions for amplitudes on the plane when they constrain the states that can appear in the partition function.

3.3 Constraints on Tensor Product Modular Invariants

In order for the tensor product partition function (3.2.3) to be invariant under the generators of modular transformations, T and S , we must have,

$$\begin{aligned} T \text{ invariance : } & h_l + h_m = h_{\bar{l}} + h_{\bar{m}} \pmod{1} \text{ if } N_{lm\bar{l}\bar{m}}^{AB} \neq 0 \\ S \text{ invariance : } & N_{lm\bar{l}\bar{m}}^{AB} = \sum_{l', m', \bar{l}', \bar{m}'} N_{l'm'\bar{l}'\bar{m}'}^{AB} S_{ll'}^A S_{mm'}^B \bar{S}_{\bar{l}\bar{l}'}^A \bar{S}_{\bar{m}\bar{m}'}^B, \end{aligned} \quad (3.3.1)$$

where h_l denotes the conformal dimension of the primary field represented by the label l , *etc.* We assume that the solutions to the corresponding equations for the

factor theories are known. That is, we are given all possibilities (labeled by i) for non-negative integer coefficients $N_{l\bar{l}}^{A,i}$ such that,

$$\begin{aligned} h_l &= h_{\bar{l}} \pmod{1} \quad \text{if } N_{l\bar{l}}^{A,i} \neq 0 \\ \text{and } N_{l\bar{l}}^{A,i} &= \sum_{l',\bar{l}'} N_{l'\bar{l}'}^{A,i} S_{l\bar{l}'}^A \bar{S}_{l'\bar{l}}^A, \end{aligned} \quad (3.3.2)$$

and similarly for $N_{m\bar{m}}^{B,j}$. We can get relations between the integer coefficients in equations (3.3.1) and (3.3.2) by multiplying (3.3.1) by $N_{l\bar{l}}^{A,i}$ and summing over l and \bar{l} ,

$$\begin{aligned} \sum_{l,\bar{l}} N_{l\bar{l}}^{A,i} N_{lm\bar{l}\bar{m}}^{AB} &= \sum_{l,l',m',\bar{l},\bar{l}',\bar{m},\bar{m}'} N_{l\bar{l}}^{A,i} N_{l'm'\bar{l}'\bar{m}'}^{AB} S_{l\bar{l}'}^A S_{m\bar{m}'}^B \bar{S}_{l'\bar{l}}^A \bar{S}_{\bar{m}'\bar{m}}^B \\ &= \sum_{l',m',\bar{l}',\bar{m}'} N_{l'\bar{l}'}^{A,i} N_{l'm'\bar{l}'\bar{m}'}^{AB} S_{m\bar{m}'}^B \bar{S}_{\bar{m}'\bar{m}}^B, \end{aligned} \quad (3.3.3)$$

where we have used (3.3.2) and the symmetry of S to simplify the right-hand side. The resulting equation is precisely of the form (3.3.2) for the B theory, therefore we must have,

$$\sum_{l,\bar{l}} N_{l\bar{l}}^{A,i} N_{lm\bar{l}\bar{m}}^{AB} = \sum_j n_j^{A,i} N_{m\bar{m}}^{B,j}, \quad (3.3.4a)$$

where $n_j^{A,i}$ are integers. This constrains some combinations of coefficients in the AB theory to be linear combinations (with integer coefficients) of the allowed coefficients in the B theory, which are presumed known. There is an analogous constraint arising from taking the appropriate traces over the B theory indices in (3.3.1),

$$\sum_{m,\bar{m}} N_{m\bar{m}}^{B,j} N_{lm\bar{l}\bar{m}}^{AB} = \sum_i n_i^{B,j} N_{l\bar{l}}^{A,i}, \quad (3.3.4b)$$

and a further constraint arising from taking appropriate traces over both sets of indices in either possible order,

$$\sum_{\substack{l,\bar{l} \\ m,\bar{m}}} N_{m\bar{m}}^{B,j} N_{l\bar{l}}^{A,i} N_{lm\bar{l}\bar{m}}^{AB} = \sum_{m,\bar{m}} N_{m\bar{m}}^{B,j} \left(\sum_j n_j^{A,i} N_{m\bar{m}}^{B,j} \right), \quad (3.3.4c)$$

$$= \sum_{l,\bar{l}} N_{l\bar{l}}^{A,i} \left(\sum_i n_i^{B,j} N_{l\bar{l}}^{A,i} \right). \quad (3.3.4d)$$

Note that the number of constraint equations increases as the factor theories become more complex (in the sense of having more possible modular invariants), and also as the tensor products theories have more factors.

These equations constrain part of the operator content of the tensor product theories, which we wish to classify. Often, this information, together with some simple consistency requirements for conveniently chosen amplitudes on the plane, serves to completely determine the allowed possibilities for the tensor product modular invariants. For concreteness, we illustrate with a simple example.

3.3.a Example: $SU(2)_{K_A} \otimes SU(2)_{K_B}$ Tensor Product Theories.

$SU(2)_K$ has $K + 1$ unitary primary fields, which we label by twice the spin, $l = 2s$, of the corresponding $SU(2)$ representation. Their conformal dimensions are $h_l = \frac{l(l+2)}{4(K+2)}$. The matrix S , implementing the modular transformation $\tau \rightarrow -1/\tau$ on the Kač-Moody characters, is

$$S_{l'l}^K = \left(\frac{2}{K+2} \right)^{1/2} \sin \left(\frac{\pi(l+1)(l'+1)}{K+2} \right). \quad (3.3.5)$$

The fusion rules, which we will make use of momentarily, are

$$\phi_l \times \phi_{l'} = \sum_{\substack{m=|l-l'| \\ m-|l-l'| \text{ even}}}^{\min(l+l', 2K-l-l')} \phi_m. \quad (3.3.6)$$

For simplicity we only consider the tensor product theories with holomorphic and anti-holomorphic chiral algebras $SU(2)_{K_A} \otimes SU(2)_{K_B}$ for both K_A and K_B odd. Then the only possible modular invariants for the factor theories are the diagonal

ones, $N_{l\bar{l}} = \delta_{l\bar{l}}$.¹⁸ Applying the constraint equations (3.3.4a-d) gives the conditions,

$$\begin{aligned} \sum_{l=\bar{l}} N_{lm\bar{l}\bar{m}}^{AB} &= a\delta_{m\bar{m}} ; & a \in \mathbb{Z}^+ \\ \sum_{m=\bar{m}} N_{lm\bar{l}\bar{m}}^{AB} &= b\delta_{l\bar{l}} ; & b \in \mathbb{Z}^+ \\ \sum_{l=\bar{l}} \sum_{m=\bar{m}} N_{lm\bar{l}\bar{m}}^{AB} &= a(K_B + 1) = b(K_A + 1) . \end{aligned} \quad (3.3.7)$$

If we label the primary operators in the tensor product theory by the corresponding l values of the factor theories, *e.g.*, $(l, m | \bar{l}, \bar{m})$, then the integer a is equal, in particular, to the number of primary operators in the theory of the form $(j, 0 | j, 0)$. These are pure A theory operators and so must form a closed operator subalgebra of the A theory. Similarly, b must equal the dimension of some closed operator algebra in the B theory. This is useful because we know (from studying the consistency of amplitudes on the plane) all consistent closed operator sub-algebras of $SU(2)$ Kač-Moody theories.^[31] For K odd these sub-algebras (labeling them by their dimensions, d) are

$$\begin{aligned} d = 1 &: \{ \Phi_0 \} \text{ (the identity)} \\ d = 2 &: \{ \Phi_0, \Phi_K \} \\ d = \frac{K+1}{2} &: \{ \Phi_l; 0 \leq \text{even } l \leq K \} \text{ (the allowed integer spin representations)} \\ d = K+1 &: \{ \Phi_l; 0 \leq l \leq K \} \end{aligned} \quad (3.3.8)$$

Thus, in the tensor product theory we know all of the possibilities for operators of the form $(j, 0 | j, 0)$ or $(0, j | 0, j)$. Given (3.3.7) and the uniqueness of the vacuum state $(0, 0 | 0, 0)$ in the tensor product theory, the multiplicities of the operators in the closed sub-algebras must be as given in (3.3.8).

We can now write down all of the possibilities for a, b, K_A and K_B that are

¹⁸For our purposes we need to consider all sets of non-negative integer coefficients $N_{l\bar{l}}$ that give rise to S and T invariant partition functions, but not necessarily only ones with a unique vacuum state ($N_{00} = 1$). Relaxing this condition in the $SU(2)$ case does not expand the space of possible solutions, aside from a trivial multiplicative constant.

consistent with (3.3.7) and (3.3.8), and consider each type of tensor product modular invariant individually:

(1) $a = K_A + 1$, $b = K_B + 1$: Here we have $N_{\bar{l}m\bar{m}}^{AB} = \delta_{\bar{l}}\delta_{m\bar{m}} + M_{\bar{l}m\bar{m}}^{AB}$, with $M_{\bar{l}m\bar{m}}^{AB}$ traceless with respect to both l, \bar{l} and m, \bar{m} . It is easy to see that M^{AB} must in fact vanish, leaving us with the simple uncorrelated tensor product of the $SU(2)_{K_A}$ and $SU(2)_{K_B}$ diagonal modular invariants. Were this not the case, then M^{AB} by itself would give rise to a modular invariant which did not include the term containing the identity operator. But this is not possible since (from (3.3.5) and quite generally in a unitary theory) $S_{0l} > 0$ for all l .

(2) $a = \frac{K_A+1}{2}$, $b = \frac{K_B+1}{2}$: In this case the operators diagonal in either of the factor theories comprise the set $\{(l, m|l, m)\}$ with l and m both odd or both even. This set contains the operators $(1, K_B|1, K_B)$ and $(K_A, 1|K_A, 1)$. The non-diagonal operators in the theory, $(i, j|m, l)$ must have a consistent operator product with these two operators, in particular at least some of the operators appearing in the naive fusion with them (using the rules (3.3.6) must have integer spins ($h - \bar{h} \in \mathbb{Z}$). This restricts the non-diagonal operators $(i, j|m, l)$, $i \neq m$, $j \neq l$, to those satisfying $i + m = K_A$ and $j + l = K_B$. For these operators, in turn, to have integer spin we have either: $i - j$ even and $K_A + K_B = 0 \pmod{4}$; or $i - j$ odd and $K_A - K_B = 0 \pmod{4}$. Taking all such operators, the former case gives the modular invariant obtained from the simple tensor product invariant of case (1) by twisting by the simple current $(K_A, K_B|0, 0)$; the latter is obtained by twisting by $(K_A, 0|0, K_B)$. An extension of the argument given in (1) using the fact that $S_{iK_A}^{K_A} S_{jK_B}^{K_B} > 0$ for all $i - j$ even, shows that these are the only possibilities in this category.

(3) $a = 1$, $b = 2$, $K_B = 2K_A + 1$ or $a = 2$, $b = 1$, $K_A = 2K_B + 1$: Take $a = 1$, $b = 2$ so $K_B = 2K_A + 1$. The model must include the states $(0, 0|0, 0)$ and $(0, K_B|0, K_B)$ but no other states of the form $(0, l|0, l)$, $(i, 0|i, 0)$ or $(j, K_B|j, K_B)$. There must also be two states of the form $(K_A, j|K_A, j)$. Demanding that the fusion products of these states with themselves are consistent with the above restriction requires $j = 0$, or

K_B , but then the states themselves are inconsistent with the restriction. Thus there are no possible consistent theories within this category.

(4) $a = 2$, $b = \frac{K_B+1}{2}$, $K_A = 3$ or $a = \frac{K_A+1}{2}$, $b = 2$, $K_B = 3$: This case differs from case (2) with $K_A = 3$ and/or $K_B = 3$, in that the $d = 2$ closed subalgebra of the $SU(2)_3$ theory consists of $\{\Phi_0, \Phi_3\}$ instead of $\{\Phi_0, \Phi_2\}$ as in (2). If $a = 2$, $b = \frac{K_B+1}{2}$ and $K_A = 3$, then the operators diagonal in either factor theory comprise the set $\{(0, l|0, l), (3, l|3, l) \text{ } l \text{ even}; (1, j|1, j), (2, j|2, j) \text{ } j \text{ odd}\}$. There must be additional non-diagonal operators, $(i, j|l, m) \text{ } i \neq l, j \neq m$, if there are to be any modular invariants in this category. If $(i, j|l, m)$ appears then $(3-i, j|3-l, m)$ appears also. For both operators to have integer spin, i and l must be both even or both odd. Thus there must be operators of the form $(0, j|2, m)$ or $(1, p|3, l)$. Fusing these with the operators $(1, K_B|1, K_B)$ from the diagonal part of the theory produces the operators $(1, K_B - j|1, K_B - m)$ and/or $(1, K_B - j|3, K_B - m)$ and $(0, K_B - p|2, K_B - l)$ and/or $(2, K_B - p|2, K_B - l)$, respectively. It is easy to see that if the former fields have integer spin then none of the possible fusion products do. Thus, there can be no consistent theories in this category.

(5) $K_A = K_B \equiv K$, $a = b = 1$ or $a = b = 2$: The situation becomes more complicated for $K_A = K_B \equiv K$. For these cases we have additional trace equations,

$$\begin{aligned} \sum_{l=\bar{m}} N_{lm\bar{l}\bar{m}}^{AB} &= a' \delta_{m\bar{l}}; & a' \in \mathbb{Z}^+ \\ \sum_{m=\bar{l}} N_{lm\bar{l}\bar{m}}^{AB} &= b' \delta_{l\bar{m}}; & b' \in \mathbb{Z}^+ . \end{aligned} \tag{3.3.9}$$

If the values of a' and b' correspond to any of cases (1)—(4), then the invariants are precisely as given above, with the factor theories permuted. Thus we only need to consider the cases: (5a) $a = b = a' = b' = 1$, (5b) $a = b = a' = b' = 2$, and (5c) $a = b = 1$, $a' = b' = 2$. Case (5b) is most quickly disposed of. The operators in the theory include the closed subalgebra $\{(0, 0|0, 0), (0, K|0, K)\}$, $\{(0, 0|0, 0), (K, 0|K, 0)\}$, $\{(0, 0|0, 0), (K, 0|0, K)\}$, and $\{(0, 0|0, 0), (0, K|K, 0)\}$. For the operator algebra with these together to be closed the chiral fields $(K, K|0, 0)$ and $(0, 0|K, K)$ must appear,

but for K odd these do not have integer conformal dimension. Therefore this case is ruled out.

Cases (5a) and (5c) can also be ruled out as follows. In both cases there must be fields $(1, j|1, j)$ and $(p, 1|p, 1)$ with a single choice for j and p in each case. Consider the four-point correlation function on the plane $\langle(1, j|1, j)(1, j|1, j)(p, 1|p, 1)(p, 1|p, 1)\rangle$. In one channel the only possible intermediate state primary fields that can appear, consistent with the restrictions of cases (5a) or (5c), are $(0, 0|0, 0)$ and $(2, 2|2, 2)$. In the cross channels only a subset of the states of the form $(p \pm 1, j \pm 1|p \pm 1, j \pm 1)$ can appear as intermediates. We know from the four-point amplitudes $\langle(1|1)(1|1)(j|j)(j|j)\rangle$ and $\langle(1|1)(1|1)(p|p)(p|p)\rangle$ in the factor theories that the chiral blocks making up the amplitudes have two-dimensional monodromy, so the blocks appearing in the tensor product theory must have four-dimensional monodromy. There is no way, then, to assemble the two chiral blocks corresponding to the allowed intermediate primaries $(0, 0|0, 0)$ and $(2, 2|2, 2)$ in such a way that the four-point function in the tensor product theory can be monodromy invariant (*i.e.*, single-valued).

To summarize: We have used the constraints (3.3.4a-3.3.4d) and the consistency of conveniently chosen fusion rules and amplitudes to find the only consistent tensor product theories of the type $SU(2)_{K_A} \otimes SU(2)_{K_B}$, with K_A, K_B odd. These turn out to be the simple (uncorrelated) product of the diagonal invariants of the factor theories and all theories obtained from them by twisting by the allowed simple current fields that can be built from the identity and fields labeled K_A and K_B .

3.4 Left-Right Asymmetric Modular Invariants

So far we have considered tensor product conformal field theories that are diagonal in the sense that for each holomorphic conformal field theory factor there is a corresponding anti-holomorphic conformal field theory factor with an isomorphic chiral algebra. While these are the relevant theories to consider for statistical mechanics applications, it is natural in the construction of heterotic string theories to consider conformal field theories that are inherently left-right asymmetric as well. For these, the methods discussed above do not apply. Nonetheless, we can exploit

known properties of left-right symmetric conformal field theories to construct modular invariants even for inherently asymmetric theories by using the following result: Given two consistent diagonal rational conformal field theories (*a priori* with different chiral algebras) with modular invariant partition functions $Z^A = \sum \chi_i^A \bar{\chi}_i^A$ and $Z^B = \sum \chi_i^B \bar{\chi}_i^B$, the left-right asymmetric partition function given by $Z^{AB} = \sum \chi_i^A \bar{\chi}_i^B$ will be modular invariant if and only if: (1) the conformal dimensions agree modulo 1, or more precisely $h_i^A - c^A/24 = h_i^B - c^B/24 \pmod{1}$; and (2) the fusion rules of the two theories coincide, $\phi_i^A \times \phi_j^A = \sum_k N_{ij}^k \phi_k^A$ and $\phi_i^B \times \phi_j^B = \sum_k N_{ij}^k \phi_k^B$.

Condition (1) is obviously necessary and sufficient for Z^{AB} to be T invariant. Condition (2) is almost immediate given Verlinde's results.^[30] Z^{AB} is invariant under the S transformation only if the S matrices implementing the modular transformations on the characters of the A and B theories coincide, $S_{ij}^A = S_{ij}^B$. As Verlinde showed, the fusion rule coefficients determine the S matrix,¹⁹ so condition (2) is required for $S^A = S^B$. In employing this relation it is crucial to define the primary fields with respect to the full A and B chiral algebras.

As a simple example, consider the theories $A = SO(31)$ level-one, and $B = (E_8)$ level-two, both with central charge $c = 31/2$. The consistent diagonal theories have partition functions,

$$Z^A = \chi_0 \bar{\chi}_0 + \chi_{\frac{1}{2}} \bar{\chi}_{\frac{1}{2}} + \chi_{\frac{31}{16}} \bar{\chi}_{\frac{31}{16}} \quad (3.4.1)$$

and

$$Z^B = \chi_0 \bar{\chi}_0 + \chi_{\frac{3}{2}} \bar{\chi}_{\frac{3}{2}} + \chi_{\frac{15}{16}} \bar{\chi}_{\frac{15}{16}} , \quad (3.4.2)$$

where the characters are labeled by the conformal dimensions of the associated primary fields. The fusion rules in both theories are analogous to those in the Ising model. The asymmetric partition function,

$$Z^{AB} = \chi_0^A \bar{\chi}_0^B + \chi_{\frac{1}{2}}^A \bar{\chi}_{\frac{3}{2}}^B + \chi_{\frac{31}{16}}^A \bar{\chi}_{\frac{15}{16}}^B , \quad (3.4.3)$$

¹⁹To be precise, Verlinde showed that the eigenvalues, $\lambda_i^{(j)}$, of the matrices $(N_i)_l^k$ satisfy $\lambda_i^{(j)} = S_{ij}/S_{0j}$ but there could be an ambiguity in the choice of superscript (j) labeling each member of the set of eigenvalues of $(N_i)_l^k$. We believe in the present case that this ambiguity is fixed given T and the requirement $(ST)^3 = 1$, but have no proof.

satisfies conditions (1) and (2), and so is itself a modular invariant. This can also be constructed by choosing appropriate boundary conditions for a collection of 31 free, real fermions.

A more interesting example, which cannot be constructed from free bosons or fermions or by twisting a known invariant by a simple current, is the following. For the A theory we take the simple tensor product of the diagonal theories for G_2 level-one and $SU(3)$ level-two; for the B theory the simple tensor product of the diagonal theories for F_4 level-one and the three state Potts model. The central charges coincide: $c^A = 14/5 + 16/5 = 6$; $c^B = 26/5 + 4/5 = 6$. The primary fields appearing in each theory are, for G_2 level-one the identity and 7 ($h = \frac{2}{5}$); for $SU(3)$ level-two the identity, 3 and $\bar{3}$ ($h = \frac{4}{15}$), 6 and $\bar{6}$ ($h = \frac{2}{3}$), and 8 ($h = \frac{3}{5}$); for F_4 level-one the identity and 26 ($h = \frac{3}{5}$); and for the Potts model, the primaries, labeled by their conformal dimensions, are 0, $\frac{2}{5}$, $\frac{2}{3}$, $\frac{\bar{2}}{3}$, $\frac{1}{15}$, and $\frac{\bar{1}}{15}$.

To economically list the fusion rules for these theories we can simply list the non-vanishing three-point amplitudes (where we represent the field by its conformal dimension). Besides the obvious ones involving the identity operator ($\langle\phi\bar{\phi}0\rangle$) these are: for G_2 level-one $\langle\frac{2}{5}, \frac{2}{5}, \frac{2}{5}\rangle$; for $SU(3)$ level-two, $\langle\frac{4}{15}, \frac{4}{15}, \frac{4}{15}\rangle$, $\langle\frac{4}{15}, \frac{4}{15}, \frac{\bar{2}}{3}\rangle$, $\langle\frac{4}{15}, \frac{4}{15}, \frac{3}{5}\rangle$, $\langle\frac{4}{15}, \frac{3}{5}, \frac{2}{3}\rangle$, $\langle\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\rangle$, $\langle\frac{3}{5}, \frac{3}{5}, \frac{3}{5}\rangle$, and the conjugates of these; for F_4 level-one, $\langle\frac{3}{5}, \frac{3}{5}, \frac{3}{5}\rangle$; and for the three state Potts model, $\langle\frac{1}{15}, \frac{1}{15}, \frac{1}{15}\rangle$, $\langle\frac{1}{15}, \frac{1}{15}, \frac{\bar{2}}{3}\rangle$, $\langle\frac{1}{15}, \frac{1}{15}, \frac{2}{5}\rangle$, $\langle\frac{1}{15}, \frac{2}{5}, \frac{2}{3}\rangle$, $\langle\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\rangle$, $\langle\frac{2}{5}, \frac{2}{5}, \frac{2}{5}\rangle$, and the conjugates of these. All of the non-zero fusion rule coefficients, N_{ijk} , for these four theories are equal to one.

