ELASTOSTATIC AND AEROELASTIC PROBLEMS
RELATING TO
THIN WINGS OF HIGH SPEED AIRPLANES

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SUMMARY

This report is concerned with the statics and dynamics of very thin wings of high speed airplanes. With the modern tendency towards sweepback, which is necessary for supersonic airplanes, the wing constructions tend more and more to an ideal structure, hence for the static problems of this report, the wing is idealized to a thin cantilever elastic plate.

Part I gives a general formulation of the fundamental equations of deformation of thin elastic plates and the direct methods of solution. For small deflection of plates, the equations and boundary conditions are derived from the three-dimensional equations of elasticity developed in power series of the thickness of the plate. It is shown that the classical Poisson-Kirchhoff theory is coincident with the first approximation in this development. These equations are then transformed into oblique coordinates for treating problems concerning swept plates. Since the problem of the cantilever plate is very difficult to solve from the standpoint of biharmonic analysis, emphasis is laid on the direct methods of solution which lead to useful approximate solutions with desired accuracy. Section 1.21 discusses the relation between plate problems and equivalent variational problems. Section 1.22 contains a systematic review of the Rayleigh-Ritz method and all methods allied to it. Section 1.23 formulates the method of relaxation of boundary conditions, including the Trefftz method as one instance.

Part II discusses the general aeroelastic problems of high speed airplanes. For airplanes accelerating or decelerating through
the transonic region, the coefficients in the aeroelasticity equations are of transient nature. Such transient perturbations are new phenomena in aeronautics but are sufficiently important to warrant detailed investigation. A general mathematical treatment is given, though due to lack of aerodynamic data at present, no specific example is included. A general solution is obtained and this solution is expanded into a generalized power series which proves to be particularly useful when the transient perturbation is small. The present result includes the ordinary small perturbation theory for finite degrees of freedom as a particular case. Several results regarding small perturbations are given in section 2.6.

The next two parts give a detailed computation on the deflection of and stresses in cantilever plates. The deflection of rectangular cantilever plates is solved both by the Rayleigh–Ritz method and the method of relaxation of boundary conditions. For swept plates the Rayleigh–Ritz method is used. A theory of stress approximation without using the intermediate deflection function is developed in Part IV, and is applied to rectangular plates.
NOTATIONS

$x, y, z$ Rectangular coordinates

$D = \frac{E t^3}{12(1-\nu^2)}$ Flexural rigidity of plates, $t =$ thickness

$M_x, M_y, M_z$ Resultant bending and twisting moments per unit length in plates. The bending moments are considered positive when they put the upper side of an element of the plate in compression. The sense of the twisting moment agrees with the couple due to the shearing stresses $\tau_{xy}$.

$Q_x, Q_y$ Resultant shearing forces per unit length in a plate, normal to middle surface. Their sign is the same as the corresponding sign of the shearing stresses.

$u, v, w$ Elastic displacements in $x, y, z$ directions respectively.

$\theta$ Sweep-back angle

$\lambda, \mu$ Elastic constants.

$\nu, E$ Poisson's ratio and Young's modulus respectively.

$\kappa = \frac{\lambda}{\mu} = \frac{2\nu}{1-2\nu}$ \hspace{1cm} $A = \frac{2(\kappa + 1)}{\kappa + 2} = \frac{1}{1-\nu}$

$q$ Intensity of distributed load per unit area of the middle surface of the plate.

$n, t$ Used in subscripts denoting outer normal and tangential directions respectively.
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I. GENERAL FORMULATIONS OF THE FUNDAMENTAL EQUATIONS OF DEFORMATION OF THIN ELASTIC PLATES AND DIRECT METHODS OF SOLUTION

1.1 Fundamental Equations of Deformation

§ 1.1.1 Introduction

An exact theory of plates should be deduced from the general equations of elasticity for a three-dimensional continuum. The classical Poisson-Love theory (Refs. 1, 2, 3) which is based upon intuitive assumptions is unsatisfactory in this respect. If the deflection is fairly large, the finite displacement must be also taken into account; since although the six components of the strain tensor are small, some of the derivatives of the components of displacement might be large. Then it is necessary to differentiate between stresses referred to the surface elements of the undeformed and the deformed system.

The following formulation is concerned with small deflections. Within this limitation the fundamental equations are deduced by starting from the conditions in a thick plate, regarded as a three-dimensional continuum, and then passing to the limit of thin plate by expanding the stresses into power series in the thickness of the plate. The validity of the three-dimensional field equations and of the boundary conditions at the faces of the plate are retained during the limiting process. This concept was suggested by P. Epstein (Ref. 6) in 1942. In its application to elastic plates, higher order terms can readily be deduced, from which comparison to the classical theory can be made.

In explicit terms the basic assumptions in the following development are:

(1) That the thickness of the plate is small compared to other dimensions of the plate, and
(2) That the deformation of the plate is small compared to the thickness of the plate.

These assumptions are of course tacitly implied in the classical theory of thin plates, which in addition postulates that any line normal to the middle surface of the plate remains straight and normal to it in any strained condition, and that the stress component normal to the middle surface is small in comparison to other stress components and may be neglected in the stress-strain relations.
§ 1.12 Fundamental Equations in Orthogonal Cartesian Coordinates

Let \((x,y,z)\) be a system of Cartesian coordinates, and let the plane \(z = 0\) coincide with the middle surface of the plate in the unstrained state. The plate is assumed to be thin, that is, the thickness \(2e \ll h(x,y)\) of the plate is small compared to other dimensions of the plate. The thickness may be variable. It is designated by \(z \in h(x,y)\) where \(2e\) is a small quantity of the dimension of the thickness and \(h(x,y)\) a function of \(x\) and \(y\) describing the thickness distribution of the plate.

Using the usual notation \(\sigma_x, \sigma_y, \tau_{xy}\) etc., for the components of the stress tensor, the general field equations are

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial y} = 0
\]

etc., by cyclic change of subscripts.

These stress components are connected to the components \((u,v,w)\) of the displacement vector by relations like

\[
\sigma_x = 2 \mu \frac{\partial v}{\partial x} + \lambda \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right),
\]

\[
\tau_{xy} = \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} \right).
\]

Expressing the field equations (1) in terms of the components of displacements, they become*

---

*In all the following formulas, there is a symmetry between \(u, v, w\) and \(x, y, z\). The third equation is usually omitted for simplicity in writing. It is always obtained by interchanging \(u\) into \(v\), and \(x\) into \(y\), and vice versa.
\[
\frac{\partial^2 w}{\partial x^2} + \nabla^4 w + (1 + \nu) \frac{\partial^2}{\partial x^2} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = 0
\]
\[
\frac{\partial^2 u}{\partial x^2} + \nabla^2 u + (1 + \nu) \frac{\partial^2}{\partial x^2} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = 0
\]
\[
\frac{\partial^2 v}{\partial y^2} + \nabla^2 v + (1 + \nu) \frac{\partial^2}{\partial y^2} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = 0
\]

where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \).

The boundary conditions for a plate under distributed traction \( p(x,y) \) over its faces applied in the direction normal to the middle plane of the plate may be written as

\[
\sigma^*_{\pm} (\pm \varepsilon h) = \pm p,
\]
\[
\tau_{2x}(\pm \varepsilon h) = \tau_{2y}(\pm \varepsilon h) = 0
\]

where "(\pm \varepsilon h)" means that the quantities attached are evaluated at \( z = \pm \varepsilon h \) respectively. By this method, a total traction of \( 2p \) per unit area of the middle surface of the plate is distributed half on top face and half on bottom face.* The conditions (4) are natural for plates of uniform thickness. But for plates of variable thickness the specifications of conditions (4) are rather artificial, as it would imply a system of loading consisting not only of normal pressure to the faces but also of shearing stress along the faces of the plate. However, the equations of deformation are developed according to the above specification first, and we will return to the more natural

*Since obviously the middle surface of a plate under the boundary conditions \( \sigma^*_{\pm}(\pm \varepsilon h) = \pm p \) will not deflect, i.e., \( w(0) = 0 \), it is evident by the principle of superposition that the differential equations involving \( w(0) \) derived from the boundary condition (4) is the same as that derived from the conditions

\[
\sigma^*_{\pm}(\pm \varepsilon h) = 2p, \quad \sigma^*_{\pm}(-\varepsilon h) = 0.
\]
boundary conditions for plates of variable thickness at the end of this section.

Assuming that the stress components are analytic throughout the thickness, one obtains, by adding and subtracting the expanded form of (4) in power series, the equations

\begin{align}
\sigma_3(o) + \frac{1}{2!} \varepsilon^4 \varepsilon_3^4 \sigma_3(o) + \frac{1}{4!} \varepsilon^4 \varepsilon_3^4 \sigma_3^{(v)}(o) + \cdots &= 0, \\
\sigma_2(o) + \frac{1}{3!} \varepsilon^4 \varepsilon_2^4 \sigma_2(o) + \frac{1}{3!} \varepsilon^4 \varepsilon_2^4 \sigma_2^{(v)}(o) + \cdots &= \frac{\nu}{\varepsilon h}, \\
\tau_{x1}(o) + \frac{1}{2!} \varepsilon^4 \varepsilon_1^1 \tau_{x1}(o) + \frac{1}{4!} \varepsilon^4 \varepsilon_1^1 \tau_{x1}^{(v)}(o) + \cdots &= 0, \\
\tau_{y1}(o) + \frac{1}{3!} \varepsilon^4 \varepsilon_1^1 \tau_{y1}(o) + \frac{1}{5!} \varepsilon^4 \varepsilon_1^1 \tau_{y1}^{(v)}(o) + \cdots &= 0.
\end{align}

(5)

and two more equations obtained from the last two by changing the subscript \(x\) into \(y\). The accents denote differentiation with respect to \(x\).

Now the stress component

\[ \tau_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]

is defined exclusively by partials with respect to \(x\) and \(y\). But \( \sigma_x \) and \( \sigma_y \) contain partial derivative with respect to \( z \). Let

\[ A = \frac{2(\kappa + 1)}{\kappa + 2} - \frac{1}{1 - \nu}, \quad \kappa = \frac{\mu}{\varepsilon}, \]

(6)

and define the two-dimensional stress components

\[ q_{xx} = 2\mu \left[ A \frac{\partial u}{\partial x} + (A-1) \frac{\partial v}{\partial y} \right], \]
\[ q_{yy} = 2\mu \left[ (A-1) \frac{\partial u}{\partial x} + A \frac{\partial v}{\partial y} \right], \]

(7)

so that

\[ \sigma_x = (A-1) \sigma_x + q_{xx}, \]
\[ \sigma_y = (A-1) \sigma_x + q_{yy}. \]

(8)

Substituting (5) and (8) into the field equations (1) which are certainly valid for the particular value of \( z = o \), one obtains, up to
fourth order in \( \varepsilon \),

\[
\frac{b}{\varepsilon h} = \frac{\varepsilon^4}{4!} \left\{ \frac{h^4}{3} \varepsilon_x^{iv} + \frac{3}{2} x (h^4 \varepsilon_{2x}^v) + \frac{3}{2} y (h^4 \varepsilon_{2y}^v) \right\} + \varepsilon^3 \left\{ \frac{h^4}{3} \varepsilon_x^{(v)} + \frac{3}{2} x (h^4 \varepsilon_{2x}^{(iv)}) + \frac{3}{2} y (h^4 \varepsilon_{2y}^{(iv)}) \right\},
\]

\[
\sigma_{xx} + \frac{3}{2} y \varepsilon_{xy} - \frac{\varepsilon^4}{4!} \left\{ (A-1) \frac{2}{2x} h^4 \varepsilon_{xx}^{iv} + \frac{1}{3} h^4 \varepsilon_{xx}^{(iv)} \right\} = 0.
\]

and a third equation by interchanging \( x \) and \( y \) in (9), where all the partial derivatives are evaluated at \( x = 0 \).

If all the partial derivatives of the stress components in (9) are expressed in terms of partials with respect to \( x \) and \( y \) of the displacement components, we will obtain the fundamental differential equations defining the deformation of the elastic plate.

In the following, the fundamental differential equation is developed up to fourth order in \( \varepsilon \). This requires that \( \varepsilon_x^{iv}, \varepsilon_{2x}^{iv}, \varepsilon_{2y}^{iv}, \varepsilon_{xx}^{iv}, \varepsilon_{2x}^{(iv)}, \varepsilon_{2y}^{(iv)} \) be accurate up to second order in \( \varepsilon \), and \( \varepsilon_x^{(iv)}, \varepsilon_{2x}^{(iv)}, \varepsilon_{2y}^{(iv)} \), etc. up to zero order. Now the definitions (2) and conditions (5) give, when \( x = 0 \),

\[
\frac{\partial W}{\partial x} = -(A-1)(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y}) - \frac{2-A}{4\mu} \varepsilon^4 h^4 \varepsilon_{xx}^{iv},
\]

\[
\frac{\partial u}{\partial x} = -\frac{\partial W}{\partial x} - \frac{\varepsilon^4 h^4}{\mu} \varepsilon_{xx}^{iv}, \quad \text{etc.}
\]

From the field equations (3) and using (10), one obtains

\[
\frac{\partial^2 W}{\partial x^2} = (A-1) \nabla^2 W + \frac{A\varepsilon^2}{\mu} \left\{ \frac{2}{2x} (h^4 \varepsilon_{2x}^v) + \frac{2}{2y} (h^4 \varepsilon_{2y}^v) \right\},
\]

\[
\frac{\partial^2 u}{\partial x^2} = -\nabla^2 u - A \frac{\partial^2 u}{\partial x^2} + \frac{A}{\mu} \varepsilon^2 \frac{\partial \varepsilon_{xx}^v}{\partial x} + \frac{A}{\mu} \varepsilon^2 \frac{\partial \varepsilon_{xx}^{(iv)}}{\partial x},
\]

(11)
But from (9) and (10), the field equations can be written as

\[ \sigma_x' + \frac{2}{\partial x} \tau_{xx} + \frac{2}{\partial y} \tau_{xy} = 0 \]

\[ \tau_{xx}' + \frac{2}{\partial x} q_{xx} + (A - 1) \frac{2}{\partial x} \tau_{xx} + \frac{2}{\partial y} \tau_{xy} = 0 \] etc. \tag{12}

The partials \( \sigma_x', \tau_{xx}' \) etc. can be found by successive differentiation of this equation with respect to \( x \). Note that \( \sigma_x', \tau_{xx}' \) etc. need only be correct to zero order in (10) and (11). Now from (2), (7), and (10),

\[ \tau_{x'y} = -2 \mu \frac{2w}{\partial x \partial y} \] \[ q_{xx} = -2 \mu \{ A \nu^w - \frac{\partial w}{\partial \nu^w} \} \] \tag{13}

Hence by differentiating (12) with respect to \( x \) and using (13), (2), and (7), one obtains

\[ \frac{1}{2\mu} \sigma_x'' = A \nu^w \left( \frac{2u}{\partial x} + \frac{2v}{\partial y} \right) \] \[ \frac{1}{2\mu} \tau_{xx}'' = A \nu^w \frac{2w}{\partial x} \] \tag{14}

from which (10) and (11) can be written in the following form:

\[ \frac{2w}{\partial x^2} = -(A - 1) \left( \frac{2u}{\partial x} + \frac{2v}{\partial y} \right) - \frac{A(2 - A)}{2} \nu^w \left( \frac{2u}{\partial x} + \frac{2v}{\partial y} \right) \] \tag{15}

\[ \frac{2u}{\partial x^2} = -\frac{2w}{\partial x} - A \nu^w \nu^x \frac{2w}{\partial x} \]

\[ \frac{2w}{\partial x^2} = (A - 1) \nu^x \nu^w + \frac{A \nu^x}{2} \left\{ \frac{2}{\partial x} (\nu^x \nu^w \frac{2w}{\partial x}) + \frac{2}{\partial y} (\nu^x \nu^w \frac{2w}{\partial y}) \right\} \] \tag{16}

\[ \frac{2u}{\partial x^2} = -\nu^x \nu^w + A \nu^x \left( \frac{2u}{\partial x} + \frac{2v}{\partial y} \right) + \frac{A \nu^x}{2} \left\{ \nu^w \left( \frac{2u}{\partial x} + \frac{2v}{\partial y} \right) \right\} \]

Substituting (15) into the results of differentiation of the definitions (2) and (7), one obtains equations similar to (13) but now correct to second order in \( \nu^w \):

\[ \frac{1}{2\mu} \tau_{x'y}'' = -\frac{2w}{\partial x \partial y} \frac{A \nu^x}{2} \left\{ \frac{2}{\partial x} (\nu^x \nu^w \frac{2w}{\partial x}) + \frac{2}{\partial y} (\nu^x \nu^w \frac{2w}{\partial y}) \right\} \] \tag{17}

\[ \frac{1}{2\mu} q_{xx}'' = -A \nu^w + \frac{2w}{\partial x^2} - A \nu^x \left\{ \frac{2}{\partial x} (\nu^x \nu^w \frac{2w}{\partial x}) + \frac{2}{\partial y} (\nu^x \nu^w \frac{2w}{\partial y}) \right\} \]
Similarly, a second differentiation of (2) and (7) with respect to \( z \) gives

\[
\frac{1}{2\mu} \tilde{C}_{x}^{y} = -\frac{1}{2} \nabla^{2} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}) - A \frac{\partial^{2} b}{\partial x^{2}} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + A \frac{\partial^{2} A}{\partial x^{2}} \left[ h^{x} \nabla^{1} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \right] ,
\]

\[
\frac{1}{2\mu} \tilde{I}_{x x}^{y} = -A (1 + A) \nabla^{2} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}) + \nabla^{2} \frac{\partial v}{\partial y} + A \frac{3}{2} \frac{\partial^{2} u}{\partial x^{2}} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + \frac{A}{2} \frac{\partial^{2} x}{\partial y^{2}} \left[ h^{x} \nabla^{1} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \right] .
\]

By successive partial differentiation of the field equations (3), (2), and (7) with respect to \( z \), and using (15) and (16), one obtains, correct to zero order in \( \varepsilon \),

\[
\frac{3}{2} \frac{\partial^{2} w}{\partial x^{2}} = (2 A - 1) \nabla^{2} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) , \quad \frac{3}{2} \frac{\partial^{2} u}{\partial x^{2}} = (1 + A) \nabla^{2} \frac{\partial w}{\partial x} ,
\]

\[
\frac{3}{2} \frac{\partial^{2} w}{\partial x^{4}} = (1 - 2 A) \nabla^{2} w , \quad \frac{3}{2} \frac{\partial^{2} u}{\partial x^{4}} = \nabla^{2} u + 2 A \nabla^{2} \frac{\partial w}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) ,
\]

and

\[
\frac{1}{2\mu} \tilde{C}_{x y} = (1 + A) \nabla^{2} \frac{\partial w}{\partial x} ,
\]

\[
\frac{1}{2\mu} \tilde{I}_{x x}^{y} = A (1 + A) \nabla^{2} w - (1 + A) \nabla^{2} \frac{\partial w}{\partial x} ,
\]

\[
\frac{1}{2\mu} \tilde{C}_{x x}^{y} = \frac{1}{2} \nabla^{2} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + 2 A \nabla^{2} \frac{\partial^{2} w}{\partial x^{2}} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})
\]

\[
\frac{1}{2\mu} \tilde{I}_{x x}^{y} = A (2 A + 1) \nabla^{2} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - \nabla^{2} \frac{\partial v}{\partial y} - 2 A \nabla^{2} \frac{\partial^{2} w}{\partial x^{2}} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})
\]

The derivatives \( \tilde{C}_{x}^{y}(\varepsilon_{x}), \tilde{C}_{x}^{y}(\varepsilon_{x}) \) etc. can now be computed.

From (12) and using (2), (7), and (15), one obtains, correct to second order in \( \varepsilon \),

\[
\frac{1}{2\mu} \tilde{C}_{x}^{y} = \frac{A}{2} \frac{\partial^{2} b}{\partial y^{2}} \left[ h^{x} \nabla^{1} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \right] ,
\]

\[
\frac{1}{2\mu} \tilde{C}_{x}^{y} = -A \frac{\partial^{2} u}{\partial x^{2}} + \frac{1}{2} \frac{\partial^{2} x}{\partial y^{2}} + \left( \frac{1}{2} - A \right) \frac{\partial^{2} v}{\partial x^{2} \partial y^{2}}
\]

\[
+ \frac{A(A-1)}{2} \frac{\partial^{2} x}{\partial x^{2}} \left[ h^{x} \nabla^{1} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \right] .
\]
Similarly, by differentiating (12) with respect to \( z \), and using (17) and (21),

\[
\frac{1}{2\mu} \sigma_z = A \mathbf{V}^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{A(A-1)}{2} \varepsilon \mathbf{V}^2 \left[ h^2 \mathbf{V} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right],
\]

(22)

\[
\frac{1}{2\mu} \tau_{xz} = A \mathbf{V}^2 \left( \frac{\partial w}{\partial y} + \frac{\partial u}{\partial x} \right) + \frac{A}{2} \left\{ (A \frac{\partial^2}{\partial y^2} + \mathbf{V}^2) (h^2 \mathbf{V} \frac{\partial w}{\partial y} + A \frac{\partial^2}{\partial y^2} (h^2 \mathbf{V} \frac{\partial w}{\partial y})) \right\}.
\]

Further differentiation of (12) leads to

\[
\frac{1}{2\mu} \sigma_{xx} = -A \mathbf{V}^4 w - \frac{A}{2} \left\{ \frac{\partial^2}{\partial y^2} \left[ (A \frac{\partial^2}{\partial x^2} + \mathbf{V}^2) (h^2 \mathbf{V} \frac{\partial w}{\partial x} + A \frac{\partial^2}{\partial y^2} (h^2 \mathbf{V} \frac{\partial w}{\partial y})) \right] + \frac{\partial}{\partial y} \left[ (A \frac{\partial^2}{\partial y^2} + \mathbf{V}^2) (h^2 \mathbf{V} \frac{\partial w}{\partial y}) + A \frac{\partial^2}{\partial y^2} (h^2 \mathbf{V} \frac{\partial w}{\partial x}) \right] \right\},
\]

(23)

and the following expressions in zero order:

\[
\frac{1}{2\mu} \sigma_{xx}^{(v)} = -2A \mathbf{V}^4 \frac{\partial w}{\partial x} ,
\]

(24)

\[
\frac{1}{2\mu} \sigma_{yy}^{(v)} = 2A \mathbf{V}^4 w .
\]

All the quantities in equation (9) have now been obtained in required form. A single step of substituting them into equation (9) gives the final result:

\[
\frac{\nu}{c_h} = \mu A \varepsilon \left\{ -\frac{h^2}{3} \mathbf{V}^4 w + \frac{\partial}{\partial y} \left[ (A \frac{\partial^2}{\partial x^2} + \mathbf{V}^2) (h^2 \mathbf{V} \frac{\partial w}{\partial x} + A \frac{\partial^2}{\partial y^2} (h^2 \mathbf{V} \frac{\partial w}{\partial y})) \right] + \frac{2A}{4} \left\{ \frac{\partial}{\partial y} \left[ (h^2 \mathbf{V} \frac{\partial w}{\partial x}) + A \frac{\partial^2}{\partial y^2} (h^2 \mathbf{V} \frac{\partial w}{\partial y}) \right] 
\right\} + \left( \frac{6}{5} \frac{\partial^2}{\partial y^2} + 4h^2 \frac{\partial^2}{\partial x^2} \right) \left[ (A \frac{\partial^2}{\partial x^2} + \mathbf{V}^2) (h^2 \mathbf{V} \frac{\partial w}{\partial x} + A \frac{\partial^2}{\partial y^2} (h^2 \mathbf{V} \frac{\partial w}{\partial y})) \right] 
\right\} + \left( \frac{2h^2}{5} \mathbf{V}^4 w - 2A \frac{\partial^2}{\partial x^2} (h^2 \mathbf{V} \frac{\partial w}{\partial x}) - 2A \frac{\partial}{\partial y} (h^2 \mathbf{V} \frac{\partial w}{\partial y}) \right) \}
\]

(25)

For a flat plate with constant thickness \( t \), \( h = 1 \), \( 2\varepsilon = t \), and
equation (25) becomes

\[
\frac{q}{D} = \nabla^4 w + \frac{E t^3}{80} \left[ 5 + \nu (1-\nu) \right] \nabla^6 w. \tag{26}
\]

where \( q = 2p \) = loading per unit area of the middle surface of the plate, and

\[
D = \frac{E t^3}{12 (1-\nu^2)} \tag{27}
\]

It is to be noted that the differential equation defining \( w \) and those defining \( u, v \) are independent of each other.

When the fourth order terms in (25) are neglected, (26) reduces to the familiar form

\[
\frac{q}{D} = \nabla^4 w \tag{28}
\]

**Boundary Conditions**

The boundary conditions at the edges of a plate should be specified in consistence with the order of approximation adopted in the differential equations of equilibrium. In a thick plate subjected to given forces, either the stresses or the displacements have prescribed values at every point of the edge. For example, if one edge of a thick plate coincides with the surface \( x = \) constant, the boundary conditions at that edge are

1. \( u = v = w = 0 \) for clamped edges,
2. \( \sigma_x = \tau_{xy} = \tau_{xx} = 0 \) for free edges, and
3. Some mixed conditions for supported edges.

In passing the field equation to the limit of a thin plate, these boundary conditions pass to their respective integrated forms. In the first approximation, when the differential equation is of fourth order, two conditions are required at each edge. These may be specified by
1. \( w = \frac{\partial^2 w}{\partial t^2} = 0 \) for clamped edges,
2. \( w \) and bending moment = 0 for simply supported edges, and
3. Bending moment = 0. Resultant reaction = 0 for free edges, where the bending moments and resultant reaction are to be calculated only up to zero order in the thickness of the plate.

In the second approximation, the differential equation is of sixth order, so three conditions are required at each edge. These may be specified, for example, by

1. \( w = 0 \), \( \frac{\partial^3 w}{\partial x^3} + A \frac{\partial w}{\partial x} = 0 \), \( \frac{\partial^2 u}{\partial x^2} = 0 \),
   for a clamped edge, \( x = \) constant.

2. \( w = 0 \), bending moment = 0, twisting moment = 0, for simply supported edges.

3. Bending moment = twisting moment = vertical shear = 0, for free edges;

while the quantities named in (3) are now to be calculated up to second order in \( \varepsilon \).

For higher order approximations, the differential equations are of still higher order and more stringent specifications of boundary conditions are necessary.

