Appendix A

Dispersion Relations for Metal-Insulator-Metal Waveguides

In this appendix, we analyze the thin film dispersion relations for a metal-insulator-metal waveguide.\(^1\) Both the transverse magnetic and transverse electric conditions will be considered. The geometry used for this derivation is shown in Figure A.1.

![Figure A.1. The coordinate system used for the following three-layer derivation.](image)

To begin, we first assume the form of the electric and magnetic fields given by:

\[
\begin{align*}
\vec{E} &= \vec{E}(z)e^{i(k_x x - \omega t)} \\
\vec{B} &= \vec{B}(z)e^{i(k_x x - \omega t)}
\end{align*}
\]

(A.1a)\hspace{1cm} (A.1b)

where we assume there is no y-dependence in either field.

For this derivation, the curl can be written in its full form for the any vector \(\vec{U}\) as:

\[
\nabla \times \vec{U} = \left[ \frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z} \right] \hat{x} + \left[ \frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x} \right] \hat{y} + \left[ \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y} \right] \hat{z}
\]

(A.2)

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\(^1\)This appendix is based on texts by Professor Heinz Raether [97], Professor Stefan Maier [71], notes and discussions with Jennifer Dionne.
for the curl components of $\vec{U}$ in the $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions respectively.

We then plug the general form of the electric and magnetic fields into Maxwell’s Equations. In the absence of space charge and currents, we have:

$$\nabla \cdot \vec{E} = 0 \tag{A.3a}$$
$$\nabla \cdot \vec{B} = 0 \tag{A.3b}$$
$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \tag{A.3c}$$
$$\nabla \times \vec{B} = \frac{1}{c \varepsilon_i(\omega)} \frac{\partial \vec{E}}{\partial t} \tag{A.3d}$$

We see that plugging (A.1a) and (A.1b) into (A.2) yields two sets of equations for either $\nabla \times \vec{E}$ or $\nabla \times \vec{B}$ which are given by:

$$\nabla \times \vec{E} :$$
$$\hat{x} : -\frac{\partial E_y}{\partial z} = \frac{i\omega}{c} B_x \tag{A.4a}$$
$$\hat{y} : -\frac{\partial E_x}{\partial z} + ik_x E_z = \frac{i\omega}{c} B_y \tag{A.4b}$$
$$\hat{z} : i k_x E_y = \frac{i\omega}{c} B_z \tag{A.4c}$$

$$\nabla \times \vec{B} :$$
$$\hat{x} : -\frac{\partial B_y}{\partial z} = -\frac{i\omega}{c} \varepsilon_i E_x \tag{A.5a}$$
$$\hat{y} : \frac{\partial B_x}{\partial z} = -ik_x B_z = -\frac{i\omega}{c} \varepsilon_i E_y \tag{A.5b}$$
$$\hat{z} : -i k_x B_y = -\frac{i\omega}{c} \varepsilon_i E_z \tag{A.5c}$$

Initially we solve for generally for each component of $\vec{E}$ and $\vec{B}$. Also note that the sets of solutions either consist of $(E_y, B_x, B_z)$ or $(E_x, E_z, B_y)$. To simplify the general solution to solve for the set of transverse-magnetic (TM) modes, we can set $E_y = 0 \rightarrow B_x = B_z = 0$. To simplify the general solution to solve for the set of transverse-electric (TE) modes, we can set $B_y = 0 \rightarrow E_x = E_z = 0$. 

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A.1 The General Solution

We begin by solving for $E_z$. This is done by combining (A.4b) and (A.5c):

$$\frac{\partial E_x}{\partial z} = -ik_x E_z = \frac{i\omega}{c} \left( -\frac{\omega}{ck_x} \right) \epsilon_i E_z$$

(A.6)

with (A.5a) and (A.5c):

$$-\frac{\partial \left( \frac{\omega}{ck_x} \cdot \epsilon_i E_z \right)}{\partial z} = \frac{i\omega}{c} \epsilon_i E_x$$

(A.7a)

$$\Rightarrow -\frac{\partial E_z}{\partial z} = ik_x E_x$$

(A.7b)

which yields:

$$E_x = \frac{1}{k_x} \frac{\partial E_z}{\partial z}$$

(A.8)

$$\frac{1}{k_x} \frac{\partial^2 E_z}{\partial z'^2} - ik_x E_z = -i \frac{\omega}{c^2} \epsilon_i E_z$$

(A.9)

$$\frac{\partial^2 E_z}{\partial z'^2} - k_x^2 E_z = -\left( \frac{\omega}{c} \right)^2 \epsilon_i E_z$$

