Chapter 4

LCS and the Perturbed Pendulum

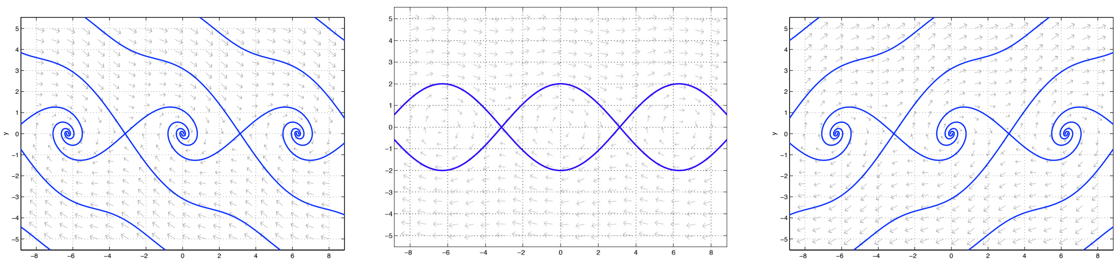
We are now at a point where we can introduce the use of the FTLE-LCS method for identifying homoclinic tangles in time-dependent flows. Our main goal will be to identify these tangle structures in atmospheric and oceanic flows, but as a first example, it is instructive to apply the method to the case of the simple pendulum.

Consider the flow generated by the coupled differential equations

$$\begin{aligned} \dot{\theta}(t) &= \omega(t)\,, \\ \dot{\omega}(t) &= -\sin t - 0.65\omega(t)\sin \pi t\,. \end{aligned} \tag{4.1}$$

This flow is identical to the simple pendulum investigated in Section 3.2, but now an extra periodic forcing term has been added. We shall see that chaotic trajectories emerge from the existence of homoclinic points and a homoclinic tangle. However, our approach to obtain this understanding will be from a different perspective than that of Poincaré.

To appreciate the effect of the periodic forcing, it is instructive to first view an animation of the velocity field and then ask the questions: "Where do trajectories go now?", "How are portions of phase space transported?", "What happens to the separatrix between windmill and oscillating motions?". In short, the time-dependent velocity field provides very little intuition in answering these questions. Furthermore, plots of instantaneous streamlines in a snapshot of the velocity field are misleading in determining the important Lagrangian structures and transport mechanisms (Figure



(a) (b) (c)

Figure 4.1: Snapshots of streamlines plotted in the time-dependent velocity field of the perturbed pendulum do not reveal the underlying transport mechanisms.

4.1).

Since the perturbation is periodic, we can follow the insight of Poincaré by considering t as a periodic variable with period 2π, and introducing a Poincaré section at $t = 0$. As described in Chapter 2, this reduces the flow to a discrete map on the two-dimensional section. Plotting iterates of this map produces the static plot in Figure 4.2 in which we recognize *resonance islands* that persist under the perturbation, and the *chaotic sea* corresponding to chaotic trajectories.

When we compute the LCS for this perturbed pendulum, we use a different approach that elucidates the time-dependent transport mechanism in the flow. Multiple intersections of the repelling and attracting LCS reveal a *homoclinic tangle*, and the time-dependence of the LCS reveals the continuous motion of the lobes defined by these intersections. Figures 4.3(a) through 4.3(d) show how lobes defined by the intersection of the repelling and attracting LCS delineate precisely the areas of phase space transported from one region to another via the mechanism of *lobe dynamics*: the green lobes are entrained into the region of oscillatory motion while yellow lobes are detrained. The action of lobe dynamics has previously been observed in laboratory flows [Shadden 2007] and in bio-propulsion flows [Franco 2007].

If we replace the periodic sinusoidal forcing in Equation (4.1) with a chaotic aperiodic forcing, the Poincaré section analysis can no longer be applied – there is no sensible definition for the Poincaré section. However, we can still proceed to compute the time-dependent LCS to understand the effect of the aperiodic perturbation