Explicit expressions of forces and moments can be easily obtained. Consider the edge \( x = \) constant. According to our fundamental assumption, the stresses are expandable into power series in \( z \) throughout the thickness of the plate. Now

\[
\begin{align*}
X &= \int_{-\varepsilon h}^{\varepsilon h} \sigma_x \, dz, \\
M_x &= \int_{-\varepsilon h}^{\varepsilon h} \tau_{xz} \, dz, \\
Q_x &= \int_{-\varepsilon h}^{\varepsilon h} \tau_{z} \, dz, \\
H_{xy} &= \int_{-\varepsilon h}^{\varepsilon h} \tau_{xy} \, dz.
\end{align*}
\]
Hence

\[ \tau = 2 \varepsilon h \sigma_x'^{(0)} + \frac{2(\varepsilon h)^2}{3!} \sigma_{xx}'' + \ldots \]

\[ M_x = 2 \frac{(\varepsilon h)^3}{3} \sigma_x'(0) + 2 \frac{(\varepsilon h)^5}{5 \times 3!} \tau_x''(0) + \ldots \]

\[ Q_x = 2 \varepsilon h \tau_{xx}'(0) + 2 \frac{(\varepsilon h)^3}{3!} \tau_{xx}''(0) + \ldots \]

\[ H_{xy} = 2 \frac{(\varepsilon h)^3}{3} \tau_y'(0) + 2 \frac{(\varepsilon h)^5}{5 \times 3!} \tau_y''(0) + \ldots \]

(30)

The quantities \( \sigma_x(0), \sigma_x'(0), \sigma_x''(0) \) etc., should be consistently evaluated according to the order of approximation used.

For second approximations (up to fourth order in \( \varepsilon \)) they are given by equations (2), (8), (17), (18), (20), (21), (22), and (23). They reduce to the familiar forms in the first approximation as follows:

\[ M_x = \frac{2(\varepsilon h)^3}{3} \sigma_x'(0) = -D \left[ \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right], \]

\[ Q_x = 2 \varepsilon h \tau_{xx}'(0) + 2 \frac{(\varepsilon h)^3}{3} \tau_{xx}''(0) = -\nu \frac{\partial w}{\partial x} \]

\[ H_{xy} = \frac{2}{3} (\varepsilon h)^3 \tau_y'(0) = D (1 - \nu) \frac{\partial^3 w}{\partial x \partial y^2}, \]

(31)

where

\[ D = \frac{Et^3}{12(1-\nu^2)} \quad t = 2 \varepsilon h = \text{the thickness of the plate} \]

The resultant reaction referred to above in the conditions for free edges leads to the well-known Kirchhoff's condition for free edges. (Refs. 9 or 10 or 11). In short, the statically equipollent resultant shearing force on the edge \( x = \text{const.} \) is a distributed load of

\[ Q_x + \frac{\partial H_{xy}}{\partial y} = -D \left\{ \frac{\partial^3 w}{\partial x^3} + (2-\nu) \frac{\partial^2 w}{\partial x \partial y^2} \right\}, \]

(32)

and a concentrated force of magnitude \( H_{xy} \) at the corners.

Similar formulas for forces and moments on other edges are easily obtained by coordinate transformations.
Calculation of Stresses

The stresses in the plate due to bending can be computed by series expansions similar to equation (30).

For the first approximation, the formulas for the maximum normal stresses are:

\[
\sigma_{x_{\text{max}}} = \frac{6 M_x}{t^2}, \quad \sigma_{y_{\text{max}}} = \frac{6 M_y}{t^2}.
\]  

(33)

The shearing stresses parallel to \(x\)- and \(y\)-axes are obtained from

\[
\tau_{x'y} = -Gt \frac{\partial^2 w}{\partial x \partial y}.
\]  

(34)

and the shearing stresses parallel to the \(z\)-axis are obtained from the fact that the shearing forces \(Q_x\) and \(Q_y\) are distributed along the thickness of the plate following the parabolic law, (cf. equations (18) and (30), and remembering that we are concerned only with the first approximation). As in the case of beams of rectangular cross section,

\[
\tau_{x'x_{\text{max}}} = \frac{3}{2h} Q_{x_{\text{max}}}, \quad \tau_{y'x_{\text{max}}} = \frac{3}{2h} Q_{y_{\text{max}}}.
\]  

(35)

Plates of Variable Thickness

Returning now to the plates of variable thickness, when the external forces are applied to the plate in the nature of air pressure, i.e. when the tractions \(\pm p(x,y)\) are applied on the faces of the plate in the direction normal to the faces, the boundary conditions become

\[
\sigma_n (\pm \epsilon h) = \pm p(x,y),
\]

\[
\tau_{n'c} (\pm \epsilon h) = 0, \quad \tau_{n't_2} (\pm \epsilon h) = 0.
\]  

(36)
where \( n \) is the outer normal to the face of the plate and \( t_1 \), the tangent to the face of the plate in the \( n-x \) plane, and \( t_2 \), that in the \( n-y \) plane. Since \( \epsilon \) is assumed to be small, the following table of direction cosines is valid: (see Fig. 1.12.1)

<table>
<thead>
<tr>
<th></th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>( -\epsilon h_x )</td>
<td>( -\epsilon h_y )</td>
<td>( \pm \left[ 1 - \frac{\epsilon^2}{2} (h_x^2 + h_y^2) \right] )</td>
</tr>
<tr>
<td>( t_1 )</td>
<td>( 1 - \frac{\epsilon^2}{2} h_x )</td>
<td>0</td>
<td>( \epsilon h_x )</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>0</td>
<td>( 1 - \frac{\epsilon^2}{2} h_y )</td>
<td>( \pm \epsilon h_y )</td>
</tr>
</tbody>
</table>

where the subscripts \( x \) and \( y \) denote partial differentiations. The "\( \pm \)" sign in the last column applies to faces on the \( \pm \) side of the \( z \)-axis respectively.

By the general tensor transformation of stress-components,

\[
\tau_{2x}(\pm \epsilon h) = \left[ 1 - \frac{\epsilon^2}{2} (h_x^2 + h_y^2) \right] \epsilon h_x \tau_{n}(\pm \epsilon h) + \epsilon^2 \left[ h_x^2 \epsilon_{n\epsilon x}(\pm \epsilon h) + h_y^2 \epsilon_{n\epsilon y}(\pm \epsilon h) + 2 h_x h_y \tau_{t_1 t_2}(\pm \epsilon h) \right],
\]

\[
\tau_{2y}(\pm \epsilon h) = \left[ 1 - \frac{\epsilon^2}{2} (h_x^2 + h_y^2) \right] \epsilon h_y \tau_{n}(\pm \epsilon h) + \epsilon^2 \left[ (1 - \frac{\epsilon^2}{2} h_x^2) \epsilon_{n\epsilon y}(\pm \epsilon h) \right. + \left. (1 - \frac{\epsilon^2}{2} h_y^2) \epsilon_{n\epsilon x}(\pm \epsilon h) \right] \tau_{t_1 t_2}(\pm \epsilon h). \tag{37}
\]

Now, up to first order in \( \epsilon \),

\[
\tau_{t_1 t_1} = \tau_{x}, \quad \tau_{t_1 t_2} = \tau_{y}, \quad \tau_{t_2 t_2} = \tau_{xy}. \tag{38}
\]

So, correct to third order in \( \epsilon \), the stress components \( \tau_{t_1 t_1} \), etc. on the right-hand side of (37) can be replaced by \( \tau_{x} \) etc.

Expand \( \sigma_{x}(\pm \epsilon h) \), etc., into a power series in \( \pm \epsilon h \), and adding and subtracting the expanded form of eqs. (37), to obtain boundary conditions on the faces of the plate similar to eqs. (5), one obtains
\[ \sigma_x^x(\phi) = \frac{\epsilon}{2} \left[ h_x^x \sigma_x^x(\phi) + \frac{\epsilon}{6} \left( h_x^h + h_y^y \right) (A-2) \right] \phi + \epsilon^2 \left[ -\frac{1}{3} \tau_x^x(\phi) + h_x^h \tau_x^x(\phi) \right] + \cdots \]

\[ \tau_{x2}(\phi) = -\epsilon h_x \phi + \epsilon^2 h \left[ -\frac{1}{2} \tau_{x2}(\phi) + h_x^x \sigma_x^x(\phi) + (h_x^h + h_y^y) \tau_{x2}(\phi) \right] + \cdots \]

\[ \tau_{x2}(\phi) = \frac{1}{h} \left[ h_x^x \sigma_x^x(\phi) + (h_x^h + h_y^y) \tau_{x2}(\phi) \right] + \frac{\epsilon^2}{2} \left[ -\frac{1}{3} \tau_{x2}^x(\phi) \right] + \cdots \]

\[ + h^2 h_x^x \sigma_x^x(\phi) + h^2 (h_x^h + h_y^y) \tau_{x2}^x(\phi) + h_x^h \sigma_x^x(\phi) + (h_x^h + h_y^y) \tau_{x2}(\phi) \]

(39)

and two similar expressions for \( \tau_{y2}(\phi) \) and \( \tau_{x2}(\phi) \). All the stress components in these equations are evaluated at \( \phi = 0 \), and primes denote differentiations with respect to \( \phi \). The definitions of \( \sigma_{xx}, \sigma_{yy} \)
are given by eqs. (7), (6).

Substituting (39) into (1), and recalling (6), one obtains the equation of equilibrium:

\[ \frac{\partial}{\partial t} \sigma_x^x = \frac{\epsilon}{6} \sigma_x^x + h_x^x \sigma_x^x + h_y^y \sigma_x^x + 2 h_x^h h_y^y \sigma_x^x + \frac{\partial}{\partial \phi} \left[ -\frac{1}{2} \tau_x^x + h_x^h \sigma_x^x \right] + h (h_x^h + h_y^y) \tau_{x2}^x + \frac{\partial}{\partial \phi} \left[ -\frac{1}{2} \tau_{x2}^x + h_x^h \sigma_x^x + h (h_x^h + h_y^y) \tau_{x2} \right] = 0. \]

(40)

where all the stress components are taken at \( \phi = 0 \). The terms \( \sigma_x^x \) etc. involved in this equation must be expressed in terms of the derivatives of \( u, v, w \) with respect to \( x \) and \( y \). Up to the first order in \( \epsilon \), equations (13) through (24) are true in this case. Hence one obtains the differential equation:
\[ 1 = D \nabla^4 w + \nabla^2 D \nabla^2 w + 2 \frac{\partial D}{\partial x} \nabla^2 \frac{\partial w}{\partial x} + 2 \frac{\partial D}{\partial y} \nabla^2 \frac{\partial w}{\partial y} \]

\[ - (1 - \mu) \left\{ \frac{\partial^4 D}{\partial x^4} \frac{\partial w}{\partial x} + \frac{\partial^4 D}{\partial y^4} \frac{\partial w}{\partial y} - 2 \frac{\partial^4 D}{\partial x^2 \partial y^2} \frac{\partial^2 w}{\partial x \partial y} \right\} 
\]

\[ + (1 - \mu) \left\{ \nabla^2 \nabla - \frac{1 - \nu}{3D} \left( \left( \frac{\partial D}{\partial x} \right)^2 + \left( \frac{\partial D}{\partial y} \right)^2 \right) \right\} \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x \partial y} \right\} \].

The boundary conditions at the edges of the plate are the same as before.
1.13 Differential Equations and Boundary Conditions in Oblique Coordinates

Oblique coordinates may be used to advantage in treating problems of swept plates. In oblique coordinates the coordinates of a point \( P \) are given by distances measured parallel to the \( \xi \) and \( \eta \) axes, respectively, from the \( \eta \) and \( \xi \) axes. (Fig. 1.13:1).

Let \( x-y \) be a rectangular coordinate system and let the \( \eta \)-axis be coincident with the \( y \) axis, and the \( \xi \)-axis be at an angle \( \theta \) from the \( x \)-axis. \( \theta \) is said to be positive when the \( \xi \)-axis lies in the first quadrant.

It is clear from the figure that the coordinate transformation is given by the following equations

\[
\begin{align*}
x &= \xi \cos \theta, \\
y &= \eta + \xi \sin \theta.
\end{align*}
\]  

(1)

From which

\[
ds^2 = dx^2 + dy^2 = d\xi^2 + d\eta^2 + 2 \sin \theta \ d\xi \ d\eta.
\]  

(2)

Because of the computational complication induced by an oblique angle, a general survey of the swept plate problem is carried out only in the first approximation.

For flat plates of constant thickness \( t \), equation (1.12:28) becomes, in oblique coordinates,

\[
\frac{\partial^2 w}{\partial \xi^2} + 4 \sin \theta \frac{\partial^2 w}{\partial \xi \partial \eta} + \left(2 + 4 \sin \theta \right) \frac{\partial^2 w}{\partial \eta^2} - 4 \sin \theta \left(\frac{\partial w}{\partial \xi} \right) \frac{\partial}{\partial \eta} + \frac{\partial w}{\partial \eta} = k,
\]  

(3)

where

\[
k = \frac{q \cos \theta}{D},
\]

(4)

\[
D = \frac{Et^3}{12 \left(1 - \nu^2\right)},
\]

(5)
This can also be written as 

\[
\left( \frac{\partial^2 w}{\partial \zeta^2} - 2 \sin \theta \frac{\partial^2 w}{\partial \zeta \partial \eta} \right) \left( \frac{\partial^2 w}{\partial \eta^2} - 2 \sin \theta \frac{\partial^2 w}{\partial \zeta \partial \eta} + \frac{\partial^2 w}{\partial \eta^2} \right) = -k \quad (3')
\]

Expressions for forces and moments in terms of the partials of \( w \) with respect to the \( \zeta, \eta \) coordinates can be obtained from equations (1.12:31) and (1.12:32). General expressions are very cumbersome in oblique coordinates. Forces and moments on sections \( \zeta = \text{const.} \) 

\( \eta = \text{const.} \) are expressed as follows:

\[ M_{\zeta} = \text{bending moment per unit length acting on sections} \]

\( \zeta = \text{constant.} \)

\[ = - \frac{D}{\cos \theta} \left\{ \frac{\partial^2 w}{\partial \zeta^2} - 2 \sin \theta \frac{\partial^2 w}{\partial \zeta \partial \eta} + (\sin^2 \theta + \nu \cos^2 \theta) \frac{\partial^2 w}{\partial \eta^2} \right\} \quad (5) \]

\[ M_{\eta} = \text{bending moment per unit length acting on sections} \]

\( \eta = \text{const.} \)

\[ = - D \left( \frac{\partial^2 w}{\partial \zeta \partial \eta} + \nu \frac{\partial^2 w}{\partial \eta^2} \right) \text{ where } n \text{ is the normal to the} \]

section \( \eta = \text{const.} \)

\[ = - \frac{D}{\cos \theta} \left\{ (\sin^2 \theta + \nu \cos^2 \theta) \frac{\partial^2 w}{\partial \zeta^2} - 2 \sin \theta \frac{\partial^2 w}{\partial \zeta \partial \eta} + \frac{\partial^2 w}{\partial \eta^2} \right\} \quad (6) \]

\[ H_{\zeta} = \text{twisting moment per unit length acting on a section} \]

\( \zeta = \text{const.} \) in the direction of the normal to the section

\[ = - \frac{D(1-\nu)}{\cos \theta} \left( \frac{\partial^2 w}{\partial \zeta \partial \eta} - \sin \theta \frac{\partial^2 w}{\partial \eta^2} \right) \quad (7) \]

\[ H_{\eta} = \text{twisting moment per unit length acting on a section} \]

\( \eta = \text{const.} \) in the direction of the normal to the section

\[ = - \frac{D(1-\nu)}{\cos \theta} \left( - \sin \theta \frac{\partial^2 w}{\partial \zeta^2} + \frac{\partial^2 w}{\partial \zeta \partial \eta} \right) \quad (8) \]

\[ Q_{\zeta} = \text{Vertical (in } \zeta \text{ direction) shearing force per unit length} \]

acting in a section \( \zeta = \text{const.} \)

\[ = - \frac{D}{\cos^2 \theta} \left\{ \frac{\partial w}{\partial \zeta} - 3 \sin \theta \frac{\partial^2 w}{\partial \zeta^2} + (1 + 2 \sin^2 \theta) \frac{\partial w}{\partial \eta} + \frac{\partial^2 w}{\partial \zeta \partial \eta} - 3 \sin \theta \frac{\partial^2 w}{\partial \zeta \partial \eta} \right\} \quad (9) \]
\[ Q_\eta = \text{Vertical (in } z \text{-direction) shearing force per unit length} \]

acting in a section \( \eta = \text{const.} \)

\[ = - \frac{\nu}{\cos^2 \theta} \left\{ - \sin \theta \frac{\partial w}{\partial \xi} + (1 + 2 \sin^2 \theta) \frac{\partial^2 w}{\partial \xi^2} + 3 \sin \theta \frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial^3 w}{\partial \eta^3} \right\}. \]  

(10)

The determination of the deflection surface of a plate now consists of integrating equation (3) with appropriate boundary conditions.

These conditions are summarized as follows:

1. **Built-in edge**: \( w \) and \( \frac{\partial w}{\partial n} \) must be zero. If the edge \( \xi = 0 \)
   is built-in, then \( (w)_\xi^\xi = 0, \quad (w)_\eta^\eta = 0 \)  
   \[ \frac{\partial w}{\partial n} = 0 \]  
   (11)

2. **Simply-supported edge**: \( w \) and the bending moment acting on the edge must be zero.

3. **Free edge**: The bending moment \( M_n \) and the reaction \( Q_n + \frac{\partial H_{nt}}{\partial t} \)
   along the edge must vanish. The twisting moment \( H_{nt} \) must also vanish at the sharp corner if that corner is unsupported.

Proper expressions for \( M_n \) are to be chosen from (5) or (6)

on edges \( \xi = \text{const.} \) or \( \eta = \text{const.} \) respectively. For the second condition on the reaction, the results are:

On the edge corresponding to \( \xi = a \),

\[ \left\{ \frac{\partial^2 w}{\partial \xi^2} - 3 \sin \theta \frac{\partial^2 w}{\partial \xi \partial \eta} + \left[ 2 + \sin \theta - \nu \cos^2 \theta \right] \frac{\partial^3 w}{\partial \xi^3} \right\}_{\xi = a} = 0 \]  

(12)

and on the edge corresponding to \( \eta = b \),

\[ \left\{ - \sin \theta \left[ 1 + (1 - \nu) \cos^2 \theta \right] \frac{\partial^2 w}{\partial \xi^2} + \left( 2 + \sin \theta - \nu \cos^2 \theta \right) \frac{\partial^3 w}{\partial \xi \partial \eta^2} \right. \]

\[ - 3 \sin \theta \frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial^3 w}{\partial \eta^3} \right\}_{\eta = b} = 0. \]  

(13)
1.2 DIRECT METHODS OF SOLUTION

§ 1.21 Equivalence Between a Differential System and a Variational Problem

A plate problem can usually be replaced by an equivalent variational problem. This equivalence is of great importance because direct methods of solving the variational problems sometimes offer powerful means to obtain useful approximate solutions. Conversely, a plate problem can be formulated according to the minimum energy theorems, which can be derived from the variation of strain energy of an elastic body either through variation in displacements or through variation in the stress tensor. From the former procedure there results the theorem of minimum potential energy (Ref. 8, § V, p. 66, or Ref. 19, § 64) that "of all displacements satisfying given boundary conditions, those that satisfy the equilibrium conditions minimize the potential energy

\[ V = U - \iiint_T F_i \, u_i \, dv - \iint_S \frac{\partial}{\partial s} \, u_i \, ds, \]  

where the surface integral is taken over that portion of the boundary on which the surface forces \( \frac{\partial}{\partial s} \) are prescribed." In this expression \( U \) is the strain energy stored in the plate

\[ U = \frac{1}{2} \iiint e_{ij} \, p_{ij} \, dv \]

and \( F_i \) is the body force vector and \( \frac{\partial}{\partial s} \) the vector of surface force acting on an element of surface with an outer normal \( \mathbf{n}_i \). \( u_i \) is, as before, the displacement vector, and \( e_{ij} \), \( p_{ij} \) the strain and stress tensor respectively. It should be emphasized that the displacements must satisfy the boundary conditions on that portion of the surface.
where the displacements are prescribed, and need to satisfy the same only. For example, the "natural" boundary conditions on free edges are satisfied as a consequence of this theorem, and need not be fulfilled in the choice of $w$.

In Kirchhoff's theory (See Ref. 2), he starts by assuming an expression for total strain energy of the bent plate which contains only terms depending on the action of bending and twisting moments, as follows:

$$U(w) = \frac{1}{2} D \iint_R \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} \, dx \, dy,$$  \hspace{1cm} (3)

where the integration is extended over the entire region $R$ of the mid-surface of the plate. The potential energy of the external forces and moments are given by the work function

$$\mathcal{W}(w) = \iint_R q \, w \, dx \, dy - \int_L M_n \ell \, n \, ds + \int_L (Q_n - \ell \frac{\partial H}{\partial s}) w \, ds$$  \hspace{1cm} (4)

where the line integral is integrated in the positive sense (so that $n$ is to $s$ as the $x$ is to the $y$ axis) along the boundary curve $L$ of the region $R$. "$n$" is the outer normal to the boundary curve $L$ of $R$, and $s$ the length of the boundary curve along the boundary $L$.

The problem is then to find a unique function $w$, which minimizes the function

$$\nabla(w) = U(w) - \nabla(w)$$ \hspace{1cm} (5)

with the auxiliary condition of continuity on $w$ and in conformity with the conditions of support at the edges $L$ of the plate.

A necessary condition that $\nabla(w)$ reaches a minimum is that the first variation $\delta \nabla(w)$ vanishes. This leads to the field equation
\[ \nabla^+ \omega = \frac{q}{D} \]  

(6)

and the boundary conditions

Either \( \delta \omega = 0 \) or

\[
D \left[ (1 - \nu) \frac{\partial}{\partial s} \left( \left( \frac{\partial^2 \omega}{\partial x^2} - \frac{\partial^2 \omega}{\partial y^2} \right) \sin 2\theta - \cos \theta \frac{\partial^2 \omega}{\partial x \partial y} \right) \right] \\
- \cos \theta \left( \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} \right) - \sin \theta \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \\
- \left( Q_n - \frac{\partial H_n}{\partial s} \right) = 0
\]

(7)

and \( \delta \frac{\partial \omega}{\partial n} = 0 \) or

\[
D \left[ (1 - \nu) \frac{\partial}{\partial s} \left( \cos^2 \theta \frac{\partial^2 \omega}{\partial x^2} + \sin^2 \theta \frac{\partial^2 \omega}{\partial y^2} + \nu \frac{\partial^2 \omega}{\partial s^2} + \nu \right) \right] + N_n = 0
\]

where \( \theta \) is the angle between the outer normal to the boundary and the x-axis, \( n \) and \( s \) are as usual the outer normal and length of the boundary curve. These results agree with those of Poisson-Love theory.

Therefore, the form of the strain energy expression (3) is compatible with the first approximation of the procedure of Section 1.1.

In such a formulation of the fundamental field equation and boundary conditions, however, one is confronted with the problem of forming a proper expression for the potential energy to begin with. The expression (3) is found satisfactory as a first approximation, but what are the proper expressions of \( U \) in higher order approximations? For a satisfactory theory one must be able to derive systematically the proper expressions for strain energy of a bent plate to any order of approximation without imposing arbitrary assumptions regarding to the deformation of the plate. In all events it seems that this problem is as complicated and is of the same nature as the problem of the last chapter.

From another point of view, suppose that a differential system for the deformation of a plate is obtained by a certain process of reasoning. But instead of proceeding as usual from the general solution
to the particular solution, which oftentimes is either impractical or impossible, one wishes to state the problem in a variational form so that useful approximate solutions can be obtained by direct methods. For example a differential system (6) and (7) can be equivalent to the problem of finding a solution of the equation

$$\delta \mathcal{V} = 0,$$

where \( \mathcal{V} \) is given by (5), (3), (4).

In general, a differential system that consists of a differential equation \( \mathcal{D}(\mathcal{W}) = 0 \) over a region \( R \) and boundary conditions \( \mathcal{E}_i(\mathcal{W}) = 0 \) on the boundary curve \( L_i \), \( i = 1, 2, \ldots \), can be represented as a problem to find a function \( \mathcal{W} \), so that

$$\iint_R \mathcal{D}(\mathcal{W}) \delta \mathcal{W} \, dx \, dy + \sum_{i=1}^\infty \int_L \mathcal{E}_i(\mathcal{W}) \delta_i \mathcal{W} \, ds = 0$$

for arbitrary \( \delta_i \mathcal{W} \). \hspace{1cm} (8)

If an integral \( \iint_R \mathcal{E}(\mathcal{W}) \, dx \, dy \) can be found of which the vanishing of the first variation leads to the above equation (8), then \( \mathcal{E} \) can be written as \( \delta \iint_R \mathcal{E}(\mathcal{W}) \, dx \, dy = 0 \). Such an integral, in general, has a simple physical meaning, as, for example, an energy integral. But when one starts from a given differential system, it is not at all evident that such an integral can be found easily. The formulation of the equivalent variational equation as (8) therefore avoids this difficulty.

Based on convergence considerations, there is another reason for not attaching too much importance on finding the energy integral. In direct methods one approaches the true solution \( \mathcal{W} \) by forming a sequence of approximating functions \( \mathcal{W}_n \) \( (n = 1, 2, 3, \ldots) \) and by passing \( n \) to infinity to get the required solution. But sometimes when an approximating sequence \( \mathcal{W}_n \) is found so that the integral \( \iint_R \mathcal{E}(\mathcal{W}_n) \, dx \, dy \)
readily converges to the true minimum value, the function \( \mathcal{W}_n \) itself, or its derivative, may not be convergent. In general, the convergence of \( \mathcal{W}_n \) or its derivatives is better if the order of the derivatives of \( w \) occurring in the integrand \( B(w) \) is higher. For example, Courant remarked (Ref. 14) that the spectacular success of the direct procedure of W. Ritz was due to a fortunate choice of the problem of the vibration of plates for his illustration rather than the seemingly easier problems of the membrane. The convergence can be improved by adding to the original minimizing integral expressions which consist of higher order derivatives of \( w \) and which vanish for the true solution. For example, the plate problem can be formulated so as to find a function \( v \) which minimizes the functional

\[
\mathcal{V}'(w) - U - \mathcal{W} = \frac{k}{2} \int_R \left( \nabla^2 w - \frac{q}{D} \right)^2 \, dx \, dy
\]

where \( U \) and \( \mathcal{W} \) are given by (3) and (4) respectively and \( k \) is any positive constant. Such additional terms make \( \mathcal{V}(w) \) more sensitive to variations of \( w \) without changing the true solution. In other words, the minimizing sequence attached to such a "sensitized" functional will be forced to be better as regards convergence. However, this "sensitization" certainly spoils the physical meaning of the original integral.

Equation (8) is not the only way to represent a differential system \( D(w) = 0 \) in a region \( R \) and \( \mathcal{B}_i(w) = 0 \), \( i = 1, 2, \ldots \) on the boundary \( L \) of \( R \). It can be as well represented as

\[
\int_R D^2(w) \, dx \, dy + \sum_i \int_{\Gamma} B^2_i(w) \, ds = 0
\]
If the functions \( D(\omega_n), B_i(\omega_n), \quad (i = 1, 2, \ldots) \) where

is an approximating functional sequence of \( \omega \), be interpreted as

error functions, then the problem is equivalent to finding a se-

quence \( \{\omega_n\} \) so that the square error

\[
\int \int_Q D^2(\omega_n) \, dx \, dy - \sum_i \int_L B_i^2(\omega_n) \, dt
\]  

(11)

be minimized.

