THE COUPLING OF GRAVITATIONAL RADIATION TO NONRELATIVISTIC SOURCES

Thesis by

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ABSTRACT

This thesis examines the problem of the coupling of gravitational radiation to its sources in the limit of weak fields and slowly-moving sources; it shows in detail how the irreversibility caused by the escape of radiation can be included in the formalism.

The usual slow-motion expansions of General Relativity (EIH and post-Newtonian) have the difficulty that they are not uniformly valid for large distances - distances where radiation becomes important and where the outgoing-wave boundary condition must be imposed. This difficulty is eliminated by using the method of matched asymptotic expansions. A second asymptotic expansion, in the same slowness parameter as enters in the near zone, is used to represent the radiation. This outer expansion provides matching conditions on the inner expansion that generate radiative corrections to the inner expansion.

Using this technique we show that the escape of radiation leads to an extraction of energy from the sources, without ever having to define the energy carried in the gravitational waves. The damping is found by calculating the work done by the fields that react back on the source. Explicit expressions are given for these fields, and these can be used to calculate, in lowest order, all the irreversible effects caused by radiation.

In this thesis the problem of calculating radiation reaction for bodies with very weak gravitational fields ($U/c^2 \ll v^2/c^2 \ll 1$) is solved definitely. The case of gravitationally bound systems

$(U/c^2 \approx v^2/c^2 \ll 1)$ is discussed and a program for dealing with this case is set up, but the calculations for this case have not yet been done.

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I. INTRODUCTION

This thesis studies the problem of coupling weak-field gravitational radiation to non-relativistic (i.e., slowly moving) sources. It shows how the irreversibility caused by the escape of radiation is to be uniquely included in slow-motion expansion such as those of Einstein, Infeld, and Hoffman[1], of Chandrasekhar[2], and of Fock[3].

The problem is approached by using the method of matched asymptotic expansions. Since this is a technique that may be unfamiliar to many readers, I have included a number of examples where the key ideas can be seen without the complcations of spherical coordinates or tensor fields.

For historical reasons I will call this approximation the "slow-motion approximation". As we work out the examples it will become clear that we are really dealing with a "long wavelength", or better, "small phase-lag" approximation (size of source)/(wavelength of radiation) $\ll 1$.

A. Motivation and Results

One studies gravitational radiation for a variety of reasons. Besides satisfying simple curiosity, gravitational radiation is

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1. Einstein, A., Infeld, L., and Hoffman, B. 1938, Ann. Math. 39,66.
2. Chandrasekhar, S. 1965, Ap. J., 142, 1488, 1513.
3. Fock, V. 1964, The Theory of Space, Time and Gravitation (Macmillan, N.Y.) 2nd ed., pp. 398 and 399.

interesting as an integral part of Einstein's General Theory of Relativity and is also a phenomenon of some astrophysical interest.

The study of gravitational radiation sheds considerable light on the role of "energy" in General Relativity. The Einstein Field Equations explicitly ignore the energy that one would have liked to ascribe to the gravitational field. Despite intensive efforts over many decades, no satisfactory definition of gravitational field energy has ever been given. There exist asymptotic definitions valid in the limit of high frequencies[4] and pseudo-tensor definitions that are not coordinate independent[5], but no definition with the simplicity of, say, the electromagnetic stress-energy tensor seems to be possible. Without ever defining the energy in the gravitational field, this thesis shows that there exist wave-like solutions of the field equations capable of transporting energy - solutions which extract mechanical energy from some systems and deposit it in other systems. The waves carry only positive energy. That is, systems subject to outgoing-wave boundary conditions damp!

Gravitational radiation plays an important role in some astrophysical situations through its introduction of small irreversible forces, principally damping forces. In this thesis I compute "resistive fields" adequate for describing the dominant

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4. Isaacson, R.A. 1968, Phys. Rev. 166, 1263.

5. Landau, L.D. and Lifshitz, E.M. 1962, Classical Theory of Fields (Addison-Wesley, Mass.) 2nd ed. page 341.

irreversible effects in the weak-field, slow-motion limit. These fields allow one to calculate the damping due to the emission of gravitational waves for systems that are not gravitational bound.

By far the most interesting question that an astrophysicist will ask of a general relativist is whether bodies moving only under the influence of gravity (such as the solar system) radiate and if so, how much? To my knowledge this question has never been rigorously answered -- nor is it answered fully by this thesis. The accepted answer comes from the use of a routine linearization of the field equations*. Unfortunately, this linearization cannot be trusted when applied to gravitationally bound systems. This approximation is not uniformly valid for long wavelengths. The point is that the strength of the field, (Newtonian potential/c^2), must remain smaller than the square of the slowness parameter ϵ, (ϵ = size of system/wavelength of radiation); and for a system that remains gravitationally bound in a weak field the kinetic energy is on the same order as the gravitational potential energy so that the fields are not weak enough to ignore the non-linearities**. It is not clear whether or not a proper treatment of this question will substantially change the accepted answers.

As an example of this limitation to very weak fields, consider gravitational radiation from the normal modes of oscillation of the

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* See, for example, Landau and Lifshitz (5) page 363 ff.

** This matter will be taken up further in section H of part IV. Perhaps the idea will be clarified if the reader compares Table I and Table II, pages 60 and 83.

earth. *The satellite period is ninety minutes and from the above* arguments this is too long for the very weak field limit to be valid. The periods of oscillation of the earth are fractions of the fifty-four minute fundamental and thus uncomfortably close to the limit. One would not be surprised to find corrections that are of the order of 20% for these modes when the non-linearities are taken into account.

The analysis presented in this thesis does not go far enough beyond the standard linearized theory to give a definitive answer to the problem of radiation for gravitationally bound systems. In order to treat that problem properly, one will have to work out a slow-motion limit, such as that used by Einstein, Infeld, and Hoffman(1), (EIH) or the post-Newtonian hydrodynamics of Chandrasekhar(2) in sufficient detail to include radiation. The approximation schemes of EIH and Chandrasekhar both consider field strengths that are related to the velocities by

$$\frac{U}{c^2} = \mathcal{O}\left(\frac{v^2}{c^2}\right) \tag{1}$$

where U = typical gravitational potential

v = typical velocity,

and hence they are appropriate to the study of free-fall motions. Previous workers have not been able to incorporate radiation into these approximation schemes. This thesis shows how this is to be done (and perhaps that is its greatest contribution!) but it does not carry the calculation out to completion.

B. Historical Resume

Early attempts to study the properties of gravitational radiation naturally proceeded by concentrating first on the simplest situations. The original paper on gravitational waves(6), by Einstein, dealt with the waves in the limit of infinitely weak fields superimposed on a flat spacetime. Later work, attempting to couple these waves to sources and to determine the motion of the sources, restricted its attention to slowly moving sources. The classic paper on the EIH method(1) dealt with the motion of slowly moving "point" sources. Unfortunately, no one has been able to satisfactorily include radiation in the approximation. In a book on the EIH method published in 1960, Infeld and Plebanski summarized the status of a study of radiation in the EIH approximation(7):

> "... The results are indeed meagre and mostly of a negative character. They show that it is hardly possible to connect any physical meaning with the flux of energy and momentum tensor defined with the help of the pseudo-energy-momentum tensor. Indeed, the radiation can be annihilated by a proper choice of coordinate system."

This confused situation led some relativists to even doubt the reality of the wave-like solutions to the linearized equations. A recent review paper by Bonnor states(8)

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6. Einstein, A. 1918, Sitzber. Preuss. Akad. Wiss. Physic-Math. Kl, 154.

7. Infeld, L. and Plebanski, J., 1960, Motion and Relativity (Pergamon Press, N.Y.) pp. 200 and 201.

8. Bonnor, W.B. 1963, Brit. J. Ap. Phys. 14, 555.

> "... a number of workers have used the EIH method on radiation problems, and their conflicting results are a monument to its unsuitability for the task."

Some workers have tried to study the problem of motion without the restriction to slow motions. In this case the equations of motion become differential-difference equations and are usually insoluble. Many recent attempts have come up with anti-damping[9,10,11]. Their methods are complicated and it is not known if any of their results are correct.

C. Brief Outline of This Thesis

This thesis shows how to uniquely incorporate radiation into a slow-motion expansion. As I mentioned above, this is one difficulty that must be surmounted before the problem of free-fall motion can be attacked. In addition, the results presented here cast new light on the usual results of the linear expansion of the field equations for systems with very weak fields ($U/c^2 \ll v^2/c^2$); in particular, they reveal that the formulae obtained using the Landau-Lifshitz pseudotensor are indeed correct.

In this thesis the damping of a source of radiation is found by explicitly calculating the damping force that acts back on the source rather than by using energy conservation and some definition of the energy in the wave field. These explicit expressions for the

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9. Havas, P. and Goldberg, J.N. 1962, Phys. Rev. 128, 398.
10. Smith, S.F. and Havas, P. 1965, Phys. Rev. 138, B495.
11. Hu, N. 1947, Proc. Roy. Ir. Acad. A51, 87.

"resistive" fields allow one to compute, in addition to energy, also other irreversible effects such as the angular momentum lost by a spinning asymmetric body. This work complements some specific numerical computations on the emission of gravitational radiation by small-amplitude oscillations of fully relativistic stars recently published by Thorne and his colleagues[12,13,14].

In this thesis we restrict our attention to systems satisfying the following conditions:

(a) the velocities are small compared to that of light:

$$\frac{v}{c} \ll 1 , \qquad (2a)$$

(b) stresses, pressures, etc., are small compared with the densities:

$$\frac{p}{\rho c^2} \ll 1 , \qquad (2b)$$

(c) the sources are confined to a region small compared with the wavelength of the radiation characteristic of the motions:

$$\frac{L}{\lambda} \ll 1 , \qquad (2c)$$

(d) the gravitational binding energy is small compared with the masses and even with the kinetic energies

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12. Thorne, K.S. and Campolattaro, A. 1967, Ap. J. 149, 591.

13. Price, R. and Thorne, K.S. 1969, Ap. J. 155, 163.

14. Further papers in preparation.

$$\frac{U}{c^2} \ll \frac{v^2}{c^2} \,, \tag{2d}$$

(although we shall give some discussion of the "post-Newtonian" case $U/c^2 = O(v^2/c^2)$. Here and throughout this thesis:

v = typical source velocity

c = speed of light

p = typical source pressure

ρ = typical source density

L = typical source dimension

λ = typical wavelength of the radiation

U = Newtonian potential

We will exploit these restrictions by approximating the solution to the Einstein equations by the first few terms of an asymptotic expansion in a small parameter ϵ, defined by

$$\epsilon \equiv \frac{L}{\lambda} \,, \tag{3}$$

studying systems for which

$$\frac{v}{c} = O(\epsilon) \tag{4a}$$

$$\frac{p}{\rho c^2} = O(\epsilon) \tag{4b}$$

$$\frac{U}{c^2} \ll O(\epsilon) \tag{4c}$$

(except for some discussion of the $U/c^2 = O(\epsilon^2)$ case).

The analysis presented here is routine if one is familiar with the recent literature on singular perturbations[15,16]. Since this material is not yet a part of most physicists' education, this thesis is more tutorial than might otherwise be appropriate. Nonetheless, it cannot substitute for the books written on the subject.

The second chapter deals with the ideas involved in the slow-motion limit viewed as a singular perturbation problem. Two simple examples will be given to illustrate the ideas, especially the ideas involved in "matching".

The third chapter discusses the slow-motion limit of electromagnetism in considerable detail.

Finally, the fourth chapter shows how the gravitational calculation is done for an arbitrary multipole, then reconsiders the calculation as a modification of the ordinary electrodynamic calculation, and finally discusses the interpretation of the results and some of the pitfalls in the calculation.

We will make extensive use of vector and tensor spherical harmonics. These will considerably simplify the analysis. To enable us to use spherical harmonics effectively we have restricted our attention to a linearization about flat space. We will use a notation for spherical harmonics that is identical with that used in the theory of angular momentum and in nuclear physics. Appendix

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15. Van Dyke, M. 1964, Perturbation Methods in Fluid Mechanics, (Academic Press, N.Y.).

16. Cole, J. 1968, Perturbation Methods in Applied Mathematics (Ginn-Blaisdell, N.Y.).

B outlines the definitions of these fields and collects some useful formula not readily available*.

The notation for General Relativity that we will use is fairly standard; but it is summarized in Appendix A lest there be any confusion. Perhaps the principal difficulty will be the signature of +2 that I am accustomed to using.

The literature references appearing throughout the text have also been collected at the end of the thesis for convenience. In addition, general references on relativity, applied mathematics, and angular momentum are given at the end.

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* A treatment of radiation reaction modeled after this treatment but using the Regge-Wheeler convention for spherical harmonics is in preparation by Thorne.

II. THE SLOW-MOTION EXPANSION AS A SINGULAR PERTURBATION PROBLEM: THE KEY IDEA

The method that I will use to solve the problem of radiation in slow-motion expansions is applicable to problems in electromagnetism, acoustics, waves on a stretched string, and to a great variety of other wave-propagation problems. This chapter will make some very general remarks on the method and will illustrate the key ideas involved in the matching and in equations of motion in general by working a model problem involving a stretched string.

A. The Expansion as an Asymptotic Expansion

The first point to be made is that we are dealing with asymptotic expansions, not with power series (or Taylor's Series) expansions, although our asymptotic sequence will turn out to be the powers. Because of this, the convergence of our expansion is irrelevant.

Roughly speaking, the difference between a power series expansion and an asymptotic expansion can be seen in the terms that must be examined. The convergence of a power series expansion is determined by the limiting behavior of the terms as they approach infinity. On the other hand, an asymptotic expansion is one in which the first few terms make an error that can in some sense be estimated from the next term in the expansion. If we are going to represent solutions by the first few terms of an expansion, then clearly we will be dealing with an asymptotic expansion.

A pertinent example is the expansion for small ϵ

of the exponential:

$$e^{(1+\varepsilon)t} = e^{t}\left(1 + \varepsilon t + \cdots\right). \tag{5}$$

While convergent for all values of t, as an asymptotic expansion it is not uniformly valid for large t. Clearly when t is as large as $1/\epsilon$ the second term is as large as the first and indeed, terms are important all the way out until the factorial finally cuts off the powers. If one only has the first few terms, the ultimate convergence is of little interest.

B. <u>Singular Problems and Non-Uniformity</u>

An asymptotic expansion for some function of x in a small parameter ϵ may not be "uniformly valid" over the entire range of x. Difficulties typically occur for special values like $x = O(\epsilon)$, or $x = O(1/\epsilon)$. In some cases a different expansion in the <u>same</u> small parameter ϵ must be used to represent the function near those singular points. To see this, consider the following example.

Suppose we want the asymptotic behavior of the following function in the limit ϵ going to zero. The function is

$$f(x,\varepsilon) = 1 + x + \frac{\epsilon}{x}. \tag{6}$$

If one assumes an expansion in the sequence*

$$f(x,\epsilon) \sim a_0(x) + \epsilon a_1(x) + \cdots \tag{7}$$

* The "~" is read "is asymptotic to".

then one can evaluate the coefficients by using the limit property of the terms in an asymptotic sequence:*

$$\lim_{\epsilon \to 0} \frac{a_{i+1}}{a_i} = 0 \tag{8}$$

The leading term is

$$a_0(x) = 1 + x , \tag{9}$$

but this is clearly a very poor approximation for x near zero; see Figure 1.

This could have been anticipated because the limit used to evaluate a_0 was not uniform in x.** If in the above limit process we had taken x to be of the order of ϵ, then we would have a different ordering of terms and a different expansion. Rewriting our function in terms of a new variable x given by

$$X = \frac{x}{\epsilon} , \tag{10}$$

and taking a new limit $\epsilon \to 0$ for fixed X, and assuming a new expansion

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* This is the definition of an asymptotic sequence.

** The definition of a limit, $\lim_{\epsilon \to 0} f(x,\epsilon) = A$, is that for any $\delta > 0$ there exists an $\bar{\epsilon}$ such that if $\epsilon < \bar{\epsilon}$, then $|f(x) - A| < \delta$. If the size of the neighborhood $\bar{\epsilon}$ must be dependent on the parameter x, then the limit is not uniform in x.

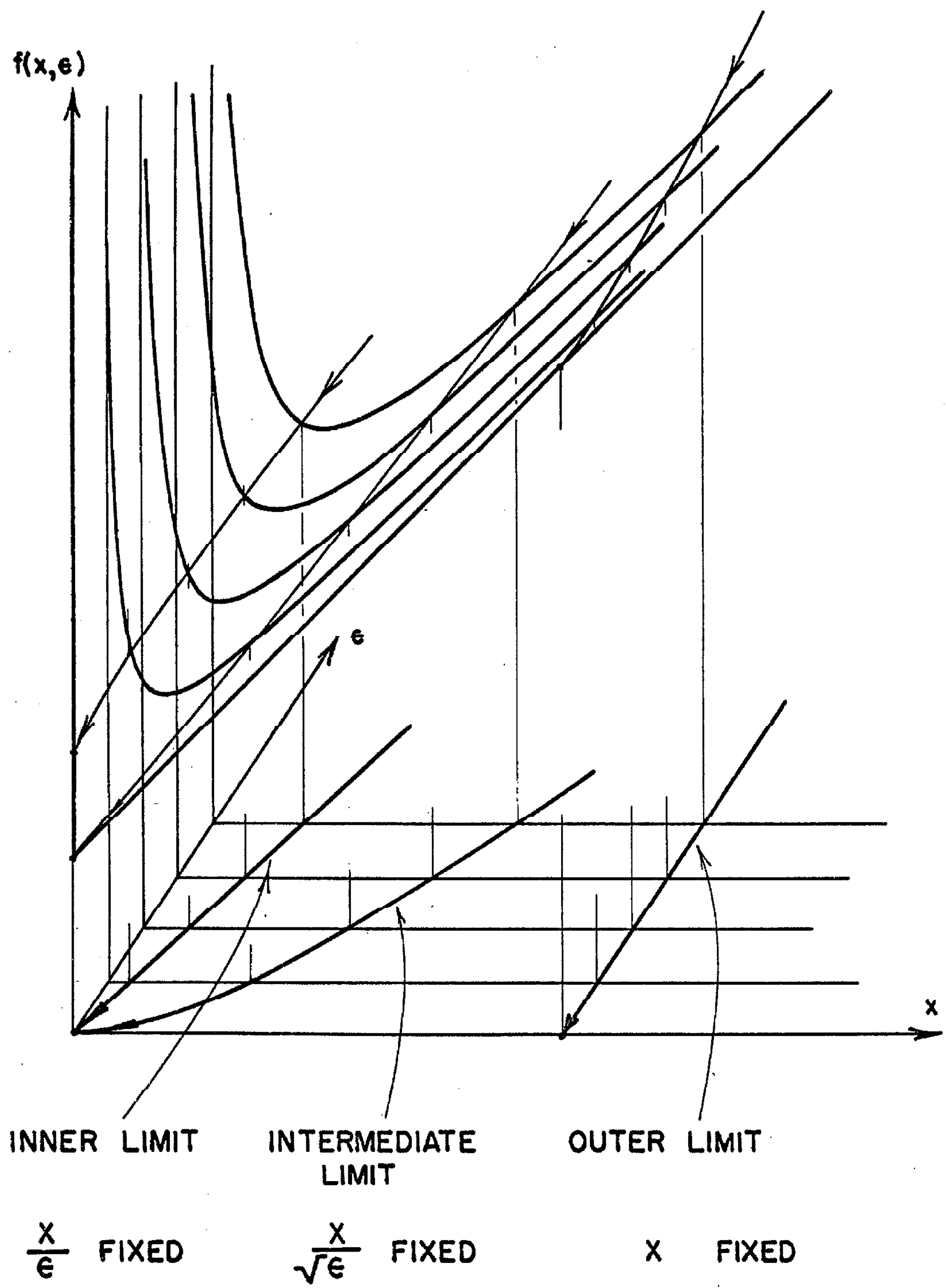

Figure 1. A plot of $f(x,\epsilon) = 1 + x + \frac{\epsilon}{x}$, showing the various limit processes described in the text.

$$f(\varepsilon X,\varepsilon) \equiv F(X,\varepsilon) \sim A_0(x) + \varepsilon A_1(x) \ \cdots \tag{11}$$

then we find for the leading term by using the above limit process

$$A_0(x) = 1 + \frac{1}{X} . \tag{12}$$

Again, the limit is non-uniform and the above expansion is not uniformly valid for $X = 0(1/\epsilon)$.

