COMPLEX ANGULAR MOMENTUM IN THREE-PARTICLE POTENTIAL SCATTERING

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ABSTRACT

The continuation of three-particle partial wave scattering amplitudes to complex values of the total angular momentum is discussed in the framework of potential scattering. We show that if there is a continuation for which a Watson-Sommerfeld transformation of the full scattering amplitude can be made, then it is unique and determines the behavior of the amplitude for large values of any single scattering angle. A non-rigorous construction of such a continuation is given for an amplitude which describes a scattering in which a given pair of the particles is bound in the initial and final states. Except for simple kinematic factors, the only singularities of this continuation are poles and possibly isolated essential singularities. The results are generalized to cases when exchange forces are present.

As a simple application of the results, we discuss a crude nuclear model to illustrate how sequences of rotational levels can be described by Regge trajectories.

The behavior of Regge trajectories near two- and three-particle thresholds is explored.

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I. INTRODUCTION

A simple description of two-particle potential scattering amplitudes at large momentum transfers has been given by Regge and others (1-4) in terms of the poles in the partial wave amplitude at complex values of the angular momentum. For a given energy E, suppose that the singularity furthest right in the angular momentum plane is a pole at position $\alpha(E)$ with residue $\beta(E)$. At large momentum transfers, t, Regge finds that the amplitude then behaves like

$$T(E,t) \sim \frac{(2\alpha+1)\beta(E)}{\sin \pi\alpha(E)} \left[1 \pm e^{i\pi\alpha}\right] \left(\frac{t}{2E}\right)^{\alpha(E)} \qquad (1.1)$$

Here, the $\pm$ sign refers to the signature, a quantum number which can be associated with every Regge trajectory in a two-particle problem (5).

The beauty of Regge's description lies in the fact that poles are the <u>only</u> angular momentum singularities of a two-particle potential scattering amplitude, so that at large momentum transfers the amplitude always has the simple form of equation 1.1. Moreover, these poles can be correlated with the bound and resonant states of the two-particle system. The energies of the bound states and resonances are the values at which Re $\alpha(E)$ is a positive integer and the width of a resonance is proportional to Im $\alpha(E)$. The question naturally arises as to whether this simplicity persists when many-particle scattering processes are considered.

This question is of interest for several reasons.

(1) The crossing relation for two-particle relativistic scattering amplitudes implies that the large momentum transfer behavior of the scattering amplitudes is the high energy behavior of the amplitude for the cross channel. This behavior is of great interest in current theories of elementary particles because, if the ideas of Regge are applicable to relativistic amplitudes, the directly observable high energy behavior of one channel will be related to the low energy particles and resonances exchanged in the cross channel (5-6). At high energies, however, two-particle channels are always coupled to channels of higher particle number through the possibility of particle production. This coupling is evident from the unitarity relation for a two-particle scattering amplitude $f_{A+B\rightarrow A'+B'}$, which can be written schematically as

$$\mathrm{Im}\, f_{A+B\rightarrow A'+B'} = \sum_{\text{(intermediate states)}''} \Big[f_{A+B\rightarrow A''+B''} f_{A''+B''\rightarrow A'+B'} + f_{A+B\rightarrow A''+B''+C''} f_{A''+B''+C''\rightarrow A'+B'} + \cdots \Big] \tag{1. 2}$$

In turn, the amplitudes $f_{A+B\rightarrow A'+B'+C'}$ will be related to the amplitudes $f_{A+B+C\rightarrow A'+B'+C'}$. If the behavior of many-particle amplitudes at complex values of the angular momentum is qualitatively different from those involving only two-particle channels, we may expect it to be reflected in the properties of the relativistic two-particle amplitudes. In particular, it is important to determine whether this coupling gives use to cuts in the angular momentum plane which would complicate the simple asymptotic behavior of equation 1.1. If cuts are present, the dynamical mechanism which produces them should be isolated and their positions and discontinuities investigated. Mandelstam (7) has recently made an important step in this direction. By applying the unitarity relation (equation 1.2) to a particular class of Feynman diagrams he has shown that angular momentum cuts can be expected in relativistic two-particle amplitudes. His work will be discussed in more detail later in the thesis.

Three-particle potential scattering provides a simple starting point for investigating the analytic properties of many particle amplitudes and a model in which some degree of rigor can presumably be obtained. If past experience is a guide, potential scattering should possess many of the features of the relativistic problem but not all of them. A potential scattering model, however, should serve to isolate the dynamical mechanism through which these relativistic features arise. In this way the results of a potential scattering model can give some direction to investigations of the analytic properties of relativistic amplitudes.

(2) The continuation of many-particle potential scattering amplitudes to complex values of the angular momentum is of interest in its own right because of possible applications to scattering processes in which nuclei are involved. For instance, one would like to understand in more detail how sequences of nuclear rotational levels can be correlated with definite Regge trajectories. Before any useful approximate descriptions can be made it is desirable to know the analytic properties of the exact amplitudes.

For these reasons, we examine in this thesis the behavior of many-particle potential scattering amplitudes at complex values of the total angular momentum. For simplicity, we have considered only amplitudes which involve spinless and non-identical particles. Where specific assumptions about the potential need to be made, we will assume the particles are interacting through two-body Yakawa forces.

In Section II some general properties of the Watson-Sommerfeld transformation for many-particle amplitudes are discussed. Here, we will make a specific but weak assumption about the behavior of the amplitude in the complex angular momentum plane. We will assume what is equivalent to the statement that the amplitude has at most a polynomial behavior at large complex values of the cosine of a scattering angle. One can then show that a continuation which satisfies this assumption is unique and determines the asymptotic behavior of the full amplitude in the cosine of <u>any</u> single scattering angle. The problem of finding the asymptotic behavior of the amplitude in this wide class of momentum transfers thus reduces to finding a single analytic continuation of the partial wave amplitude in the complex angular momentum plane.

In this section we also discuss the continuation of the partial wave Schrödinger equation to complex values of the angular momentum. If the two-particle case is a guide, the solution of the analytically continued equation will yield the partial wave amplitude from which the large momentum transfer behavior of the full amplitude can be determined. For integer values of the angular momentum, L, the partial wave Schrödinger equation is a set of $(2L+1)$ coupled equations, one for each allowed value of the non-conserved projection of L on an axis fixed in the system of particles. For complex values of L we find a much larger set of coupled equations in which the projection can assume any integer value.

The results of Section II are generalized to the case when exchange forces are present in Section III.

Although the discussion of these sections is couched in the language of potential scattering,it is applicable to the Watson-Sommerfeld transformation of the relativistic amplitude provided its partial wave amplitudes obey the weak analyticity assumption.

Section IV contains a discussion of the analytic properties of a class of potential scattering amplitudes which are at once interesting from the point of view of real processes and simple to calculate. These are the amplitudes which describe three-particle scattering in which initially and finally a given pair of the particles is bound. A proof of analyticity is outlined which, although non-rigorous, indicates that with a proper continuation these amplitudes will have only dynamical poles and possibly isolated essential singularities in the angular momentum. A Watson-Sommerfeld transformation of the full amplitude can be performed to yield a simple description of the large momentum transfer behavior analogous to equation 1.1. This justifies the assumptions used earlier in deriving the general properties of the Watson-Sommerfeld transformation for this class of amplitudes.

The particular continuation explored here is compared with some work of Newton (8) and Drummond (9) who give a different continuation for these amplitudes in which cuts appear in the angular momentum plane. Their continuation, however, will not give the asymptotic behavior of the full amplitude in any momentum transfer. Indeed, if the proof given here can be fully justified, the continuation given in this thesis is the unique one from which the large momentum transfer behavior can be determined.

The results of Mandelstam for the relativistic problem are shown not to be in conflict with the analytic properties given here since they explicitly depend on the possibility of particle production.

Sections V and VI contain two simple applications of the previous work. In Section V a crude nuclear model, which consists of a single particle outside of a rigid core, is considered at complex values of the total angular momentum. The presence of the larger set of equations discussed above and the corresponding larger set of amplitudes allows the theory to describe sequences of nuclear rotational levels whose ground state spin is greater than zero or one-half. At every integer the trajectory can choose to appear either in a physical or unphysical amplitude. If it appears in the unphysical amplitudes for low spins and in the physical quantities for higher values, a Regge trajectory describes such a sequence of levels.

In Section VI the behavior of Regge trajectories at certain thresholds is determined. Specifically, we consider the behavior at the lowest threshold in the case when it is for a) the scattering of a single particle off a bound system, and b) three free particle scattering.

II. THE PARTIAL WAVE EXPANSION OF A MANY-PARTICLE SCATTERING AMPLITUDE

Every many-particle system has three degrees of freedom which correspond to total rotations. Invariance of the Hamiltonian under these rotations implies the conservation of the total angular momentum L and its projection on a space-fixed axis M. The coordinates which specify rotations of the entire system we may take to be three Euler angles φ, θ, ψ, relating a "space-fixed" and a "body-fixed" set of Cartesian coordinates.* There are many ways of specifying these angles. Each way corresponds to a definite convention as to how the body-fixed axes are fixed in the system of particles.**

A complete set of commuting observables conjugate to the three Euler angles are $\vec{L}^2$, L_z and L_z', the total angular momentum and its projection on the space-fixed and body-fixed z-axes respectively. The eigenfunctions of these quantities are discussed in Appendix A and defined by:

$$\begin{aligned}\vec{L}^2 D^L_{MK}(\varphi,\theta,\psi) &= L(L+1) D^L_{MK}(\varphi,\theta,\psi)\\ L_z D^L_{MK}(\varphi,\theta,\psi) &= M D^L_{MK}(\varphi,\theta,\psi)\\ L_z' D^L_{MK}(\varphi,\theta,\psi) &= K D^L_{MK}(\varphi,\theta,\psi)\end{aligned} \tag{2.1}$$

*If L_x, L_y, L_z are components of $\vec{L}$, a rotation R through φ, θ, ψ is given by

$$R(\varphi,\theta,\psi) = e^{iL_z\varphi} e^{iL_y\theta} e^{iL_z\psi}$$

The conventions used in this thesis are those of Ref. 10.

**For some examples see Section IV and Ref. 11.

Consider the scattering of N particles whose initial momenta $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_N$ will be denoted collectively by $\vec{p}$. The scattering wave function is a function of the coordinates $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$, denoted collectively by $\vec{r}$ and the momenta of the incoming wave; $\psi = \psi(\vec{r}, \vec{p})$. Denoting the Euler angles of the coordinates by Ω_r, and those of the momenta by Ω_p, we may expand

$$\psi(\vec{r},\vec{p}) = (8\pi^2)^{-1}\sum (2L+1)\psi^L_{K'K}(r,p)D^L_{MK'}(\Omega_r)D^{L^*}_{MK}(\Omega_p)$$

$$= (8\pi^2)^{-1}\sum (2L+1)\psi^L_{K'K}(r,p)D^L_{KK'}(\Omega_{rp}) \qquad (2.2)$$

The sum ranges over positive integral values of L and integer M, K, K' such that $|M| \leq L$, $|K| \leq L$, $|K'| \leq L$. The last line follows from the addition theorem (10) for the eigenfunctions of $\vec{L}^2, L_z, L'_z$, where Ω_{rp} are the Euler angles of $\vec{r}$ with the body-fixed axes for $\vec{p}$ used as the space-fixed axes. Similarly, the scattering amplitude may be expanded as

$$\langle\vec{p}\,'|T|\vec{p}\rangle = (8\pi^2)^{-1}\sum(2L+1)\,T^L_{K'K}(p',p)D^L_{KK'}(\Omega_{p'p}) \qquad (2.3)$$

If $\mathcal{J}$ is the kinetic energy, the Schrödinger equation is

$$[\mathcal{J}(\vec{r}) - E + V(r)]\,\psi(\vec{r},\vec{p}) = 0 \qquad (2.4)$$

Since the kinetic energy is at most quadratic in the components of $\vec{L}$, it is straightforward using equation A.1 to project out the angles ψ and φ obtaining a partial wave equation

$$\sum_{K''} [\mathcal{J}^{L}_{K'K''}(r) + (V(r) - E)\, \delta_{K'K''}]\, \psi^{L}_{K''K}(r, p) = 0 \qquad (2.5)$$

The potential energy and energy terms are rotational scalars and hence diagonal in K. Again the sum is over $|K''| \leq L$.

For the case of two-particle potential scattering, the scattering amplitude for complex values of the angular momentum is found by writing a radial Schrödinger in which L appears as a parameter and then solving for the scattering solution at complex values of this parameter. The amplitude can be read off of the scattering solution and is the unique one for which a Watson-Sommerfeld transformation of the full amplitude can be made to determine its large momentum transfer behavior.

We will try and follow this program for the case of many particles. A difficulty is that in equation 2.5, L not only appears as a parameter but also determines the number of coupled equations. Specifically, there are $(2L+1)^2$ equations, one for each value of K and K'. How can L become complex if it determines the number of equations?

The difficulty can be overcome by defining additional unphysical equations and unphysical wave functions which are not coupled to the physical quantities for integral values of L and such that the number of equations does not depend on L. Indeed, since equation 2.5 contains only the matrix elements from equation A.1, we can obtain a sensible set of equations at complex L simply by ignoring the restrictions on K, K', and K" in equation 2.5 and allowing the sum to range over all integral values.

An equivalent way of doing this is to continue the $D^L_{KK'}$ to complex L so that they form a matrix with K and K' ranging over all integers from $-\infty$ to $+\infty$. This is discussed in Appendix A. Since $\vec{L}^2$ commutes with the Hamiltonian, we can demand that

$$\psi^L(\vec{r},\vec{p}) = (8\pi^2)^{-1} \sum_{KK'=-\infty}^{\infty} (2L+1)\psi^L_{K'K}(r,p)D^L_{KK'}(\Omega_{rp}) \qquad (2.6)$$

solve the full Schrödinger equation 2.4. Projecting out the angles ψ and φ, we arrive at the infinite set of coupled equations mentioned above. Since the $D^L_{KK'}$ at integral values of L are non-zero only if K and K' are simultaneously greater than or less than L in absolute value, the equations with $|K| \leq L$, $|K'| \leq L$ will decouple from the rest and coincide with the physical ones. This can also be seen from equation A.1. The presence of unphysical wave functions is familiar from the problems involving the scattering of particles with spin (12, 2). L now occurs only as a parameter in the larger set of equations and the solutions can be examined at its complex values.

The asymptotic behavior of the full amplitude can be determined from the singularities in the angular momentum plane if the amplitude has the following properties:

(i) $T^L_{K'K}D^L_{K'K}(\Omega)$ is an analytic function of L with singularities consisting of poles, isolated essential singularities, and cuts confined to the region Re $L < L_o$ for some L_o.

(ii) $T^L_{K'K}$ decreases sufficiently fast as $|L| \to \infty$ so that the integral in the Watson-Sommerfeld transformation over a large semi-circle in the right-hand plane tends to zero as its radius becomes large.

In Section IV we will outline a proof that by solving the Schrödinger equation continued to complex L as above, an analytic continuation of certain amplitudes can be obtained with properties (i) and (ii), and, moreover, that the possibility of cuts can be dispensed with.

Under assumption (i), the partial wave expansion can be written

$$\langle \vec{p}\,' | T | \vec{p} \rangle = \frac{1}{16\pi^2 i} \sum_{KK'=-\infty}^{\infty} \int_{C_{KK'}} \frac{(2L+1)\,dL}{\sin \pi L} \; T^L_{-K'K} D^L_{K'K}(\omega)$$

Here, we have used the symmetry relations of the $D^L_{KK'}(\Omega)$ for integral L (10), so that if $\Omega_{p'p}$ denotes the angles φ, θ, ψ, ω denotes $\pi-\psi, \pi-\theta, \varphi$. The contour $C_{KK'}$ encloses the real axis $\mathrm{Re}\, L > [\max(|K|, |K'|) - \frac{1}{2}]$. It is assumed to exclude all poles but those of $\sin \pi L$. If a cut crosses the real axis in this region, it would be necessary to subtract out its contribution.

Under assumption (ii), the contour can be deformed to a line Γ parallel to the imaginary axis at $\mathrm{Re}\, L = -\frac{1}{2}$, yielding

$$\langle \vec{p}\,' | T | \vec{p} \rangle = \frac{1}{16\pi^2 i} \int_{\Gamma} \frac{(2L+1)\,dL}{\sin \pi L} \sum_{K'K} T^L_{-K'K} D^L_{K'K}(\omega)$$
$$+ \sum_i \int \frac{d\tau}{\sin \pi \alpha_i(\tau)} \sum_{K'K} \rho^i_{K'K}(\tau) D^{\alpha_i(\tau)}_{K'K}(\omega)$$
$$+ \sum_n \frac{1}{\sin \pi \alpha_n} \sum_{K'K} B^{\alpha_n}_{K'K} D^{\alpha_n}_{K'K}(\omega) \qquad (2.7)$$

where $\rho^i_{K'K}$ and $B^{\alpha_n}_{K'K}$ are simply related to the discontinuities and residues of $T^L_{K'K}$ respectively.