Given the obvious similarities of the fusion rules and conformal dimensions of these theories, it is not difficult to verify that the asymmetric partition function given by,

$$\begin{aligned}
Z^{AA'BB'} = & \chi_0^A \chi_0^{A'} \bar{\chi}_0^B \bar{\chi}_0^{B'} + \chi_0^A \chi_{\frac{3}{5}}^{A'} \bar{\chi}_{\frac{3}{5}}^B \bar{\chi}_0^{B'} + \chi_0^A \chi_{\frac{2}{3}}^{A'} \bar{\chi}_0^B \bar{\chi}_{\frac{2}{3}}^{B'} + \chi_0^A \chi_{\frac{2}{3}}^{A'} \bar{\chi}_0^B \bar{\chi}_{\frac{2}{3}}^{B'} + \\
& \chi_0^A \chi_{\frac{4}{15}}^{A'} \bar{\chi}_{\frac{3}{5}}^B \bar{\chi}_{\frac{2}{3}}^{B'} + \chi_0^A \chi_{\frac{4}{15}}^{A'} \bar{\chi}_{\frac{3}{5}}^B \bar{\chi}_{\frac{2}{3}}^{B'} + \chi_{\frac{2}{5}}^A \chi_0^{A'} \bar{\chi}_0^B \bar{\chi}_{\frac{2}{5}}^{B'} + \chi_{\frac{2}{5}}^A \chi_{\frac{3}{5}}^{A'} \bar{\chi}_{\frac{3}{5}}^B \bar{\chi}_{\frac{2}{5}}^{B'} + \quad (3.4.4) \\
& \chi_{\frac{2}{5}}^A \chi_{\frac{2}{3}}^{A'} \bar{\chi}_0^B \bar{\chi}_{\frac{1}{15}}^{B'} + \chi_{\frac{2}{5}}^A \chi_{\frac{2}{3}}^{A'} \bar{\chi}_0^B \bar{\chi}_{\frac{1}{15}}^{B'} + \chi_{\frac{2}{5}}^A \chi_{\frac{4}{15}}^{A'} \bar{\chi}_{\frac{3}{5}}^B \bar{\chi}_{\frac{1}{15}}^{B'} + \chi_{\frac{2}{5}}^A \chi_{\frac{4}{15}}^{A'} \bar{\chi}_{\frac{3}{5}}^B \bar{\chi}_{\frac{1}{15}}^{B'} ,
\end{aligned}$$

satisfies the two conditions for modular invariance. Here A , A' , B , and B' denote the G_2 , $SU(3)$, F_4 , and Potts theories, respectively.

An alternative sewing of the operators in these four conformal field theories gives rise to the diagonal E_6 level-one modular invariant

$$\begin{aligned}
Z^{E_6} = & \chi_0^A \chi_0^{A'} \bar{\chi}_0^B \bar{\chi}_0^{B'} + \chi_{\frac{2}{5}}^A \chi_{\frac{3}{5}}^{A'} \bar{\chi}_0^B \bar{\chi}_0^{B'} + \chi_0^A \chi_0^{A'} \bar{\chi}_{\frac{3}{5}}^B \bar{\chi}_{\frac{2}{5}}^{B'} + \chi_{\frac{2}{5}}^A \chi_{\frac{3}{5}}^{A'} \bar{\chi}_{\frac{3}{5}}^B \bar{\chi}_{\frac{2}{5}}^{B'} + \\
& \chi_0^A \chi_{\frac{2}{3}}^{A'} \bar{\chi}_0^B \bar{\chi}_{\frac{2}{3}}^{B'} + \chi_0^A \chi_{\frac{2}{3}}^{A'} \bar{\chi}_0^B \bar{\chi}_{\frac{2}{3}}^{B'} + \chi_0^A \chi_{\frac{2}{3}}^{A'} \bar{\chi}_{\frac{3}{5}}^B \bar{\chi}_{\frac{1}{15}}^{B'} + \chi_0^A \chi_{\frac{2}{3}}^{A'} \bar{\chi}_{\frac{3}{5}}^B \bar{\chi}_{\frac{1}{15}}^{B'} + \quad (3.4.5) \\
& \chi_{\frac{2}{5}}^A \chi_{\frac{4}{15}}^{A'} \bar{\chi}_{\frac{3}{5}}^B \bar{\chi}_{\frac{1}{15}}^{B'} + \chi_{\frac{2}{5}}^A \chi_{\frac{4}{15}}^{A'} \bar{\chi}_{\frac{3}{5}}^B \bar{\chi}_{\frac{1}{15}}^{B'} + \chi_{\frac{2}{5}}^A \chi_{\frac{4}{15}}^{A'} \bar{\chi}_0^B \bar{\chi}_{\frac{2}{3}}^{B'} + \chi_{\frac{2}{5}}^A \chi_{\frac{4}{15}}^{A'} \bar{\chi}_0^B \bar{\chi}_{\frac{2}{3}}^{B'} .
\end{aligned}$$

It is natural to suppose that the asymmetric modular invariant, (3.4.4), can be obtained from the symmetric one, (3.4.5), by twisting by the appropriate field or fields. This intuition is correct, but the twisting is not by a simple current operator, and correspondingly there is no definite algorithm for achieving it. In the symmetric theory the chiral algebra is enlarged (to $E_6 \otimes E_6$). Twisting by a simple current^[24] cannot reduce the chiral algebra, and here gives back the same theory. There is, however, a candidate field that is primary under the *smaller* chiral algebra of the asymmetric theory, and which has simple fusion rules when defined with respect to this algebra, namely the field $(0, \frac{2}{3}|0, \frac{2}{3})$. Twisting Z^{E_6} by this operator, that is throwing out those operators which when fused with $(0, \frac{2}{3}|0, \frac{2}{3})$ give T noninvariant states while adding those T invariant operators which result from fusing, gives only a subset of the characters in the asymmetric theory. To get the full set we must add the operators formed by fusing $(\frac{2}{5}, \frac{3}{5}|\frac{3}{5}, \frac{2}{5})$ with itself under the now modified fusion rules of the new theory (*i.e.*, those preceding eq. (3.4.4)), (which *a priori* is an ambiguous procedure). Similarly, twisting the asymmetric invariant by any combinations of simple currents in that theory gives back the same invariant. In order to obtain Z^{E_6} we have to twist by the non-simple current $(\frac{2}{5}, \frac{3}{5}|0, 0)$, with suitably modified fusion rules, which again is an ambiguous procedure.

3.5 Concluding Comments on MIPFs

The techniques introduced in sections 3.3 and 3.4 make the classification of modular invariants for tensor product theories built from a small number of factors feasible. A complete classification of the invariants for $SU(2)_{K_A} \otimes SU(2)_{K_B}$ theories, that is the straightforward extension of the results of section 3.2 to even K , may, in particular, prove interesting if there is some generalization of the ADE classification found for the single theories. Nonetheless, a complete classification for tensor product theories built with many factors is not likely to be found, given the enormous number of possibilities. For the purposes of string model building a procedure for constructing any new class of invariants, such as the one known for free field constructions, would be progress. Perhaps a generalization of the twisting procedure to operators with nontrivial (or altered) fusion rules, as suggested by the example in section 3.3, could produce one. In this regard, the results of [29,32], (which have been extensively exploited recently by Gannon^[33]) are an intriguing step though not totally satisfactory. In these works, new tensor product modular invariants are obtained by shifting the momentum lattice of a free boson theory, but at the cost of sacrificing positivity of the coefficients in the partition function.

Finally we must stress that the condition of (one-loop) modular invariance alone is insufficient to guarantee a consistent conformal field theory; for constructions not based on free fields we must still check that there is a consistent operator algebra.

Chapter 4: Fractional Superstrings

4.1 Introduction to Fractional Superstrings

In the last few years, several generalizations of standard (supersymmetric) string theory have been proposed.^[34,28,26,35] One of them^[36–40] uses the (fractional spin) parafermions introduced from the perspective of 2-D conformal field theory (CFT) by Zamolodchikov and Fateev^[41] in 1985 and further developed by Gepner^[42] and Qiu.²⁰ In a series of papers, possible new string theories with local parafermionic world-sheet currents (of fractional conformal spin) giving critical dimensions $D = 6, 4, 3,$ and 2 have been proposed.^[36–40]

At the heart of these new “fractional superstrings” are \mathbb{Z}_K parafermion conformal field theories (PCFT’s) with central charge $c = \frac{2(K-1)}{K+2}$. (Equivalently, these are $SU(2)_K/U(1)$ conformal field theories.) The (integer) level- K PCFT contains a set of unitary primary fields ϕ_m^j , where $0 \leq j, |m| \leq K/2$; $j, m \in \mathbb{Z}/2$, and $j - m = 0 \pmod{1}$. These fields have the identifications

$$\phi_m^j = \phi_{m+K}^j = \phi_{m-\frac{K}{2}}^{\frac{K}{2}-j}. \quad (4.1.1)$$

In the range $|m| \leq j$, the conformal dimension is $h(\phi_m^j) = \frac{j(j+1)}{K+2} - \frac{m^2}{K}$. At a given level the fusion rules are

$$\phi_{m_1}^{j_1} \otimes \phi_{m_2}^{j_2} = \sum_{j=|j_1-j_2|}^r \phi_{m_1+m_2}^j, \quad (4.1.2)$$

where $r \equiv \min(j_1 + j_2, K - j_1 - j_2)$. This CFT contains a subset of primary fields,

$$\{\phi_i \equiv \phi_i^0 \equiv \phi_{-K/2+i}^{K/2}; 0 \leq i \leq K-1\} \quad (4.1.3)$$

($\phi_i^\dagger \equiv \phi_{K-i}$) which, under fusion, form a closed subalgebra possessing a \mathbb{Z}_K Abelian symmetry:

$$\phi_i \otimes \phi_j = \phi_{(i+j)} \pmod{K}. \quad (4.1.4)$$

²⁰This is not to be confused with the original definition of “parafermions.” The term “parafermion” was introduced by H. S. Green in 1953.^[43] Green’s parafermions are defined as spin-1/2 particles that do not obey standard anticommutation rules, but instead follow more general trilinear relations.^[44–46]

The conformal dimensions, $h(\phi_i)$, of the fields in this subgroup have the form

$$h(\phi_i) = \frac{i(K-i)}{K}. \quad (4.1.5)$$

It has been proposed that string models based on tensor products of a level- K PCFT are generalizations of the Type II $D = 10$ superstring.^[36–40] In these potential models, the standard $c = \frac{1}{2}$ fermionic superpartner of the holomorphic world sheet scalar, $X(z)$, is replaced by the “energy operator,” $\epsilon \equiv \phi_0^1$, of the \mathbb{Z}_K PCFT.²¹ ϵ has conformal dimension (spin) $\frac{2}{K+2}$, which is “fractional” (*i.e.*, neither integral nor half-integral), for $K \neq 2$. This accounts for to the name of these models. Each $\epsilon - X$ pair has a total conformal anomaly (or central charge) $c = \frac{3K}{K+2}$.

The naive generalization of the (holomorphic) supercurrent (SC) of the standard superstring, $J_{\text{SC}}(z) = \psi(z) \cdot \partial X(z)$ (where ψ is a real world sheet fermion), to $J_{\text{FSC}} = \phi_0^1(z) \cdot \partial X(z)$ proves to be inadequate.^[40] Instead, the proposed “fractional supercurrent” (FSC) is

$$J_{\text{FSC}}(z) = \phi_0^1(z) \cdot \partial_z X(z) + : \phi_0^1(z) \phi_0^1(z) : . \quad (4.1.6)$$

$: \phi_0^1(z) \phi_0^1(z) :$ (which vanishes for $K = 2$ since $\phi_0^1 = \psi$ at $K = 2$) is the first descendent field of ϕ_0^1 . $J_{\text{FSC}}(z)$ is the generator of a local “fractional” world sheet supersymmetry between $\epsilon(z)$ and $X(z)$, extending the Virasoro algebra of the stress-energy tensor $T(z)$. This local current of spin $h(J_{\text{FSC}}) = 1 + \frac{2}{K+2}$ has fractional powers of $\frac{1}{(z-w)}$ in the OPE with itself, implying a non-local world-sheet interaction and, hence, producing cuts on the world sheet. The corresponding chiral “fractional superconformal algebra”^[40] is,

$$T(z)T(w) = \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \dots \quad (4.1.7a)$$

$$T(z)J_{\text{FSC}}(w) = \frac{hJ_{\text{FSC}}(w)}{(z-w)^2} + \frac{\partial J_{\text{FSC}}(w)}{(z-w)} + \dots \quad (4.1.7b)$$

$$J_{\text{FSC}}(z)J_{\text{FSC}}(w) = \frac{1}{(z-w)^{2h}} + \frac{\frac{2h}{c}T(w)}{(z-w)^{2h-2}} + \frac{\lambda_K(c_0)J_{\text{FSC}}(w)}{(z-w)^h} + \frac{\frac{1}{2}\lambda_K(c_0)\partial J_{\text{FSC}}(w)}{(z-w)^{h-1}} + \dots \quad (4.1.7c)$$

²¹Note that ϵ is not in the \mathbb{Z}_K Abelian subgroup, and thus is not a \mathbb{Z}_K parafermion, except for the degenerate $K = 2$ superstring case where $\phi_0^1 \equiv \phi_1^0$.

where $c = Dc_0$. D is the critical dimension, $c_0 = \frac{3K}{K+2}$ is the central charge for one dimension, and λ_K is a constant.^[47]

The relationship between critical dimension, D , and the level, K , of the parafermion CFT may be shown to be

$$D = 2 + \frac{16}{K}, \quad (4.1.8)$$

for $K = 2, 4, 8, 16$, and ∞ . In [36–40] the relationship (4.1.8) is derived by requiring a massless spin-1 particle in the open string spectrum, produced by $\phi_0^1(z)^\mu$ (where μ is the spacetime index) operating on the vacuum.

4.1.a Parafermion Characters

Before we present the computer generated fractional superstring partition functions of refs. [36–38,48] and follow with a new derivation of these these partition functions, as a prerequisite we wish to discuss the characters $Z(\phi_m^j)$ for the Verma modules, $[\phi_m^j]$,²² for $j, |m| < K/2$. Each verma module contains a single (holomorphic) parafermionic primary field $\phi_m^j(z)$ and its parafermion descendents. The form of the characters is

$$\begin{aligned} Z(\phi_m^j) &= q^{-c/24} \text{tr } q^{L_0} \\ &= \eta(\tau) c_{2m}^{2j}(\tau), \end{aligned} \quad (4.1.9)$$

where $q = e^{2\pi\tau}$ and η is the Dedekind eta-function,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \quad (4.1.10)$$

$c_{2m}^{2j}(\tau)$ is a string function^[49] defined by

$$c_{2m}^{2j}(\tau) = \frac{1}{\eta^3(\tau)} \sum_{x,y} \text{sign}(x) q^{x^2(K+2) - y^2 K} \quad (4.1.11a)$$

$$\begin{aligned} &= q^{h_m^j + \frac{1}{4(K+2)}} \frac{1}{\eta^3} \sum_{r,s=0}^{\infty} (-1)^{r+s} q^{r(r+1)/2 + s(s+1)/2 + rs(K+1)} \times \\ &\quad \{ q^{r(j+m) + s(j-m)} - q^{K+1-2j+r(K+1-j-m) + s(K+1-j+m)} \} \end{aligned} \quad (4.1.11b)$$

$$= q^{h_m^j - \frac{c(SU(2)_K)}{24}} (1 + \dots) \quad (4.1.11c)$$

²²From here on, we do not distinguish between the primary field ϕ_m^j and its complete Verma module $[\phi_m^j]$. Thus, ϕ_m^j can represent either, depending on the context.

where in (4.1.11a) the conditions

1. $-|x| < y < |x|$,
2. either $x = \frac{2j+1}{2(K+2)} \pmod{1}$ or $(\frac{1}{2} - x) = \frac{2j+1}{2(K+2)} \pmod{1}$; and
3. either $y = \frac{m}{K} \pmod{1}$ or $(\frac{1}{2} + y) = \frac{m}{K} \pmod{1}$

must be met simultaneously. ($h_m^j \equiv h(\phi_m^j)$ and $c(SU(2)_K) = \frac{3K}{K+2}$.) These string functions obey the same equivalences as their associated primary fields ϕ_m^j :

$$c_{2m}^{2j} = c_{2m+2K}^{2j} = c_{2m-K}^{K-2j}. \quad (4.1.12a)$$

Additionally, since $\phi_m^j = (\phi_{-m}^j)^\dagger$,

$$c_{2m}^{2j} = c_{-2m}^{2j}. \quad (4.1.12b)$$

Since the $K = 2$ theory is the standard Type II superstring theory,²³ expressing its partition function in terms of string functions rather than theta-functions can be accomplished simply using the following set of identities:

$$K = 2 : \begin{cases} 2\eta^2(c_1^1)^2 = \vartheta_2/\eta; \\ \eta^2(c_0^0 + c_0^2)^2 = \vartheta_3/\eta; \\ \eta^2(c_0^0 - c_0^2)^2 = \vartheta_4/\eta. \end{cases} \quad (4.1.13)$$

For each spacetime dimension in these theories, a term in the partition function of the form (4.1.9) is tensored with the partition function $Z(X)$ for an uncompactified chiral boson $X(z)$. Since

$$Z(X) \propto \frac{1}{\eta(\tau)}, \quad (4.1.14)$$

the $\eta(\tau)$ factors cancel out in $Z(\phi_m^j) \times Z(X)$. Similar cancellation of $\bar{\eta}(\bar{\tau})$ occurs in the antiholomorphic sector. In the following partition functions, we suppress the trivial factor of $(\text{Im } \tau)^{-8/K}$ contributed together by the $D - 2$ holomorphic and antiholomorphic world sheet boson partition functions.

The purpose of this chapter is to examine a number of issues relating to these models: In section 4.2 we derive the partition functions of the $D = 6, 4$, and 3

²³The $K = 2$ parafermion model is a $c = \frac{1}{2}$ CFT that corresponds to a critical Ising (free fermion) model.

theories (corresponding to $K = 4, 8$ and 16 respectively), using the factorization method of Gepner and Qiu,^[42] as well as demonstrating a new approach to obtaining the superstring partition function. In section 4.3 we consider other necessary elements of string theory. In particular, we propose a generalization of the GSO projection that applies to the fractional superstring and we address the question of whether similar theories at different Kač-Moody levels can be constructed. Additionally, a comparison with the superstring is made and we attempt to elucidate its features in the current, more general context.

4.2 Fractional Superstring Partition Functions

Computerized searches demonstrated that for each (and only those) K listed above, there is a unique one-loop partition function (written in light-cone gauge) that is (1) modular invariant, (2) contains a term, $(c_0^0)^{D-3}(c_0^2)$, which is the character for a massless spacetime spin-2 particle generated by an untwisted non-chiral $\phi_0^1(z)\bar{\phi}_0^1(\bar{z})$ field acting on the vacuum, and (3) has no characters for tachyonic states.^[36-38,48] Partition functions with these properties were found to exist only in 10, 6, 4, and 3 dimensions and were presented as:

$$D = 10 \quad (K = 2): \quad Z = |A_2|^2, \text{ where}$$

$$\begin{aligned} A_2 &= 8(c_0^0)^7(c_0^2) + 56(c_0^0)^5(c_0^2)^3 + 56(c_0^0)^3(c_0^2)^5 + 8(c_0^0)(c_0^2)^7 - 8(c_1^1)^8 \\ &= \frac{1}{2}\eta^{-12}(\vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4) \end{aligned} \quad (4.2.1)$$

$$D = 6 \quad (K = 4): \quad Z = |A_4|^2 + 3|B_4|^2, \text{ where}$$

$$A_4 = 4(c_0^0 + c_0^4)^3(c_0^2) - 4(c_0^2)^4 - 4(c_2^2)^4 + 32(c_2^2)(c_2^4)^3 \quad (4.2.2a)$$

$$B_4 = 8(c_0^0 + c_0^4)(c_0^2)(c_2^4)^2 + 4(c_0^0 + c_0^4)^2(c_2^2)(c_2^4) - 4(c_0^2)^2(c_2^2)^2 \quad (4.2.2b)$$

$$D = 4 \quad (K = 8): \quad Z = |A_8|^2 + |B_8|^2 + 2|C_8|^2, \quad \text{where}$$

$$A_8 = 2(c_0^0 + c_0^8)(c_0^2 + c_0^6) - 2(c_0^4)^2 - 2(c_4^4)^2 + 8(c_4^6 c_4^8) \quad (4.2.3a)$$

$$B_8 = 4(c_0^0 + c_0^8)(c_4^6) + 4(c_0^2 + c_0^6)(c_4^8) - 4(c_0^4 c_4^4) \quad (4.2.3b)$$

$$C_8 = 4(c_2^2 + c_2^6)(c_2^8 + c_8^8) - 4(c_2^4)^2 \quad (4.2.3c)$$

$$D = 3 \quad (K = 16): \quad Z = |A_{16}|^2 + |C_{16}|^2, \quad \text{where}$$

$$A_{16} = c_0^2 + c_0^{14} - c_0^8 - c_8^8 + 2c_8^{14} \quad (4.2.4a)$$

$$C_{16} = 2c_4^2 + 2c_4^{14} - 2c_4^8. \quad (4.2.4b)$$

These closed-string partition functions $Z(\text{level}-K)$ all have the general form

$$Z(\text{level}-K) = |A_K|^2 + |B_K|^2 + |C_K|^2. \quad (4.2.5)$$

The $D = 10$ partition function, in string function format, was obtained by the authors of refs. [36–38] as a check of their program, both by computer generation and by the $K = 2$ string functions/Jacobi ϑ -functions equivalences.

In the above partition functions, the characters for the massless graviton (spin-2 particle) and gravitino (spin- $\frac{3}{2}$) are terms in the A_K -sector, $|A_K|^2$. The $D < 10$ fractional superstrings have a new feature not present in the standard $D = 10$ superstrings. This is the existence of the massive B_K - and C_K -sectors. These additional sectors were originally derived in the computer program of the authors of refs. [36–38] by applying S transformations to the A_K -sector and then demanding modular invariance of the theory.

An obvious question with respect to these partition functions is how to interpret the relationship between the spacetime spin of the physical states and the subscripts of the corresponding characters in the partition functions. The solution is not immediately transparent for general K . $K = 2$ is, of course, the exception. Based

on the aforementioned identities (4.1.13), we see that terms with all $n_i \equiv 2m_i = 0$ correspond to spacetime bosons, while those with all $n_i \equiv 2m_i = K$ correspond to spacetime fermions. This rule also seems to be followed by terms in A_K for all K . In the B_K - and C_K -sectors, interpretation is much less clear. There have been two suggested meanings for the terms in B_K and C_K . The first hypothesis is that these terms correspond to massive spacetime anyons, specifically spin- $\frac{1}{4}$ particles for B_K , and massive spin- $\frac{1}{8}$ particles for C_K . The second alternative is that the B_K -sector particles are spacetime fermions and bosons, but with one (for $K = 8$) or two (for $K = 4$) spatial dimensions compactified.^[36-38,48] Along this line of thought, the C_K -sector particles are still interpreted to be spacetime anyons, but with spin- $\frac{1}{4}$ rather than spin- $\frac{1}{8}$.

Until present, the general consensus has been that spacetime anyons can presumably exist only in three or less uncompactified dimensions. This would seem to contradict suggestions that the $D = 4$ or $D = 6$ models may contain spacetime anyons unless at least one or two dimensions, respectively, are compactified. Based on other, independent, reasons we will suggest this compactification does automatically occur in fractional superstring models. Further, we will also show that there are possibly no physical states in the C_K -sector for $K = 8$.

At each level of K , the contribution of each sector is separately zero. This is consistent with spacetime SUSY and suggests cancellation between bosonic and fermionic terms at each mass level. This leads to the following identities:^[36]

$$A_2 = A_4 = B_4 = A_8 = B_8 = C_8 = A_{16} = C_{16} = 0 . \quad (4.2.6)$$

In this section, we will introduce a new method for generating these partition functions that reveals (1) new aspects of the relationship between the B_K - and C_K -sectors and the A_K -sector, and (2) the evidence for spacetime supersymmetry in all sectors. (Specifically, these type II models should have $N = 2$ spacetime SUSY, with the holomorphic and antiholomorphic sectors each effectively contributing an $N = 1$ SUSY. Hence, heterotic fractional superstrings would only possess $N = 1$ SUSY.) We will demonstrate that cancellation suggestive of spacetime SUSY results from the

action of a simple twist current used in the derivation of these partition functions. Only by this twisting can cancellation between bosonic and fermionic terms occur at each mass level in the A_K - and B_K -sectors. The same twisting results in a “self-cancellation” of terms in the C_K -sector, and does, indeed, suggest the anyonic spin- $\frac{1}{4}$ interpretation of the C_K -sector states.