Such singular behavior will occur in the expansions that we will make for the waves driven by our slowly-moving sources. An important feature of such pairs of expansions is that they have a common region of validity. By "matching" coefficients in such a common region of validity, any undetermined coefficients in our solutions can be evaluated.

In our example we could have considered an intermediate limit where the variable x_η, defined by

$$x_\eta \equiv \frac{x}{\eta(\varepsilon)} , \tag{13}$$

was held fixed as $\epsilon \to 0$. (For example, $\eta(\epsilon)$ could be $\sqrt{\epsilon}$.) If we apply such a limit with the properties

$$\lim_{\varepsilon \to 0} \eta(\varepsilon) = 0, \tag{14a}$$

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\eta(\varepsilon)} = 0 \tag{14b}$$

to our function, we get a leading term

$$f(\eta x_\eta, \varepsilon) \sim 1. \tag{15}$$

In terms of x_η, we have

$$x = \eta x_\eta \tag{16a}$$

$$X = \frac{\eta}{\varepsilon} x_\eta \tag{16b}$$

and so our intermediate limit corresponds to

$$x \to 0 \quad , \quad X \to \infty \tag{17a,b}$$

and taking the limit $x \to 0$ of the x expansion gives us the intermediate expansion as does taking the limit $X \to \infty$ of the X expansion. Thus the two expansions are said to "match".

For the simple problems that we will have to deal with, the matching can be done in the above, relatively unsophisticated manner*.

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* The modern theory and technique of matching was developed by Saul Kaplun in the mid-fifties. See Lagerstrom, P.A., Howard, L.N., and Liu, C. 1967, Fluid Mechanics and Singular Perturbations (Academic Press, N.Y.).

C. A Non-Trivial Example of Matching

The previous section displayed the essentials of the idea of "matching". To better understand the method, let us work a dynamical problem. This example will also demonstrate how I intend to handle the problem of equations of motion.

Let us consider a stretched, infinite, elastic string. For small slopes the transverse displacement of the string obeys the equation

$$T\frac{\partial^2 y}{\partial x^2} - \rho\frac{\partial^2 y}{\partial t^2} = f(x,t), \tag{18}$$

where: y = transverse displacement of the string

x = longitudinal coordinate along the string

ρ = mass/unit length, assumed constant

T = tension, assumed constant

f = transverse force, a function of position and time.

Let us further consider two identical spring-mass oscillators weakly coupled to the string and separated by a distance much smaller than a wavelength. We define ϵ by

$$\varepsilon \equiv \frac{L}{\lambda}, \tag{19}$$

where: L = spacing of oscillators; λ = wavelength of resulting radiation, and look at the limit where ϵ is very small. The assumption of weak coupling is

$$\kappa \ll \varepsilon, \tag{20}$$

where: κ = ratio of coupling spring constant to main spring constant, and it is not essential but it does keep the computations from obscuring the key ideas. The geometry is sketched in Figure 2.

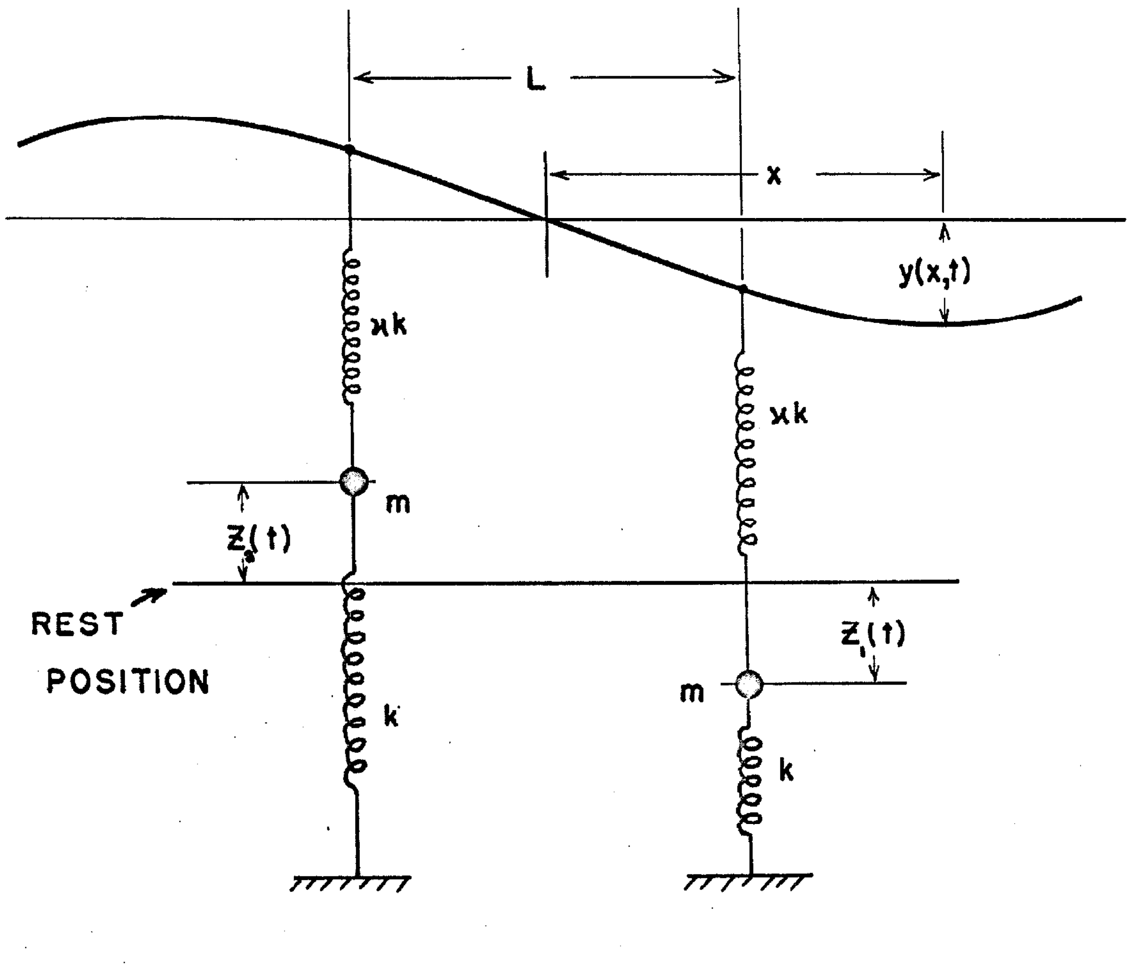

Figure 2. Geometry of the oscillators coupled to the elastic string.

This problem has <u>two characteristic lengths</u>, L and λ, and we can expect (especially in the light of the previous example) that there will be different asymptotic expansions in the two non-dimensional coordinates defined by

$$X^* = 2\pi \frac{x}{L} \tag{21a}$$

$$x^* = 2\pi \frac{x}{\lambda} \ ; \tag{21b}$$

these coordinates are obviously related by

$$X^* = \frac{x^*}{\varepsilon} \ . \tag{22}$$

We will solve for the motion of this system by using the following technique. We will: (i) assume that we have a given motion for the masses; (ii) we will compute the motion of the string caused by the weak coupling to the masses for their assumed (as yet unspecified) motion; and (iii) find the force of the string on the masses and use this to write an equation of motion for the oscillators.

Let us concentrate on the odd mode, having the property

$$z_1(t) = - z_2(t) , \tag{23}$$

where $z_1(t), z_2(t)$ are the transverse positions of the two masses relative to their equilibrium positions.

We ignore the even mode because it is much more strongly damped than the odd mode and the damping is independent of L to first order.

Let us introduce a non-dimensional time, t*, defined by

$$t^* \equiv \omega t = \frac{2\pi c}{\lambda} t = \sqrt{\frac{k}{m}}\, t, \tag{24}$$

and let us write the mechanical equation describing the motion of each mass

$$\frac{d^2 z_1}{dt^{*2}} + z_1 - \varkappa\left[y\left(\frac{L}{2}, \frac{\lambda t^*}{2\pi c}\right) - z_1\left(\frac{\lambda t^*}{2\pi c}\right)\right] = 0. \tag{25}$$

The wavelength of the radiation is given by

$$\lambda = \frac{2\pi c}{\omega} = 2\pi\sqrt{\frac{Tm}{pk}} \tag{26}$$

In the outer, (x*,t*) coordinates the wave equation for the string reads

$$\frac{\partial^2 y}{\partial x^{*2}} - \frac{\partial^2 y}{\partial t^{*2}} = 0. \tag{27}$$

In the limit $\epsilon \to 0$ at fixed x* the oscillators shrink into the origin, leaving us with homogeneous "outer equations". In the inner, (X*,t*) coordinates the wave equation for the string reads

$$\frac{\partial^2 y}{\partial X^{*2}} - \varepsilon^2 \frac{\partial^2 y}{\partial t^{*2}} = -\frac{d}{\pi}(z - \xi)\left[\delta(X^* - \pi) - \delta(X^* + \pi)\right] \tag{28a}$$

where

$$\xi(t^*) = y\left(\frac{L}{Z},\ \frac{\lambda t^*}{2\pi c}\right) \tag{28b}$$

and

$$d = \frac{\varkappa k L}{2T}\ , \tag{28c}$$

thus: Zd = the displacement of the string under the force caused by moving the masses a distance Z from equilibrium. The dimensionless parameter d measures the stiffness of the coupling spring relative to the stiffness of the string. To keep the model simple, I will consider only the case where $d \ll \epsilon$.

We assume that there is an asymptotic expansion for y(x,t) valid in the inner limit:

$$y\left(\frac{L}{2\pi}X^*,\ \frac{t^*}{\omega}\right) \sim A(X^*, t^*) + \epsilon B + \cdots \tag{29}$$

Inserting this expansion into the inner equation (equation 26a) and using the limit process to collect terms of the same order, we have the following equations for the terms in the inner expansion

$$\frac{\partial^2 A}{\partial X^{*2}} = -\frac{d}{\pi}(Z - \xi)\left[\delta(X^* - \pi) - \delta(X^* + \pi)\right], \tag{30a}$$

$$\frac{\partial^2 B}{\partial X^{*2}} = 0\ , \tag{30b}$$

$$\frac{\partial^2 C}{\partial X^{*2}} = \frac{\partial^2 A}{\partial t^{*2}} , \tag{30c}$$

Looking at the leading equation we can see why this is sometimes called the "quasi-static" limit. Time enters this equation only as a parameter. As far as this first term is concerned, the solution at any time is independent of the solution at any other time.

The solution for A is easily found, either from a Green's function or by guessing. It is

$$A = \begin{cases} d(z - \xi) & X^* > \pi \\ \frac{d}{\pi} X^* (z - \xi) & -\pi < X^* < \pi \\ -d(z - \xi) & X^* < -\pi \end{cases} \tag{31}$$

where we have chosen the constant of integration so that A will be antisymmetric about the center of symmetry, x = 0 -- i.e. $A(0,t^*) = 0$. From this expression and equations (28b) and (29) we find that a first approximation for the string displacement at the point of coupling is

$$\xi(t^*) \sim d\,z\left(\frac{\lambda t^*}{2\pi c}\right) \tag{32}$$

and

$$A \sim \begin{cases} d\mathcal{Z} & X^* > \pi \\ d\mathcal{Z}\,\dfrac{X^*}{\pi} & -\pi < X^* < \pi \\ -d\mathcal{Z} & X^* < -\pi \, . \end{cases} \tag{33}$$

This result for the displacement of the string is sketched in Figure 3.

We have partially solved our problem. Given the motion of the masses, Z(t*), we have found the first term in the inner expansion for the motion of the string. Since this term is time-symmetric, there is no information about the damping in the expansion so far.

What about the function B? The only homogeneous solution with odd parity is given by

$$B = \alpha(t^*)\, X^*, \tag{34}$$

and one is tempted to dismiss this solution because of its divergent behavior for large X*.

On the other hand, we might expect, on the basis of our previous discussion that the expansion thus far obtained might not be uniformly valid in the limit $X^* \to \infty$, especially since that limit takes us into the other non-dimensional coordinates where indeed other phenomena dominate the equation. One can only determine whether such a term is present by examining the problem in the outer limit and then matching back to find the effect of the outer expansion on the inner expansion.

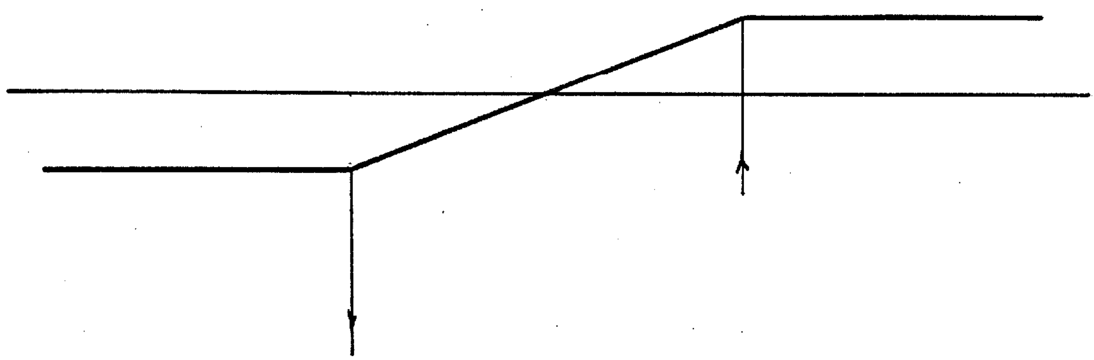

Figure 3. The quasi-static solution for the string; a plot of A(X*).

We assume that there is an asymptotic expansion valid in the outer limit:

$$y\left(\frac{\lambda}{2\pi}x^*, \frac{\lambda}{2\pi c}t^*\right) \sim F(x^*, t^*) + \varepsilon G(x^*, t^*) + \cdots . \tag{35}$$

All the terms of this expansion must satisfy equations of the same form:

$$\frac{\partial^2 F}{\partial x^{*2}} - \frac{\partial^2 F}{\partial t^{*2}} = 0 . \tag{36}$$

The solutions of these equations in the right-hand region are just free outgoing waves:

$$F(x^*, t^*) = C(t^* \mp x^*) , \tag{37}$$

where the function C can be chosen arbitrarily, the upper sign corresponds to outgoing waves at infinity (physically reasonably), and the lower sign corresponds to incoming waves (physically unreasonable).

The escape of radiation introduces irreversibility into our problem. The terms sensitive to this irreversibility are those that contain a ± sign. These terms are called "time-odd". Once the leading time-odd terms have been identified, we will use the upper sign corresponding to physically reasonable boundary conditions.

What do these solutions look like for small x*? Their form, which is sketched in Figure 4, is given by a Taylor's series expansion of the above solution:

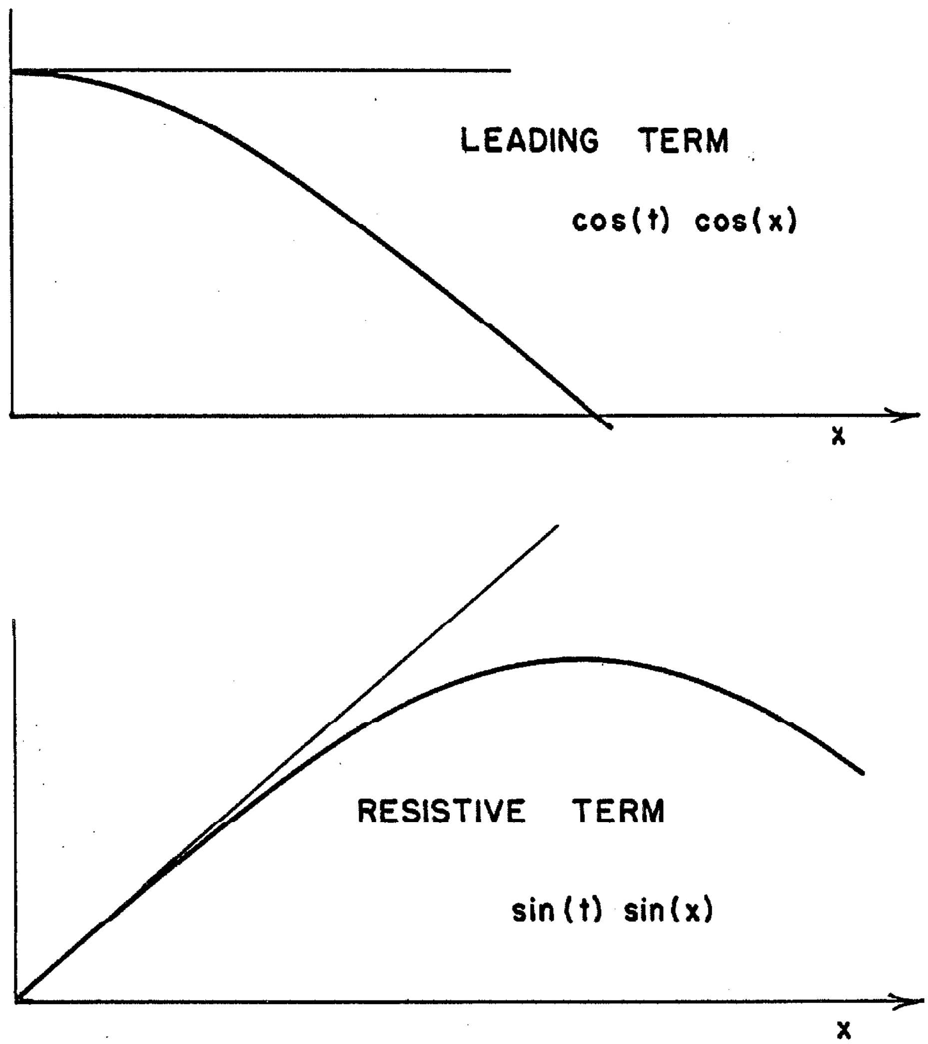

Figure 4. Behavior of the outer solution for free waves on the string. Case where $C(t^* - x^*) = \cos(t^* - x^*)$.

$$y \longrightarrow C(t^*) \mp x^* C'(t^*) + \cdots \tag{38}$$

Writing this expansion in inner coordinates we have

$$y_{\text{OUT}} \longrightarrow C(t^*) \mp \varepsilon X^* C'(t^*) + \cdots \tag{39}$$

Expanding the inner solution (eqs. (29), (33), and (34)) for large X* we find

$$y_{\text{IN}} \longrightarrow d\,Z(t^*) + \varepsilon\,\alpha(t^*)\,X^* + \cdots . \tag{40}$$

These two expansions will match provided that we take

$$C(t^*) = d\,Z(t^*), \tag{41}$$

thereby matching the zero-order terms, and then take

$$\alpha(t^*) = \mp d \frac{dZ(t^*)}{dt^*}, \tag{42}$$

thereby matching the first order terms. (For outgoing waves we would take the upper sign.)