For large values of $z = \cos\theta$, $D^L_{K'K}(\omega)$ behaves like (2),

$$D^L_{K'K}(\omega) \sim g^L_{K'K}(\varphi, \psi) z^L \tag{2.8}$$

for some $g^L_{K'K}$. The asymptotic behavior of the amplitude is then determined by the position of the singularity furthest right in the L-plane. For instance, suppose it is a pole at position α, then

$$< \vec{p}\,' | T | \vec{p} > \sim \left(\sum_{K'K} B^\alpha_{K'K} \, g^\alpha_{K'K} \right) z^\alpha \tag{2.9}$$

The continuation which satisfies conditions (i) and (ii) and thus determines the asymptotic behavior is unique. The uniqueness may be established by considering the partial wave expansion of

$$\int d\psi \int d\varphi \, e^{-iM\varphi} e^{-iN\psi} < \vec{p}\,' | T | \vec{p} > \tag{2.10}$$

and applying the discussion of Squires (13). Briefly, suppose there are two continuations $T^L_{MN}(1)$ and $T^L_{MN}(2)$ which satisfy the conditions (i) and (ii) and agree with the physical values on the integrals, then

$$T^L_{MN}(1) + (L - L_1)^{-1} \left[T^L_{MN}(1) - T^L_{MN}(2) \right] \tag{2.11}$$

will also. The quantity in equation 2.10 has a unique asymptotic behavior which we may say is weaker than z^{L_o}. If we take $\mathrm{Re}\, L_1 > \mathrm{Re}\, L_o$, we must have

$$T^{L_1}_{MN}(1) = T^{L_1}_{MN}(2) \tag{2.12}$$

in order for the continuation 2.11 to give the correct asymptotic behavior. Since this holds for any L_1 for which $\mathrm{Re}\, L_1 > \mathrm{Re}\, L_o$, it must hold everywhere.

In a two-particle scattering problem, there is essentially only one scattering angle. For three-particle scattering, however, there are many. Each may be characterized as an angle between z-axes fixed in the initial and final systems of particles. As we have mentioned before, there are many ways of choosing how the body-fixed axes are fixed in the system of particles and hence many scattering angles. We will now show that if a continuation exists which determines the asymptotic behavior in one scattering angle, then it determines the behavior in all,

From a single choice of body-fixed axes, all others can be found by rotations in the body-fixed frames of the coordinates and momenta. Let Ω_1 and Ω_2 be two choices of Euler angles and W the rotation which sends Ω_1 and Ω_2. For integral values of L one knows how the D^L_{MK} transform under this rotation since they form a representation of the rotation group

$$D^L_{MK}(\Omega_2) = D^L_{MK}(W\Omega_1) = \sum_N D^L_{MN}(W) D^L_{NK}(\Omega_1) \qquad (2.13)$$

In Appendix A it is shown that the D^L_{MK} for complex L also obey equation 2.13 (except for some angles where it becomes singular) and thus also form a representation of the rotation group. To determine how the partial wave functions $\psi^L_{K'K}$ transform under a new choice of body-fixed axes, consider the rotational scalar $\psi^L(\vec{r}, \vec{p})$ of equation 2.6.

Denote by W_1 the rotation in the body-fixed coordinate frame and by W_2 the rotation in the body-fixed momentum frame. These will be functions of the internal variables r and p respectively. Setting $\Omega_{rp} = W_2\Omega'_{rp}W_1$ we have

$$\psi^L(\vec{r},\vec{p}) = (8\pi^2)^{-1}\sum_{K'K}(2L+1)\psi^L_{K'K}(r,p)D^L_{KK'}(W_2\Omega'_{rp}W_1)$$

$$= (8\pi^2)^{-1}\sum_{K'K}(2L+1)\psi^L_{K'K}(r,p)\sum_{N'N}D^L_{KN}(W_2)D^L_{NN'}(\Omega'_{rp})D^L_{N'K'}(W_1) \tag{2.14}$$

The wave functions therefore transform like

$$\psi'^L_{N'N}(r,p) = \sum_{K'K}D^L_{N'K'}(W_1)\psi^L_{K'K}(r,p)D^L_{KN}(W_2) \tag{2.15}$$

Similarly, if rotations are performed in the initial and final momenta so that $\omega = Y_2\omega'Y_1$ the amplitude transforms like

$$T'^L_{-N'N} = \sum_{K'K}D^L_{K'N'}(Y_2)T^L_{-K'K}D^L_{NK}(Y_1) \tag{2.16}$$

Since terms with different complex angular momenta L do not mix under rotations, a singularity which determines the asymptotic behavior in one representation will also determine it in any other. For instance, if the amplitude has a large z behavior given by equation 2.9, then the asymptotic behavior in any other scattering angle z' would be given by

$$\langle\vec{p}'|T|\vec{p}\rangle \sim \Big(\sum_{K'K}B'^{\alpha}_{K'K}g^{\alpha}_{K'K}\Big)z'^{\alpha} \tag{2.17}$$

where

$$B'^{\alpha}_{K'K} = \sum_{NN'} D^{\alpha}_{N'K'}(Y_2) B^{\alpha}_{N'N} D^{\alpha}_{KN}(Y_1) \qquad (2.18)$$

and Y_1 and Y_2 are the rotations appropriate to the changes of body-fixed axes in the initial and final states respectively.

Since the continuation which determines the asymptotic behavior in a given set of scattering angles is unique, we conclude that there is one unique continuation (if it exists) which determines the asymptotic behavior in any scattering angle. This conclusion does not depend on the non-relativistic model being considered here. It is a general property of the Watson-Sommerfeld transformation and will hold for the relativistic amplitudes if they satisfy assumptions (i) and (ii).

III. EXCHANGE FORCES

In the non-relativistic limit of a problem in which particles can be created and destroyed at a vertex, certain non-local potentials can be expected to occur. In particular, there is the class of potentials which rearrange the positions of the particles but otherwise act in a local way. Such potentials have the form

$$V = \sum_n \lambda V^n(\vec{r}_1, \ldots, \vec{r}_N) P_n \tag{3.1}$$

where P_n is a member of the permutation group on N objects. We will now consider the simple generalization of the preceeding section necessary to continue to complex values of the angular momentum amplitudes produced by such forces.

The full wave Schrödinger equation 2.4 can be written

$$[\mathcal{T}(\vec{r}) - E + \lambda \sum_n V^n(\vec{r}) P_n] \psi(\vec{r}, \vec{p}) = 0 \tag{3.2}$$

Operate on this equation with the permutation P_i. Since the permutations form a group, we have

$$P_i P_n = \sum_j \mathcal{P}^n_{ij} P_j \tag{3.3}$$

where $\mathcal{P}^n_{ij}$ is a representation of P_n. The indicated operation then gives:

$$\sum_j [(\mathcal{T}_i - E)\delta_{ij} + \lambda \sum_n V^n_i(\vec{r}) \mathcal{P}^n_{ij}] \psi_j(\vec{r}, \vec{p}) = 0 \tag{3.4}$$

where

$$\mathcal{J}_i(\vec{r}) = \mathcal{J}(P_i\vec{r}), \qquad V_i^n(\vec{r}) = V^n(P_i\vec{r}), \qquad \psi_i(\vec{r},\vec{p}) = \psi(P_i\vec{r},\vec{p})$$

This is a set of six coupled equations on the six unknowns $\psi_i(\vec{r})$ involving only local potentials. The corresponding set of partial wave equations can then be continued to complex L as discussed in Section II, and every coefficient will be bounded by a polynomial for large $|L|$. The presence of exchange operators, on the other hand, could entail factors with asymptotic behaviors like $e^{\pm i\pi L}$ and perhaps lead to a violation of assumption (ii).

The original equation 3.2 was solved with the boundary condition that it approach a plane wave ϕ as the potential tends to zero.

$$\psi(\vec{r},\vec{p}) \to \phi(\vec{r},\vec{p}), \quad \lambda \to 0 \tag{3.5}$$

The same solution may be generated by solving equation 3.4 with the boundary conditions

$$\psi_i(\vec{r},\vec{p}) \to \phi_i(\vec{r},\vec{p}) = \phi(P_i\vec{r},\vec{p}), \quad \lambda \to 0 \tag{3.6}$$

If $P_1 = 1$ the scattering solution is then given by $\psi(\vec{r},\vec{p}) = \psi_1(\vec{r},\vec{p})$. Equivalently, we can write

$$\psi(\vec{r},\vec{p}) = \sum_j \psi_{1j}(\vec{r},\vec{p}) \tag{3.7}$$

where ψ_{ij} has the boundary condition

$$\psi_{ij}(\vec{r},\vec{p}) \to \delta_{ij}\phi_i(\vec{r},\vec{p}), \quad \lambda \to 0 \tag{3.8}$$

For a plane wave, a permutation acting on the coordinates is the same as the conjugate permutation acting on the momenta

$$\phi_i(\vec{r},\vec{p}) = \phi(P_i\vec{r},\vec{p}) = \phi(\vec{r}, P_i^+\vec{p}) \tag{3.9}$$

Every permutation of the momenta can be written as the product of a transformation Q_i which changes the lengths of the relative momentum vectors p and a rotation of the body-fixed axes R_i.

$$P_i = Q_i R_i \tag{3.10}$$

The partial wave expansion of $\phi_i(\vec{r},\vec{p})$ can then be written

$$\phi_i(\vec{r},\vec{p}) = (8\pi^2)^{-1} \sum_{LK'K} (2L+1)\phi^L_{K'K}(r, Q_i^+p) D^L_{KK'}(R_i^+\Omega_{rp}) \tag{3.11}$$

The boundary condition for the partial wave functions of $\psi_{ij}(\vec{r},\vec{p})$ are then

$$\psi^L_{iK',jK}(r,p) \to \delta_{ij} \sum_{K''} \phi^L_{K'K''}(r, Q_j^+p) D^L_{K''K}(R_j^+) \tag{3.12}$$

Consider the wave functions $\tilde{\psi}^L_{iK',jK}$ defined by the boundary conditions

$$\tilde{\psi}^L_{iK',jK} \to \delta_{ij}\phi^L_{K'K}(r, Q_j^+p) \tag{3.13}$$

Since the Schrödinger equation is linear, multiplication of the boundary condition by the constant matrix $D^L_{K'K}(R_j^+)$ only multiplies the solution by the same matrix.

$$\psi^L_{iK',jK} = \sum_{K''} \tilde{\psi}^L_{iK',jK''} D^L_{K''K}(R_j^+) \tag{3.14}$$

The scattering solution, Equation 3.7, can then be written

$$\psi(\vec{r},\vec{p}) = (8\pi^2)^{-1} \sum_{LK'Kj} (2L+1)\, \tilde{\psi}^{L}_{1K',jK} D^{L}_{KK'}(R_j^{+}\Omega_{rp}) \qquad (3.15)$$

Similarly, the partial wave expansion of the amplitude becomes

$$<\vec{p}\,'|T|\vec{p}> = (8\pi^2)^{-1} \sum_{LK'Kj} (2L+1)\, \tilde{T}^{L}_{1K',jK}(p',p) D^{L}_{KK'}(R_j^{+}\Omega_{rp}) \qquad (3.16)$$

where the amplitudes $\tilde{T}^{L}_{iK',jK}$ are computed from the scattering solutions $\tilde{\psi}^{L}_{iK',jK}$.

Equation 3.4 has only local potentials. The boundary condition of equation 3.13 has the same large $|L|$ behavior as that for a partial wave expansion of equation 3.5. If there is an analytic continuation of the problem of a single three-particle channel with local potentials which satisfies criteria (i) and (ii) of Section II, it is not unreasonable to expect that the same result holds for the six coupled three-particle channels of equation 3.4. $\tilde{T}^{L}_{1K',jK}$ can therefore be assumed to satisfy conditions (i) and (ii) and the Watson-Sommerfeld transformation of equation 3.16 can be performed.

For non-identical particles, equation 3.4 may not be further decoupled and a given pole will appear in all the analytically continued partial wave amplitudes. However, if

$$m_1 = m_2 = m_3 \qquad \text{and} \qquad [P_i, V] = 0 \qquad (3.17)$$

as in the case of identical particles, a further decomposition of the set of equations may be obtained by reducing the regular representation $\mathcal{P}^{n}_{ij}$. The amplitude can be decomposed into a sum of terms (linear combinations of the arguments of equation 3.16) for each irreducible representation of the permutation group. The poles in one of these terms need not appear in any other. This is the analog of signature in the two-particle case. If the particles are truly identical, (and not, for example, differently charged and hence distinguishable pions interacting by nuclear forces) only the poles of the symmetric or anti-symmetric amplitudes could correspond to physical states.

Let us consider the simple example of three particles using the choice of Euler angles discussed by Blatt and Derrick (6). In these coordinates the body fixed z-axis is taken normal to the triangle formed by the three particles and directed so that a right handed screw will advance along it if turned successively through particles 1, 2, 3, and back to 1. Orthogonal x- and y-axes are defined invariantly as discussed by these authors, for instance by taking the x-axis to lie along the principle axis of the triangle with greatest moment of inertia.

With these definitions an interchange of particles changes only the sign of the z-axis

$$
\begin{aligned}
P_i D^L_{K'K}(\pi-\psi, \pi-\theta, \varphi) &= D^L_{K'K}(\pi-\psi, \pi-\theta, \varphi), \quad \delta_i = -1 \\
&= D^L_{K'K}(\pi-\psi, \theta, \varphi), \quad \delta_i = +1 \qquad (3.18)
\end{aligned}
$$

where δ_i is +1 or -1 as the number of interchanges in P_i is even or odd. The remaining variables p may be taken to be the relative momenta p_{12}, p_{23}, p_{13}. A Regge pole term in the full amplitude then has the form

$$
\begin{aligned}
(\sin \pi\alpha)^{-1} \sum_{K'K} [\, & F^{\alpha}_{K'K}(p', p) D^{\alpha}_{K'K}(\pi-\psi, \pi-\theta, \varphi) \\
& + G^{\alpha}_{K'K}(p', p) D^{\alpha}_{K'K}(\pi-\psi, \theta, \varphi)] \qquad (3.19)
\end{aligned}
$$

If conditions of equation 3.17 are satisfied there will be three classes of poles for the symmetric, anti-symmetric and mixed representations of the permutation group on three objects. Equation 3.19 will then have the corresponding symmetry.

IV. THE ANALYTIC PROPERTIES OF A POTENTIAL SCATTERING AMPLITUDE

1. The Radial Schrödinger Equation

Regge was able to obtain a simple description of the large momentum transfer behavior of two-particle potential scattering amplitudes because the only singularities in the complex angular momentum plane were poles. In this Section we will discuss the specific analytic properties of a many-particle potential scattering amplitude by solving the analytically continued Schrödinger equation discussed in Section II with scattering boundary conditions. We will find that here also there are no cuts in the angular momentum plane and a simple expression for the asymptotic behavior of the scattering amplitude can be obtained.

Since any multi-particle system has only three degrees of rotational freedom, all of the angular momentum features of the many-particle problem are already present in a system of three particles. We will therefore be concerned in the bulk of this section with three-particle scattering and later indicate how the generalization to non-relativistic many-particle amplitudes can be made. For simplicity, we are considering spinless and non-identical particles interacting by means of two-body Yukawa potentials.

In a three-particle scattering process there are several amplitudes that can be discussed depending on how much of the interaction is turned off in the asymptotic states. One can have either three free particles or a bound pair with the third free in the initial and final states. Each possible amplitude represents a different class of scattering boundary conditions (14). To avoid discussing these various amplitudes at length, we will concentrate on the particular class of amplitudes in which initially and finally particles 1 and 2 are bound. This particular class is at once interesting from the point of view of real processes and simple to calculate.

To determine these amplitudes we will exploit the kinematical similarity that three-particle scattering bears to the scattering of two particles with spin. If initially and finally two of the three particles are in a definite state of their relative angular momentum ℓ, then the scattering is kinematically the same as the scattering of a composite particle with a spin ℓ and certain other internal degrees of freedom. Of course, this spin is not conserved and there is a continuous infinity of internal degrees of freedom corresponding to the energy of the composite object.