The classical method of solving a variational problem by form-

ing its Euler's equation and then solving it with appropriate bound-

ary conditions is an indirect method. The alternative idea of looking

for a single sequence of functions which solves the given minimal

problem is called the direct method. First conceived by J. Bernoulli

and later by Riemann (Ref. 15), it was developed by the fundamental

work of Hilbert (Ref. 16). Its utilization to numerical calculations

of the solutions was independently envisaged by Lord Rayleigh (Ref. 5)

and Walther Ritz (Ref. 17, 18). Since then numerous applications and

improvements have been made by many authors. (See Refs. 10, 11, and

19).

Stated in general terms, the problem is the following: Given

an integral expression \( V[\Psi] \) over a given domain \( R \) of the independent

variables, whose boundary satisfies all the desired continuity assump-

tions; to find a function \( \Psi = \omega \) for which \( V[\omega] = \min \), the lower

limit of \( V[\Psi] \) for the totality of all \( \Psi \)'s which satisfy the continuity

conditions and boundary conditions.

It is assumed that such a lower limit exists, and that the

integrand satisfies all the continuity and regularity conditions.
The assumption on the definability implies the existence of a so-called minimal sequence \( \varphi_1, \varphi_2, \varphi_3, \ldots \) of allowable functions, for which

\[
\lim_{n \to \infty} \varphi_n = d \tag{12}
\]

The fundamental conception of all direct methods is such that each \( \varphi_n \) be obtained from a definite elementary minimal problem, and through the passing to the limit \( N \to \infty \) to get the required solution \( w \) of the given minimal problem. These two principal steps, the construction of the minimal sequence on the basis of common minimal problems and the passing to the limit, characterize the direct methods.
§ 1.22 RAYLEIGH-RITZ METHOD AND ITS ALLIED METHODS

The Rayleigh-Ritz method consists in narrowing down the choice of \( \psi \) (see the last paragraph of §1.21) into a smaller class of functions. It is assumed that \( \psi \) can be written as functions that satisfy the boundary conditions and involve a number of parameters. These parameters are then determined so as to make \( \mathcal{W}^{(j)} \) a minimum. The approximating functions must satisfy the boundary conditions, but in general do not the differential equations.

The deflection \( \psi \) of an elastic plate minimizes the potential energy integral (1.21:5). In order to obtain an approximate solution, one assumes that \( \psi \) can be represented with sufficient accuracy by a series of the form

\[
\mathcal{W}_N = \sum_{i=1}^{N} c_i \psi_i(x,y)
\]

(1)

where \( \{ \psi_i \} \) is so chosen that \( \psi_n \) satisfies the same continuity conditions of \( \psi \) and boundary conditions on the edges where displacements are prescribed. If (1) is inserted for \( \psi \) into the integral \( \mathcal{V} \), equation (1.21:6), the latter becomes a quadratic function of the parameters \( c_i \) \( (i = 1, 2, \ldots, N) \). The minimizing conditions

\[
\frac{\partial \mathcal{W}_N}{\partial c_i} = 0 \quad \text{ (} i = 1, 2, \ldots, N \text{)}
\]

(2)

provide, therefore, \( N \) linear equations to determine the \( N \) unknown constants \( c_i \). The approximate deflection surface \( \psi_n \) is thus determined. This is the Rayleigh-Ritz procedure (Refs. 5, 17, 18).

Galerkin’s method (Ref. 20) is more closely related to equation (1.21:8). If \( \psi_n \) satisfies all the boundary conditions \( B_i(\psi), i = 1, 2, \ldots, \ldots \).
then the last member of (1.21:3) vanishes automatically when \( w_n \) is substituted into it, which now becomes

\[
\iint_D (w_n) \delta w \, dx \, dy = 0
\]  

(3)

This cannot be satisfied by arbitrary variations \( \delta w \), otherwise \( w_n \) would be an exact solution. The constants \( c_i \) in (2) can, however, be so chosen that (3) is satisfied for a set of values of \( \delta w \), \( \delta w(x,y) \), where \( \{ \varphi_i(x,y) \} \), \( i = 1, 2 \ldots N \) are the coordinate functions of \( w_n \), as defined by equation (1), and \( \lambda_i \) are positive constants.

If now, \( N \) becomes infinite and the functions \( \{ \varphi_i(x,y) \} \) form a complete system of functions, then the set of all relations

\[
\iint_D (w_n) \varphi_i \, dx \, dy = 0 \quad , \quad i = 1, 2 \ldots N
\]  

(4)

becomes equivalent to the relation (3).

A generalization due to L. V. Kantorovitch (Ref. 21) consists in replacing the constants \( c_i \) by unknown functions of one variable \( c_i(x) \), for example, and an application of the minimum principle leads to a system of ordinary differential equations for the functions \( c_i(x) \).

The collocation method of Bieszeno and Koch (Refs. 22, 23) consists in specifying that \( D(w_n) \) be equal to zero at \( N \) points of the region, and that of Courant (Ref. 24) in demanding that

\[
\delta w(x,y) = \begin{cases} 
1 \quad \text{in } R_i \quad , \quad (i = 1, 2 \ldots N) \\
0 \quad \text{elsewhere in } R
\end{cases}
\]  

(5)

where \( R_i \), \( (i = 1, 2 \ldots N) \) is a subdivision of the entire region \( R \).
The $N$ equations

$$\iint_{\mathcal{D}} D(\omega_i) \, dx \, dy = 0 \quad (i = 1, 2, \ldots, N) \quad (6)$$

are then just enough to determine the constants $c_i$.

The method of least squares (Refs. 26, 27, and 28), is related to eq. (1.21:10). The constants $c_i$ are to be determined by requiring that the mean square error be as small as possible. That is,

$$\iint_{\mathcal{R}} \left[ D(\omega_i) \right]^2 \, dx \, dy = \text{Min.} \quad (7)$$

Hence there are $N$ equations

$$\iint_{\mathcal{R}} D(\omega_i) \frac{\partial D(\omega_i)}{\partial c_i} \, dx \, dy = 0 \quad (i = 1, 2, \ldots, N) \quad (8)$$

to determine the constants $c_i$.

These methods, although sometimes easier to apply, are narrower in scope than the original Rayleigh-Ritz method inasmuch as all the boundary conditions $\delta_i(\omega)$ should be satisfied by $\omega_n$; while in the Rayleigh-Ritz method, which starts from the original energy integral, only "rigid" boundary conditions (in contrast to the "natural" boundary conditions) are required to be satisfied by $\omega_n$. In cantilever plates, it is the "natural" boundary conditions which cause great difficulty.

All the methods outlined above are characterized by the fact that the potential energy $\mathcal{V}$ is approached from above by the approximating functions $\omega_n \sum_{i=1}^{n} c_i \varphi_i$. In other words, if $\mathcal{V}(\omega)$ is the value of $\mathcal{V}$ when the true solution $w$ is replaced by $\omega_n$, then

$$\mathcal{V}(\omega_n) \geq \mathcal{V}(\omega) \quad (9)$$
This follows directly from the general minimum energy theorem. Using Green's formula, it can be verified directly, and one obtains the following result:

\[ \nabla (\mathcal{W}_n) = \nabla (\mathcal{W}) + \mathcal{U} (\varepsilon_n) \]  \hspace{1cm} (10)

where \( \mathcal{U}(\varepsilon_n) \) is the integral (1.21:3) when \( \mathcal{W} \) is substituted by \( \varepsilon_n \), and \( \varepsilon_n \) is the error function

\[ \varepsilon_n(x, y) = \mathcal{W}(x, y) - \mathcal{W}_n(x, y) \]  \hspace{1cm} (11)

\( \mathcal{U}(\varepsilon_n) \) is the strain energy stored in a plate when it is subjected to a fictitious deflection \( \mathcal{W} = \varepsilon_n(x, y) \) and is of course a positive quantity.
1.23 METHODS OF RELAXATION OF BOUNDARY CONDITIONS

Equation (1.21.9) suggests immediately another method to obtain an approximate solution for \( w \). As before, assume that \( w \) can be represented by an approximating series

\[
W_N = \sum_{i=1}^{\infty} b_i j_i(x, y) \quad (N = 1, 2, 3 \ldots) \quad (1)
\]

where \( b_i \) are unknown constants and \( j_i(x, y) \), \( (i = 1, 2, \ldots) \) the so-called coordinate functions, satisfying the field equation (1.21.8). Hence \( W_N \) is a solution of the field equation and the first part of equation (1.21.8) vanishes. For the plate problem, there remains an equation in the form of a line integral

\[
\int \mathcal{B}_1(w) \, \frac{\delta w}{\delta n} \, ds + \int \mathcal{B}_2(w) \, \delta w \, ds = 0, \quad (2)
\]

where

\[
\begin{align*}
\mathcal{B}_1(w) & = D \left\{ (1-\nu) \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] \right\} + \nu \frac{\partial \pi}{\partial x} \\
& = \frac{P}{L} \frac{\partial w}{\partial n} & \text{when the edge is free to rotate}
\end{align*}
\]

or

\[
\begin{align*}
\mathcal{B}_2(w) & = M_n \left\{ \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2} \right\} + \nu \pi \frac{\partial \pi}{\partial x} \\
& = \frac{P}{L} \frac{\partial w}{\partial n} & \text{when the corresponding edge is clamped}
\end{align*}
\]

When (1) is substituted for \( w \) into (2), this equation cannot be satisfied unless \( w \) is an exact solution. However, the coefficients \( \{b_i\}, \ i = 1, 2, 3 \ldots \) can be determined so that this equation is
best approximately satisfied. This method was suggested by Courant (Ref. 15) and Trefftz (Ref. 29) who derived it from the condition of least square error on the boundary, i.e., in specifying particularly
\[ \delta w = B_2(w), \quad \delta \frac{\partial w}{\partial n} = B_1(w) \]
(5)
in equation (2) above. Then the left hand side of (2) is a quadratic function in the constants \( \{ b_i \} \), \( i = 1, 2, \ldots, N \). Regarding this as a mean square error, an ordinary maximum-minimum process leads to a set of \( N \) linear simultaneous equations:
\[ \int L B_1(w_n) B_1(j_i) ds + \int L B_2(w_n) B_2(j_i) ds = 0, \quad (i = 1, 2, \ldots) \]
(6)
from which the \( N \) unknown constants \( \{ b_i \} \) can be solved.

By assigning to \( \delta w \) and \( \delta \frac{\partial w}{\partial n} \) suitable forms, many approximating procedures can be formulated. Every method enumerated in the last part of section (1.22) has a counterpart in the present case. It is needless to state them individually in detail. For example, an analogy to Galerkin's method can be obtained by taking
\[ \delta w = g_i(x, y), \quad \delta \frac{\partial w}{\partial n} = \frac{\partial g_i(x, y)}{\partial n}, \quad (i = 1, 2, \ldots, N) \]
(7)
Collocation method by taking \( \delta w \) and \( \delta \frac{\partial w}{\partial n} \) to be unity at \( N \) points and zero everywhere else; Courant's modified collocation method, by dividing the boundary into \( N \) parts, and consider \( \delta w \) and \( \delta \frac{\partial w}{\partial n} \) to be unity in one part at a time, and zero in others; and so on. The practical value of each depends on the nature of the particular problem.

For Dirichlet's problems, the method of relaxation of boundary conditions always approaches the minimizing integral from below, that
is, if the functional \( I(\psi) \) is being minimized, then

\[
I(\psi_{\text{approx.}}) \leq I(\psi_{w_N}),
\]

for any \( N \). \hfill (8)

Hence it is a true counterpart of the Rayleigh-Ritz procedure, which approaches \( V \) from above. In the plate problems the situation is different. Let us again define an error function \( \varepsilon_N \) by

\[
\varepsilon_N = \omega(x,y) - w_N(x,y).
\]

Then

\[
V(\omega) = V(w_N + \varepsilon_N) = V(w_N) + U(\varepsilon_N)
\]

\[
+ \int_{R} \left[ \partial^2 w_N \partial^2 \varepsilon_N + (1-\mu) \left( \frac{\partial w_N}{\partial x} \frac{\partial^2 \varepsilon_N}{\partial y^2} + \frac{\partial w_N}{\partial y} \frac{\partial^2 \varepsilon_N}{\partial x^2} \right) \right] dxdy - \int_{L} M_{n}\frac{\partial \varepsilon_N}{\partial n} ds + \int_{L} \left( Q_{n} - \frac{\partial H_{N}}{\partial s} \right) \varepsilon_{N} ds \hfill (10)
\]

where \( V(\omega) \) is given by (1.21:5) and \( U(\varepsilon_N) \) is given by (1.21:3).

Transforming the integrals on the right hand side of (10) by Green's formula and integrating by parts, and remembering that

\[
\nabla^2 w_N - \frac{9}{D} = 0 \quad \text{in} \quad R,
\]

one obtains:

\[
V(\omega) = V(w_N) + U(\varepsilon_N) + \int_{L} B_1(\omega_N) \frac{\partial \varepsilon_N}{\partial n} ds + \int_{L} B_2(\omega_N) \varepsilon_N ds. \hfill (11)
\]

The last two line integrals are minimized in the process of determining the unknown constants \( \{ k_i \} \) of \( w_N \), but, in general, they do not vanish. If all the edges are clamped, then the true solution \( \omega \) and \( \frac{\partial \omega}{\partial n} \) will have the boundary values zero, and the last integrals become identical with eq. (6) or corresponding to the analogy to Gaderkin's solution, and they vanish by the process of determining \( \{ k_i \} \).

In this case, the tendency of approaching \( V \) from below is certain. However, for plates with free edges, the true values of \( \omega \) and \( \frac{\partial \omega}{\partial n} \) are unknown, and the determination of the sign and magnitude of these integrals are uncertain.
II. AEREOELASTIC PROBLEMS OF HIGH SPEED AIRPLANES

§ 2.1 Introduction

In the field of aeroelasticity concerned with thin swept wings of high speed airplanes, the aeronautical engineer is encountering a new phenomenon: the transient nature of aerodynamic coefficients. A supersonic airplane must reach its full speed or recover from its full speed to landing speed in a relatively short time. Thus a very important part of the flight is essentially accelerating or decelerating. Now in the transonic region, the effect of compressibility is particularly pronounced, and the nature of aerodynamic response to an oscillating wing changes in a rapid and subtle manner: from a phase lag in the subsonic case to a phase lead in the supersonic case. The aeroelastic phenomena in this important region can be studied only if the transient nature of the aerodynamic coefficients are taken into account. If the engine power is properly scheduled, so that from performance computations the velocity of flight as a function of time is known, then all the coefficients can be regarded as known functions of time. A particularly important case is that the transient perturbations are small, i.e., that the part of a coefficient which varies with time is small compared to the coefficient itself. In this case the computation is relatively simple.

This report treats the basic mathematical development of the problem. In aeroelasticity the determination of physical constants involved in the problem might be a more difficult task. But granting that these coefficients are known, there is still a need for a systematic method of attack. At present it seems that no work has been
published in this field; on problems such as the effect of transient perturbations on flutter, dynamic stability, etc.

This transient perturbation theory includes the ordinary small perturbation theory as a particular case. The perturbation theory extends the meaning of a particular solution which is usually obtained for a simplified picture of the actual case. By the fact that the solution of a boundary value problem depends continuously on the parameters of the differential system, it is natural to look for the possibility of treating problems in which the variation of parameters from the solved case is infinitesimal. For example in the later part of this report it is often necessary to construct a sequence of orthogonal functions with preassigned boundary conditions. This can usually be done by finding the natural modes of free vibrating elastic bodies. If \( \psi_m \) is the \( m \)th mode of a vibrating string, or bar, or membrane, or plate, with density distribution \( \delta \), then

\[
\int \delta \psi_m \psi_n \, d\sigma = \begin{cases} \text{constant} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}
\]

\( \{ \psi_m \} \) can usually be determined when \( \delta \) is a constant \( \delta_0 \). The perturbation theory then gives \( \{ \psi_m \} \) when \( \delta \) is slightly different from \( \delta_0 \). This problem has been treated by Lord Rayleigh (Ref. 5, p. 113), H. Schmidt (Ref. 39, Teil II, Kap. 3, §6) and W. Meyer zur Capellen (Ref. 40.)

§ 2.2 A General Solution

Assume that the dynamical system can be described with sufficient accuracy by \( n \) generalized coordinates \( q_1, q_2, \ldots, q_n \), \( n \) being finite. The equilibrium configuration is taken as \( (0, 0, 0, \cdots, 0) \). If the system
is conservative, the set of displacements $q_1, q_2, \ldots, q_n$ will produce forces $Q_1, Q_2, \ldots, Q_n$ that are derivatives of the potential energy of the system. In the vicinity of the equilibrium position they can be expressed by the linear expressions

$$Q_i = -\frac{\partial U}{\partial q_i} = -\sum_{j=1}^{n} k_{ij} \dot{q}_j,$$

where $i = 1, 2, \ldots, n$ and $k_{ij} = k_{ji}$. On the other hand, in a non-conservative system, the forces may still depend on the coordinates, and, in the neighbourhood of the equilibrium position, are given by linear functions of coordinates, but in this case $k_{ij} \neq k_{ji}$. In aeroelasticity, the aerodynamic forces are usually linear functions of the velocity and acceleration components in the neighbourhood of the equilibrium position. The part due to acceleration, the apparent-mass forces, which have their origin in the inertia of the surrounding air, can be included in the inertia forces. The part due to velocity can be written as

$$Q_i = -\sum_{j=1}^{n} \beta_{ij} \dot{q}_j,$$

the dot indicating differentiation with respect to time. To this may be added a disturbing force $\gamma_i$ due to gust or control movements. Hence in general

$$Q_i = -\sum_{j=1}^{n} k_{ij} \ddot{q}_j - \sum_{j=1}^{n} \beta_{ij} \dot{q}_j + \gamma_i \quad (i = 1, 2, \ldots, n). \quad (1)$$

Substituting into Lagrange's equations, one obtains the equations of motion

$$\sum_{j=1}^{n} a_{ij} \ddot{q}_j = -\sum_{j=1}^{n} k_{ij} \ddot{q}_j - \sum_{j=1}^{n} \beta_{ij} \dot{q}_j + \gamma_i \quad (i = 1, 2, \ldots, n). \quad (2)$$
\( a_{ij}, k_{ij}, q_{ij}, \text{ and } \gamma \) are functions of airplane geometry and speed of flight, they are either constants or continuous functions of time.

When the coefficients are all constants, the classical approach is to assume

\[ q_j = c_j e^{i\omega t} \]

substitute into eqs. (2) and determine \( \omega \) from the determinantal equation \( 4(\omega) = 0 \). The constants \( c_j \) are then determined from the initial conditions.

For stability problems, an investigation of the nature of the roots of the determinantal equation will be enough to determine the stability characteristics. For problems of disturbed motion, such as rolling power, controllability, maneuver, divergence, gust loading problems and so on, it is the motion following the disturbance that must be computed. In such cases it is simpler to take care of the initial conditions at the start. This can be done either by the following method or by an application of operational calculus (Ref. 41, 42).

When the coefficients are functions of time, the above method must be modified. It is simpler to reduce eq. (2) into a first-order system. Write

\[ x_1 - 2_1, x_2 - 2_2, \ldots, x_n - 2_n \quad (3) \]

then eqs. (2) reduces into the form

\[ \sum_{j=1}^{2n} b_{ij}(t) x_j = \sum_{j=1}^{2n} c_{ij}(t) x_j + \xi_i(t) \quad (4) \]

The set of equations (4) is expressible as the single matrix equation

\[ \left[ \begin{array}{c} b_{ij} \end{array} \right] \{ x_i \} = \left[ \begin{array}{c} c_{ij} \end{array} \right] \{ x_i \} + \{ \xi_i \} \quad (4) \]
or when no ambiguity can result, even more concisely as

$$b\dot{X} = cX + \xi. \quad (4)b$$

If the matrix $b$ is non-singular, $|b| \neq 0$, so that $b^{-1}$ exists, then eq. (4b) can be premultiplied by $b^{-1}$ to give

$$X = AX + B. \quad (5)a$$

If the matrix $b$ is singular, of rank $r$, then (4) will be reducible to a system of $r$ independent equations of the above form and $2n-r$ purely algebraic relations connecting the $2n$ variables. These $2n-r$ linear relations can be used to eliminate $2n-r$ of the variables $x_i$ from the remaining equations of the system.

Therefore it is assumed that the differential system is

\[
\begin{aligned}
\dot{X} &= A(t)X + B(t) \\
X(t=0) &= X_0
\end{aligned}
\quad (5)
\]

where $X$ is a column matrix of the variables of motion, $X_0$ the initial conditions, $B(t)$ a column matrix representing the disturbing forces and $A(t)$ a square matrix consists of aerodynamic and inertia and stiffness coefficients.

The classical theory deals with the case when $A$ and $B$ are matrices of constants. The fundamental problem of this section is the case when $A$ and $B$ are functions of time. Special attention is paid to the effect of a small change in the elements of $A$ on the final solution. In other words, if $X(A,t)$ is a solution of equation (5), we look for the solution of the equation

$$\dot{X} = (A + \delta A)X + B, \quad X(t=0) = X_0. \quad (b)$$
as a perturbation to $X(A(t))$.

The general solution of (5) is obtained as follows. Integrating eq. (5) from $0$ to $t$, one obtains

$$X(t) = \int_0^t A(\tau) X(\tau) d\tau + \int_0^t B(\tau) d\tau + X_0.$$  

Let

$$f(t) = \int_0^t B(\tau) d\tau + X_0.$$  

Then (7) can be written as

$$f(t) = X(t) - \int_0^t A(\tau) X(\tau) d\tau.$$  

This is the equation of motion following the disturbance. It is a well-known matrix Volterra integral equation of the second kind. Since $A(t)$ is, by nature of the problem, a matrix valued continuous function of time, the existence of a unique continuous solution is assured. As a well-known result we can write down the solution explicitly at once:

$$X(A(t)) = f(t) + \int_0^t A(\tau) f(\tau) d\tau + \sum_{k=0}^{\infty} \int_0^t A(\tau) d\tau \cdots \int_0^t A(\tau_k) d\tau_k X_0.$$  

At this point a slight generalization may be introduced. The kernel $A(\tau)$, as $\tau \leq t$, ($t$ is the point of time in question), can always be expressed as a power series in $\tau$. But sometimes it is more convenient to express $A$ as a function of both $t$ and $\tau$, for example, as $e^{-(t-\tau)^2}$. This case occurs in problems of determining the disturbed motion at the end of an accelerating period of the airplane. We may then write $A(t')$. But to distinguish from the former case let it be denoted by $K(t,\tau)$, reserving the notation $A(t)$ to functions of $\tau$ only.
Equation (9) then corresponds to
\[ f(x) = \chi(t) - \int_0^t K(t, \tau) \chi(\tau) d\tau \] (11)

where \( K(t, \tau) \) is a fixed square matrix of continuous functions of two real variables \( t \) and \( \tau \) in the triangle \( T \); \( 0 \leq \tau \leq t \); \( f(x) \) is a fixed column matrix of continuous functions of \( t \) in \( 0 \leq t \leq b \), defined by eq. (8); and \( \chi \) is, of course, also matrix valued.

The unique continuous solution of (11) is given by the formula
(Ref. 43)
\[ \chi(t) = f(t) + \int_0^t K(t, \tau) f(\tau) d\tau \] (12)

where the resolvent kernel \( K(t, \tau) = K(t, \tau) + K^2(t, \tau) + K^3(t, \tau) + \cdots \) (13)
is a matrix valued continuous function of \( t \) and \( \tau \) in \( T \), with the following notations:
\[
\begin{align*}
K^1(t, \tau) &= \int_0^t K(t, \xi) K(\xi, \tau) d\xi \\
K^2(t, \tau) &= \int_0^t K(t, \xi) K^1(\xi, \tau) d\xi \\
&\quad \ldots \\
K^m(t, \tau) &= \int_0^t K(t, \xi) K^{m-1}(\xi, \tau) d\xi \\
&\quad \ldots
\end{align*}
\] (14)

From the general property of the Volterra integral equation of the second kind, \( \chi(t, \tau) \) is a continuous function of \( t \) and \( \tau \); the series \( K + K^2 + K^3 + \cdots \) converges uniformly and absolutely for any continuous kernel \( K(t, \tau) \).

From the general solutions (10) and (12), it follows that for any dynamical system with a finite degree of freedom, the solution is a continuous function of the initial conditions and of the coefficients of the differential equations, provided that the coefficients are continuous functions of time. This proves the statement quoted in the introduction, § 2.1.
2.3 A Generalized Taylor's Series

The following is a shorter derivation of a series expansion of A.D. Michal (Refs. 31, 44, and 45) in abstract space.

Let us define a particular Banach space $E$, that is, a complete normed linear space closed under multiplication by real or complex numbers, which forms a normed linear ring. The elements $K_i(t, \tau)$ of $E$ are matrix valued continuous functions of two real variables $t$ and $\tau$ in the domain $T := [a, b] \times [c, d]$. The addition of elements and multiplication by numbers are defined in the usual way, and the norm of $K_i(t, \tau)$ is defined by any of the equivalent definitions of the norm of a matrix. The ring multiplication is defined by the following:

$$K_1 \ast K_2 \equiv \int_T K_1(t, \tau) K_2(\tau, \xi) \, d\tau, \quad (K_1, K_2 \in E). \quad (1)$$

In order to simplify the notation for an abstract series, an element $I$ is added to $E$, which has the properties: $\|I\| = 1$, $K_1 \ast I = I \ast K_1 = K_1$, $I \ast K_2 = K_2 \ast I = K_2$, $(K_1 + I) \ast K_2 = K_1 \ast (I + K_2)$, $(K_1 + I) \ast (K_2 + I) = K_1 \ast K_2 + I \ast K_2$, and $K_1 \ast (I + K_2) = K_1 \ast I + K_1 \ast K_2$; for any $K_1, K_2$ in $E$. Let this new space be called $E'$.

The composition multiplication defined above is in general not commutative, that is, $K_1 \ast K_2 \neq K_2 \ast K_1$. That $E'$ is really a normed linear ring can be easily verified. For example, the associative law of composition multiplication is verified in the next section, § 2.4.

As a consequence of the associative law

$$K_1 \ast (K_2 \ast K_3) = (K_1 \ast K_2) \ast K_3, \quad (2)$$

the continued multiplication of a finite number of elements in $E'$ is
uniquely defined. In particular, all the powers of \( K \) are uniquely defined. Similarly, terms like \( K_i^{*i} K_j^{*j} K_k^{*k} \) have unique interpretations.

Now let us consider a power series of \( K \) in \( E' \). From a general property of the Volterra resolvent kernel, the series \( 1 + K + K^{*2} + \ldots \) converges absolutely in \( E' \). Denote the sum by \( \Omega(kt, \tau) \) to signify the functional dependence of \( \Omega(t, \tau) \) on the kernel \( K \), and so obtain the formula

\[
1 + K + K^{*2} + K^{*3} + \ldots \xrightarrow{\text{abs.}} \Omega(kt, \tau) \text{ in } E', \quad \text{(any } K \text{ in } E\text{).} \quad (13)
\]

In other words, \( \Omega(kt, \tau) \) is an entire functional of \( K \) in \( E' \).