(A.10)

$$\frac{\partial^2 E_z}{\partial z'^2} - \left( k_x^2 - \left( \frac{\omega}{c} \right)^2 \epsilon_i \right) E_z = 0$$

(A.11)

Here we use the definition: $k_{zi}^2 \equiv k_x^2 - \left( \frac{\omega}{c} \right)^2 \epsilon_i$ to yield:

$$\frac{\partial^2 E_z}{\partial z'^2} - k_{zi}^2 E_z = 0 \rightarrow \lambda^2 - k_{zi}^2 = 0 \rightarrow \lambda = \pm k_{zi}$$

(A.12)

$$E_z = Ae^{k_{zi}z} \pm Be^{-k_{zi}z}$$

(A.13)

Where the $A = B$ condition yields the even (symmetric) solutions and the $A = -B$ conditions yields the odd (anti-symmetric) solutions. To solve for $E_x$, we combine this result with that of (A.7b) to get:

$$\frac{\partial E_z}{\partial z} = -ik_x E_x = Ak_{zi}e^{k_{zi}z} \mp Bk_{zi}e^{-k_{zi}z}$$

(A.14)

$$E_x = \frac{k_{zi}}{ik_x} \left( Ae^{k_{zi}z} \mp Be^{-k_{zi}z} \right)$$

(A.15)
Finally, to solve for $E_y$, we combine (A.4a), (A.5b), and (A.4c) to obtain:

$$B_x = -\frac{c}{i\omega} \frac{\partial E_y}{\partial z}$$  \hspace{1cm} (A.16)

$$\frac{\partial B_x}{\partial z} - ik_x B_z = -\frac{i\omega}{c} \epsilon_i E_y$$  \hspace{1cm} (A.17)

$$\frac{\partial}{\partial z} \left( -\frac{c}{i\omega} \frac{\partial E_y}{\partial z} \right) - ik_x \left( \frac{ck_x}{\omega} \right) E_y = -\frac{i\omega}{c} \epsilon_i E_y$$  \hspace{1cm} (A.18)

$$-\frac{c}{i\omega} \frac{\partial^2 E_y}{\partial z^2} + \frac{c}{i\omega} \cdot k_x^2 E_y + \frac{i\omega}{c} \epsilon_i E_y = 0$$  \hspace{1cm} (A.19)

$$\frac{\partial^2 E_y}{\partial z^2} - \left( k_x^2 + \left( \frac{i\omega}{c} \right)^2 \epsilon_i \right) E_y = 0$$  \hspace{1cm} (A.20)

$$\frac{\partial^2 E_y}{\partial z^2} - \left( k_x^2 - \left( \frac{\omega}{c} \right)^2 \epsilon_i \right) E_y = 0$$  \hspace{1cm} (A.21)

Here we use the definition $\beta^2_i \equiv -k_x^2 + \left( \frac{\omega}{c} \right)^2 \epsilon_i \rightarrow \beta_i \equiv ik_{zi}$ to obtain:

$$\frac{\partial^2 E_y}{\partial z^2} + \beta^2_i E_y = 0$$  \hspace{1cm} (A.22)

$$E_y = Ce^{i\beta_i z} \pm De^{-i\beta_i z} = Ce^{-k_z z} \pm De^{k_z z}$$  \hspace{1cm} (A.23)

To obtain the $z$-component of $\vec{B}$, we use the fact that $ik_x E_y = \frac{i\omega}{c} B_z$ from (A.4c). It then follows that:

$$B_z = \frac{ck_x}{\omega} \left( Ce^{i\beta_i z} \pm De^{-i\beta_i z} \right) = \frac{ck_x}{\omega} \left( Ce^{-k_z z} \pm De^{k_z z} \right)$$  \hspace{1cm} (A.24)

and from (A.4a) it follows that:

$$-\frac{\partial E_y}{\partial z} = -C\beta_i e^{i\beta_i z} \pm D\beta_i e^{-i\beta_i z} = \frac{i\omega}{c} \cdot B_x$$  \hspace{1cm} (A.25)

$$B_x = \frac{c}{\omega} \beta_i \left( -C e^{i\beta_i z} \mp D e^{-i\beta_i z} \right)$$  \hspace{1cm} (A.26)

Finally, using the fact that $ik_x B_y = -\frac{i\omega}{c} \cdot \epsilon_i E_z$ from (A.5c) we obtain:

$$B_y = -\frac{\omega}{c} \left( \frac{1}{k_x} \right) \epsilon_i \left( A e^{k_z z} \mp B e^{-k_z z} \right)$$  \hspace{1cm} (A.27)

Now that we have each component of $\vec{E}$ and $\vec{B}$, we introduce the boundary conditions necessary
for the metal-insulator-metal waveguide structure. We know that outside the waveguide, both \( \vec{E} \) and \( \vec{B} \) must decay to 0 as \( z \to \infty \); however, within the waveguide, no such restrictions exist. For the general solution, we have \( A = B \) for the even (symmetric) solutions and \( A = -B \) for the odd (anti-symmetric) solutions. Combining these assumptions we get two sets of equations for the general solution for waves either inside, or outside of the layered, waveguide structure.