In order to study the solutions of Schrödinger's equation, we will introduce a specific coordinate system and make explicit the procedures outlined in Section II. We begin by supressing the three degrees of freedom corresponding to the total center of mass. The wave function then depends on two position vectors which may be taken to be $\vec{r}$, the relative coordinate of particles 1 and 2, and $\vec{R}$ the coordinate of their center of mass relative to the third particle (see figure 1). These coordinates were denoted collectively by $\vec{r}$ in Section II. The scattering wave function also depends on the quantum numbers which label the incoming wave (denoted by $\vec{p}$ in Section II). These will be chosen to be $\vec{P}$, the total momentum of the composite object 1 and 2, and the quantum numbers which characterize its internal wave function. The latter labels

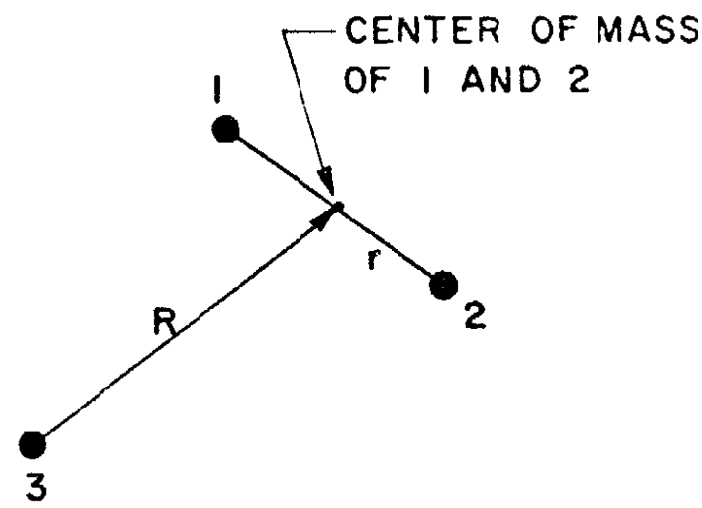

Fig. 1. Position vectors for a three-particle system in the center of mass frame.

will be taken to be ℓ, the relative angular momentum of particles 1 and 2, their "helicity", $\eta = \vec{\ell}\cdot\vec{P}/|\vec{P}|$, and their energy, $p^2/2m$, in their own center-of-mass system. Since $\vec{P}\cdot(\vec{L}-\vec{\ell}) = \vec{P}\cdot(\vec{R}\times\vec{P}) = 0$, the helicity is also the projection of $\vec{L}$ on $\vec{P}$. The total conserved energy is

$$E = \frac{p^2}{2m} + \frac{\vec{P}^2}{2m'}, \tag{4.1}$$

Here, if m_1, m_2, m_3 are the particle masses, we have

$$m = \frac{m_1 m_2}{m_1+m_2} \qquad m' = \frac{(m_1+m_2)m_3}{m_1+m_2+m_3} \tag{4.2}$$

Explicitly, the wave function is written: $\Psi = \Psi[(\ell p \eta)\vec{P};\vec{r},\vec{R}]$. This choice of variables is clearly suitable for pursuing the analogy with the scattering of particles with spin.

Now introduce polar coordinates R, θ, φ of $\vec{R}$ relative to some arbitrary polar axis, and r, β, α of $\vec{r}$ defined with $\vec{R}$ as a polar axis. It is also convenient to introduce a fixed and arbitrary angle α' to define the origin of α (see figure 2).

The three angles $\varphi, \theta, \upsilon = \alpha + \alpha'$ are the Euler angles discussed in Section II. The corresponding body-fixed z-axis lies along $\vec{R}$. The potential, which depends only on the interparticle distances, is a function of the remaining coordinates, R, r, β.

We will now write down the Schrödinger equation in these coordinates and make the partial wave expansion of Section II. The full Schrödinger equation for the wave function $\Phi(\vec{r},\vec{R}) = \Psi(\vec{R},\vec{r})/Rr$ is (with $\hbar = 1$)

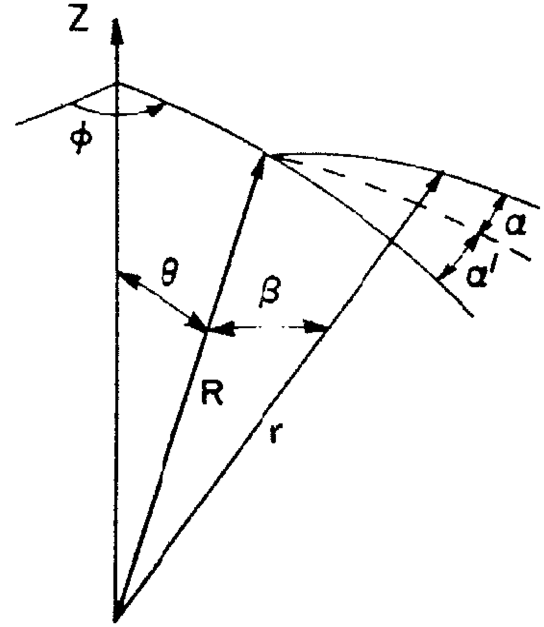

Fig. 2. Polar coordinates for the vectors $\vec{r}$, $\vec{R}$. Z is a fixed polar axis.

$$\left[\frac{1}{2m} \left(\frac{\partial^2}{\partial r^2} - \frac{\vec{\ell}^{\,2}}{r^2} \right) + \frac{1}{2m'} \left(\frac{\partial^2}{\partial R^2} - \frac{(\vec{L} - \vec{\ell}\,)^2}{R^2} \right) + E - V(R, r, \beta) \right] \Phi(\vec{R}, \vec{r}\,) = 0 \tag{4.3}$$

Here, $\vec{L}$ and $\vec{\ell}$ are regarded as differential operators in the angles.

The wave function can now be expanded in a complete set of functions for the variables r, β, α, φ, θ obtaining a set of coupled differential equations in the coordinate R. To do this, we make the partial wave expansion of equation 2. 2 taking $\vec{P}$ to be the space-fixed z-axis. We will thus be expanding in eigenfunctions labelled by L, its projection on the space-fixed axis η, and the projection on the body-fixed axis η'

$$\eta' = \vec{L} \cdot \vec{R} / |\vec{R}\,| \tag{4.4}$$

We also expand in the eigenfunctions of the relative angular momentum $\vec{\ell}^{\,2}$ and its projection on $\vec{R}$. Since $\vec{L} \cdot \vec{R} = \vec{\ell} \cdot \vec{R}$, the combined orthogonal eigenfunctions may be written

$$\left(\frac{2L+1}{4\pi} \right) D^{L}_{\eta\eta'}(\varphi, \theta, \alpha') Y^{\eta'}_{\ell}(\beta, \alpha) \tag{4.5}$$

Finally, we will expand in a complete set of solutions for the variable r. These will be the solutions $\psi_\ell(p, r)$ of the two-particle problem

$$\left[\frac{d^2}{dr^2} + p^2 - \frac{\ell(\ell+1)}{r^2} - 2mV(r) \right] \psi_\ell(p, r) = 0 \tag{4.6}$$

The complete expansion then becomes

$$\Phi\left[(\ell p \eta)\vec{P};\vec{r},\vec{R}\right] = \sum_{L}\sum_{\ell' p' \eta'} \phi^{L}_{\ell' p' \eta', \ell p \eta}(R)\,\psi_{\ell'}(p', r)$$

$$\times D^{L}_{\eta\eta'}(\varphi, \theta, \alpha')\, Y^{\eta'}_{\ell'}(\beta, \alpha) \qquad (4.7)$$

The sum is over the discrete and continuous spectrum of the two-particle problem in p, over integral values of ℓ with $\ell \geq |\eta'|$, and over η' with $|\eta'| \leq L$.

In projecting out the equations which govern the $\phi^{L}_{\ell' p' \eta', \ell p \eta}(R)$, only the matrix elements of $\vec{\ell} \cdot \vec{L}$ and $V(R, r, \beta)$ are not readily evaluated. In order to evaluate the former it is convenient when considering rotations generated by $\vec{L}$ to maintain α fixed and let α' vary, while when considering rotations generated by $\vec{\ell}$, we keep α' fixed and let α vary. In terms of the spherical components (10) of $\vec{L}$ and $\vec{\ell}$,

$$\vec{L}(\varphi, \theta, \alpha') \cdot \vec{\ell}(\beta, \alpha) = - L_{+}\ell_{-} - L_{-}\ell_{+} + L_{o}\ell_{o} \qquad (4.8)$$

The matrix elements of equation 4.8 are then given by equation A.1, remembering that $\vec{\ell}$ is an angular momentum like $\vec{L}$, and that $Y^{\eta}_{\ell}(\beta, \alpha) = D^{L}_{\eta 0}(\alpha, \beta, 0)$.

Define

$$a_{\eta} = \left[(L-\eta)(L+\eta+1)(\ell-\eta)(\ell+\eta+1) \right]^{1/2} \qquad (A.9)$$

$$V_{\ell' p' \eta', \ell p \eta} = 2m'\delta_{\eta'\eta} \int d^3r\, \psi^{*}_{\ell'}(p', r)\, Y^{\eta' *}_{\ell'}(\beta, \alpha) \left[V(R, r, \beta) - V(r) \right]$$

$$\times \quad \psi_{\ell}(p, r)\, Y^{\eta}_{\ell}(\beta, \alpha)$$

where $V(r)$ is the interaction between particles 1 and 2. The equation becomes (15):

$$\left[\frac{d^2}{dR^2} + p'^2 - \frac{L(L+1)+\ell(\ell+1)-2\eta'^2}{R^2}\right]\phi^L_{\ell' p' \eta', \ell p \eta}(R)$$

$$+\frac{1}{R^2}\left[a_{\eta'}\phi^L_{\ell' p' \eta'+1, \ell p \eta}(R) + a_{-\eta'}\phi^L_{\ell' p' \eta'-1, \ell p \eta}\right]$$

$$-\sum_{\ell'' p'' \eta''} V_{\ell' p' \eta', \ell'' p'' \eta''}(R)\,\phi^L_{\ell'' p'' \eta'', \ell p \eta}(R) = 0 \qquad (4.11)$$

This can be written in a matrix form as

$$\left[\frac{d^2}{dR^2} + P^2 - \frac{\Lambda^L}{R^2} - V(R)\right]\phi^L(R) = 0 \qquad (4.12)$$

The intermediate η'' sum remains restricted to $|\eta''| \leq L$.

This equation displays the formal equivalence of the three-particle problem with a problem having many two-body channels. In equation 4.7 we can regard $\psi_\ell(p, r)Y^\eta_\ell(\beta, \alpha)$ as the internal wave function of a composite particle composed of particles 1 and 2 which scatters off particle 3. The equation which governs such a set of two-body channels is equation 4.12.

2. The Equations and Solutions for Complex Angular Momentum

We continue the equations to complex L as discussed in Section II. The restriction that $|\eta'| \leq L$ is removed, and the coupled equations are considered for arbitrary integral values of η'. When L is an integer,

a_L vanishes thus guaranteeing that the physical equations will decouple from the unphysical ones.

In order to examine the analytic properties of the solutions, it is convenient to introduce a wave function, $\hat{\phi}^L$, which obeys an equation all of whose coefficients are entire functions of L.

$$\phi^L = \rho\hat{\phi}^L\rho$$

where*

$$\rho_{\ell p\eta,\ell' p'\eta'} = \delta_{\ell\ell'}\delta_{pp'}\delta_{\eta\eta'}[(L-\eta)!(L+\eta)!]^{-1/2} \tag{4.13}$$

The potential, a function of the interparticle distances, is independent of the angle α and hence its matrix elements are diagonal in η (see equation 4.10). The equation governing $\hat{\phi}^L$ can then be written

$$\left[\frac{d^2}{dR^2} + P^2 - \frac{\hat{\Lambda}^L}{R^2} - V(R)\right]\hat{\phi}^L(R) = 0 \tag{4.14}$$

Here, $\hat{\Lambda}^L$ is defined in the same way as Λ^L with a_η replaced by

$$\hat{a}_\eta = (L-\eta)[(\ell-\eta)(\ell+\eta+1)]^{1/2} \tag{4.15}$$

Every element of $\hat{\Lambda}^L$ and therefore every element of the matrix of equations is an entire function of L.

We will now examine the analytic properties of the solutions to equation 4.14 for two simple classes of boundary conditions. From these solutions, the solution to the scattering problem can be constructed and

*For complex z we define $z! = \Gamma(z+1)$.

the analytic properties of the amplitude determined.

In the succeeding paragraphs, many questions of convergence will of necessity be left unanswered. We will treat the coupled set of equations in equation 4.14 as though it were a finite matrix of equations. This can be done by introducing a cut-off in the intermediate p and ℓ sums. The true S-matrix will be the limit as the cut-off tends to infinity of the S-matrices computed from these truncated equations. The analytic properties we derive are those of each term in the sequence. The potential which couples the equations together is independent of the angular momentum, so it is perhaps plausible that the limit has the same analytic properties as each term in the sequence. We are well aware that mathematically this may not be the case, and we have been unable to find a rigorous proof for the statement.

Certain intermediate steps in the proof, such as the convergence of the several power series used to define the solutions, will also not be discussed in detail in this thesis. Some of them, however, have been proved rigorously and we will indicate in a later paragraph which these are.

We first obtain solutions of equation 4.14 specified by boundary conditions at the origin. In order to examine their analytic properties, we employ the standard power series technique (16, 2) and look for solutions of the form

$$\hat{\phi}^L(R) = R^\sigma \sum_{n=0}^{\infty} a(n,\sigma) R^n \qquad (4.16)$$

Here, σ and a are matrices. R^σ is a matrix whose diagonal elements have the form R^{σ_ξ} when σ is diagonal with eigenvalues σ_ξ.

Near the origin, the matrix elements of a Yukawa potential increase no faster than R^{-1} and, consequently, one can expand the expression $V(R) - P^2$ as

$$V(R) - P^2 = \sum_{m=-1}^{\infty} b(m)R^m \tag{4.17}$$

The Schrödinger equation then determines a recursion relation for the coefficients of the power series which is

$$[(\sigma+n+2)(\sigma+n+1) - \hat{\Lambda}^L]\, a(n+2, \sigma) = \sum_{k=0}^{n+1} b(n-k)a(k, \sigma) \tag{4.18}$$

The first term in the recursion relation is

$$[\sigma(\sigma-1) - \hat{\Lambda}^L]\, a(0, \sigma) = 0 \tag{4.19}$$

We demand that σ and $\hat{\Lambda}^L$ be simultaneously diagonal. $\hat{\Lambda}^L$ is the orbital angular momentum of particles 1 and 2, $\vec{L} - \vec{\ell}$. Its eigenvalues, as discussed in Appendix B, are given by

$$(L+\xi)(L+\xi+1) \qquad \xi \text{ integral,} \qquad |\xi| \leq \ell \tag{4.20}$$

including the case of complex L. The allowed values of σ_ξ are thus from equation 4.19

$$\sigma_\xi = L + \xi + 1, \quad \sigma'_\xi = -L - \xi \qquad |\xi| \leq \ell \tag{4.21}$$

Since successive values of σ_ξ differ by integers, the set of solutions $\phi^L(R, \sigma)$ will not be linearly independent. A similar statement holds for $\phi^L(R, \sigma')$. Linearly independent solutions are obtained by differentiating the equation 3.16 with respect to the eigenvalue σ_ξ an appropriate number of times, and evaluating the derivative at one of the allowed values of σ_ξ (17).

Each column of $a(0,\sigma)$ must be an eigenfunction of $\hat{\Lambda}^L$ corresponding to one of the eigenvalues in equation 4.20. They are then linear combinations of the column vectors of the transformation which diagonalizes $\hat{\Lambda}^L$. This is discussed in Appendix B and is denoted by:

$$\hat{U}_{\ell'p'\eta',\ell p\xi} = \delta_{\ell\ell'}\delta_{pp'}\hat{U}_{\eta'\xi} \tag{4.22}$$

The index ξ refers to the representation in which $\hat{\Lambda}^L$ is diagonal.

For the set of indices σ_ξ it is convenient to choose $a(0,\sigma)$ as

$$a_{\ell p\eta,\ell'p'\eta'}(0,\sigma) = \delta_{\ell\ell'}\delta_{pp'}\frac{1}{2}\sum_{\xi}\hat{U}_{\eta\xi}R^{\sigma_\xi}[(L+\xi+\tfrac{1}{2})!]^{-1}\hat{U}^{-1}_{\xi\eta'} \tag{4.23}$$

The $\hat{U}_{\eta\xi}$ are meromorphic functions of L (see Appendix B). From equation 4.18 we can conclude that $a(n,\sigma)$ are also meromorphic functions of L with additional poles due to those of

$$[(\sigma+n+2)(\sigma+n+1) - \hat{\Lambda}^L]^{-1} \tag{4.24}$$

These poles are fixed and kinematic and do not depend on R. Formally, the same analytic properties then hold for the sum, equation 4.16. We denote this solution by $J(R)$.

Similar properties hold for the solutions with indices σ'_ξ, for which we choose

$$a_{\ell p\eta,\ell'p'\eta'} = \delta_{\ell\ell'}\delta_{pp'}\sum_{\xi}\hat{U}_{\eta\xi}R^{\sigma'_\xi}(L+\xi-\tfrac{1}{2})!\;\hat{U}^{-1}_{\xi\eta'} \tag{4.25}$$

This solution is denoted by $N(R)$.

The general solution of the Schrödinger equation may be expressed as a matrix linear combination of $J(R)$ and $N(R)$. For physical values of L, $J(R)$ is the solution regular near $R = 0$. We will, therefore, require that the scattering solution be a multiple of $J(R)$ as a boundary condition for complex L. We note that $J(R)$ contains irregular solutions for general complex L.