From this it follows that

\[
1 + (A + B) + (A + B)^{*2} + (A + B)^{*3} + \ldots \xrightarrow{\text{abs.}} \Omega(A + B, t, \tau) \text{ in } E', \quad \text{(any } A, B \text{ in } E\text{).}
\]

Evidently the following series, obtained by expanding all the brackets above,

\[
1 + A + B + A^{*2} + A * B + B * A + B^{*2} + A^{*3} + A^{*2} * B + A * B * A + \ldots
\]

also converges absolutely in \( E' \) for any \( A \) and \( B \) in \( E \).

But an absolutely convergent series in \( E' \) can be rearranged without changing its sum. Now rearrange the above series into a double series:

\[
\begin{align*}
1 + A + A^{*2} + A^{*3} & + \ldots \\
& + (1 + A + A^{*2} + \ldots) * B * (1 + A + A^{*2} + \ldots) \\
& + (1 + A + A^{*2} + \ldots) * B * (1 + A + A^{*2} + \ldots) * B * (1 + A + A^{*2} + \ldots) \\
& + \ldots
\end{align*}
\]

sum by rows and obtain the second formula.
\( \Omega(A \oplus B \mid t, \tau) = \Omega(A \mid t, \tau) + \Omega(A) \ast B \ast \Omega(A) + \Omega(A) \ast \hat{\Omega}(A) \ast B \ast \hat{\Omega}(A) + \ldots \) for any \( A, B \in E, \Omega \in E' \). (4)

This is the fundamental expansion theorem. It means that the functional \( \Omega(A \oplus B \mid t, \tau) \) with kernel \( A(t, \tau) + B(t, \tau) \) can be expanded into a generalized Taylor's series in "powers" of \( B(t, \tau) \) with "coefficients" the functionals \( \Omega(A \mid t, \tau) \).

Apply this result to the matrix integral equation

\[ f(t) = \chi(t) - \int_0^t K(t, \tau) \chi(\tau) \, d\tau \quad \text{(2.2:11)} \]

which has the unique continuous solution

\[ \chi(K \mid t) = f(t) + \int_0^t k(t, \tau) f(\tau) \, d\tau \quad \text{(2.2:12)} \]

where \( k(t, \tau) = K(t, \tau) + K^{(2)}(t, \tau) + K^{(3)}(t, \tau) + \ldots. \) (2.2:13)

Recall that \( \Omega(K \mid t, \tau) \) is I plus a continuous function of \( t, \tau \) in \( T \), which is the sum of an absolutely convergent series with the general term a continuous function of \( t, \tau \) in \( T \), so it can be postmultiplied by a column matrix \( f(t) \) and integrated from \( 0 \) to \( t \) term-by-term. Thus the solution (2.2:12) of the integral eq. (2.2:11) can be written as

\[ \chi(K \mid t) = \Omega(K) \ast f, \]

and therefore, following (4),

\[ \chi(K + \delta K \mid t) = \Omega(K + \delta K) \ast f \]

\[ = [ \Omega(K) + \Omega(K) \ast \delta K \ast \Omega(K) + \ldots ] \ast f \]

\[ = \chi(K) + \Omega(K) \ast \delta K \ast \chi(t), \quad \sum \Omega(K) \ast \delta K \ast \ldots \ast \Omega(K) \ast \delta K \ast \chi(K). \]

This expansion is valid for all continuous \( \delta K(t, \tau) \) in \( T; 0 \leq \tau \leq t \leq b \).

Further explanation is given in the next Section, \( \text{§} 2.4. \)
2.4 Elementary Derivation of the Series Expansion of Solutions

The results of the preceding section will be derived again in elementary way without using the notion of abstract space.

In section 2.2, the composition powers of \( K(t, \tau) \) are defined by equations (2.2.14). A generalization of this definition to products of two functions \( K_1(t, \tau) \) and \( K_2(t, \tau) \) is as follows:

\[
K_1 \star K_2(t, \tau) = \int_c^t K_1(t, \xi) K_2(\xi, \tau) \, d\xi. \tag{1}
\]

The star product is a convenient shorthand for writing integrals. \( K_1(t, \tau) \) and \( K_2(t, \tau) \) are square matrices of continuous functions of two variables \( t \) and \( \tau \) in the domain \( T; \) \( 0 \leq \tau \leq t \leq b \), each matrix in particular may consist of a single function. The formal work in the following is the same whether one thinks of \( K \) as a single continuous function or a square matrix of continuous functions.

Now for three functions \( K_1(t, \tau), K_2(t, \tau) \) and \( K_3(t, \tau) \), it can be verified that

\[
K_1 \star (K_2 \star K_3) = (K_1 \star K_2) \star K_3. \tag{2}
\]

For let \( L(t, \tau) = K_1 \star K_2 \), \( M(t, \tau) = K_2 \star K_3 \). Then

\[
(K_1 \star K_2) \star K_3 \quad \text{def} \quad \int_c^t L(t, \xi) K_3(\xi, \tau) \, d\xi
\]

\[
= \int_c^t d\xi \int_c^t K_1(t, \eta) K_2(\eta, \xi) K_3(\xi, \tau) \, d\eta
\]

which, by Dirichlet's lemma on the change of order of integration,

\[
= \int_c^t d\eta \int_c^\eta K_1(t, \eta) K_2(\eta, \xi) K_3(\xi, \tau) \, d\xi
\]

\[
= \int_c^t K_1(t, \eta) M(\eta, \tau) \, d\eta = K_1 \star (K_2 \star K_3).
\]
As a consequence there is no ambiguity in writing the star product of three continuous functions \( K_1, K_2, K_3 \) as \( K_1 * K_2 * K_3 \), which can be interpreted either as \( K_1 \ast (K_2 \ast K_3) \) or \( (K_1 \ast K_2) \ast K_3 \). In particular, if \( K_1 = K_2 = K_3 \), then \( K_1 \ast \) is uniquely defined as \( K_1 \ast K_1 \ast K_1 \). Similar definitions apply to higher powers of \( K \), and to terms like \( K_1 \ast K_2 \ast K_1 \ast K_2 \). The star product is, in general, non-commutative, i.e. \( K_1 \ast K_2 \neq K_2 \ast K_1 \).

For convenience, let us define a function \( I \), which has the value unity regarded as a function of \((t, \tau)\), but its star multiplication with other functions is defined by the special rule

\[
I \ast K = K = K \ast I
\]  

(3)

instead of an integration. Obviously the associative law holds for continued multiplication of \( I \) and \( K \)'s.

Now consider the series

\[
I + K(t, \tau) + K^{-1}(t, \tau) + \cdots + K^{-n}(t, \tau) + \cdots,
\]  

(4)

or

\[
I + K(t, \tau) + \int_{t}^{\infty} K(t, \xi) K(\xi, \tau) d\xi + \int_{t}^{\infty} K(t, \xi) d\xi \int_{\xi}^{\infty} K(\xi, \eta) K(\eta, \tau) d\eta + \cdots.
\]

It will be shown that this series converges uniformly and absolutely for \( t, \tau \) in \( T : a < t < T < b \). For this purpose, let us define the norm of a matrix \( [K_{ij}(t, \tau)] \) as \( \| K_{ij}(t, \tau) \| = \sup_{a \leq t \leq b} \sqrt{\sum_{ij} K_{ij}(t, \tau)^2} \).

Let

\[
\mathcal{M} = \| K(t, \tau) \|,
\]

then

\[
\| K \ast Z(t, \tau) \| = \left\| \int_{t}^{\infty} K(t, \xi) K(\xi, \tau) d\xi \right\| \leq \int_{t}^{\infty} \mathcal{M}^2 d\xi \leq \mathcal{M}^2 (t - \tau).
\]
By induction \[ K^{N}(t, \tau) \| = \frac{M^K}{(N-1)!} (t - \tau)^{N-1} \]

Therefore the series (4) is dominated term-by-term by the series

\[ |f| + \sum_{n=1}^{\infty} \frac{M^n}{(n-1)!} (b - \tau)^{n-1} \]

which is convergent for all finite values of \( b \) and \( M \). Hence the series (4) converges uniformly and absolutely in \( \mathfrak{K} \), \( c < t < b \).

Let the sum of the series (4) be denoted by \( \Omega(t, \tau) \), to signify the functional dependence of \( \Omega(t, \tau) \) on the kernel \( K \). Then

\[ \Omega(t, \tau) = I + K(t, \tau) + K^{*2}(t, \tau) + K^{*2}(t, \tau) + \cdots \quad (5) \]

Since (5) is true for any continuous function \( K(t, \tau) \), so it is true when the kernel is a sum of two continuous functions \( K(t, \tau) \) and \( \delta K(t, \tau) \).

Hence

\[ \Omega(t + \delta K)(t, \tau) = I + (K + \delta K) + (K + \delta K)^{*2} + (K + \delta K)^{*2} + \cdots \quad (6) \]

But

\[ (K + \delta K)^{*2}(t, \tau) = \int_{t}^{\tau} \left[ K(t, \xi) + \delta K(t, \xi) \right] K(\xi, \tau) + \delta K(\xi, \tau) \, d\xi \]

\[ = \int_{t}^{\tau} \left[ K(t, \xi) K(\xi, \tau) + \delta K(t, \xi) K(\xi, \tau) + K(t, \xi) \delta K(\xi, \tau) \right. \]

\[ + \left. \delta K(t, \xi) \delta K(\xi, \tau) \right] d\xi \]

\[ = K^{*2} + \delta K \times K + K \delta K + \delta K^{*2} \]

A similar relation holds for higher powers of \( K + \delta K \). The expansion is formally the same as in ordinary non-commutative algebra. Now expand all the terms in the right hand side of (6) as above.
\[
\Omega(K + \delta K) = I + K + \delta K + K^2 + K \delta K + \delta K K + \delta K^2 + K^3 + \delta K^3 + \delta K^4 + \ldots
\]

The series on the right converges absolutely, and hence can be re-arranged in any manner whatsoever without changing its value. Re-arrange into a double series

\[
I + K + K^2 + K^3 + \ldots
\]

\[
+ (I + K + K^2 + \ldots) \delta K \ast (I + K + K^2 + \ldots)
\]

\[
+ (I + K + K^2 + \ldots) \delta K \ast (I + K + K^2 + \ldots) \delta K \ast (I + K + K^2 + \ldots)
\]

\[
+ \ldots \ldots
\]

and sum by rows, to obtain

\[
\Omega(K + \delta K(t, \tau)) = \Omega(K) + \Omega(K) \ast \delta K \ast \Omega(K) + \Omega(K) \ast \delta K \ast \Omega(K) \ast \delta K \ast \Omega(K)
\]

\[
+ \ldots \ldots
\]

(7)

This is the fundamental expansion theorem. It is analogous to the Taylor’s series. The functional \(\Omega(K + \delta K)\) can be expanded into a series in “powers” of \(\delta K\) with “coefficients” as functions \(\Omega(K)\).

Apply this result to the metric integral equation (2.2:11) which has the unique continuous solution

\[
\chi(K|t) = f(t) + \int_0^t K(t, \tau) f(\tau) d\tau
\]

(2.2:12)$_{bis}$

where

\[
\delta K(t, \tau) = K(t, \tau) + K^2(t, \tau) + K^3(t, \tau) + \ldots
\]

Notice that

\[
\Omega(K|t, \tau) = I + \delta K(t, \tau),
\]

(8)
hence by defining the star product of \( k \) or \( \mathfrak{I} \) to a column matrix \( \{ f_i(\tau) \} \)
as
\[
I * f = f, \quad k * f = \int_0^t \sum_j k_{ij}(t, \tau) f_j(\tau) d\tau, \tag{9}
\]
then the solution (2.3;12) can be written as
\[
\chi(kt) = \Omega(\mathfrak{I}) * f \tag{10}
\]
and therefore from (7):
\[
\chi(k+\delta k) = \Omega(\mathfrak{I}+\delta k) * f
= \chi(k) + \Omega(\mathfrak{I}) * \delta k * \chi(k) + \sum \Omega(\mathfrak{I}) * \delta k * \cdots * \delta k * \chi(k), \tag{11}
\]
This expansion is valid for all continuous \( \delta k(t, \tau) \) in \( T: a < \tau < b \).

The result can be stated in the following theorem:

Let \( \Omega(\mathfrak{I}, t, \tau) \) be a functional of \( \mathfrak{I}(t, \tau) \) defined by the series (5), and \( \chi(\mathfrak{I}) \) be the unique continuous solution of the Volterra integral equation of the second kind (2.2;11) with continuous kernel \( \mathfrak{I}(t, \tau) \); then the unique continuous solution of the equation (2.2;11) when the kernel becomes \( \mathfrak{I}(\tau, \tau) + \delta \mathfrak{I}(t, \tau) \) can be written as
\[
\chi(k+\delta k) = \chi(k) + \Omega(\mathfrak{I}) * \delta k * \chi(k) + \sum \Omega(\mathfrak{I}) * \delta k * \cdots * \delta k * \chi(k). \tag{11}_{\text{bis}}
\]
The star products in this formula are defined by eqs. (1), (3), and (9).

More explicitly
\[
\chi(k+\delta k) = \chi(k) + \sum_{s=0}^{\infty} \sum_{\eta_0, \eta_1, \ldots, \eta_s} \delta \mathfrak{I}(t, \tau_0, \ldots, \tau_s) \chi(k) \ldots \chi(k \eta_s) d\tau_0 \ldots d\tau_s \tag{12}
\]
where for shortness \( \Omega(t, \tau) \) has been written in place of \( \Omega(\mathfrak{I}t, \tau) \).

It should be noted, however, that the expression (11) is in fact
easier to interpret than (12). The letter is a particular way of expressing (11), and may not be the simplest. For example, it may be desirable to interpret the general term of (11) as (cf. eq. 17 below)

\[ \Omega (N) \ast \xi K \ast \{ \Omega (N) \ast \xi K \ast \cdots \ast \Omega (N) \ast \xi K \} = X (N). \]

Returning now to the original form of the dynamical equation (2.2:6) or (2.2:9) of which the solution is given by (2.2:10), it is useful to have the explicit formulas:

When \( R (t) = 0 \), i.e. when no guest or control forces and moments are acting, the matrix equation of motion becomes

\[ \dot{X}_i = \sum_{j=1}^{2n} A_{ij} (t) X_j (t), \quad X_i (t=0) = X_{0i}, \quad (i = 1, 2, \ldots, 2n). \]  \( \text{(13)} \)

The solution is given by

\[ X_i (A | t) = X_{0i} + \int_0^t A (\tau) d\tau + \int_0^t A (\tau) d\tau + \int_0^t A (\tau) d\tau + \cdots + \int_0^t A (\tau) d\tau. \]  \( \text{(14)} \)

The so-called "matrizzant functional" of \( A (t) \) is defined as follows:

\[ R^x_0 (A) = I + \int_h^t A (s) ds + \int_h^t A (s) ds + \int_h^t A (s) ds + \cdots + \int_h^t A (s) ds, \]  \( \text{(15)} \)

where \( I \) is the idemfactor.

With this notation, (14) can be written as

\[ X (A | t) = R^x_0 (A) X_0. \]  \( \text{(16)} \)

The generalized Taylor's expansion then leads to the following relation:
\[ \dot{X}(A+\delta A/t) = X(A(t)) + \int_0^t R(A) \delta A(s) X(A(s)) \, ds \]
\[ + \sum_{i=0}^{\infty} \int_0^t R(A) \delta A(s) ds \int_0^{s_1} R(A) \delta A(s_2) ds_2 \int_0^{s_2} \cdots \int_0^{s_{i-1}} R(A) \delta A(s_i) ds_i \, X(A(s_i)). \]  

Then \( B(t) \neq 0 \), i.e., the gust or control forces or other exciting forces exist, the differential system

\[ \dot{X} = A(t)X(t) + B(t), \quad X(t=0) = X_o \]  

where \( A(t) \) is a square matrix of continuous functions of \( t \), and \( B(t) \) a column matrix of functions of \( t \), (integrable, but not necessarily continuous) has the unique continuous solution (2.2:10) which can be written as

\[ \dot{X}(A(t)) = f(t) + A \ast R(A) \ast f(t), \]

where

\[ f(t) = \int_0^t B(t) \, dt + X_o. \]

The generalized Taylor's series of \( X(A+\delta A/t) \), where \( \delta A(t) \) is another square matrix of continuous function of \( t \), now takes exactly the same form as (17).

§ 2.5 The Solution in Canonical Form

The nature of the solution can be best recognized by reducing the result into canonical form. The dynamical system is represented by the differential equation:

\[ \dot{X} = A(t)X(t) + B(t), \quad X(t=0) = X_o, \]  

where \( A(t) \) is a square matrix of continuous functions of \( t \), \( B(t) \) a column matrix of integrable functions of \( t \), and \( X(t) \) a column matrix of the variables in question. Now let the variables \( X \) be subjected
to a linear transformation \( P \), so that a column matrix of new variables \( W \) is defined by
\[
WX = W,
\]
where \( P \) is a non-singular square matrix of constants. The differential system (2.2:5) will be transformed into:
\[
\dot{W} = \Lambda W + E, \quad W_0 = PX_0,
\]
where
\[
\Lambda = P XP^{-1}, \quad E = P E_P.
\]
A collineatory transformation \( P \) which renders the new kernel \( \Lambda \) to the simplest form is to be found.

(I) Consider first the case when \( A \) is a square matrix of constants.

From the general properties of \( \lambda \)-matrices and elementary divisors, it follows that: (Ref. 16, Chap. 20)

\( A \lambda a_1 \) and \( a_2 \) are two matrices independent of \( \lambda \), then the necessary and sufficient condition for the existence of a non-singular matrix \( P \), such that
\[
a_2 = P a_1 P^{-1}
\]
is that the characteristic matrices of \( a_1 \) and \( a_2 \) have the same invariant factors, or if one prefers, the same elementary divisors.

Now the elementary divisors of the characteristic matrix \( A - \lambda I \) of \( A \) depend on the nature of the latent roots of \( A \), (i.e., the roots of the determinantal equation \( |A - \lambda I| = 0 \), which are also called characteristic roots). Naturally one looks for the condition of existence of a collineatory transformation which renders \( A \) into a diagonal matrix. It is easily verified by the above general theorem,
that this collinear transformation is always possible when the latent roots of \( A \) are all distinct. However, it is not always possible to reduce \( A \) into a diagonal matrix by collinear transformation when there are repeated latent roots. It is found (Chap. VI, Ref. 47, and Chap. 3, Ref. 48), that the general canonical form is a diagonal matrix with a certain number of unit elements added in the superdiagonal (i.e., the elements immediately to the right of the principal diagonal). The rule is that unit element is certainly absent if the place in the superdiagonal is adjacent horizontally and vertically to distinct elements in the principal diagonal, but that it may be present when the adjacent elements in the principal diagonal are the same. Thus the typical canonical form is

\[
\begin{pmatrix}
\lambda_1 & 1 \\
1 & \lambda_1 \\
\lambda_1 & 1 \\
\lambda_1 & \\
\lambda_2 & 1 \\
1 & \lambda_2 \\
\lambda_2 & 1 \\
\lambda_2 & \\
\end{pmatrix}
\]

where the elements in the principal diagonal are, of course, the latent roots. It is clear that the canonical matrix can be partitioned into a diagonal matrix of square submatrices whose rows and columns do not overlap, and which are all of the simple classical type:

\[
\begin{pmatrix}
\lambda_1 & 1 \\
1 & \lambda_1 \\
\lambda_1 & 1 \\
1 & \\
\end{pmatrix}
\]

(all the rest of places are zeros)

The order of the simple classical submatrices can range from 1 to \( n \), depending on the degree of multiplicity of the particular latent root \( \lambda_i \) and the rank of the \( \lambda \)-matrix \( A - \lambda_i I \).
This being established, several particular cases follow:

Case Ia. All latent roots distinct.

Let \( \{ \lambda_i \} \) be the latent roots of \( A \), then there exists a non-singular matrix \( P \) such that \( PAP^{-1} = \Lambda \) where \( \Lambda \) is a diagonal matrix:

\[
\Lambda = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix}
\] (6)

the order of \( A \) being \( n \).

Write (6) as a sum:

\[
\Lambda = \lambda_i \ z_i
\] (7)

where \( z_i \) is a diagonal matrix with all elements zero except the one occupying the \( i \) th column which is unity; the summation convention for the subscripts is used in (7).

Since

\[
z_i \ z_j = \begin{cases} 
0 & \text{if } i \neq j \\
z_i & \text{if } i = j
\end{cases}
\]

it follows that \( \Lambda^n = \lambda_i^n \ z_i \).

Therefore substituting \( \Lambda \) as kernel into (2.4:15), one obtains the

matriscant functional

\[
\mathcal{R}_n^{x}(A) = e^{A^{x_{\times}}_{\times}} = e^{\Lambda \times \times^{(\times \times)}} z_i .
\] (8)

Hence from (2) and (2.4:15), it follows that when external forces

(gust or control) are absent, the solution is

\[
\mathcal{W} (A^t) = \sum_{i=1}^{n} e^{\lambda_i^t} z_i \). \mathcal{W}_0.
\]

Hence by (1),

\[
\mathcal{X}^r(A^t) = r^{-1} \left( \sum_{i=1}^{n} e^{\lambda_i^t} z_i \right) \mathcal{F}_0.
\] (9)
When external forces do exist, one must use (2.4.10) or its equivalent (2.4.18), the solution is then

\[ \chi(A(t)) = \int_0^t \beta(t) \, dt + \chi_0 + \sum_{\lambda_i} \lambda_i e^{\lambda_i(t)} \left( \int_0^t e^{-\lambda_i(s)} \, ds \right). \]  

where

\[ f(t) = P \int_0^t \beta(t) \, dt + \dot{P} \chi_0. \]

It is at once recognized that this is identical with the classical solution. However, the advantage of the Heaviside Operational method is preserved in the way the initial conditions are taken care of. In fact, the amount of work necessary in evaluating the transformation \( P \) is just the same as that needed by the classical or Heaviside solution. The advantage of the present method lies in the fact that in many cases one is not interested in reducing the result into its canonical form; and then it is simpler to work directly with the method developed in Section 2.4.

Case IIb. With repeated latent roots.

Consider a typical simple classical submatrix.

\[ \Lambda_i = \begin{bmatrix} \lambda_i & 1 \\ \lambda_i & 0 \end{bmatrix}. \]  

(11)

The series (2.4.15) gives

\[ R_n^t(\Lambda_i) = I + \Lambda_i(t) + \Lambda_i^2 (t^2) + \cdots + \Lambda_i^n (t^n) \]

\[ = \begin{bmatrix} e^{\lambda_i(t-n)} & (t-n) e^{\lambda_i(t-n)} & \frac{(t-n)^2}{2!} e^{\lambda_i(t-n)} \\ \frac{t}{n!} e^{\lambda_i(t-n)} & \frac{(t-n)}{n!} e^{\lambda_i(t-n)} & 0 \end{bmatrix} \]  

(12)

This shows that the solution includes terms like \( t e^{\lambda_i t} \) and \( t^2 e^{\lambda_i t} \) which is to be expected from considerations on the familiar
solution of a system of linear ordinary differential equations
with constant coefficients when the auxiliary equation has multiple
roots.

The above result is typical, and generalization to any order of
the classical type is obvious. Recall that the general form of the
canonical matrix (4) can be partitioned into a diagonal matrix of
simple classical submatrices \( \Lambda_i \), so write

\[
\Lambda = \begin{pmatrix}
\Lambda_1 & \\
& \Lambda_2 & \\
& & \cdots & \\
& & & \Lambda_k
\end{pmatrix},
\]

then

\[
R^t_\Lambda (\Lambda) = \begin{pmatrix}
R^t_\Lambda (\Lambda_1) \\
& R^t_\Lambda (\Lambda_2) \\
& & \ddots \\
& & & R^t_\Lambda (\Lambda_k)
\end{pmatrix}
\]

and as before

\[
\chi (A(t)) = f(t) + \int_0^t A \int_0^t R^s_\Lambda (\Lambda) P f(s) \, ds \, d\tau,
\]

where

\[
f(t) = \int_0^t B(t) \, dt + \lambda_0.
\]

(II). When \( A \) is a square matrix of functions of time.

Although the solution of the dynamical equation given in Section
2.4 is an entire analytic functional of the kernel \( A(t) \), the behaviour
of the solution is in general complicated. However, in aeroelastic
problems, the variation with time of the elements of the matrix \( A(t) \)
is in general small; that is, if \( A(t) \) is written as

\[
A(t) = A_0 + \varepsilon A^{(t)},
\]

(15)
where $A_\circ$ is a matrix of constants, then in general

$$\| \delta A_{ij}^{(4)} \| \ll \| A_{ij} \| .$$

(16)

Such small perturbation problems are discussed in section 2.6.
§ 2.6 Small Perturbations

Although the results derived in section 2.4 are perfectly general so that there is no limitation to the magnitude of the norm of \( \delta K(t, \tau) \) (or \( \delta A(t) \)) in the series expansions (2.4:11) or (2.4:17), these expansions are of practical interest only when \( \| \delta K \| \) is small.

An upper bound to the remainder after \( n \) terms of the expansion (2.4:17) can be obtained as follows: the norm of the \( n \)th term (the term with \( \delta A(t) \) occurring \( n \) times) is bounded by \( \| \Lambda(A(t)) \frac{M^n}{n!} \) where \( M = T \| R(K) \| \| \delta K \| \), \( T \) being the time interval in question.

Recall that the ordinary Taylor's expansion of \( e^u \) with remainder is:

\[
e^u = 1 + u + \frac{u^2}{2!} + \cdots + \frac{u^{n-1}}{(n-1)!} + R_n,
\]

where

\[
R_n = \frac{u^n}{(n-1)!} \int_0^1 (1-x)^{n-1} e^{ux} dx.
\]

So similarly we have,

\[
\| \Lambda(K + \delta K) - \Lambda(K) - \sum_{i=1}^{2^n-1} \int \cdots \int \| \leq \| \Lambda(K) \| \frac{M^n}{n!} \int_0^1 \int_{x_1} x_2^n e^{Mz} dz.
\]

\[
\leq \| \Lambda(K) \| \frac{M^n}{(n+1)!} e^M.
\]

(1)

Hence the expansion converges at least as fast as an exponential series.

According to the assumption of small perturbation, if

\[
A(t) = A_0 + \delta A(t)
\]

(2)

where \( A_0 \) is a square matrix of constants, then
for $t$ in the time interval in question. The solution with the kernel $A_0$ may be regarded as the initial solution, and that with the kernel $A_0 + \delta A$ the perturbed solution, where $\delta A$ is the perturbation kernel.

From the general expansion theorem, if only the first few terms are taken, the perturbed solution is

$$\chi(A_0 + \delta A | t) = \chi(A_0 | t) + \int_0^t R^\dagger_s(A_0) \delta A(s) \chi(A_0 | s) ds + \text{remainder},$$

where $R^\dagger_s(A_0)$ is the matrization functional with kernel $A_0$, given by eq. (6.4.15).