4.2.a New Derivation of the Partition Functions

We find the computer generated partition functions listed above not to be in the most suggestive form. By using the string function equivalences, (4.1.12a-b), the partition functions for the level- K fractional superstrings in refs. [36–38,48] with critical spacetime dimensions $D = 2 + \frac{16}{K} = 10, 6, 4,$ and 3 can be rewritten (in light-cone gauge) in the form below.

$$D = 10 \quad (K = 2): \quad Z = |A_2|^2, \text{ where}$$

$$\begin{aligned} A_2 &= \frac{1}{2} \left\{ (c_0^0 + c_0^2)^8 - (c_0^0 - c_0^2)^8 \right\}_{\text{boson}} - 8(c_1^1)_{\text{fermion}}^8 \\ &= 8 \left\{ (c_0^0)^7 c_0^2 + 7(c_0^0)^5 (c_0^2)^3 + 7(c_0^0)^3 (c_0^2)^5 + c_0^0 (c_0^2)^7 \right\}_{\text{boson}} - 8(c_1^1)_{\text{fermion}}^8 \end{aligned} \quad (4.2.7)$$

$$D = 6 \quad (K = 4): \quad Z = |A_4|^2 + 3|B_4|^2, \text{ where}$$

$$\begin{aligned} A_4 &= 4 \left\{ (c_0^0 + c_0^4)^3 (c_0^2) - (c_0^2)^4 \right\} \\ &\quad + 4 \left\{ (c_2^0 + c_2^4)^3 (c_2^2) - (c_2^2)^4 \right\} \end{aligned} \quad (4.2.8a)$$

$$\begin{aligned} B_4 &= 4 \left\{ (c_0^0 + c_0^4)(c_2^0 + c_2^4)^2 (c_0^2) - (c_0^2)^2 (c_2^2)^2 \right\} \\ &\quad + 4 \left\{ (c_2^0 + c_2^4)(c_0^0 + c_0^4)^2 (c_2^2) - (c_2^2)^2 (c_0^2)^2 \right\} \end{aligned} \quad (4.2.8b)$$

$$D = 4 \quad (K = 8): \quad Z = |A_8|^2 + |B_8|^2 + 2|C_8|^2, \text{ where}$$

$$\begin{aligned} A_8 = & 2 \{ (c_0^0 + c_0^8)(c_0^2 + c_0^6) - (c_0^4)^2 \} \\ & + 2 \{ (c_4^0 + c_4^8)(c_4^2 + c_4^6) - (c_4^4)^2 \} \end{aligned} \quad (4.2.9a)$$

$$\begin{aligned} B_8 = & 2 \{ (c_0^0 + c_0^8)(c_4^2 + c_4^6) - (c_0^4 c_4^4) \} \\ & + 2 \{ (c_4^0 + c_4^8)(c_0^2 + c_0^6) - (c_4^4 c_0^4) \} \end{aligned} \quad (4.2.9b)$$

$$\begin{aligned} C_8 = & 2 \{ (c_2^0 + c_2^8)(c_2^2 + c_2^6) - (c_2^4)^2 \} \\ & + 2 \{ (c_2^0 + c_2^8)(c_2^2 + c_2^6) - (c_2^4)^2 \} \end{aligned} \quad (4.2.9c)$$

$$D = 3 \quad (K = 16): \quad Z = |A_{16}|^2 + |C_{16}|^2, \text{ where}$$

$$\begin{aligned} A_{16} = & \{ (c_0^2 + c_0^{14}) - c_0^8 \} \\ & + \{ (c_8^2 + c_8^{14}) - c_8^8 \} \end{aligned} \quad (4.2.10a)$$

$$\begin{aligned} C_{16} = & \{ (c_4^2 + c_4^{14}) - c_4^8 \} \\ & + \{ (c_4^2 + c_4^{14}) - c_4^8 \} . \end{aligned} \quad (4.2.10b)$$

The factorization method of Gepner and Qiu^[42] for string function partition functions allows us to rederive the fractional superstring partition functions in this new form systematically. Using this approach we can express a general parafermion partition function (with the level of the string functions henceforth suppressed),

$$Z = |\eta|^2 \sum N_{l,n,\bar{l},\bar{n}} c_n^l \bar{c}_{\bar{n}}^{\bar{l}}, \quad (4.2.11a)$$

in the form

$$Z = |\eta|^2 \sum \frac{1}{2} L_{l,\bar{l}} M_{n,\bar{n}} c_n^l \bar{c}_{\bar{n}}^{\bar{l}}, \quad (4.2.11b)$$

(with $c_{n=2m}^{l=2j} = 0$ unless $l - n \in 2\mathbb{Z}$ since $\phi_m^j = 0$ for $j - m \notin \mathbb{Z}$). As a result of the factorization,

$$N_{l,n,\bar{l},\bar{n}} = \frac{1}{2} L_{l,\bar{l}} M_{n,\bar{n}}, \quad (4.2.12)$$

we can construct all modular invariant partition functions (MIPF's) for parafermions from a tensor product of modular invariant solutions for the (l, \bar{l}) and (n, \bar{n}) indices separately. This results from the definition of level- K string functions, c_n^l , in terms of the $SU(2)_K$ characters χ_l and the Jacobi theta-function, $\vartheta_{n,K}$:²⁴

$$\chi_l(\tau) = \sum_{n=-K+1}^K c_n^l(\tau) \vartheta_{n,K}(\tau), \quad (4.2.13)$$

where the theta-function is defined by

$$\vartheta_{n,K}(\tau) = \sum_{p \in \mathbb{Z} + \frac{n}{2K}} e^{2\pi i K p^2 \tau}, \quad (4.2.14)$$

and χ_l is the character for the spin- $\frac{l}{2}$ representation of $SU(2)_K$,

$$\chi_l(\tau) = \frac{\vartheta_{l+1, K+2}(\tau) - \vartheta_{-l-1, K+2}(\tau)}{\vartheta_{1,2}(\tau) - \vartheta_{-1,2}(\tau)}. \quad (4.2.15)$$

This factorization is seen in the transformation properties of c_n^l under the modular group generators S and T ,

$$S : c_n^l \rightarrow \frac{1}{\sqrt{-i\tau K(K+2)}} \sum_{l'=0}^K \sum_{\substack{n'=-K+1 \\ l'-n' \in 2\mathbb{Z}}}^K \exp\left\{\frac{i\pi n n'}{K}\right\} \sin\left\{\frac{\pi(l+1)(l'+1)}{K+2}\right\} c_{n'}^{l'} \quad (4.2.16a)$$

$$T : c_n^l \rightarrow \exp\left\{2\pi i \left(\frac{l(l+2)}{4(K+2)} - \frac{n^2}{4K} - \frac{K}{8(K+2)}\right)\right\} c_n^l. \quad (4.2.16b)$$

Thus, (4.2.11b) is modular invariant if and only if the $SU(2)$ affine partition function

$$W(\tau, \bar{\tau}) = \sum_{l, \bar{l}=0}^K L_{l, \bar{l}} \chi_l(\tau) \bar{\chi}_{\bar{l}}(\bar{\tau}) \quad (4.2.17)$$

and the $U(1)$ partition function

$$V(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{n, \bar{n}=-K+1}^K M_{n, \bar{n}} \vartheta_{n,K}(\tau, \bar{\tau}) \bar{\vartheta}_{\bar{n}, K}(\tau, \bar{\tau}) \quad (4.2.18)$$

²⁴The associated relationship between the level- K $SU(2)$ primary fields Φ^j and the parafermionic ϕ_m^j is

$$\Phi^j = \sum_{m=-j}^j \phi_m^j : \exp\left\{i \frac{m}{\sqrt{K}} \varphi\right\} :$$

where φ is the $U(1)$ boson field of the $SU(2)$ theory.

are simultaneously modular invariant. That is, $N_{l,n,\bar{l},\bar{n}} = \frac{1}{2}L_{l,\bar{l}}M_{n,\bar{n}}$ corresponds to a MIPF (4.2.11a) if and only if $L_{l,\bar{l}}$ and $M_{n,\bar{n}}$ correspond to MIPF's of the forms (4.2.17) and (4.2.18), respectively.

This factorization is also possible for parafermion tensor product theories, with matrices \mathbf{L} and \mathbf{M} generalized to tensors. Any tensor \mathbf{M} corresponding to a MIPF for p factors of $U(1)$ CFT's can be written as a tensor product of p independent matrix \mathbf{M} solutions to (4.2.18) twisted by simple currents \mathcal{J} .^[50] This approach greatly simplifies the derivation of the fractional superstring partition functions, while simultaneously suggesting much about the meaning of the different sectors, the origin of spacetime supersymmetry, and related “projection” terms. We now proceed with the independent derivations of \mathbf{L} and \mathbf{M} for the PCFT's.

4.2.b Affine Factor and “W” Partition Function

In the A_K -sectors defined by eqs. (4.2.8a, 4.2.9a, 4.2.10a) the terms inside the first (upper) set of brackets carry “ $n \equiv 2m = 0$ ” subscripts and can be shown, as our prior discussion suggested, to correspond to spacetime bosons; while the terms inside the second (lower) set carry “ $n \equiv 2m = K/2$ ” and correspond to spacetime fermions. (See eqs. (4.2.36a-b).) Expressing the A_K -sector in this form makes a one-to-one correspondence between bosonic and fermionic states in the A_K -sector manifest. If we remove the subscripts on the string functions in the bosonic and fermionic subsectors (which is analogous to replacing c_n^l with χ_l), we find the subsectors become equivalent. In fact, under this operation of removing the “ n ” subscripts and replacing each string function by its corresponding affine character (a process we denote by $\xrightarrow{\text{affine}}$), all sectors become the same up to an integer coefficient:

$D = 6$ ($K = 4$):

$$A_4, B_4 \quad \xrightarrow{\text{affine}} \quad A_4^{\text{aff}} \equiv (\chi_0 + \chi_K)^3 \chi_{K/2} - (\chi_{K/2})^4 \quad (4.2.19a)$$

$D = 4 \quad (K = 8):$

$$A_8, B_8, C_8 \xrightarrow{\text{affine}} A_8^{\text{aff}} \equiv (\chi_0 + \chi_K)(\chi_2 + \chi_{K-2}) - (\chi_{K/2})^2 \quad (4.2.19b)$$

$D = 3 \quad (K = 16):$

$$A_{16}, C_{16} \xrightarrow{\text{affine}} A_{16}^{\text{aff}} \equiv (\chi_2 + \chi_{K-2}) - \chi_{K/2} . \quad (4.2.19c)$$

We see that the B - and C -sectors are not arbitrary additions, necessitated only by modular invariance, but rather are naturally related to the physically motivated A -sectors. Thus, the affine factor in each parafermion partition function is:

$$Z_{\text{affine}}(K) = |A_K^{\text{aff}}|^2 , \quad (4.2.20)$$

where we see that eqs. (4.2.19a-c) all have the general form

$$A_K^{\text{aff}} \equiv (\chi_0 + \chi_K)^{D-3}(\chi_2 + \chi_{K-2}) - (\chi_{K/2})^{D-2} . \quad (4.2.21)$$

(Note that the modular invariance of W requires that A_K^{aff} transforms back into itself under S .)

The class of partition functions (4.2.20) is indeed modular invariant and possesses special properties. This is easiest to show for $K = 16$. The $SU(2)_{16}$ MIPF's for $D = 3$ are trivial to classify, since at this level the A-D-E classification forms a complete basis set of modular invariants, even for MIPF's containing terms with negative coefficients. The only free parameters in $K = 16$ affine partition functions $Z(SU(2)_{16})$ are integers a , b , and c , where

$$Z(SU(2)_{K=16}) = a \times Z(A_{17}) + b \times Z(D_{10}) + c \times Z(E_7) . \quad (4.2.22)$$

Demanding that neither a left- nor a right-moving tachyonic state be in the Hilbert space of states in the $K = 16$ fractional superstring when the intercept v , defined by

$$L_0|\text{physical}\rangle = v|\text{physical}\rangle , \quad (4.2.23)$$

is positive, removes these degrees of freedom and requires $a = -(b + c) = 0$, independent of the possible $U(1)$ partition functions. These specific values for a , b , and c give us (4.2.20) for this level:

$$W(K = 16) = Z(D_{10}) - Z(E_7) = |A_{16}^{\text{aff}}|^2 . \quad (4.2.24)$$

Though not quite as straightforward a process, we can also derive the affine partition functions $W(K)$ for the remaining levels. The affine factors in the $K = 4$ and 8 partition functions involve twisting by a non-simple current. (See footnote p. for the definition of non-simple current.) These cases correspond to theories that are the difference between a $\bigotimes_{\text{factors}}^{D-2} D_{\frac{K}{2}+2}$ tensor product model and a $\bigotimes_{\text{factors}}^{D-2} D_{\frac{K}{2}+2}$ tensor product model twisted by the affine current

$$J_{\text{non-simple}}^{K, \text{affine}} = (\Phi^{\frac{K}{4}})^{D-2} \bar{\Phi}^1 (\bar{\Phi}^0)^{D-3} . \quad (4.2.25)$$

The equivalent parafermionic twist current is obvious,

$$J_{\text{non-simple}}^{K, \text{parafermion}} = (\phi_0^{\frac{K}{4}})^{D-2} (\bar{\phi}_0^1) (\bar{\phi}_0^0)^{D-3} . \quad (4.2.26)$$

(This derivation applies to the $K = 16$ case also.)²⁵

²⁵We have left off the spacetime indices on most of the following currents and fields. We are working in light-cone gauge so only indices for transverse modes are implied. The $D-2$ transverse dimensions are assigned indices in the range 1 to $D-2$ (and are generically represented by lowercase Greek superscripts.) When spacetime indices are suppressed, the fields, and their corresponding characters in the partition function, acting along directions 1 to $D-2$ are ordered in equations from left to right, respectively, for both the holomorphic and antiholomorphic sectors separately. Often, we will be still more implicit in our notation and will express r identical factors of ϕ_m^j along consecutive directions (when these directions are either all compactified or uncompactified) as $(\phi_m^j)^r$. Thus, eq. (4.2.26) for $K = 8$ means

$$J_{\text{non-simple}}^{K=8, \text{parafermion}} \equiv (\phi_0^{K/4})^{\mu=1} (\phi_0^{K/4})^{\nu=2} (\bar{\phi}_0^1)^{\bar{\mu}=1} (\bar{\phi}_0^0)^{\bar{\nu}=2} .$$

4.2.c Theta-Function Factor and the “V” Partition Function

We now consider the theta-function factors, \mathbf{M} , carrying the (n, \bar{n}) -indices in the fractional superstring partition functions. Since all A_{K^-} , B_{K^-} , C_{K^-} -sectors in the level- K fractional superstring partition function (and even the boson and fermion subsectors separately in A_K) contain the same affine factor, it is clearly the choice of the theta-function factor which determines the spacetime supersymmetry of the fractional superstring theories. That is, spacetime spins of particles in the Hilbert space of states depend upon the \mathbf{M} 's that are allowed in tensored versions of eq. (4.2.18). In the case of matrix \mathbf{M} , rather than a more complicated tensor, invariance of (4.2.18) under S requires that the components $M_{n\bar{n}}$ be related by

$$M_{n', \bar{n}'} = \frac{1}{2K} \sum_{n, \bar{n} = -K+1}^K M_{n, \bar{n}} e^{i\pi n n' / K} e^{i\pi \bar{n} \bar{n}' / K} , \quad (4.2.27a)$$

and T invariance demands that

$$\frac{n^2 - \bar{n}^2}{4K} \in \mathbb{Z} , \quad \text{if } M_{n, \bar{n}} \neq 0 . \quad (4.2.27b)$$

At every level- K there is a unique modular invariant function corresponding to each factorization^[42], $\alpha \times \beta = K$, where $\alpha, \beta \in \mathbb{Z}$. Denoting the matrix elements of $\mathbf{M}^{\alpha, \beta}$ by $M_{n, \bar{n}}^{\alpha, \beta}$, they are given by²⁶

$$M_{n, \bar{n}}^{\alpha, \beta} = \frac{1}{2} \sum_{\substack{x \in \mathbb{Z}_{2\beta} \\ y \in \mathbb{Z}_{2\alpha}}} \delta_{n, \alpha x + \beta y} \delta_{\bar{n}, \alpha x - \beta y} . \quad (4.2.28)$$

²⁶By eq. (2.23), $M_{n, \bar{n}}^{\alpha, \beta} = M_{n, -\bar{n}}^{\beta, \alpha}$. Hence, $\mathbf{M}^{\alpha, \beta}$ and $\mathbf{M}^{\beta, \alpha}$ result in equivalent fractional superstring partition functions. To avoid this redundancy, we choose $\alpha \leq \beta$.

Throughout this subsection we will view the n as representing, simultaneously, the holomorphic $\vartheta_{n, K}$ characters for $U(1)$ theories and, in some sense, the holomorphic string functions, c_n^0 , for parafermions. (\bar{n} represents the antiholomorphic parallels.) However, we do not intend to imply that the string functions can actually be factored into $c_n^l \times c_n^0 = c_n^l$. Rather, we mean to use this in eqs. (2.31b, 2.33b, 2.35b) only as an artificial construct for developing a deeper understanding of the function of the parafermion primary fields (Verma modules) ϕ_m^0 in these models. In the case of the primary fields, ϕ_m^j , factorization is, indeed, valid: $\phi_0^j \otimes \phi_m^0 = \phi_m^j$ (for integer j, m).

Thus, for $K = 4$ the two distinct choices for the matrix $M^{\alpha,\beta}$ are $M^{1,4}$ and $M^{2,2}$; for $K = 8$, we have $M^{1,8}$ and $M^{2,4}$; and for $K = 16$, the three alternatives are $M^{1,16}$, $M^{2,8}$, and $M^{4,4}$. $M^{1,K}$ represents the level- K diagonal, *i.e.* $n = \bar{n}$, partition function. $M^{\alpha,\beta=\frac{K}{\alpha}}$ corresponds to the diagonal partition function twisted by a \mathbb{Z}_α symmetry. (Twisting by \mathbb{Z}_α and $\mathbb{Z}_{K/\alpha}$ produce isomorphic models.) Simple tensor products of these $M^{\alpha,\beta}$ matrices are insufficient for producing fractional superstrings with spacetime SUSY (and, thus, without tachyons). We have found that twisting by a special simple $U(1)$ current is required to achieve this. Of the potential choices for the $U(1)$ MIPF's, $V(\text{level } K)$, the following are the only ones that produce numerically zero fractional superstring partition functions:

$$D = 6 \quad (K = 4):$$

The $M = M^{2,2} \otimes M^{2,2} \otimes M^{2,2} \otimes M^{2,2}$ model twisted by the simple $U(1)$ current²⁷

$$\mathcal{J}_4 \equiv \phi_{K/4}^0 \phi_{K/4}^0 \phi_{K/4}^0 \phi_{K/4}^0 \bar{\phi}_0^0 \bar{\phi}_0^0 \bar{\phi}_0^0 \bar{\phi}_0^0 \quad (4.2.29)$$

results in the following $U(1)$ partition functions:

$$\begin{aligned} V(K=4) = & [(\vartheta_{0,4} + \vartheta_{4,4})^4 (\bar{\vartheta}_{0,4} + \bar{\vartheta}_{4,4})^4 + (\vartheta_{2,4} + \vartheta_{-2,4})^4 (\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^4 \\ & + (\vartheta_{0,4} + \vartheta_{4,4})^2 (\vartheta_{2,4} + \vartheta_{-2,4})^2 (\bar{\vartheta}_{0,4} + \bar{\vartheta}_{4,4})^2 (\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^2 \\ & + (\vartheta_{2,4} + \vartheta_{-2,4})^2 (\vartheta_{0,4} + \vartheta_{4,4})^2 (\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^2 (\bar{\vartheta}_{0,4} + \bar{\vartheta}_{4,4})^2]_{\text{untwisted}} \end{aligned}$$

²⁷Recall that the parafermion primary fields ϕ_m^0 have simple fusion rules,

$$\phi_m^0 \otimes \phi_{m'}^0 = \phi_{m+m'}^0 \pmod{K}$$

and form a \mathbb{Z}_K closed subalgebra. This fusion rule, likewise, holds for the $U(1)$ fields : $\exp\{i\frac{m}{K}\varphi\}$:. This isomorphism makes it clear that any simple $U(1)$ current, \mathcal{J}_K , in this subsection that contains only integer m can be expressed equivalently either in terms of these parafermion fields or in terms of $U(1)$ fields. (We specify integer m since $\phi_m^0 = 0$ for half-integer m .) In view of the following discussion, we define all of the simple twist currents, \mathcal{J}_K , as composed of the former. (Please note, to distinguish between simple $U(1)$ currents and affine currents, the $U(1)$ currents appear in calligraphy style, as above.)

(4.2.30a)

$$\begin{aligned}
& + [(\vartheta_{2,4} + \vartheta_{-2,4})^4(\bar{\vartheta}_{0,4} + \bar{\vartheta}_{4,4})^4 + (\vartheta_{4,4} + \vartheta_{0,4})^4(\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^4 \\
& + (\vartheta_{2,4} + \vartheta_{-2,4})^2(\vartheta_{4,4} + \vartheta_{0,4})^2(\bar{\vartheta}_{0,4} + \bar{\vartheta}_{4,4})^2(\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^2 \\
& + (\vartheta_{4,4} + \vartheta_{0,4})^2(\vartheta_{2,4} + \vartheta_{-2,4})^2(\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^2(\bar{\vartheta}_{0,4} + \bar{\vartheta}_{4,4})^2]_{\text{twisted}} .
\end{aligned}$$

Writing this in parafermionic form, and then using string function identities, followed by regrouping according to A_4 and B_4 components, results in

$$Z(\text{theta factor}, K = 4) = |(c_0^0)^4 + (c_2^0)^4|_{(A_4)}^2 + |(c_0^0)^2(c_2^0)^2 + (c_2^0)^2(c_0^0)^2|_{(B_4)}^2 . \quad (4.2.30b)$$

$D = 4$ ($K = 8$):

The $M = M^{2,4} \otimes M^{2,4}$ model twisted by the simple $U(1)$ current

$$\mathcal{J}_8 \equiv \phi_{K/4}^0 \phi_{K/4}^0 \bar{\phi}_0^0 \bar{\phi}_0^0 \quad (4.2.31)$$

results in

$$\begin{aligned}
V(K = 8) = & [(\vartheta_{0,8} + \vartheta_{8,8})(\bar{\vartheta}_{0,8} + \bar{\vartheta}_{8,8}) + (\vartheta_{4,8} + \vartheta_{-4,8})(\bar{\vartheta}_{4,8} + \bar{\vartheta}_{-4,8})]_{\text{untwisted}}^2 \\
& + [(\vartheta_{2,8} + \vartheta_{-6,8})(\bar{\vartheta}_{2,8} + \bar{\vartheta}_{-6,8}) + (\vartheta_{-2,8} + \vartheta_{6,8})(\bar{\vartheta}_{-2,8} + \bar{\vartheta}_{6,8})]_{\text{untwisted}}^2 \\
& + [(\vartheta_{4,8} + \vartheta_{-4,8})(\bar{\vartheta}_{0,8} + \bar{\vartheta}_{8,8}) + (\vartheta_{0,8} + \vartheta_{8,8})(\bar{\vartheta}_{4,8} + \bar{\vartheta}_{-4,8})]_{\text{twisted}}^2 \\
& + [(\vartheta_{6,8} + \vartheta_{-2,8})(\bar{\vartheta}_{2,8} + \bar{\vartheta}_{-6,8}) + (\vartheta_{2,8} + \vartheta_{-6,8})(\bar{\vartheta}_{-2,8} + \bar{\vartheta}_{6,8})]_{\text{twisted}}^2 .
\end{aligned} \quad (4.2.32a)$$

Hence,

$$Z(\text{theta factor}, K = 8) = |(c_0^0)^2 + (c_4^0)^2|_{(A_8)}^2 + |(c_0^0)(c_4^0) + (c_4^0)(c_0^0)|_{(B_8)}^2 + 4|(c_2^0)^2|_{(C_8)}^2 . \quad (4.2.32b)$$

$D = 3 \quad (K = 16)$:

The $M = M^{4,4}$ model twisted by the simple $U(1)$ current

$$\mathcal{J}_{16} \equiv \phi_{K/4}^0 \bar{\phi}_0^0 \quad (4.2.33)$$

produces,

$$\begin{aligned} V(K = 16) = & |(\vartheta_{0,16} + \vartheta_{16,16}) + (\vartheta_{8,16} + \vartheta_{-8,16})|_{\text{untwisted}}^2 \\ & + |(\vartheta_{4,16} + \vartheta_{-4,16}) + (\vartheta_{12,16} + \vartheta_{-12,16})|_{\text{untwisted}}^2 . \end{aligned} \quad (4.2.34a)$$

Thus,

$$Z(\text{theta factor}, K = 16) = |c_0^0 + c_8^0|_{(A_{16})}^2 + 4|c_4^0|_{(C_{16})}^2 . \quad (4.2.34b)$$

(In this case the twisting is trivial since \mathcal{J}_{16} is in the initial untwisted model.)