The scattering solution consists of the free solution plus a scattered part which contains only outgoing or decaying waves. As a second class of boundary conditions, we will consider elementary solutions of this type. Solutions having outgoing waves in the open channels $(p^2/2m > E)$ and decaying waves in the closed ones $(p^2/2m < E)$ are generated by the free Green's function:

$$G^L(R, R') = P^{-1} j(R_<) h^{(1)}(R_>) \tag{4.26}$$

Here, $j(R)$ is the free solution regular at the origin for physical L and $h^{(1)}(R)$ is a free solution corresponding to pure outgoing waves in the open channels or decaying waves in the closed channels where P is defined with positive imaginary part. They are given explicitly in Appendix C. In Appendix C it is shown that both $j(R)$ and $h^{(1)}(R)$ are entire functions of L so that the Green's function is also.

The free solution corresponding to $h^{(1)}(R)$ but defined with incoming or expanding waves is denoted by $h^{(2)}(R)$ and given explicitly in Appendix C. We introduce the free solution $h^{(3)}(R)$ defined by:

$$\begin{aligned} h^{(3)}_{\ell p\eta, \ell' p' \eta'}(R) &= \delta_{pp'} h^{(2)}_{\ell p\eta, \ell' p' \eta'}(R) \qquad p^2/2m < E \\ &= 0 \qquad\qquad\qquad\qquad\quad\; p^2/2m > E \end{aligned} \tag{4.27}$$

$h^{(3)}(R)$ is an entire function of L containing only incoming and no expanding waves. Solutions of the complete Schrödinger equation (equation 4.14) are then defined by:

$$H^{(1,3)}(R) = h^{(1,3)}(R) + \int_{R_o}^{\infty} dR' P^{-1} j(R_<) h^{(1)}(R_>) V(R') H^{(1,3)}(R') \quad (4.28)$$

$$R \geq R_o > 0$$

The Fredholm method (18) may be used to construct solutions to these equations. One obtains

$$H^{(1,3)}(R) = h^{(1,3)}(R) + \frac{\Delta^{(1,3)}(R)}{D(E,L)} \quad (4.29)$$

Here, $\Delta^{(1,3)}(R)$ is a matrix and $D(E,L)$ a function both expressed as Fredholm series in the kernal of equation 4.28. These series will be discussed in more detail later, but for our present purpose it suffices to recall their form. The n^{th} term in the series for $D(E,L)$ consists of an integral over a determinant of an $n \times n$ matrix whose entries are the kernal of equation 4.28. A similar statement holds for $\Delta^{(1,2)}(R)$ except that there is an additional integration over $h^{(1,3)}(R)$. Since the kernal is an entire function of L and $h^{(1,3)}(R)$ is also, both n^{th} order terms are entire functions of L. With the qualificiations on rigor mentioned above, we can then conclude that both $D(E,L)$ and every element of $\Delta^{(1,3)}(R)$ are entire functions of L. The solutions $H^{(1,3)}$ are thus analytic in the complex angular momentum plane except at dynamical poles arising from the zeros of $D(E,L)$.

The Green's function, equation 4.26, is a diagonal matrix in the indices ℓ and p. A given diagonal element $G^L_{\ell p\eta, \ell p'\eta'}(R,R')$ will be irregular for small values of $R_<$ if Re $L < \ell - 1$ as can be seen from the explicit expression for $j(R)$ in Appendix C. Since arbitrarily large values of ℓ are coupled by equation 4.28, there is no value of L for which every element of the kernal of equation 4.28 is regular for small $R_<$. The Fredholm method thus cannot be straightforwardly applied unless R_0 is different from zero.

Every solution of Schrödingers equation cannot be expressed as a matrix linear combination of the two solutions $H^{(1,3)}(R)$ since in equation 4.27 we have omitted the expanding waves. The scattering solution, however, contains no expanding waves and can be expressed as a linear combination of these solutions for $R > R_0$.

3. The Scattering Amplitude

The asymptotic states which contain particles 1 and 2 bound and particle 3 free can be characterized by the quantun numbers $\vec{P}(\ell p \eta)$. The scattering amplitude will, therefore, be denoted by $\langle (\ell' p' \eta') \vec{P}\,|T|\,\vec{P}(\ell p \eta) \rangle$, and is a function of the angles $\cos\theta = \vec{P}\cdot\vec{P}'/|\vec{P}|\,|\vec{P}'|$ and φ an azimuthal angle of $\vec{P}'$ about $\vec{P}$. Since there are only two vectors which characterize the amplitude, the angle ψ of the partial wave expansion discussed in Section II is superfluous and may be chosen as $-\varphi$. The partial wave expansion is then*

$$\langle (\ell' p' \eta') \vec{P}'\,|T|\,\vec{P}(\ell p \eta) \rangle$$

$$= (4\pi)^{-1} \sum_L (2L+1) T^L_{\ell' p' \eta', \ell p \eta}(E) D^L_{\eta \eta'}(\varphi, \theta, -\varphi) \tag{4.30}$$

The sum ranges are integral values of $L \geq (|\eta|, |\eta'|)$.

*Compare with Ref. 19.

At large R the scattering solution is a sum of two parts. First, there is the product of an incoming plane wave and a bound state wave function characterized by ℓ, p and $\eta = \vec{\ell}\cdot\vec{P}/|\vec{P}|$. The second part consists of outgoing spherical waves times states of definite ℓ', p' and $\eta' = \vec{\ell}\cdot\vec{R}/|\vec{R}|$.

$$\Psi[(\ell p\eta)\vec{P};\vec{r},\vec{R}] \rightarrow e^{i\vec{P}\cdot\vec{R}}\psi_\ell(p,r)\sum_{\eta'} D^{\ell}_{\eta'\eta}(\varphi,\theta,-\varphi)Y_\ell^{\eta'}(\beta,\alpha)$$

$$+ e^{iP'R}/R\sum_{\ell'p'\eta'} f_{\ell'p'\eta',\ell p\eta}(\vec{P}',\vec{P})\psi_{\ell'}(p',r)Y_{\ell'}^{\eta'}(\beta,\alpha) \qquad (4.31)$$

where

$$\vec{P}'/|\vec{P}'| = \vec{R}/|\vec{R}|, \qquad E = \frac{P^2}{2m'} + \frac{p^2}{2m} \qquad (4.32)$$

The rotation matrices in the incoming part result from expressing states of definite $\vec{\ell}\cdot\vec{P}/|\vec{P}|$ as superpositions of states of definite $\vec{\ell}\cdot\vec{R}/|\vec{R}|$. Asymptotically, final states of definite $\vec{\ell}\cdot\vec{R}/|\vec{R}|$ becomes states of definite helicity $\vec{\ell}\cdot\vec{P}'/|\vec{P}'|$. The sum over final states is taken over all accessible $\psi_{\ell'}(p',r)$ including those in the continuum. In the case where p and p' correspond to bound states, we may take this expression to define the scattering amplitude

$$<(\ell'p'\eta')\vec{P}'|T|\vec{P}(\ell p\eta)> = \frac{(PP')^{1/2}}{2\pi} f_{\ell'p'\eta',\ell p\eta}(\vec{P}',\vec{P}) \qquad (4.33)$$

These expressions resemble those of the scattering of particles with spin where $\psi_\ell(p,r)Y_\ell^\eta(\beta,\alpha)$ play the part of internal wave functions (20).

The asymptotic behavior of $\hat{\phi}^L(R)$ may be projected out of equation 4.31. This involves replacing the plane wave by its familiar Rayleigh expansion evaluated at large R and performing some Clebsch-Gordan sums arising from the angular integration. Apart from a constant, one has

$$\hat{\phi}^{L}_{\ell'p'\eta',\ell p\eta}(R) \xrightarrow[R\to\infty]{} \delta_{pp'}\delta_{-\eta\eta'}\delta_{\ell\ell'}e^{-iPR}(P)^{-1/2}$$

$$- e^{-i\pi(L+\ell)}e^{iP'R}(P')^{-1/2}S^{L}_{\ell'p'\eta',\ell p\eta}(E) \qquad (4.34)$$

where we have introduced the S-matrix

$$S = I + iT \qquad (4.35)$$

Equation 4.34 is taken as the definition of the partial wave amplitude for complex L.

The amplitude thus defined can be found by constructing the scattering solution from the elementary solutions discussed earlier. To do this, we use a method which differs slightly from that employed by Regge and others for the case of two-body channels (1-3). That approach would involve the knowledge of solutions which behave like expanding waves in the closed channels, and these we have not computed.

The scattering solution, equation 4.34, has a unit amount of incoming wave. Noting the asymptotic properties of $H^{(3)}(R)$ implicit in equation C.4, it can be written as a superposition of $H^{(1)}(R)$ and $H^{(3)}(R)$ in the form

$$\hat{\phi}^{L}(R) = H^{(3)}(R) + H^{(1)}(R)\,X(E,L) \qquad (4.36)$$

The matrix coefficient X is to be determined by the boundary condition at the origin. Since $J(R)$ and $N(R)$ are also a complete set of solutions to the Schrödinger equation, we may express $H^{(1,3)}(R)$ in terms of them.

$$H^{(1,3)}(R) = J(R)A^{(1,3)} + N(R)B^{(1,3)} \tag{4.37}$$

In order to determine the constant matrices $A^{(1,3)}$ and $B^{(1,3)}$, we may introduce an analog of the Wronskian in a many-channel problem. If χ and ψ are solutions of the Schrödinger equation, equation 4.14, then

$$W[\chi,\psi] = \chi^T \rho^2 \frac{\partial\psi}{\partial R} - \frac{\partial\chi}{\partial R}^T \rho^2 \psi \tag{4.38}$$

as independent of R. This is a simple consequence of the symmetry of the potential and Λ^L. The boundary conditions on the function $J(R)$, $N(R)$ imply

$$\begin{aligned} W[N,J] &= I \\ W[J,J] &= W[N,N] = 0 \end{aligned} \tag{4.39}$$

R_o can always be chosen to be less than the radii of convergence of the power series for J and N. In the region where these converge, we may evaluate

$$\begin{aligned} A^{(1,3)} &= W[N, H^{(1,3)}] \\ B^{(1,3)} &= - W[J, H^{(1,3)}] \end{aligned} \tag{4.40}$$

The scattering solution can then be written

$$\hat{\phi}^L(R) = J(R)[A^{(1)}X + A^{(3)}] + N(R)[B^{(1)}X + B^{(3)}] \tag{4.41}$$

The boundary condition at the origin is that the scattering solution be a multiple of $J(R)$. This condition determines the matrix X.

$$X = - [B^{(1)}]^{-1} B^{(2)} \tag{4.42}$$

The S-matrix may be expressed in terms of X by taking the asymptotic limit of equation 4.36 and comparing with equation 4.34. This can be done by consulting Appendix C and defining

$$Z^{(1,3)} = \int_{R_o}^{\infty} dR' P^{-1} j(R') V(R') H^{(1,3)}(R') dR' \tag{4.43}$$

Putting $S = \rho \hat{S} \rho^{-1}$ we have for $\hat{S}$

$$\hat{S} = [I + Z^{(1)}] X + Z^{(3)} \tag{4.44}$$

This formula holds only for those elements $S_{\ell p \eta, \ell' p' \eta'}$ such that p and p' correspond to open channels.

The analytic properties of $\hat{S}$ can now be read from equation 4.44 knowing those of X and $Z^{(1,3)}$. From the properties of j and $H^{(1,3)}$ discussed in the previous section, we have that $Z^{(1,3)}$ are meromorphic functions of L with only the dynamical poles of $H^{(1,3)}$ as singularities.

J and $H^{(1)}$ can both be written as the free solution which is diagonal in p plus a term involving the potential which is not. Since the Wronskian of the corresponding free solutions is a constant multiple of the unit matrix, $B^{(1)}$ has the form of a multiple of the unit matrix, containing a delta function in the momentum p, plus terms involving the potential whose matrix elements have a smooth behavior in p and p'.

$$B^{(1)} = cI + M \tag{4.45}$$

In this situation, we can apply the Fredholm theory to invert $B^{(1)}$.

The matrices $B^{(1)}$ and $B^{(2)}$ will have the isolated kinematic singularities at half-integer values of L possessed by the solution J. For the truncated set of equations, these are poles which cancel in the formation of X.

The dynamical singularities of X come from three sources. First, there are the poles of $B^{(2)}$. Second, there are poles at det $B^{(1)}$ where the inverse of $B^{(1)}$ does not exist. Finally, since $H^{(1)}$ has poles from the zeros of D which occur at the same position in every matrix element, the matrix M will have them also. In constructing the Fredholm series for the inverse of $B^{(1)}$, the matrix M will be iterated an arbitrarily large number of times. There is therefore the possibility that $[B^{(1)}]^{-1}$ and hence X will possess isolated essential singularities at these points, although a detailed study of the convergence of these series would be necessary to rigorously establish their existence.

The important point is that all these singularities of X are isolated. Since they are all dynamical, we will presume that they are confined in a region $\mathrm{Re}\, L < L_o$ for some L_o. Even if the isolated singularity is essential, it will still produce the characteristic simple power law dependence of the amplitude at large momentum tramsfers.

Combining the information about $Z^{(1,3)}$ and X, we may summarize the properties of $\hat{S}$ by saying that it possesses only dynamical, isolated singularities.

Using the analytic properties of the amplitude outlined above, the partial wave expansion can be written as an integral over the contour C (see figure 3)

$$< (\ell' p' \eta') \vec{P}' \,|\, T \,|\, \vec{P} (\ell p \eta) > = \frac{1}{8\pi i} \int_C \frac{dL (2L+1)}{\sin \pi L} \hat{D}^L_{-\eta' \eta} (\pi + \varphi, \pi - \theta, \varphi) \times \hat{T}^L_{\ell' p' \eta', \ell p \eta} (E) \qquad (4.46)$$

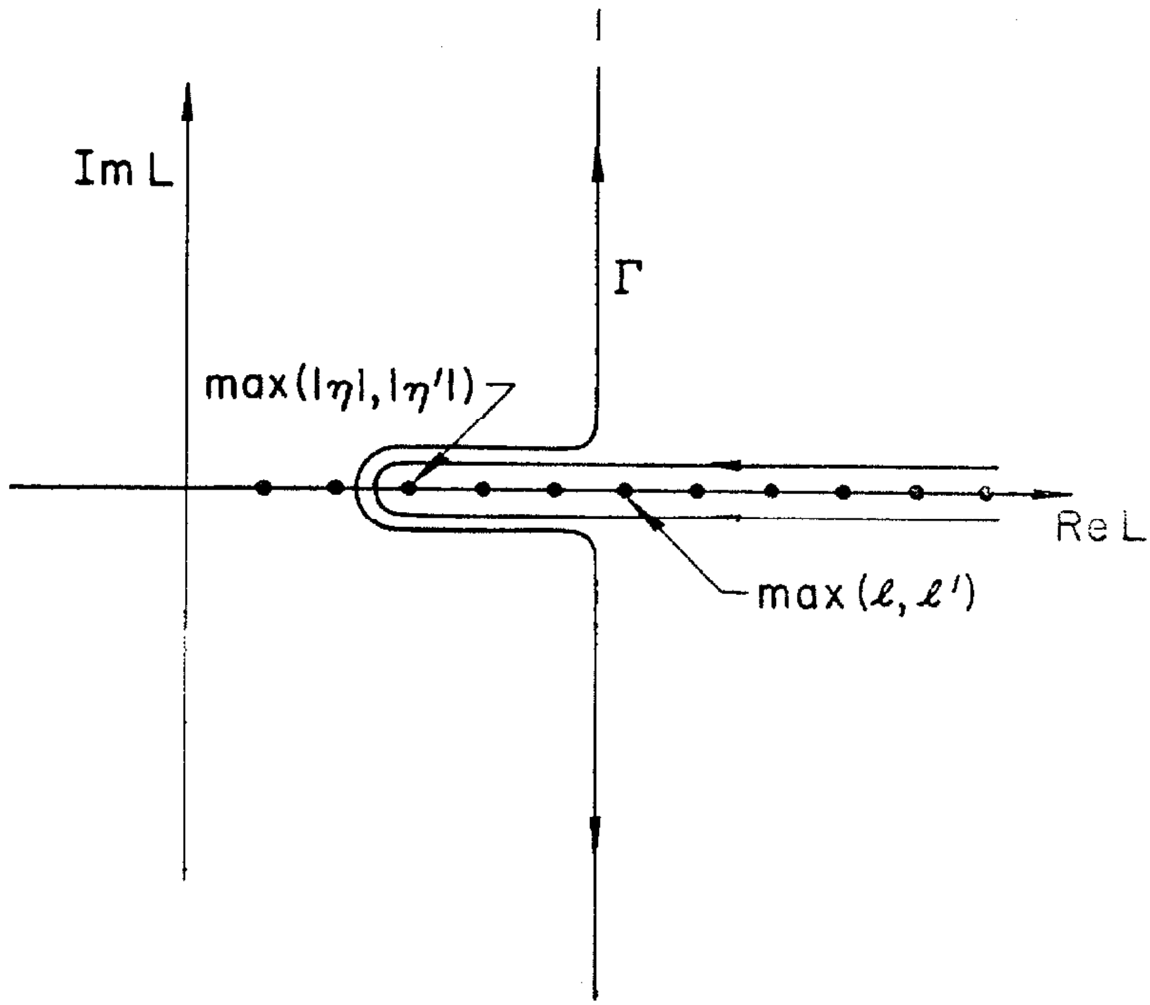

Fig. 3. Contours for making the Watson-Sommerfeld transformation.

where

$$\hat{D}^L_{\eta\eta'} = \rho_{\eta'} D^L_{\eta'\eta} \rho^{-1}_{\eta} \tag{4.47}$$

The $\hat{D}^L_{\eta\eta'}$ are entire functions of L as shown in Appendix A. The integrand of equation 4.41 is therefore an analytic function with only isolated singularities.