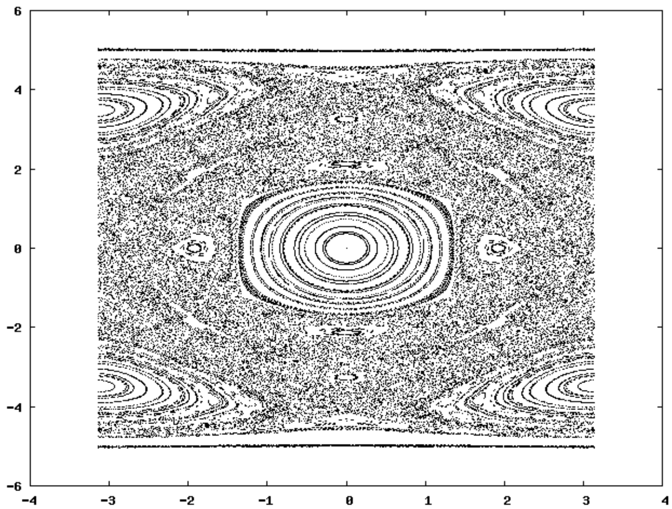


Figure 4.2: The Poincaré section for the periodically-forced pendulum reveals resonance islands and the chaotic sea.

on transport. Figure 4.4 shows the lobe structures present when the pendulum is perturbed with forcing obtained by coupling the pendulum to chaotic trajectories in the Lorenz equations. The lobes are now irregularly spaced in accordance with the irregularity of the time-dependence in the forcing, but they nevertheless persist as the important structures that indicate the passages to transport.

In this way, the FTLE-LCS method provides a technique for revealing homoclinic trajectories in aperiodic flows. The identification of lobes, and their subsequent entrainment and detrainment, will be a repeated theme in the study of geophysical flows in the remaining chapters.

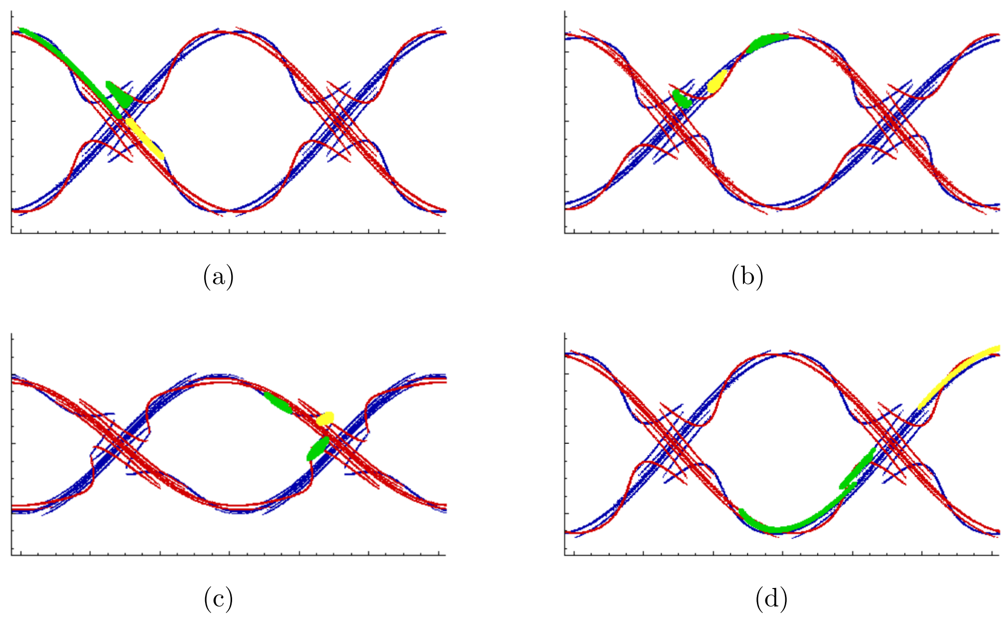


Figure 4.3: LCS analysis of the perturbed pendulum reveals that transport occurs through the action of *lobe dynamics*. Panes (c) through (f) illustrate the entrainment and detrainment of lobes: green lobes are entrained, yellow lobes are detrained.

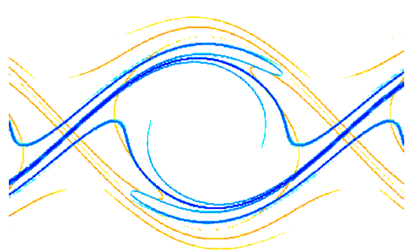


Figure 4.4: The LCS method can be applied to a chaotically forced aperiodic flow, and reveals the aperiodic lobe structures that govern transport. Here, the pendulum is chaotically forced by coupling to the Lorenz attractor.