The partition function for the standard $D = 10$ superstring can also be factored into affine and theta-function parts:

$D = 10 \quad K = 2$:

$$A_2 \xrightarrow{\text{affine}} \sum_{i \text{ odd}=1}^7 \binom{8}{i} (\chi_0)^i (\chi_K)^{8-i} - (\chi_{K/2})^8 . \quad (4.2.35a)$$

The accompanying $U(1)$ factor is

$$\begin{aligned} Z(\text{theta factor}, K = 2) = & |(\vartheta_{0,2})^8 + (\vartheta_{1,2})^8 + (\vartheta_{-1,2})^8 + (\vartheta_{2,2})^8|^2 \\ & + 35|(\vartheta_{0,2} + \vartheta_{2,2})^4 (\vartheta_{1,2} + \vartheta_{-1,2})^4|^2 \\ & + 35[(\vartheta_{1,2} + \vartheta_{-1,2})^4 (\vartheta_{0,2} + \vartheta_{2,2})^4][(\bar{\vartheta}_{0,2} + \bar{\vartheta}_{2,2})^4 (\bar{\vartheta}_{1,2} + \bar{\vartheta}_{-1,2})^4] \end{aligned} \quad (4.2.35b)$$

which²⁸ originates from the

$$M = M^{2,1} \otimes M^{2,1} \otimes M^{2,1} \otimes M^{2,1} \otimes M^{2,1} \otimes M^{2,1} \otimes M^{2,1} \otimes M^{2,1}$$

²⁸Note that the effective $Z((n, \bar{n}), K = 2)$ contributing to eq. (4.2.7) reduces to just the first mod-squared term in eq. (4.2.35b) since $c_n^l \equiv 0$ for $l - n \neq 0 \pmod{2}$.

model twisted by the (simple) current

$$\mathcal{J}_2^{\text{theta}} \equiv (: \exp\{i\varphi/2\} :)^8 . \quad (4.2.36)$$

The difference between the factorization for $K = 2$ and those for $K > 2$ is that here we cannot define an actual parafermion twist current $(\phi_{K/4}^0)^8$ since $\phi_{K/4}^0 = 0$ for $K = 2$.

All of the above simple $U(1)$ twist currents are of the general form

$$\mathcal{J}_K = (\phi_{K/4}^0)^{D-2} (\bar{\phi}_0^0)^{D-2} \text{ for } K > 2 . \quad (4.2.37)$$

We believe this specific class of twist currents is the key to spacetime supersymmetry in the parafermion models.²⁹ Without its twisting effect, numerically zero fractional superstring MIPF's in three, four, and six dimensions cannot be formed and, thus, spacetime SUSY would be impossible. This twisting also reveals much about the necessity of non- A_K -sectors. Terms from the twisted and untwisted sectors of these models become equally mixed in the $|A_K|^2$, $|B_K|^2$, and $|C_K|^2$ contributions to the level K partition function. Further, this twisting keeps the string functions with $n \not\equiv 0, K/2 \pmod{K}$ from mixing with those possessing $n \equiv 0, K/2 \pmod{K}$. This is especially significant since we believe the former string functions in the C_K -sector likely correspond to spacetime fields of fractional spin-statistics (*i.e.*, anyons) and the latter in both A_K and B_K to spacetime bosons and fermions. If mixing were allowed, normal spacetime SUSY would be broken and replaced by a fractional supersymmetry, most-likely ruining Lorentz invariance for $D > 3$.

Since in the antiholomorphic sector \mathcal{J}_K acts as the identity, we will focus on its effect in the holomorphic sector. In the A_K -sector the operator $(\phi_{K/4}^0)^{D-2}$ transforms the bosonic (fermionic) nonprojection fields into the fermionic (bosonic) projection fields and vice-versa.³⁰ For example, consider the effect of this twist current on the

²⁹ $\bar{\mathcal{J}}_K = (\phi_0^0)^{D-2} (\bar{\phi}_{K/4}^0)^{D-2}$ is automatically generated as a twisted state.

³⁰We use the same language as the authors of refs. [48]. Nonprojection refers to the bosonic and fermionic fields in the A_K^{boson} and A_K^{fermion} subsectors, respectively, corresponding to string functions with positive coefficients, whereas projection fields refer to those corresponding to string

fields represented in

$$A_4 = A_4^{\text{boson}} - A_4^{\text{fermion}} , \quad (4.2.38a)$$

where

$$A_4^{\text{boson}} \equiv 4 \{ (c_0^0 + c_0^4)^3 (c_0^2) - (c_0^2)^4 \} \quad (4.2.38b)$$

$$A_4^{\text{fermion}} \equiv 4 \{ (c_2^2)^4 - (c_2^0 + c_2^4)^3 (c_2^2) \} . \quad (4.2.38c)$$

Twisting by $(\phi_{K/4}^0)^{D-2}$ transforms the related fields according to

$$(\phi_0^0 + \phi_0^2)^3 (\phi_0^1) \xleftrightarrow{(\phi_{K/4}^0)^{D-2}} (\phi_1^2 + \phi_1^0)^3 (\phi_1^1) \quad (4.2.39a)$$

$$(\phi_0^1)^4 \xleftrightarrow{(\phi_{K/4}^0)^{D-2}} (\phi_1^1)^4 . \quad (4.2.39b)$$

Although the full meaning of the projection fields is not yet understood, the authors of refs. [37] and [48] argue that the corresponding string functions should be interpreted as “internal” projections, *i.e.*, cancellations of degrees of freedom in the fractional superstring models. Relatedly, the authors show that when the A_K -sector is written as $A_K^{\text{boson}} - A_K^{\text{fermion}}$, as defined above, the q -expansions of both A_K^{boson} and A_K^{fermion} are all positive. Including the fermionic projection terms results in the identity

$$\eta^{D-2} A_K^{\text{fermion}} = (D-2) \left(\frac{(\vartheta_2)^4}{16\eta^4} \right)^{\frac{D-2}{8}} . \quad (4.2.40a)$$

Eq. (4.2.40a) is the standard theta-function expression for $D-2$ worldsheet Ramond Major-ana-Weyl fermions. Further,

$$\eta^{D-2} A_K^{\text{boson}} = (D-2) \left(\frac{(\vartheta_3)^4 - (\vartheta_4)^4}{16\eta^4} \right)^{\frac{D-2}{8}} . \quad (4.2.40b)$$

Now consider the B_K -sectors. For $K=4$ and 8, the operator $(\phi_{K/4}^0)^{D-2}$ transforms the primary fields corresponding to the partition functions terms in the first set of brackets on the RHS of eqs. (4.2.8b, 4.2.9b) into the fields represented by the partition functions terms in the second set. For example, in the $K=4$ ($D=6$) case

functions with negative signs. With this definition comes an overall minus sign coefficient on A_K^{fermion} , as shown in eq. (4.2.38a). For example, in (4.2.38b), the bosonic non-projection fields are $(\phi_0^0 + \phi_0^2)^3 (\phi_0^1)$ and the bosonic projection is $(\phi_0^1)^4$. Similarly, in (4.2.38c) the fermionic non-projection field is $(\phi_1^1)^4$ and the projections are $(\phi_1^0 + \phi_1^2)^3 (\phi_1^1)$.

$$(\phi_0^0 + \phi_0^2)(\phi_0^1)(\phi_1^0 + \phi_1^2)^2 \xleftrightarrow{(\phi_{K/4}^0)^{D-2}} (\phi_1^0 + \phi_1^2)(\phi_1^1)(\phi_0^2 + \phi_0^0)^2 \quad (4.2.41a)$$

$$(\phi_0^1)^2(\phi_1^1)^2 \xleftrightarrow{(\phi_{K/4}^0)^{D-2}} (\phi_1^1)^2(\phi_0^1)^2 . \quad (4.2.41b)$$

Making an analogy with what occurs in the A_K -sector, we suggest that $(\phi_{K/4}^0)^{D-2}$ transforms bosonic (fermionic) nonprojection fields into fermionic (bosonic) projection fields and vice-versa in the B_K -sector also. Thus, use of the twist current \mathcal{J}_K allows for bosonic and fermionic interpretation of these fields³¹:

$$B_4 = B_4^{\text{boson}} - B_4^{\text{fermion}} , \quad (4.2.42a)$$

where

$$B_4^{\text{boson}} \equiv 4 \{ (c_0^0 + c_0^4)(c_0^2)(c_2^0 + c_2^4)^2 - (c_0^2)^2(c_2^2)^2 \} \quad (4.2.42b)$$

$$B_4^{\text{fermion}} \equiv 4 \{ (c_2^2)^2(c_0^2)^2 - (c_2^0 + c_2^4)(c_2^2)(c_0^0 + c_0^4)^2 \} . \quad (4.2.42c)$$

What appears as the projection term, $(c_0^2)^2(c_2^2)^2$, for the proposed bosonic part acts as the nonprojection term for the fermionic half, when the subscripts are reversed. One interpretation is this implies a compactification of two transverse dimensions.³² The spin-statistics of the physical states of the $D = 6$ model as observed in four-dimensional uncompactified spacetime would be determined by the (matching) n subscripts of the first two string functions³³ (corresponding to the two uncompactified transverse dimensions) in each term of four string functions, $c_n^{l_1} c_n^{l_2} c_{n'}^{l_3} c_{n'}^{l_4}$. The B_8 terms can be interpreted similarly when one dimension is compactified.

However, the C_K -sectors are harder to interpret. Under $(\phi_{K/4}^0)^{D-2}$ twisting, string functions with $K/4$ subscripts are invariant, transforming back into themselves.

³¹Similar conclusions have been reached by K. Dienes and P. Argyres for different reasons. They have, in fact, found theta-function expressions for the B_K^{boson} - and B_K^{fermion} - subsectors.^[51]

³²This was also suggested in ref. [37] working from a different approach.

³³Using the subscripts n' of last two string functions to define spin-statistics in $D = 4$ uncompactified spacetime corresponds to interchanging the definitions of B_4^{boson} and B_4^{fermion} .

Thus, following the pattern of A_K and B_K we would end up writing, for example, C_{16} as

$$C_{16} = C_{16}^a - C_{16}^b \quad (4.2.43a)$$

where,

$$C_{16}^a \equiv (c_4^2 + c_4^{14}) - c_4^8 \quad (4.2.43b)$$

$$C_{16}^b \equiv c_4^8 - (c_4^2 + c_4^{14}) . \quad (4.2.43c)$$

The transformations of the corresponding primary fields are not quite as trivial, though. $(\phi_2^1 + \phi_2^7)$ is transformed into its conjugate field $(\phi_{-2}^7 + \phi_{-2}^1)$ and likewise ϕ_2^4 into ϕ_{-2}^4 , suggesting that C_{16}^a and C_{16}^b are the partition functions for conjugate fields. Remember, however, that $C_{16} = 0$. Even though we may interpret this sector as containing two conjugate spacetime fields, this (trivially) means that the partition function for each is identically zero. We refer to this effect in the C_K -sector as “self-cancellation.” One interpretation is that there are no states in the C_K -sector of the Hilbert space that survive all of the internal projections. If this is correct, a question may arise as to the consistency of the $K = 8$ and 16 theories. Alternatively, perhaps anyon statistics allow two (interacting?) fields of either identical fractional spacetime spins $s_1 = s_2 = \frac{2m}{K}$, or spacetime spins related by $s_1 = \frac{2m}{K} = 1 - s_2$, where in both cases $0 < m < \frac{K}{2} \pmod{1}$, to somehow cancel each other’s contribution to the partition function.

Using the $\phi_m^j \equiv \phi_{m+K}^j \equiv \phi_{m-\frac{K}{2}}^{\frac{K}{2}-j}$ equivalences at level $K \in 4\mathbb{Z}$, a PCFT has $K/2$ distinct classes of integer m values. If one associates these classes with distinct spacetime spins (statistics) and assumes m and $-m$ are also in the same classes since $(\phi_m^0)^\dagger = \phi_{-m}^0$, then the number of spacetime spin classes reduces to $\frac{K}{4} + 1$. Since $m = 0$ ($m = \frac{K}{4}$) is associated with spacetime bosons (fermions), we suggest that general m correspond to particles of spacetime spin $\frac{2|m|}{K}$, $\frac{2m}{K} + \mathbb{Z}^+$, or $\mathbb{Z}^+ - \frac{2m}{K}$. If this is so, most likely $\text{spin}(m) \in \{\frac{2m}{K}, \mathbb{Z}^+ + \frac{2m}{K}\}$ for $0 < m < K/4 \pmod{K/2}$ and $\text{spin}(m) \in \mathbb{Z}^+ - \frac{2|m|}{K}$ for $-K/4 < m < 0 \pmod{K/2}$. This is one of the few spin assignment rules that maintains the equivalences of the fields ϕ_m^j under $(j, m) \rightarrow (\frac{k}{2} - j, m - \frac{K}{2}) \rightarrow (j, m + K)$ transformations. According to this rule,

the fields in the C_K -sectors have quarter spins (statistics), which agrees with prior claims.^[36-38]

Also, we do not believe products of primary fields in different m classes in the B_K -sectors correspond to definite spacetime spin states unless some dimensions are compactified. Otherwise by our interpretation of m values above, Lorentz invariance in uncompactified spacetime would be lost. In particular, Lorentz invariance requires that either all or none of the transverse modes in uncompactified spacetime be fermionic spinors. Further, B -sector particles apparently cannot correspond to fractional spacetime spin particles for a consistent theory. Thus, the $D = 6(4)$ model must have two (one) of its dimensions compactified.³⁴

The B_8 -sector of the $D = 4$ model appears necessary for more reasons than just modular invariance of the theory. By the above spacetime spin assignments, this model suggests massive spin-quarter states anyons in the C_K -sectors, which presumably cannot exist in $D > 3$ uncompactified dimensions. However, the B_K -sector, by forcing compactification to three dimensions where anyons are allowed, would save the model, making it self-consistent. Of course, anyons in the $K = 16$ theory with $D_{\text{crit}} = 3$ are physically acceptable. (Indeed, no B_K -sector is needed and none exists, which would otherwise reduce the theory to zero transverse dimensions.) Thus, $K = 8$ and $K = 16$ models are probably both allowed solutions for three uncompactified spacetime dimensional models. If this interpretation is correct then it is the B_K -sector for $K = 8$ which makes that theory self-consistent.

An alternative, less restrictive, assignment of spacetime spin is possible. Another view is that the m quantum number is not fundamental for determining spacetime spin. Instead, the transformation of states under $\phi_{K/4}^j$ can be considered to be what divides the set of states into spacetime bosonic and fermionic classes. With this interpretation, compactification in the B_K -sector is no more necessary than in the A_K -sector. Unfortunately, it is not *a priori* obvious, in this approach, which group of states is bosonic, and which fermionic. In the A_K -sector, the assignment can also be

³⁴This implies the $D = 6, 4$ partition functions are incomplete. Momentum (winding) factors for the two compactified dimensions would have to be added (with modular invariance maintained).

made phenomenologically. In the B_K -sector, we have no such guide. Of course, using the m quantum number to determine spacetime spin does not truly tell us which states have bosonic or fermionic statistics, since the result depends on the arbitrary choice of which of the two (one) transverse dimensions to compactify.

A final note of caution involves multiloop modular invariance. One-loop modular invariance amounts to invariance under S and T transformations. However modular invariance at higher orders requires an additional invariance under U transformations: Dehn twists mixing cycles of neighboring tori of $g > 1$ Riemann surfaces.^[12-14] We believe neither our new method of generating the one-loop partitions, nor the original method of Argyres *et al.* firmly proves the multi-loop modular invariance that is required for a truly consistent theory.

4.3 Beyond the Partition Function: Additional Comments

In the last section, we introduced a new derivation of the fractional superstring partition functions. However, this previous discussion did not fully demonstrate the consistency of the fractional superstrings. Further comparisons to the $K = 2$ superstring are of assistance for this. Here in this section, we comment on such related aspects of potential string theories. We consider the analog of the GSO projection and the uniqueness of the “twist” field $\phi_{K/2}^{K/2}$ for producing spacetime fermions. First, however we investigate bosonized representations of the fractional superstrings and what better understanding of the models, this approach might reveal.

4.3.a Bosonization of the $K = 4$ Theory.

Several papers^[52] have examined the issue of bosonization of \mathbb{Z}_K parafermion CFTs. Since $0 \leq c(K) \leq 2$ for these theories, generically a \mathbb{Z}_K model can be bosonized using two distinct free bosonic fields, with one carrying a background charge. The chiral energy-momentum tensor for a free bosonic field X with background charge α_0 is

$$T(z) = \frac{1}{2}[\partial_z X(z)]^2 - \frac{\alpha_0}{2}\partial_z^2 X(z) , \quad (4.3.1)$$

which results in

$$c(X) = 1 - 3(\alpha_0)^2. \quad (4.3.2)$$

For $2 < K < \infty$,³⁵ only two \mathbb{Z}_K theories (those at $K = 3, 4$) do not require two free real bosonic fields in the bosonized version and only for $K = 4$ is a background charge unnecessary since $c(K = 4) = 1$. The bosonization process for the \mathbb{Z}_4 parafermion CFT is straightforward since $c = 1$ CFTs have only three classes of solutions, corresponding to a boson propagating on (1) a torus of radius R , (2) a \mathbb{Z}_2 orbifold of radius R , or (3) discrete orbifold spaces defined on $SU(2)/\Gamma_i$, where Γ_i are discrete subgroups of $SU(2)$.^[53]

The \mathbb{Z}_4 parafermion CFT is identical^[54] to the \mathbb{Z}_2 orbifold at radius $R = \sqrt{6}/2$ (and $R = 1/\sqrt{6}$ by duality). The \mathbb{Z}_4 primary fields with their conformal dimensions and corresponding partition functions for the Verma modules are listed in Table 4.1.

Table 4.1 \mathbb{Z}_4 Primary Fields

<u>Primary Fields</u>	<u>Conformal Dimension h</u>	<u>Partition Fn.</u>
$\phi_0^0 \equiv \phi_0$	0	ηc_0^0
$\phi_{-1}^2 = \phi_1^0 \equiv \phi_1 = \phi_3^\dagger$	$\frac{3}{4}$	ηc_2^4
$\phi_0^2 = \phi_2^0 \equiv \phi_2$	1	ηc_0^4
$\phi_1^2 = \phi_3^0 \equiv \phi_3 = \phi_1^\dagger$	$\frac{3}{4}$	ηc_2^4
$\phi_0^1 \equiv \epsilon$	$\frac{1}{3}$	ηc_0^2
$\phi_1^1 = \phi_{-1}^1$	$\frac{1}{12}$	ηc_2^2
$\phi_{-1/2}^{1/2}$	$\frac{1}{16}$	ηc_{-1}^1
$\phi_{1/2}^{1/2}$	$\frac{1}{16}$	ηc_1^1
$\phi_{-1/2}^{3/2}$	$\frac{9}{16}$	ηc_{-1}^3
$\phi_{1/2}^{3/2}$	$\frac{9}{16}$	ηc_1^3

³⁵The only primary field for $K = 1$ is the vacuum and, as discussed prior, the $K = 2$ theory is the $c = \frac{1}{2}$ critical Ising (free fermion) model.

An S^1/\mathbb{Z}_2 orbifold at radius R has the partition function

$$Z_{\text{orb}}(R) = \frac{1}{2} \left\{ Z(R) + \frac{|\eta|}{|\vartheta_2|} + \frac{|\eta|}{|\vartheta_3|} + \frac{|\eta|}{|\vartheta_4|} \right\} \quad (4.3.3a)$$

$$= \frac{1}{2} \left\{ Z(R) + \frac{|\vartheta_3\vartheta_4|}{|\eta|^2} + \frac{|\vartheta_2\vartheta_4|}{|\eta|^2} + \frac{|\vartheta_2\vartheta_3|}{|\eta|^2} \right\} \quad (4.3.3b)$$

where $\vartheta_{i=1 \text{ to } 4}$ are the classical Jacobi theta-functions.

$$Z(R) = \frac{1}{\eta\bar{\eta}} \sum_{m,n=-\infty}^{\infty} q^{\{\frac{m}{2R}+nR\}^2/2} \bar{q}^{\{\frac{m}{2R}-nR\}^2/2} \quad (4.3.3c)$$

is the partition function for a free scalar boson compactified on a circle of radius R . For $R = \frac{\sqrt{6}}{2}$ the generalized momentum states $p = \frac{m}{\sqrt{6}} + \frac{n\sqrt{6}}{2}$ can be categorized into four classes based on the value of $\frac{p^2}{2} \pmod{1}$. The classes are $\frac{p^2}{2} = 0, \frac{1}{12}, \frac{1}{3},$ and $\frac{3}{4} \pmod{1}$. $p = \frac{m}{\sqrt{6}} + \frac{n\sqrt{6}}{2}$ and $\bar{p} = \frac{m}{\sqrt{6}} - \frac{n\sqrt{6}}{2}$ belong to the same class. That is,

$$\frac{1}{2}(p^2 - \bar{p}^2) \equiv 0 \pmod{1}, \quad (4.3.4)$$

(as required by modular invariance or, equivalently, by level matching.)^[55,56]

The untwisted sector of the model corresponds to the first two terms on the right-hand side of eq. (4.3.3c) and the twisted sector, the remaining two terms. The factor of $\frac{1}{2}$ is due to the GSO projection from the \mathbb{Z}_2 orbifolding, requiring invariance of states under $g : X(z, \bar{z}) \rightarrow -X(z, \bar{z})$. In the untwisted sector this invariance requires pairing together of momentum states $|m, n\rangle$ and $| -m, -n\rangle$ and so projects out half the original number of states in the untwisted sector. The second term in (4.3.3a) and (4.3.3b) correspond to states antiperiodic along the “time” loop and thus can only be states built from (net even numbers) of $\alpha(z)$ and $\bar{\alpha}(\bar{z})$'s acting on $|m = n = 0\rangle$. The twisted sector states correspond to a total even number of α_r and $\bar{\alpha}_r, r \in \mathbb{Z} + \frac{1}{2}$, oscillations acting on the $|m = n = 0\rangle$ twisted vacuum with $h = \bar{h} = \frac{1}{16}$. Thus the twisted states have conformal dimensions of the form $(h, \bar{h}) \in (\frac{1}{16} + \mathbb{Z}, \frac{1}{16} + \mathbb{Z})$ or $(\frac{1}{16} + \mathbb{Z} + \frac{1}{2}, \frac{1}{16} + \mathbb{Z} + \frac{1}{2})$.

