Chapter 6

Conclusions

In this thesis, we developed discrete Routh reduction, discrete exterior calculus, discrete connections on principal bundles, and generalized variational integrators, which can be classified into two categories, discrete geometry, and discrete mechanics, which form the basis for computational geometric mechanics.

Computational Geometric Mechanics	
Discrete Geometry	**Discrete Mechanics**
Discrete Exterior Calculus (DEC)	Discrete Routh Reduction (DRR)
Discrete Connections on Principal Bundles (DCPB)	Generalized Variational Integrators (GVI)

The new machinery that has been developed will aid in the systematic development of computational algorithms that are motivated by the techniques of geometric mechanics. Some of the links between the material in the various chapters are summarized below.

DRR and DEC. The curvature term in the discrete Routh equations can be though of as arising from the discrete exterior derivative applied to the connection 1-form, in the case whereby the spatial discretization goes to the continuum limit.

DRR and DCPB. Discrete connections provide the separation of the space $Q \times Q$ into horizontal (shape) and vertical (group) components, thereby providing the coordinates necessary to realize a discrete theory of reduction.

DRR and GVI. Generalized variational integrators provide a framework for the construction of G-invariant discrete Lagrangians, using G-equivariant natural charts, which are necessary to apply discrete reduction theory.

DEC and DCPB. Discrete connections and discrete exterior calculus provide the necessary tools to make sense of the discrete Levi-Civita connection, and to realize its curvature as a $\mathrm{SO}(n)$-valued discrete dual 2-form, that arises from the exterior derivative of a discrete connection expressed as a discrete dual 1-form.

DEC and GVI. Discrete exterior calculus provides a means of discretizing the action integral, and it can be combined with multiscale or spectral discretizations in time to yield hybrid variational schemes that capture the spatial geometry.

DCPB and GVI. Lie group variational integrators could provide an efficient method of constructing discrete connections that approximate continuous connections to a prescribed degree of accuracy.

Unifying Application. The primary motivation for developing accurate simulations is to provide numerical results that are reliable, and minimizes the use of arbitrary parameters in the simulation in order to obtain a correspondence with reality. In providing a link between physical models across scales, one is in a position to accurately predict large-scale and long-term behavior, and in so doing close the simulation and design cycle.

The next stage of evolution for numerical computation is not simply to predict what happens given a set of initial conditions, but rather to enable simulation driven design through the use of adjoint sensitivity techniques. This can be applied to an optimal design problem in computational electromagnetism, which is to modify the shape of an aircraft wing by homotopy methods so as to minimize its radar cross section. Computational electromagnetism is an application area wherein a large number of the techniques I have been working on converge.

Reduction provides a general framework to remove the gauge symmetries in Maxwell's equations, and discrete exterior calculus discretizes the equations in a geometrically exact fashion. The variational framework of discrete mechanics provides a means of deriving numerical schemes that have good structure-preserving properties, and adaptive mesh movement allow for the resolution of shocks while minimizing computational cost. And incorporating multiscale and numerical homogenization techniques provide the bridge to simulating the effect of complex hybrid materials on the far field scattered wave. More generally, reduction, adaptivity, and multiscale methods increase the efficiency of computations, while discrete exterior calculus and discrete mechanics increase the accuracy. Only by having accurate and efficient numerical methods can one hope to realize the promise of simulation driven design.