The effect of perturbation can be appreciated in a few simple cases as follows:

Let $P$ be a collinear transformation which transforms $A_0$ into the canonical form $\Lambda_0$. Assuming that all the roots of the characteristic equation of $A_0$ are distinct, then $\Lambda_0$ is a diagonal matrix:

$$PA_0 P^{-1} = \Lambda_0 = \sum \lambda_i \mathbf{Z}_i,$$

where $\{\lambda_i\}$ are the roots of the characteristic equation $|A_0 - \lambda I| = 0$ and $\mathbf{Z}_i$ the unitary matrix defined by (2.5:7). Then, according to (2.5:8),

$$R^\dagger_s(A_0) = \sum \lambda_i e^{\lambda_i (s-a)} \mathbf{Z}_i.$$

Write also

$$\zeta \Lambda = P \delta A P^{-1}.$$
and let the elements of $\delta A$ be denoted by $\delta a_{ij}, \ i = 1, 2, \ldots, n$. The variables $X$ are reduced to $W$ by $FX = W$ as in the last section.

Consider the simplest case when the differential system consists of

$$\dot{W} = (\Lambda + \delta \Lambda) W ; \quad W(t = 0) = W^{(0)},$$

and $\delta \Lambda$ is a matrix of constants. Then

$$\int_{s_0}^{t} R_s^T (\Lambda_s) \delta \Lambda R_s (\Lambda_s) \, ds = \left[ \int_{s_0}^{t} e^{\Lambda_s (t-s)} + \sum_{s} \frac{\lambda_j^s}{\lambda_j - \lambda_i} \delta a_{ij} \, ds \right] = \text{a matrix with elements:}$$

$$\left\{ \begin{array}{l}
e^{\lambda_j^i t} - e^{-\lambda_i^i t} \delta a_{ij} & \text{if } i \neq j, \\
t e^{\lambda_i^i t} \delta a_{ii} & \text{if } i = j,
\end{array} \right.$$  

and the solution of disturbed motion is approximately:

$$W(\Lambda + \delta \Lambda) = \sum_{i=1}^{n} e^{\lambda_i^i t} W_i^{(0)} + \sum_{i=1}^{n} t e^{\lambda_i^i t} \delta a_{ii} W_i^{(0)}$$

$$+ \sum_{j \neq i}^{n} \frac{e^{\lambda_j^i t} - e^{\lambda_i^i t}}{\lambda_j - \lambda_i} \delta a_{ij} W_j.$$  

In general $\{\lambda_i^i\}$ are imaginary numbers or complex numbers. The formula above of course holds only for small values of $t$.

The case discussed above has an important engineering application. It tells the effect of a slight change in design on the final behaviour of an airplane. As particular examples, problems in dynamic stability, controllability, and maneuverability, etc., may be mentioned.

If $\delta A(t)$ is proportional to $e^{-\alpha t}$, with elements $\delta a_{ij} e^{-\alpha t}$, the corresponding result will be

$$W(\Lambda + \delta \Lambda) = \sum_{i=1}^{n} e^{\lambda_i^i t} W_i^{(0)} + \sum_{i=1}^{n} t e^{\lambda_i^i t} \frac{\delta a_{ii} W_i}{\lambda_j - \lambda_i - \alpha}.$$  

(10)
the summation over \( i, j \) excludes those terms for which \( \lambda_j - \lambda_i = \omega - \omega' \),
for such terms the coefficients before \( \delta_{i,j} \) should be replaced by
\( te^{\lambda_i t} \). In the reduced form \( \mathcal{W}_i(\lambda_i) \) are normal coordinates: \( \mathcal{W}_i^{(n)} \)
excites the motion of \( \mathcal{W}_i \) alone. For the perturbed case, however, \( \mathcal{W}_i^{(n)} \)
excites all the coordinates.
III. DEFLECTION OF CANTILEVER PLATES

§ 3.1. PRELIMINARY DISCUSSIONS

In second order (first) approximation the problem of bending or flat plates consists in integrating the equation

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D} q(x, y) \quad (1)$$

where \(w(x, y)\) satisfies along the boundary \(\partial \) of a region \(R\) certain boundary conditions defined by the problem, and \(w\) is continuous in \(R\) and on \(L\), and \(z\) satisfies desirable continuity properties.

There are four kinds of problems, classified according to the nature of prescribed boundary conditions:

(A) Built-in edge problems

\[ w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial, \text{ where } n \text{ is the outer normal to } \partial. \quad (2) \]

(B) Supported edge problems

\[ w = 0, \quad \frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2} = f(x, y) \quad \text{on } L, \quad (3) \]

where \(n\) and \(t\) are outer normal and tangent to \(L\) respectively, the sense of \(n\) is to \(t\) as \(x\) is to \(y\)-axis.

Subcase \((B_1)\). Simply-supported edge problems, \(f = 0\).

(C) Unsupported edge problems

\[ \frac{\partial^3 w}{\partial n^3} + \nu \frac{\partial^3 w}{\partial t^3} = g(x, y) \quad (4) \]

\[ \frac{\partial^3 w}{\partial x^3} + (2 - \nu) \frac{\partial^3 w}{\partial n \partial t^2} = h(x, y) \quad \text{on } L. \]

Subcase \((C_1)\). Free-edge problems: \(g(x, y) = h(x, y) \equiv 0\).

(D) Mixed edge-condition problems.

This can be further classified into
\( (D_1) \) BS. When \( L \) is partly simply-supported and partly built-in.

\( (D_2) \) BU. When \( L \) is partly built-in and partly unsupported.

\( (D_3) \) SU. When \( L \) is partly supported and partly unsupported.

\( (D_4) \) SEU. When \( L \) is partly built-in, partly supported, and partly unsupported.

Problem (A) is often made a subject of mathematical investigation. Existence theorems and solutions in fairly general forms have been obtained by Almansi, Leuricelle, J. Hadamard, A. Korn, and others. (Refs. 32, 33, 34, 35). Approximate solutions for rectangular plates have been given by W. Ritz, Timoshenko and others (Refs. 17, 10).

Problem (B) is extensively studied by Timoshenko and others. (Ref. 10). For rectangular plates an application of Fourier series makes the problem particularly simple. Both problems (A) and (B) can be reduced to a set of first boundary value problems of the potential theory. (Ref. 36, Chapter 19).

Problem (C) is much more difficult than the first two and few existing results can be quoted. The problem of vibration of a rectangular plate with free edges which is akin to \( (C_1) \) was studied by Ritz in his 1909 paper in which he illustrated a spectacular success of his energy method.

Of problem (D), a few cases of \( (D_1), (D_3), \) and \( (D_4) \) have been studied. (See Ref. 10 for references to original works). The solved cases are:

1. Rectangular plates with two opposite edges simply supported and the other two edges built-in.

2. Rectangular plates with three edges simply supported and one edge built-in.
(3) Rectangular plates with two opposite edges simply supported, the third edge free and the fourth edge built-in or simply supported.

(4) Rectangular plates with two opposite edges simply supported and the other two edges supported elastically.

(5) Plate of the form of a parallelogram with two opposite edges simply supported. (Ref. 54).

In all the above cases, the solution can be put in the form of an infinite series.

\[ W = \sum_{m=1,2,\ldots} Y_m(y) \sin \frac{m\pi x}{a} \]

where \( \{Y_m(y)\} \) are functions of \( y \) alone, provided that the edges \( x \) equal to constants are simply supported.

(6) Circular plates supported at several points along the boundary.

(7) Circular plates in the form of a sector, the straight edges of which are simply supported.

For other problems, there is little available information.

The swept wing problem is a problem (D2) with BU boundary. Exact solutions are in general hard to get from the standpoint of biharmonic analysis but approximate solutions or engineering significance can be found by the various direct methods of solution.
Solution of the Differential Equation

In the application of the method of relaxation of boundary conditions, it is usually convenient to write the solution of eq. (3.1.1) in the form of a sum of a particular integral and a biharmonic function:

\[ w(x,y) = w_p(x,y) + w_c(x,y) \]  \hspace{1cm} (1)

so that

\[ \frac{\partial^4 w_p}{\partial x^4} + 2 \frac{\partial^4 w_p}{\partial x^2 \partial y^2} + \frac{\partial^4 w_p}{\partial y^4} = \frac{1}{D} q(x,y), \]

and

\[ \frac{\partial^4 w_c}{\partial x^4} + 2 \frac{\partial^4 w_c}{\partial x^2 \partial y^2} + \frac{\partial^4 w_c}{\partial y^4} = 0. \]  \hspace{1cm} (2)

The particular integral \( w_p \) can be easily obtained if \( q(x,y) \) is simple, say, being a constant, a function of single variable \( x \) or \( y \), or a polynomial in \( x, y \). For other cases it was shown by Mathieu (Ref. 37), that a general expression for \( w_p \) can be obtained by an extension of the potential theory. It can be verified that if

\[ w_p(x,y) = -\frac{1}{8 \pi D} \int_R \left( \tau_R^2 (e_1 + \frac{r^2}{2}) \right) q(a,b) da db, \]  \hspace{1cm} (3)

where

\[ r^2 = (x-a)^2 + (y-b)^2, \]

then

\[ \nabla^4 w_p(x,y) = \begin{cases} 0 & \text{if the point } (x,y) \text{ is outside of the region } R, \\ \frac{q(x,y)}{D} & \text{if the point } (x,y) \text{ is inside of the region } R. \end{cases} \]

The biharmonic function \( w_c(x,y) \) can be constructed conveniently from harmonic functions, for which there is a very extensive store of information. The following results are well-known:
If \( \psi_1 \) and \( \psi_2 \) are two harmonic functions in a region \( R \), then

\[
\psi = x\psi_1 + \psi_2
\]  

(4)

is a biharmonic function in the same region. Conversely, if the boundary \( L \) of \( R \) intersects every line parallel to the \( x \)-axis in at most two points, then for every biharmonic function \( \psi \) in \( R \), there exist two harmonic functions \( \psi_1 \) and \( \psi_2 \), so that \( \psi \) can be represented by the formula (4).

Similar result is true when \( x \) in the above theorem is replaced by \( y \). And furthermore, if the origin is enclosed in \( R \) and every radius vector intersects \( L \) in only one point, then every biharmonic function \( \psi \) in \( R \) can be represented by functions of the form

\[
\psi = (r^2 - r_0^2) \psi_1 + \psi_2,
\]

where

\[
r^2 = x^2 + y^2,
\]

\( \psi_1 \) and \( \psi_2 \) are two harmonic functions in \( R \), and \( r_0 \) is any preassigned constant. Conversely, every function \( \psi \) in this form is a biharmonic function in \( R \).

In the application of the method of relaxation of boundary conditions to swept plate problems, a suitable form of particular solution \( \psi_p \) is first obtained. This \( \psi_p \), while balancing the distributed normal loads over the faces of the plate, will induce bending moments and shear reactions along the edges of the plate. The next step is then to find a solution for a plate loaded along the edges, and to this, the Trefftz method is applied by taking the biharmonic functions as elementary solutions.

A remark on the places where the maximum mean stresses occur may
be useful. When the function \( q(x,y) \) is zero over a subregion \( R_4 \) of \( R \), the differential equation (3.1.1) becomes
\[
\nabla^2 (\nabla^2 w) = 0 \text{ in } R_4.
\]

(1)

Hence \( \nabla^2 w \) is a harmonic function over \( R_4 \). But a harmonic function in \( R_4 \) cannot have a maximum or minimum at a point in the interior of the region \( R_4 \), (Ref. 13, or Ref. 37). Its extremum is reached only on the boundary of the region. Now \( \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \) is the mean curvature of the bent middle surface of the plate. Hence the mean curvature of a bent plate reaches its maximum or minimum either on the edges of the plate or under the points of loading.

Thus when the external load is applied along the edges of the plate only, the mean curvature is assured to reach its maximum only on the edges of the plate. The mean curvature is proportional to the mean maximum bending stresses (e.g., \( Q_x + Q_y \)) in the plate. Hence the mean maximum bending stresses occur either under the load or on the boundary of the plate.

Similar reasoning leads to corresponding results regarding to shearing stresses. If, in a certain subregion \( R_4 \), \( q \) is a function of \( y \) alone, then
\[
\frac{\partial}{\partial x} \nabla^2 (\nabla^2 w) = \nabla^2 (\nabla^1 \frac{\partial w}{\partial x}) = \frac{1}{b} \frac{\partial^2 G(x,y)}{\partial x} = 0.
\]

But
\[
-D \frac{\partial^2 w}{\partial x^2} - Q_{xx}, \quad \text{the vertical shearing force on } x = \text{ const. sections}
\]

Hence
\[
\nabla^2 Q_{xx} = 0 \quad \text{where } q = q(y)
\]
In the same way,

\[ \nabla^2 Q_y = 0 \quad \text{where } q = q(x). \]

\(Q_x\) and \(Q_y\) are then harmonic functions in the respective cases.

Hence \(Q_x\) (or \(Q_y\)) reaches its maximum or minimum either on the edges of the plate or at points where \(\frac{\partial q}{\partial x}\) (or \(\frac{\partial q}{\partial y}\)) is different from zero.

In the application of the method of relaxation of boundary conditions, an error function is defined by the relation

\[ \varepsilon_N = w - w_N \]

where \(w_N\) is an approximating function to the true solution \(w\).

Since \(w\) and \(w_N\) both satisfy the differential equation

\[ \nabla^4 w = \frac{q}{D}, \]

so

\[ \nabla^4 \varepsilon_N = 0. \]

over the whole plate. Therefore the mean curvature of the error function reaches the maximum only on the boundary and hence the maximum mean error in stresses occurs only on the boundary of the plate.
§ 3.2 Rectangular Cantilever Plates

The deflection of a rectangular cantilever plate can be obtained by the Rayleigh-Ritz method and the method of relaxation of boundary conditions. In this section both procedures are carried out in detail. It is shown that the homogeneous part of the system of linear equations for the determination of the unknown coefficients is independent of the external loading, and can be formulated once and for all. The loading condition effects the constant terms of these equations only. Hence the sample computation is carried out only for the case of distributed shear load at the tip section.

For simplicity the coordinates \((x,y)\) are expressed in dimensionless form. The length is measured in terms of \(c\), the half-chord of the plate (Fig. 3.2:1). Without modifying the dimensions of \(w\) and \(q\), the flexural rigidity is here defined as

\[
D = \frac{4Eh^3}{3(1-\nu^2)c^4}
\]

where \(t\) is the thickness of the plate. \(E\) and \(\nu\) are the Young's modulus and Poisson's ratio of the material of the plate. With this modification, the plate dimension is shown in Fig. 3.2:2.
§ 3.21 Solution by Rayleigh-Ritz Method

The Rayleigh-Ritz method can be applied with success to find the deflection surface of a cantilever plate. According to the energy principle, the deflection \( w(x,y) \) of the plate is one of the "admissible" functions which minimizes the potential energy of the plate:

\[
V = \frac{1}{2} \int \int D \left\{ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right\} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} dx \, dy \\
- \int \int \omega \omega \, dx \, dy + \int \min \frac{\partial w}{\partial n} \, ds - \int \left( Q_n - \frac{3}{2} H \right) \omega ds,
\]

(1)

where the line integral is extended over the free edges of the plate.

A function \( w \) is called admissible if and only if it satisfies the "rigid" support conditions on the boundary, for example, \( w = 0 \) on a simply supported edge and \( \frac{\partial w}{\partial n} = 0 \) on a built-in edge.

It is assumed that \( w \) can be represented by a series of admissible functions in the form

\[
w = w_0(x,y) + \sum_{m,n} a_{mn} w_{mn}(x,y),
\]

(2)

where the coefficients \( \{ a_{mn} \} \) are determined by the minimization of the energy integral (1).

The simplest procedure is to choose \( \{ w_{mn}(x,y) \} \) which possess the best orthogonality properties, such as the normal modes of free vibration of a plate of the same form and same supporting conditions as the given plate. Since, however, the vibration modes of a cantilever plate are not conveniently expressible in elementary functions, their use is impractical.

In the following, \( w_{mn}(x,y) \) is taken in the form

\[
w_{mn}(x,y) = f_m(x) \cdot g_n(y)
\]

(3)
Where \( f_m(x) \) is the \( n^{th} \) free vibration mode of a thin elastic bar of length \( L \), clamped at the end \( x = 0 \) and free at the end \( x = L \); and \( g_n(y) \) is the \( n^{th} \) mode of a bar of length 2, with both ends \( y = \pm 1 \) free. The \( f_m \) and \( g_n \) are normalized so that

\[
\int_0^L f_m(x) f_n(x) \, dx = S_{mn}, \quad \int_{-1}^1 g_m(y) g_n(y) \, dy = S_{mn} \quad (4)
\]

where \( S_{mn} = 1 \) if \( m = n \), and \( S_{mn} = 0 \) if \( m \neq n \).

Substituting (2), (3) into (1), one obtains a quadratic function of the constants \( \{ a_{mn} \} \). When the \( \{ a_{mn} \} \) is so chosen as to minimize \( V \), the conditions

\[
\frac{\partial V}{\partial a_{mn}} = 0, \quad \text{(any } m, n \text{)} \quad (5)
\]

lead to a system of simultaneous linear equations from which \( \{ a_{mn} \} \) can be determined.

To obtain explicit formulas, let us introduce the following notations:

\[
\lambda_{mn} = \int_0^L f_m''(x) f_n''(x) \, dx, \quad \lambda_{mn} = \int_{-1}^1 g_m''(y) g_n''(y) \, dy,
\]

\[
\beta_{mn} = \int_0^L f_m'(x) f_n'(x) \, dx = \beta_{nm} , \quad \lambda_{mn} = \int_{-1}^1 g_m'(y) g_n'(y) \, dy = \lambda_{nm},
\]

\[
\gamma_{mn} = \int_0^L f_m''(x) f_n(x) \, dx, \quad \gamma_{mn} = \int_{-1}^1 g_m''(y) g_n(y) \, dy. \quad (6)
\]

and

\[
\frac{\partial V}{\partial a_{mn}} = \int_{-1}^1 \int_0^L \left\{ \frac{\partial^2 W}{\partial x^2} (f_m'' g_n + \gamma f_m g_n') + \frac{\partial^2 W}{\partial y^2} (f_m g_n'' + \gamma f_m' g_n') \right\} \, dx \, dy
\]

\[
- \int_{-1}^1 \int_0^L \frac{g_m}{f_m} g_n \, dx \, dy + \int M_n \frac{\partial}{\partial n} \left( \frac{f_m g_n}{f_n} \right) \, ds + \int (Q_n - \frac{\partial H_n}{\partial s}) f_m g_n \, ds \quad (7)
\]
Where the primes denote differentiation with respect to the independent variables, and in (7), the line integrals are extended over that part of the free edges on which external shear and moments are applied.

The aforesaid system of linear equations (5) now takes the form:

$$a^m_{nm} \left[ \alpha_{mm} + \kappa_{mn} + 2 \gamma \eta_{mm} \right] \lambda_{mn} + \sum_{ij \neq nn} a_{ij} \left[ 2 \gamma \eta_{m1} \lambda_{jn} + 2 (1 - \gamma) \beta_{mi} \lambda_{jn} \right] = -\frac{2}{B} \frac{\delta V_0}{\delta e_{nm}},$$

(any $m, n$). \hspace{1cm} (8)

The left hand side of the equations (8) are definite linear forms in $\{ a_{nm} \}$, and can be determined once and for all for a given aspect ratio of the plate. For different loading conditions, only the terms $\{ \frac{\delta V_0}{\delta e_{nm}} \}$ are different.

The functions $\{ f_m (x) \}$ and $\{ g_n (y) \}$ are solutions of the differential equation

$$\frac{d^4 f}{dx^4} = p u$$

with the boundary conditions $u = \frac{\partial u}{\partial x} = 0$ for a clamped end and $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^4 u}{\partial x^4} = 0$ for free ends. They are found to be as follows: (Ref. 5)

(I) $f_m (x)$. Normal mode of a clamped-free bar of length L, clamped at $x = 0$, free at $x = L$:

$$f_m (x) = \int \frac{1}{L} \left( -\cos \frac{P_n x}{L} + \cosh \frac{P_n x}{L} \right) - \cos P_n + \cosh P_n \left( -\sin \frac{P_n x}{L} + \sinh \frac{P_n x}{L} \right),$$

Where $\{ P_n \}$ are the roots of the equation

$$\cos P_n \cosh P_n + 1 = 0.$$ 

$P_1 = 1.875104$, \hspace{1cm} $P_2 = 4.694091$, \hspace{1cm} $P_3 = 7.854757$

$P_4 = 10.995541$, \hspace{1cm} $P_5 = 14.137168$. 
72.

\[ n > 5, \quad \Phi_n = \frac{\pi}{2} (2n - 1) \] to 6 decimal places.

For \( n > 2, \)

\[
\sqrt{L} f_n(x) = - \cos \frac{\Phi_n}{L} + [1 - (-1)^{n+1}] \cdot 2e^{-\Phi_n} \sin \frac{\Phi_n}{L} + (-)^{n+1} \cdot e^{-\Phi_n}(1- \frac{X}{L})
\]

\[
+ [1 - (-1)^{n+1}] e^{-\Phi_n} \sqrt{\frac{\Phi_n}{L}}
\]

within 5 significant figures.

(II). \( s_n(y), \) normal mode of a free-free beam of length 2, with origin at the middle-point.

If \( n \) is even:

\[
s_n(y) = \frac{\cosh q_n \cos q_n y + \cos q_n \cosh q_n y}{\sqrt{\cosh 2q_n + \cos 2q_n}}
\]

where \( \tan q_n + \tanh q_n = 0.\) (10a)

If \( n \) is odd:

\[
s_n(y) = \frac{\sinh q_n \sin q_n y + \sin q_n \sinh q_n y}{\sqrt{\sinh 2q_n - \sin 2q_n}}
\]

where \( \tan q_n - \tanh q_n = 0.\) (10b)

The roots \( q_0, \) and \( q_1 = 0 \) correspond to \( g_0(y) = \frac{1}{\sqrt{2}}, \) and \( g_1(y) = \sqrt{\frac{3}{2}} y. \) The other roots are

\[ q_2 = 2.365020, \quad q_3 = 3.926602, \quad q_4 = 5.497804, \]

\[ n > 5, \quad q_n = (n - \frac{1}{2}) \frac{\pi}{2} \] to 6 decimal places.

In general, for \( n \geq 2, \) up to four decimal places,

\[
s_n(y) = \cos \left( \frac{n}{2} - \frac{1}{4} \right) \pi y + \frac{(-)^{n/2} \cosh \left( \frac{n}{2} - \frac{1}{4} \right) \pi y}{\sqrt{2} \cos \left( \frac{n}{2} - \frac{1}{4} \right) \pi} \quad \text{for even } n. \] (10c)

\[
s_n(y) = \sin \left( \frac{n}{2} - \frac{3}{4} \right) \pi y + \frac{(-)^{n/2} \sinh \left( \frac{n}{2} - \frac{3}{4} \right) \pi y}{\sqrt{2} \sinh \left( \frac{n}{2} - \frac{3}{4} \right) \pi} \quad \text{for odd } n. \] (10d)

For small values of \( y \) and large \( n, \) the hyperbolic part of these expressions becomes negligible, and \( s_n(y) \) behaves like trigonometric
functions.