The first six primary fields of the \mathbb{Z}_K PCFT listed in Table 4.2 have representations in the untwisted sector of $Z(\text{orbifold}, R = \frac{\sqrt{6}}{2})$ and the latter four have

representations in the twisted sector.³⁶ From the classes of $\frac{p^2}{2} = 0, \frac{1}{12}, \frac{1}{3}, \frac{3}{4} \pmod{1}$ states we find the following identities for string functions:

$$|\eta c_0^0|^2 + |\eta c_0^4|^2 = \frac{1}{2} \left\{ \frac{1}{|\eta|^2} \sum_{\frac{p^2}{2} \equiv 0 \pmod{12}} q^{\left(\frac{m}{2R} + nR\right)^2/2} \bar{q}^{\left(\frac{m}{2R} - nR\right)^2/2} + \frac{|\vartheta_3 \vartheta_4|}{|\eta|^2} \right\} \quad (4.3.5a)$$

$$|\eta c_2^2|^2 = \frac{1}{2} \frac{1}{|\eta|^2} \sum_{\frac{p^2}{2} \equiv 1 \pmod{12}} q^{\left(\frac{m}{2R} + nR\right)^2/2} \bar{q}^{\left(\frac{m}{2R} - nR\right)^2/2} \quad (4.3.5b)$$

$$|\eta c_0^2|^2 = \frac{1}{2} \frac{1}{|\eta|^2} \sum_{\frac{p^2}{2} \equiv 4 \pmod{12}} q^{\left(\frac{m}{2R} + nR\right)^2/2} \bar{q}^{\left(\frac{m}{2R} - nR\right)^2/2} \quad (4.3.5c)$$

$$|\eta c_2^4|^2 = |\eta c_{-2}^4|^2 = \frac{1}{4} \frac{1}{|\eta|^2} \sum_{\frac{p^2}{2} \equiv 9 \pmod{12}} q^{\left(\frac{m}{2R} + nR\right)^2/2} \bar{q}^{\left(\frac{m}{2R} - nR\right)^2/2} \quad (4.3.5d)$$

$$|\eta c_1^1|^2 + |\eta c_1^3|^2 = |\eta c_{-1}^1|^2 + |\eta c_{-1}^3|^2 = \frac{1}{4} \left\{ \frac{|\vartheta_2 \vartheta_4|}{|\eta|^2} + \frac{|\vartheta_2 \vartheta_3|}{|\eta|^2} \right\}. \quad (4.3.5e)$$

Identities for the primary fields partition functions are possible from this bosonization. Since only the ϕ_m^j with integer j, m are in the \mathbb{Z}_4 model, we will henceforth concentrate on the untwisted sector of the S/\mathbb{Z}_2 model. We make the following identifications between the primary fields of the two models:

Table 4.2 Primary Field Representation From Orbifold Bosonization

\mathbb{Z}_4 Primary Field	S/\mathbb{Z}_2	h
$\phi_0(z)$	1	0
$\phi_1(z) + \phi_{-1}(z)$	$e^{i\frac{3}{\sqrt{6}}X(z)} + e^{-i\frac{3}{\sqrt{6}}X(z)}$	$\frac{3}{4}$
$\phi_2(z)$	$i\partial X$	1
$\epsilon(z)$	$e^{i\frac{2}{\sqrt{6}}X(z)} + e^{-i\frac{2}{\sqrt{6}}X(z)}$	$\frac{1}{3}$
$\phi_1^1(z)$	$e^{i\frac{1}{\sqrt{6}}X(z)} + e^{-i\frac{1}{\sqrt{6}}X(z)}$	$\frac{1}{12}$

(ϕ_1 and ϕ_{-1} must be paired together since the S/\mathbb{Z}_2 physical state is $e^+ + e^-$, where

³⁶We note that independent of the choice of the affine factor in the partition functions of section 4.2, the required (n, \bar{n}) partition functions of (4.2.30a, 4.2.32a, 4.2.34a) effectively remove from the theory the primary fields with half integer j, m . The only theory which uses the twisted sector is the $K = 2$ superstring. The significance of this observation is under investigation.

$\epsilon^+ \equiv e^{i\frac{3}{\sqrt{6}}X}$, $\epsilon^- \equiv e^{-i\frac{3}{\sqrt{6}}X}$.) Perhaps the first aspect that becomes apparent is how to represent the fractional supercurrent, J_{FSC} as $J_{\text{FSC}}^+ + J_{\text{FSC}}^-$:

$$J_{\text{FSC}, -\frac{4}{3}} = \epsilon \partial X + : \epsilon \epsilon : \quad (4.3.6a)$$

$$= \epsilon^+ \partial X + : \epsilon^+ \epsilon^+ : + \epsilon^- \partial X + : \epsilon^- \epsilon^- : \quad (4.3.6b)$$

$$= J_{\text{FSC}, -\frac{4}{3}}^+ + J_{\text{FSC}, -\frac{4}{3}}^- \quad (4.3.6c)$$

with $e^{\pm i\frac{4}{\sqrt{6}}}$ the only candidates for $: \epsilon^\pm \epsilon^\pm :$.³⁷

Since the identities (4.3.5a–4.3.5e) involve $|\eta c_{2m}^{2j}|^2$ rather than just ηc_{2m}^{2j} , they do not necessarily imply the exact equivalence of the parafermion and orbifold models. However, more fundamental identities for the string functions do exist. Since none of the \mathbb{Z}_4 parafermion fields connected with the twisted orbifold sector appear in the $K = 4$ FSC model, we can look just at a left-moving (holomorphic) boson compactified on a circle with $R = \sqrt{6}$, but not \mathbb{Z}_2 twisted.

$$Z(z, R = \sqrt{6}) = \frac{1}{\eta} \sum_{m=-\infty}^{\infty} q^{[\frac{m}{R}]^2/2} . \quad (4.3.7a)$$

If we change summation using $m = 6n + i$, $i = 0$ to 5 , then the partition function can be split into³⁸

$$Z(z, R = \sqrt{6}) = \frac{1}{\eta} \sum_{\substack{n=-\infty \\ i=0 \text{ to } 5}}^{\infty} q^{[\frac{(6n+i)}{R}]^2/2} . \quad (4.3.7b)$$

This suggests the following more succinct identities:³⁹

$$\eta c_2^2 = \frac{1}{\eta} q^{\frac{1}{12}} \sum_{n=-\infty}^{\infty} q^{3n^2+n} \quad (4.3.8a)$$

$$\eta c_0^2 = \frac{1}{\eta} q^{\frac{1}{3}} \sum_{n=-\infty}^{\infty} q^{3n^2+2n} \quad (4.3.8b)$$

$$\eta c_2^4 = \eta c_{-2}^4 = \frac{1}{2\eta} q^{\frac{3}{4}} \sum_{n=-\infty}^{\infty} q^{3n^2+3n} \quad (4.3.8c)$$

$$\eta(c_0^0 + c_0^4) = \frac{1}{\eta} \sum_{n=-\infty}^{\infty} q^{3n^2} . \quad (4.3.8d)$$

³⁷Subsequently, [48] has shown that closure of the fractional current and energy-momentum OPEs requires $: \epsilon^\pm \epsilon^\pm := e^{\pm i\frac{4}{\sqrt{6}}}$ be the descendent term in G^\mp , respectively.

³⁸Note that $m = i \pmod{6}$ terms are equivalent to $m = -i \pmod{6}$ terms, so if we include a factor of two, we need only sum over $i = 0$ to 3 .

³⁹These were verified up to q^{1300} using mathematica.

The corresponding free boson representations of the parafermion primary fields are given in Table 4.3:

Table 4.3 Primary Field Representation From $R = \sqrt{6}$ Bosonization

\mathbb{Z}_4 Parafermion	$R = \sqrt{6}$ Boson Rep.	Verma Module	Boson Rep.
ϕ_0	1	$[\phi_0]$	$\{1, e^{i\frac{6n}{\sqrt{6}}X} + e^{-i\frac{6n}{\sqrt{6}}X};$ $n > 0\}$
ϕ_1	$e^{i\frac{3}{\sqrt{6}}X}$	$[\phi_1]$	$\{e^{i\frac{6n+3}{\sqrt{6}}X}\}$
ϕ_{-1}	$e^{-i\frac{3}{\sqrt{6}}X}$	$[\phi_{-1}]$	$\{e^{-i\frac{6n+3}{\sqrt{6}}X}\}$
ϕ_2	$i\partial X$	$[\phi_{-2}]$	$\{\alpha_{-n}\}$
ϵ	$e^{i\frac{2}{\sqrt{6}}X}, e^{-i\frac{2}{\sqrt{6}}X}$	$[\epsilon]$	$\{e^{i\frac{6n+2}{\sqrt{6}}X}, \{e^{-i\frac{6n+2}{\sqrt{6}}X}\}$
ϕ_1^1	$e^{i\frac{1}{\sqrt{6}}X}, e^{-i\frac{1}{\sqrt{6}}X}$	$[\phi_1^1]$	$\{e^{i\frac{6n+1}{\sqrt{6}}X}, \{e^{-i\frac{6n+1}{\sqrt{6}}X}\}$

In this representation ϕ_1 and ϕ_{-1} need not be paired together. Also, ϵ and ϕ_1^1 have double representations. For ϵ , this allows the fractional supercurrent, J_{FSC} , to be expressed as $J_{\text{FSC}}^+ + J_{\text{FSC}}^-$. For ϕ_1^1 , this should correspond to the two spin modes, call them (+) and (-). The zero mode of one representation of ϵ should act as a raising operator between these spin states and the other as a lowering operator:

$$\begin{aligned}
\epsilon_0^+(+) &= (-) \\
\epsilon_0^+(-) &= 0 \\
\epsilon_0^-(-) &= (+) \\
\epsilon_0^-(+) &= 0 .
\end{aligned} \tag{4.3.9}$$

The free boson/orbifold representations of the \mathbb{Z}_4 parafermion CFT should be a valuable tool for better understanding the $K = 4$ FSC model, especially its associated partition function.

4.3.b Generalized Commutation Relations and the GSO Projection

One of the major complications of generalizing from the $K = 2$ fermion case to $K > 2$ is that the parafermions (and bosonic field representations) do not have simple commutation relations.^[41] What are the commutation relations for non-(half) integral spin particles? Naively, the first possible generalization of standard (anti-)commutation relations for two fields A and B with fractional spins seems to be:

$$AB - e^{[i4\pi \text{spin}(A) \text{spin}(B)]} BA = 0 \quad (4.3.10)$$

(which reduces to the expected result for bosons and fermions). This is too simple a generalization, however.^[57] Fractional spin particles must be representations of the braid group.^[57] Zamolodchikov and Fateev^[41] have shown that worldsheet parafermions (of fractional spin) have complicated commutation relations that involve an infinite number of modes of a given field. For example:

$$\begin{aligned} \sum_{l=0}^{\infty} C_{(-1/3)}^{(l)} \left[A_{n+(1-q)/3-l} A_{m-(1-q)/3+l}^{\dagger} + A_{m-(2-q)/3-l}^{\dagger} A_{n-(2-q)/3+l} \right] = \\ - \frac{1}{2} \left(n - \frac{q}{3} \right) \left(n + 1 - \frac{q}{3} \right) \delta_{n+m,0} + \frac{8}{3c} L_{n+m} \end{aligned} \quad (4.3.11a)$$

and

$$\sum_{l=0}^{\infty} C_{(-2/3)}^{(l)} \left[A_{n-q/3-l} A_{m+(2-q)/3+l} - A_{m-q/3-l} A_{n+(2-q)/3+l} \right] = \frac{\lambda}{2} (n - m) A_{(2-2q)/3+n+m}^{\dagger}, \quad (4.3.11b)$$

where A is a parafermion field, and L_n are the generators of the Virasoro algebra. λ is a real coefficient, n is integer, and $q = 0, 1, 2 \pmod{3}$ is a \mathbb{Z}_3 charge of Zamolodchikov and Fateev that can be assigned to each primary field in the $K = 4$ model. The coefficients, $C_{(\alpha)}^{(l)}$, are determined by the power expansion

$$(1 - x)^{\alpha} = \sum_{l=0}^{\infty} C_{(\alpha)}^{(l)} x^l. \quad (4.3.12)$$

As usual, $c \equiv \frac{2(K-1)}{K+2}$ is the central charge of the level- K PCFT. These commutation relations were derived from the OPE of the related fields.^[41] (Hence more terms in

a given OPE result in more complicated commutation relations.) Similar relations between the modes of two different primary fields can also be derived from their OPE's. The significance of these commutation relations is that they severely reduce the number of distinct physical states in parafermionic models. There are several equivalent ways of creating a given physical state from the vacuum using different mode excitations from different parafermion primary fields in the same CFT. Thus, the actual Hilbert space of states for this $K = 4$ model will be much reduced compared to the space prior to moding out by these equivalences.⁴⁰

Although the fields in the PCFT do not (anti-)commute, but instead have complicated commutation relations, some insight can be gained by comparing the $D = 6$, $K = 4$ FSC model to the standard $D = 10$ superstring. We can, in fact, draw parallels between ϵ and the standard fermionic superpartner, ψ , of an uncompactified boson X . In the free fermion approach, developed simultaneously by Kawai, Lewellen and Tye and by Antoniadis, Bachas and Kounas, generalized GSO projections based on boundary conditions of the world sheet fermions are formed.^[12-14] Fermions with half-integer modes (NS-type) are responsible for \mathbb{Z}_1 (trivial) projections; fermions with integer modes (R-type) induce \mathbb{Z}_2 projections. In the non-Ramond sectors these \mathbb{Z}_2 projections remove complete states, while in the Ramond sector itself, eliminate half of the spin modes, giving chirality. Fermions with general complex boundary conditions,

$$\psi(\sigma_1 = 2\pi) = -e^{i\pi x} \psi(\sigma_1 = 0), \quad (4.3.13)$$

where $x \equiv \frac{a}{b}$ is rational with a and b coprime and chosen in the range $-1 \leq x < 1$, form in the non-Ramond sector \mathbb{Z}_{2b} projections if a is odd and \mathbb{Z}_b projections if a is even. For free-fermion models, the GSO operator, originating from a sector where the world sheet fermions ψ^i have boundary conditions

$$\psi^i(2\pi) = -e^{i\pi x^i} \psi^i(0), \quad (4.3.14a)$$

and that acts on a physical state $|\text{phys}\rangle_{\mathcal{F}}$ in a sector where the same fermions have

⁴⁰These equivalences have subsequently been explicitly shown and the distinct low mass² fields determined in Argyres *et al.*^[48]

boundary conditions

$$\psi^i(2\pi) = -e^{i\pi y^i} \psi(0), \quad (4.3.14b)$$

takes the form,

$$\left\{ e^{i\pi \vec{x} \cdot \vec{F}_{\vec{y}}} = \delta_{\vec{y}} C(\vec{y} | \vec{x}) \right\} |\text{phys}\rangle_{\vec{y}} \quad (4.3.15)$$

for states surviving the projection. Those states not satisfying the demands of the GSO operator for at least one sector \vec{x} will not appear in the partition function of the corresponding model.⁴¹

The boundary conditions are encoded in the mode expansions of the complex fermion field, ψ^+ , and its complex conjugate, ψ^- , on a torus. These have the following form for a general twist by $x \equiv \frac{a}{b}$:

$$\begin{aligned} \psi^+(\sigma_1, \sigma_2) = & \sum_{n=1}^{\infty} [\psi_{n-1/2-x/2}^{\alpha} \exp \{-i(n-1/2-x/2)(\sigma_1 + \sigma_2)\} \\ & + \psi_{1/2-n-x/2}^{\beta} \exp \{-i(1/2-n-x/2)(\sigma_1 + \sigma_2)\}] \quad (4.3.16a) \end{aligned}$$

$$\begin{aligned} \psi^-(\sigma_1, \sigma_2) = & \sum_{n=1}^{\infty} [\psi_{1/2-n+x/2}^{\alpha} \exp \{-i(1/2-n+x/2)(\sigma_1 + \sigma_2)\} \\ & + \psi_{n-1/2+x/2}^{\beta} \exp \{-i(n-1/2+x/2)(\sigma_1 + \sigma_2)\}] \quad (4.3.16b) \end{aligned}$$

(where $\psi_r^{\alpha\dagger} \equiv \psi_{-r}^{\alpha}$ and $\psi_r^{\beta\dagger} \equiv \psi_{-r}^{\beta}$). ψ_r^{α} and ψ_r^{β} are independent modes. Thus,

$$\psi^+(\sigma_1 + 2\pi) = e^{+i2\pi(1/2)} e^{i\pi x} \psi^+(\sigma_1) \quad (4.3.17a)$$

$$\psi^-(\sigma_1 + 2\pi) = e^{-i2\pi(1/2)} e^{-i\pi x} \psi^-(\sigma_1). \quad (4.3.17b)$$

The specification of the fields is completed by stating the commutation relation that the modes obey,

$$\{\psi_c^{\alpha\dagger}, \psi_d^{\alpha}\} = \{\psi_c^{\beta\dagger}, \psi_d^{\beta}\} = \delta_{cd}, \quad (4.3.18a)$$

⁴¹In eq. (4.3.15), $\vec{F}_{\vec{y}}$ is the (vector) fermion number operator for states in sector \vec{y} . $\delta_{\vec{y}}$ is -1 if either the left-moving or right-moving $\psi^{\text{spacetime}}$ are periodic and 1 otherwise. $C(\vec{y} | \vec{x})$ is a phase with value chosen from an allowed set of order $g_{\vec{y}, \vec{x}} = GCD(N_{\vec{y}}, N_{\vec{x}})$, where $N_{\vec{y}}$ is the lowest positive integer such that

$$N_{\vec{y}} \times \vec{y} = \vec{0} \pmod{2}.$$

$$\left\{ \psi_c^{\alpha\dagger}, \psi_d^\beta \right\} = \left\{ \psi_c^{\beta\dagger}, \psi_d^\alpha \right\} = 0 . \quad (4.3.18b)$$

A similar analysis can be done with the $\epsilon = \phi_0^1$ fields of the $K = 4$ parafermion theory. The normal untwisted (*i.e.*, Neveu-Schwarz) modes of ϵ are $\epsilon_{-\frac{1}{3}-n}^+$ and $\epsilon_{\frac{1}{3}-n}^-$ where $n \in \mathbb{Z}$. That is, untwisted $\epsilon = (\epsilon^+, \epsilon^-)$ has the following normal-mode expansions.

N-S Sector:

$$\begin{aligned} \epsilon^+(\sigma_1, \sigma_2) = & \sum_{n=1}^{\infty} [\epsilon_{n-1/3}^\alpha \exp \{-i(n-1/3)(\sigma_1 + \sigma_2)\} \\ & + \epsilon_{2/3-n}^\beta \exp \{-i(2/3-n)(\sigma_1 + \sigma_2)\}] \end{aligned} \quad (4.3.19a)$$

$$\begin{aligned} \epsilon^-(\sigma_1, \sigma_2) = & \sum_{n=1}^{\infty} [\epsilon_{1/3-n}^\alpha \exp \{-i(1/3-n)(\sigma_1 + \sigma_2)\} \\ & + \epsilon_{n-2/3}^\beta \exp \{-i(n-2/3)(\sigma_1 + \sigma_2)\}] \end{aligned} \quad (4.3.19b)$$

(where $\epsilon_r^{\alpha\dagger} = \epsilon_{-r}^\alpha$ and $\epsilon_r^{\beta\dagger} = \epsilon_{-r}^\beta$). Similarly, the associated boundary conditions in this sector are

$$\epsilon^+(\sigma_1 + 2\pi) = e^{+i2\pi/3} \epsilon^+(\sigma_1) \quad (4.3.20a)$$

$$\epsilon^-(\sigma_1 + 2\pi) = e^{-i2\pi/3} \epsilon^-(\sigma_1) . \quad (4.3.20b)$$

Like the standard fermion, the ϵ operators at $K = 4$ can be in twisted sectors, where the normal-mode expansions have the following form.

General Twisted Sector:

$$\begin{aligned} \epsilon^+(\sigma_1, \sigma_2) = & \sum_{n=1}^{\infty} [\epsilon_{n-1/3-x/2}^{\alpha} \exp \{-i(n-1/3-x/2)(\sigma_1 + \sigma_2)\} \\ & + \epsilon_{2/3-n-x/2}^{\beta} \exp \{-i(2/3-n-x/2)(\sigma_1 + \sigma_2)\}] \end{aligned} \quad (4.3.21a)$$

$$\begin{aligned} \epsilon^-(\sigma_1, \sigma_2) = & \sum_{n=1}^{\infty} [\epsilon_{1/3-n+x/2}^{\alpha} \exp \{-i(1/3-n+x/2)(\sigma_1 + \sigma_2)\} \\ & + \epsilon_{n-2/3+x/2}^{\beta} \exp \{-i(n-2/3+x/2)(\sigma_1 + \sigma_2)\}] . \end{aligned} \quad (4.3.21b)$$

The associated boundary conditions are

$$\epsilon^+(\sigma_1 + 2\pi) = e^{+i2\pi(1/3)} e^{i\pi x} \epsilon^+(\sigma_1) \quad (4.3.22a)$$

$$\epsilon^-(\sigma_1 + 2\pi) = e^{-i2\pi(1/3)} e^{-i\pi x} \epsilon^-(\sigma_1) . \quad (4.3.22b)$$

The complicated commutation relations of the modes of ϵ have already been discussed. (See eq. (4.3.11a).)

From the analogy of free-fermion models, we suggest that in $K = 4$ parafermion models the presence of a sector containing twisted ϵ fields with boundary conditions (4.3.22a) or (4.3.22b) will result in \mathbb{Z}_b or $\mathbb{Z}_2 \times \mathbb{Z}_b$ GSO projections, depending on whether a is even or odd respectively. (We assume as before that a and b are relative primes but now use the range $-2/3 \leq x \equiv a/b < 4/3$.) Zero-modes correspond to a twist by $x = -2/3$. Whatever the GSO projection is, states resulting from D factors of ϵ_0 acting on the fermionic vacuum must survive, in order to have spacetime fermions. Thus, we conjecture that the presence of these (twisted) zero-modes ϵ_n , $n \in \mathbb{Z}$ in a model, results in a generalized \mathbb{Z}_3 GSO projection. Likewise for $K = 8$ and 16, one might expect \mathbb{Z}_5 and \mathbb{Z}_9 projections, respectively. Such projections for $K = 8$ and 16 could be significantly altered though, by the effects of the non-Abelian braiding of the non-local interactions.

