# STABILITY AND DYNAMICS OF SPHERICALLY SYMMETRIC MASSES IN GENERAL RELATIVITY

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## Abstract

The Einstein equations for a spherically symmetric distribution of matter are recast in comoving (Lagrangian) coordinates in a form similar to that of the classical hydrodynamic equations, thus facilitating the physical interpretation of the equations. The jump conditions for a shock wave in an ideal fluid are found in these coordinates. The equation of radiative transfer in general relativity is derived, and analyzed with respect to the non-gravitational interaction of the radiation and the matter as reflected in the equations arising from the zero covariant divergence of the energy-momentum tensor. The radiative transfer equation is solved assuming the radiation is in local thermodynamic equilibrium with the matter.

We present some examples of numerical calculations of the equilibrium, stability, and dynamics near equilibrium of spherically symmetric masses for a simple, although physically reasonable, type of equation of state, in which the thermal energy density is given solely in terms of the pressure and an adiabatic index $\Gamma_1$ that is independent of density and pressure. One numerical method follows the growth of instabilities to all orders in the fractional change in

radius away from equilibrium. Our formulation of the Einstein equations is applied to the analytical study of the stability, and we prove that, regardless of the equation of state, a maximum or minimum of the binding energy as a function of central density along a sequence of masses in hydrostatic and convective equilibrium with constant number of baryons and constant rest mass per baryon implies some mode of radial oscillation has zero frequency there. As a result of this theorem, the stability properties of models in convective equilibrium can be read off a plot of fractional binding energy against the ratio of gravitational radius to radius. One example is presented of a numerical difference equation calculation of the collapse of a large mass inside the gravitational radius.

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## Introduction

Developments in astronomy, in particular the discovery and investigation of the quasi-stellar radio sources or quasars (1), have led to a renewal of interest in the application of general relativity to astrophysics on a local, rather than cosmological, scale. Hoyle and Fowler (2, 3) have proposed models of quasars in which, because of the predominance of radiation pressure, general relativity plays an important role due to its effect on stability, even though the effect on the structure may be small. The apparent need for the sudden release of tremendous amounts of energy in the formation of the extended double radio sources (4) associated with some large elliptical galaxies has stirred speculation (5) that gravitational collapse to near the gravitational radius $GM/c^2$ might be a means of providing this energy. An entirely different area in which general relativity may be important is in the collapse of supernovae cores (6).

Our purpose in this paper is not to analyze in detail any particular physical model, but rather to lay the groundwork for numerical calculations based on the exact equations of general relativity. What calculations we do present are for the most part based on a simple ansatz for the equation of state in terms of the adiabatic index relating pressure and specific volume at constant entropy. For certain values of the adiabatic index this may be a good approximation to a particular physical model, but we wish to analyze the

effects of general relativity, not those of some peculiarity of the equation of state.

The foundation of our work is a reformulation of the Einstein equations for a spherically symmetric distribution of matter. We neglect rotation because of the great complication it produces in the equations of general relativity, even though it may be very important in astrophysical applications. Our reformulation is particularly suited to comoving coordinates, and the resulting equations bear a close resemblence to those of Newtonian Lagrangian hydrodynamics if the energy-momentum tensor is that of a perfect fluid. We look upon the freedom of coordinate transformation in general relativity not as a reason to maintain a covariant form of the equations but as a license to find that coordinate system which puts the equations in the form which has the simplest and most direct physical interpretation. Comoving coordinates have the advantage that it is for an observer moving with the matter that the local properties of the matter, as described by the energy-momentum tensor, are the simplest. It is also possible to make a consistent assignment of the "potential energy" of the gravitational field in terms of a factor multiplying the local energy density of the matter which also contains the "kinetic energy" of the matter relative to the origin. This localization of the energy is possible in a spherically symmetric problem because of an unambiguous geometrical radius which measures the circumference of a circle or the area of a sphere centered at the origin.