The integrals \( \alpha_{\mu n}, \beta_{\mu n}, \text{etc.} \), are found as follows:

\[ (I). \quad \alpha_{\mu n} = \begin{cases} \frac{P_m}{L^2} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \quad (11) \]

\[ (II). \quad \beta_{\mu n} = \frac{P_m P_n}{L^2} \left\{ \frac{(1+\sigma_m \sigma_n)}{2} \left[ \frac{\sin (P_m-P_n)}{P_m-P_n} + \frac{\sinh (P_m+P_n)}{P_m+P_n} \right] - \frac{(1-\sigma_m \sigma_n)}{2} \left[ \frac{\sin (P_m+P_n)}{P_m+P_n} + \frac{\sinh (P_m-P_n)}{P_m-P_n} \right] \right. \right. 
\]  
\[ + \frac{(\sigma^m - \sigma^n)}{2} \frac{\cos (P_m-P_n) + \cosh (P_m-P_n) - \sigma^m + \sigma^n}{P_m-P_n} \frac{\cos (P_m+P_n) + \cosh (P_m+P_n)}{P_m+P_n} \]
\[ + \frac{2 \sigma^m P_n + \sigma^n P_m}{P_m^2 + P_n^2} + \frac{2 \sigma^n P_m - \sigma^m P_n}{P_m^2 - P_n^2} \right\} , \]
\[ + \frac{1}{P_m^2 + P_n^2} \left[ P_m (\cos P_m \sinh P_n - \cosh P_m \sin P_n) + P_n (\cos P_n \sinh P_m - \cosh P_n \sin P_m) \right] \]
\[ + \frac{\sigma^m}{P_m^2 + P_n^2} \left[ P_m (\sin P_m \sinh P_n - \sin P_n \sinh P_m) - P_n (\cos P_m \cosh P_n - \cos P_n \cosh P_m) \right] \]
\[ + \frac{\sigma^n}{P_m^2 + P_n^2} \left[ P_n (\sin P_n \sinh P_m - \sin P_m \sinh P_n) - P_m (\cos P_n \cosh P_m - \cos P_m \cosh P_n) \right] \]
\[ + \frac{\sigma^m \sigma^n}{P_m^2 + P_n^2} \left[ P_m (\cos P_m \sinh P_n + \cosh P_m \sin P_n) + P_n (\cos P_n \sinh P_m + \cosh P_n \sin P_m) \right] \right\} , \quad (12) a \]

\[ (3) \quad \beta_{mn} = \frac{P_m}{L^2} \left\{ \frac{(1+\sigma_m^2) \sinh 2\frac{P_m}{4} - (1-\sigma_m^2) \sin 2\frac{P_m}{4} + \sigma_m^2 P_m}{4} \right. \right. 
\[ - (1-\sigma^m_m) \sin P_m \cosh P_m + (1+\sigma^m_m) \sinh P_m \cos P_m \]
\[ + \sigma^m_m \left( \frac{1}{2} - \frac{2}{2} \right) \frac{\sinh^2 P_m + \sin^2 P_m}{2} \right\} ; \quad (12) b \]

where

\[ \sigma^m_m = \frac{\cos P_m + \cosh P_m}{\sin P_m + \sinh P_m} . \quad (12) c \]
These exact formulae are needed for small values of $m$ and $n$. For moderate values of $m$, $n$, the following approximate formulae are valid to four significant figures:

For $m, n > 2$: \[
\frac{L^2}{\text{PmPn}} \beta_{mn} = \sin \left( \frac{\text{Pm-Pn}}{\text{Pm+Pn}} \right) + 2 \left( \frac{m+n}{\text{Pm+Pn}} \right) + \left[ \frac{m}{\text{Pm+Pn}} \right] e^{-\frac{2}{\text{Pm+Pn}}} \\
\frac{1}{\text{Pm}^2+\text{Pn}^2} \left\{ \left[ 1 + (-)^{m+n+1} \right] \left[ \text{Pm+Pn} + \left[ 1 + (-)^n \right] \text{Pm} + \left[ 1 + (-)^m \right] \text{Pn} \right] \right\} \\
\left[ (-)^m (-)^n \right] \left[ e^{-\text{Pn}} P_m + e^{-\text{Pm}} P_n \right] \\
\left[ (-)^m (-)^n \right] \left[ e^{-\text{Pn}} P_m - e^{-\text{Pm}} P_n \right]\]

\[ (12d) \]

$m, n > 4, \frac{L^2}{\text{Pm}} \beta_{mn} = 2 + \text{Pm} + (-)^m \left[ \frac{4}{\text{Pm}+4} \right] \text{Pm} e^{-\text{Pm}} \] \[ (12e) \]

(III). \[ \gamma_{mn} = \frac{L}{L^2} \alpha \frac{P_m}{P_m} - \beta_{mn} \] \[ (13) \]

(IV). \[ \kappa_{mn} = \int_y \frac{g_m''(y)}{g_n''(y)} \text{dy} = \begin{cases} q_m^h & \text{if } m = n, \\ 0 & \text{if } m \neq n \end{cases} \] \[ (14a) \]

(V). \[ \lambda_{mi} = \lambda_{im} = \int_y g_m'(y) g_i'(y) \text{dy} \] \[ (15a) \]

When $m$ and $i$ are both even:

\[ \lambda_{m,i} = \frac{q_m q_i}{\cosh^2 q_i} \left\{ \frac{\sin (q_m - q_i)}{q_m - q_i} - \frac{\sin (q_m + q_i)}{q_m + q_i} \right\} \]

\[ + \cos q_m \cos q_i \left[ \frac{\tanh q_m + \tanh q_i}{q_m + q_i} - \frac{\tanh q_m - \tanh q_i}{q_m - q_i} \right] \]
\[
- \frac{2 \cos q_i}{q_{m2} + q_i^2} \left[ q_{m000} q_{m} \tanh q_i + q_i \sin q_m \right] + \frac{2 \cos q_m}{q_{m2} + q_i^2} \left[ q_{m000} q_{m} \tanh q_m + q_m \sin q_i \right]. \quad (15)_b
\]

When \( m \) and \( i \) are both odd: (except for index 1, which can be easily integrated)

\[
\lambda_{m, i} = \frac{q_m q_i}{\sinh^2 q_m} \sqrt{(1 - \sin^2 q_m)(1 - \sin^2 q_i)} \left\{ \frac{\sin (q_m - q_i)}{q_m - q_i} + \frac{\sin (q_m + q_i)}{q_m + q_i} \right\}
\]

\[
+ \frac{\sin q_i}{q_{m2} + q_i^2} \left[ q_m \sin q_m \coth q_i + q_i \cos q_m \right]
\]

\[
+ \frac{2 \sin q_m}{q_{m2} + q_i^2} \left[ q_i \sin q_i \coth q_m + q_m \cos q_i \right]. \quad (15)_c
\]

\[
\lambda_{m, \text{even}} = \frac{q_m^2}{\cos \frac{q_m}{\cosh q_m}} \left\{ 1 - \frac{\sin 2 q_m}{2 q_m} + \cos^2 q_m \left[ \tanh q_m - \frac{1}{\cosh^2 q_m} \right] \right\}
\]

\[
+ \frac{2 \cos q_m}{q_m} \left[ \cos q_m \tanh q_m - \sin q_m \right]. \quad (15)_d
\]

\[
\lambda_{m, \text{odd}} = \frac{q_m^2}{1 - \sin^2 q_m} \left\{ 1 + \frac{\sin 2 q_m}{2 q_m} + \sin^2 q_m \left[ \coth q_m - \frac{1}{\sinh^2 q_m} \right] \right\}
\]

\[
+ \frac{2 \sin q_m}{q_m} \left[ \sin q_m \coth q_m + \cos q_m \right]. \quad (15)_e
\]

When one of \( m \) and \( \lambda \) is even and the other is odd, \( \lambda_{m, \lambda} = 0. \quad (15)_f
\]

\[
(\text{VI}) \quad \left( \mu_{m, i} = \int_{-1}^{1} g_m(\gamma) g_i(\gamma) d\gamma = 2 \quad g_m(1) g_i(1) - \lambda_{i, m} \right). \quad (16)
\]
The values of the constants $\alpha_{mn}$, $\beta_{mn}$ etc., are computed according to formulas (11) through (16) and listed in Table 3.21:1 through Table 3.21:6, for a range of $m$ and $n$ from 0 to 4. Coefficients with small subscripts are the more difficult ones to compute; for $m, n > 4$, approximate calculations of these coefficients can be carried out relatively easily.

Values of $f_m (L)$, $f_m (L)$, $g_n (1)$ and $g_n (1)$ together with $q_m$ and $p_m$, $q_m$, are given in Tables 3.21:7 and 3.21:8. $q_m$ is defined by equation (12)c.
Example 1. Cantilever square plate loaded by uniform shear along the tip section.

As an example consider the deflection of a square cantilever plate loaded by a uniform shear load along the tip section. It is well-known that the classical solution of St. Venant's problem of flexure of a beam of rectangular cross-section holds only for a particular distribution of the shear load along the tip, and that the ordinary engineering beam theory holds only for comparatively long beams with the width and depth of its cross-section of the same order of magnitude. The present method solves the case when the thickness of the beam is much smaller than the width under various loading conditions, which is taken as uniform shear in this example.

By symmetry, only even indices of \( g_n(y) \) will appear in the deflection function. Let us take six indeterminate coefficients:

\[
\begin{align*}
\alpha_{10}, \quad \alpha_{12}, \quad \alpha_{14}, \\
\alpha_{20}, \quad \alpha_{22}, \quad \alpha_{24},
\end{align*}
\]

so that the deflection is expressed as:

\[
w = w_0(x,y) + \alpha_{10} f_1(x)g_0(y) + \alpha_{12} f_1(x)g_2(y) + \cdots + \alpha_{24} f_2(x)g_4(y) (17)
\]

The function \( w_0(x,y) \) is taken as the deflection function given by the ordinary beam theory, so that

\[
w_0(x,y) = \frac{g}{(1-y^2)D} \left\{ L \frac{x^2}{2} - \frac{x^3}{6} \right\}. (18)
\]
where \( q \) is the load per unit length applied along the edge of the plate.

Substituting (18) into (7) and integrating, one obtains

\[
\frac{\partial V}{\partial a_{mn}} = \frac{q}{(1 - \nu^2)} \int_0^1 \int_{-1}^1 (1 - x + \nu f_m(x) g_n(y) + \nu f_m(x) g_n''(y) \int_1^1 g_n(y) \, dy - \frac{1}{p_m} \int_1^1 f_m(x) \, dx \, dy.
\]

\[
= \begin{cases} 
\frac{\sqrt{2}}{1 - \nu^2} f_m(L) q & \text{if } n = 0, \\
\frac{\nu}{1 - \nu^2} g_n'(1) \left\{ 2L \int_1^L \sigma_m - L^2 \int_1^1 f_m(x) \, dx \right\} \frac{1}{p_m} q & \text{if } n \neq 0.
\end{cases}
\]

(19)

where \( p_m \) is the root corresponding to \( f_m(x) \). Values of \( g_n'(1), f_m(L), \) and \( \sigma_m, p_m \) are listed in Table 3,21,7 and 8. For the particular case of \( L = 2 \), and \( \nu = .3 \),

\[
\frac{\partial V}{\partial a_{00}} = .19583 \, q, \quad \frac{\partial V}{\partial a_{20}} = -.19701 \, q,
\]

\[
\frac{\partial V}{\partial a_{02}} = 1.6153 \, q, \quad \frac{\partial V}{\partial a_{22}} = 5.2599 \, q,
\]

\[
\frac{\partial V}{\partial a_{34}} = -0.0817 \, q, \quad \frac{\partial V}{\partial a_{44}} = 12.445 \, q.
\]

Substituting these into equations (8), with \( m \) ranges from 1 and 2, and \( n \) from 0, 2, 4, one obtains the system of linear simultaneous equations as follows, where \( \nu \) is taken to be 0.3:

\[
\begin{align*}
1.5453 \, a_{10} + 2.7586 \, a_{12} - 6.5130 \, a_{14} - 8.1768 \, a_{22} + 19.3488 \, a_{24} &= -.19583 \, 2a_D, \\
+158.469 \, a_{10} - 32.431 \, a_{14} + 14.327 \, a_{22} - 15.302 \, a_{24} &= -1.6153 \, 2a_D, \\
-33.674 \, a_{10} + 21.8997 \, a_{14} - 11.4798 \, a_{22} + 44.8341 \, a_{24} &= 3.8217 \, 2a_D, \\
+60.690 \, a_{20} + 1.2880 \, a_{12} - 3.00006 \, a_{14} + 13.705 \, a_{22} + 121.067 \, a_{24} &= 1.9701 \, 2a_D, \\
+34.745 \, a_{12} + 12.329 \, a_{14} + 518.313 \, a_{22} - 134.445 \, a_{24} &= 5.2599 \, 2a_D, \\
-24.507 \, a_{12} + 132.12 \, a_{14} - 140.69 \, a_{22} + 3399.2 \, a_{24} &= -12.445 \, 2a_D.
\end{align*}
\]

This can be solved by iteration or otherwise. The solution is:
\[a_{10} = -.003965 \frac{D}{D}, \quad a_{12} = -.01108 \frac{D}{D}, \quad a_{14} = -.001703 \frac{D}{D},\]
\[a_{20} = .006075 \frac{D}{D}, \quad a_{22} = .01009 \frac{D}{D}, \quad a_{24} = -.003391 \frac{D}{D}\]

Hence the deflection surface is given by:

\[w(x,y) = \frac{D}{2} \left\{ \frac{1}{1 - \frac{D}{E}} \left( -0.07918 f_1(x) - 0.02216 f_1(x) g_3(y) + 0.003406 f_1(x) g_4(y) + 0.01615 f_2(x) + 0.02018 f_2(x) g_2(y) - 0.006782 f_2(x) g_4(y) \right) \right\},\]

where \(f(x)\) and \(g(y)\) are functions given by (9) and (10).
3.22. Solution by Relaxation of Boundary Conditions

The problem of the deflection of rectangular cantilever plates can be solved by the method of relaxation of boundary conditions. This is essentially a modification of biharmonic analysis to obtain approximate solutions. The deflection \( w(x,y) \) is represented by a sum of a particular integral and a complementary function. The latter is then taken as a series of biharmonic functions, the coefficients of this series are determined by minimizing a weighted mean error on the boundary of the plate. The principle of this method is discussed in Section 1.23.

The procedure is simplified if one can construct a convenient sequence of biharmonic functions which possess orthogonality properties along the boundary curve of the plate. However, the classical Schmidt process of normalization (p. 131, Ref. 25) is rather tedious for the present case. Hence a more direct, though less systematic procedure is adopted in the following.

Biharmonic Functions

Since the functions

\[
\sinh \lambda x \sin \lambda y \quad \text{and} \quad \sin \lambda x \sinh \lambda y
\]

are harmonic functions, where \( \lambda \) is any real constant, it follows that (Cf. section 3.12):

\[
(A_1 - A_2 x - A_3 y) \left\{ \begin{array}{c}
\sin \lambda x \sinh \lambda y \\
\cos \lambda x \cosh \lambda y - C \cosh \lambda x \cos \lambda y
\end{array} \right\}
\]

are biharmonic functions, where \( A, B, C \) are arbitrary constants.

The notation on the possible combinations of trigonometric and hyperbolic functions seems to be obvious.
The following functions are also biharmonic:

\[(a+bx) p^2(y), \quad (a+by) p^2(x)\]

where \(p^2(y)\) and \(p^2(x)\) are polynomials of degree not higher than third in \(y\) and \(x\) respectively. It can be verified that the most general form of biharmonic functions when the variables are separable is given by the above two types.

**Approximating Sequence**

Let \(w_0(x,y)\) be a particular integral of the differential equation

\[\nabla^4 w_0 = \frac{q}{D}\]

where \(q\) is the distributed load over the plate. This function in general does not satisfy the boundary conditions along the edges of the plate (otherwise \(w_0\) is an exact solution and the problem is solved).

For convenience, however, assume that \(w_0\) satisfies the boundary conditions along the built-in edge of the plate.

The following examples are carried out for the case when the loading and hence the deflection of the plate is symmetric about the \(x\)-axis. The anti-symmetric case can be similarly worked out.

The deflection surface is assumed to be expressible in series of the form:

\[w = w_0 + \sum_{\lambda} A_{\lambda} x \sinh \lambda x \cos \lambda y + \sum_{\mu} B_{\mu} x \sin \mu x \cosh \mu y, \quad (1)\]

where \(\lambda, \mu\) are constants. In this form the boundary conditions along the edge \(x = 0\),

\[w = \frac{\partial w}{\partial x} = 0,\]

are satisfied. Since this is the edge where the maximum stress occurs, it seems reasonable to have boundary conditions on this edge satisfied exactly to start with,
Boundary Functions.

The boundary functions $B_1(w)$ and $B_2(w)$ (Cf. section 1.23) are defined as functions along the boundary:

$$B_1(w) = \begin{cases} \mathcal{D} \left[ \frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2} \right] + M_n & \text{on unsupported edges}, \\ \mathcal{P} \frac{\partial w}{\partial n} & \text{on built-in edges} \end{cases}$$

$$B_2(w) = \begin{cases} -\mathcal{D} \left[ \frac{\partial^2 w}{\partial n^2} + (1-\nu) \frac{\partial^2 w}{\partial s^2} \right] - \left( Q_n + \frac{\partial H_n}{\partial n} \right) & \text{on free edges}, \\ \mathcal{P} w & \text{on supported edges} \end{cases}$$

where $n$ and $t$ denote the normal and tangential directions on the boundary and $M_n$, $Q_n$, $H_{nt}$ the external moment, shear, and twisting couple acting on the boundary respectively. Specifically,

$$B_1^{x_1}(w) = B_1(w) \quad \text{on free edge perpendicular to x-axis}$$

$$= \mathcal{D} \left[ \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right] - M_x$$

$$B_1^{y_1}(w) = B_1(w) \quad \text{on free edges perpendicular to y axis}$$

$$= \mathcal{D} \left[ \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right] - M_y$$

and so on.

Similarly let the following notations be used:

$$B_1^{x_{i_1}}(w) = \begin{cases} B_1^{x_1}(w) \end{cases}_{x=1} \quad \text{and} \quad B_2^{x_{i_1}}(w) = \begin{cases} B_2^{x_1}(w) \end{cases}_{x=1}$$

$$B_1^{y_{i_1}}(w) = \begin{cases} B_1^{y_1}(w) \end{cases}_{y=1} \quad \text{and} \quad B_2^{y_{i_1}}(w) = \begin{cases} B_2^{y_1}(w) \end{cases}_{y=1}$$

$$B_1^{side_{i_1}}(w) = \begin{cases} B_1^{side}(w) \end{cases}_{y=\pm 1} \quad \text{and} \quad B_2^{side_{i_1}}(w) = \begin{cases} B_2^{side}(w) \end{cases}_{y=\pm 1}$$

By symmetry \( B_1^{y_{i_1}}(w) \) means \( B_1^{y_1}(w) \) on either side of the plate: \( y = \pm 1 \).

Of course, all the boundary functions vanish for exact solutions.

When \( w \) is not an exact solution, the boundary functions represent the error on the boundary.

According to (1) and (2), by substitution, one obtains
\[ B_{1}^{\text{tip}}(w) = B_{1}^{\text{tip}}(w_{0}) + D \sum_{\lambda} A_{\lambda} \left[ (1-\nu) \lambda^{2} \sinh \lambda L - 2 \lambda \cosh \lambda L \right] \cos \gamma \]
\[ + D \sum_{\mu} B_{\mu} \left[ (\nu-1) \mu^{2} \lambda \sin \mu L - 2 \mu \cosh \mu L \right] \cos \mu \gamma \]
\[ B_{2}^{\text{tip}}(w) = B_{2}^{\text{tip}}(w_{0}) - D \sum_{\lambda} A_{\lambda} \left[ (\nu-1) \lambda^{2} \cosh \lambda L + (1+\nu) \lambda \sinh \lambda L \right] \cos \lambda \gamma \]
\[ - D \sum_{\mu} B_{\mu} \left[ (1-\nu) \mu^{2} \lambda \cosh \mu L - (1+\nu) \mu \sin \mu L \right] \cos \mu \gamma \]
\[ B_{1}^{\text{side}}(w) = B_{1}^{\text{side}}(w_{0}) + D \sum_{\lambda} A_{\lambda} \left[ (\nu-1) \lambda^{2} \sinh \lambda x + 2 \nu \cos \lambda \cosh \lambda x \right] \]
\[ - D \sum_{\mu} B_{\mu} \left[ (1-\nu) \mu \lambda \sin \mu x + 2 \mu \cos \mu \cosh \mu x \right] \]
\[ B_{2}^{\text{side}}(w) = B_{2}^{\text{side}}(w_{0}) - D \sum_{\lambda} A_{\lambda} \left[ (\nu-1) \lambda^{2} \sin \lambda x - 2(1-\nu) \lambda \sin \lambda \cosh \lambda x \right] \]
\[ - D \sum_{\mu} B_{\mu} \left[ (\nu-1) \mu^{2} \sin \mu x - 2 (1-\nu) \mu \sin \mu \cosh \mu x \right] \]

Equations for the Determination of the Unknown Coefficients

The equations for the determination of the unknown coefficients \{A_{\lambda}\} and \{B_{\mu}\} can be formed in various ways according to section 1.23. The following is one of the simplest ways for the present problem:

\[ \int_{0}^{1} B_{1}^{\text{tip}}(w, L) \sinh \lambda L \cdot \cos \lambda y dy + \int_{0}^{L} B_{1}^{\text{side}}(w) \times \sinh \lambda x (\cos \lambda) dx = 0 \]
\[ \int_{0}^{1} B_{2}^{\text{tip}}(w, L) \sinh \lambda L \cdot \cos \gamma dy + \int_{0}^{L} B_{2}^{\text{side}}(w) \times \sin \lambda x \cdot \cosh \lambda \cdot dx = 0 \]
\[ \int_{0}^{1} B_{1}^{\text{tip}}(w, L) \sin \mu L \cdot \cosh \mu y dy + \int_{0}^{L} B_{1}^{\text{side}}(w) \times \sin \mu x \cdot \cosh \mu \cdot dx = 0 \]
\[ \int_{0}^{1} B_{2}^{\text{tip}}(w, L) \sin \mu L \cdot \cosh \mu y dy + \int_{0}^{L} B_{2}^{\text{side}}(w) \times \sin \mu x \cdot \cosh \mu \cdot dx = 0 \]

This gives a system of linear simultaneous equations in \{A_{\lambda}\} and \{B_{\mu}\} from which these constants can be determined.

Example 4. Rectangular Plates with Uniform Shear Load at the Tip Section

For a cantilever plate loaded at the tip section (See Fig. 3,3:1) with a uniform load of \( w \) per unit length, the particular solution \( w_{0}(x,y) \) may be obtained by considering the plate as a cantilever beam:

\[ w_{0}(x,y) = \frac{q}{6D} \left\{ (L-x)^{3} - L^{2} (L-3x) \right\} \]
This satisfies all the boundary conditions except the bending moment along the edges \( y = \pm 1 \). that is, 
\( B_1^{\text{clp}}(\nu_0) = B_2^{\text{clp}}(\nu_0) = B_1^{\text{side}}(\nu_0) = B_2^{\text{side}}(\nu_0) = 0 \), but 
\( B_1^{\text{side}}(\nu_0) = \frac{\nu_0 (L-x)}{D} \).

Let four indeterminate constants be taken for the first approximation. Let \( \{ \lambda \} \) take the values \( \{ n \pi \} \) and \( \{ \rho \} \) the values \( \{ \frac{n \pi}{L} \} \), where \( \{ n \} \) are positive integers. Then the approximating deflection function is

\[
w = \frac{q}{6p} \left\{ (L-x)^3 - \frac{3}{2} (L-3x)^2 \right\} + A_1 x \sinh \frac{\pi x}{L} \cos \frac{\pi y}{L} + A_2 x \sinh \frac{2 \pi x}{L} \cos \frac{2 \pi y}{L} \\
+ B_1 \sin \frac{\pi x}{L} \cosh \frac{\pi y}{L} + B_2 \sin \frac{2 \pi x}{L} \cosh \frac{2 \pi y}{L} \).
\]

(7)

\( A_1, A_2, B_1, B_2, \) are to be determined from the boundary conditions.

The condition that there is no resultant force at the free corners demands that 

\[
\frac{3}{x \sinh \frac{\pi y}{L}} = 0 \quad \text{for} \quad x = L , \ y = \pm 1 .
\]

This condition imposes the restriction that

\[
B_2 = \frac{\sinh \frac{\pi y}{L}}{4 \sinh \frac{2 \pi y}{L}} B_1 .
\]

(8)

Hence in fact, only three unknown constants are to be determined.

The linear equations for the determination of the constants \( \{ \lambda \} , \{ \rho \} \) are formed according to equations (5). In the present example they are the following:

\[
L \sin \pi L \int_0^L B_1^{\text{clp}}(w) \cos \pi y \cdot dy + \int_0^L B_1^{\text{side}}(w) x \sinh \pi x \cdot dx = 0 ,
\]

\[
\int_0^L B_1^{\text{side}}(w) x \sin \frac{\pi y}{L} dx = 0 , \quad (9)
\]

\[
L \sin \pi L \int_0^1 B_2^{\text{clp}}(w) \cos \pi y \cdot dy + \int_0^L B_2^{\text{side}}(w) x \sinh \pi x \cdot dx = 0 .
\]
Let us write

\[
\frac{1}{D} B_{2}^{\text{cl}}(\nu) = C_{A1} A_{1} \cos y + C_{A2} A_{2} \cos 2\pi y + C_{B1} B_{1} \cosh \frac{\nu}{L} + C_{B2} k B_{1} \cosh \frac{2\nu}{L},
\]

\[
\frac{1}{\nu} B_{2}^{\text{cl}}(\nu) = C_{A1}' A_{1} \cos y + C_{A2}' A_{2} \cos 2\pi y + C_{B1}' B_{1} \cosh \frac{\nu}{L} + C_{B2}' k B_{1} \cosh \frac{2\nu}{L},
\]

\[
\frac{1}{D} B_{1}^{\text{side}}(\nu) = \frac{\nu}{D} (L-x) + k_{A1} A_{1} x \sinh \pi x + 2 \nu A_{1} \cosh \pi x + k_{A2} A_{2} x \sinh 2\pi x + 2 \nu A_{2} \cosh 2\pi x + k_{B1} B_{1} x \sinh \frac{\nu}{L} + 2 \nu B_{1} \cosh \frac{\nu}{L} + l_{B1} B_{1} \cos \frac{\nu}{L} + k_{B2} k B_{1} \cos \frac{2\nu}{L} + \frac{\nu}{D} (L-x) + k_{B1} B_{1} x \sinh \frac{\nu}{L} + 2 \nu B_{1} \cosh \frac{\nu}{L} + k_{B2} k B_{1} \cos \frac{2\nu}{L} + \frac{\nu}{D} (L-x) + k_{B1} B_{1} x \sinh \frac{\nu}{L} + 2 \nu B_{1} \cosh \frac{\nu}{L} + k_{B2} k B_{1} \cos \frac{2\nu}{L},
\]

where

\[
C_{A1} = (1-\nu) L \pi^2 \sinh \pi L + 2 \pi \cosh \pi L,
\]

\[
C_{A2} = (1-\nu) 4 \pi^4 L \sinh 2\pi L + 4 \pi \cosh 2\pi L,
\]

\[
c_{A1}' = (1-\nu) \pi^3 L \cosh \pi L - (1+\nu) \pi^3 \sinh \pi L,
\]

\[
c_{A2}' = (1-\nu) \pi^3 L \cosh 2\pi L - (1+\nu) \pi^3 \sinh 2\pi L,
\]

\[
c_{B1} = -2 \frac{\pi}{L}, \quad c_{B2} = 4 \frac{\pi}{L}, \quad c_{B1}' = (1-\nu) \frac{\pi^3}{L}, \quad c_{B2}' = -(1-\nu) \frac{8 \pi^3}{L^2},
\]

\[
k_{A1} = (1-\nu) \pi^3, \quad k_{A2} = -4 (1-\nu) \pi^3, \quad k_{B1} = (1-\nu) \cosh \frac{\nu}{L}, \quad k_{B2} = -(1-\nu) \cosh \frac{2\nu}{L},
\]

\[
k_{B1}' = (1-\nu) \frac{\pi^3}{L} \sinh \frac{\nu}{L}, \quad k_{B2}' = (1-\nu) \frac{8 \pi^3}{L^2} \sinh \frac{2\nu}{L},
\]

\[
l_{B1} = 2 \nu \frac{\pi}{L} \cosh \frac{\nu}{L}, \quad l_{B2} = 4 \nu \frac{\pi}{L} \cosh \frac{2\nu}{L}, \quad l_{B1}' = -2 (2-\nu) \frac{\pi^3}{L} \sinh \frac{\nu}{L},
\]

\[
l_{B2}' = -2 (2-\nu) \frac{8 \pi^3}{L^2} \sinh \frac{2\nu}{L}, \quad k = \sinh \frac{\nu}{L} \sqrt{4 \sinh \frac{2\nu}{L}}.
\]

Then the explicit expression of the system of equations (9) becomes:

\[
\frac{\nu}{D} (I) + \left[ L \sinh (\pi L) \frac{C_{A1}}{2} + k_{A1} (\nu) - 2 \nu (\nu) \right] A_{1} + \left[ k_{A2} (\nu) + 2 \nu (\nu) \right] A_{2}
\]

\[
+ \left[ L \sinh (\pi L) \left\{ C_{B1} (\alpha) + C_{B2} L (\beta) \right\} + k_{B1} (\nu) + l_{B1} (\nu) \right] B_{1} = 0,
\]

\[
\frac{\nu}{D} (II) + \left[ k_{A1} (\nu) - 2 \nu (\nu) \right] A_{1} + \left[ k_{A2} (\nu) + 2 \nu (\nu) \right] A_{2}
\]

\[
+ \left[ k_{B1} (\nu) + l_{B1} (\nu) + k_{B2} L (\nu) + l_{B2} L (\nu) \right] B_{1} = 0,
\]
\[ L \sinh \frac{\pi L}{c_1} A_1 + \int L \sinh \frac{\pi L}{c_1} \left\{ c_{an} (\alpha) + c_{bn} k (\beta) \right\} + k'_{bl} (\gamma) \]
\[ + l_{bl} (\gamma) + k'_{bl} k (\gamma) + l_{bl} k (\gamma) \right\} B_i = 0, \]

where \((\alpha), (\beta), (I), (II), (III)\) etc. are integrals listed in Table 3.22.1.

**Sample Computation of a Square Cantilever Plate.**

As a particular case consider a square cantilever plate, \(L = 2\).