One other aspect to notice is that within the range $-2/3 \leq x < 4/3$ there are actually two distinct N-S sectors, corresponding not just to $x = 0$, but also to $x = 2/3$. This is associated with the \mathbb{Z}_2 symmetry that interchanges ϵ^+ and ϵ^- . This

symmetry may explain the origin of the additional \mathbb{Z}_2 GSO-type projection we will shortly discuss.

For the $K = 4$ FSC model, one expects a GSO projection to depend on a generalization of fermion number. However, the naive generalization to parafermion number, $F(\phi_0^1)$, is insufficient. We find that we must also consider the multiplicities of the twist field, ϕ_1^1 , and the field ϕ_1^0 , which increases the m quantum number by one while keeping j constant. In order to derive the MIPF we discovered that, indeed, a \mathbb{Z}_3 projection must be applied to both the left-moving modes (LM) and right-moving modes (RM) independently. Survival of a physical state, $|\text{phys}\rangle$, in the Hilbert space under this \mathbb{Z}_3 projection requires⁴²

$$\left\{ e^{\left\{ i\pi\frac{2}{3} \cdot [\bar{F}_{\text{LM(RM)}}(\phi_0^1) + \bar{F}_{\text{LM(RM)}}(\phi_1^1)] \right\}} = e^{i\pi\frac{2}{3}} \right\} |\text{phys}\rangle, \quad (4.3.23a)$$

or equivalently

$$\left\{ Q_{3,\text{LM(RM)}} \equiv \sum_i F_{i,\text{LM(RM)}}(\phi_0^1) + \sum_i F_{i,\text{LM(RM)}}(\phi_1^1) = 1 \pmod{3} \right\} |\text{phys}\rangle, \quad (4.3.23b)$$

where $F_i(\phi_m^j)_{\text{LM(RM)}}$ is the number operator for the field ϕ_m^j along the i direction for left-moving (right-moving) modes. Prior to projection by this extended GSO operator, we consider all physical states associated with the LM partition function terms in the expansion of $(c_0^0 + c_0^4 + c_0^2)^4$ or $(c_2^2 + c_2^4)^4$ to be in the A_4 -sector. Similarly, we initially place in the B_4 -sector all the LM physical states associated with the partition function terms in the expansion of $(c_2^2 + c_2^4)^2(c_0^0 + c_0^4 + c_0^2)^2$ or $(c_0^0 + c_0^4 + c_0^2)^2(c_2^2 + c_2^4)^2$. There is however, a third class of states; let us call this the “ D_4 ” class. This last class would be present in the original Hilbert space if not for an additional \mathbb{Z}_2 GSO projection. Left-moving states in D class, would have partition functions that are terms in the expansion of $(c_0^0 + c_0^4 + c_0^2)^3(c_2^2 + c_2^4)$ or $(c_2^2 + c_2^4)^3(c_0^0 + c_0^4 + c_0^2)$. The thirty-two D_4

⁴²Note, this projection alone does not prevent mixing holomorphic A_4 -sector and antiholomorphic B_4 -sector terms. This is prevented by the standard requirement $M_{\text{LM}}^2 = M_{\text{RM}}^2$, i.e., $L_0 = \bar{L}_0$, which here results in the RM factors in the partition function being the complex conjugates of the LM, giving only mod-squared terms in the partition functions. This allows us to examine only the left-movers in detail in the following.

terms in the expansions are likewise divisible into subclasses based on their associated \mathbb{Z}_3 charges, Q_3 .⁴³ Twelve have charge 0 (mod 3), twelve have charge 1 (mod 3) and eight have charge 2 (mod 3). Without the \mathbb{Z}_2 projection, it is impossible to keep only the wanted terms in the A_4 - and B_4 -sectors, while projecting away all of the D_4 -sector terms. Simple variations of the projection (4.3.23a) cannot accomplish this. All D_4 terms can be eliminated, without further projections on the A and B terms, by a \mathbb{Z}_2 projection defined by

$$\left\{ \sum_i F_{i, \text{LM(RM)}}(\phi_1^1) + \sum_i F_{i, \text{LM(RM)}}(\phi_{\pm 1}^0) = 0 \pmod{2} \right\} |\text{phys}\rangle. \quad (4.3.23c)$$

(Note that for $K = 2$, ϕ_1^1 is equivalent to the identity field and ϕ_0^1 is indistinguishable from the usual fermion, ϕ_0^1 . Thus for $K = 2$ there is no additional \mathbb{Z}_2 GSO projection.)

Consideration of these D_4 class states reveals some physical meaning to our particular \mathbb{Z}_3 charge and the additional \mathbb{Z}_2 projection. First, in all sectors the charge Q_3 commutes with $(\phi_{K/4}^0)^{D-2}$, which transforms between non-projection and projection states of opposite spacetime statistics in the A_4 - and B_4 -sectors. Second, the values of this charge are also associated with specific $\text{mass}^2 \pmod{1}$ levels. Third, only for the A_4 - and B_4 -sector states does $\text{mass}^2 \pmod{1}$ commute with the same twist operator $(\phi_{K/4}^0)^{D-2}$. Recall, in section 4.2 we suggested that twisting by this latter field was the key to spacetime SUSY. Without any of our projections the $\text{mass}^2 \pmod{1}$ of states present would be $\text{mass}^2 = 0, \frac{1}{12}, \frac{2}{12}, \dots, \frac{11}{12}$. When acting on D_4 -sector fields, $(\phi_{K/4}^0)^{D-2}$ transforms $\text{mass}^2 = \frac{i}{12} \pmod{1}$ states into $\text{mass}^2 = \frac{i+6}{12} \pmod{1}$ states. Thus, if present in a model, states in the D_4 -sector paired by the supersymmetry operator $(\phi_{K/4}^0)^{D-2}$ would have to be associated with different mod-squared terms of the partition function, in order to preserve T invariance. As a result, the paired contributions to the partition function could not cancel, proving that D terms cannot be part of any supersymmetric theory. Although $\text{mass}^2 \pmod{1}$ commutes with $(\phi_{K/4}^0)^{D-2}$ in the $A(Q_3 = 0)$, $A(Q_3 = -1)$, $B(Q_3 = 0)$, $B(Q_3 = -1)$ subsectors, within these subsectors (1) there is either a single bosonic state or fermionic state of

⁴³For clarity, we are always pairing c_0^0 and c_0^4 in these partition functions, rather than expanding $(c_0^0 + c_0^4)^n$, as is done in Table 4.4.

state or fermionic state of lowest mass without superpartner of equal mass, and/or (2) the lowest mass states are tachyonic. (See Table 4.4.)⁴⁴ Thus, our specific GSO projections in terms of our \mathbb{Z}_3 charge projection and our \mathbb{Z}_2 projection equate to spacetime SUSY, uniquely so.

⁴⁴Our assignments of states as spacetime bosons or fermions in the B -sector, uses an additional projection that we believe distinguishes between the two. Following the pattern in eq. (4.2.8b) with bosonic/fermionic assignment of related states defined in eqs. (4.2.33a-c), we suggest that for these states the two primary fields, $\phi_{m_3}^{j_3}$ and $\phi_{m_4=m_3}^{j_4}$ (implicitly) assigned compactified spacetime indices must be the same, *i.e.*, $j_3 = j_4$, or else must form a term in the expansion of $(\phi_0^0 + \phi_0^2)^2$. This second case is related to ϕ_0^0 and ϕ_0^2 producing the same spacetime fermion field, ϕ_1^2 , when separately twisted by $\phi_{K/4}^0$. (Note however that $\phi_1^2 \otimes \phi_{K/4}^0 = \phi_0^0$ only.) Following this rule, neither the states corresponding to $(c_0^2)(c_0^0)(c_2^2)(c_2^4)$ and $(c_2^2)(c_2^4)(c_0^2)(c_0^0)$, (which transform between each other under twisting by $\phi_{K/4}^0 \phi_{K/4}^0 \phi_{K/4}^0 \phi_{K/4}^0$) nor those associated with $(c_0^2)(c_0^4)(c_2^2)(c_2^4)$ and $(c_2^2)(c_2^4)(c_0^2)(c_0^4)$, survive the projections as either spacetime bosons or fermions. However, for completeness we include these partition functions in the B-sector columns of Table 4.4. We define the associated states as either spacetime bosons or fermions based on the value of $m_3 = m_4$. This is academic, though, because the states do not survive the \mathbb{Z}_3 projection.

Table 4.4 Masses of $K = 4$ Highest Weight States
(Represented by Their Associated Characters)

A-Sector			Survives					B-Sector	
Boson	Mass ²	Fermion	Q_3	GSO	Boson ?	Mass ²	Fermion ?		
$(c_0^4)^2(c_0^4)^2$	$3\frac{2}{3}$		0	No	$(c_0^4)^2(c_2^4)^2$	$3\frac{1}{6}$	$(c_2^4)^2(c_0^4)^2$		
$c_0^2 c_0^4 (c_0^4)^2$	3		1	Yes	$c_0^2 c_0^4 (c_2^4)^2$	$2\frac{1}{2}$	$c_2^2 c_2^4 (c_0^4)^2$		
$c_0^0 c_0^4 (c_0^4)^2$	$2\frac{2}{3}$	$(c_2^4)^2(c_2^4)^2$	0	No	$c_0^0 c_0^4 (c_2^4)^2$	$2\frac{1}{6}$	$(c_2^4)^2 c_0^0 c_0^4$		
$(c_0^4)^2(c_0^2)^2$	$2\frac{1}{3}$		-1	No	$(c_0^4)^2(c_2^2)^2$	$1\frac{5}{6}$	$(c_2^4)^2(c_0^2)^2$		
$(c_0^2)^2(c_0^4)^2$					$(c_0^2)^2(c_2^4)^2$		$(c_2^2)^2(c_0^4)^2$		
					$c_0^2 c_0^4 c_2^2 c_2^4$		$c_2^2 c_2^4 c_0^2 c_0^4$		
$c_0^2 c_0^0 (c_0^4)^2$	2	$c_2^2 c_2^4 (c_2^4)^2$	1	Yes	$c_0^2 c_0^0 (c_2^4)^2$	$1\frac{1}{2}$	$c_2^2 c_2^4 c_0^0 c_0^4$		
$c_0^2 c_0^4 (c_0^2)^2$	$1\frac{2}{3}$		0	No	$c_0^2 c_0^4 (c_2^2)^2$	$1\frac{1}{6}$	$c_2^2 c_2^4 (c_0^2)^2$		
$(c_0^0)^2(c_0^4)^2$					$(c_0^0)^2(c_2^4)^2$				
$(c_0^4)^2(c_0^0)^2$							$(c_2^4)^2(c_0^0)^2$		
$c_0^0 c_0^4 (c_0^2)^2$	$1\frac{1}{3}$	$(c_2^2)^2(c_2^4)^2$	-1	No	$c_0^0 c_0^4 (c_2^2)^2$	$\frac{5}{6}$			
		$(c_2^4)^2(c_2^2)^2$			$c_0^0 c_0^2 c_2^2 c_2^4$		$c_2^2 c_2^4 c_0^0 c_0^2$		
$(c_0^2)^2(c_0^2)^2$	1		1	Yes	$(c_0^2)^2(c_2^2)^2$	$\frac{1}{2}$	$(c_2^2)^2(c_0^2)^2$		
$c_0^2 c_0^4 (c_0^0)^2$							$c_2^2 c_2^4 (c_0^0)^2$		
$c_0^2 c_0^0 (c_0^2)^2$	$\frac{2}{3}$	$c_2^2 c_2^4 (c_2^2)^2$	0	No	$c_0^2 c_0^0 (c_2^2)^2$	$\frac{1}{6}$			
$c_0^0 c_0^4 (c_0^0)^2$									
$(c_0^0)^2(c_0^2)^2$	$\frac{1}{3}$		-1	No	$(c_0^0)^2(c_2^2)^2$	$-\frac{1}{6}$			
$(c_0^2)^2(c_0^0)^2$							$(c_2^2)^2(c_0^0)^2$		
$c_0^2 c_0^0 (c_0^0)^2$	0	$(c_2^2)^2(c_2^2)^2$	1	Yes					
$(c_0^0)^2(c_0^0)^2$	$-\frac{1}{3}$		0	No					

Table 4.5 Mass Sectors as Function of \mathbb{Z}_3 Charge

Lowest M^2	$M^2 \bmod 1$	Sector	\mathbb{Z}_3 Charge	Sector	$M^2 \bmod 1$	Lowest M^2
0	0	A	$Q_3 = 1$	B	$\frac{6}{12}$	$\frac{6}{12}$
$-\frac{1}{12}$	$\frac{11}{12}$	D	$Q_3 = 0$	D	$\frac{5}{12}$	$\frac{5}{12}$
$-\frac{2}{12}$	$\frac{10}{12}$	B	$Q_3 = -1$	A	$\frac{4}{12}$	$\frac{4}{12}$
$-\frac{3}{12}$	$\frac{9}{12}$	D	$Q_3 = 1$	D	$\frac{3}{12}$	$\frac{3}{12}$
$-\frac{4}{12}$	$\frac{8}{12}$	A	$Q_3 = 0$	B	$\frac{2}{12}$	$\frac{2}{12}$
$\frac{7}{12}$	$\frac{7}{12}$	D	$Q_3 = -1$	D	$\frac{1}{12}$	$\frac{1}{12}$

(In Table 4.5, columns one and seven give the lowest mass² of a state with center column \mathbb{Z}_3 charge in the appropriate sector. For the D sector states, under $(\phi_{K/4}^0)^{D-2}$ twistings, mass² values in column two transform into mass² values in column six of the same row and vice-versa.)

Unlike in the $K = 2$ case, for $K = 4$ the \mathbb{Z}_3 projection in the Ramond sector wipes out complete spinor fields, not just some of the modes within a given spin field. This type of projection does not occur in the Ramond sector for $K = 2$ since there are no fermionic states with fractional mass² values in the $D = 10$ model. Note also that our \mathbb{Z}_3 GSO projections relate to the \mathbb{Z}_3 symmetry pointed out in [41] and briefly commented on following eqs. (4.3.11a-b).

For $K = 8$, a more generalized \mathbb{Z}_5 projection holds true for all sectors. For the $K = 16$ theory, there are too few terms and products of string functions to determine if a \mathbb{Z}_9 projection is operative. In the $K = 4$ case, the value of our LM (RM) Q_3 charges for states surviving the projection is set by demanding that the massless spin-

2 state $\epsilon_{-\frac{1}{3}}^\mu \bar{\epsilon}_{-\frac{1}{3}}^{\bar{\nu}} |0\rangle$ survives. In the A_{K^-} , B_{K^-} , (and C_{K^-} for $K = 8, 16$) sectors, these projections result in states with squared masses of $0 + \text{integer}$, $\frac{1}{2} + \text{integer}$, and $\frac{3}{4} + \text{integer}$, respectively.

4.3.c The Unique Role of the Twist Field, $\phi_{K/4}^{K/4}$.

In this subsection we examine whether other consistent models are possible if one generalizes from the twist field, $\phi_{K/2}^{K/2}$ to another that could fulfill its role. When it is demanded that the standard twist and $\epsilon \equiv \phi_0^1$ fields of reference [36–38], be used we can derive the critical dimensions of possible models simply by observing that $K = 2, 4, 8$, and 16 are the only levels for which

$$h(\phi_0^1)/h(\phi_{K/4}^{K/4}) \in \mathbb{Z} . \quad (4.3.24a)$$

If we assume (as in [36]) that the operator $(\phi_{K/4}^{K/4})^\mu$ acting on the (tachyonic) vacuum produces a massless spacetime spinor vacuum along the direction μ , and $(\phi_0^1)^\mu$ produces a massless spin-1 state, then for spacetime supersymmetry (specifically $N = 2$ SUSY for fractional type II theories and $N = 1$ for fractional heterotic) $h(\phi_0^1)/h(\phi_{K/4}^{K/4})$ must equal the number of transverse spin modes, *i.e.*,

$$\begin{aligned} h(\phi_0^1) &= (D - 2)h(\phi_{K/4}^{K/4}) \\ \frac{2}{K + 2} &= (D - 2)\frac{K/8}{K + 2} . \end{aligned} \quad (4.3.24b)$$

Hence,

$$D = 2 + \frac{16}{K} \in \mathbb{Z} . \quad (4.3.24c)$$

Thus, from this one assumption, the possible integer spacetime dimensions are determined along with the associated levels. Perhaps not coincidentally, the allowed dimensions are precisely the ones in which classical supersymmetry is possible. This is clearly a complementary method to the approach for determining D followed in refs. [36–38].

Demanding eq. (4.3.24a) guarantees spin-1 and spin-1/2 superpartners in the open string a (spin-2 and spin-3/2 in the closed string) with

$$\text{mass}^2 = \text{mass}^2(\text{vacuum}) + h(\phi_0^1) = \text{mass}^2(\text{vacuum}) + (D - 2) * h(\phi_{K/4}^{K/4}) . \quad (4.3.25)$$

(Double the total mass² for the closed string.) *A priori* simply demanding the ratio be integer in eq. (4.3.24a) is not sufficient to guarantee local spacetime supersymmetry in the closed string. However, in the previous subsections it proved to be; for the $K = 4$ model the masslessness of the open string (spin-1, spin-1/2) pair occurred automatically and hence also in the closed string for the (spin-2, spin-3/2) pair.

Figure 4.1 Supersymmetry of Lowest Mass States of Fractional Open String

$$\frac{m^2(\text{spin} - 1)}{h(\phi_0^1) \uparrow} = \frac{m^2(\text{spin} - 1/2)}{(D - 2) \times h(\phi_{K/4}^{K/4}) \uparrow}$$

$$m^2(\text{vacuum})$$

In fractional superstrings, the primary field $\phi_{K/4}^{K/4} \equiv \phi_{-K/4}^{K/4}$ for $K = 4, 8,$ and $16,$ and its associated character $Z_{K/4}^{K/4} = \eta c_{K/2}^{K/2}$, are viewed as the generalizations of $\phi_{1/2}^{1/2}$ at $K = 2$ and $(\vartheta_2/\eta)^{1/2}$. Are there any other parafermion operators at additional levels K that could be used to transform the bosonic vacuum into a massless fermionic vacuum and bring about local spacetime supersymmetric models? The answer is that by demanding masslessness⁴⁵ of the (spin-1, spin-1/2) pair, there is clearly no other choice for $K < 500$. (We believe this will generalize to $K < \infty$.)

The proof is short. We do not start from the assumption that the massless spin-1 fields are a result of the ϕ_0^1 fields. Rather, to the contrary, the necessity of choosing

⁴⁵Masslessness of at least the left- (right-)moving spin-1 spacetime fields (whose tensor product forms the massless spin-2 graviton in a closed string) is of course required for a consistent string theory. Consistent two-dimensional field theories with

lowest mass of left – (right–)moving spacetime spin – 1 fields =

$$\text{lowest mass of left – (right–)moving spacetime spin – 1/2 fields} \equiv M_{\min} > 0$$

may exist (as we discuss below) but, the physical interpretation of such models is not clear, (other than to say they would not be theories with gravity.

ϕ_0^1 appears to be the result of the uniqueness of $\phi_{K/4}^{K/4}$.

Proof: Assume we have a consistent (modular invariant) closed fractional superstring theory at level- K with supersymmetry in D dimensional spacetime, ($N = 2$ for type-II and $N = 1$ for heterotic). Let the massless left- (right-)moving spin-1 field be $(\phi_{m_1}^{j_1})^\mu|\text{vacuum}\rangle$. This requires that $\phi_{m_1}^{j_1}$ have conformal dimension

$$h(\phi_{m_1}^{j_1}) = c_{\text{eff}}/24 = (D - 2) \frac{K}{8(K + 2)}. \quad (4.3.26)$$

Thus, the twist field $\phi_{m_2}^{j_2}$ that produces the spinor vacuum along one of the $D - 2$ transverse dimensions must have conformal dimension

$$h(\phi_{m_2}^{j_2}) = \frac{K}{8(K + 2)}. \quad (4.3.27)$$

For $K < 500$ the only primary fields with this dimension are the series of $\phi_{K/4}^{K/4}$ for $K \in 2\mathbb{Z}$, and the accidental solutions ϕ_0^2 for $K = 48$, ϕ_0^3 for $K = 96$, and $\phi_{7/2}^{9/2}$ for $K = 98$. Being fields with $m = 0$, neither ϕ_0^2 nor ϕ_0^3 at any level cannot be used to generate spacetime fermions. The $\phi_{7/2}^{9/2}$ alternative is not acceptable either because at $K = 98$ there is not an additional field to replace ϵ . In other words, there is not a field to be paired with $\phi_{7/2}^{9/2}$ whose conformal dimension is an integer multiple of $\phi_{7/2}^{9/2}$'s. (A proof of the uniqueness of $\phi_{K/4}^{K/4}$ for all K is being prepared by the author.)

Confirmation of $\phi_{K/4}^{K/4}$ as the spin-1/2 operator, though, does not immediately lead one to conclude that ϵ is the only possible choice for producing massless boson fields. Table 4.6 shows alternative fields at new levels $K \neq 2, 4, 8, \text{ or } 16$ whose conformal dimension is one, two, or four times the conformal dimension of $\phi_{K/4}^{K/4}$. (Note that successful alternatives to ϵ would lead to a relationship between level and spacetime dimension differing from eq. (4.3.24c).) However, nearly all alternatives are of the form $\phi_0^{j>1}$ and we would expect that modular invariant models using $\phi_0^{j>1}$ to create massless bosons, would necessarily include (at least) the tachyonic state, $(\phi_0^1)^\mu|\text{vacuum}\rangle$. That is, we do not believe valid GSO projections exist which can project away these tachyons while simultaneously keeping the massless graviton and gravitino and giving modular invariance. Further, the remaining fields on the list

have $m \neq 0 \pmod{K}$. Each of these would not have the correct fusion rules with itself, nor with $\phi_{K/4}^{K/4}$ to be a spacetime boson.

Table 4.6 Fields $\phi_{m_1}^{j_1} \neq \phi_0^1$ with Conformal Dimensions in Integer Ratio with $h(\phi_{K/4}^{K/4})$

K	ϕ_m^j	$h(\phi_m^j)/h(\phi_{K/4}^{K/4})$
12	ϕ_0^2	4
24	ϕ_0^2	2
	ϕ_0^3	4
36	ϕ_6^7	4
40	ϕ_0^4	4
48	ϕ_0^2	1
	ϕ_0^3	2
60	ϕ_0^5	4
80	ϕ_0^4	2
84	ϕ_0^6	4
96	ϕ_0^3	1
112	ϕ_0^7	4
120	ϕ_0^5	2
\vdots	\vdots	\vdots

Lastly, we want to consider the possibility that there is meaning to (non-stringy) two-dimensional field theories that contain neither supergravity nor even gravity. Instead let a model of this type contain only a global supersymmetry. The lowest mass spin-1 ($(\phi_0^1)^\mu|\text{vacuum} \rangle$) and spin-1/2 ($(\phi_{m_3}^{j_3})^{D-2}|\text{vacuum} \rangle$) left- or right-moving fields would be related by

$$\text{mass}^2(\text{vacuum}) + h(\phi_0^1) = \text{mass}^2(\text{vacuum}) + (D - 2) \times h(\phi_{m_3}^{j_3}). \quad (4.3.28)$$

In PCFT's there is only a very small number (12) of potential candidates for $\phi_{m_3}^{j_3}$. (Like $\phi_{K/4}^{K/4}$ these twelve are all of the form $\phi_{\pm j_3}^{j_3}$.) We are able to reduce the number of

candidates down to this finite number very quickly by proving no possible candidate could have $j_3 > 10$, independent of the level. We demonstrate this as follows:

Any potential level- K candidate $\phi_{m_3}^{j_3}$ must satisfy the condition of

$$\frac{K}{K+2} [j_3(j_3+1) - 2] \leq (m_3)^2 \leq (j_3)^2 \leq K^2/4. \quad (4.3.29)$$

By parafermion equivalences (4.1.12a-b), $|m| \leq j \leq K/2$ can be required for any level- K fields. The other half of the inequality, $\frac{K}{K+2} [j_3(j_3+1) - 2] \leq m^2$ results from the weak requirement that the conformal dimension of the candidate (spacetime) spin-1/2 field, $\phi_{m_3}^{j_3}$, creating the fermion ground state along one spacetime direction cannot be greater than the conformal dimension of ϵ , *i.e.*, $h(\phi_{m_3}^{j_3}) \leq h(\phi_0^1)$.