In order to deform the contour and display the Regge pole terms explicitly, the asymptotic behavior of the amplitude for large $|L|$ must be established. We will assume that this can be estimated from the Born approximation which is appropriate here since for large $|L|$ the potential can be neglected in comparison with the centrifugal barrier.

For sufficiently large Re L the Born approximation is given by

$$T^B = \frac{1}{2}\int_0^\infty dR\, \phi^{oT}(R) V(R) \phi^o(R) \tag{4.48}$$

where ϕ^o is the solution of the free Schrödinger equation

$$\phi^o_{\ell'p'\eta',\ell p\eta} = \delta_{p'p}\delta_{\ell'\ell} \sum_{\xi} U_{\eta'\xi}\left(\tfrac{1}{2}\pi R\right)^{1/2} J_{L+\xi+\frac{1}{2}}(PR) U^{-1}_{\xi\eta} \tag{4.49}$$

The integral converges for Re $L \geq (\ell + \ell' - 1)/2$.

$$<(\ell'p'\eta')|T^B|(\ell p\eta)> = \frac{1}{2}\sum_{\xi\xi'\eta_1\eta_2} U_{\eta'\xi'} \int_0^\infty dR\, R J_{L+\xi'+\frac{1}{2}}(P'R)$$

$$\times U_{\xi'\eta_1} V_{\ell'p'\eta_1,\ell p\eta_2} U^{-1}_{\eta_2\xi} J_{L+\xi+\frac{1}{2}}(PR) U^{-1}_{\xi\eta} \tag{4.50}$$

Now if $\ell p\eta$ and $\ell'p'\eta'$ are quantum numbers of bound states, the corresponding wave functions $\psi_\ell(p, r)$ and $\psi_{\ell'}(p', r)$ decrease exponentially for large r. If we assume interparticle Yukawa potentials, a typical integral involved in computing $V_{\ell' p' \eta_1, \ell p \eta_2}(R)$ is (for equal mass particles)

$$\int d^3 r \, \psi_{\ell'}(p', r) Y_{\ell'}^{\eta_1}(\beta, \alpha) \frac{e^{-\mu[R^2 + \frac{r^2}{4} - Rr\cos\beta]^{1/2}}}{[R^2 + \frac{r^2}{4} - Rr\cos\beta]^{1/2}} \psi_{\ell}(p, r) Y_{\ell}^{\eta}(\beta, \alpha) \tag{4.51}$$

The integrand decreases exponentially with R for all values of r and $\cos\beta$, and it is therefore reasonable to assume the integral does likewise. Since it is also a continuous function of R, and has a singularity no worse than R^{-1} at the origin, it is plausible that it is a superposition of Yukawa potentials.

$$RV_{\ell p \eta, \ell' p' \eta'}(R) = \int m_{\ell' p' \eta', \ell p \eta}(\mu) e^{-\mu R} d\mu \tag{4.52}$$

Indeed, all these statements can be justified in detail but we will not reproduce them here.

Knowing equation 4.52, we can immediately take over the results of Charap and Squires (2) for contour integrals such as equation 4.46. The contour C can be deformed into a curve Γ (see figure 3) for all $\mathrm{Re}\ \theta > 0$ with the integral along the large semi-circle vanishing. The resulting expansion is:

$$< (\ell' p' \eta') \vec{P} \,|\, T \,|\, \vec{P} (\ell p \eta) > = \sum_n \frac{B^{\alpha_n}_{\ell' p' \eta', \ell p \eta}(E)}{\sin \pi \alpha_n(E)} \hat{D}^{\alpha_n}_{\eta' \eta}(\pi+\varphi, \pi-\theta, \varphi)$$

$$+ \frac{1}{8\pi i} \int_\Gamma \frac{dL(2L+1)}{\sin \pi L} \hat{T}^L_{\ell' p' \eta', \ell p \eta}(E) \hat{D}^L_{-\eta' \eta}(\pi+\varphi, \pi-\theta, \varphi) \quad (4.58)$$

The $B^{\alpha_n}_{\ell' p' \eta', \ell p \eta}$ are simply related to the residues of the isolated singularities of $\hat{T}^L_{\ell' p' \eta', \ell p \eta}$. The asymptotic behavior in $\cos \theta$ can now be read off the above expression. With a more stringent estimate on the asymptotic behavior of the amplitude at large L the contour could be moved further to the left.

The generalization of the preceding discussion to the scattering of a single particle from a bound state of N particles is straightforward. Such an analysis is useful in a discussion of the scattering from nuclei. Again, a formal reduction of the three-particle equation can be made to a set of coupled two-particle equations except that the number of internal variables specifying the composite object is now much larger. The same analytic properties can be derived.

The extension to the scattering of particles with spin can also be made. The behavior of the system under rotation is now more complex but the above analysis should differ in no essential way. Wick (21) has shown how to make the partial wave expansion appropriate to this case.

4. Discussion

The partial wave Schrödinger equation continued to complex L, equation 2.8, does not uniquely determine the scattering solutions until appropriate boundary conditions are imposed. For the physical scattering problem at an integral value of L, there is only one way to do this. For complex L, however, there are many choices of the boundary conditions which reduce to the physical boundary conditions at integral values of L. Each choice of the boundary conditions will result in a different continuation of the amplitude to complex values of L. We can illustrate the situation with an example.

The three-particle kinetic energy expressed in terms of r, R, ℓ, and Λ, the orbital angular momentum of the center of mass of particles 1 and 2, can be written in its diagonal form:

$$\mathcal{J}^{L}_{\ell\Lambda} = -\frac{1}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right) - \frac{1}{2m'}\left(\frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} - \frac{\Lambda(\Lambda+1)}{R^2}\right) \tag{4.54}$$

For small values of r and R, we must impose the following boundary condition on the wave function in order that the solutions be regular

$$\begin{aligned} \psi^{L}_{\ell\Lambda} &\rightarrow R^{\Lambda}F(r) \quad \text{as} \quad R \rightarrow 0 \\ \psi^{L}_{\ell\Lambda} &\rightarrow r^{\ell}G(R) \quad \text{as} \quad r \rightarrow 0 \end{aligned} \tag{4.55}$$

Recalling that $\vec{\ell} + \vec{\Lambda} = \vec{L}$, this can be written, with L still an integer, as

$$\begin{aligned} \psi^{L}_{\ell\Lambda}(r, R) &\rightarrow R^{L+\xi}F(r) \qquad |\xi| \le \max(L,\ell) \\ \psi^{L}_{\ell\Lambda}(r, R) &\rightarrow r^{\ell}G(R) \end{aligned} \tag{4.56}$$

or

$$\psi^{L}_{\ell\Lambda}(r,R) \to R^{\Lambda}F(r) \qquad |\zeta| \leq \max(L,\Lambda)$$
$$\psi^{L}_{\ell\Lambda}(r,R) \to r^{L+\zeta}G(R) \tag{4.57}$$

Still at integral values of L, we introduce the unphysical wave functions corresponding to $L < |\xi| \leq \ell$ or $L < |\zeta| \leq \Lambda$. Two ways of specifying the boundary conditions on these functions which will be smooth in L when L becomes complex are to require either equation 4.56 or 4.57. These alternative boundary conditions will lead to solutions of a fundamentally different character for complex L since one choice specifies some solutions irregular at small R and regular for small r, while the other does the opposite. In comparing these particular alternatives, we may put the matter another way by saying that when we continue to complex L we can make either of the internal angular momenta, Λ or ℓ, complex at the same time.

The boundary condition of equation 4.56 is the one used in this work. Recently, Newton (9) and Drummond (8) have given another continuation of the three-particle scattering amplitude in which initially and finally two of the particles are bound. Along with the total angular momentum, Newton continues an orbital angular momentum associated with an interparticle distance. This corresponds to the boundary condition of equation 4.57. The resulting amplitude has cuts in the L-plane extending infinitely far to the right. These cuts correspond to the positions of the poles in the two-body amplitudes of particles 1 and 2 but smeared out because their two-particle energy is not conserved. Mathematically these cuts arise from the presence of poles in the diagonal part of a matrix which must be inverted (compare equation 4.45). This mechanism would be present even if the equations were truncated in the way discussed here.

The continuation given by Newton does not solve the problem of determining asymptotic behavior of the amplitude in any angle. This follows from the fact that there is no right-hand most singularity and the considerations of Section II.

If there is an analytic continuation from which the asymptotic behavior can be deduced from the position of the right-hand most singularity, it is unique and thus there must be a unique boundary condition for the scattering solution.*

The considerations given here show that for a certain class of amplitudes, at least, such a continuation does exist. Moreover, this continuation maintains the simplicity of the Regge prescription for large momentum transfer behavior in that it has only isolated singularities.

Mandelstam (7) has demonstrated that there are cuts in the relativistic two-particle amplitudes due to the coupling with three and higher particle states. His results are not in conflict with those presented here since the particular diagrams he considers are not present in the non-relativistic case. They are a type of diagram which arise in the relativistic theory from the possibility of creating virtual pairs of particles. The simple inelasticity of a three-particle scattering does not seem sufficient to complicate the large momentum transfer behavior characteristic of poles and isolated essential singularities in the complex angular momentum plane.

*The scattering solution for another type of process, for example the scattering of three free particles into three free particles, could require a different boundary condition in order to yield a proper continuation.

5. A Further Discussion of the Proof of Analyticity

It should be emphasized that the above discussion does not exhaust the attention which must be given to the proof of analyticity for three-particle amplitudes. The treatment given here could be rigorously justified if the intermediate p and ℓ sums were truncated. We have tried, however, to develop a method for which recourse to truncation need not be made. In the next paragraphs we will discuss an approach to a rigorous discussion of the problem and briefly indicate how it can be used to justify some of the intermediate steps of the above discussion.

Instead of attempting to examine the limit of the truncated S-matrices as the cut-off tends to infinity, it is more fruitful to deal with equation 4.14 expressed as a partial differential equation. We introduce

$$\hat{\phi}_{\ell p}(R, \vec{r}) = \sum_{\ell' p'} \hat{\phi}_{\ell' p' \eta', \ell p \eta}(R)\, \psi_{\ell'}(p', r)\, Y_{\ell'}^{\eta'}(\beta, \alpha) \qquad (4.58)$$

This obeys the equation

$$\left[\frac{1}{2m} \nabla_{\vec{r}}^2 + \frac{1}{2m'}\left(\frac{\partial^2}{\partial R^2} - \frac{\hat{\Lambda}^L}{R^2}\right) + E - V(R, r, \beta)\right] \hat{\phi}_{\ell p \eta}(R, \vec{r}) = 0$$

$$(4.59)$$

Here, $\hat{\Lambda}^L$ is the differential operator given by

$$\hat{\Lambda}^L = L(L+1) + \vec{\ell}^{\,2} - 2\ell_o^2 - 2\ell_+(L+\ell_o) - 2\ell_-(L-\ell_o) \qquad (4.60)$$

where $\vec{\ell}$ is to be interpreted as the usual differential operator in the coordinates β and α. This equation is not the same as the partial wave Schrödinger equation. Indeed, the partial wave Schrödinger equation, equation 2.5, is an infinite matrix of coupled equations in the index η while this is a single partial differential equation.

The proof outlined in Section 3 can be transcribed into this representation where the convergence of the series for $H^{(1,3)}(R)$ and $J(R)$ are more easily investigated. The integral equations for $H^{(1,3)}(R)$ take the form

$$H^{(1,3)}_{\ell p\eta}(R,\vec{r}) = h^{(1,3)}_{\ell p\eta}(R,\vec{r}) + \int_{R_o}^{\infty} dR' \int d^3r' \times G^L(R,\vec{r};R',\vec{r}\,')H^{(1,3)}_{\ell p\eta}(R,\vec{r}) \qquad (4.61)$$

Here, $G^L(R,\vec{r};R',\vec{r}\,')$ is given by

$$G^L(R,\vec{r};R',\vec{r}\,') = \sum_{\ell p\eta,\ell' p'\eta'} \psi_\ell(p,r)Y^\eta_\ell(\beta,\alpha) \times G^L_{\ell p\eta,\ell' p'\eta'}(R,R')\psi_{\ell'}(p',r')Y^{\eta'}_{\ell'}(\beta,\alpha) \qquad (4.62)$$

Since $G^L_{\ell p\eta,\ell' p'\eta'}$ is an entire function of L we may assume that $G^L(R,\vec{r};R',\vec{r}\,')$ is also. Strictly speaking this requires a detailed study of the convergence of equation 4.61. However, it is plausible that the singularities of the free Green's function correspond to those values of L for which solutions of free equation with the appropriate boundary condition exist. If there is a one to one correspondence between the solutions in the two representations given by equation 4.57, then it is not unreasonable that the singularities in the two representations are the same.

From equation 4.60 an integral equation for $[V(R,\vec{r})]^{\frac{1}{2}}H^{(1,3)}(R,\vec{r})$

can be deduced. This equation has as its kernal

$$[V(R,\vec{r})]^{\frac{1}{2}} G^L(R,\vec{r};R',\vec{r}')[V(R',\vec{r}')]^{\frac{1}{2}} \qquad (4.63)$$

From the asymptotic properties of the differential equation, equation 4.58, one can show that G^L is bounded for large values of its arguments. One can also deduce the form of its singularity at $R = R'$, $\vec{r} = \vec{r}'$ and find that equation 4.63 is bounded for $R > R_o$ by

$$K[V(R,\vec{r})]^{\frac{1}{2}}[(R-R')^2 + (\vec{r}-\vec{r}')^2]^{-2}[V(R,\vec{r})]^{\frac{1}{2}} \qquad (4.64)$$

for some K independent of L. This kernal is integrable to the power 3/2. This suffices to guarantee the convergence of the Fredholm series which solve equation 4.61. Moreover, equation 4.64 is independent of L so the series converge uniformly in the angular momentum. The analytic properties of the sum are, therefore, the same as the analytic properties of each term. This justifies the discussion of Section 2 so that the only singularities of $H^{(1,3)}_{\ell p\eta}(R,\vec{r})$ are the dynamical poles from the zeros of the Fredholm determinant.

The convergence of the power series for J(R) and N(R) can also be investigated in this representation. In the representation in which σ is diagonal the power series expansion, equation 4.16, becomes

$$\hat{\phi}_{\ell p\xi}(R,\vec{r}) = R^{\sigma_\xi} \sum_{n=0}^{\infty} a_{\ell p\xi}(\vec{r}, n, \sigma)R^n \qquad (4.65)$$

Supressing the indices $\ell p\xi$, the recursion relation, equation 4.18, can be written

$$[(\sigma+n+2)(\sigma+n+1)-\hat{\Lambda}^L]\,a(\vec{r},n+2,\sigma) = \frac{m'}{m}\Bigg[\left(\frac{\partial^2}{\partial r^2}-\frac{\vec{\ell}^{\,2}}{r^2}\right)a(\vec{r},n,\sigma) + \sum_{k=0}^{n+1} b(n-k,\vec{r})a(\vec{r},k,\sigma)\Bigg] \qquad (4.66)$$

where $b(k,\vec{r})$ is the transcription of equation 4.17 to this representation.

The Green's function of equation 4.21 will be denoted by $G^{L\sigma}_{n+2}(\Omega,\Omega')$

$$G^{L\sigma}_{n+2}(\Omega,\Omega') = \sum_{\ell\eta\eta'} Y^{\eta}_{\ell}(\beta,\alpha)[(\sigma+n+2)(\sigma+n+1)-\hat{\Lambda}^L]^{-1}_{\eta\eta'}\,Y^{\eta'}_{\ell}(\beta',\alpha') \qquad (4.67)$$

Again, we will assume that the analytic properties of the Green's function are the same in both representations. $G^{L\sigma}_{n+2}(\Omega,\Omega')$ then has simple fixed poles at integral and half-integral values of L.