Inside the waveguide:

\[
\begin{align*}
E_x &= -\frac{k_{z1}}{ik_x} \left( e^{k_{z1}z} \mp e^{-k_{z1}z} \right) \\
E_y &= e^{-k_{z1}z} \pm e^{k_{z1}z} \\
E_z &= e^{k_{z1}z} \pm e^{-k_{z1}z} \\
B_x &= \frac{ic}{\omega} k_{z1} \left( -e^{-k_{z1}z} \pm e^{k_{z1}z} \right) \\
B_y &= -\frac{\omega}{c} \left( \frac{1}{k_x} \right) \epsilon_l \left( e^{k_{z1}z} \mp e^{-k_{z1}z} \right) \\
B_z &= \frac{ck_x}{\omega} \left( e^{-k_{z1}z} \pm e^{k_{z1}z} \right)
\end{align*}
\]  

(A.28a) \hspace{1cm} (A.28b) \hspace{1cm} (A.28c) \hspace{1cm} (A.28d) \hspace{1cm} (A.28e) \hspace{1cm} (A.28f)

Outside the waveguide:

\[
\begin{align*}
E_x &= -\frac{k_{z2}}{ik_x} \left( \mp \beta e^{-k_{z2}z} \right) \\
E_y &= \beta e^{-k_{z2}z} \\
E_z &= \pm \beta e^{-k_{z2}z} \\
B_x &= -\frac{ic}{\omega} k_{z2} \beta e^{-k_{z2}z} \\
B_y &= -\frac{\omega}{c} \left( \frac{1}{k_x} \right) \epsilon_l \left( \mp \beta e^{-k_{z2}z} \right) \\
B_z &= \frac{ck_x}{\omega} \beta e^{-k_{z2}z}
\end{align*}
\]  

(A.29a) \hspace{1cm} (A.29b) \hspace{1cm} (A.29c) \hspace{1cm} (A.29d) \hspace{1cm} (A.29e) \hspace{1cm} (A.29f)

A.2 Boundary Conditions

For all solutions, we assume that:

- \( E_x \) and \( D_z \) are continuous at \( z = \pm \frac{d}{2} \)
• \( E_y \) is continuous at \( z = \pm \frac{d}{2} \)

• \( B_z \) is continuous at \( z = \pm \frac{d}{2} \)

• \( \frac{1}{\mu} B_x \) is continuous at \( z = \pm \frac{d}{2} \)

• \( \frac{1}{\mu} B_y \) is continuous at \( z = \pm \frac{d}{2} \)

### A.2.1 \( E_x \) and \( D_z \) are continuous:

From (A.28a) and (A.29a) we have:

\[
\frac{i k_{z1}}{k_x} (e^{k_{z1} z} \mp e^{-k_{z1} z}) = \pm \frac{k_{z2}}{ik_x} (Be^{-k_{z2} z})
\]

(A.30)

From (A.28c) and (A.29c) we have:

\[
\epsilon_1 (e^{k_{z1} z} \pm e^{-k_{z1} z}) = \pm \epsilon_2 (Be^{-k_{z2} z})
\]

(A.31)

and combining the two we derive:

\[
\frac{i k_{z1}}{k_x} (e^{k_{z1} z} \mp e^{-k_{z1} z}) = \frac{k_{z2}}{ik_x} \left( \frac{\epsilon_1}{\epsilon_2} \right) (e^{k_{z1} z} \mp e^{-k_{z1} z})
\]

(A.32)

\[- \epsilon_2 k_{z1} (e^{k_{z1} z} \mp e^{-k_{z1} z}) = \epsilon_1 k_{z2} \left( e^{k_{z1} z} \pm e^{-k_{z1} z} \right)
\]

(A.33)

\[- \epsilon_2 k_{z1} = \epsilon_1 k_{z2} \left\{ \begin{array}{c}
\coth(k_{z1} d/2) \\
\tanh(k_{z1} d/2)
\end{array} \right\}
\]