From eq. (4.3.29), we can determine both the minimum and maximum values of K , for a given j_3 , (independent of m_3). These limits are $K_{\min} = 2j_3$ and $K_{\max} = \text{int} \left(\frac{2(j_3)^2}{j_3-2} \right)$. Thus, the number of different levels that can correspond to the field $\phi_{m_3}^{j_3}$ is $\text{int} \left(\frac{5j_3-2}{j_3-2} \right)$. This number quickly decreases to six as j_3 increases to ten and equals five for j_3 greater than ten. For a given j_3 , we will express the levels under consideration as $K_i = 2j_3 + i$. Also, since $K_{\min} = 2j_3$, the weak constraint on m_3 implies that we need only consider $\phi_{m_3=\pm j_3}^{j_3}$ fields. Thus, our search reduces to finding fields $\phi_{\pm j_3}^{j_3}$ whose conformal dimensions satisfy

$$\frac{h(\phi_0^1)}{h(\phi_{\pm j_3}^{j_3})} = \frac{\frac{2}{K_i+2}}{\frac{j_3(j_3+1)}{K_i+2} - \frac{(j_3)^2}{K_i}} \in \mathbb{Z}. \quad (4.3.30)$$

Clearly, there are no solutions to eq. (4.3.30) for $i = 0$ to 4 and $j_3 > 10$. Hence, our range of possible alternative sources for fermionic ground states reduces to only considering those $\phi_{\pm j_3}^{j_3}$ with $0 < j_3 \leq 10$. Within this set of j_3 's, a computer search reveals the complete set of fields that obey eq. (4.3.30), as shown in Table 4.7.

Table 4.7 Potential Alternatives, $\phi_{\pm m_3}^{j_3}$, to $\phi_{K/4}^{K/4}$ for Spin Fields

j_3	$\pm m_3$	K	i	$h(\phi_0^1)$	$h(\phi_{m_3}^{j_3})$	D
1/2	1/2	2	1	1/2	1/16	10 **
		3	2	2/5	1/15	8
		5	4	2/7	2/35	7
1	1	3	1	2/5	1/15	8
		4	2	1/3	1/12	6 **
		6	4	1/4	1/12	5
3/2	3/2	9	6	2/11	1/11	4
2	2	5	1	2/7	2/35	7
		6	2	1/4	1/12	5
		8	4	1/5	1/10	4 **
5/2	5/2	25	20	2/27	2/27	3
3	3	9	3	2/11	1/11	4
		18	12	1/10	1/10	3
4	4	16	8	1/9	1/9	3 **
6	6	18	6	1/10	1/10	3
10	10	25	5	2/27	2/27	3

The sets of solutions for $j_3 = \frac{1}{2}, 1,$ and 2 are related. The existence of a set $\{i = 1, 2,$ and $4\}$ of solutions for any one of these j_3 implies identical sets $\{i\}$ for the remaining two j_3 as well. The known $\phi_{K/4}^{K/4}$ solutions (marked with a **) correspond to the $i = 1, 2,$ and 4 elements in the $j_3 = \frac{1}{2}, 1,$ and 2 sets respectively. Whether this pattern suggests anything about the additional related $\phi_{\pm j_3}^{j_3}$ in these sets, other than explaining their appearance in the above table, remains to be seen.

The set of distinct fields can be further reduced. There is a redundancy in the above list. Among this list, for all but the standard $\phi_{K/4}^{K/4}$ solutions, there are two fields at each level, with different values of j_3 . However, these pairs are related by the field equivalences (4.1.1):

$$\phi_{\pm 1/2}^{1/2} \equiv \phi_{\mp 1}^1 \quad \text{at level } K = 3 \quad (4.3.31a)$$

$$\phi_{\pm 1/2}^{1/2} \equiv \phi_{\mp 2}^2 \quad \text{at level } K = 5 \quad (4.3.31b)$$

$$\phi_{\pm 1}^1 \equiv \phi_{\mp 2}^2 \quad \text{at level } K = 6 \quad (4.3.31c)$$

$$\phi_{\pm 3/2}^{3/2} \equiv \phi_{\mp 3}^3 \quad \text{at level } K = 9 \quad (4.3.31d)$$

$$\phi_{\pm 3}^3 \equiv \phi_{\mp 6}^6 \quad \text{at level } K = 18 \quad (4.3.31e)$$

$$\phi_{\pm 5/2}^{5/2} \equiv \phi_{\mp 10}^{10} \quad \text{at level } K = 25 . \quad (4.3.31f)$$

Because ϕ_m^j and ϕ_{-m}^j have identical partition functions and $\phi_{-m}^j \equiv (\phi_m^j)^\dagger$ we can reduce the number of possible alternate fields in half, down to six. (Note that we not been distinguishing between \pm on m anyway.)

If we want models with *minimal* (global) super Yang-Mills Lagrangians we can reduce the number of the fields to investigate further. Such theories exist classically only in $D_{\text{SUSY}} = 10, 6, 4, 3,$ (and 2) spacetime. Thus we can consider only those $\phi_{\pm j_3}^{j_3}$ in the above list that have integer conformal dimension ratios of $D_{\text{SUSY}} - 2 = h(\phi_0^1)/h(\phi_{j_3}^{j_3}) = 8, 4, 2,$ and 1 . This would reduce the fields to consider to just the two new possibilities for $D = 4,$ and 3 since there there are no new additional for $D = 10$ or 6 .

4.4 Concluding Discussion

A consistent generalization of the superstring would be an important development. Our work has shown that the fractional superstring has many intriguing features that merit further study. The partition functions for these theories have simple origins when derived systematically through the factorization approach of Gepner and Qiu. Furthermore, using this affine/theta-function factorization of the parafermion partition functions, we have related the A_K -sector containing the graviton and gravitino with the massive sectors, B_K and C_K . A bosonic/fermionic interpretation of the B_K -subsectors was given. Apparent “self-cancellation” of the C_K -sector was shown, the meaning of which is under investigation. A possible GSO projection was found, adding hope that the partition functions have a natural physical interpretation.

Nevertheless, fundamental questions remain concerning the ghost system and current algebra, which prevent a definite conclusion as to whether or not these are consistent theories. Perhaps most important are arguments suggesting that fractional superstrings in D dimensions are not formed from tensor products of D separate $SU(2)_K/U(1)$ CFT's. Rather, a tensor product CFT may be an illusion of the tensor product form of the partition function. Instead of having a total conformal anomaly contribution of $c = 12$ from matter, the appearance in the six-dimensional ($K = 4$) theory of extra null states at the critical value suggests that $c = 10$. This would require a non-tensor product representation of the fractional superconformal algebra (4.1.7a-4.1.7c). However, even if the theories are ultimately shown to be inconsistent, we believe that this program will at least provide interesting identities and new insight into the one case that we know is consistent, $K = 2$. In other words, viewed in this more general context, we may understand better what is special about the usual superstring.

On the other hand, fractional superstrings may eventually prove to be a legitimate class of solutions to a new string theory. This class would then join the ranks of bosonic, type-II, and heterotic string theories. Further, it is claimed that MIPF's for heterotic fractional superstrings with left-movers at level- K_1 and right-movers at

level- K_2 are also possible.^[36] Let us call these heterotic $_{(K_1, K_2)}$ models. The simplest of this class, the heterotic $_{(1, K)}$ model, should have $SO(52 - 2D) = SO(48 - 16/K)$ gauge symmetry.

The dominant view holds that the bosonic, type-II, and (standard) heterotic theories are uniquely defined by their underlying extended Virasoro algebras, with many solutions (models) existing for each theory.^{footnote}The number of uncompactified dimensions is regarded as a parameter in the space of solutions. An alternate view^[24] suggests that heterotic and type-II strings are related to subregions in the parameter space of bosonic strings. Specifically, the claim is that for *any* heterotic (type-II) string there exists a bosonic string that faithfully represents its properties. This means that the classification of heterotic and type-II strings is contained within that of the bosonic strings,

$$\text{Bosonic Strings} \supset \text{Heterotic Strings} \supset \text{Type II Strings} . \quad (4.4.32)$$

Thus, theoretically, once the conditions for modular invariance of bosonic strings are known, determination of them for heterotic or type-II is transparent. The basis for this mapping is that the non-unitary supersymmetric ghost system can be transformed into a unitary conformal field theory. This transformation preserves both conformal and modular invariance. The partition functions of a new unitary theory satisfies all the consistency conditions to serve as a partition function for a bosonic string compactified on a lattice.^[24]

Where might (heterotic) fractional superstrings, if proven consistent, fit in this scheme? If their ghost system is finally understood, perhaps the same mapping technique can be applied. If so, can fractional superstrings be represented by subclasses of bosonic strings? We suspect that the answer is “yes,” given that fractional (heterotic) strings are found to be consistent. Further, we would expect the fractional heterotic superstrings to correspond specifically to a subset of heterotic (or type-II if $K_1 = K_2$) strings. This is suggested by the apparent spacetime SUSY of heterotic $_{(K_1, K_2)}$ strings, even though the local world sheet symmetry is “fractional.”

Appendix A: Dynkin Diagrams for Lie Algebras and Kač-Moody Algebras

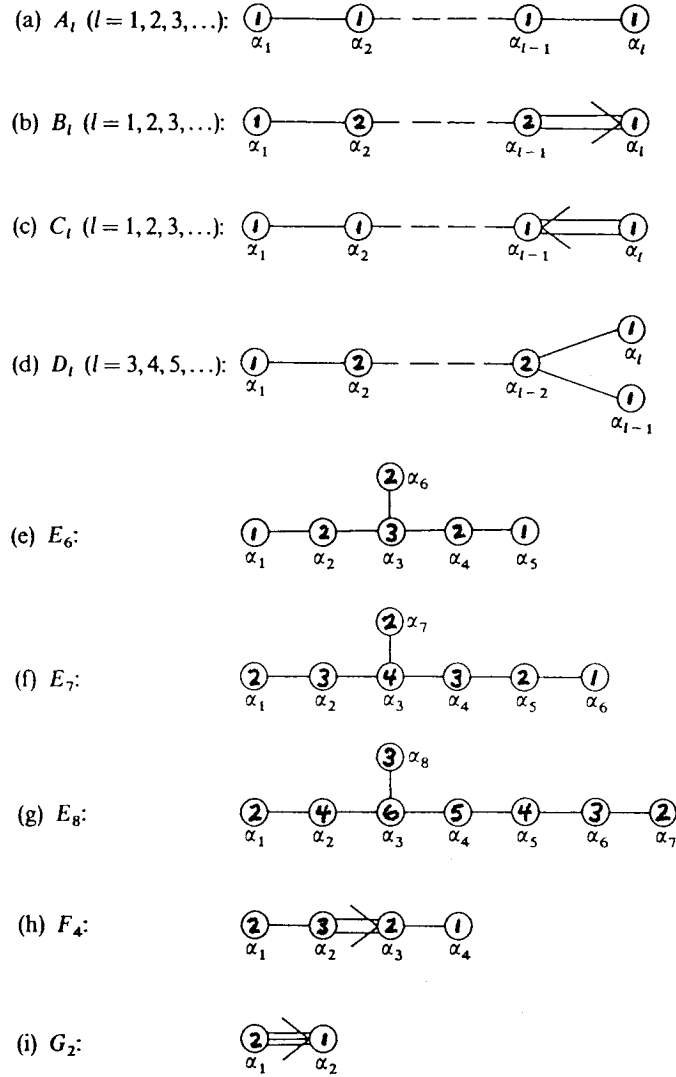


Figure A.1 Generalized Dynkin diagrams of the finite KM algebras (*i.e.*, of the compact simple Lie algebras)

Each simple root of a Lie algebra or a Kač-Moody algebra is represented by a circle. Two circles not directly connected by at least one line imply two orthogonal roots. One line connecting circles signifies a 120° angle between corresponding roots, two lines 135° , three lines 150° , and four lines 180° . For non-simply-laced algebras, arrows point towards the shorter of two connected roots. The co-marks associated with the simple roots of a Lie algebra appear inside the circles.

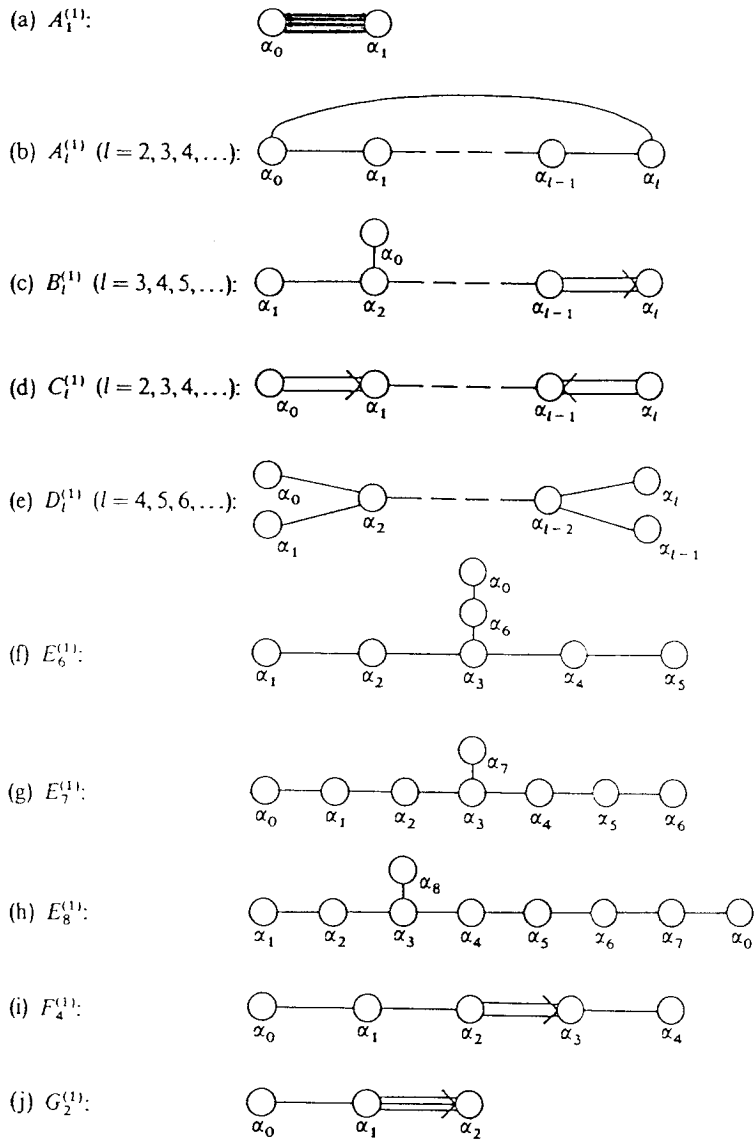


Figure A.2 Generalized Dynkin diagrams of the untwisted affine KM algebras

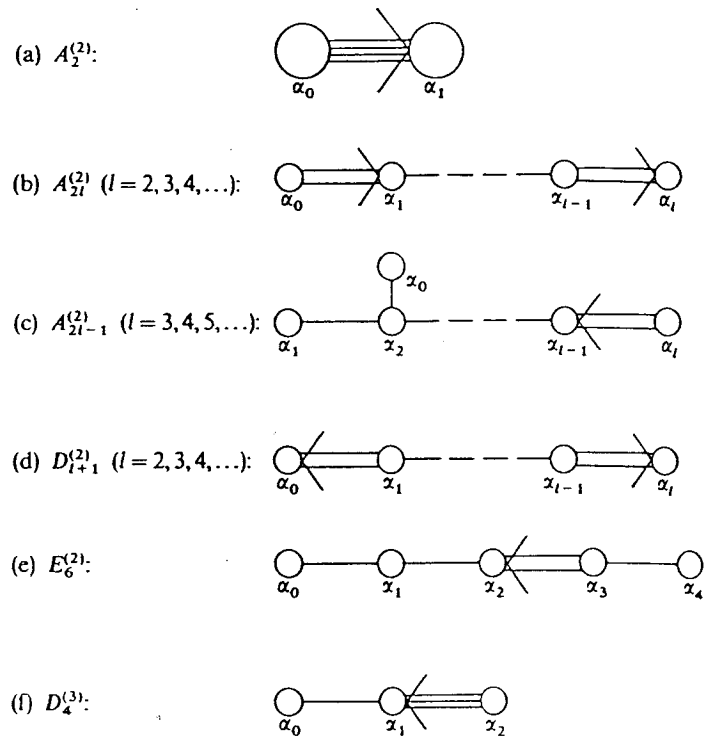


Figure A.3 Generalized Dynkin diagrams of the twisted affine KM algebras

Appendix B: Proof that completeness of the A-D-E classification of modular invariant partition functions for $SU(2)_K$ is unrelated to uniqueness of the vacuum

In this appendix we prove that relaxing the condition of uniqueness of the vacuum does not allow new solutions to $SU(2)_K$ MIPFs. The allowed solutions are still just the A-D-E classes. We prove this through a review of Cappelli, Itzykson, and Zuber's (CIZ's) derivation of the A-D-E classification.^[23] We treat the coefficients $N_{\vec{l}}$ in the partition function of eq. (B.5) as the components of a symmetric matrix, \mathbf{N} , that operates between vectors of characters $\vec{\chi}$:

$$\begin{aligned} Z &= \sum_{\vec{l}} N_{\vec{l}} \bar{\chi}_{\vec{l}}^{(K)}(\bar{\tau}) \chi_{\vec{l}}^{(K)}(\tau) \\ &\equiv \langle \vec{\chi} | \mathbf{N} | \vec{\chi} \rangle \end{aligned} \tag{B.1}$$

In this notation, under $\mathbf{V} \in \mathbf{S}, \mathbf{T}$, $\mathbf{V}^\dagger \mathbf{V} = 1$, the partition function transforms as

$$Z_{\mathbf{V}} = \langle \vec{\chi} | \mathbf{V}^\dagger \mathbf{N} \mathbf{V} | \vec{\chi} \rangle, \tag{B.2}$$

where \mathbf{S} and \mathbf{T} are the matrix representations of the standard S and T modular transformations for a specific conformal field theory (CFT). Thus, a partition function Z for a specific CFT is modular invariant iff \mathbf{N} commutes with \mathbf{S} and \mathbf{T} in the given CFT. CIZ proved that for a general $SU(2)_K$ algebra the commutant of \mathbf{S} and \mathbf{T} is generated by a set of linearly independent symmetric matrix operators, Ω_δ , labeled by δ , a divisor of $n \equiv K + 2$. Thus, for (B.1) to be a MIPF \mathbf{N} must be formed from the basis set of Ω_δ . In [23], CIZ showed that the additional requirements of (1) uniqueness of the vacuum ($N_{11} = 1$) and (2) absence of $\bar{\chi}_{\vec{l}} \chi_{\vec{l}}$ with coefficients $N_{\vec{l}} < 0$ constrain MIPFs for $SU(2)_K$ to the A-D-E classification. This is apparent from the limited possibilities for $\sum_\delta c_\delta \Omega_\delta$ with $c_\delta \in \mathbb{Z}$ that produce MIPF's satisfying both (1) and (2). (See Table B.1 below.)

Table B.1 A–D–E Classification in Terms of Ω_δ Basis Set

<u>$n \equiv \text{level} + 2$</u>	<u>Basis Elements For MIPF's</u>	<u>A–D–E Classification</u>
$n \geq 2$	Ω_n	(A_{n-1})
n even	$\Omega_n + \Omega_2$	$(D_{n/2+1})$
$n = 12$	$\Omega_{n=12} + \Omega_3 + \Omega_2$	(E_6)
$n = 18$	$\Omega_{n=18} + \Omega_3 + \Omega_2$	(E_7)
$n = 30$	$\Omega_{n=30} + \Omega_5 + \Omega_3 + \Omega_2$	(E_8)

In actuality, relaxing requirement (1) to $N_{11} \geq 1$ does not enlarge this solution set. Rather, the solutions in column two of Table B.1 are simply multiplied by an overall constant, $N_{11} = c_n$. Our proof proceeds along the lines of [23]:

Let $\alpha(\delta) = \text{GCF}(\delta, \bar{\delta} \equiv n/\delta)$ and $N = 2(K + 2)$. Next we define $p \equiv \bar{\delta}/\alpha$ and $p' \equiv \delta/\alpha$. Then we choose a pair of integers ρ, σ such that $\rho p - \sigma p' = 1$. From these we form $\omega(\delta) = \rho p + \sigma p' \pmod{N/\alpha^2}$, which leads to the following equations:⁴⁶

$$\begin{aligned}
\omega^2 - 1 &= 4\rho\sigma pp' = 0 \pmod{2N/\alpha^2}; \\
\omega + 1 &= 2\rho p \pmod{N/\alpha^2}; \\
\omega - 1 &= 2\sigma p' \pmod{N/\alpha^2}.
\end{aligned}
\tag{B.3}$$

An $N \times N$ -dimensional matrix Ω_δ operates on an enlarged set of characters χ_λ , with λ defined mod N . The “additional” characters carry indices in the range $-(K + 2) < \lambda \pmod{N} < 0$ and are trivially related to the customary $SU(2)_K$ characters, carrying positive indices \pmod{N} , by $\chi_\lambda = -\chi_{-\lambda}$ and $\chi_{\xi(K+2)} = 0$ for $\xi \in \mathbb{Z}$.⁴⁷ This means the overall sign coefficient in the q -expansion of (χ_λ) is positive

⁴⁶Note for future reference that interchanging the roles of δ and $\bar{\delta}$ in these equations amounts to replacing ω by $-\omega$.

⁴⁷The character χ_l , for $0 < l < K + 1$, is associated here with the primary field that is an l -dimensional representation of $SU(2)$. This notation differs from that used in the rest of the thesis. Outside of this appendix, the character corresponding to the primary field in the l -dimensional representation is denoted by χ_{l-1} . In particular, while here the character for the singlet representation is denoted by χ_1 , elsewhere it is denoted by χ_0 .

for $0 < \lambda \pmod{N} < n$ and negative for $n < \lambda \pmod{N} < N$. The components of a matrix, Ω_δ , are defined to be:

$$(\Omega_\delta)_{\lambda,\lambda'} = \begin{cases} 0, & \text{if } \alpha \not\equiv \lambda \text{ or } \alpha \not\equiv \lambda'; \\ \sum_{\xi \pmod{\alpha}} \delta_{\lambda', \omega\lambda + \xi N/\alpha}, & \text{otherwise.} \end{cases} \quad (\text{B.4})$$

Thus, Ω_n is the $N \times N$ -dimensional identity matrix.