Since $\vec{\ell}^{\,2}$ is hermetian with the boundary conditions imposed on β and α, equation 4.66 can be written

$$a(\vec{r},n+2,\sigma) = \frac{m'}{m}\int d\Omega' G^{L\sigma}_{n+2}(\Omega,\Omega')\Bigg[\frac{\partial^2 a(\vec{r},n,\sigma)}{\partial r^2} + \sum_{k=0}^{n+1} b(n-k,\vec{r})a(\vec{r},k,\sigma)\Bigg]$$

$$+\frac{m'}{m}\int d\Omega' \vec{\ell}^{\,\prime 2} G^{L\sigma}_{n+2}(\Omega,\Omega')a(\vec{r},n,\sigma)/r^2 \qquad (4.68)$$

Because all integrations are over bounded regions in β,α each coefficient $a(\vec{r},n,\sigma)$ is an analytic function of L except for the poles arising from the Green's functions.

The boundary condition $a_{\ell p\xi}(\vec{r},0,\sigma)$ depends on r through the wave function $\psi_\ell(p, r)$ (see equation 4.20). This is an entire function of r (20). The coefficients $b(k, \vec{r})$ of the power series expansion of a Yukawa potential can be established to be analytic in the region $0 < |r| < \infty$. Since only differentiation, multiplication by analytic functions, and integration over finite ranges of β and α are involved in the recursion relation, the $a(\vec{r}, n, \sigma)$ are also analytic in the region $0 < |r| < \infty$.

Choose a disc centered on the real r axis, $|r - r_o| < c$. Also choose a finite region in the L plane which deletes the singularities of $G_n^{L\sigma}$. In these regions $a(\vec{r}, n, \sigma)$ and $b(n, \vec{r})$ are bounded by

$$\begin{aligned} |a(\vec{r}, n, \sigma)| &\leq A_n \\ |b(n, \vec{r})| &\leq B_n \end{aligned} \tag{4.69}$$

where A_n and B_n do not depend on L, σ, or $\vec{r}$. From Cauchy's inequality we also have

$$\left| \frac{\partial^2 a(\vec{r}, n, \sigma)}{\partial r^2} \right| \leq M A_n \tag{4.70}$$

where M depends on r_o and c alone. With a similar constant M', the recursion relation implies

$$A_{n+2} = G^1_{n+2} \left[M A_n + \sum_{k=0}^{n+1} B_{n-k} A_k \right] + G^2_{n+2} M' A_k \tag{4.71}$$

with

$$G_n^1 = \max_{\Omega} \int d\Omega' \, |G_{n'}(\Omega, \Omega')| \, \frac{m'}{m}$$

$$G_n^2 = \max_{\Omega} \int d\Omega' \, |\vec{\ell}\,'^2 G_n(\Omega, \Omega')| \, \frac{m'}{m} \qquad (4.72)$$

We will now assume

$$G_n^{1,2} \leq P \qquad n = 0, 1, 2, \ldots \qquad (4.73)$$

This assumption, while plausible from the form of equation 4.66, should strictly be justified by a discussion of the convergence of that series.

The power series, equation 4.64, will converge uniformly in L and σ, if the convergence of the dominating series defined by the following recursion relation can be guaranteed.

$$\tilde{A}_{n+2} = P\left[(M+M')\tilde{A}_n + \sum_{k=0}^{n+1} B_{n-k}\tilde{A}_k\right] \qquad (4.74)$$

To show this define the functions

$$\tilde{A}(R) = \sum_{n=0}^{\infty} \tilde{A}_n R^n$$

$$B(R) = \sum_{n=-1}^{\infty} B_n R^n \qquad (4.75)$$

The second series converges since the $b(n,\vec{r})$ are bounded in n being the coefficients of a series expansion of the potential. The recursion relation becomes

$$[\tilde{A}(R) - \tilde{A}_0 - \tilde{A}_1 R]\, R^{-2} = P[(M+M')\tilde{A}(R) + B(R)\tilde{A}(R)] \qquad (4.76)$$

which has the solution

$$\tilde{A}(R) = \frac{\tilde{A}_o - \tilde{A}_1 R}{1 - PR^2(M + M' + B(R))} \qquad (4.77)$$

The coefficients $\tilde{A}_o$ and $\tilde{A}_1$ are determined by the boundary condition. A power series expansion of equation 4.76 clearly exists for some $R < R_1$. Since $A_n \leq \tilde{A}_n$, equation 4.64 converges uniformly in L and σ in a region which excludes the singularities of $G_n^{L\sigma}$. Since the $a(\vec{r}, n, \sigma)$ are analytic in this region we may conclude that J(R) and N(R) are also. This justifies the discussion of Section 2.

To determine the nature of the kinematic singularities of J(R) a more detailed study of the convergence of the series is necessary. From the recursion relation, however, it can be seen that $a(\vec{r}, n, \sigma)$ has poles of order n so that it is possible that the sum has essential singularities. It is important to show that these singularities cancel when the matrix X is formed. This is trivial for the truncated problem where the singularities are only poles of order $\sim \ell_{\text{cut-off}}$, but is less straightforward if they are really essential singularities.

By making some simple assumptions about the properties of the free Green's functions we have been able to demonstrate carefully the analytic properties of the elementary solutions J(R), N(R), and $H^{1,3}(R)$. The remaining problem is to construct the matrix X. In order to do this it is necessary to invert the matrix $B^{(1)}$ by Fredholm theory. This can be done if the matrix M of equation 4.45 can be shown to be summable to some power $p > 1$ (18). We have not succeeded in doing this rigorously.

V. A SIMPLE NUCLEAR MODEL

In this section, we discuss a simple consequence of the decoupling of the physical and unphysical solutions to Schrodinger's equation at integral values of L. On the basis of a crude model, nuclear rotational levels are correlated with definite Regge trajectories. The presence of the unphysical solutions at integral values of L enables the theory to incorporate sequences of rotational levels whose lowest state possesses a spin greater than zero.

Let us first examine the implications of the unitarity condition for the behavior of a trajectory as it crosses an integral value of L. For integral values of L the unitarity condition can be written from equation 2.3 as

$$T^{L}_{KK'}(p,p') - T^{L^*}_{KK'}(p,p') = i\int dp'' \sum_{K''=-L}^{L} T^{L}_{KK''}(p,p'')\theta(p'')T^{L^*}_{K''K'}(p'',p') \qquad (5.1)$$

$$|K| \leq L, \quad |K'| \leq L$$

where $\theta(p'')$ is a phase space factor independent of L. Since there is no coupling between the physical and unphysical solutions of Schrödinger's equation at the integers, the amplitudes which connect these states must vanish

$$T^{L}_{KK'} = T^{L}_{K'K} = 0 \qquad |K| \leq L, \quad |K'| \leq L, \quad L \text{ integral} \qquad (5.2)$$

The sum in equation 5.1 may therefore be extended over all values of K".

We can now use a standard argument to continue the unitarity condition to complex L. Under assumption (i) of Section II and the discussion of Appendix A, $\hat{T}^{L}_{KK'} = \rho_K^{-1} T^{L}_{KK'}\rho_{K'}$ is an analytic function of L for Re $L > L_o$. Defining the analytic function $\hat{T}^{*L}_{KK'} = (\hat{T}^{L*}_{KK'})^*$, the following expression is also an analytic function of L for Re $L > L_o$:

$$\hat{T}^{L}_{KK'}(p, p') - \hat{T}^{*L}_{KK'}(p, p')$$

$$- i\int dp'' \sum_{K''=-\infty}^{\infty} \hat{T}^{L}_{KK''}(p, p'')\theta(p'')\hat{T}^{*L}_{K''K'}(p'', p') \qquad (5.3)$$

This expression vanishes on the integral values of L in the region Re $L > \max(|K|, |K'|)$ from equation 5.1. By virtue of assumption (ii) of Section II it also vanishes strongly for large $|L|$ in the right half-plane. Carlson's theorem (22) then implies that it vanishes identically giving the unitarity relation for complex L.

One of the consequences of this unitarity relation is that the residues of a Regge pole factor (23)

$$B^{\alpha}_{K'K}(p', p) = b^{\alpha}_{K'}(p')b^{\alpha}_{K}(p) \qquad (5.4)$$

Suppose a trajectory $\alpha(E)$ passes through a positive integral α_o at energy E_o. If the pole appears in one physical amplitude, it must appear in all of them, and similarly for the unphysical amplitudes. Factorization and equation 5.2 then imply that either $b_K^{\alpha_o} = 0$ for all K such that $|K| > \alpha_o$ or $b_K^{\alpha_o} = 0$ for all K such that $|K| \leq \alpha_o$. In the first case, the pole appears in the physical and not the unphysical amplitudes. In the second case, the opposite is true. Every Regge trajectory as it passes through an integer thus has a choice. It can either appear in the physical or unphysical amplitudes but not both.

At integral values of L, only the poles in the physical amplitudes correspond to bound states. Thus, as a trajectory crosses an integral value it does not always correspond to a physical state. Which choice it makes at a given integer depends on the particular dynamics at hand.

To illustrate the application of these alternative possibilities for a trajectory at the integrals, we will consider a standard model from nuclear physics (24). The model consists of a single particle scattering off a rigid rotator core.* For simplicity, we will take the particle outside of the core to be spinless. The wave function is then a function of the three Euler angles which specify the orientation of the core φ, θ, ψ, and the position of the external particle, $\vec{r}$. If this position is specified in the body-fixed system, then the three Euler angles are conjugate to the total angular momentum $\vec{L}$. The partial wave expansion is again

$$\psi(\vec{r}, \varphi, \theta, \psi) = \sum_{LMK} \psi_K^L(\vec{r}) D_{MK}^L(\varphi, \theta, \psi) \tag{5.5}$$

If $\vec{I}$ is the angular momentum of the rotator, J_a $(a = 1, 2, 3)$ its moments of inertia, and $\vec{\ell}$ the angular momentum of the particle, the Hamiltonian for the system can be written:

$$\begin{aligned} H &= H_p + H_{rot} \\ H_p &= -\frac{1}{2m}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\vec{\ell}^{\,2}}{r^2}\right) + V(\vec{r}) \\ H_{rot} &= \sum_a I_a^2/2J_a \end{aligned} \tag{5.6}$$

*This model has also been considered for the case of complex angular momentum and separable potentials by E. Kazes (25).

In the rotational part of the Hamiltonian we can replace $\vec{I}$ by $\vec{L} - \vec{\ell}$. We first make the standard approximation that the nucleus is spheroidal

$$J_1 = J_2 = J \tag{5.7}$$

and that the potential is symmetric under rotations about the body-fixed z-axis and reflections in the body-fixed x-y plane. It is conventional to demand that wave function must be invariant under rotations of the rotator about its figure axis, or equivalently that the figure axis component of the rotator angular momentum $(L_3 - \ell_3)$ is zero, so that this term in the Hamiltonian may be neglected.

We can then write

$$H_{rot} = \frac{1}{2J}(\vec{L} - \vec{\ell})^2 = \frac{1}{2J}(\vec{L}^2 + \vec{\ell}^2 - 2L_3\ell_3) + H_{coup}$$

$$H_{coup} = \frac{1}{J}(L_+\ell_- + L_-\ell_+) \tag{5.8}$$

In the first approximation, we neglect the coupling term so that L_3 is a good quantum number. The partial wave Schrödinger equation is then

$$\left[\frac{1}{2m}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) - \left(\frac{1}{2mr^2} + \frac{1}{2J}\right)\vec{\ell}^2 - V(\vec{r}) + \mathcal{E}\right]\psi_K^L(\vec{r}) = 0 \tag{5.9}$$

with

$$\mathcal{E} = E - \frac{L(L+1) - 2K^2}{2J} \tag{5.10}$$

We wish to continue the amplitude to complex L. Clearly, this amounts to continuing the solutions of equation 5.9 to complex values of $\mathcal{E}$ with scattering boundary conditions. It is not important to consider the resulting analytic properties in detail. The model is an approximation to a many-particle system only for energies E near the bound state energies; that is, only for complex L near those values for which bound states exist at the given E. The bound states of equation 5.9 are reflected as poles in the scattering amplitude in the quantity $\mathcal{E}$. Let $\mathcal{E}_K^o$ be the position of one such pole which will, in general, depend on K if the potential is only axially symmetric. To each such pole, we have a trajectory in the right-half plane.

$$L_K(E) + \frac{1}{2} = + \left[\frac{1}{4} + 2K^2 + 2J(E - \mathcal{E}_K^o) \right]^{1/2} \qquad (5.11)$$

The levels which lie on a single trajectory are all members of a given rotational band.

Since K is a good quantum number for this approximate problem, the amplitude is diagonal in it. Each diagonal element T_{KK}^L has its own trajectory $L_K(E)$. At the integers, the physical amplitudes are those for which $|K| \leq L$ and thus as $L_K(E)$ crosses an integral it represents a physical bound state only for $L_K(E) \geq |K|$. Each trajectory thus represents a sequence of rotational levels with spin values $L = |K|, |K|+1, |K|+2, \ldots$. The trajectory itself, however, crosses all integral values.

A requirement usually imposed on this model is that the wave function be invariant under reflections in the x-y plane of the body-fixed system. The results of this assumption are in good agreement with experiment where the model applies. For $K \neq 0$, the character of the spectrum remains the same; there is sequence of levels with spins $L = |K|, |K|+1, |K|+2, \ldots$. For $K = 0$, however, a new feature arises.

Let R denote a rotation about the body-fixed y axis by π. For $K = 0$, $\psi_0^L(\vec{r})$ is invariant under rotations about the body-fixed z axis so that

$$R\psi_0^L(\vec{r}) = \nu\,\psi_0^L(\vec{r}) \tag{5.12}$$

where ν is the parity of $\psi_0^L(\vec{r})$. Because of the axial symmetry of the problem, the requirement of invariance under reflections is equivalent to a requirement of invariance under R. Taking $M = 0$ for simplicity, the wave function satisfying this requirement is

$$\psi(\vec{r},\varphi,\theta,\psi) = \psi_0^L(\vec{r})\,[P_L(\cos\theta) + \nu P_L(-\cos\theta)] \tag{5.13}$$

A given Regge trajectory in $\psi_0^L(\vec{r})$ will now represent a physical state only at every other integral value of L. It thus describes a sequence of levels with $L = 0^+, 2^+, 4^+, \ldots$ or $L = 1^-, 3^-, 5^-, \ldots$.

Suppose, now, K is no longer a good quantum number. This happens when we consider the effect of H_{coup} or introduce small asymmetries in the moments of inertia J_1, J_2. The trajectories can no longer be computed exactly, but we can assume that for small symmetry breaking terms they do not depart strongly from equation 5.11. Each trajectory is characterized by a number $|K|$ which determines the diagonal amplitude T_{KK}^L in which it appears as the symmetry breaking coupling disappears. As the coupling is turned on, the pole will appear in the amplitudes to which T_{KK}^L is coupled. In general this will be all the amplitudes except at the integers where the pole will appear in the physical amplitudes if K is physical ($|K| \leq L$) and in the unphysical amplitudes if K is unphysical ($|K| > L$).

Thus, even in the case when L_3 is not conserved, we have a sequence of rotational levels or bands correlated with a definite trajectory characterized by a number $|K|$ which is the spin of the lowest physical state in the band. For integral spins greater than $|K|$, the trajectory appears in the physical amplitudes, and in the unphysical amplitudes for spin values less than $|K|$.

VI. THRESHOLD BEHAVIOR OF REGGE TRAJECTORIES

Physical quantities usually have a simple energy dependence near scattering thresholds. In this section we will explore the behavior of Regge trajectories near certain thresholds of the three-particle scattering problem. For applications, it is useful to know how trajectories move in the complex angular momentum plane, and their behavior at threshold is one of the most easily obtained general features.

If pairs of particles can form bound states the lowest threshold of a three-particle scattering problem is at $E_o = p_o^2/2m$, the energy of the lowest two-particle bound state. Just above this threshold the scattering of a free particle off a bound pair can take place. If no two-particle bound states exist then the lowest threshold is that for the scattering of three free particles, $E = 0$. The threshold behavior of Regge trajectories near these two classes of lowest thresholds will be investigated here.

Below the lowest threshold $\alpha(E)$ lies on the real axis since there the trajector corresponds to three-particle bound states. Slightly above the threshold energy it describes a resonance whose width is proportional to $\text{Im}\ \alpha(E)$. Our object in this section is to determine the energy dependence of $\text{Im}\ \alpha(E)$ near threshold. We will use a slight generalization of the methods used to determine threshold behavior in ordinary quantum mechanics (26) which have been successfully applied to determining the threshold behavior of Regge trajectories when only two-particle channels are present (27).