Thus we see that although B satisfies a homogeneous equation (eq. 30b) whose solutions (eq. 34) look disastrous, B is non-zero; in fact it is uniquely determined by the wave-zone properties of the solution, including the outgoing wave boundary condition. We see that over large distances the small inertial term in the wave equation for the string "bends" the seemingly divergent solution of the homogeneous inner equation into a well-behaved, outgoing wave solution. This action of a small parameter over a large distance is typical of non-uniformity. Note that if the size of ϵ is reduced, the inertial term in the string equation is smaller, but the wavelength is longer, so the smaller effect of the inertial term accumulates over a larger distance.

If we insert the terms given in equations (33) and (34,42) into the inner expansion (eq. 29), we can find a more accurate expression for $\xi(t^*)$ than that given in equation (32):

$$\xi(t^*) \sim d\, z(t^*) \mp \varepsilon \pi d \frac{dz(t^*)}{dt^*} . \tag{43}$$

This correction to $\xi(t^*)$ is time-odd and includes the effects of the irreversible loss of radiation down the string. For any given motion Z(t*) we can compute the approximation (43) for $\xi(t^*)$. Now we just have to insert this expression into our force law (equation 25) to arrive at an equation of motion. Care must be taken to convert the t*'s back into t's using

$$\frac{d}{dt^*} = \frac{1}{\omega} \frac{d}{dt} . \tag{44}$$

The resulting equation of motion is:

$$m \frac{d^2 z}{dt^2} \pm \varepsilon \varkappa d \pi \sqrt{km}\, \frac{dz}{dt} + k(1+\varkappa) z = 0. \tag{45}$$

Ignoring the first-order frequency shift due to the coupling spring κk, one can easily show that for outgoing waves the small damping term causes the motion to decay with a "Q" (mean fractional energy loss per radian) given by

$$Q \approx \frac{1}{\pi \varepsilon \kappa d} = \frac{2T\lambda}{\pi \kappa^2 k L^2} \; . \tag{46}$$

The actual string problem conceals a host of further singular perturbations which would have to be considered if the results of the problem were to be used for other than tutorial purposes. Besides making sure that the driving forces did not generate slopes that were too large, one would also have to check that the stiffness of the wire was unimportant. The stiffness will manifest itself as a boundary layer around the point of support with a thickness

$$\Delta = \sqrt{\frac{YI}{T}} \; , \tag{47}$$

where: Y = Young's Modulus

I = second moment of the cross section

Δ = scale size of the stiffness boundary layer.

Clearly we must have $L \gg \Delta$. This stiffness problem is also an interesting problem in singular perturbations, but we shall not pursue it here.

At this point we have covered all the ideas of singular perturbations that we will need. <u>The key idea is that the ordinary slow-motion expansion is not valid for large distances where the outgoing</u>

boundary condition is to be imposed. This difficulty is resolved by using a different expansion to represent the solution in the outer region. In the outer region we generate an approximation to the solution of the field equation, and by matching that solution to the inner solution we generate an approximate equation of motion suitable for describing the motion of the sources of the field.

In the next two sections we shall apply this technique to the study of electromagnetic and gravitational waves.

III. SLOW-MOTION ELECTROMAGNETISM

There is a great formal similarity between the equations of weak-field gravity and those of classical electromagnetism. We can exploit this similarity by using classical electromagnetism to develop the analytical techniques needed for our study of gravitational waves*. The discussion of gravity can then concentrate on the problems peculiar to gravity, especially the physical interpretation of the results. In addition, the formalism of slow-motion electromagnetism is interesting and useful in itself.

After describing the formalism of slow-motion electromagnetism, we will consider the problem of motion for the classical electron - specifically, we will find the radiative corrections to the motion of a point** charge moving slowly along the z-axis. Motions in three dimensions can be obtained by superposition and their inclusion would contribute nothing further to the example. This example will show how the matching is done in spherical coordinates, how vector spherical harmonics can be used to simplify the problem, and finally, how the "runaway" solutions that have disturbed many writers(17) should be handled.

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* Analytical calculations, like computer programs, must be checked and debugged.

** Point charge here means just a bound charge distribution smaller than the length scale of the motions.

17. For example, see Rohrlich, F. 1965, "Classical Charged Particles" (Addison-Wesley, Mass.).

Next we will discuss systems where the charge to mass ratio is constant for all the particles. Such systems can emit only quadrupole radiation, and they have many features in common with gravitational radiators. This example will show how the damping due to quadrupole radiation behaves and also how to simplify the calculations by a skillful use of gauge transformations.

A. The Equations of Electromagnetism

The most convenient representation of the electromagnetic field for these problems will be the one using the vector potential in Lorentz gauge. In this representation we have field equations*

$$A^{\mu\ \ \nu}_{\cdot;\cdot\nu} = -4\pi j^{\mu}, \tag{48}$$

a gauge condition (initial-value equation!)

$$A^{\nu}_{\cdot;\nu} = 0, \tag{49}$$

and a force law

$$ma^{\mu} = -q\, g^{\mu\nu}(A_{\nu;\sigma} - A_{\sigma;\nu})u^{\sigma}, \tag{50}$$

where: u^{a} = the 4-velocity of the test charge q

a^{μ} = the 4-acceleration of q.

In the slow-motion limit the space and time components of the

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* We take c = 1 for simplicity.

vector potential are of different orders. To make the size of our terms explicit, we shall represent our 4-vectors by 3-vectors and 3-scalars. Thus, from our 4-potential A^{μ}, we form a 3-vector A^{a}, (using the same kernel to denote 3 and 4-vectors and summing a,b,c over only space-like indices) and a scalar φ, according to

$$A^{a} \equiv A^{a}, \tag{51}$$

and

$$\phi \equiv A^{t}. \tag{52}$$

We proceed similarly with the 4-current, 4-acceleration, etc., always using the contravariant time components for the scalars. We will write three vectors in boldface. In terms of our 3 + 1 split, the field equations (48) become

$$\nabla^2 \phi - \partial_t^2 \phi = -4\pi\rho, \tag{53}$$

$$\nabla^2 \mathbb{A} - \partial_t^2 \mathbb{A} = -4\pi \mathbb{J}, \tag{53b}$$

the gauge condition (49) becomes

$$\nabla \cdot \mathbb{A} + \partial_t \phi = 0 \tag{54}$$

and the spatial part of the force law* becomes

$$m\mathbf{a} = q\,\mathbf{v}\times(\nabla\times\mathbf{A}) - q\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\phi\right) + \mathcal{O}(v^2). \tag{55}$$

These are the customary equations of electromagnetism with c = 1.

B. The Slow-Motion Limit

Having set up the formalism, we now turn our attention to a system of slowly moving charges. We will study its motion by using an asymptotic expansion in the slow-motion limit. As in the model problem of the last part there are two characteristic lengths, the length typical of the motion of the source, L, and the length typical of the radiation**, λ. Also as before, we introduce two non-dimensional coordinate systems; and in each we construct a different asymptotic expansion in the small parameter ϵ, which we define as before

$$\varepsilon \equiv \frac{L}{\lambda}. \tag{56}$$

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* Because of the identity $a^{\mu}u_{\mu} = 0$, there are only three independent equations in the force law.

** The length λ here is a scale length and is a constant. The wavelength of the radiation is some numerical factor (which may be time-dependent) times λ. Of course, the radiation need not be sinusoidal at all, as long as it stays on a length scale of the order of λ. Discontinuities are forbidden. They correspond to the introduction of a new small length and a new limit process.

The "outer coordinates" are defined by

$$r^* = \frac{r}{\lambda} \quad , \tag{57a}$$

$$t^* = \frac{t}{\lambda} \quad , \tag{57b}$$

and the process of letting $\epsilon \to 0$ such that r^* and t^* are held constant is referred to as the "outer limit". In this limit the source appears to shrink into the point at the origin. Since all the sources are assumed to lie within a region that goes to zero like ϵ, the "outer equations," that is, the exact equations written in outer coordinates, are homogeneous equations.

The inner coordinates typical of the source are defined by

$$\bar{r} \equiv \frac{r}{L} = \frac{r^*}{\varepsilon} \quad , \tag{58a}$$

$$\bar{t} \equiv \frac{t}{\lambda} = t^* , \tag{58b}$$

and the inner limit is $\bar{\epsilon} \to 0$ for fixed $\bar{r}, \bar{t}$.

For example, for radiation from the earth-sun system, the inner length is the astronomical unit, the outer length is the light year, and the small parameter is about 10^{-5}.

The outer equations will be

$$\nabla^{*2}\phi - \partial_{t^*}^2 \phi = 0, \tag{59a}$$

$$\nabla^{*2}\mathbb{A} - \partial_{t^*}^2 \mathbb{A} = 0, \tag{59b}$$

$$\nabla^* \cdot \mathbb{A} + \partial_{t^*}\phi = 0, \tag{59c}$$

and assuming that there is an outer expansion

$$\phi \sim \varepsilon^{\delta}(p + \varepsilon q + \cdots) \tag{60a}$$

$$\mathbb{A} \sim \varepsilon^{\delta}(\mathbb{P} + \varepsilon \mathbb{Q} + \cdots) \tag{60b}$$

we find that the various terms satisfy the following equations

$$\nabla^{*2} p - \partial_{t^*}^2 p = 0, \tag{61a}$$

$$\nabla^{*2}\mathbb{P} - \partial_{t^*}^2 \mathbb{P} = 0, \tag{61b}$$

$$\nabla^* \cdot \mathbb{P} + \partial_{t^*} p = 0. \tag{61c}$$

In outer coordinates the simplication arising from the presence of the small parameter lies in the simplification of the boundary conditions, leaving us with homogeneous equations.

The inner equations will be

$$\bar{\nabla}^2\phi - \varepsilon^2\partial_{t^*}^2\phi = -4\pi L^2\rho, \tag{62a}$$

$$\overline{\nabla}^2 \mathbb{A} - \varepsilon^2 \partial_{t*}^2 \mathbb{A} = -4\pi\lambda^2 \mathbb{J}, \tag{62b}$$

$$\overline{\nabla} \cdot \mathbb{A} + \varepsilon \partial_{t*} \phi = 0. \tag{62c}$$

As a consequence of our assumption of slow motion the current densities will be an order smaller than the charge densities. Thus the leading term in the vector potential expansion will be an order smaller than the leading term in the scalar potential expansion.

If there is an inner expansion

$$\phi \sim a + \varepsilon b + \varepsilon^2 c + \varepsilon^3 d + \cdots \tag{63a}$$

$$\mathbb{A} \sim \varepsilon \mathbb{M} + \varepsilon^2 \mathbb{N} + \cdots \tag{63b}$$

then the various terms satisfy the equations

$$\overline{\nabla}^2 a = -4\pi\lambda^2 \rho, \tag{64a}$$

$$\overline{\nabla}^2 b = 0, \tag{64b}$$

$$\overline{\nabla}^2 c = \partial_{t*}^2 a, \tag{64c}$$

$$\vdots$$

$$\overline{\nabla}^2 \mathbb{M} = -4\pi\lambda^2 \mathbb{J}, \tag{65a}$$

$$\overline{\nabla}^2 \mathbb{N} = 0, \tag{65b}$$

$$\vdots$$

To lowest order we have an "instantaneous" potential a, and an "instantaneous" Coulomb force $- \nabla a$. These form an analog of Newtonian Gravitation.

The relation between the inner expansion (equation 63) and the outer expansion (equations 60) is provided by demanding that the limit of the inner expansion as $\bar{r} \to \infty$ match the limit of the outer expansion as $r^* \to 0$.

Two points should be mentioned about the procedure set out above. The usual slow-motion expansion (inner expansion) is often written as an expansion in inverse powers of the speed of light. This is an unfortunate practice. One usually expands in a dimensionless parameter if one is able. Expansions in dimensioned parameters are bound to be not uniformly valid in space or time because the terms in the expansion must be dimensionless, and so x's or t's must come in along with the dimensioned parameter when it is squared, cubed, etc. Because of this, one must be extremely cautious when dealing with expansions in a dimensioned parameter -- they are quite different from asymptotic expansions. Here we have a genuine asymptotic expansion and that fact should not be concealed.

The practice of expanding in powers of "1/c" is also confusing since the outer expansion is an expansion in the same small parameter, yet in outer coordinates the 1/c terms are not considered small compared with the other terms. Further, ϵ has a definite value, while one cannot look at the value of 1/c and decide whether it is small or not. Finally, it is more convenient to work in units where $c = 1$.

Therefore, I have abandoned the historical practice (EIH, Chandrasekhar, etc.) of treating the expansion as one in the parameter "1/c".

A second point concerns our strategy with regard to the basis vectors. One could have also transformed the vector components when going from inner to outer coordinates. This would use basis vectors of length ϵ in inner coordinates and the sizes of terms would not be explicit. The scheme used here is to refer the vectors to basis vectors that are unit vectors in the physical coordinates r,t. I have worked with both conventions and this is by far the simplest.

C. The Motion of a Classical Electron

The last section described the formal structure of the slow-motion expansion of electromagnetism. As an aid to better understanding and to illustrate some features of the resulting equations of motion, let us now turn to a specific problem. This section will work the problem of finding the fields and forces caused by the slow motion of a point charge along the z-axis. This is a classic problem in electrodynamics, in vogue when people thought that the electron could be represented as a truly "point" charge distribution. It is called the "Problem of the Classical Electron". The discussion, given later (section D of this part), of the validity of our results will bring out some interesting aspects of this problem, especially with regard to the so-called "runaway" solutions.

Let the position of the charge be given by Z(t). Then the charge and current distributions are given by

$$\rho = \frac{q}{L^3}\,\delta(\bar{x})\,\delta(\bar{y})\,\delta(\bar{z}-\mho(t^*)), \tag{66a}$$

$$\mathcal{J} = \varepsilon\frac{q}{L^3}\,e_z\,\mho'(t^*)\;\delta(\bar{x})\,\delta(\bar{y})\,\delta(\bar{z}-\mho(t^*)), \tag{66b}$$

where we have gone to non-dimensional coordinates taking

$$\mho(t^*) \equiv \frac{1}{L}\,z(\lambda t^*), \tag{67}$$

and we have introduced the notation*

$$\mho' \equiv \frac{d\mho}{dt^*}\;. \tag{68}$$

Using these distributions, we can integrate easily the equations given in the previous section for a and $\mathbb{M}$. The result is

$$a = \frac{q}{L\left[\bar{r}^2+\mho^2-2\bar{r}\mho\cos\theta\right]^{1/2}}\;, \tag{69a}$$

$$\mathbb{M} = \frac{q\,\mho'(t^*)\,e_z}{L\left[\bar{r}^2+\mho^2-2\bar{r}\mho\cos\theta\right]^{1/2}}\;. \tag{69b}$$

These potentials lead to the usual "action-at-a-distance" electric and magnetic fields around the charge.

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* This is just the derivative with respect to the argument. I try to follow the conventions of mathematics rather than thermodynamics in this regard but often that is clumsy.

For matching we need the behavior of these solutions for large $\bar{r}$. We have

$$a \to \frac{q}{L\bar{r}} + \frac{q\mho(t^*)\cos\theta}{L\bar{r}^2}, \tag{70a}$$

$$\mathbb{M} \to \frac{q\mho'(t^*)\, \mathbf{e}_z}{L\bar{r}}. \tag{70b}$$

The static, monopole part of this matches to a static, monopole outer solution with no difficulty. The time-dependent dipole terms will radiate, as we shall see, so we will need an outer solution corresponding to electric-dipole radiation. Note one advantage to using two different expansions: the multipole decomposition need only be made in the outer zone.

Using spherical harmonics(18) we can write down the potentials representing electric-dipole radiation directly. The scalar potential must be proportional* to Y_{10}. The vector potential could be either $\mathbb{Y}_{100}$ or $\mathbb{Y}_{120}$. ($\mathbb{Y}_{110}$ has the wrong parity and belongs to the magnetic-dipole solution). The outer limit of the inner expansion of the vector potential $\mathbb{M}$, (70b) has the angular dependence** $\mathbb{Y}_{100}$; so the outer solution that matches to it must go like $\mathbb{Y}_{100}$. The coefficient relating the size of the vector

18. See, for these, Edmonds, A.R. 1960, Angular Momentum in Quantum Mechanics, (Princeton).

* Axisymmetry implies that M = 0.

** Recall that $\mathbb{Y}_{100} = 1/\sqrt{4\pi}\ \mathbf{e}_z$.

potential to the scalar potential in the outer region can be determined from the gauge condition. The radial dependence of the outer solution is determined by the wave equations and the boundary conditions at $r^* = \infty$. These wave equations and the spherical harmonics are discussed in Appendix B.

The electric-dipole solution is given by

$$p = \Upsilon_{10} \left\{ \pm \frac{C'(t^* \mp r^*)}{r^*} + \frac{C(t^* \mp r^*)}{r^{*2}} \right\}, \tag{71a}$$

$$\mathbb{P} = +\sqrt{3}\, \mathbf{\Upsilon}_{100} \frac{C'(t^* \mp r^*)}{r^*} . \tag{71b}$$

where the function C will be determined by matching and the upper sign* corresponds to out-going waves at large r^*.

For matching we need the behavior of this solution for small r^*. Recalling the form of the outer expansion (60), we examine

$$\varepsilon^{\delta} p \longrightarrow \sqrt{\frac{3}{4\pi}} \cos\theta \; \varepsilon^{\delta} \left[\frac{C(t^*)}{r^{*2}} - \frac{1}{2} C''(t^*) \pm \frac{1}{3} r^* C'''(t^*) + \cdots \right] \tag{72a}$$

* The escape of radiation introduces irreversibility into our problem. The terms sensitive to this irreversibility are those that contain a ± sign. These terms are called time-odd. Once the leading time-odd terms have been identified, we will use the upper sign corresponding to physically reasonable boundary conditions.

$$\overset{\delta}{\varepsilon}\mathbb{P} \to \sqrt{\frac{3}{4\pi}}\, e_z \overset{\delta}{\varepsilon} \left[\frac{C'(t^*)}{r^*} \mp C''(t^*) + \cdots \right] \tag{72b}$$

and writing this in inner coordinates (the matching must be done in some consistent coordinate system), we have

$$\overset{\delta}{\varepsilon}p \to \sqrt{\frac{3}{4\pi}} \cos\theta \left[\overset{\delta-2}{\varepsilon} \frac{C(t^*)}{\bar{r}^2} - \frac{1}{2} \overset{\delta}{\varepsilon} C''(t^*) \pm \frac{1}{3} \overset{\delta+1}{\varepsilon} \bar{r}\, C'''(t^*) + \cdots \right] \tag{73a}$$

$$\overset{\delta}{\varepsilon}\mathbb{P} \to \sqrt{\frac{3}{4\pi}}\, e_z \left[\overset{\delta-1}{\varepsilon} \frac{C'(t^*)}{\bar{r}} \mp \overset{\delta}{\varepsilon} C''(t^*) + \cdots \right] \tag{73b}$$

The leading term in this small r* limit of the outer expansion will match the leading term in the large $\bar{r}$ limit of the inner expansion (eq. 63) provided that we take

$$\delta = \ell \,, \tag{74}$$

$$C(t^*) = \frac{q}{h} \sqrt{\frac{4\pi}{3}}\, \mho(t^*) \,. \tag{75}$$

Having determined the function C(t*), we can find the leading terms of expansions valid in the radiation zone from equations (60) and (71). Thus we have found the radiation emitted for a given motion of the charge. Matching higher terms will show us how the radiation field acts back on the charge, extracting the energy that appears in the radiation.

The term $-\frac{1}{2}C''(t^*)$ in the expansion given in equation (73a) leads to effects that are even in time. These time-even terms cannot lead to damping and I will ignore them. The first odd terms are the $\pm \frac{1}{3} \epsilon^3 \bar{r} C'''(t^*) \Upsilon_{10}$ term in the scalar potential and the $\mp \epsilon^2 \mathbf{\Upsilon}_{100} C''(\overset{*}{t})$ term in the vector potential. We can find the leading terms in the radiation resistance by finding the terms in the inner expansion that match these.