The precise definition of comoving coordinates and how the metric is related to the thermodynamic properties of the matter is treated with some care in Chapters II and III. We accept a hydrodynamic description of the matter, going to kinetic theory only for the discussion of radiative transfer in Chapter IV. The equation of radiative transfer is derived for arbitrary coordinates, and in the case of spherical symmetry and comoving coordinates used to interpret physically the equations arising from the zero covariant divergence of the energy-momentum tensor. The equation is solved assuming the radiation is in local thermodynamic equilibrium with the matter, obtaining results in agreement with those of Misner and Sharp (7), who consider the process of thermal diffusion in general relativity directly. R. W. Lindquist (8) in independent unpublished work has formulated the problem of radiative transfer in a way similar to ours. An elaborate formalism for treating kinetic theory in general relativity has been constructed by Tauber and Weinberg (9); however, our approach, which emphasizes the role of an observer comoving with the matter, has a more direct physical interpretation and takes advantage of the local flatness of space-time.

The hydrodynamic equations are used to study intensively the stability of a mass in equilibrium and the growth of instabilities. An important result is a proof that a maximum or minimum of the binding energy as a function of central density, keeping the number of nucleons constant, in a sequence of equilibrium models,

each in convective equilibrium, implies some mode of radial oscillation is in neutral equilibrium there. We discuss a numerical method which finds how the kinetic energy changes assuming a certain shape for the velocity as a function of radius. One application is to investigate the stability under second order pertubations when the fundamental mode of radial oscillation is in neutral equilibrium with respect to first order pertubations of the equilibrium equation.

Only a single example is presented of a numerical solution of the hydrodynamic equations using a difference equation method, but we discuss in a general way the advantages and disadvantages of alternative formulations of the difference equations. A difference scheme which is accurate when the size of the body is much larger than its gravitational radius might not be suitable when the body contracts inside its gravitational radius. An asymptotic solution, under certain assumptions about the variation of pressure with density, is obtained analytically for the limit that the radius of a mass shell goes to zero.

In the final chapter we discuss our numerical and analytical results and their implications for models of quasars. We conclude that if a quasar is to be a single coherent mass, a mass large enough to meet the energy requirements cannot be stabilized at central temperatures high enough for nuclear reactions unless rotation is invoked. A crude estimate of the effects of rotation is made by treating both the general relativity and the rotation as pertubations on a spherically symmetric classical model. The

question of the reversibility of gravitational collapse near and inside the gravitational radius also receives attention.

Other investigators have pursued lines of research parallel to some of those reported here. In particular, Podurets (10) and Misner and Sharp (11) have made use of essentially the same reformulation of the Einstein equations and at least in the case of Misner and Sharp have interpreted them in a similar way. Chandrasekhar (12, 13) has derived the equations governing radial oscillations and stability, and he and others have applied them to various physical models. Difference equation numerical integrations of gravitational collapse inside the gravitational radius have been made by White and May (14). Since we have developed an independent approach to the interpretation and application of the hydrodynamic equations we will not refer to parallel work in the body of the thesis except to compare results.

It should be realized that the only general relativistic corrections to the Newtonian theory tested by experiment are first order in the ratio of gravitational radius to radius. Thus the post-Newtonian analyses of Chandrasekhar (13) for the stability of a class of polytropes and of Fowler (5) for the properties of large masses have some experimental foundation. The application of general relativity when the radius is close to the gravitational radius is a considerable theoretical extrapolation, but it is the best theory we have.

## I. The Einstein Equations

The assumption of spherical symmetry allows drastic simplifications to be made in the metric tensor and in the energy-momentum tensor. The detailed arguments justifying the simplifications may be found in Synge (15), and will not be repeated here. The result for the metric tensor is that coordinates can be found so the only non-zero components are $g_{00} = a^2$, $g_{11} = -b^2$, $g_{22} = -R^2$, and $g_{33} = -R^2\sin^2\theta$, where $x^0 = t$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \varphi$. a, b, and R are functions of r and t only. The line element is

$$ds^2 = a^2\,dt^2 - b^2\,dr^2 - R^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2) \quad . \qquad (1.01)$$

We use throughout this paper units in which the gravitational constant and the speed of light are unity.