Let us consider first the case where the lowest threshold corresponds to a two-particle bound state. This state will have $\ell = 0$. We denote the indices $\ell = 0$, $p_o^2, \eta = 0$ collectively by the index 0. If E is only slightly above the lowest threshold there is only one open channel. Here, the Schrödinger equation, equation 4.14, can be written

$$\left(\frac{d^2}{dR^2} + P_o^2 - V_{o,o} - \frac{L(L+1)}{R^2} \right) \phi_{o,\ell p\eta}(R)$$

$$- \sum_{\ell' p' \eta'} V_{o,\ell' p' \eta'} \phi_{\ell' p' \eta', \ell p\eta}(R) = 0 \tag{6.1}$$

where $P_o^2 = 2m'(E - p_o^2/2m)$. For large values of R, $V_{o,\ell p\eta}$ decreases exponentially like $\exp[-|p_o|R]$ (compare the discussion of Section IV.3). For $R >> 1/|p_o|$ we may, therefore, neglect the potential terms in equation 6.1 provided the closed channel wave functions do not increase exponentially with large R. If $R << 1/|p_o|$, we may neglect the P_o^2 term in the equation. For E sufficiently close to threshold there is a region of values of R where both criteria are satisfied simultaneously and both potential and energy terms may be neglected. In this circumstance a solution in the open channel can be written

$$\chi_o(R) = AR^{L+1} + BR^{-L} \tag{6.2}$$

where A and B are constants.

Consider the solution $\chi_{\ell p\eta}(R)$ with a boundary condition that there are no expanding waves in the closed channels and in the open channel

$$\chi_o(R) \to R^{L+1} \quad \text{as} \quad R \to 0 \tag{6.3}$$

This is a boundary condition which is independent of P_o^2 and hence the constants A and B of equation 6.2 must also be independent of P_o^2. The scattering solution will be a multiple of this solution since it is regular for small R.

For values of R comparable or greater than $|P_o|^{-1}$, the P_o^2 term in the equation can no longer be neglected. The potential, however, is negligible if E is close to threshold, and the solutions can be expressed as linear combinations to the solutions of the free equation. The scattering amplitude may be defined by the solution which has the following form in the open channel

$$\phi_{oo}(R) = (\tfrac{1}{2} P_o R)^{1/2} J_{L+\frac{1}{2}}(P_o R) + T_{oo}(E,L)(\tfrac{1}{2} P_o R)^{1/2} H^{(1)}_{L+\frac{1}{2}}(P_o R) \tag{6.4}$$

This follows from equation 4.30 and the large R behavior of the Bessel functions (32). For small values of R equation 6.4 becomes (32)

$$\phi_{oo}(R) = \frac{(\frac{1}{2} P_o R)^{L+1}}{(L+\frac{1}{2})!} \left[1 + i \frac{T_{oo} e^{-i\pi(L+\frac{1}{2})}}{\sin \pi(L+\frac{1}{2})} \right] - i \frac{(\frac{1}{2} P_o R)^{-L}}{(-L-\frac{1}{2})!} \quad \frac{T_{oo}}{\sin \pi(L+\frac{1}{2})} \tag{6.5}$$

Since the solution in equation 6.5 must be a multiple of that in equation 6.2, we can write T_{oo} in the form

$$T_{oo}(E,L) = \frac{i P_o^{2L+1} \cos \pi L}{F(E,L) + P_o^{2L+1} e^{-i\pi(L+\frac{1}{2})}} \tag{6.6}$$

where

$$F(E_o, L) = \frac{A(-L-\frac{1}{2})!}{B(L+\frac{1}{2})!}\; 2^{2L+1} \qquad (6.7)$$

The poles of the amplitude then correspond to the solutions of

$$F(E, \alpha(E)) = -P_o^{2\alpha+1} e^{-i\pi(\alpha+\frac{1}{2})} \qquad (6.8)$$

Suppose there is a pole at threshold for $\mathrm{Re}\,\alpha > -\frac{1}{2}$, then

$$F(E_o, \alpha(E_o)) = 0 \qquad (6.9)$$

Equation 6.8 can be expanded as

$$\frac{\partial F(E_o, \alpha(E_o))}{\partial \alpha}\,[\alpha(E) - \alpha(E_o)] + \frac{\partial F(E_o, \alpha(E_o))}{\partial E}\,(E-E_o) + \ldots$$

$$= -P_o^{2\alpha+1} e^{-i\pi(\alpha+\frac{1}{2})} \qquad (6.10)$$

The function $F(E, L)$ is a real function of the energy and angular momentum, as can be seen from the following argument. For values of E away from threshold equation 6.7 will still relate F to the coefficients A and B of equation 6.2, these coefficients now having higher order terms in P_o. The Schrödinger equation is a real equation in E and L. Since the boundary condition of equation 6.3 is a real function of E and L, A and B must be also. From equation 6.7 the desired result for F follows.

If F is a real function of E and L then at threshold, where $\alpha(E_o)$ is real, each of the coefficients in equation 6.10 will be real. Taking the imaginary part of both sides we find near threshold

$$\text{Im } \alpha(E) \approx CP_o^{2\alpha+1}$$

$$\approx C(E - p_o^2/2m)^{\alpha+\frac{1}{2}} \qquad \text{Re } \alpha > -\frac{1}{2} \qquad (6.11)$$

where C is a constant given by $\cos \pi L(\partial F/\partial \alpha)^{-1}$

This is the threshold behavior which characterizes two-body amplitudes. This is hardly surprising since the process we are considering is the kinematically similar one of elastic scattering of a particle off a bound level.

Now we will consider the threshold behavior of a Regge trajectory for a three-particle system in which no two-particle bound states are present. The lowest threshold is then at $E = 0$ and the process is the scattering of three free particles into three free particles. For simplicity we will assume equal mass particles. The analyticity of the amplitudes which describe such a process has not been considered in this work, but that does not prevent us from examining the threshold behavior of its trajectories.

The poles of the scattering amplitude arise from the poles of the scattering solution to Schrödinger's equation. In order to examine the poles of this solution one must first find the boundary conditions which determine it. In terms of the outgoing wave Green's function $G(\vec{R}, \vec{r}; \vec{R}', \vec{r}')$ the scattering solution may be written

$$\Psi(\vec{R}, \vec{r}) = e^{i(\vec{P} \cdot \vec{R} + \vec{p} \cdot \vec{r})}$$

$$+ \int G(\vec{R}, \vec{r}; \vec{R}', \vec{r}') V(\vec{R}', \vec{r}') \Psi(\vec{R}', \vec{r}') \qquad (6.12)$$

The Green's function can be expressed, up to constant factors, as

$$G(\vec{R},\vec{r};\vec{R}',\vec{r}') \propto \int d^3P\, d^3p\, \frac{e^{i[\vec{P}\cdot(\vec{R}-\vec{R}')+\vec{p}\cdot(\vec{r}-\vec{r}')]}}{E-\frac{P^2}{2m'}-\frac{p^2}{2m}+i\varepsilon} \tag{6.13}$$

Introducing the vectors in a six-dimensional space

$$\begin{aligned} \vec{\rho} &= (\sqrt{2m'}\,\vec{R},\ \sqrt{2m}\,\vec{r}) \\ \vec{K} &= \left(\frac{\vec{P}}{\sqrt{2m'}},\ \frac{\vec{p}}{\sqrt{2m}}\right) \end{aligned} \tag{6.14}$$

this can be written*

$$G(\vec{\rho},\vec{\rho}\,') \propto \frac{H_2^{(1)}(K\,|\vec{\rho}-\vec{\rho}\,'|)}{|\vec{\rho}-\vec{\rho}\,'|} \tag{6.15}$$

For large values of ρ this can be approximated by

$$G(\vec{\rho},\vec{\rho}\,') \approx \frac{e^{iK\rho}}{\rho^2}\, e^{i\vec{K}\cdot\vec{\rho}\,'} \tag{6.16}$$

Substituting this in equation 6.12, we see that the appropriate boundary condition for the scattered wave is that it behave like $e^{iK\rho}/\rho^2$ for $\vec{\rho}$ in those directions which correspond to a complete separation of all three particles (large $|\vec{r}|$ and large $|\vec{R} \pm \frac{1}{2}\vec{r}|$ for equal mass particles.

The angular variables $\varphi, \theta, \beta, \alpha$ may be separated out of the Schrödinger equation as in Section IV. In the representation in which Λ^L is diagonal we may write

*The formalism we are following the latter part of this section is that of ref. 28.

$$\left[\frac{1}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{\ell(\ell+1)}{r^2}\right)+\frac{1}{2m'}\left(\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}-\frac{(L+\xi)(L+\xi+1)}{R^2}\right)\right.$$

$$\left.+K^2-V(r)\right]\phi^L_{\ell\xi}(R,r)-\sum_{\ell'\xi'}V_{\ell\xi,\ell'\xi'}(R,r)\phi_{\ell'\xi'}(R,r)=0 \qquad (6.17)$$

The potential is negligible in the region where the particles are separated by large distances

$$r > 1/\mu$$
$$\left|R-\frac{1}{2}r\right| > 1/\mu \qquad (6.18)$$

Here, $1/\mu$ is the longest range of the interparticle Yukawa potentials. In this region the scattering solution may be written as a superposition of free solutions whose scattered part satisfies the boundary condition mentioned above.

For small values of K the energy term in the equation can be neglected provided

$$\rho=(2mr^2+2m'R^2)^{1/2} << K^{-1} \qquad (6.19)$$

For sufficiently small K, there is a region where both the potential and energy terms can be neglected.

In order to discuss the solutions in regions in which the potential can be neglected it is convenient to follow Morse and Feshbach and introduce the coordinates ρ and ζ

$$\rho\sin\zeta=\sqrt{2m'}\,R$$
$$\rho\cos\zeta=\sqrt{2m}\,r \qquad 0\le\zeta\le\frac{\pi}{2} \qquad (6.20)$$

The free solutions can then be further separated into products of functions of ρ and functions of ζ.

Consider a region $0 \leq \zeta < \zeta_0$, $\rho > \rho_0$ where the conditions of equation 6.18 are satisfied. Small values of ζ correspond to small values of R. In this region we recall that the boundary condition was that the wave function behave like $R^{L+\xi}$. A complete set of angular functions which have this property are (29)

$$f_{\ell n\xi}(\zeta) = (\sin\zeta)^{L+\xi}(\cos\zeta)^{\ell} \times F(-n, L+\xi+n+\ell+2, L+\xi+\tfrac{3}{2}; \sin^2\zeta)$$

$$g_{\ell n\xi}(\zeta) = (\sin\zeta)^{L+\xi}(\cos\zeta)^{-\ell-1} \times F(-n, L+\xi+n-\ell+1, L+\xi+\tfrac{3}{2}; \sin^2\zeta) \qquad (6.21)$$

where $n = 0, 1, 2, \ldots$ and the F's are hypergeometric functions. In terms of Bessel functions typical elementary free solutions can be written

$$\rho^{-2} J_{\nu(\ell n\xi)}(K\rho) f_{\ell n\xi}(\zeta)$$

$$\rho^{-2} H^{(1)}_{\nu(\ell n\xi)}(K\rho) f_{\ell n\xi}(\zeta) \qquad (6.22)$$

$$\rho^{-2} H^{(1)}_{\nu'(\ell n\xi)}(K\rho) g_{\ell n\xi}(\zeta)$$

where

$$\nu(\ell n\xi) = L + \xi + 2n + \ell + 2$$

$$\nu'(\ell n\xi) = L + \xi + 2n - \ell + 1 \qquad (6.23)$$

The first of these solutions is regular for small ρ. These are the solutions which are involved in the expansion of a plane wave (29). The second solutions have pure outgoing waves in ρ. The scattering solution can, therefore, be written in this region as

$$\rho^2 \phi_{\ell\xi,\ell'n'\xi'}(\rho,\zeta) = \sum_{n=0}^{\infty} \Big[J_{\nu(\ell n\xi)}(K\rho) f_{\ell n\xi}(\zeta) + H^{(1)}_{\nu(\ell n\xi)}(K\rho) f_{\ell n\xi}(\zeta) W_{\ell n\xi,\ell'n'\xi'} + H^{(1)}_{\nu'(\ell n\xi)}(K\rho) g_{\ell n\xi}(\zeta) Y_{\ell n\xi,\ell'n'\xi'} \Big] \tag{6.24}$$

$$0 \leq \zeta < \zeta_o, \quad \rho_o < \rho$$

where the indices $\ell'n'\xi'$ refer to the boundary condition of the incoming wave and W and Y are constant matrices. The remaining boundary conditions which determine the matrices W and Y are the requirements that the solution ϕ be regular for small ρ and at $\zeta = \frac{\pi}{2}$ and that the scattered wave possess the asymptotic behavior generated by equation 6.12 at large ρ and $\zeta_o < \zeta \leq \frac{\pi}{2}$.

Imagine that the potential is divided into two parts, the first non-zero in the region $0 < \rho < \rho_o$ and the second in the region $\rho_o < \rho < \infty$. Let the second part be characterized by a coupling constant λ. For $\lambda = 0$, Y must vanish since equation 6.24 is then valid for all ζ and must be regular at $\zeta = \frac{\pi}{2}$ (see equation 6.21).

Every bound state and resonance pole will occur in the matrix W. Physically, this is because the part of the potential which can produce a resonance must attract all three particles at once. It must, therefore, be significant for small ρ. The matrix Y is due mainly to the large ρ part of the potential in which only two particles act simultaneously. In considering the behavior of the Regge trajectories we may, therefore, restrict our attention to W.

For $\rho << K^{-1}$, the K^2 term in the Schrödinger equation can be neglected. The boundary condition that ϕ be regular for small ρ does not depend on K. The ratio of the coefficients of $\rho^{\nu(\ell n\xi)} f_{\ell n \xi}(\zeta)$ and $\rho^{-\nu(\ell n\xi)} f_{\ell n\xi}(\zeta)$ is therefore a constant to lowest order in K. In an argument which follows that of the scattering of a single particle off a composite object discussed above, we find for the form of W

$$W = i \sin \pi\nu K^{2\nu} [F(K, L) + K^{2\nu} e^{-i\pi\nu}]^{-1} \qquad (6.25)$$

Here, F is a matrix and ν a diagonal matrix with elements $\nu(\ell n\xi)$.

The poles of W are given by

$$\det [F(K, L) + K^{2\nu} e^{-i\pi\nu}] = 0 \qquad (6.26)$$

The slowest K dependence of the diagonal term in equation 6.26 is given by those elements with $\xi = -\ell$ and $n = 0$. The matrix relation

$$\det [A_{ij} + B_{ij}] = \det [A_{ij} + B'_{ij}] + B_{ii} \{A+B\}_{ii} \qquad (6.27)$$

where B' is the matrix with B_{ii} deleted and $\{A+B\}_{ii}$ is the cofactor of element $A_{ii} + B_{ii}$, can be applied successively to these elements to write equation 6.26 as

$$G(E, L) + K^{2L+4} e^{-i\pi(L+2)} = 0 \qquad (6.28)$$

for some new function G.

The analysis which followed equation 6.8 can now be applied to equation 6.28 to yield the threshold behavior of a Regge trajectory at $E = 0$, $\text{Re}\ \alpha > -2$ when no two-particle bound states are present. We find

$$\text{Im}\ \alpha(E) \approx CE^{\alpha+2} \ , \qquad \text{Re}\ \alpha > -2 \tag{6.29}$$

for some constant C.

This behavior can be compared with a qualitative estimate for the width of the three-particle decay of a particle X with spin L.

$$\Gamma = \sum_{\ell\xi\eta} \int p^2\,dp \int P^2\,dP < \ell p\eta\xi\ |M(E, L)|\,X > \times \delta\left(\frac{p^2}{2m} + \frac{P^2}{2m'} - E \right) \tag{6.30}$$

We have taken the final states to be characterized by ℓ, p, η and $L + \xi$ the orbital angular momentum of the center of mass of emitted particles 1 and 2. For these angular momentum states the matrix element has the following form for small P and p

$$< \ell p\eta\xi\ |M(E, L)|\,X > = \text{const.} \times P^{L+\xi} p^{\ell} \tag{6.31}$$

Substituting this in equation 6.30 and counting powers of the energy we find

$$\Gamma = \text{const.} \times E^{L+2} \tag{6.32}$$

The qualitative estimate is thus in agreement with the result calculated in equation 6.29.

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APPENDIX A.