The resistive term $\pm \frac{1}{3} \epsilon^3 \bar{r} \cos\theta\, C'''(t^*)$ will match the d term in the inner expansion (eq. 63); the time-even term $-\frac{1}{2} C''(t^*)$ will match the c term; and the b term in the inner expansion will be identically zero (homogeneous equation and nothing to match it). Thus, d satisfies the equation

$$\bar{\nabla}^2 d = 0, \tag{76}$$

and the asymptotic condition

$$d \rightarrow \pm \frac{1}{3}\sqrt{\frac{3}{4\pi}}\ \bar{r} \cos\theta\ C'''(t^*). \tag{77}$$

The solution for d is in fact

$$d = \pm \frac{1}{3}\sqrt{\frac{3}{4\pi}}\ \bar{r} \cos\theta\ C^{IV}(t^*). \tag{78}$$

The fact that the small r* limit of the outer expansion satisfies the inner equation could have been expected. Since the outer equations are really the exact equations, their inner limit yields the inner equations; consequently, the inner limit of the outer solution is indeed the inner solution. This simplification does not always occur in matching problems.

The leading resistive term in the vector potential can be found in the same way; and we have

$$\phi_R = \frac{\varepsilon^2}{\sqrt{12\pi}} \; r^* \cos\theta \; C'''(t^*) \,, \tag{79a}$$

$$\mathbb{A}_R = -\varepsilon^2 \sqrt{\frac{3}{4\pi}} \; \mathbb{e}_z \; C''(t^*). \tag{79b}$$

The force on a charge due to these resistive fields is given by

$$\mathbb{F}_R = -\frac{q'}{\lambda} \frac{\partial \mathbb{A}_R}{\partial t^*} - \frac{q'}{\lambda} \nabla^* \phi_R \,. \tag{80}$$

By evaluating this at the position of the charge we find

$$\mathbb{F}_R = \frac{2}{3} q^2 \left(\frac{L}{\lambda^3}\right) \mathfrak{v}''(t^*) \,, \tag{81}$$

or

$$\mathbb{F}_R = \frac{2}{3} q^2 \frac{d^3 z(t)}{dt^3} \,. \tag{82}$$

for the resistive force on the charge caused by the escape of radiation. As one expects, the scale lengths L, λ have dropped out. They are used only to achieve the correct ordering of terms.

D. Discussion of the Solution: Validity, Runaways

We have derived an approximate equation of motion for a point charge which takes radiative effects into account at lowest order. To better appreciate this solution, one could study a specific example. In this section I will report on the results of working such an example. By studying the solution of the example in detail, we will be able to see what really must be small for our expansion to be valid. Finally, we will resolve the difficulty of the "runaway" solutions that come up here and in the case of gravity.

As our example we use the expression for the damping force (equation 82) to evaluate the damping of a charge mass on a spring undergoing simple harmonic motion. In a detailed calculation we would write an equation of motion for the mass:

$$m \frac{d^2 z}{dt^2} + k z - \frac{2}{3} q^2 \frac{d^3 z}{dt^3} = 0, \tag{83}$$

and from this we would find that the "Q" (energy loss per radian) of the system for small damping is given by

$$Q \approx \frac{3}{2} \frac{\sqrt{km}}{q^2 \omega^2} = \frac{3}{2} \frac{\lambda}{2\pi r_e} , \tag{84}$$

where: $r_e = q^2/m$. The length r_e is an important factor in radiation damping (for an electron it is called the "classical" radius). The corresponding length in gravitation is the "gravitational radius" given by

$$r_g = \frac{Gm}{c^2} . \tag{85}$$

To exhibit more explicitly the nature of our approximation scheme, we can write out the form of the higher terms in the equation of motion. The resultant, more accurate equation of motion is

$$\mho''(t^*) + \left(\frac{\lambda^2 k}{m}\right)\mho + \frac{r_e}{\lambda}\left\{\mho'''(t^*) + \alpha\left(\frac{L}{\lambda}\right)\mho''''(t^*) + \cdots\right\}, \tag{86}$$

and the criterion of validity is clearly that

$$\frac{L}{\lambda} \ll 1. \tag{87}$$

Using this criterion of validity, we can now examine a spurious solution of the equation of motion (equation 83), often called the "runaway" solution. This solution exists even for k = 0, and comes from balancing the inertial term directly against the damping term. The runaway solution behaves like

$$z \propto e^{t/r_e}, \tag{88}$$

and the characteristic time scale for the motion is r_e. Thus unless we have

$$\frac{L}{r_e} \ll 1, \tag{89}$$

i.e., unless the body is much smaller than its "electromagnetic radius", the higher terms in the equation of motion are <u>not</u> small and the first few terms do not give a good approximation to a solution. Since realistic bodies are always larger than their electromagnetic radius, these solutions have no physical interest.

Interestingly enough, the gravitational "runaway solutions" can be dismissed completely since bodies must always be larger than their "gravitational radii".

The presence of these runaway solutions as solutions of the approximate equations would cause us a good deal of trouble were we to have to integrate them numerically. As long as we deal with them analytically, however, there will be no difficulty. Recall that we can only consider forcing terms that act on the slow time scale. Thus there is no problem with the generation of runaway solutions by impulsive forces. For these reasons, we can ignore the runaway solutions in our calculations.

Consider the general method of the last section. The expressions for the radiation field and the leading term in the resistive field are determined using only the distribution of charge density, ρ. The current density $\mathbb{J}$ was nowhere needed in the calculation. This will occur for all multipoles for the following reason.

The vector potential in outer coordinates will always go like $\mathbb{Y}_{L,L-1,M}$ at lowest order. To see this, note that in general the currents will be combinations of the two vector spherical harmonics having the correct parity:

$$\mathbb{J} = \alpha\, \mathbb{Y}_{L,L-1,M} + \beta\, \mathbb{Y}_{L,L+1,M}\,, \tag{90}$$

and these lead through the source equation to terms in $\mathbb{A}$ that go like

$$\mathbb{A} \longrightarrow \alpha' \frac{\mathbb{Y}_{L,L-1,M}}{\bar{r}^{L}} + \beta' \frac{\mathbb{Y}_{L,L+1,M}}{\bar{r}^{(L+2)}}\,, \tag{91}$$

in inner coordinates. These match terms in the outer expansion that go like

$$\mathbb{A} \longrightarrow \alpha' \varepsilon^{L} \frac{\mathbf{\Upsilon}_{L,L-1,M}}{r^{*L}} + \beta' \varepsilon^{L+2} \frac{\mathbf{\Upsilon}_{L,L+1,M}}{r^{*(L+2)}} \tag{92}$$

and we see that, indeed, the $\mathbf{\Upsilon}_{L,L-1,M}$ term comes in at a lower order than the $\mathbf{\Upsilon}_{L,L+1,M}$.

Knowing from these arguments that the vector potential in the outer region goes like $\mathbf{\Upsilon}_{L,L-1,M}$ and the scalar potential goes like Υ_{LM}, in lowest order, we can find the scalar potential by matching onto the outer limit of the induction field of the charges and then we can find the vector potential by using the gauge condition. The radiation fields are thus determined solely by the charge density $\rho(r,t)$.

Since the radiation fields are determined only by the charge-density distribution, it must be the case that the radiation-resistance fields are also determined only by the charge-density, since the radiation resistance extracts the energy that appears in the radiation. The resistive fields associated with the (L,L-1,M) multipole are two orders stronger than those associated with the (L,L+1,M) multipole, and so it is only the (L,L-1,M) multipole that contributes, and this can be found from $\rho(r,t)$. (See part IV, section E for detailed calculation, especially equation (153).)

E. <u>Quadrupole Problems</u>

The calculation of the radiative corrections to the motion of a single charge discussed in the last section was useful for showing the essentials of the method and in particular, showed how the spherical harmonics simplified the problem and how the runaway solutions could be ignored. Now let us consider briefly a system which is much closer to those of interest in gravity -- a system of particles whose charge-to-mass ratio is a universal constant. Momentum conservation prevents such a system from emitting dipole radiation, so the dominant term in the radiation damping comes from quadrupole radiation. Viewing such a system as a collection of particles, we see that the dipole self-force on each particle, calculated in section C, is cancelled by the radiation from other particles, leaving a radiation resistance that is of higher order -- a resistance which depends on the quadrupole moment of the entire collection of particles. Because of this, <u>I do not feel that it is fruitful to try to postulate one-particle force laws in general relativity.</u>

Let me just sketch the highlights of the electric-quadrupole calculation. The asymptotic behavior of the near field is given by

$$a_M \rightarrow \frac{4\pi}{5} \frac{q_{2M}(t^*)}{r^3} Y_{2M}, \tag{93}$$

where the quadrupole moment is conventionally defined by

$$q_{LM} \equiv \int Y_{LM}^* r^L \rho(r,\Omega)\, dV. \tag{94}$$

The outer solution that matches this is given by

$$\mathcal{P}_M = \frac{4\pi}{15} \Upsilon_{2M} \left[\frac{q''_{2M}(t^* \mp r^*)}{r^*} \pm 3 \frac{q'_{2M}}{r^{*2}} + 3 \frac{q_{2M}}{r^{*3}} \right] \tag{95a}$$

$$\mathbb{P}_M = \pm \frac{4\pi}{15} \sqrt{\frac{5}{2}}\, \mathbf{\Upsilon}_{21M} \left[\frac{q''_{2M}}{r^*} \pm \frac{q'_{2M}}{r^{*2}} \right] \tag{95b}$$

Following the same method of computation as used in section C, we expand these outer potentials for small r*, take the leading time-odd terms, and find that the "resistive electric field" is given by

$$\mathbb{E}_R = -\frac{4\pi}{15\sqrt{10}} \sum_M{}' \left[\overset{(5)}{q}_{2M}(t) \; r\, \mathbf{\Upsilon}_{21M} \right]. \tag{96}$$

This resistive electric field can be used to compute the radiation damping of systems that emit only quadrupole radiation at lowest order. Note that the quadrupole moment is a non-linear function of the amplitude and the equations of motion will be non-linear.

The calculation of the resistive electric field (96) can be simplified by making use of the invariance of the result under a gauge transformation

$$\mathbb{A} \rightarrow \tilde{\mathbb{A}} = \mathbb{A} + \nabla \chi \,, \tag{97a}$$

$$\phi \rightarrow \tilde{\phi} = \phi - \frac{\partial \chi}{\partial t} \,. \tag{97b}$$

Only a gauge function proportional to Υ_{2M} can couple with our solution . If we take a gauge function given by

$$\chi = \frac{5}{2}\ \frac{4\pi}{15} \sum_M{}' \left[\frac{q'_{2M}(t^* \mp r^*)}{r^*} \pm 3\frac{q_{2M}}{r^{*2}} + 3\frac{q^{-1}_{2M}}{r^{*3}} \right] Y_{2M}, \tag{98}$$

then the new form of the solution is

$$\tilde{\mathbb{A}} = \frac{2\pi}{\sqrt{15}} \sum_M{}' \mathbb{Y}_{23M} \left[\frac{q''_{2M}}{r^*} + \cdots \right], \tag{99a}$$

$$\tilde{\phi} = -\frac{3}{2}\phi. \tag{99b}$$

The leading resistive term in the vector potential will now come into the inner expansion in the ϵ^6 order, rather than the ϵ^4 order because its radial dependence corresponds to L = 3 rather than L = 1. For this reason the vector potential will not contribute to the resistive field in lowest order; it can now be computed solely from the scalar potential φ.

A similar gauge transformation will be used in the gravity calculation. There it will both simplify the complicated force law and also simplify the physical interpretation of the results.

IV. *SLOW-MOTION GRAVITY*

The previous section developed the ideas of the slow-motion approximation in considerable detail. To apply this formalism to gravitational radiation we have to deal with three problems:

First, we must write the equations describing weak-field gravitational waves in the form of wave equations and derive the gravitational analog of the Lorentz force law. This is routine and appears in textbooks such as Landau and Lifshitz[5], although some care needs to be taken with the force law. Also, we must separate the wave equations and force law into their space and time components.

Second, we must solve these equations using the matching techniques developed earlier. This will proceed quite in analogy with the electromagnetic examples worked in detail in the last chapter.

Third, we must interpret the results physically. There are many subtle points involved here which require careful attention, especially if one wishes to extend these results.

A. <u>Weak-Field Gravity</u>

The most concise method for carrying out the linearization of the field equations is to consider a space with two metrics defined on it. The background metric is a previously known solution of the Einstein field equations (the vacuum, flat-space solution in our case). The exact metric is obtained from the background metric by a small perturbation. In this manner one has manifest covariance at every step of the derivation.

In terms of some small parameter, κ, we make an expansion

$$^*g_{\mu\nu} \sim g_{\mu\nu} + \kappa h_{\mu\nu} + \cdots \tag{100}$$

where: $g_{\mu\nu}$ = metric of flat space,

$^*g_{\mu\nu}$ = metric of the exact solution,

$h_{\mu\nu}$ = perturbation,

and we must decide on the relative sizes of the weakness parameter κ and the slowness parameter ϵ. The choice $\kappa \ll \epsilon^2$ leads us to the weak-field limit. The choice $\kappa = \epsilon^2$ leads us to the EIH and post-Newtonian limits. For simplicity we shall restrict ourselves to the weak-field limit, though we shall discuss the EIH and post-Newtonian limits later (section H).

Using the Einstein Field Equations we can find the energy-momentum tensor for the exact metric, $^*g_{\mu\nu}$. To first order in κ we have

$$-16\pi\, {}^*T_{\mu\nu} = \kappa \left[\bar{h}_{\mu\nu;}{}^{\alpha}{}_{\alpha} - \bar{h}^{\alpha}{}_{\mu;\nu\alpha} - \bar{h}^{\alpha}{}_{\nu;\mu\alpha} + g_{\mu\nu} \bar{h}^{\alpha\beta}{}_{;\alpha\beta} \right] \tag{101}$$

where:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2} h^{\alpha}{}_{\alpha} g_{\mu\nu} \tag{102}$$

and the semicolon (;) denotes a covariant derivative with respect to the background metric*, that is

* Although the background space is flat, we will use curvilinear coordinates and so need covariant derivatives.

$$g_{\mu\nu;\sigma} = 0. \tag{103}$$

Later we will use the stroke ($|$) to indicate a covariant derivative with respect to the exact metric, thus

$$^*g_{\mu\nu|\sigma} = 0. \tag{104}$$

We will use the background metric $g_{\mu\nu}$ to raise and lower the indices of unstarred tensors, and the exact metric $*g_{\mu\nu}$ to raise and lower indices of starred tensors such as $*T^{\mu\nu}$.

To "solve" these equations, we want to find functions $h_{\mu\nu}$ such that the resulting $*T_{\mu\nu}$ and $*g_{\mu\nu}$ correspond to a physically interesting situation. Two different approaches come to mind for this.

One approach is to specify $h_{\mu\nu}$ and $*T_{\mu\nu}$ on some initial hypersurface and to continue the solution forward in time using the Einstein field equations (101), the mechanical equations:

$$^*T^{\mu\nu}{}_{|\nu} = 0, \tag{105}$$

(which follow as identities from the Einstein field equations), an equation of state for the matter, and four arbitrary coordinate conditions.

This approach has the difficulty that one doesn't really know how, in practice, to pose the typical problem in gravitational radiation as an initial-value problem. Most choices of initial-value data will include a certain amount of incoming radiation and will not be physically acceptable.

Another approach which will generate interesting metrics is to solve simultaneously the slightly different equations

$$\bar{h}_{\mu\nu;\cdot\alpha}^{\quad\;\alpha} = -16\pi t_{\mu\nu} \quad , \tag{106}$$

and

$$t^{\mu\nu}_{\cdot\cdot|\nu} = 0 \tag{107}$$

using retarded potentials. Since we have

$$t^{\mu\nu}_{\cdot\cdot;\nu} = O(\varkappa), \tag{108}$$

we can find solutions satisfying

$$\bar{h}^{\mu\nu}_{\cdot\cdot;\nu} = O(\varkappa), \tag{109}$$

in which case we have

$${}^{*}T_{\mu\nu} = \varkappa t_{\mu\nu} + O(\varkappa^2) \tag{110}$$

and the last three terms in equation (101) are now included with the higher order terms involving the squares of h.

In our model problems we were able to solve for the fields that would arise from an arbitrary motion of the sources. Here we will be able to find the gravitational field corresponding to any motions that satisfy

$$t^{\mu\nu}_{\cdot\cdot;\nu} = O(\varkappa) \tag{111}$$

(this insures that the $h^{\mu\nu}{}_{;\nu} = O(\kappa)$), and then from these fields and the force law (107) we will be able to write down "equations of motion" for the sources that will include radiative corrections. The resulting spacetime will have $*T_{\mu\nu} = \kappa t_{\mu\nu}$ (but of course $*T^{\mu\nu} \neq \kappa t^{\mu\nu}$). One hopes to be able to pick a suitable $t_{\mu\nu}$ such that when it is finally interpreted in terms of $*g_{\mu\nu}$, the situation is of physical interest.

We are going to have to use the analog of the gauge invariance of electromagnetism (equations 97) to simplify our calculations. In General Relativity the invariance of the solution under certain transformations reflects the fact that a particular solution can be expressed in any coordinate system whatsoever. For the gauge transformations of importance to us, we need consider only the infinitesimal coordinate transformations.

Any solution $h_{\mu\nu}$ could be made to look <u>functionally</u> different by introducing an infinitesimal coordinate transformation:

$$x^\mu \longrightarrow \tilde{x}^\mu = x^\mu + \kappa \chi^\mu . \tag{112}$$

Under such a transformation, the field $h_{\mu\nu}$ goes to a new functional form of the new coordinates:

$$h_{\mu\nu} \longrightarrow \tilde{h}_{\mu\nu} = h_{\mu\nu} + \chi_{\mu;\nu} + \chi_{\nu;\mu} . \tag{113}$$

One can check directly that if $h'_{\mu\nu}$ is substituted into the expression for $*T_{\mu\nu}$ (equation 101), the χ_μ terms cancel out identically. In view of the analogies with electrodynamics, this is also called a gauge transformation.

B. Space-Time Separation

We need to separate the space and time components of our equations in order to make the size of the terms explicit. We decompoze the metric perturbation in the form $\bar{h}_{\mu\nu}$ into a 3-scalar, a 3-vector, and a symmetric 3-dyadic defined by

$$\psi \equiv \bar{h}^{tt} , \tag{114a}$$

$$V^a \equiv \bar{h}^{at} , \tag{114b}$$

$$H^{ab} \equiv \bar{h}^{ab} . \tag{114c}$$

In terms of this 3 + 1 split the wave equations (equations 106) read

$$\nabla^2 \psi - \partial_t^2 \psi = -16\pi \rho , \tag{115a}$$

$$\nabla^2 \mathbb{V} - \partial_t^2 \mathbb{V} = -16\pi \mathbb{J} , \tag{115b}$$

$$\nabla^2 \mathbb{H} - \partial_t^2 \mathbb{H} = -16\pi \mathbb{S} , \tag{115c}$$

and the gauge condition (109) reads

$$\nabla \cdot \mathbb{V} + \partial_t \psi = 0, \tag{116a}$$

$$\nabla \cdot \mathbb{H} + \partial_t \mathbb{V} = 0 . \tag{116b}$$

Here the stress energy tensor has been decomposed in the obvious fashion

$$\varkappa\rho \equiv T^{tt}, \tag{117a}$$

$$\varkappa J^a \equiv T^{at}, \tag{117b}$$

$$\varkappa S^{ab} \equiv T^{ab}, \tag{117c}$$

and we have taken G = 1 as well as c = 1 to simplify the notation.