When the Poisson's ratio \(\nu\) is taken as 0.3, the value of the constants listed in equations (11) are then as follows:

\[
\begin{align*}
  k &= .048395, \\
  c_{an} &= 10.0503 e^{2\pi}, \quad c_{bn} = 92.9181 e^{4\pi}, \quad c_{an}' = 16.2891 e^{2\pi}, \quad c_{bn}' = 147.9742 e^{4\pi} \\
  c_{bl} &= -3.14159, \quad c_{bn} = 6.28318, \quad c_{bn}' = 5.42569, \quad c_{bn}' = -4.34087, \\
  k_{al} &= 6.90872, \quad k_{al} = -27.6349, \quad k_{bl} = 1.75643, \quad k_{bl} = 8.11443, \\
  k_{bl}' &= 6.24534, \quad k_{bl}' = 250.6602, \\
  l_{bl} &= 2.35485, \quad l_{bl} = 21.8505, \quad l_{bl}' = 19.3069, \quad l_{bl}' = -387.5397.
\end{align*}
\]

Then equations (12) become, when the numerical values are substituted from Table 3.22.1,

\[
\begin{align*}
  5.919634 e^{2\pi} A_1 &= 2.607414 e^{4\pi} A_2 + .116980 B_1 = -.0690695 \frac{\pi^2}{D}, \\
  .963228 e^{4\pi} A_1 &= 1.402525 e^{4\pi} A_2 + .501483 B_1 = -1.03205 \frac{\pi^4}{D}, \\
  7.64455 e^{2\pi} A_1 &= 0 + .731412 B_1 = 0.
\end{align*}
\]

The solution is easily found to be

\[
\begin{align*}
  e^{2\pi} A_1 &= .0044641 \frac{\pi}{D}, \\
  e^{4\pi} A_2 &= .207136 \frac{\pi}{D} \\
  B_1 &= -.0466575 \frac{\pi}{D}, \\
  B_2 &= .022580 \frac{\pi}{D}.
\end{align*}
\]
Hence the deflection surface is approximately:

\[
\begin{align*}
\omega &= \frac{1}{D} \left\{ \frac{L}{3} + \frac{L^3}{3} \right\} - 0.044641 e^{-2\kappa} x \sinh \kappa x \cos \kappa y \\
&\quad + 0.207136 e^{-2\kappa} x \sin 2\kappa x \cos 2\kappa y \\
&\quad - 0.0466575 x \sin \frac{\kappa \pi}{L} \cos \frac{\kappa \pi}{L} - 0.0028580 x \sin \frac{2\kappa \pi}{L} \cos \frac{2\kappa \pi}{L}.
\end{align*}
\]

At the tip section, \( x = L \):

\[
\omega = \frac{1}{D} \left\{ \frac{L^3}{3} + 0.0044641 \cos \pi y + 0.207136 \cos 2\pi y \right\}.
\]

This is perhaps less accurate than the previous result obtained by the Rayleigh-Ritz method, since only three independent coefficients have been taken. As a conclusion we may say that in solving the plate problem the method of relaxation of boundary conditions is less accurate and more tedious than the procedure described in Section 3.21. Accordingly the Rayleigh-Ritz method is used in the next section to solve the problems of deflection of swept plates.
3.3 Swept Cantilever Plates

With the coordinate system and notations used in Section (1,13), the potential energy integral (3.21:1) becomes

\[
V = \frac{1}{2} \int_D \int_{\frac{\pi}{2}}^{\pi} \left( \frac{2\pi}{2\pi} - \sin \theta \frac{2\pi}{2\pi} + \frac{2\pi}{2\pi} \right) \left( \sin \left( -\frac{2\pi}{2\pi} \right) \cos \theta \frac{2\pi}{2\pi} \right) \frac{1}{2} d\theta d\phi
- \int_{\frac{\pi}{2}}^{\pi} q \cos \phi d\theta d\phi + \int_{\frac{\pi}{2}}^{\pi} \int_{\frac{\pi}{2}}^{\pi} \left( \mathcal{M}_n \frac{\partial w}{\partial n} + \mathcal{Q}_n - \frac{\partial \mathcal{H}_n}{\partial n} \right) \psi d\theta d\phi
\]

where \( R \) is extended over the whole plate and \( L \) over the free edges; \( \mathcal{M}_n, \mathcal{H}_n \), and \( \mathcal{Q}_n \) being the externally applied couples and shear on the edges. The sweep angle is \( \theta \).

The same method used in Section (a,21) can be applied here. The boundary conditions on displacements at the restrained edge are

\[
w = 0 \quad \text{and} \quad \frac{\partial w}{\partial \phi} = 0 \quad \text{when} \quad \phi = 0
\]

Any function \( w \) continuous (D,4) and which satisfies (2) is called "admissible". The solution of a loaded swept cantilever plate is given by one of the admissible functions which minimises the potential energy.

As before, write

\[
w (r, \theta) = \tilde{w} (r, \theta) + \sum_{m,n} w_{mn} (r, \theta)
\]

where \( \tilde{w} (r, \theta) \) is an approximate solution. Of course all \( \tilde{w} \) and \( \{ w_{mn} \} \) must be admissible functions. Furthermore, write

\[
w_{mn} (r, \theta) = f_m (r) q_n (\theta)
\]

where \( \{ f_m (r) \} \) and \( \{ q_n (\theta) \} \) are the normal modes of a vibrating bar as defined in Section (3.21) and given by equations (3.21:4) and 3,21:10).

Obviously \( w_{mn} \) in this form satisfies the supporting conditions (3.3:2).

Using the definitions adopted in Section (3.21) for the symbols \( \zeta_{mn}, \beta_{mi} \).
\[ \gamma_{mi}, \kappa_{mi}, \lambda_{mi}, \mu_{mi}, \] and in addition, let
\[ \psi_{mi} = \int_0^L f_m f_i \, dx, \quad \pi_{mi} = \int_0^L g_m g_i \, dy, \]
\[ \tau_{mi} = \int_0^L f_m f_i \, dx, \quad \xi_{mi} = \int_0^L g_m g_i \, dy. \]
(5)

Substituting (3.3.3) into (3.3.1) and equating its partial derivatives with respect to \(a_{mn}\) to zero, one obtains a system of linear simultaneous equations for the determination of \(\{a_{mn}\}\) as follows:
\[ 2 A_{mn} \left[ \alpha_{mn} + \kappa_{mn} + 2 (1 + \sin \theta - \nu \cos \phi) \beta_{mn} \lambda_{mn} + 2 (\sin \theta + \nu \cos \phi) \gamma_{mn} \right] 
- 4 \sin \theta \left[ \psi_{mn} + \pi_{mn} \right] 
+ \sum_{i \neq \pm n} a_{ij} \left[ 2 (1 + \sin \theta - \nu \cos \phi) \beta_{mi} \lambda_{jn} + 2 (\sin \theta + \nu \cos \phi) \psi_{mi} \psi_{jn} \right] 
- 4 \sin \theta \left[ \psi_{mn} \lambda_{jn} + \pi_{mn} \pi_{jn} \right] = - \frac{2 \cos \theta}{\nu} \frac{\partial^2 \nu}{\partial a_{mn}}, \] (6)

where
\[ \frac{\partial \nu}{\partial a_{mn}} = \frac{D}{\cos \theta} \int_0^L \int_0^L f_m g_i \left[ \frac{\partial \nu}{\partial x} - 2 \sin \frac{\partial \nu}{\partial y} + (\sin \theta + \nu \cos \phi) \frac{\partial^2 \nu}{\partial y^2} \right] 
+ 2 f_m \frac{\partial g_i}{\partial y} \left[ - \sin \theta \frac{\partial \nu}{\partial x} \right] + (1 + \sin \phi - \nu \cos \phi) \frac{\partial^2 \nu}{\partial y^2} \right] 
+ f_m \frac{\partial g_i}{\partial y} \left[ (\sin \theta + \nu \cos \phi) \frac{\partial^2 \nu}{\partial y^2} - 2 \sin \theta \frac{\partial \nu}{\partial y} \right] 
- \cos \theta \left[ \int_0^L f_m g_i \, dx \right] \frac{\partial^2 \nu}{\partial x^2} + \int_0^L f_m \frac{\partial^2 \nu}{\partial x^2} \, dx - \int_0^L \left[ \frac{\partial \nu}{\partial y} \right] f_m \frac{\partial g_i}{\partial y} \, dx \] (7)

These equations reduce correctly to (3.2.17) and (3.2.18) when \(\theta = 0\).

The expressions for \(\alpha_{mn}, \beta_{mi}, \gamma_{mi}, \kappa_{mn}, \lambda_{mi}, \mu_{mi}\) are given before in Section 3.21. Those for \(\psi_{mi}, \tau_{mi}, \pi_{mi}, \xi_{mi}\) are given below.

(1) \[ \psi_{mi} = \int_0^L f_m f_i \, dx = f_m (L) f_i (L) - \psi_{im}, \] (8a)
\[ \frac{L^3}{p_m^2} \psi_{mm} = \frac{1 + \sigma_m^2 \cos \frac{2\pi m}{L}}{2} - \frac{1 - \sigma_m^2 \cos \frac{2\pi m}{L}}{2} \cos \frac{2\pi m}{L} + \frac{3\sigma_m^2}{2} + \frac{\sigma_m}{2} \left( \sin \frac{\pi m}{L} - \sin \frac{2\pi m}{L} \right) \\
+ \sin p_m \sinh p_m \sigma_m \left( \sinh p_m \cos p_m - \sin p_m \cosh p_m \right) \]  
\text{for } m \neq 1 \tag{8}_c \]

For \( m > 4 \), to four significant figures: \[
\frac{L^3}{p_m^2} \psi_{mm} = 2 + (-1)^m e^{-p_m}.
\]

\[ (\Pi) \quad \tau_{\ell m} = \int f_j \phi \ dx = f_{\ell m} (L) \frac{\sinh \phi}{\cosh \phi} - \tau_{\ell m} \]

\[ \frac{1}{p_{i+i}^{+}} \tau_{i+i} = \frac{1 + \sigma_m^2 \cos \left( \frac{\pi}{L} - \frac{p_{i+i}}{p_{i+i}^+} \right)}{2} - \frac{1 - \sigma_m^2 \cos \left( \frac{\pi}{L} + \frac{p_{i+i}}{p_{i+i}^+} \right)}{2} \cos \left( \frac{\pi}{L} - \frac{p_{i+i}}{p_{i+i}^+} \right) \]

\[ + \frac{1}{p_{i+i}^{+}} \left\{ \begin{array}{l}
\sin \phi \cos \phi \cosh \phi \sinh \phi + \cos \phi \sin \phi \cosh \phi - \cosh \phi \sin \phi \cosh \phi \cosh \phi - 2 \\
- \sigma_m \left[ \sin \phi \cosh \phi \sin \phi \cosh \phi \cosh \phi \right] + \sigma_m \left[ \cos \phi \sinh \phi \cosh \phi \sinh \phi \cosh \phi \right] + \sigma_m \left[ \cos \phi \cosh \phi \sin \phi \cosh \phi \cosh \phi \right]
\end{array} \right\} \]

\[ \frac{L}{p_{i+i}^{+}} \tau_{i+i} = \frac{1 + \sigma_m^2 \cos \frac{2\pi m}{L}}{4} - \frac{1 - \sigma_m^2 \cos \frac{2\pi m}{L}}{4} \cos \frac{2\pi m}{L} + \frac{3\sigma_m^2}{2} \left[ \sin \frac{\pi m}{L} + \sin \frac{2\pi m}{L} \right] \\
+ \sigma_m \left( \cos \frac{\pi m}{L} + \cos \frac{2\pi m}{L} \right) \]  
\text{for } m > 4, \quad \tau_{i+i} = \frac{2}{L}. \tag{10}_a \]

\[ (\text{III}) \quad \gamma_{i+i} = \int_{a}^{b} g_0 \gamma_0^i \ dy = 2 \ell_{0} (L) \frac{\phi_i (1) \phi_i (1)}{g_{i+i}^2} - \tau_{i+i} \]  
\text{when } m \text{ and } i \text{ are both even numbers or both odd numbers, } \tau_{i+i} = 0. \tag{10}_b \]

When \( m \) is even and \( i \) is odd,

\[ \frac{1}{g_{i+i}} \tau_{i+i} \gamma_{i+i} = \frac{l}{\sqrt{(1 + \cosh \frac{2\pi m}{L}) (1 - \sin \frac{2\pi m}{L})}} \left\{ \begin{array}{l}
\frac{\sin (q_{m-2i})}{q_{m-2i}} - \frac{\sin (q_{m+2i})}{q_{m+2i}} \\
\frac{\sinh (q_{m-2i})}{q_{m-2i}} + \frac{\cosh (q_{m+2i})}{q_{m+2i}} - 1
\end{array} \right\} + \cos q_{m-2i} \left[ \tanh q_{m+2i} + \frac{1}{\tanh q_{m+2i} - 1} \right] \]
\[ \frac{2q_m}{q_m^2 + q_i^2} \left( \cos q_i \cos q_m \tanh q_m - \sin q_i \sin q_m \coth q_i \right) \]  \hspace{2cm} (10b)

(IV) \[ \zeta_{m,i} = \int_{-h}^{h} q_m q_i \, dy \quad = 2 \frac{q_m}{p} (1) \frac{q_i}{(1)} - \tilde{\zeta}_{m,i} \] \hspace{2cm} (11a)

Similar to \( \Pi_{m,i} \), \( \zeta_{m,i} \) vanishes when \( m, i \) are both even or both odd. When \( m \) is even and \( i \) is odd,

\[ \frac{1}{q_i} \zeta_{m,i} = \frac{1}{\sqrt{1 + \cos^2 q_m \cosh^2 q_i}} \sqrt{1 - \sin^2 q_i \sinh^2 q_i} \left\{ \frac{\sin (2m + 2i)}{2m - 2i} + \frac{\sin (2m + 2i)}{2m + 2i} \right\} \]

\[ + \frac{2}{p + h} \left[ 2 q_i \sin q_i \cos q_m + q_i \left( \cos q_m \cos q_i \tanh q_m + \sin q_m \sin q_i \coth q_i \right) \right] \]

\[ + \cos q_m \sin q_i \left( \frac{-\tanh q_m \coth q_i + 1}{2m - 2i} + \frac{-\tanh q_m \coth q_i - 1}{2m + 2i} \right) \]. \hspace{2cm} (11b)

The values of the constants \( \Psi_{m,i}, \tau_{m,i} \) etc., are computed according to the above formulas and listed in Table 3.3.1 through Table 3.3.4, for a range of \( m \) and \( i \) from 0 to 4. Coefficients with larger subscripts can be computed relatively easily.

**Empirical Results**

The choice of the first approximation \( w_o(x,y) \) is guided by two principles: It should be simple and general enough so that \( w_o \) can be easily constructed for various loading conditions, and it should be powerful enough to take care of a major part of the deflection. For this purpose any empirical information would be invaluable.

From the test results obtained at GARCIT Structural Laboratory on the deflection of thin elastic swept plates (Ref. 52), it has been pointed out that the deflection of a swept cantilever plate at a distance fairly far away from the built-in edge can be approximated by assuming
the plate to be built-in along a fictitious line A-A perpendicular
to the longitudinal axis of the plate (Fig. 3.3:1). The point of
intersection (point D in Fig. 3.3:1) of this line A-A with the actual
built-in edge BC varies with the sweep angle and loading conditions.
This variation, however, is not conveniently predictable.

The following approximate theory, together with an experimentally
determined empirical factor, seems to give desirable results.

Consider the swept cantilever plate as being formed by a series
of elementary symmetrical rectangular beams of length $dx$ each, set one
against the other so that in the limit they approach the form of the
given plate (Fig. 3.4:2). Assume that, for each of these elementary
beams, the ordinary beam formulae for bending and torsion hold. In
the swept beam, however, it is evident that the stress distribution is
no longer uniform across the width of the beam. To take care of the
effect of this non-uniform stress distribution, an effectiveness factor $k$
is assumed, so that when the actual width $c$ of the elementary beam is
reduced to $kc$, the result will give a correct estimation of deflection.

With the notations and sign conventions illustrated in Fig. 3.3:2,
and according to the assumption on torsion formula above, the torsional
deflection of any elementary beam is given by

$$d\varphi = \frac{T}{GJ'} \, dx$$

where $GJ'$ is the modified torsional rigidity of the beam. The deflec-
tion at $(x, y)$ due to the twist of the element at $\xi$ is

$$dw = (y - \eta) \, d\varphi = (y - \eta) \, \frac{T}{GJ'} \, d\xi.$$  

The total deflection at $(x, y)$ due to the torsion of the whole
plate is then
\[ w = \int_{\xi}^{\eta} \frac{\xi \gamma}{T} \frac{M(\xi')}{E I(\xi')} \, d\xi' \]

The deflection due to bending moment is given by
\[ \frac{\partial w}{\partial \xi} = -\frac{M}{E I}, \]

where EI' is the modified rigidity of the beam and M the bending moment at the section. Since the slope and deflection of the beam at the built-in edge \( x = 0 \) are both zero, so a repeated integration gives
\[ w = -\int_{\xi}^{\eta} ds \int_{\xi}^{s} \frac{M(\xi')}{E I'(\xi')} \, d\xi'. \]

Therefore, the deflection of the plate is given by superposing (12) and (13):
\[ w(x, y) = -\int_{\xi}^{\eta} ds \int_{\xi}^{s} \frac{M(\xi')}{E I'(\xi')} \, d\xi' + \int_{\xi}^{\eta} \left[ y - q(\xi') \right] \frac{T(\xi')}{E J'(\xi')} \, d\xi'. \]

The function \( w(x, y) \) obtained above is linear in \( y \). The second term on the right-hand side of (14) in general does not satisfy the boundary condition \( \left( \frac{\partial w}{\partial \xi} \right)_{\xi=0} = 0 \) at the built-in edge. This is due to the fact that the free torsion formula is used in computing the torsional deflection. Actually the built-in end is restrained from warping. However, from the results of Timoshenko (Ref. 49), Sanft (Ref. 50), and Böpple (Ref. 51) it appears that the effect of end-restraint dies out exponentially as the distance from that end increases. The effect of end-restraint amounts approximately to multiplying the second term in (14) by a factor
\[ (1 - e^{-\lambda x}), \]

where \( \lambda \) is a constant, and then \( w(x, y) \) given by equation (14) satisfies all the boundary conditions at the built-in edge. The correction
due to the factor $e^{-\lambda x}$ is, however, quite negligible at finite distances from the built-in edge. Hence, with this understanding, the function $w(x,y)$ given by equation (14) may be taken as the first approximation $w_0(x,y)$ in the Rayleigh-Ritz procedure.

Returning to the effectiveness factor, the modified rigidities for narrow rectangular cross-sections are given by

$$EI' = E \frac{k c t^3}{12}, \quad \text{and}$$

$$GJ' = G \frac{k c t'}{3}.$$  \hspace{1cm} (15)

By comparison with experimental results an approximate value

$$k = \cos^2 \theta$$  \hspace{1cm} (16)

is obtained, where $\theta$ represents the sweep angle. That this formula gives fairly good results is illustrated by Fig. (3.3:1).

Several particular cases for cantilever plates of uniform thickness are listed below:

(I) Pure bending, \hspace{1cm} \( w = - \int_0^x ds \int_0^s \frac{M(z')}{Ei'} \frac{dz'}{t} \). \hspace{1cm} (17)

If $M$ = const., then $w = - \frac{M x^2}{2 E i'}$.

(II) Pure torsion \hspace{1cm} \( w = \int_0^y \frac{G J'}{G J} \frac{T(z)}{y} dz \). \hspace{1cm} (18)

If $T$ = const., \hspace{1cm} \( w = \frac{T}{G J'} \left[ y x - \frac{4 a \theta x^3}{2} \right] \).

(III) Concentrated load at the tip section:

If the coordinates of the point of application of the concentrated load $P$ is $(X,Y)$, then

$$w(x,y) = \frac{P}{E I'} \left\{ \frac{x^2}{2} - \frac{x^3}{6} \right\} + \frac{P}{G J'} \left\{ -a \theta \frac{x^3}{3} - (Y + \gamma) \frac{4 a \theta x^3}{2} + \gamma y \right\}. \hspace{1cm} (19)$$

(IV) Uniformly loaded

If $q$ = load per unit area, and $b$ and $c$ the span and chord respectively, then
\[ w(x,y) = \frac{3c_x^4}{24E1} + \frac{3c_x^4}{24E1} \left[ -\frac{\tan \theta}{4} \left( x^4 - \frac{8}{9} b x^3 + 2b^2 x^2 \right) + \left( \frac{x}{3} - b^2 x + b^2 x \right) \right]. \] (20)

In actually carrying out the computation, it is often advantageous to expand \( w_0(x,y) \) and its derivatives into series in terms of functions \( \{f_m(x)\} \), \( \{g_n(y)\} \) as defined in connection with equation (4).

**Example. Cantilever Swept Plate Under Uniform Shear Load Along the Tip Section.**

As an application of this method, the deflection of a cantilever swept plate under uniform shear load along the tip section is carried out. The angle of sweep is denoted by \( \theta \), and the load is applied along the edge EF (Fig. 3,3:1). The length EF is taken as 2 and that of the edge CF is \( L \). The approximate solution \( w_0(x,y) \) is constructed according to the elementary theory given above, taking \( k = \cos^2 \theta \), and according to equations (1b) and (19):

\[ w_0(x,y) = \frac{q}{(1-\nu^2)\cos \theta} \left\{ \frac{x^2}{2} \left( \frac{1}{2} + \frac{1}{2} \right) + \frac{\tan \theta}{2} \left[ \frac{1}{2} r + h \right] \right\}. \] (21)

where \( q \) is the total shear load acting on the tip section. The coordinates \( x \) and \( y \) should now be taken as \( L \cos \theta \) and \( L \sin \theta \), respectively.

Transformed into the oblique coordinates \((\xi, \eta)\) by the relation

\[ x = \frac{\xi}{2} \cos \theta \]
\[ y = \eta + \frac{1}{2} \sin \theta \]

equation (21) becomes

\[ w_0(\xi, \eta) = \frac{q}{(1-\nu^2)\cos \theta} \left\{ \cos \theta \left[ \frac{1}{2} \xi^2 - \frac{\xi^2}{2} \right] + \frac{1+\nu}{2} \sin \theta \left[ -\sin \theta \frac{\xi^2}{2} + \frac{1}{2} L \sin \theta \frac{\xi^2}{2} \right] \right\}. \] (23)
Substituting (23) into equation (7), one obtains

\[
\frac{\partial V_e}{\partial a_{mn}} = \frac{9}{(1+\nu^2)\cos^2 \theta} \int_0^1 \int_0^1 f_m g_n (A + B \xi + C \eta) + f_m g_n' (D + E \xi + F \eta) + f_m g_n'' (G + H \xi + \kappa \eta) \, d\xi \, d\eta + 2 \int_0^1 f_m (\xi) \int_0^1 g_n (\eta) \, d\eta,
\]

where

\[A = \frac{3 \nu^2}{4} \sin^2 \theta, \quad B = -A/L, \quad C = -\frac{1+\nu}{2} \sin \theta, \quad D = -\frac{EI}{L}, \quad E = -2 \sin \theta \left[ \frac{1+\nu}{2} \sin^2 \theta \right], \quad F = (1+\nu) \sin \theta.

\[G = L \left[ (\sin^2 \theta + \nu \cos \theta) (\cos^2 \theta + \frac{1+\nu}{2} \sin^2 \theta) \right] \sin \theta H = -\frac{G}{L} \quad X = -\frac{1+\nu}{2} \sin \theta \left( \sin^2 \theta + \nu \cos \theta \right) \]
now be written as

\[ \frac{\partial V}{\partial a_{mn}} = \frac{8}{(1-x^2)\cos^4 \theta} \sum_{mn} - 8 f_m(x) \int_{-1}^{1} g_n(\eta) \, d\eta, \quad (27) \]

where \[ \sum_{mn} = A(I)_{mn} + B(II)_{mn} + C(III)_{mn} + D(IV)_{mn} + E(V)_{mn} + F(VI)_{mn} \]
\[ + G(VII)_{mn} + H(VIII)_{mn} + K(IX)_{mn}, \]
and \[ \int_{-1}^{1} \varepsilon_n(\eta) \, d\eta = \sqrt{2} \text{ if } n = 0; \quad \text{and } = 0 \text{ if } n \neq 0. \]

Sample Computation: Swept Plate, \( L = 4, \theta = 45^\circ \)

Let six indeterminate coefficients be taken in this case, so that the deflection function is represented by

\[ w(x, \tau) = w_0(x, \tau) + a_{10} f_1(x) g_0(\tau) + a_{11} f_1(x) g_1(\tau) + a_{21} f_2(x) g_1(\tau) \]
\[ + a_{20} f_2(x) g_0(\tau) + a_{21} f_2(x) g_1(\tau) + a_{21} f_2(x) g_1(\tau). \quad (28) \]

Now the equations for the determination of the coefficients \( a_{11}, a_{12} \) etc. are, according to equation (6), and taking \( \gamma = 0.3, \theta = 45^\circ, L = 4, \)

\begin{align*}
2a_{mn} \left[ \alpha_{mn} + \chi_{mn} + 2.7 \beta_{mn} \lambda_{mn} + 1.3 \gamma_{mn} \kappa_{mn} - 2.82843 (\psi_{mm} \gamma_{mn} + \tau_{mm} \kappa_{mn}) \right] \\
+ \sum_{ij \neq mn} a_{ij} \left[ 2.7 \beta_{mi} \lambda_{jn} + 1.3 \gamma_{mi} \kappa_{jn} - 2.82843 (\psi_{mi} \gamma_{jn} + \tau_{mi} \kappa_{jn}) \right] = -\frac{2\cos^2 \theta}{p} \frac{\partial V}{\partial a_{mn}} \quad (29)
\end{align*}

Substituting the numerical values in, one obtains:

\[ \begin{align*}
.09658a_{10} &- .3032a_{11} + 1.491a_{12} + 1.292a_{21} - 4.429a_{22} = -.29091 \\
33.10a_{11} &+ .6064a_{12} - 2.584a_{21} - 2.584a_{22} = +.566476 \\
-45.10a_{12} &- 285.4a_{22} = -2.4454 \\
3.794a_{20} &+ .6873a_{11} + .6869a_{12} - 3.516a_{21} + 7.425a_{22} = +.29264 \quad (30) \\
-23.13a_{11} &- 1.375a_{12} + 65.74a_{21} + 7.032a_{22} = +1.19170 \\
+193.7a_{12} &+ 111.1a_{22} = -.91876
\end{align*} \]
where the right-hand-side of these equations are written in terms of
\[
- \frac{2 \cos \theta \eta}{(1 - \nu^2) \phi}
\]

The solution is
\[
\begin{align*}
\alpha_{10} &= 1.863, & \alpha_{11} &= -0.01412, & \alpha_{12} &= 0.007508, \\
\alpha_{20} &= -0.05989, & \alpha_{21} &= -0.01685, & \alpha_{22} &= -0.007245,
\end{align*}
\]
expressed in terms of \( \frac{\phi}{(1 - \nu^2) \phi} \), i.e., \( \alpha_{10} = 1.863 \frac{\phi}{(1 - \nu^2) \phi} \), etc.