PROPERTIES OF THE ANGULAR MOMENTUM EIGENFUNCTIONS

Denote by φ, θ, ψ the three Euler angles conjugate to the total angular momentum $\vec{L}(\varphi, \theta, \psi)$. $\vec{L}$ is a differential operator on functions of these angles whose Cartesian components we denote by $L_i'(\varphi, \theta, \psi)$ in the body-fixed system and by $L_i(\varphi, \theta, \psi)$ in the space-fixed system.* The $D^L_{MK}(\varphi, \theta, \psi)$ are then defined for complex L by:

$$D^L_{00}(\varphi, \theta, \psi) = P_L(\cos\theta)$$

$$\pm\left[\frac{1}{2}(L \pm M)(L \mp M + 1)\right]^{1/2} D^L_{M\mp 1, K} = L_\mp(\varphi, \theta, \psi)\, D^L_{MK}$$

$$\mp\left[\frac{1}{2}(L \pm K)(L \mp K + 1)\right]^{1/2} D^L_{M, K\mp 1} = L'_\pm(\varphi, \theta, \psi)\, D^L_{MK} \qquad \text{(A. 1)}$$

where $L_\pm$, L_0 are the spherical components of $\vec{L}$. One choice of the branch of the square root must be made, say by taking it real for large real L. Since L^2, $L_\pm$, L_0 satisfy the usual commutation relations, the D^L_{MK} defined in this way also satisfy the differential equations implied by equation 2.1. The solutions of these equations can be expressed in terms of hypergeometric functions. It can be checked from the definition, equation A.1, that the D^L_{MK} can be written:

$$D^L_{MK}(\varphi, \theta, \psi) = e^{iM\varphi} d^L_{MK}(\cos\theta) e^{iK\psi}$$

$$D^L_{MK}(z) = \left[\frac{(L+M)!(L-K)!}{(L+K)!(L-M)!}\right]^{1/2} \left(\frac{1+z}{2}\right)^{(M+K)/2} \left(\frac{1-z}{2}\right)^{(M-K)/2}$$

$$\times \frac{1}{(M-K)!} F(-L+M, L+M+1, M-K+1, \frac{1-z}{2}) \qquad \text{(A. 2)}$$

---

*For explicit expressions see ref. 10.

for $M - K \geq 0$, other cases being obtained from the symmetry relations (10) for the $d^L_{MK}(z)$. In particular, we have

$$D^L_{M0}(\varphi, \theta, \psi) = P^M_L(\cos\theta)\, e^{iM\varphi} \left[\frac{(L-M)!}{(L+M)!}\right]^{1/2}$$

$$D^L_{0K}(\varphi, \theta, \psi) = (-1)^K D^L_{K0}(\psi, \theta, \varphi) \qquad M, K > 0 \qquad \text{(A. 3)}$$

The D^L_{MK} thus defined reduce to the usual values for integral L and $|M| \leq L$, $|K| \leq L$.

To prove that the D^L_{MK} as defined for complex L by equation A.1 form a representation of the rotation group, we begin with the addition theorem for Legendre functions (28)

$$P_L(x) = P_L(x_2)P_L(x_1) + 2\sum_{N=1}^{\infty} \frac{(L-N)!}{(L+N)!} P^N_L(x_2)P^N_L(x_1) \cos N(\varphi_1 + \psi_2 - \pi) \qquad \text{(A. 4)}$$

Here, $x = \cos\theta$, $x_1 = \cos\theta_1$, $x_2 = \cos\theta_2$, where φ, θ, ψ are the angles of the rotation resulting from successive rotations through $\varphi_1\theta_1\psi_1$ and $\varphi_2\theta_2\psi_2$. Applying equation A. 3, this can be written

$$D^L_{00}(\varphi, \theta, \psi) = \sum_{N=-\infty}^{\infty} D^L_{0N}(\varphi_2, \theta_2, \psi_2) D^L_{N0}(\varphi_1, \theta_1, \psi_1) \qquad \text{(A. 5)}$$

Consider the operator $L_+(\varphi_1, \theta_1, \psi_1)$ on functions φ, θ, ψ with $\varphi_2, \theta_2, \psi_2$ remaining constant. $L_+(\varphi_1, \theta_1, \psi_1)$ may be expressed as a function of the new variables φ, θ, ψ as $L_+(\varphi_1(\varphi, \theta, \psi), \theta_1(\varphi, \theta, \psi), \psi_1(\varphi, \theta, \psi))$. To do this, regard the transformation of variables as a change of coordinates under which $\vec{L}$ transforms like a vector

$$L_i(\varphi,\theta,\psi) = \sum_j D^{(1)}_{ij}(\psi_2,\theta_2,\varphi_2)L_j(\varphi_1(\psi,\theta,\varphi),\theta_1(\psi,\theta,\varphi),\psi_1(\psi,\theta,\varphi)) \quad (A.6)$$

The left-hand side of equation A. 6 gives the components of $\vec{L}$ along the rotated axes while we need them along the old axes as functions of the new variables. To find these, multiply the new components by the matrix $[D^{(1)}(\psi_2,\theta_2,\varphi_2)]^{-1}$, thus obtaining

$$L_+(\varphi,\theta,\psi) = L_+(\varphi_1(\psi,\theta,\varphi),\theta_1(\psi,\theta,\varphi),\psi_1(\psi,\theta,\varphi)) \quad (A.7)$$

Thus, applying $(L_+)^M$ to both sides of equation A. 6, and multiplying both sides by a common normalization factor, we find

$$D^L_{M0}(\varphi,\theta,\psi) = \sum_{N=-\infty}^{\infty} D^L_{MN}(\varphi_2,\theta_2,\psi_2)D^L_{N0}(\varphi_1,\theta_1,\psi_1) \quad (A.8)$$

A similar procedure with L_-, L'_+, L'_- gives the desired result, equation 2.13. The differentiation under the summation is justified since equation A. 5 converges like $(\tan\theta_1/2 \tan\theta_2/2)^N$ and hence uniformly in θ_1, θ_2 provided

$$0 \le \theta_1 + \theta_2 < \pi,\ 0 \le \theta_1 < \pi,\ 0 \le \theta_2 < \pi \quad (A.9)$$

We may remark that these representations of the rotation group are not further reducible to block form except at integral L. Suppose this were not true; then there would be a matrix commuting with all D^L_{MK} and not a multiple of the unit matrix. Suppose, however, M_{NK} commutes with all the D^L_{MK}

$$\sum_N M_{MN} D^L_{NK} = \sum_N D^L_{MN} M_{NK} \qquad \text{(A.10)}$$

multiply by $e^{-i(P\psi+Q\varphi)}$ and project out the angles to find

$$d^L_{PQ} M_{MP} \delta_{KQ} = \delta_{PM} M_{QK} d^L_{PQ} \qquad \text{(A.11)}$$

Provided d^L_{PQ} is not in block form, which case only for integral L, we have

$$M_{MP} = \delta_{MP} M_{QQ} \qquad \text{(A.12)}$$

Thus, M_{MP} is a multiple of the unit matrix, and we have a contradiction. For integral L, the elements of D^L_{MK} are non-zero only if

$$|M| > L \quad \underline{\text{and}} \quad |K| > L$$

or

$$|M| \leq L \quad \underline{\text{and}} \quad |K| \leq L$$

The second class of elements corresponds to the usual D^L_{MK}. At integrals, the matrix D^L_{MK} is thus in block form.

Using equation A. 2 we can define

$$\hat{D}^L_{MK} = \rho_M D^L_{MK} \rho_K^{-1} = \frac{(L-K)!}{(L-M)!} \left(\frac{z+1}{2}\right)^{(M+K)/2} \left(\frac{z-1}{2}\right)^{(M-K)/2}$$

$$\times\, e^{iM\varphi} e^{iK\psi} \frac{1}{(M-K)!} F(M-L, M+L+1, M-K+1, \frac{z-1}{2}) \qquad \text{(A. 14)}$$

for $M-K \geq 0$, and similarly for other values. Since $[\Gamma(c)]^{-1} F(a, b, c, z)$ is an entire function of a, b, c, (29) we have that $\hat{D}^L_{KK'}$ is an entire function of L.

Further properties of the D^L_{MK} such as their asymptotic behavior for large L and large z have been discussed by Charap and Squires (2). Their properties for integral values of L are given in ref. 10.

APPENDIX B

DIAGONALIZATION OF Λ^L

Define $\hat{\Lambda}^{L,\ell}_{\eta\eta'}$ by

$$\hat{\Lambda}^{L}_{\ell p\eta,\ell' p'\eta'} = \delta_{\ell\ell'}\delta_{pp'}\hat{\Lambda}^{L,\ell}_{\eta\eta'} \tag{B.1}$$

$\hat{\Lambda}^{L,\ell}$ is a $(2\ell+1) \times (2\ell+1)$ matrix representing the orbital angular momentum of the center of mass of particles 1 and 2. As such, it has eigenvalues $(L+\ )(L+\ +1)$ with $-\ell \leq \xi \leq \ell$ for integral values of L such that $L \gtrsim \ell$.

Consider the function

$$\det\left[(L+\xi)(L+\xi+1)I - \hat{\Lambda}^{L,\ell}\right] \tag{B.2}$$

It is clearly an entire function of L with a polynomial behavior at large $|L|$. Since it vanishes on the integers $L \geq \ell$, Carlson's theorem (24) requires that it vanish identically. The eigenvalues of $\hat{\Lambda}^L$ for complex L are, therefore, also given by $(L+\xi)(L+\xi+1)$ with $-\ell \leq \xi \leq \ell$.

When L is an integral greater than ℓ, the elements of the unitary transformation $U_{\eta\xi}$ which diagonalizes $\Lambda^{L,\ell}$ are easily computed. Denote by λ an eigenvalue of $\Lambda^{L,\ell}$, then

$$<LM\ell\eta|\lambda\ell LM> = \sum_{m_1 m_2} <LM\ell\eta|\lambda m_1 \ell m_2><\lambda m_1 \ell m_2|\lambda\ell LM> \tag{B.3}$$

Now,

$$<LM\ell\eta|\ell m_1 \lambda m_2> = \left(\frac{2L+1}{4\pi}\right)^{1/2}\int d\Omega_{\theta\varphi}\int d\Omega_{\beta a}$$

$$\times D^{L*}_{M\eta}(\varphi,\theta,0)Y^{\eta *}_{\ell}(\beta,a)Y^{m_2}_{\ell}(\beta_1,a_1)Y^{m_1}_{\lambda}(\theta,\varphi) \tag{B.4}$$

where

$$Y_\ell^{m_2}(\beta_1, \alpha_1) = \sum_{\eta'} D^\ell_{\eta' m_2}(0, \theta, \varphi) Y_\ell^{\eta'}(\beta, \alpha)$$

via a rotation of coordinates. Performing the integration, one finds in terms of Clebsch-Gordan coefficients

$$< LM\ell\eta \,|\, \lambda m_1 \ell m_2 > = (-1)^{\eta+M} \left(\frac{2\lambda+1}{2L+1}\right)^{1/2}$$

$$\times\ C(\lambda \ell L; 0\eta\eta)\ C(\lambda\ell L; m_1 m_2 M) \qquad \text{(B. 5)}$$

The sum in equation B. 4 may be explicitly performed and an arbitrary phase chosen to give

$$< LM\ell\eta \,|\, \lambda\ell LM > = (-1)^{\eta} \left(\frac{2\lambda+1}{2L+1}\right)^{1/2} C(\lambda\ell L; 0\eta\eta) \qquad \text{(B. 6)}$$

Writing $\lambda = L + \xi$, we have for integral values of L

$$U_{\eta\xi} = (-1)^{\eta} \left[\frac{2L+2\xi+1}{2L+1}\right]^{1/2} C(L+\xi, \ell, L; 0\eta\eta) \qquad \text{(B. 7)}$$

The presence of the Clebsch-Gordan coefficient has an obvious significance in terms of the addition $(\vec{L} - \vec{\ell}) + \vec{\ell} = \vec{L}$ and the relation $\vec{L} \cdot \vec{R} = \vec{\ell} \cdot \vec{R}$. The transformation which diagonalizes $\hat{\Lambda}^{L,\ell}$ is then $\hat{U}_{\eta\xi} = \rho_\eta^{-1} U_{\eta\xi}$.

When L is complex, or real and less than ℓ, $\Lambda^{L,\ell}$ can no longer be diagonalized if a unitary transformation. Since $\Lambda^{L,\ell}$ is symmetric, this can, however, be effected with a complex orthogonal transformation

$$UU^T = 1 \qquad \text{(B. 8)}$$

or equivalently if $U(L)$ is real for real L

$$U^{+}(L)U(L) = 1 \tag{B.9}$$

where $U^{+}(L) = [U(L^{*})]^{+}$. Let us examine the following continuation of $U_{\eta\xi}$ to complex L, obtained by replacing the factorials in Wigner's expression (31) by Γ-functions.

$$U_{\eta\xi} = \left[\frac{(2L+2\xi+1)(2L+\xi-\ell)!(\ell-\xi)!(\ell+\xi)!(L+\eta)!(L-\eta)!}{(2L+\xi+\ell+1)!(\ell+\eta)!(\ell-\eta)!} \right]^{1/2}$$

$$\times \frac{1}{(L+\xi)!} \sum_{\nu} \frac{(-1)^{\nu+\ell}}{\nu!} \frac{(L+\ell-\nu)!(L+\xi+\nu)!}{(\ell-\xi-\nu)!(L+\eta-\nu)!(L+\xi-\ell-\eta+\nu)!} \tag{B.10}$$

Using this, form the product

$$\hat{U}_{\eta\xi} \hat{U}^{-1}_{\xi\eta'} = \rho^{-1}_{\eta} U_{\eta\xi} U^{-1}_{\xi\eta'} \rho_{\eta'} \tag{B.11}$$

For $\mathrm{Re}\, L > \ell$ this expression is analytic in L. Since $U_{\eta\xi}$ as defined by equation B.10 is bounded for large $|L|$, (2) it is easily checked that equation B.11 has at most a polynomial behavior for $|L| \to \infty$. An application of Carlson's theorem implies both equation B.9 for this choice of continuation and

$$\sum_{\xi} \hat{U}_{\eta\xi}(L+\xi)(L+\xi+1)\hat{U}^{-1}_{\xi\eta'} = \hat{\Lambda}^{L,\ell}_{\eta\eta'} \tag{B.12}$$

which shows $\hat{U}$ diagonalizes $\hat{\Lambda}^{L}$ for complex L also. In the sense that it diagonalizes $\hat{\Lambda}^{L}$ for complex L, equation B.10 is the unique continuation of the Clebsch-Gordan coefficients.

APPENDIX C

ANALYTIC PROPERTIES OF THE FREE SOLUTIONS

We may introduce solutions to the free Schrödinger equation, equation 4.14, by

$$h^{(1,2)}_{\ell p\eta,\ell' p'\eta'}(R) = \delta_{\ell\ell'}\delta_{pp'}\sum_{\xi}\hat{U}_{\eta\xi}\left(\frac{1}{2}\pi R\right)^{1/2} e^{i\pi\left[\frac{1}{2}(\xi-L-1)-\ell\right]}$$

$$\times H^{(1,2)}_{L+\xi+\frac{1}{2}}(PR)\,\hat{U}^{-1}_{\xi\eta'} \tag{C.1}$$

$$j_{\ell p\eta,\ell' p'\eta'}(R) = \delta_{\ell\ell'}\delta_{pp'}\sum_{\xi}\hat{U}_{\eta\xi}\left(\frac{1}{2}\pi P^2 R\right)^{1/2} e^{-i\pi\left[\frac{1}{2}(\xi-L-1)-\ell\right]}$$

$$\times J_{L+\xi+\frac{1}{2}}(PR)\,\hat{U}^{-1}_{\xi\eta'}$$

where $J_{L+\xi+\frac{1}{2}}$, $H^{(1,2)}_{L+\xi+\frac{1}{2}}$ are Bessel functions (32).

The matrix δ is defined with elements

$$\delta_{\ell p\eta,\ell' p'\eta'} = \delta_{\ell\ell'}\delta_{pp'}\sum \hat{U}_{\eta\xi}e^{-i\pi(\xi-\ell)}\,\hat{U}^{-1}_{\xi\eta'}$$

$$= \delta_{\ell\ell'}\delta_{pp'}\delta_{-\eta\eta'} \tag{C. 2}$$

The last line follows from the symmetry properties of the Clebsch-Gordan coefficients at integral values of L and an application of Carlson's theorem (24). If ℓ denotes the diagonal matrix with elements $\ell\,\delta_{\ell\ell'}$, then

$$j(R) = \frac{1}{2}P^2\left[h^{(1)}(R) + h^{(2)}(R)\right]\delta e^{i\pi(L+\ell+1)} \tag{C. 3}$$

For $\text{Im}\, P \leq 0$, $h^{(2)}(R)$ satisfies the integral equation

$$h^{(2)}(R) = P^{-1/2} e^{-iPR} \delta + \int_R^{\infty} dR' \frac{\sin P(R-R')}{P} \frac{\hat{\Lambda}^L}{R^2} h^{(2)}(R) \qquad (C.4)$$

Equation C.4 can be iterated and converges uniformly* for all finite values of L and for $R > 0$. Since $\hat{\Lambda}^L$ is an entire function of L, it follows that $h^{(2)}(R)$ is also. $h^{(1)}(R)$ obeys the same integral equation but with an inhomogenous term $e^{iPR} P^{-1/2} I e^{-i\pi(L+\ell+1)}$ where I is the unit matrix. A similar statement can, therefore, be made about the analyticity of $h^{(1)}(R)$ but with the restriction $\text{Im}\, P \geq 0$. By equation C.3, $j(R)$ is then an entire function of R if $\text{Im}\, P = 0$. The coefficient of each term in a power series expansion of $j(R)$ must, therefore, also be an entire function of L when $\text{Im}\, P = 0$. However, this series converges absolutely (compare equation C.1 and the known expansion of the Bessel function) and $j(R)$ is thus an entire function of L for all values of P.

*Compare the treatment of ref. 3.