The gauge transformation (equation 113) can be written in terms of a 3-scalar χ and a 3-vector $\boldsymbol{\chi}$:

$$\psi \longrightarrow \tilde{\psi} = \psi + \nabla\cdot\boldsymbol{\chi} - \frac{\partial\chi}{\partial t}, \tag{118a}$$

$$\mathbb{V} \longrightarrow \tilde{\mathbb{V}} = \mathbb{V} + \nabla\chi - \partial_t\boldsymbol{\chi}. \tag{118b}$$

$$\mathbb{H} \longrightarrow \tilde{\mathbb{H}} = \mathbb{H} + \underset{\smile}{\nabla\boldsymbol{\chi}} - \mathbb{I}\left[\nabla\cdot\boldsymbol{\chi} + \frac{\partial\chi}{\partial t}\right]. \tag{118c}$$

where we have used dyadic symbols defined by

$$(\nabla\boldsymbol{\chi})_{ab} \equiv \chi_{a;b} + \chi_{b;a}, \tag{119a}$$

$$(\mathbb{I})_{ab} \equiv g_{ab}. \tag{119b}$$

The metric that we are considering is diagrammed in Table I.

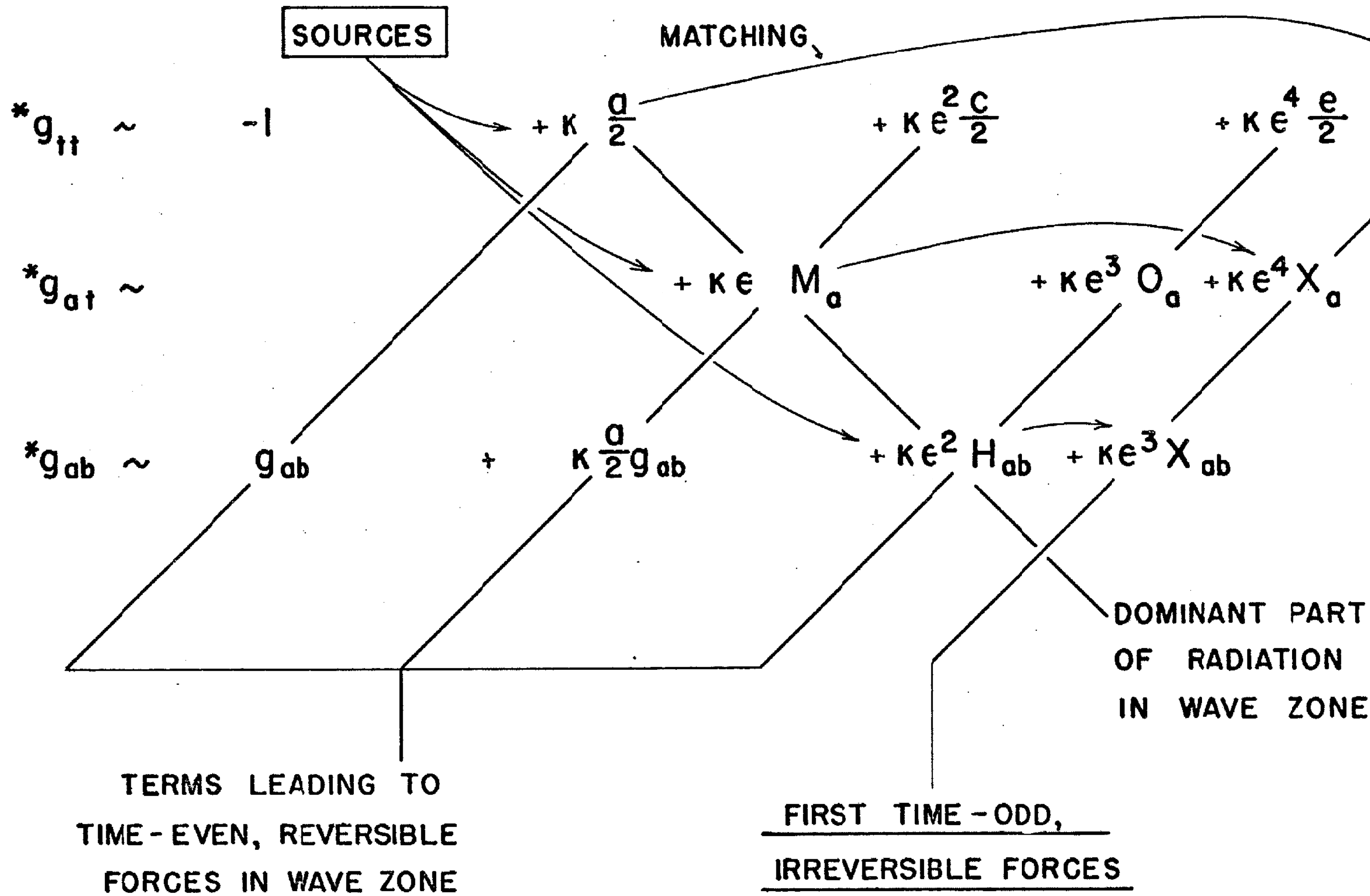

TABLE I. THE FORM OF THE METRIC IN THE VERY-WEAK-FIELD LIMIT, $\kappa \ll \epsilon^2$.

C. The Force Law

The conservation law (equation 107)

$$ {}^{*}T^{\mu\nu}{}_{|\nu} = 0 \tag{120} $$

implies that particles travel along geodesics of the exact metric. We can derive a force law by rewriting this to show the "apparent" coordinate acceleration of the particle. Let me emphasize that this acceleration is not measurable: an accelerometer carried with a freely falling particle would show that it was, of course, in free fall. This force is a "pseudoforce", like the familiar coriolis and centrifugal forces.

Using this force law one can set up equations of motion and integrate them to find the trajectories of the particles with respect to some particular coordinate system. Then one must use the exact metric to interpret the coordinate motions in terms of observables.

The force law is written using the fully covariant expression for the 4-acceleration of a particle's world line with respect to the background metric. The expression is easily derived by translating the expression for a geodesic of the exact metric into terms involving the background metric and the perturbation.

Let the world line be given in terms of a special affine parameter in the background space

$$ x^{\mu} = z^{\mu}(s) , \tag{121} $$

with a 4-velocity given by

$$u^{\mu} = \frac{dz^{\mu}}{ds} , \tag{122}$$

satisfying

$$g_{\mu\nu} u^{\mu} u^{\nu} = -1 , \tag{123}$$

and similarly an "apparent" 4-acceleration*

$$a^{\mu} = \frac{du^{\mu}}{ds} + \Gamma^{\mu}_{\alpha\beta} u^{\alpha} u^{\beta} . \tag{124}$$

In terms of the exact metric

$$^{*}g_{\mu\nu} = g_{\mu\nu} + \varkappa h_{\mu\nu} , \tag{125}$$

we have a new 4-velocity $^{*}u^{\mu}$ and a new special affine parameter $^{*}S$, proportional to the old 4-velocity u^{μ}:

$$^{*}u^{\mu} = \frac{dz^{\mu}}{ds^{*}} = k u^{\mu} . \tag{126}$$

Using the relation

$$^{*}g_{\mu\nu} \, ^{*}u^{\mu} \, ^{*}u^{\nu} = -1 , \tag{127}$$

we can find k and hence $^{*}u^{\mu}$ in terms of $h_{\mu\nu}$ and u^{μ}

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* Here the $\Gamma^{\mu}_{\nu\sigma}$ are the affinity components for the flat background space in some curvilinear coordinate system.

$$^*u^\mu = \left(1 + \tfrac{1}{2}\varkappa h_{\alpha\beta} u^\alpha u^\beta\right) u^\mu + \mathcal{O}(\varkappa^2). \tag{128}$$

The affinity for the metric $^*g_{\mu\nu}$ can be written in terms of the background space affinity and the covariant derivatives of the perturbation:

$$^*\Gamma^\mu_{\nu\sigma} - \Gamma^\mu_{\nu\sigma} = \frac{\varkappa}{2}\, {}^*g^{\mu\alpha}\left(h_{\alpha\nu;\sigma} + h_{\alpha\sigma;\nu} - h_{\nu\sigma;\alpha}\right) \tag{129}$$

(This is easily calculated in coordinates where $\Gamma^\alpha_{\beta\gamma} = 0$.) Both sides of the above equation are tensors.

The equation for a geodesic of the exact metric is

$$\frac{d\,{}^*u^\mu}{ds^*} + {}^*\Gamma^\mu_{\alpha\beta}\, {}^*u^\alpha\, {}^*u^\beta = 0, \tag{130}$$

and from this and equations (116) and (117) one can derive a "pseudo-force" law:

$$a^\mu = -\frac{\varkappa}{2} g^{\mu\sigma}\left(2h_{\sigma\alpha;\beta} - h_{\alpha\beta;\sigma}\right)u^\alpha u^\beta - \frac{\varkappa}{2} u^\mu h_{\alpha\beta;\gamma} u^\alpha u^\beta u^\gamma, \tag{131}$$

correct for high velocities, but linearized in the h's.

For the work here we are interested in only the leading time-odd term of the resistive force. The corrections for high velocities involve v^2 etc., and are time-even. They will be ignored here.

The metric tensor can be split into space and time parts using our previously defined quantities (equations 114)

$$h^{tt} = \frac{1}{2}\psi, \tag{132a}$$

$$h^{ta} = V^{a}, \tag{132b}$$

$$h^{ab} = H^{ab} + \frac{1}{2}\psi g^{ab}. \tag{132c}$$

Here we have assumed, as it will turn out, that H^{ab} is traceless.

Using these quantities and dropping corrections that are $O(v^2)$ but not making any assumptions on the relative sizes of ψ, $\mathbb{V}$, and $\mathbb{H}$, we have

$$\begin{aligned} a_a = \kappa \Big[& \frac{1}{4}\psi_{,a} + \frac{\partial V_a}{\partial t} \\ & - \frac{\partial H_{ab}}{\partial t} v^b + \left(V_{a;b} - V_{b;a}\right) v^b \\ & - H_{ab;c} v^b v^c + \frac{1}{2} H_{bc;a} v^b v^c \Big] , \end{aligned} \tag{133}$$

where we define the 3-velocity as usual

$$\frac{v_a}{\sqrt{1-v^2}} \equiv u_a, \tag{134}$$

and

$$v^2 \equiv v^a v_a , \tag{135}$$

and use it instead of u_a, correct to $O(v^2)$ in our force law, for the spatial part of the acceleration. This is the gravitational analog of the Lorentz force law, although the interpretation is by no means as straightforward.

D. Multipole Solutions

We have now set up the formalism for weak-field gravity in parallel with the formalism for electromagnetism. Aside from the complications of the additional dyadic field, one can nearly copy the electromagnetic results line for line.

Now that we understand the machinery of the matching there is no need to actually use two non-dimensional coordinate systems. All of the manipulations can be performed in the usual physical coordinates. In this section we will derive the general L-pole solution, find the radiation fields, give the gauge transformations that simplify the problem, and compute the resistive fields for gravitational L-pole radiation.

Using the tensor analogs of the vector spherical harmonics, $\mathbb{T}_{JLM}$ -- which are traceless and symmetric representations of the rotation group of total spin J, spatial spin L, and z-component of total spin M (see Appendix B), we can easily write down an electric parity multipole solution:

$$\psi = \Upsilon_{LM}\left\{C^{(L)}\right\}_{L} , \tag{136a}$$

$$\mathbb{V} = \pm\sqrt{\frac{2L+1}{L}}\;\mathbb{Y}_{L,L-1,M}\left\{C^{(L)}\right\}_{L-1} , \tag{136b}$$

$$\mathbb{H} = \sqrt{\frac{(2L-1)(2L+1)}{L\,(L-1)}}\ \mathbb{T}_{L,L-2,M} \left\{C^{(L)}\right\}_{L-2} , \tag{136c}$$

where $\{C^{(P)}\}_Q$ denotes an $L = Q$ solution of the radial equation going like $C^{(P)}(t \mp r)/r$ for $r \to \infty$. For harmonic motions these are just the spherical Hankel functions. These solutions are discussed in Appendix B.

As in the previous examples, the function C is determined by matching in the limit $r \to 0$. Expanding the scalar potential, we have*

$$\psi \longrightarrow \mathcal{Y}_{LM} \frac{C(t)}{r^{L+1}} (2L-1)!! \tag{137}$$

If we follow the conventional definition of multipole moments

$$q_{LM}(t) \equiv \int \mathcal{Y}^{*}_{LM}\, r^{L} \rho(r,\Omega,t)\, dV , \tag{138}$$

we have the induction zone field behaving asymptotically like

$$\psi \longrightarrow \frac{4\pi}{2L+1} \quad \frac{4\, q_{LM}}{r^{L+1}} \ \mathcal{Y}_{LM} . \tag{139}$$

By matching equations (137) and (139), we determine the function C in terms of the mass density multipole moment

---

* $(2n - 1)!! \equiv (2n - 1)(2n - 3) \ldots 1$. These formulae are given in Appendix B.

$$C(t) = \frac{16\pi\, q_{LM}(t)}{(2L+1)!!} \quad . \tag{140}$$

This determines the radiation field caused by a given motion of the masses.

An alternative solution representing electric-parity L-pole radiation is

$$\tilde{\Psi} = \Upsilon_{LM} \left\{ D^{(L)}(t \mp r) \right\}_L \quad , \tag{141a}$$

$$\tilde{\mathbb{V}} = \mp\sqrt{\frac{2L+1}{L+1}}\; \mathbb{Y}_{L,L+1,M} \left\{ D^{(L)} \right\}_{L+1} \quad , \tag{141b}$$

$$\tilde{\mathbb{H}} = \sqrt{\frac{(2L+1)(2L+3)}{(L+1)(L+2)}}\; \mathbb{T}_{L,L+2,M} \left\{ D^{(L)} \right\}_{L+2} \quad . \tag{141c}$$

This must be related to the previous solution (equation 136) by a gauge transformation.

Using the formulae derived in the appendix and picking gauge functions with the proper symmetry, that is

$$\mathbb{X} = \alpha\, \mathbb{Y}_{L,L-1,M} + \beta\, \mathbb{Y}_{L,L+1,M} \quad , \tag{142a}$$

$$\chi = \gamma\, \Upsilon_{LM} \quad , \tag{142b}$$

we can find the relation between these two representations of the electric-parity, L-pole solution. The only result that we will need is the relation between the new scalar potential and the old, which is

$$\tilde{\psi} = \frac{(L+1)\,(L+2)}{L\,\,(L-1)}\,\psi\,. \tag{143}$$

E. Resistive Fields

As in electromagnetism, the damping force comes solely from the leading time-odd term in the expansion of $\tilde{\psi}$ in the small r limit*. This term is

$$\tilde{\psi}_R = \left[\frac{(L+1)(L+2)}{L\,\,(L-1)}\right](-)^{L+1}\,\overset{(2L+1)}{C}(t)\,\frac{r^L\,\mathbb{Y}_{LM}}{(2L+1)!!}\,, \tag{144}$$

and the resistive force per unit volume is given by

$$\mathbb{F}_R = \frac{\rho}{4}\,\nabla\,\tilde{\psi}_R\,, \tag{145}$$

which, for $\tilde{\psi}_R$ given by equation (144) is

$$\mathbb{F}_R = (-)^{L+1}\left[\frac{(L+1)(L+2)}{L\,(L-1)}\right]\frac{\sqrt{L\,(2L+1)}}{[(2L+1)!!]^2}\,\rho\,\overset{(2L+1)}{q}_{LM}(t)\,r^{L-1}\,\mathbb{Y}_{L,L-1,M}\,. \tag{146}$$

The case with L = 2 is the only one with large enough resistive forces to be physical interest;

$$\mathbb{F}_R = -\,\frac{8\pi\sqrt{10}}{75}\sum_M \rho\,\overset{(5)}{q}_{2M}(t)\;r\,\mathbb{Y}_{21M}\,. \tag{147}$$

---

* The gauge transformation from ψ to $\tilde{\psi}$ was done so that the resistive effects of $\tilde{\mathbb{V}}$ and $\hat{\mathbb{H}}$ would be of lower order than those of $\tilde{\psi}$ which is not true for ψ, $\mathbb{V}$, and $\mathbb{H}$.

The direction of this resistive force leads to an extraction of energy from the source. The work done by the force is given by

$$\frac{dE}{dt} = \int \mathbf{v} \cdot \mathbf{F}_R \, dV \propto \int \mathbf{J} \cdot \mathbf{F}_{R/\rho} \, dV \propto \int \mathbf{J} \cdot \mathbf{Y}_{L,L-1,M} r^{L-1} \, dV \tag{148}$$

which is proportional to the (L,L-1,M) vector multipole moment of the mass current. This, in turn, is determined by the mass density multipole moment as follows.

Our motions satisfy continuity (at least to zeroth order):

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{149}$$

Multiply this by $r^L \, Y_{LM}$ and integrate over volume:

$$\int Y_{LM} r^L (\nabla_a J_a) \, dV + \frac{\partial}{\partial t} q_{LM} = 0 \tag{150}$$

Integrate the first term by parts to get

$$-\int J_a \, \nabla_a (r^L Y_{LM}) \, dV + \frac{d}{dt} q_{LM}(t) = 0 \tag{151}$$

A standard definition of the current multipole moment is

$$a_{LM} = \int \mathbf{Y}^*_{L,L-1,M} \cdot \mathbf{J} \, r^{L-1} \, dV. \tag{152}$$

The differentiation of the spherical harmonic in equation (151) is routine, producing a vector spherical harmonic, which in turn gives us just the vector multipole moment. The result is

$$a_{LM} = \frac{\frac{d}{dt} q_{LM}}{\sqrt{L(2L+1)}} \tag{153}$$

This relation can be used to write the energy-loss relation strictly in terms of the mass-density multipole moment q_{LM}:

$$\frac{dE}{dt} = (-)^{L+1} \frac{4\pi \, (L+1)(L+2)}{L(L-1)[(2L+1)!!]^2} \sum_M \overset{(2L+1)}{q}_{LM} \; \overset{\prime}{q}_{LM} \, , \tag{154}$$

and using the identity

$$\overset{(2L+1)}{q} \; \overset{\prime}{q} = -\frac{d}{dt} \sum_{k=1}^{L} (-)^k \, \overset{(2L+1-k)}{q} \; \overset{k}{q} + (-)^L \left(\overset{(L+1)}{q}\right)^2 , \tag{155}$$

this can be written

$$\frac{dE}{dt} = -\frac{4\pi \, (L+1)(L+2)}{L(L-1)[(2L+1)!!]^2} \left(\overset{(L+1)}{q}(t)\right)^2 + \frac{d}{dt} X . \tag{156}$$

If the motions are periodic (or at least bounded) the time derivative term averages to zero over the long run and energy is thus extracted from the source as a consequence of the escape of radiation. Remember that here L is the first non-zero, time-dependent multipole moment of the mass distribution.