A general MIPF for $SU(2)$ at level K can be written as

$$Z(\tau, \bar{\tau}) = \frac{1}{2} \sum_{\lambda, \lambda' \pmod{N}} \bar{\chi}_\lambda(\bar{\tau}) \left(\sum_{\delta} c_\delta \Omega_\delta \right)_{\lambda, \lambda'} \chi_{\lambda'}(\tau). \quad (\text{B.5})$$

We divide the integers $\lambda \neq 0 \pmod{N}$ into two disjoint sets U and L with $\lambda \in U$ if $1 \leq \lambda \leq n-1$ and $\lambda \in L$ if $n+1 \leq \lambda \leq 2n-1$. Therefore, $L \equiv -U \pmod{N}$ and we choose U as the fundamental domain over which λ is varied for χ_λ .

The matrices Ω_δ have the following properties between their elements:

$$(\Omega_\delta)_{\lambda,\lambda'} = (\Omega_\delta)_{-\lambda,-\lambda'} \quad (\text{B.6})$$

$$\begin{aligned} (\Omega_\delta \chi)_\lambda &= (\Omega_{n/\delta} \chi)_{-\lambda} \\ &= -(\Omega_{n/\delta} \chi)_\lambda. \end{aligned} \quad (\text{B.7})$$

We use these relationships to reexpress the partition function (B.5) as

$$Z = \sum_{\lambda \in U, \lambda' \pmod{N}} \bar{\chi}_\lambda(\bar{\tau}) \sum_{\delta|n} (c_\delta \Omega_\delta)_{\lambda, \lambda'} \chi_{\lambda'}(\tau) \quad (\text{B.8})$$

$$= \sum_{\lambda, \lambda' \in U} \bar{\chi}_\lambda(\bar{\tau}) \left\{ \sum_{\delta|n} c_\delta [(\Omega_\delta)_{\lambda, \lambda'} - (\Omega_\delta)_{\lambda, -\lambda'}] \chi_{\lambda'}(\tau) \right\} \quad (\text{B.9})$$

$$= \frac{1}{4} \sum_{\lambda, \lambda' \in U, L} \bar{\chi}_\lambda(\bar{\tau}) \left\{ \sum_{(\delta, \delta') \text{ pairs}, \delta|n} (c_\delta - c_{\delta'}) [(\Omega_\delta)_{\lambda, \lambda'} - (\Omega_{\delta'})_{\lambda, \lambda'}] \right\} \chi_{\lambda'}(\tau) \quad (\text{B.10})$$

with $c_\delta \geq 0$ and $c_\delta > 0$ implying $c_{\delta=n/\delta} = 0$ by convention. Two properties of these partition functions become apparent: (i) either Ω_n or Ω_1 contribute to the coefficient of the vacuum state $\bar{\chi}_1 \chi_1$ but not both, and (ii) the Ω_δ corresponding to $\delta^2 = n$ makes no net contribution to the partition function.⁴⁸

⁴⁸CIZ shows (ii) from a different approach.

The coefficient of $\bar{\chi}_1$ is (choosing $c_n \geq 1$ and, therefore, $c_1 = 0$)

$$c_n \chi_1 + \sum_{\delta \neq n, 1; \alpha(\delta)=1} c_\delta \chi_{\omega(\delta)} \quad (\text{B.11})$$

(since $(\Omega_\delta)_{\lambda, \lambda'} = 0$ unless $\alpha | \lambda'$). For $1 < \delta < n$, $\omega(\delta) \in (\mathbb{Z}/N\mathbb{Z})^*$ are all distinct from ± 1 .⁴⁹ Additionally, $\omega(\delta) = -\omega(\delta')$ implies $\delta' = n/\delta$ for $\alpha(\delta') = \alpha(\delta)$. This forces $\omega(\delta)$ to belong to U for $c_\delta > 0$ and $\alpha(\delta) = 1$. Otherwise, if $\omega(\delta) \in L$, then $c_\delta \chi_{\omega(\delta)} = -c_\delta \chi_{-\omega(\delta)}$ would contribute negatively to the coefficient of $\bar{\chi}_1$. This would require a positive $c_{\delta'} \chi_{\omega'(\delta')}$ contribution from a δ' , such that $\alpha(\delta') = 1$ and $\omega(\delta') = -\omega(\delta)$. But this implies that $\delta' = n/\delta$. Hence, c_δ and $c_{n/\delta}$ must both be positive definite, which is excluded.

Lemma 1: (i) $\alpha_{\min} = 1$ or 2, (ii) if $\alpha_{\min} = 2$, then the unique partition function (other than the diagonal A-type) is

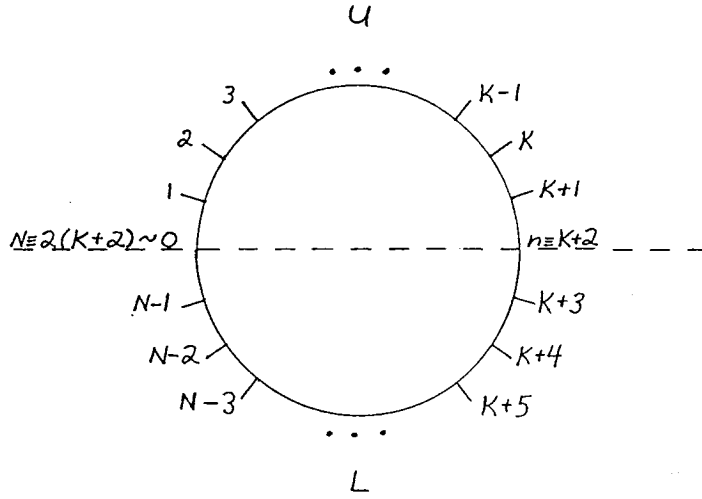
$$\Omega_n + \Omega_2 \text{ with } n = 0 \pmod{4}. \quad (\text{B.12})$$

Proof: α_{\min} is defined as the lowest $\alpha(\delta)$ of those $\delta \neq n, 1$ with $c_\delta > 0$. The coefficient of $\bar{\chi}_{\lambda=\alpha_{\min}}$ is

$$c_n \chi_{\lambda'=\alpha_{\min}} + \sum_{\delta \neq 1; \alpha(\delta)=\alpha_{\min}} \sum_{\xi=0}^{\alpha_{\min}-1} c_\delta \chi_{\lambda'=\omega(\delta)\alpha_{\min}+\xi N/\alpha(\delta)}. \quad (\text{B.13})$$

For $\alpha(\delta) > 1$ the $\lambda' = \omega(\delta)\lambda + \xi N/\alpha(\delta)$ of Ω_δ in eq. (B.8) correspond to vertices of an α -sided polygon. (Eq. (B.8) is the form of the partition function that we use from here on. These λ' can be viewed as points on a circle of radius $N/2\pi$, where one half of the circle corresponds to the region U and the opposite half to the region L . Any λ' in L must be compensated by a vertex point $-\lambda' \pmod{N}$ in U either (1) from the same polygon, (2) a different polygon, or (3) from the point corresponding to $\chi_{\lambda'=\lambda}$ from Ω_n .

⁴⁹Here $(\mathbb{Z}/N\mathbb{Z})^* \equiv$ integers \pmod{N} that are prime to N . We also define $U^* \equiv (\mathbb{Z}/N\mathbb{Z})^* \cap U$ and $L^* \equiv (\mathbb{Z}/N\mathbb{Z})^* \cap L$.

Figure B.1 The integers (mod N) mapped to a circle of radius $N/2\pi$ 

Independent of the value of $c_n \geq 1$, only the negative contribution to the coefficient of $\bar{\chi}_\lambda$ from at most one point, $\lambda' \in L$, on the α -gon of a generic Ω_δ can be compensated by the contribution from the single point $-\lambda' > 0 \pmod{N}$ from Ω_n . If $\lambda' < 0 \pmod{N}$ from Ω_δ must be cancelled by a $\lambda' > 0 \pmod{N}$ from Ω_n then $c_n \geq c_\delta$.

Any Ω_δ with $\alpha \geq 4$ must have at least one point of its related α -gon in L . The case of only one point in L corresponds to $\alpha(\delta) = 4$ and a set of indices

$$\lambda' = \omega\alpha + \xi N/\alpha \in 0, \frac{n}{2}, n, \frac{3n}{2} \pmod{N}. \quad (\text{B.14})$$

This set of indices only exists at $n = \alpha^2$, *i.e.*, $n = \delta^2$. Therefore, the corresponding $\Omega_\delta = \Omega_{\sqrt{n}}$ cannot contribute to the partition function. Consider then partition functions with $\alpha_{\min} \geq 4$. Based on the above, the coefficient of $\bar{\chi}_{\alpha_{\min}}$ in any of these models will contain at least two different negative $\chi_{\lambda' \in L}$ terms. Hence at least one of these terms must be cancelled by methods (1) or (2). First consider method (2). Assume $\Omega_{\delta'}$ can compensate a negative term from Ω_δ . This requires that

$$\omega'(\delta')\alpha_{\min} \equiv -\omega(\delta)\alpha_{\min} \pmod{N/\alpha_{\min}}. \quad (\text{B.15})$$

Equivalently, $\omega'(\delta') \equiv -\omega(\delta) \pmod{N/\alpha_{\min}^2}$, which again implies $\delta\delta' = n$, in contradiction to $c_{n/\delta} = 0$ if $c_\delta > 0$. Hence method (2) cannot be used. Similarly, cancelling

the negative terms from Ω_δ with positive ones from the same α -gon of Ω_δ implies that $2\omega \equiv 0 \pmod{N/\alpha_{\min}}$, which again is only possible for $n = \alpha_{\min}^2 = \alpha^2(\delta)$. Thus, neither methods (1) nor (2) can be used in the cancellation of negative terms in partition functions. Thus, MIPFs with $\alpha_{\min} \geq 4$ cannot exist for any value of c_n .

We now consider potential MIPFs with $\alpha_{\min} = 3$. In this case it is possible for just one point, λ' , from a Ω_δ to be in L . This can be cancelled by a χ_3 term from Ω_n . However, then $\omega \equiv -1 \pmod{N/9}$ and the terms from Ω_δ would be $\chi_{-3} + \chi_{-3+N/3} + \chi_{-3-N/3}$, where $\pm N/3 - 3 \in U \cup \{0, n\}$. This implies $n \leq 9$ while $9|n$. Hence, we have another case of $n = \alpha_{\min}^2$. Therefore, MIPFs cannot have $\alpha_{\min} > 2$, independent of the coefficient c_n . The 2-gon (line) resulting from a Ω_δ with $\alpha = 2$ has as its vertices the two points 2ω and $2\omega + n$. The coefficient of χ_2 for a model with $\alpha_{\min} = 2$ is

$$\chi_2 + \sum_{\alpha(\delta)=2} c_\delta (\chi_{\lambda'=2\omega} + \chi_{\lambda'=2\omega+n}). \quad (\text{B.16})$$

Three sets of values are possible for $\lambda' = 2\omega, 2\omega + n$. One corresponds to the excluded case $n = \alpha_{\min}^2$, the second to another excluded by contradiction (requiring both $2\omega = 0 \pmod{n}$ and $\omega^2 = 1 \pmod{n}$). The remaining case corresponds to a unique δ with $\alpha(\delta) = 2$. Choosing $\omega \pmod{N/4}$ $2\omega = -2 \pmod{N}$, $2\omega + n \in U$, we find the negative term, $\chi_{2\omega}$, is compensated by χ_2 from Ω_n . This requires $n \equiv 4m \in 4\mathbb{Z}$, and $\omega(\delta) \equiv -1 \pmod{2m}$, which is only satisfied by $\delta = 2$. Since there is one-to-one cancellation between χ_2 and $\chi_{2\omega}$, $c_n > c_2$ is also mandatory. If $c_n = c_2$ and all other $c_\delta = 0$ for even n , then the resulting partition function is simply an integer multiple of the D-type solution. Further, $c_n > c_2$, with all other $c_\delta = 0$, can be expressed as $(c_n - c_2)Z(A) + (c_2)Z(D)$. Thus, for $\alpha_{\min} = 2$, the freedom to have $c_n \geq 1$ for even n only increases the solution set of MIPFs beyond the A-D-E class if and only if (iff) additional Ω_δ 's (not in E class for $n = 12, 18, 30$) with $\alpha(\delta) \geq 3$ can be included with $c_n(\Omega_n + \Omega_2)$. We now show that is not possible to include such terms and keep the partition function positive.

For $\lambda \in U$, $(\Omega_n + \Omega_2)\chi_\lambda$ equals χ_λ if λ is odd and $\chi_{n-\lambda}$ if λ is even. That no other Ω_δ 's are allowed in MIPFs with $\alpha_{\min} = 2$ is evident by repetition of prior arguments after replacing χ_λ with $\chi_{n-\lambda}$ if λ is even. The only remaining candidate for giving

non-zero c_δ are those Ω_δ corresponding to α -gons with exactly 2 vertices in L . This limits consideration to Ω_δ with $\alpha(\delta) = 3, 4, 5, 6$. The negative contributions from any such Ω_δ must be cancelled by positive from Ω_n and $(\Omega_n + \Omega_2)$. First we consider the coefficients of $\bar{\chi}_3$ ($\bar{\chi}_5$) coming from a Ω_δ with $\alpha(\delta) = 3$ (5), respectively. In these cases $(\Omega_n + \Omega_2)$ acts identically to Ω_n since λ is odd. The prior arguments that eliminated partition functions with $\alpha_{\min} = 3, 5$ likewise exclude any Ω_δ with $\alpha(\delta) = 3, 5$. The two $\delta' \in L$ for a Ω_δ with $\alpha(\delta) = 4$ that contribute negative coefficients $\chi_{\delta' \in L}$ to $\bar{\chi}_4$ can be cancelled by a combination of Ω_n and $(\Omega_n + \Omega_2)$ iff $\delta' = N - 4, n + 4$. However, this only occurs for $N = 32$, *i.e.*, $n = \alpha^2$. By similar argument the last possibility, which is including a Ω_δ with $\alpha(\delta) = 6$, requires $N = 36$, which by factorization of n only allows $\alpha = 1, 3$. Thus no additional $\Omega_{\delta \neq n, 2}$ are allowed for $\alpha_{\min} = 2$ even when $c_n > 1$. So *lemma 1* of CIZ is independent of the restriction of $c_n = 1$.

Lemma 2: If n is odd, then the unique possibility is Ω_n .

Proof: We show this by contradiction. Assume, contrary to *lemma 2*, that MIPFs can be formed from additional combinations of Ω_δ 's. Recall that *lemma 1* requires that $\alpha_{\min} = 1$ for odd n and consider specifically the coefficient of $\bar{\chi}_{2^\gamma}$ for $2^\gamma < n$. Since an off n limits the Ω_δ 's, that contribute terms to the coefficient of $\bar{\chi}_{2^\gamma}$, to those with $\alpha(\delta) = 1$, this coefficient is:

$$c_n \chi_{2^\gamma} + \sum_{\alpha(\delta)=1} c_\delta \chi_{\omega(\delta)2^\gamma} . \quad (\text{B.17})$$

By prior argument all of these $\omega(\delta) \in U$. That is, $(0 < \omega < n)$. Consider the case of $2^\gamma \omega \in L$. The resulting negative contribution to the coefficient would require an additional $\omega'(\delta')$ from some $\Omega_{\delta'}$ (including the possibility $\omega' = 1$) such that $2^\gamma(\omega + \omega') \equiv 0 \pmod{N}$. For $\gamma = 1$, $2(\omega + \omega') \equiv 0 \pmod{N}$. However, $\omega + \omega' < N$ implies $2(\omega + \omega') < 2N$. This leads to $2(\omega + \omega') = N$, *i.e.* $\omega + \omega' = n$; but ω and ω' are both odd by $(\omega^{(\prime)})^2 \equiv 1 \pmod{2N}$ (from $\omega^2 - 1 \equiv 0 \pmod{2N/\alpha^2}$). Thus $\omega + \omega'$ is even while n is by assumption odd. To resolve this potential contradiction requires $2\omega \in U$ and $\omega < n/2$. This argument can be iterated to prove that any γ , with the property that $2^\gamma \in U$, requires $2^\gamma \omega \in U$ with $0 < \omega < n/2^\gamma$.

Now consider γ_{\max} defined by $2^{\gamma_{\max}} < n < 2^{\gamma_{\max}+1}$. Based on above arguments, we require that $0 < \omega < n/2^{\gamma_{\max}}$. From the defining eq. of $\omega(\delta)$ we can show that $\omega(\delta)\bar{\delta} \equiv \bar{\delta} \pmod{N}$ for $\bar{\delta} = n/\delta$. Since n has been chosen odd, $\delta > 2$. So $\bar{\delta} < 2^{\gamma_{\max}} \frac{1}{\delta/2} < 2^{\gamma_{\max}}$ while $\omega\bar{\delta} < \frac{n}{2^{\gamma}} 2^{\gamma_{\max}}$. Thus $\omega\bar{\delta} \equiv \bar{\delta} \pmod{N}$ implies $\omega(\delta) = 1$, in contradiction to the assumption that $1 < \delta < n$ (for which $|\omega(\delta)| > 1$). The only solution is that $c_{\delta \neq n} = 0$ in eq. (B.17). Thus, *lemma 2* is also true independent of whether $c_n = 1$.

Since *lemmas 1* and *2* are still valid for $c_n > 1$, the only remaining possibility for new types of MIPFs corresponds to n even and $\alpha_{\min} = 1$ (the latter implying $\omega^2 \equiv 1 \pmod{2N}$ and $\omega \in U$). Henceforth we assume these values for n and α_{\min} and consider the last *lemma* of CIZ.

Lemma 3: For n even, $n \neq 12, 30$, $\omega \in U^*$, $\omega^2 = 1 \pmod{2N}$, $\omega \neq 1$, and $\omega \neq n-1$ if $n = 2 \pmod{4}$, there exists $\lambda \in U^*$ such that $\omega\lambda \in L^*$.

The $\{\delta, \bar{\delta}\}$ pairs excluded from the claims of *lemma 3* by the conditions imposed within it are $\{n, 1\}$ for any even n since $w(n) = 1$, $w(1) = -1 \notin U^*$; $\{2, n/2\}$ for $n = 2 \pmod{4}$ where $\alpha = 1$ and $\omega(2) = n-1$; $\{3, 4\}$ for $n = 12$; $\{2, 15\}$, $\{3, 10\}$, $\{5, 6\}$ for $n = 30$. As *lemma 3* in no way involves the coefficients c_δ , generalizing $c_n = 1$ to $c_n \geq 1$ clearly does not invalidate it. So we assume *lemma 3* and show that the conclusions based upon it do not alter if $c_n > 1$.

Consider the coefficient of $\bar{\chi}_{\lambda \in U^*}$ (for n odd and $\alpha_{\min} = 1$). Only Ω_δ with $\alpha(\delta) = 1$ can contribute and the $\omega(\delta)$ corresponding to $c_\delta \neq 0$ must have the property that $\omega\lambda \in U$, for all $\lambda \in U$. This can be shown by contradiction: Assume instead that $\omega\lambda \in L^*$. Cancellation of this term requires another $\omega'(\delta')$ such that $\omega'\lambda \in U^*$ and $(\omega' + \omega)\lambda \equiv 0 \pmod{N}$, which requires that $\omega' + \omega \equiv 0 \pmod{N}$, since λ is invertible \pmod{N} . However, $\omega' \equiv -\omega \pmod{N}$ implies $\delta' = n/\delta$ in contradiction to $c_\delta > 0$ implying $c_{n/\delta} = 0$.

Now apply *lemma 3*: For $n \equiv 0 \pmod{4}$ and $n \neq 12$, all Ω_δ , with $\delta \neq n$ are $\alpha(\delta) = 1$, are excluded by this *lemma* since no ω exists that always giving $\omega\lambda \in U$

for all λ . For $n \equiv 2 \pmod{4}$, if $n \neq 30$, the only possible allowed Ω_δ with $\alpha(\delta) = 1$ is Ω_2 ($\omega = n - 1$). From prior arguments c_2 must be less than c_n . The matrix, \mathbf{N} , for MIPFs with even n , $\alpha_{\min} = 1$ and $c_n, c_2 \neq 0$ can be expressed as:

$$\mathbf{N} = c_n^A \Omega_n + c_n^D (\Omega_n + \Omega_2) + [c_n^E (\Omega_n + \Omega_2) + \sum_{\substack{\text{odd } \alpha(\delta) \geq 3 \\ \delta \neq n, 2}} c_{\delta \neq n, 2} \Omega_\delta]. \quad (\text{B.18})$$

The first term on the RHS is just a multiple of the (diagonal) A-type and the second term a multiple of the D-type. In (B.18), all $c_{\delta \neq n, 2}$ for $n \neq 12, 18, 30$ must vanish. The arguments demanding this parallel those for *lemma 1*. Let $\hat{\alpha}_{\min}$ be the smallest of odd $\alpha(\delta) \geq 3$ associated with $c_\delta > 0$ and consider the coefficients of $\bar{\chi}_{\hat{\alpha}_{\min}}$.⁵⁰ $c_n^E (\Omega_n + \Omega_2)_{\lambda, \lambda'} \chi_{\lambda'}$ contributes the coefficients $c_n^D (\chi_{\hat{\alpha}_{\min}} + \chi_{n - \hat{\alpha}_{\min}})$ while $c^A \Omega_n$ contributes additional $c^A \chi_{\hat{\alpha}_{\min}}$.

Any $\lambda' \in L$ from an Ω_δ with $c_\delta > 0$ and $\alpha(\delta) = \hat{\alpha}_{\min}$ can only be compensated by the positive $\lambda' = \hat{\alpha}_{\min}, n - \hat{\alpha}_{\min}$ coming from $\Omega_{\delta=n}$ or $\Omega_{\delta=2}$. Odd α -gons with just one or two vertices in L are limited to $\alpha = 3, 5$. In the event of two vertices in L , $\hat{\alpha}_{\min} = 3$ requires $\frac{2n}{3} = (n + \hat{\alpha}_{\min}) - \frac{n}{2}$, *i.e.*, $n = 18$. Thus $\delta = 3$ and $c_{\delta=3} \leq c^E$. $c_3 = c^E$ for $n = 18$ forms the standard E_7 invariant.⁵¹ By the same logic, $\hat{\alpha}_{\min} = 5$ requires $\frac{4n}{5} = (n - \hat{\alpha}_{\min}) - \frac{n}{2}$, *i.e.* $n = \frac{50}{3}$, which is not allowed since $n \in \mathbb{Z}$). The last possibility, that of a 3-gon having only one vertex in L , that is compensated by either $\lambda' = 3$ from Ω_n or $\lambda' = n - 3$ from Ω_2 was shown not possible in discussion of *lemma 1*.

The only remaining cases not covered by any of the *lemmas* are $n = 12, 30$ with $\alpha_{\min} = 1$. It is straightforward to show that in these cases too, $c_n > 1$ does not lead to new MIPFs, only to multiples of A, D, or E classes. Therefore, we conclude that relaxing the condition of uniqueness of the vacuum does not enlarge the solution space of MIPFs beyond the A-D-E classification of Cappelli, Itzykson, and Zuber. Whether this rule can be applied to MIPFs of other Kač-Moody algebras, we do not know.

⁵⁰We consider only odd α since $4 \nmid (n \equiv 2 \pmod{4})$.

⁵¹We let $c_3 = c^E$ since $c^E - c_3$ can be redefined as a contribution to c^D .

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