(A.34)

which yields the transverse-magnetic dispersion relation:

\[
\epsilon_1 k_{z2} + \epsilon_2 k_{z1} \left\{ \begin{array}{c}
\coth(k_{z1} d/2) \\
\tanh(k_{z1} d/2)
\end{array} \right\} = 0
\]

(A.35)

Here, the “coth” function represents the symmetric plasmon modes and the “tanh” function represents the antisymmetric modes.
A.2.2 \( E_y \) is continuous:

From (A.28b) and (A.29b) we have:

\[
e^{-k z_1 z} + e^{k z_1 z} = Ce^{-k z_2 z}
\]

(A.36)

which yields:

\[
C = e^{k z_2 d/2} \left( e^{-k z_1 d/2} \pm e^{k z_1 d/2} \right)
\]

(A.37)

A.2.3 \( H_z, H_y \), and \( B_z \) are continuous:

From (A.28e) and (A.29e) we have:

\[
-\omega c \left( \frac{1}{k x} \varepsilon_1 \right) \left( e^{k z_1 d/2} \mp e^{-k z_1 d/2} \right) = -\omega \varepsilon_2 \left( \frac{1}{k x} \mu_2 \right) \left( \mp B e^{-k z_2 d/2} \right)
\]

(A.38)

\[
\varepsilon_1 \mu_2 \left( e^{k z_1 d/2} \mp e^{-k z_1 d/2} \right) = \varepsilon_2 \mu_1 \left( \mp B e^{-k z_2 d/2} \right)
\]

(A.39)

which yields:

\[
\mathcal{B} = \frac{\varepsilon_1 \mu_2}{\varepsilon_2 \mu_1} e^{k z_2 d/2} \left( e^{k z_1 d/2} \mp e^{-k z_1 d/2} \right)
\]

(A.40)

From (A.28f) and (A.29f) we have:

\[
\frac{ck x}{\omega} \left( e^{-k z_1 d/2} \pm e^{k z_1 d/2} \right) = \frac{ck x}{\omega} e^{-k z_2 d/2}
\]

(A.41)

\[
\mathcal{C} = e^{k z_2 d/2} \left( e^{-k z_1 d/2} \pm e^{k z_1 d/2} \right)
\]

(A.42)

From (A.28d) and (A.29d) we have:

\[
\frac{ick z_1}{\omega \mu_1} \left( -e^{-k z_1 d/2} \pm e^{k z_1 d/2} \right) = -\frac{ick z_2}{\omega \mu_2} Ce^{-k z_2 z}
\]

(A.43)

\[
\mathcal{C} = -\frac{k z_1 \mu_2}{k z_2 \mu_1} e^{k z_2 d/2} \left( -e^{-k z_1 d/2} \pm e^{k z_1 d/2} \right)
\]

(A.44)

From this we can infer that:

\[
-\frac{k z_1}{k z_2} \left( -e^{-k z_1 d/2} \pm e^{k z_1 d/2} \right) = e^{-k z_1 d/2} \pm e^{k z_1 d/2}
\]

(A.45)
\[
\frac{k_{z1}\mu_2}{k_{z2}\mu_1} \left( e^{-k_{z1}d/2} \mp e^{k_{z1}d/2} \right) = \left( e^{-k_{z1}d/2} \pm e^{k_{z1}d/2} \right) 
\]
(A.46)

\[
\mp k_{z1}\mu_2 \left( e^{k_{z1}d/2} \mp e^{-k_{z1}d/2} \right) = \pm k_{z2}\mu_1 \left( e^{k_{z1}d/2} \pm e^{-k_{z1}d/2} \right) 
\]
(A.47)

\[
\mp k_{z1}\mu_2 = \pm k_{z2}\mu_1 \left\{ \begin{array}{c}
\coth(k_{z1}d/2) \\
\tanh(k_{z1}d/2)
\end{array} \right\} 
\]
(A.48)

which yields the transverse-electric dispersion relation:

\[
\pm \mu_1 k_{z2} \pm \mu_2 k_{z1} \left\{ \begin{array}{c}
\tanh(k_{z1}d/2) \\
\coth(k_{z1}d/2)
\end{array} \right\} = 0 
\]
(A.49)

Thus, by comparing (A.35) and (A.49) we see that the sets of solutions either consist of \((E_y, B_x, B_z)\) or \((E_x, E_z, B_y)\) as was stated at the beginning of the Appendix. From here it can be shown, [71], that there are no non-zero modal solutions to the transverse-electric modes operating within this structure. As a result, we can state that surface plasmon polaritons are strictly transverse-magnetic.