For the special case of harmonic motions, we define an amplitude by the equation

$$q_{LM}(t) = \hat{q}_{LM} \cos \omega t , \tag{157}$$

and we can easily compute the average energy loss over a cycle. The mean energy loss per radian is

$$\frac{1}{\omega}\left\langle\frac{dE}{dt}\right\rangle = \frac{4\pi\,(L+1)(L+2)}{L\,(L-1)} \frac{\omega^{2L+1}}{[(2L+1)!!]^2} \sum_M{}' \hat{q}_{LM}^{\;2} \,, \tag{158}$$

which for L = 2 becomes

$$\frac{1}{\omega}\left\langle\frac{dE}{dt}\right\rangle = -\frac{4\pi\ \omega^5}{75} \sum_M{}' \hat{q}_{2M}^{\;2} \,. \tag{159}$$

This is a potentially useful formula and it is probably worthwhile to put the G's and c's back into it:

$$\frac{1}{\omega}\left\langle\frac{dE}{dt}\right\rangle = -\frac{4\pi}{75}\ \frac{G\omega^5}{c^5} \sum_M{}' (\hat{q}_{2M})^2 \,. \tag{160}$$

An interesting form of this expression can be found by introducing a typical length L and a typical mass M such that

$$\sum_M{}' (\hat{q}_{2M})^2 = \beta^2 M^2 L^4 \tag{161}$$

where β is some numerical factor of order unity. In this case we can write the energy lost as

$$\frac{1}{\omega}\left\langle\frac{dE}{dt}\right\rangle = -\frac{4\pi\,\beta^2}{75} \left(\frac{GM^2}{L}\right)\left(\frac{2\pi L}{\lambda}\right)^5 . \tag{162}$$

The second term in this expression is an energy of the size of the gravitational binding energy. Thus we see that the system radiates a small fraction of its gravitational binding energy per radian.

If we combine the previous expression with an expression for the kinetic energy in the motion, we can derive an expression for the damping of the source assuming that there is no energy input. Ignoring numerical coefficients, we find that the "Q" is

$$Q \propto \left(\frac{L}{r_g}\right)\left(\frac{\lambda}{L}\right)^3 \tag{163}$$

where: r_G = geometrical mass GM/c^2

M = typical mass

L = typical source length

λ = typical wavelength.

All of the energy loss formulae derived in this section are the same as the formulae derived using the Landau-Lifshitz pseudo-tensor.

F. The Damping Result

In the previous calculation we have not been too careful about the sign of the damping. We can check that the resistive forces actually lead to damping, rather than to antidamping, by comparing this calculation with the similar electromagnetic calculation.

In electromagnetism the near zone source equation is

$$\nabla^2 \phi = -4\pi\rho, \tag{164}$$

while in gravity it is

$$\nabla^2\left(\frac{\psi}{4}\right) = -4\pi\rho. \tag{165}$$

The force law in electromagnetism is

$$\mathbb{F} = -q\nabla\phi \tag{166}$$

while in gravity it is

$$\mathbb{F} = m\,\nabla\left(\frac{\psi}{4}\right). \tag{167}$$

From this we can see that both theories have inverse square static forces, although in gravity likes attract because of the sign difference.

The gauge transformation that is used to eliminate resistive fields from all but the scalar potential at lowest order has the form for electromagnetism (see equation A64 in Appendix B)

$$\tilde{\phi} = -\frac{L+1}{L}\,\phi, \tag{168}$$

while for gravity it is

$$\tilde{\psi} = \frac{L+1}{L}\,\frac{L+2}{L-1}\,\psi. \tag{169}$$

The sign difference between these equations cancels the sign difference in the force law so it is clear that any gravitational multipole has a damping field which points in the same direction as, but is $(L + 2)/(L - 1)$ times as large as that of the corresponding electromagnetic multipole*. <u>If electromagnetism always damps, then so must gravity</u>.

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* Thus equation (96) and equation (147), both for $L = 2$, differ numerically only by a factor of 4.

For completeness we should also consider the magnetic-parity multipoles. In some cases these can compete at lowest order with the electric-parity modes. The same type of reduction can be carried through to show that the gravitational damping of these modes is just L/(L + 1) times as strong as the electromagnetic damping of the corresponding modes (see Appendix B).

If we define a mass-current multipole moment d_{LM}

$$d_{LM} = \int \mathbb{Y}^{*}_{LLM} \cdot \mathbb{J}\, r^{L}\, dV, \tag{170}$$

then we have resistive fields

$$\mathbb{F}_R = (-)^{L+1} \frac{4\pi \quad L}{(L+1)\left[(2L+1)!!\right]^2} \sum_M{}' \overset{(2L+2)}{d_{LM}}(t)\, r^L\, \mathbb{Y}_{LLM}, \tag{171}$$

and power extracted per radian for harmonic motions

$$\frac{1}{\omega}\left\langle\frac{dE}{dt}\right\rangle = -8\pi \frac{L}{L+1} \frac{\omega^{2L+1}}{\left[(2L+1)!!\right]^2} \sum_M{}' (\hat{d}_{LM})^2. \tag{172}$$

The explicit expressions for the resistive fields (equations 146, 147, and 171) can be used to calculate all time-odd secular effects. An example of such a time-odd effect is one affecting the motion of an axi-symmetric body spinning about a non-symmetry axis. The angle between the spin axis and the symmetry axis should change slowly in time, and this could be calculated using the formulae for the resistive field given above. Note that the perihelion precession of a planet is a reversible phenomenon, not depending upon the escape of radiation. It is time-even and is caused by non-linear terms neglected in our very weak-field limit.

G. Interpretation and Pitfalls

The above derivations have been carefully routed to avoid several sources of confusion. These will be discussed here so that people interested in extending the results will not have to waste time on them.

One obvious point is that the formulae are only valid for the first multipole moment of a given parity which has a non-zero time derivative. Otherwise the small corrections to the lower multipoles create errors as large as the effect of the higher multipole.

A certain amount of caution must be used when these energy loss formulae are used for anything but electric-quadrupole radiation. The even corrections to the lower multipoles that are $O(v^2)$ or smaller can lead to radiation from the lower L multipoles that will compete with the higher multipole.

As an example of this, consider the magnetic quadrupole radiation emitted by two counter-oscillating mass shells as diagrammed in Figure 5. Now in electromagnetism a moving charge has the same total charge as it had at rest, the $\sqrt{1 - v^2}$ coming from charge density being the t-component of a vector is cancelled by the $\sqrt{1 - v^2}$ in the Lorentz contraction of the volume. For gravity there is an extra $\sqrt{1 - v^2}$ since mass (energy) density is the tt-th component of a tensor. Expanding this shows that the source for gravity is the energy content: mc^2 rest-mass plus $1/2\ mv^2$ kinetic energy.

As the mass shells counter-oscillate, this kinetic energy is

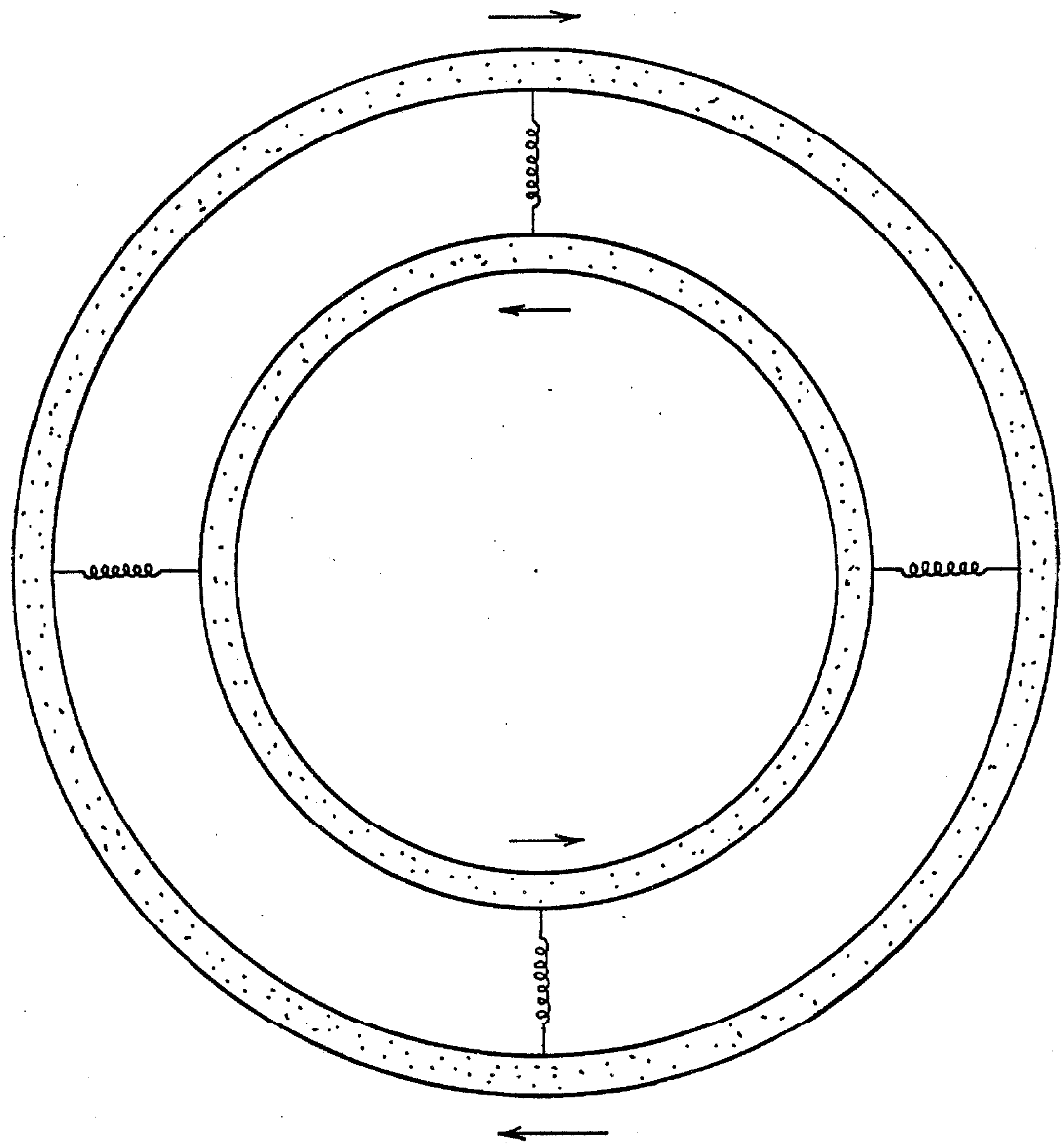

Figure 5. Counter-oscillating mass shells -- a typical magnetic quadrupole source.

unequally distributed between the shells for part of the cycle, and resides in the springs for other parts of the cycle. This periodic redistribution of the oscillation energy leads to electric-quadrupole radiation that is smaller by ϵ^2 than the usual electric-quadrupole radiation, and which is thus of the same size as the magnetic quadrupole radiation.

Another point to be made is that one cannot always compute the resistive force from the force law

$$(F_R)_a = \frac{m}{4}(\psi_R)_{,a} \quad . \tag{173}$$

Before the gauge transformation was used to simplify the problem, the lower L dependence of the vector and tensor potential allowed them to compete with the scalar potential, so one had to use the complete force law:

$$(F_R)_a = m \left[\frac{1}{4}(\psi_R)_{,a} + (V_R)_{a,t} + \left\{(V_R)_{a;b} - (V_R)_{b;a}\right\} v^b - \frac{\partial (H_R)_{ab}}{\partial t} v^b - (H_R)_{ab;c} v^b v^c + \frac{1}{2}(H_R)_{bc;a} v^b v^c \right] \tag{174}$$

By far the most serious source of confusion lies in incorrect physical interpretation. The results presented so far have been written in terms of pseudoforces. The integration of the equations of motion resulting from these pseudoforces allows one to write down the development of the system referred to some arbitrary* coordinate

* Arbitrary to order κ.

system. Until this has been converted into observables, the solution is incomplete.

A big difference between gravity and electromagnetism is seen at this point. The electromagnetic field produces effects only through its force law. On the other hand, not only does the gravitational field affect the coordinate motion of the system, but also the <u>potentials themselves</u> determine the clock rates and the behavior of rigid bodies. Until one knows the $\mathfrak{H}$ field, one cannot convert coordinate differences into proper length without making errors that are $O(\kappa)$ *. For the most part, since our formulae for the decay are only accurate to ϵ^2 themselves, we will not need amplitudes any more accurately. On the other hand, there are situations where such an error can make a big effect, and these have come up as I will now describe.

The gauge transformation that was used to simplify the force law represents only a redefinition of coordinates and should not affect invariants such as damping rates. Thus it should be possible to use the original electric-parity solutions in conjunction with the complete force law to derive the same results. If one attempts to check this, working as an exercise of the problem of the small-amplitude oscillations of a spring-mass system, and if one works in the untransformed gauge (i.e., ψ not $\tilde{\psi}$) with the original potentials and the complete force law, taking as the equation of motion

* Remember that we have calculated only the leading time-odd term in $\mathfrak{H}$. There are much larger time-even corrections that have been ignored.

$$m\ddot{d} + kd - F_R = 0 \tag{175}$$

where the geometry is indicated in Figure 6, then one finds that the resistive forces do work on the system*. What is the explanation for this apparent paradox?

The equation of motion written down for the system (equation 175) implicitly assumes that the rest position of the spring is at $z = L$. This is incorrect. The rest position is at $z = L\sqrt{g_{zz}}$. Neglecting this amounts to considering a problem in which the support of the spring is moved back and forth in such a manner that the rest position always has the coordinate position $z = L$. Now oscillating the support of any oscillator at its resonant frequency is dangerous. Here the errors due to the time-odd terms are 90° out of phase with the motion and are capable of putting energy into the system (parametric excitation!). The gauge transformation not only simplifies the force law but it reduces the size of the time-odd terms in $\mathbb{H}$ (i.e., in g_{zz}) by a factor of ϵ^4, more than enough to eliminate this difficulty.

One would like to check the above interpretation by working the problem taking into account the change in coordinate location of the rest position of the spring, but this is an involved calculation. An easier problem that will check this interpretation and also yield a useful result is the problem of the radiation damping of a rotating,

* These calculations are available from the author.

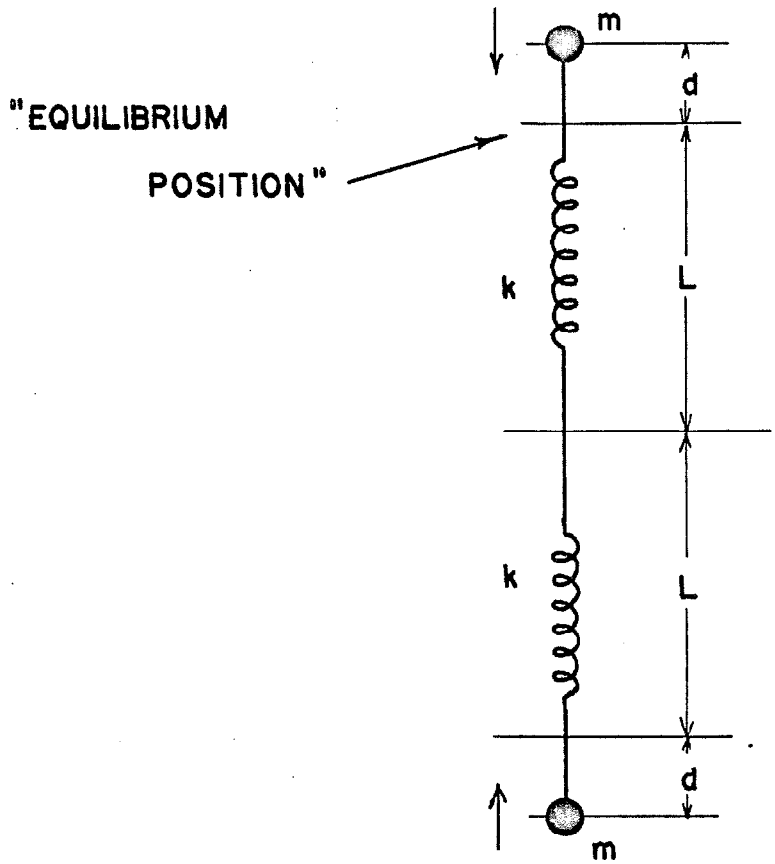

Figure 6. Geometry of the spring/mass system discussed on page 79, d and L are coordinate lengths.

ideal dumbbell, i.e., two point masses connected by a line stress. Clearly a small error in the length of the dumbbell should make little difference in the damping. This is borne out by the calculation, and one gets the same damping whether one works in the transformed system where the force is simple given by

$$\mathbb{F}_R = \frac{m}{4} \nabla \tilde{\psi}_R , \tag{176}$$

or in the original system, where there are non-zero contributions from the following terms in the force law (including a velocity-dependent term*)

$$\mathbb{F}_R = \frac{m}{4} \nabla \psi_R + m \frac{\partial \mathbb{V}_R}{\partial t} - m \frac{\partial \mathbb{H}_R}{\partial t} \cdot \mathbb{v} . \tag{177}$$

The agreement between these two calculations gives us some support for our resolution of the difficulty with the mass-spring system.

H. Radiation in the Post-Newtonian Limit.

If the result of the work presented in this thesis were only to justify the formulae already in common use among relativists, then there would be little interest in it beyond noticing its existence. On the other hand, the methods worked out here should allow one to finally solve the problem of incorporating radiation into a slow-motion formalism such as a post-Newtonian expansion, and hence to solve finally the problem of gravitational radiation from gravi-

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* Calculations available from the author.

tationally bound systems.

A vigorous program aimed at extending the post-Newtonian expansion to high orders is being actively pursued by Chandrasekhar and his colleagues. At present they have stopped just short of the level at which radiative effects would come in[(19)]:

> "... it is expected that the effects of gravitational radiation on the behavior of the system will first manifest itself when the next half-step is successfully taken. But it appears that entirely new considerations will be needed before we can properly take this next half-step."

The approach used in this thesis should be just what is needed to overcome the difficulty, I feel. Let me just sketch out a program for dealing with the problem and discuss a few of the difficulties that I can foresee.

Table II presents the form of the metric that is used in the post-Newtonian expansion. Nearly everything that is important can be seen in this table. Remember that ϵ is the <u>slowness</u> parameter (L/λ), and that the <u>weakness</u> parameter has been taken to be ϵ^2.

A look at the force law (equation 133) shows that the vector potential enters the force law with either a time derivative or a velocity and that the dyadic potential enters with either two time derivatives or two velocities. For this reason the vector potential is needed to one less order than the scalar potential, and the dyadic potential to two less orders for computing forces. The terms con-

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19. Chandrasekhar, S. and Nutku, Y. 1969, Ap. J. (Oct.) in press.