Hence the correction to \( w_e(x, y) \) is quite small. The deflection surface is given by equation (28).
IV. **Stresses in Rectangular Cantilever Plates**

§ 4.1 **Introduction**

The stresses in a cantilever plate can be obtained from the deflection function according to equations (1.12:33) through (1.12:35) of section (1.12) for rectangular plate and equations (1.13:5) through (1.13:10) of section (1.13) for swept plate.

Hence the strength computation can be carried out easily if the deflection problem is solved.

However, an exact solution for deflection of a cantilever plate can be obtained only in a very limited number of particular cases. In general, direct procedures such as those developed in the last chapter are often used to obtain practical approximate solutions within desired accuracy. These procedures are satisfactory for the determination of the deflection $w$ as far as convergence is concerned. But in order to obtain stresses from the approximate expression of $w$, one has to differentiate $w$ twice or thrice. Unfortunately, in many cases this differentiation is not permissible. In differentiating the approximating series for $w$, either the convergence of the resulting series is no longer guaranteed or the degree of convergence becomes so poor that the result is of no practical use. The approximating function always oscillates about the true solution. Indeed, the very proof of the possibility of an approximating procedure (Ref. 19) lies in the fact that the function $(w_{\text{true}} - w_{\text{approx}})$ possesses a large number of zeroes somewhat uniformly distributed over the domain in question, and that the number of zeroes increases without limit as the number of terms in the approximating series $w_{\text{approx}}$ tends to infinity. If $w$ and $w_{\text{approx}}$ are continuous functions, this guarantees the convergence of
w_{\text{approx.}} \to w$. But apparently the derivatives of $w_{\text{approx.}}$ may not converge to the derivatives of $w$. As an example one may take Ritz' solution on the normal mode of a vibrating square plate: here the $w_{\text{approx.}}$ series behaves like the series $\sum_{n=0}^{\infty} \frac{2}{n-2} \frac{1}{n^3} n^3$, and obviously would not permit differentiation higher than the second order. Hence a theory of direct stress approximation without using intermediate deflection functions is desirable. For rectangular plates this can be carried out by a procedure somewhat similar in nature to the Rayleigh-Ritz method. A sequence of solutions is first constructed which satisfy the equations of equilibrium and boundary condition (on stresses), but not necessarily the equation of compatibility. Then it is assumed that the state of stresses can be represented by a linear combination of elements of this sequence of functions. To choose from the infinitude of possible linear combinations the one that satisfies (in the limit) the equation of compatibility, the basic minimum energy principle is used from which the coefficients in the approximating series can be determined.

§ 4.2 Method of Solution

The method of solution is as follows.

Let $M_x$ and $M_y$ be the bending couples and $H_y$ the twisting couple per unit length in the plate, $Q_x$ and $Q_y$ the transverse shear-stress resultants per unit length, and $p$ the surface load per unit area acting on the plate. (See Fig. 4.1:1) The equations of equilibrium are

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0. \tag{1}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial H_{xy}}{\partial y} - Q_x = 0,$$

$$\frac{\partial H_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0.$$
These equations are independent of the way in which the stresses are distributed over the thickness of the plate. In terms of the stresses,

\[ M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, dx \quad \quad M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y \, dx \quad \quad H_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} \, dx \]  

\[ Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{x2} \, dx \quad \quad Q_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{y2} \, dx \]  

Equations (1) consist of three equations in five unknowns. Solutions can be obtained by superposing a particular integral and a complementary function. The complementary function satisfies the homogeneous equations

\[ \frac{\partial^2 Q_x}{\partial x^2} + \frac{\partial^2 Q_y}{\partial y^2} = 0, \quad \quad \frac{\partial^2 M_x}{\partial y^2} + \frac{\partial H_{xy}}{\partial y} - Q_x = 0, \quad \quad \frac{\partial^2 H_{xy}}{\partial y^2} + \frac{\partial^2 M_y}{\partial x^2} - Q_y = 0. \]  

(3)

Let a stress function \( \chi \) be defined by the equations

\[ Q_x = \frac{\partial \chi}{\partial y}, \quad \quad Q_y = -\frac{\partial \chi}{\partial x} \]  

(4)

Then the first equation of (3) is satisfied by an arbitrary function \( \chi \). The second and third equations of (3) now become

\[ \frac{\partial M_x}{\partial y} + \frac{\partial (H_{xy} - \chi)}{\partial y} = 0, \]  

\[ \frac{\partial M_y}{\partial x} - \frac{\partial (H_{xy} + \chi)}{\partial x} = 0. \]  

(5)

Define two more stress functions \( \phi \) and \( \psi \) as follows:

\[ M_x = \frac{\partial \phi}{\partial y}, \quad \quad M_y = -\frac{\partial \psi}{\partial x}, \quad \quad H_{xy} + \chi = \frac{\partial \psi}{\partial y}, \quad \quad H_{xy} - \chi = -\frac{\partial \phi}{\partial x}. \]  

(6)
and hence  
\[ H_{x} = \frac{1}{2} \left( \frac{\phi}{\gamma} - \frac{\psi}{\eta} \right), \]
\[ \chi = \frac{1}{2} \left( \frac{\phi}{\gamma} + \frac{\psi}{\eta} \right). \]

Then the system of equations (5) are satisfied by arbitrary continuous (D,1) functions \( \phi \) and \( \psi \). The boundary conditions on the free edges are obviously

\[ \chi = \text{const. and } M_{nt} = M_{n} = 0, \]  \hspace{1cm} (7)

where \( n \) denotes direction of outer normal to the edge in question. The system of moments and shears derived from the two arbitrary functions \( \phi \) and \( \psi \) according to equations (6) may be called a "self-equilibrating system".

To determine the particular self-equilibrating system which in conjunction with the particular integral solves the problem, use has to be made of the basic energy principle for the stresses (Castigliano's theorem of least work), according to which "the true state of stress is distinguished from all statically correct states of stress by the condition that the complementary energy be a minimum".

For materials obeying Hooke's law, and for given surface stresses or displacements, the complementary energy is the difference of the strain-energy \( U \) and the work \( W_c \) which the surface stresses do over that portion of the surface where the displacements are prescribed.

In the particular case of a cantilever plate, that portion of the surface where the displacements are prescribed is the built-in edge, where the deflection and rotations are zero. Hence the function \( W_c \) vanishes in all cases and the complementary energy is identical with the strain-energy in the plate.
§ 4.3 Applications

For plates of uniform thickness and isotropic material, the stress-strain relations are

\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x - \nu (\sigma_y + \sigma_z) \right], \]
\[ \gamma_{xy} = \frac{1}{E} (1 - \nu) \tau_{xy}, \]

etc., by cyclic substitution of indices. The strain energy stored in the plate is then

\[ U = \frac{1}{2} \int_{-t}^{t} \int_{-t}^{t} \left\{ \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2 \nu (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x) + 2 (1 + \nu) (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right\} dx \, dy. \]

Assuming that the normal stresses are distributed linearly over the thickness of the plate and that the contribution to the total energy due to \( \sigma_z \) and \( \tau_{xz}, \tau_{yz} \) are negligible (being of order \( t^2 / \lambda \) compared to other terms), so that

\[ \sigma_x = \frac{M_x}{t^2/6} \frac{z}{t/2}, \quad \sigma_y = \frac{M_y}{t^2/6} \frac{z}{t/2}, \quad \tau_{xy} = \frac{H_{xy}}{t^2/6} \frac{z}{t/2}, \]

(1)

and hence

\[ U = \frac{1}{2} \iint_D \left\{ M_x^2 + M_y^2 - 2 \nu M_x M_y + 2 (1 + \nu) H_{xy}^2 \right\} dx \, dy. \]

(2)

Let a particular solution of the system of equations of equilibrium for the given load distribution be denoted by \( M_{x0}, M_{y0}, H_{xy0}, Q_{x0}, \) and \( Q_{y0}, \) and let a system of self-equilibrating stresses derived from the stress-funcions \( \phi \) and \( \psi \) according to eq (4.1.16) be added to them so that

\[ M_x = M_{x0} + \frac{\partial \phi}{\partial y}, \quad M_y = M_{y0} - \frac{\partial \psi}{\partial x}, \]
\[ H_{xy} = H_{xy0} + \frac{1}{2} \left( \frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial x} \right). \]

(3)
Then the strain energy \( U \) in the plate can be written as

\[
U = U_o + U_1 + U_2
\]

where

\[
U_o = \frac{1}{2(1-\nu^2)} \int_0^L \left\{ M_{xx} \frac{\partial^2 \phi}{\partial y^2} + M_{yy} \frac{\partial^2 \psi}{\partial x^2} + 2\nu M_{xx} \frac{\partial \psi}{\partial y} + 2(1+\nu) M_{yy} \frac{\partial \phi}{\partial x} \right\} \, dx \, dy,
\]

\[
U_1 = \frac{1}{2(1-\nu^2)} \int_0^L \left\{ 2M_x \frac{\partial \psi}{\partial y} + 2M_y \frac{\partial \psi}{\partial x} + 2\nu (M_{xx} \frac{\partial^2 \psi}{\partial x^2} + M_{yy} \frac{\partial^2 \psi}{\partial y^2}) + 2(1+\nu) M_{yy} \frac{\partial \phi}{\partial x} \right\} \, dx \, dy,
\]

\[
U_2 = \frac{1}{2(1-\nu^2)} \int_0^L \left\{ \left( \frac{\partial \phi}{\partial y} \right)^2 + \left( \frac{\partial \psi}{\partial x} \right)^2 + 2\nu \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial x} + \frac{1+\nu}{2} \left( \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right)^2 \right\} \, dx \, dy.
\]

According to the eqs. (4.2:7), the boundary conditions to be imposed on \( \phi \) and \( \psi \) are

\[
\begin{align*}
\phi &= \alpha, & \therefore \frac{\partial \psi}{\partial y} &= \alpha, & \frac{\partial \phi}{\partial x} &= \alpha \quad (5a) \\
\phi &= \beta, & \therefore \frac{\partial \psi}{\partial y} &= \beta, & \frac{\partial \phi}{\partial x} &= \beta
\end{align*}
\]

This can be replaced by

\[
\begin{align*}
x &= L : & \phi &= \gamma = 0, & \frac{\partial \phi}{\partial x} &= 0, & \text{and} \quad (5) \\
y &= \pm 1 : & \phi &= \gamma = 0, & \frac{\partial \phi}{\partial x} &= 0.
\end{align*}
\]

\( \phi \) and \( \psi \) are otherwise arbitrary.

To construct the functions \( \phi \) and \( \psi \), use is made of the vibration modes of thin elastic rods. Let \( u_n(y) \) be the amplitude of the \( n \)th mode of a free vibrating thin elastic bar of length 2 clamped at both ends; and let \( v_n(x) \) be the corresponding function for a bar of length \( L \) which is clamped at the end \( x = L \) and free at the end \( x = 0 \). Let \( w_n(x) \) be that for a bar of length \( L \) and simply supported at its ends \( x = -L \) and \( x = L \), and \( v_n(y) \) be that for a bar of length 2 and simply supported at both ends \( y = \pm 1 \).

Then the set of functions \( \{ u_n(x) \} \) etc., are orthogonal sequences.
Let them be normalized so that relations like
\[ \int u_m(x) u_n(x) \, dx = \delta_n^m = \begin{cases} 0 & \text{when } m \neq n; \\ 1 & \text{when } m = n. \end{cases} \] (6)

hold, \( \delta_n^m \) being the Kronecker delta. The derived sequences \( \{ u_m'(x) \} \)
and \( \{ v_n'(x) \} \), where primes indicate differentiation with respect
to \( x \), are not orthogonal, since although \( u_m'(x) \), \( v_n'(x) \)
satisfy the same differential equation as \( u_m(x) \) and \( v_n(x) \)
respectively, their corresponding boundary conditions are not admissible.
On the other hand, the sequences \( \{ r_n'(x) \} \) and \( \{ w_n'(x) \} \) are orthogonal,
since the derived boundary conditions are admissible. Let the following notations be used:
\[ \omega_{mn} = \omega_{nm} = \int_{-1}^{1} u_m'(y) u_n'(y) \, dy, \quad \varphi_{mn} = \varphi_{nm} = \int_{0}^{L} v_m'(x) v_n'(x) \, dx, \]
\[ \theta_{mn} \delta_n^m = \int_{-1}^{1} r_m'(y) r_n'(y) \, dy, \quad \epsilon_{mn} \delta_n^m = \int_{0}^{L} w_m'(x) w_n'(x) \, dx, \] (7)
\[ \phi_{mn} = \int_{-1}^{1} u_m(y) r_n'(y) \, dy, \quad \varphi_{mn} = \int_{0}^{L} v_m(x) w_n'(x) \, dx, \]
\[ \eta_{mn} = \int_{-1}^{1} u_m'(y) r_n(y) \, dy, \quad \xi_{mn} = \int_{0}^{L} v_m'(x) w_n(y) \, dx. \]

Now a set of stress functions \( \phi \) and \( \gamma \), which satisfy the boundary conditions (5) can be written as
\[ \phi = \sum_{m,n} a_{mn} v_m(x) r_n(y), \]
\[ \gamma = \sum_{m,n} b_{mn} w_m(x) u_n(y). \] (8)

Substituting (8) into (4), one obtains an expression for \( U \) as a quadratic function of the constants \( \{ a_{mn} \} \) and \( \{ b_{mn} \} \). The constants \( \{ a_{mn} \}, \{ b_{mn} \} \) must be so chosen as to make \( U \) a minimum.
By ordinary maximum minimum process this requires that

\[ \frac{\partial U_1}{\partial a_{mn}} + \frac{\partial U_2}{\partial a_{mn}} = 0, \quad \frac{\partial U_1}{\partial b_{mn}} + \frac{\partial U_2}{\partial b_{mn}} = 0, \quad \text{for any } m, n, \]

(9)

\[ U \] being independent of \( \{a_{mn}\} \) and \( \{b_{mn}\} \). Now

\[ \frac{\partial U_1}{\partial a_{mn}} = \frac{1}{(1+\nu)^2} \int \left\{ \left( M_{x_0} + M_{y_0} \right) v_m(x) r_n(y) \right\} dx dy. \]

(10a)

\[ \frac{\partial U_2}{\partial a_{mn}} = \frac{1}{(1+\nu)^2} \int \left\{ \left( M_{y_0} + M_{x_0} \right) w_m(x) u_n(y) \right\} dx dy. \]

(10b)

\[ \frac{\partial U_1}{\partial a_{mn}} = \left[ 2 \theta_{mn} + (1+\nu) \phi_{mn} \right] a_{mn} + \frac{1+\nu}{2} \sum_{j \neq m} \phi_{mj} a_{jn} \]

\[ + \sum_{j \neq k} \left[ 2 \nu \phi_{mj} \psi_{kn} - (1+\nu) \xi_{mj} \eta_{kn} \right] b_{jk}. \]

(10c)

\[ \frac{\partial U_2}{\partial b_{mn}} = \left[ 2 \psi_{mn} + (1+\nu) \omega_{mn} \right] b_{mn} + \frac{1+\nu}{2} \sum_{k \neq n} \omega_{nk} b_{mk} \]

\[ + \sum_{j \neq k} \left[ 2 \nu \phi_{jm} \phi_{nk} - (1+\nu) \xi_{jm} \eta_{nk} \right] a_{jk}. \]

(10d)

It should be noted that the right hand side of (10a) and (10b) are known quantities and that of (10c) and (10d) are linear in \( \{a_{mn}\} \) and \( \{b_{mn}\} \). Hence equations (9) lead to a system of linear equations from which \( \{a_{mn}\}, \{b_{mn}\} \) can be determined.

The linear forms (10c) and (10d) are functions of plate geometry and independent of the external loading conditions, hence once tabulated can be used for all loading conditions.

Convergence can be established from the fact that the sequences
\{ u_n(x) \} etc., are complete systems. Physically, good convergence can be expected since good estimations on \( M_xo, M_yo, \) etc., from the point of view of statics can usually be made, and hence those due to \( \phi \) and \( \psi \) are only small correction terms.

It remains to write out the explicit formulae for \( u_n(y), v_n(x), \)
\( u_{mn}, \bar{v}_{mn} \) etc. The several functions \( u_n(x), v_n(x), r_n(x), w_n(x) \)
all satisfy the differential equation

\[
\frac{d^4 u}{dx^4} = \beta^4 u.
\]  

and the boundary conditions \( u = \frac{\partial u}{\partial x} = 0 \) for clamped ends, \( u = \frac{\partial^3 u}{\partial x^3} = 0 \)
for supported ends and \( \frac{\partial^3 u}{\partial x^3} = \frac{\partial^3 v}{\partial x^3} = 0 \) for free ends. A fourth set
of admissible boundary conditions are \( \frac{\partial u}{\partial x} = \frac{\partial^3 u}{\partial x^3} = 0 \)
which are the conditions satisfied by \( \{ r_n(x) \} \) and \( \{ w_n(x) \} \).

The functions \( u, v, \omega \) etc., are found to be as follows:

(I) \( u_n(y) \), normal mode of a clamped-clamped bar of length 2, with
origin at the middle point. This is similar to the function \( q_n(y) \)
of section 3.21.

If \( n \) is even:

\[
u_n(y) = \frac{-\cosh p_n \cos p_n y + \cos p_n \cosh p_n y}{\sqrt{\cosh^2 p_n + \cos^2 p_n}},
\]

where \( \tan p_n + \tanh p_n = 0 \).

If \( n \) is odd:

\[
u_n(y) = \frac{-\sinh p_n \sin p_n y + \sin p_n \sinh p_n y}{\sqrt{\sinh^2 p_n - \sin^2 p_n}},
\]

where \( \tan p_n - \tanh p_n = 0 \).

The roots are approximately equal to \( (n - \frac{1}{2}) \frac{\pi}{2} \). Their exact
values are given in Section 3.21.
(II) \( V_n(x) \): Normal mode of a free-clamped bar of length \( L \), free at \( x = 0 \), clamped at \( x = L \).

\[
V_n(x) = \frac{1}{\sqrt{L}} \left\{ \cos \frac{p_n x}{L} + \cosh \frac{p_n x}{L} - \frac{\cos p_n + \cosh p_n}{\sin p_n + \sinh p_n} \left( \sin \frac{p_n x}{L} + \sinh \frac{p_n x}{L} \right) \right\},
\]

where \( p_n \) is the root of \( \cos p_n \cosh p_n + 1 = 0 \).

This is equal to the function \( f_n(x) \) of Section 3.21 with the origin changed to \( x = L \). The roots \( p_n \) are given in Section 3.21.

(III) \( W_n(x) \): Normal mode of a supported-supported bar of length \( L \), with ends at \( x = 0 \) and \( x = L \).

\[
W_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n \pi x}{L} \quad n, \text{ any integer.}
\]

(IV) \( r_n(y) \): Normal mode of a supported-supported bar of length \( 2 \), with ends at \( y = \pm 1 \).

\[
r_n(y) = (-1)^{\frac{n}{2} + 1} \sin \frac{n \pi y}{2} \quad \text{if } n \text{ is even,}
\]

\[
r_n(y) = (-1)^{\frac{n+1}{2}} \cos \frac{n \pi y}{2} \quad \text{if } n \text{ is odd.}
\]

The integrals \( \omega_{mn}, \zeta_{mn} \) etc., are then found to be as follows:

(1). \[
\omega_{ml} = \omega_{lm} = \int_{-1}^{1} u_m(y) u_l(y) dy.
\]

When \( m \neq l \), \( m \) and \( l \) both even:

\[
\omega_{ml} = \frac{\sin (p_m - p_l)}{\sqrt{(1 + \cos \gamma_p m)(1 + \cosh \gamma_p l)}} \left\{ \frac{\sin \gamma_p}{p_m - p_l} - \frac{\sin \gamma_p}{p_m + p_l} \right\}.
\]
\[ + \cos \beta \cos \beta \left[ \frac{\tanh \beta \cos \beta + \tanh \beta \cos \beta}{\beta + \beta} - \frac{\tanh \beta - \tanh \beta}{\beta - \beta} \right] \]
\[ + \frac{2}{\beta + \beta} \left[ \beta \cos \beta \cos \beta + \beta \cos \beta \cos \beta \tanh \beta + \beta \cos \beta \cos \beta \tanh \beta \right] \]

When \( \beta \neq \lambda \), and \( \beta \) and \( \lambda \) are both odd:
\[ \omega_{\beta \lambda} = \frac{\beta \beta}{\sqrt{(1 - \sin^2 \beta \sinh^2 \lambda)(1 - \sin^2 \lambda \sinh^2 \beta)}} \left\{ \frac{\sin (\beta - \lambda)}{\beta - \lambda} + \frac{\sin (\beta + \lambda)}{\beta + \lambda} \right\} \]
\[ + \sin \beta \sin \lambda \left[ \frac{\cosh \beta + \cosh \lambda}{\beta + \lambda} - \frac{\cosh \beta - \cosh \lambda}{\beta - \lambda} \right] \]
\[ - \frac{2 \sin \beta}{\beta + \beta} \left[ \beta \cos \beta \cosh \lambda + \beta \cos \beta \cosh \beta \right] - \frac{2 \sin \lambda}{\lambda + \lambda} \left[ \lambda \sin \beta \cosh \beta + \lambda \sin \beta \cosh \lambda \right] \]  

When \( \beta = \lambda \), and \( \beta \) is an even number:
\[ \omega_{\beta \beta} = \beta \beta \left\{ - \frac{\sin^2 \beta}{\lambda \lambda} + \cos^2 \beta \left[ \frac{\tanh \beta}{\beta} - \frac{1}{\cosh^2 \beta} \right] \right\} \]
\[ + \frac{2 \cos \beta}{\beta \beta} \left[ - \cos \beta \tanh \beta + \sin \beta \right] \]  

When \( \beta = \lambda \), and \( \beta \) is an odd number:
\[ \omega_{\beta \beta} = \beta \beta \left\{ 1 + \frac{\sin^2 \beta}{\lambda \lambda} + \sin^2 \beta \left[ \frac{\coth \beta}{\beta} - \frac{1}{\sinh^2 \beta} \right] \right\} \]
\[ - \frac{2 \sin \beta}{\beta \beta} \left[ \sin \beta \cosh \beta + \cos \beta \right] \]  

When one of \( \beta \) and \( \lambda \) is even and the other is odd, \( \omega_{\beta \lambda} = 0 \). (16e)

The p's in the above formulas are the roots corresponding to \( u_\beta(y) \).

(II). \( \beta_{\beta \lambda} = \beta_{\lambda \beta} \) is given in Section 3.21.

(III). \( \theta_{\beta \beta} = \left( \frac{\beta \pi}{2} \right)^2 \)  

(IV). \( \varepsilon_{\beta \beta} = \left( \frac{\beta \pi}{2} \right)^2 \)  

(17)

(18)
\[ J_{mn} = \frac{4 \pi \n \n^2 P_m}{\sqrt{1 + \frac{\cos^2 P_m}{\cosh^2 P_m}}} \left[ -\frac{\sin P_m}{4 P_m^4} - \frac{\cos P_m \tanh P_m}{4 P_m^4 + n^2 \pi^2} \right], \] if \( m, n \) are both even.  
(19a)

\[ J_{mn} = \frac{4 \pi \n \n^2 P_m}{\sqrt{1 - \frac{\sin^2 P_m}{\sinh^2 P_m}}} \left[ \frac{\cos P_m}{4 P_m^4} + \frac{\sin P_m \coth P_m}{4 P_m^4 + n^2 \pi^2} \right], \] if \( m, n \) are both odd.  
(19b)

\[ J_{mn} = 0, \] if one of the \( m, n \) is even and the other is odd.  
(19c)

The \( P_m \) in these formulas are the roots belonging to \( u_n(y) \).

\[ J_{mn} = \frac{(-1)^{n}}{L} \sqrt{2 \pi n \n^2 P_m} \left[ \frac{\sin P_m + \sigma_m \cos P_m}{P_m^2 - n^2 \pi^2} + \frac{(\tanh P_m - \sigma_m) \coth P_m}{P_m^2 + n^2 \pi^2} + (-1)^n \frac{2 P_m^2 \sigma_m}{n^2 \pi^2 - P_m^4} \right], \] where \( \sigma_m = \frac{\cos P_m + \cosh P_m}{\sin P_m + \sinh P_m} \), and \( P_m \) are the roots belonging to \( v_m(x) \).

For \( m > 2 \), \( \sigma_m = 1 - 2 e^{-P_m} \) to five decimal places, and

\[ J_{mn} = \frac{(-1)^{n}}{L} \sqrt{2 \pi n \n^2 P_m} \left[ \frac{\sin P_m + \sigma_m \cos P_m}{P_m^2 - n^2 \pi^2} + \frac{1}{P_m^2 + n^2 \pi^2} + (-1)^n \frac{2 \sigma_m P_m^2}{n^2 \pi^2 - P_m^4} \right]. \] (20)

(VII). \( \eta_{mn} \) is the same as \( J_{mn} \) except that the term \( \sin P_m \) should be changed into \( -\sin P_m \).

(VIII). \( \bar{J}_{mn} \) is the same as \( -J_{mn} \) except that the last term in the bracket should be changed into \( (-1)^n \frac{2 \sigma_m}{(n \pi)^2 - P_m^4} \).

The numerical values of these functions are given in Tables 4.3:1 through 4.3:7. No practical example is given since the application of this method is quite similar to the Rayleigh-Ritz method of sections 3.2:1 and 3.3. **

** At the time of writing this thesis, an elaborate experimental program is going on at the GALCIT structural laboratory. When such experimental data become available, then it will be interesting to compare the numerical solutions of particular cases.
References


(x, y, z) -- right-handed coordinate system.

(x, y) --- coordinates of field point.

(ξ, η) --- running coordinates of the points along the line of the centers of twist.

The positive sense of torsional couple coincides with the x-axis, and the twisting angle ψ is measured in the same sense as the torsional couple.
The Deflection of a 40 Degree Swept Plate

Loading: Concentrated, 0.35 lb, applied at the mid-point of the tip chord.

Material: 241T Dural.

Elastic constants: $E = 10.3 \times 10^6$ psi,
$G = 3.8 \times 10^6$ psi.

Dimensions: Thickness = 0.085 in.

Solid line: Deflections calculated according to equation (3.3.15).

+ GALCIT Experimental data, from Ref. 52.
Table 3.21: \( \alpha_{mn} \times L^4 \)

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Table 3.21: \( \beta_{mn} \times L^2 \)

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Table 3.21: \( \gamma_{mn} \times L^2 \)

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Table 3.21 : 6 \( \mu_{mn} \)

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Table 3.21: $\gamma \ f_m(x)$

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Table 3.21: $8 \ \varphi_m(\mu)$

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Table 3.3 : 1 $\Psi_{mn} L^3$

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Table 3.3 : 2 $\tau_{mn} L$

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Table 3.3 : C

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<td>((-3.70141)^\frac{1}{L})</td>
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Table 4.2:1 \( \omega_{mn} \)

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Table 4.2:2 \( \theta_{mn} \)

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Table 4.2:3 \( \phi_{mn} \)

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Table 4.2:4 \( \eta_{mn} \)

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### Table 4.2:5 \( \varepsilon_{mn} \times L^2 \)

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### Table 4.2:6 \( g_{mn} \times L \)

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### Table 4.2:7 \( \xi_{mn} \)

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