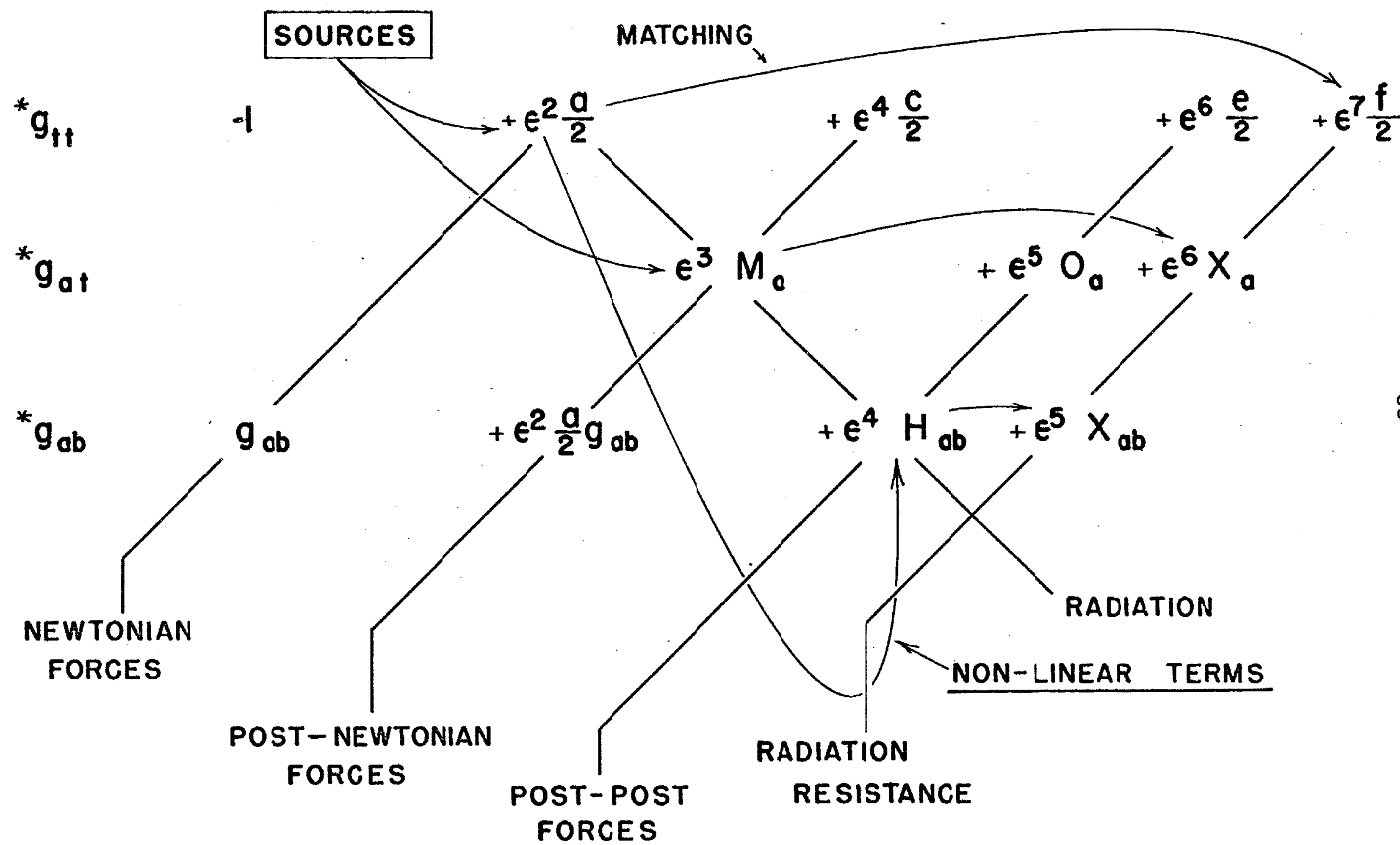

TABLE II. THE FORM OF THE METRIC IN THE POST-NEWTONIAN LIMIT, $\kappa = \epsilon^2$.

tributing to Newtonian gravity and to the first post-Newtonian correction are indicated by diagonal lines in the table.

Now look at the radiation solution that we used earlier (equation 122). The vector potential corresponding to electric-parity quadrupole radiation had the radial dependence corresponding to $L = 1$ and the dyadic potential had a radial dependence corresponding to $L = 0$. (In the language of angular momentum, the vector potential makes total spin 2 by adding the intrinsic spin 1 of a vector to an orbital spin 1, and the dyadic potential adds intrinsic spin 2 to orbital spin 0.) For these reasons, for the radiation we need the vector potential to one higher order (and one lower L-value) than the scalar potential and the dyadic potential to two higher orders (and a two lower L-value). This is indicated by the negative slope diagonal line in the table.

The resistive fields caused by the escape of radiation of this size are indicated by the dashed line in the figure. These forces are completely dominated in the short run by the time-even post-Newtonian and second post-Newtonian forces. In the long run these small time-odd resistive forces lead to damping and hence large and interesting effects.

For a body in free fall motion, we have mass density and momentum flux but no stresses in the sense that Einstein's equations have stresses as their source. The functions ψ and V_a can be found as usual, but the function H_{ab} has its source only in the non-linear correction terms like $\psi_{,a}\psi_{,b}$, $\delta_{ab}\psi_{,c}\psi_{,c}$ etc. Only if the resulting

H_{ab} has the correct relation to ψ and V_a will we be able to make the gauge transformation that eliminates H_{ab} from the resistive fields to lowest order. If this occurs, then the ordinary formulae will apply to free-fall motion.

The term H_{ab} is a part of the second post-Newtonian correction and has been explicitly calculated by Chandrasekhar(19). There is one difficulty that I can foresee that will prevent one from just grinding out the H_{ab} field from his formulae and using it to compute the radiation and the radiation reaction. The sources of the H_{ab} field are not confined to a region smaller than a wavelength but come from the gravitational field everywhere. To avoid enormous phase errors, the integration for the H_{ab} field should use retarded potentials even in the near-zone equations. Even if this is possible and successful, I expect the results of such an integration along the lightcone to produce the usual (log r) divergences that invalidate the usual linear expansion when it is carried to higher orders. This difficulty will probably require that one work in a coordinate system similar to the one proposed by Bondi(20) where the null rays are not changed by the perturbation.

One can hope that these difficulties will be overcome and that we will soon know whether or not the generally accepted results for radiating, gravitationally bound systems are correct.

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20. Bondi, H., van der Burg, M.G.J., and Metzner, A.W.K. 1962, Proc. Roy. Soc. 269, 21.

J. Conclusions

The calculations presented in this thesis have shown how the irreversible escape of radiation can be included in a slow-motion expansion, and that the escape of gravitational radiation leads to a loss of energy from the sources. The explicit formulae for the resistive fields that we have found should allow other interesting effects to be calculated, such as the problem of the spinning asymmetrical body mentioned earlier.

The results of the calculations can be easily summarized. To calculate the energy loss for an electric-parity motion, compute the energy lost by radiation from an equivalent distribution of electric charge,

$$\rho\,[\mathrm{e.s.u.}] = \sqrt{G}\;\rho\,[\mathrm{gm./cm.^3}]\,, \tag{178}$$

and multiply the results by (L + 2)/(L - 1). For a magnetic-parity motion, multiply the electromagnetic results by L/(L - 1).

Also, hopefully, the techniques presented here will allow radiation to be included in the EIH and post-Newtonian expansions. A study of these approximations will finally produce answers to the questions: "Does a gravitationally bound system emit gravitational waves? How much? Does it experience a reaction force?" These questions can be studied now that the slow-motion limit has been extended to include irreversibility.

The key ideas are the realization that the usual slow-motion limit is not uniformly valid for large distances and that this difficulty can be eliminated through the use of an additional asymptotic expansion valid in the radiation zone.

APPENDICES

A. Summary of Notation

There are several different conventions commonly used in General Relativity. I follow the one used by Synge[21]. I use a metric with signature +2, that is, one for which $g_{\mu\nu}$ may be put in the form: Diag: (1, 1, 1, -1). The squared length of a time-like vector is negative. The affine connection is defined by

$$\Gamma^{\mu}_{\nu\sigma} \equiv \frac{1}{2} g^{\mu\alpha}\left(g_{\alpha\nu,\sigma} + g_{\alpha\sigma,\nu} - g_{\nu\sigma,\alpha}\right), \tag{A1}$$

and a covariant derivative by either

$$V_{\mu;\nu} \equiv V_{\mu,\nu} - \Gamma^{\alpha}_{\mu\nu} V_{\alpha} \tag{A2}$$

or

$$V^{\mu}{}_{;\nu} \equiv V^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\nu\alpha} V^{\alpha}. \tag{A3}$$

The Riemann Curvature tensor is defined by

$$R^{\mu}_{\cdot\nu\sigma\tau} \equiv \Gamma^{\mu}_{\nu\tau,\sigma} - \Gamma^{\mu}_{\nu\sigma,\tau} + \Gamma^{\gamma}_{\nu\tau}\Gamma^{\mu}_{\gamma\sigma} - \Gamma^{\gamma}_{\nu\sigma}\Gamma^{\mu}_{\gamma\tau}, \tag{A4}$$

and is contracted on the first and last indices to form the Ricci tensor:

$$R_{\nu\sigma} \equiv R^{\alpha}_{\cdot\nu\sigma\alpha}. \tag{A5}$$

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21. Synge, J.L. 1960, Relativity: The General Theory (North-Holland, Amsterdam).

The commutation formulae for covariant derivatives read

$$V_{\mu;\alpha\beta} - V_{\mu;\beta\alpha} = R^{\gamma}_{\cdot\mu\alpha\beta} V_{\gamma} , \tag{A6a}$$

$$T_{\mu\nu;\alpha\beta} - T_{\mu\nu;\beta\alpha} = R^{\gamma}_{\cdot\nu\alpha\beta} T_{\mu\gamma} + R^{\gamma}_{\cdot\mu\alpha\beta} T_{\gamma\nu} , \tag{A6b}$$

and field equations are

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = -8\pi T_{\mu\nu} . \tag{A7}$$

Several notations for derivatives are used in this thesis. Partial derivatives are indicated either by a comma (,) preceding the subscript or by the operator ∂_a. Covariant derivatives are indicated by a semicolon (;), and when a second covariant derivative is to be defined on the same space, the vertical stroke (|) is used to indicate this.

B. Tensor Spherical Harmonics and Multipole Radiation

This appendix contains material useful for solving the tensor wave equation. The conventions of Edmonds[(18)] have been followed, and that book contains all the formulae needed to supplement the material presented here*.

In the study of radiation from confined systems, we can introduce a great deal of simplification by separating the sources and the solutions into "multipoles" having simple behavior under rotations. The ordinary wave equation does not mix the multipoles, and for slow motions only a few multipoles will dominate the radiation field.

The angular functions that will be defined here have the following features. They behave simply under rotations and form irreducible representations of the rotation group. They have simple formulae for their gradients, curls, and divergences. Finally, they can be computed with relative ease.

Scalar Harmonics. The rotations of a scalar field are generated by the operators

$$L_a = \frac{1}{i}\,\varepsilon_{abc}\,x_b\,\partial_c \qquad \begin{pmatrix} a,b,c \text{ summed} \\ \text{over } x,y,z \end{pmatrix}, \tag{A8}$$

and the scalar harmonics are eigenfunctions of the operators

* The material in this Appendix is presented not because it is original but because it is relatively inaccessible. Tensor spherical harmonics were developed from the rotation group by Jon Mathews, J. Soc. Indust. Appl. Math. 10, 768 (1962).

$L^2 = L_a L_a$ and L_z:

$$L^2 \Upsilon_{LM} = L(L+1) \Upsilon_{LM} , \tag{A9a}$$

$$L_z \Upsilon_{LM} = M \Upsilon_{LM}. \tag{A9b}$$

These Υ_{LM} are useful for solving the wave equation

$$\nabla^2 \phi - \partial_t^2 \phi = f, \tag{A10}$$

which in spherical coordinates can be written

$$\frac{1}{r} \partial_r^2 r\phi - \frac{L^2 \phi}{r^2} - \partial_t^2 \phi = f \tag{A11}$$

where we have taken

$$r^2 \equiv x_a x_a \quad ; \qquad r\partial_r \equiv x_a \partial_a . \tag{A12}$$

Note that we treat both ∂_r and r as operators and omit the redundant parenthesis. If we write both the solution and the source as a sum of multipoles:

$$\phi = \sum_{L,M} \phi_{LM}(r,t) \Upsilon_{LM}(\Omega) , \tag{A13a}$$

$$f = \sum_{L,M} f_{LM}(r,t) \Upsilon_{LM}(\Omega), \tag{A13b}$$

we find that the multipole coefficients (now only functions of r and t) must satisfy

$$\frac{1}{r}\partial_r^2 r\phi_{LM} - \frac{L(L+1)}{r^2}\phi_{LM} - \partial_t^2\phi_{LM} = f_{LM} \tag{A14}$$

which I refer to as the "radial equation".

The vector and tensor spherical harmonics will be combinations of constant unit vectors and unit tensors with these scalar harmonics. Since ∇^2 will then act only on the spatial variables, the radial dependence of the vector and tensor spherical harmonics will be given by the same radial equation (A14) as for scalar spherical harmonics.

The Radial Equation. As mentioned in the preceding section, the only equation that we will have to solve is the "spherical wave equation" (A14). For harmonic motions the radial solutions are the Hankel functions (spherical Bessel functions). For our purposes no real simplification results from the consideration of harmonic motions and here we find the general solutions of the radial equation.

We define an operator W_L by

$$W_L\phi \equiv \frac{1}{r}\partial_r^2 r\phi - \frac{L(L+1)}{r^2}\phi - \partial_t^2\phi\,, \tag{A15}$$

and consider the equation

$$W_L\phi = 0. \tag{A16}$$

One can easily derive the following commutation relations

$$[\partial_r, r] = 1 \tag{A17a}$$

$$[W_L, \partial_r] = \frac{2}{r^2}\partial_r - \frac{2L(L+1)}{r^3} \tag{A17b}$$

$$\left[\mathcal{W}_L , \frac{1}{r} \right] = - \frac{2}{r^2} \partial_r , \tag{A17c}$$

from which one can verify that if φ_L satisfies

$$\mathcal{W}_L \phi_L = 0 \tag{A18}$$

then

$$\mathcal{W}_{L+1} \left(\partial_r - \frac{L}{r} \right) \phi_L = 0 \tag{A19a}$$

$$\mathcal{W}_{L-1} \left(\partial_r + \frac{L+1}{r} \right) \phi_L = 0 . \tag{A19b}$$

These relations give us raising and lowering operators for our solutions. We can abbreviate these as

$$D_L^+ = \partial_r - \frac{L}{r} , \tag{A20a}$$

$$D_L^- = \partial_r + \frac{L+1}{r} . \tag{A20b}$$

There is no problem in finding the solution of the radial equation for L = 0. If we multiply the equation by r, we obtain the usual one-dimensional wave equation for the quantity $r\varphi_0$. Thus the general L = 0 solutions are given by

$$\phi_0 = \frac{C(t \mp r)}{r} , \tag{A21}$$

where C can be any function of one variable. The upper sign corresponds to outgoing waves at infinity.

Solutions for higher L values may be obtained by using D^+ repeatedly. We have, for example,

$$\phi_1 = \frac{C'(t \mp r)}{r} \pm \frac{C(t \mp r)}{r^2} , \tag{A22a}$$

$$\phi_2 = \frac{C''(t \mp r)}{r} \pm 3 \frac{C'(t \mp r)}{r^2} + 3 \frac{C(t \mp r)}{r^3} , \tag{A22b}$$

(normalizing the 1/r term). We can conveniently abbreviate these radial solutions by writing just the leading term and the L value. Thus we can write the L = 1 solution

$$\phi_1 \longleftrightarrow \left\{ C'(t \mp r) \right\}_1 \tag{A23}$$

and our raising and lowering operators applied to these radial functions yield new radial functions according to

$$D_L^+ \left\{ C \right\}_L = \mp \left\{ C' \right\}_{L+1} \tag{A24a}$$

$$D_L^- \left\{ C \right\}_L = \mp \left\{ C' \right\}_{L-1} \tag{A24b}$$

These raising and lowering operators will appear throughout the formulae for the gradient, divergence, etc. of our spherical harmonics.

These functionals $\{C\}_L$ have been normalized by the size of their radiation (that is, the size of their 1/r terms). We will also be interested in the behavior of these functions for small r. This can be found from the Taylor's Series expansion of

$$\left\{ C^{(p+L)}(t \mp r) \right\}_L = (\mp)^L \prod_{q=0}^{L-1} D_q^+ \left\{ C^{(p)}(t \mp r) \right\}_0 \tag{A25}$$

which gives us

$$\left\{ C^{(Q)}(t \mp r) \right\}_L = \sum_{p=0}^{\infty} (\mp)^{p+L} \frac{(p-1)(p-3)\cdots(p-2L+1)}{p!} r^{(p-L-1)} C^{(p+Q-L)}(t). \tag{A26}$$

The leading term in the small r limit is the p = 0 term*:

$$\left\{ C^{(Q)} \right\}_L \longrightarrow (\pm)^L (2L-1)!! \frac{C^{(Q-L)}(t)}{r^{L+1}} , \tag{A27}$$

and the leading time-odd term in the small r limit is the p = 2L + 1 term:

$$\left\{ C^{(Q)} \right\}_L \Big|_R = (\mp)^{L+1} \frac{r^L C^{(L+Q+1)}(t)}{(2L+1)!!} . \tag{A28}$$

This term is time-odd relative to the induction field p = 0 term, and determines the time-odd effects in the small r limit.

Vector Spherical Harmonics. One can easily generalize these spherical harmonics to vector fields. To do so, however, one must augment the operator $\mathbb{L}$ with an operator that shifts the components of vectors among themselves during the rotation. The rotation of a vector field is generated by the operator $\mathbb{J}$ defined by

* Note that (2L + 1)!! ≡ (2L + 1)(2L - 1) ... (1), is the "odd factorial".

$$J_a V^c \equiv L_a V^c + i\,\varepsilon_{abc}\, e_a V^b \tag{A29}$$

$$\left(a, b, c = x, y, z \;;\; e_a = e_x, e_y, e_z = \hat{x}, \hat{y}, \hat{z} \right),$$

which is formed from the Lie derivatives along the Killing congruences generating the rotations.

We will define special vector fields, called vector spherical harmonics, that are simultaneous eigenfunctions of J^2, L^2, and J_z, that is

$$J^2\, \mathbf{Y}_{JLM} = J(J+1)\, \mathbf{Y}_{JLM}, \tag{A30a}$$

$$L^2\, \mathbf{Y}_{JLM} = L(L+1)\, \mathbf{Y}_{JLM}, \tag{A30b}$$

$$J_z\, \mathbf{Y}_{JLM} = M\, \mathbf{Y}_{JLM}. \tag{A30c}$$

The first step is to combine the basis vectors so that they form a representation of spin 1 under the rotation generators $\mathbf{S}$ defined by

$$\mathbf{S} \equiv i\, \mathbf{e}\, \times \tag{A31}$$

The combinations can be written down by inspecting Y_{1M}, which is also a representation of spin 1,

$$e_0 = \hat{z}$$

$$e_{\pm 1} = \frac{1}{\sqrt{2}}\left(\mp \hat{x} - i\,\hat{y} \right); \tag{A32}$$

and they indeed satisfy

$$S^2 \mathbb{e}_m = 2 \mathbb{e}_m , \tag{A33a}$$

$$S_z \mathbb{e}_m = m \mathbb{e}_m . \tag{A33b}$$

Our vector spherical harmonics are formed by adding the intrinsic spin 1 of the basis vectors to the spin L of the spatial dependence of the Y_{LM} according to the rules of angular momentum:

$$\mathbb{Y}_{JLM} \equiv \sum_s \begin{bmatrix} L & 1 & J \\ M-s & s & M \end{bmatrix} Y_{L,M-s} \mathbb{e}_s , \tag{A34}$$

where $\begin{bmatrix} L_1 & L_2 & J \\ M_1 & M_2 & M \end{bmatrix}$ is a Clebsch-Gordon coefficient. Many of the $\mathbb{Y}_{JLM}$ are identically zero because of the selection rules for the Clebsch-Gordon coefficients. For example, J and L can differ by at most one unit.

We can easily form solutions of the vector wave equation from these vector spherical harmonics and the radial functions defined earlier by taking

$$\mathbb{V}_{JLM} = \left\{ C^{(L)} \right\}_L \mathbb{Y}_{JLM}. \tag{A35}$$

The most important relation for the manipulation of spherical harmonics is the gradient relation:

$$\nabla\left(\phi(r,t)\, Y_{LM}\right) = -\sqrt{\frac{L+1}{2L+1}}\; D_L^+\phi\;\; \mathbf{Y}_{L,L+1,M}$$

$$+\sqrt{\frac{L}{2L+1}}\; D_L^-\phi\;\; \mathbf{Y}_{L,L-1,M}\,, \tag{A36}$$

which is proved in Edmunds and other places. From this and the rules for adding three angular momenta one can deduce the divergence relations:

$$\nabla\cdot\left(\phi\, \mathbf{Y}_{L,L+1,M}\right) = -\sqrt{\frac{L+1}{2L+1}}\; D_{L+1}^-\phi\;\; Y_{LM}\,, \tag{A37a}$$

$$\nabla\cdot\left(\phi\, \mathbf{Y}_{L,L,M}\right) = 0\,, \tag{A37b}$$

$$\nabla\cdot\left(\phi\, \mathbf{Y}_{L,L-1,M}\right) = +\sqrt{\frac{L}{2L+1}}\; D_{L-1}^+\phi\;\; Y_{LM}\,, \tag{A37c}$$

and the curl relations

$$\nabla\times\left(\phi\, \mathbf{Y}_{L,L+1,M}\right) = i\sqrt{\frac{L}{2L+1}}\; D_{L+1}^-\phi\;\; \mathbf{Y}_{LLM}\,, \tag{A38a}$$

$$\nabla\times\left(\phi\, \mathbf{Y}_{LLM}\right) = i\sqrt{\frac{L}{2L+1}}\; D_L^+\phi\;\; \mathbf{Y}_{L,L+1,M} \tag{A38b}$$

$$+i\sqrt{\frac{L+1}{2L+1}}\; D_L^-\phi\;\; \mathbf{Y}_{L,L-1,M}\,,$$

$$\nabla\times\left(\phi\, \mathbf{Y}_{L,L-1,M}\right) = i\sqrt{\frac{L+1}{2L+1}}\; D_L^+\phi\;\; \mathbf{Y}_{LLM}. \tag{A38c}$$

Manipulations with the vector spherical harmonics are often simplified if one uses the spherical vectors $\mathbb{e}_0$, $\mathbb{e}_{\pm 1}$ instead of the Cartesian vectors $\mathbb{e}_x$, $\mathbb{e}_y$, $\mathbb{e}_z$ as basis vectors. One can define the spherical components of a vector $\mathbb{V}$ by

$$(\mathbb{V})_q \equiv \mathbb{e}_q \cdot \mathbb{V} \qquad (q = -1, 0, +1). \tag{A39}$$

This can be inverted

$$\mathbb{V} = \sum_q (-)^q V_q \, \mathbb{e}_{-q} \,, \tag{A40}$$

using the normalization of the spherical basis vectors

$$\mathbb{e}_q \cdot \mathbb{e}_{q'} = (-)^q \, \delta_{q,-q'}. \tag{A41}$$

The spherical components of the vector spherical harmonics are

$$(\mathbb{Y}_{JLM})_s = \begin{bmatrix} L & 1 & J \\ M+s & -s & M \end{bmatrix} Y_{L,M+s} (-)^s. \tag{A42}$$

The dot product of two vectors can be thought of as the multiplication of a spin 1 representation by another spin 1 representation to form a spin 0 representation. One can thus write

$$\mathbb{A} \cdot \mathbb{B} = -\sqrt{3} \sum_s \begin{bmatrix} 1 & 1 & 0 \\ s & -s & 0 \end{bmatrix} A_s B_{-s} = A_0 B_0 - A_{-1} B_1 - A_1 B_{-1}. \tag{A43}$$

Similarly, the curl can be thought of as adding spin 1 to spin 1 to get spin 1

$$(\mathbb{A} \times \mathbb{B})_r = \frac{\sqrt{2}}{i} \sum_s \begin{bmatrix} 1 & 1 & 1 \\ r-s & s & r \end{bmatrix} A_{r-s} B_s \,. \tag{A44}$$

The gradient relation can be written in terms of spherical components

$$\left[\nabla(\phi \Upsilon_{LM})\right]_S = -\sqrt{\frac{L+1}{2L+1}}\, D_L^+\phi \begin{bmatrix} L+1 & 1 & L \\ M+S & -S & M \end{bmatrix} (-)^S\, \Upsilon_{L+1,\,M+S}$$

$$+\sqrt{\frac{L}{2L+1}}\, D_L^-\phi \begin{bmatrix} L-1 & 1 & L \\ M+S & -S & M \end{bmatrix} (-)^S\, \Upsilon_{L-1,\,M+S}\,. \tag{A45}$$

The vector spherical harmonics are normalized such that

$$\int_0^{2\pi}\!\!\int_0^{\pi} \mathbf{\Upsilon}^*_{JLM} \cdot \mathbf{\Upsilon}_{J'L'M'} \sin\theta\, d\theta\, d\varphi = \delta_{JJ'}\,\delta_{LL'}\,\delta_{MM'} \tag{A46}$$

and their complex conjugates are given by

$$\mathbf{\Upsilon}^*_{JLM} = (-)^{J+L+M+1}\, \mathbf{\Upsilon}_{J,L,-M}\,. \tag{A47}$$

<u>Tensor Spherical Harmonics</u>. The general tensor is not an irreducible representation of spin-2 at a point. The trace behaves like a scalar under rotations and the antisymmetric part like a vector. The spin-2 part of a tensor is its traceless, symmetric part.

We can easily generate a suitable set of basis tensors by combining the basis vectors used in the previous section, We define five basis tensors $\mathbf{t}_p$, ($p = -2, -1, 0, 1, 2$) by

$$\mathfrak{t}_p = \sum_s \begin{bmatrix} 1 & 1 & 2 \\ p-s & s & p \end{bmatrix} \mathscr{e}_{p-s}\, \mathscr{e}_s\,, \tag{A48}$$

which is equivalent to

$$\mathfrak{t}_0 = \frac{1}{\sqrt{6}}\left(2\hat{z}\hat{z} - \hat{x}\hat{x} - \hat{y}\hat{y}\right), \tag{A49a}$$

$$\mathfrak{t}_{\pm 1} = \frac{1}{2}\left(\mp \hat{x}\hat{z} \mp \hat{z}\hat{x} - i\hat{y}\hat{z} - i\hat{z}\hat{y}\right), \tag{A49b}$$

$$\mathfrak{t}_{\pm 2} = \frac{1}{2}\left(\hat{x}\hat{x} - \hat{y}\hat{y} \pm i\hat{x}\hat{y} \pm i\hat{y}\hat{x}\right). \tag{A49c}$$

Using these basis tensors, we can construct tensor fields that are simultaneous eigenfunctions of J^2, L^2 and J_z just as we constructed the vector spherical harmonics. Now, of course, we are using the operator $\mathfrak{J}$ suitable for rotating a tensor field. This can be found from the Lie derivative, but we will not need to be explicit about it. We define tensor spherical harmonics $\mathbb{T}_{JLM}$ by

$$\mathbb{T}_{JLM} = \sum_s \begin{bmatrix} L & 2 & J \\ M-s & s & M \end{bmatrix} Y_{L,M-s}\, \mathfrak{t}_s\,. \tag{A50}$$

Perhaps the most useful formulae for these tensor fields relates the divergence of a tensor spherical harmonic to vector spherical harmonics. One can write this in the form:

$$\nabla\cdot\left(\phi\,\mathbb{T}_{JLM}\right) = (-)^{J+L+1}\sqrt{5}\left[-\sqrt{L+1}\begin{Bmatrix} L+1 & 1 & L \\ 2 & J & 1 \end{Bmatrix}\Psi_{J,L+1,M}\,D_L^{+}\phi\right.$$

$$\left. +\sqrt{L}\begin{Bmatrix} L-1 & 1 & L \\ 2 & J & 1 \end{Bmatrix}\Psi_{J,L-1,M}\,D_L^{-}\phi\right], \qquad \text{(A51)}$$

where the symbol $\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}$ is a 6-j symbol, tabulated in several places[22], or one can use the formulae for the 6-j symbols given in Edmunds to write out the 5 cases explicitly.

$$\nabla\cdot\left(\phi\,\mathbb{T}_{L,L-2,M}\right) = \sqrt{\frac{(L-1)}{(2L-1)}}\;D_{L-2}^{+}\phi\;\Psi_{L,L-1,M}\;, \qquad \text{(A52a)}$$

$$\nabla\cdot\left(\phi\,\mathbb{T}_{L,L-1,M}\right) = \sqrt{\frac{(L-1)}{2(2L+1)}}\;D_{L-1}^{+}\phi\;\Psi_{LLM}\;, \qquad \text{(A52b)}$$

$$\nabla\cdot\left(\phi\,\mathbb{T}_{LLM}\right) = \sqrt{\frac{L(2L-1)}{6(2L+1)(2L+3)}}\;D_L^{+}\phi\;\Psi_{L,L+1,M}$$

$$-\sqrt{\frac{(L+1)(2L+3)}{6(2L-1)(2L+1)}}\;D_L^{-}\phi\;\Psi_{L,L-1,M}\;, \qquad \text{(A52c)}$$

$$\nabla\cdot\left(\phi\,\mathbb{T}_{L,L+1,M}\right) = -\sqrt{\frac{(L+2)}{2(2L+1)}}\;D_{L+1}^{-}\phi\;\Psi_{LLM}\;, \qquad \text{(A52d)}$$

$$\nabla\cdot\left(\phi\,\mathbb{T}_{L,L+2,M}\right) = -\sqrt{\frac{(L+2)}{(2L+3)}}\;D_{L+2}^{-}\phi\;\Psi_{L,L+1,M}\;. \qquad \text{(A52e)}$$

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22. E.g., Rotenberg, M. et al., 1959, "The 3-j and 6-j Symbols" (M.I.T.).

In discussing the gauge transformations useful in weak-field gravity we will need the symmetrized gradients of our vector spherical harmonics expressed as tensor spherical harmonics. We define these by

$$\left(\underbrace{\nabla \mathbf{V}}\right)_{ab} \equiv \partial_a V_b + \partial_b V_a \qquad (a, b = x, y, z). \tag{A53}$$

A straightforward computation gives us

$$\underbrace{\nabla \phi \mathbf{Y}_{JLM}} - \tfrac{2}{3}\mathbb{I}\left(\nabla \cdot \phi \mathbf{Y}_{JLM}\right) =$$

$$(-)^{J+L+1}\, 2\sqrt{5}\left[-\sqrt{(L+1)}\; D_L^{+}\phi \begin{Bmatrix} L+1 & 1 & L \\ 1 & J & 2 \end{Bmatrix} \mathbb{T}_{J,L+1,M}\right. \tag{A54}$$

$$\left. + \sqrt{L}\; D_L^{-}\phi \begin{Bmatrix} L-1 & 1 & L \\ 1 & J & 2 \end{Bmatrix} \mathbb{T}_{J,L-1,M}\right],$$

where

$$(\mathbb{I})_{ab} \equiv \delta_{ab}. \tag{A55}$$

<u>Multipole Fields</u>. Using these results we can write down multipole solutions to vector and tensor wave equations practically by inspection. The equations of electromagnetism in the wave zone (no sources!) with Lorentz gauge can be written:

$$\Box \mathbb{A} = 0 \tag{A56a}$$

$$\Box \phi = 0 \tag{A56b}$$

$$\nabla\cdot\mathbb{A} + \partial_t\phi = 0. \tag{A56c}$$

We can find solutions of these equations corresponding to a definite parity. Solutions whose parity is the same as that of a current form what is called magnetic multipole radiation. These solutions can be written

$$\begin{aligned} \mathbb{A} &= \left\{C^{(L)}(t\mp r)\right\}_L \mathbb{Y}_{LLM} \\ \phi &= 0. \end{aligned} \tag{A57}$$

The solutions that couple to charge density form what is called electric multipole radiation, and can be written in two ways:

$$\begin{aligned} \mathbb{A} &= \pm\sqrt{\frac{2L+1}{L}}\left\{C^{(L)}(t\mp r)\right\}_{L-1} \mathbb{Y}_{L,L-1,M} \\ \phi &= \left\{C^{(L)}(t\mp r)\right\}_L Y_{LM}, \end{aligned} \tag{A58}$$

or

$$\begin{aligned} \tilde{\mathbb{A}} &= \mp\sqrt{\frac{2L+1}{L+1}}\left\{D^{(L)}(t\mp r)\right\}_{L+1} \mathbb{Y}_{L,L+1,M} \\ \tilde{\phi} &= \left\{D^{(L)}(t\mp r)\right\}_L Y_{LM} \end{aligned} \tag{A59}$$

These are different representations of the same solution and must be related by a gauge transformation

$$\tilde{\mathbb{A}} = \mathbb{A} + \nabla\chi \tag{A60a}$$

$$\tilde{\phi} = \phi - \partial_t\chi \tag{A60b}$$

for some function χ.

To find χ note that $\partial_t\chi$ must be proportional to $\{C^{(L)}(t \mp r)\}_L\, Y_{LM}$. Thus we are led to try

$$\chi = \alpha\left\{C^{(L-1)}(t \mp r)\right\}_L Y_{LM}\,, \tag{A61}$$

and to determine α so as to cancel the $\mathbb{Y}_{L,L-1,M}$ term in the vector potential. Now we have (from equation A60a)

$$\tilde{\mathbb{A}} = \pm\alpha\sqrt{\frac{L+1}{2L+1}}\left\{C^{(L)}\right\}_{L+1}\mathbb{Y}_{L,L+1,M} + \left(\mp\alpha\sqrt{\frac{L}{2L+1}} \pm \sqrt{\frac{2L+1}{L}}\right)\left\{C^{(L)}\right\}_{L-1}\mathbb{Y}_{L,L-1,M}\,, \tag{A62}$$

where we have used the gradient relation (equation A36). Comparing this with equation (A59) we see that

$$\alpha = \frac{2L+1}{L}\,. \tag{A63}$$

With this form for χ we have

$$\tilde{\phi} = \left(1 - \frac{2L+1}{L}\right)\left\{C^{(L)}(t \mp r)\right\}_L Y_{LM} = -\frac{L+1}{L}\,\phi\,. \tag{A64}$$

This relation will be useful in the problem of radiation reaction for electromagnetism.

Gravitational multipoles can be written in an analogous manner. The equations of weak-field gravity (equations 106 a-e) read

$$\Box \mathbb{H} = 0 \tag{A65a}$$

$$\Box \mathbb{V} = 0 \tag{A65b}$$

$$\Box \Psi = 0 \tag{A65c}$$

$$\nabla \cdot \mathbb{H} + \partial_t \mathbb{V} = 0 \tag{A65d}$$

$$\nabla \cdot \mathbb{V} + \partial_t \Psi = 0 \tag{A65e}$$

Electric parity (i.e., those which couple to mass density) multipole solutions of these may be written in two different ways:

$$\mathbb{H} = \sqrt{\frac{(2L+1)(2L-1)}{L(L-1)}} \left\{ C^{(L)}(t \mp r) \right\}_{L-2} \mathbb{T}_{L,L-2,M} , \tag{A66a}$$

$$\mathbb{V} = \pm \sqrt{\frac{(2L+1)}{L}} \left\{ C^{(L)}(t \mp r) \right\}_{L-1} \mathbb{Y}_{L,L-1,M} , \tag{A66b}$$

$$\Psi = \left\{ C^{(L)}(t \mp r) \right\}_{L} \mathbb{Y}_{LM} \quad ; \tag{A66c}$$

or

$$\tilde{\mathbb{H}} = \sqrt{\frac{(2L+1)(2L+3)}{(L+1)(L+2)}} \left\{ D^{(L)}(t \mp r) \right\}_{L+2} \mathbb{T}_{L,L+2,M} , \tag{A67a}$$

$$\tilde{\mathbb{V}} = \mp\sqrt{\frac{2L+1}{L+1}}\left\{D^{(L)}(t\mp r)\right\}_{L+1} \mathbb{Y}_{L,L+1,M}\ , \tag{A67b}$$

$$\tilde{\psi} = \left\{D^{(L)}(t\mp r)\right\}_{L} Y_{LM}. \tag{A67c}$$

There must be a transformation relating these two representations (cf. equations 107.2)

$$\tilde{\mathbb{H}} = \mathbb{H} + \underbrace{\nabla\mathbb{X}} - (\nabla\cdot\mathbb{X} - \partial_t\chi)\mathbb{I}\ , \tag{A68a}$$

$$\tilde{\mathbb{V}} = \mathbb{V} + \nabla\chi - \partial_t\mathbb{X}\ , \tag{A68b}$$

$$\tilde{\psi} = \nabla\cdot\mathbb{X} + \partial_t\chi + \psi\ . \tag{A68c}$$

If we take our gauge 3-vector in the form

$$\mathbb{X} = A\left\{C^{(L-1)}(t\mp r)\right\}_{L-1} \mathbb{Y}_{L,L-1,M} + B\left\{C^{(L-1)}(t\mp r)\right\}_{L+1} \mathbb{Y}_{L,L+1,M} \tag{A69}$$

dictated by parity considerations, then we can determine the constants A,B as follows. Using the formulae for the symmetrized gradient of a vector spherical harmonic equations (A54) and using the formulae in Edmunds (1960) to work out the 6-j symbols in terms of L, we can write equation (A68a)

$$\tilde{\mathbb{H}} = \left[\sqrt{\frac{(2L+1)(2L-1)}{L(L-1)}} - 2A\sqrt{\frac{L-1}{2L-1}} \right] \mathbb{T}_{L,L-2,M}$$

$$+ \left[A\sqrt{\frac{(2L+2)(2L+3)}{3(2L-1)(2L+1)}} - B\sqrt{\frac{2L(2L-1)}{3(2L+1)(2L+3)}} \right] \mathbb{T}_{L,L,M}$$

$$+ 2B\sqrt{\frac{L+2}{2L+3}}\, \mathbb{T}_{L,L+2,M} \qquad \text{(A70)}$$

$$+ \left[\text{terms in } \nabla\cdot\chi,\ \partial_t\chi,\ \psi \right] \mathbb{I}.$$

The various terms in this equation must vanish separately.

Noting that $\tilde{\mathbb{H}}$ has neither a $\mathbb{T}_{L,L-2,M}$ nor a $\mathbb{T}_{L,L,M}$ component, we can find A by setting the coefficient of $\mathbb{T}_{L,L-2,M}$ equal to zero, and then B by setting the coefficient of $\mathbb{T}_{L,L,M}$ equal to zero. Knowing the size of B gives us the size of the $\mathbb{T}_{L,L+2,M}$ term. Finally we must have $D(t \mp r)$ proportional to $C(t \mp r)$, and the ratio can be found by comparing the size of the $\mathbb{T}_{L,L+2,M}$ term resulting from the above gauge transformation with the $\mathbb{T}_{L,L+2,M}$ term appearing in equation (A67a). This constant of proportionality can then be used to find that

$$\tilde{\psi} = \frac{L+1}{L} \frac{L+2}{L-1} \psi \tag{A71}$$

which is the result of interest.

Magnetic parity (i.e., those which couple to mass currents) multipole solutions may also be written in two ways:

$$\mathbb{H} = \pm \sqrt{\frac{2(2L+1)}{L-1}} \left\{ C^{(L)}(t \mp r) \right\}_{L-1} \mathbb{T}_{L,L-1,M} \quad , \tag{A72a}$$

$$\mathbb{V} = \left\{ C^{(L)}(t \mp r) \right\}_{L} \mathbb{Y}_{LLM} \quad ; \tag{A72b}$$

or

$$\tilde{\mathbb{H}} = \mp \sqrt{\frac{2(2L+1)}{L+2}} \left\{ D^{(L)}(t \mp r) \right\}_{L+1} \mathbb{T}_{L,L+1,M} \tag{A73a}$$

$$\tilde{\mathbb{V}} = \left\{ D^{(L)}(t \mp r) \right\}_{L} \mathbb{Y}_{LLM} . \tag{A73b}$$

Analysis similar to that carried out above shows that

$$\tilde{\mathbb{V}} = - \frac{L}{L+1} \mathbb{V} . \tag{A74}$$

These formulae are all that we will need to develop the energy loss formalism for gravitational radiation.

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