When formal use of subscripts and superscripts is convenient, we will follow the usual summation convention for repeated indices, with Latin indices running from zero to three and Greek indices from one to three. If F is a function of r and t, let

$$\dot{F} = \left(\frac{\partial F}{\partial t}\right)_r , \qquad F' = \left(\frac{\partial F}{\partial r}\right)_t .$$

Letting $Y = 1/a$ and $X = 1/b$, the proper time derivative is $\dot{F}Y$ and the proper radial derivative is $F'X$.

The coordinate system is not yet completely specified in that for any $r^* = r^*(r,t)$ we may find a $t^* = t^*(r,t)$ such that the line element still has the form (1.01). Of course, there is also invariance

of form under independent scale transformations of $r$ and $t$, with $r^* = r^*(r)$ and $t^* = t^*(t)$. Independent of all these transformations $R$ has the definite geometric significance that $4\pi R^2$ is the area of a sphere centered at the origin. The variable $r$ has no direct geometrical significance, although it is possible to choose coordinates in which $r = R$.

The spherical symmetry restricts the non-zero components of the energy-momentum tensor to $T^0{}_0$, $T^1{}_1$, $T^1{}_0$, $T^0{}_1$, $T^2{}_2$, and $T^3{}_3$, which is equal to $T^2{}_2$. The symmetry of the contravariant form of the energy-momentum tensor allows us to define a quantity $K$ such that

$$T^1{}_0 bY = -T^0{}_1 aX = K \quad . \tag{1.02}$$

We will work with $T^0{}_0$, $T^1{}_1$, $T^2{}_2$, and $K$, since they are independent of scale transformations of $r$ and $t$.

Requiring invariance under scale transformations is a useful guide to physically relevant forms of the equations, since the physics cannot depend on the units chosen to measure the time coordinate and the radial coordinate. The physically important derivatives of the components of the metric tensor will be the scale invariant quantities

$$Z = R'X \quad , \tag{1.03}$$

$$U = \dot{R}Y \quad , \tag{1.04}$$

$$f = \frac{a'}{a}X = -\frac{Y'}{Y}X \quad , \tag{1.05}$$

$$q = \frac{\dot{b}}{b}Y = -\frac{\dot{X}}{X}Y \quad . \tag{1.06}$$

Although the Riemann-Christoffel symbols associated with the metric (1.01) are well-known and may be found, for instance, in Synge (14), we list the non-zero ones in Appendix I. In terms of the scale invariant quantities defined above, the Riemann tensor has the non-zero components

$$R^{10}{}_{01} = f'X + f^2 - \dot{q}Y - q^2 \quad , \tag{1.07}$$

$$R^{20}{}_{02} = R^{30}{}_{03} = -\frac{\dot{U}Y}{R} + \frac{Z}{R} f \quad , \tag{1.08}$$

$$R^{21}{}_{12} = R^{31}{}_{13} = \frac{Z'X}{R} - \frac{U}{R} q \quad , \tag{1.09}$$

$$R^{32}{}_{23} = -(1 + U^2 - Z^2)/R^2 \quad , \tag{1.10}$$

$$bY\, R^{12}{}_{20} = bY\, R^{13}{}_{30} = \frac{\dot{Z}Y}{R} - \frac{U}{R} f = \frac{U'X}{R} - \frac{Z}{R} q \quad . \tag{1.11}$$

The Ricci tensor is defined by $R^i{}_j = R^{mi}{}_{jm}$ .

The Einstein equations,

$$R^i{}_j - \tfrac{1}{2} g^i{}_j R^k{}_k = 8\pi T^i{}_j \quad , \tag{1.12}$$

which may be found in somewhat different notation in Landau and Lifschitz (16), are

$$8\pi T^0{}_0 = -\frac{2Z'X}{R} + \frac{2U}{R} q + (1 + U^2 - Z^2)/R^2 \quad , \tag{1.13}$$

$$8\pi T^1{}_1 = \frac{2\dot{U}Y}{R} - \frac{2Z}{R} f + (1 + U^2 - Z^2)/R^2 \quad , \tag{1.14}$$

$$8\pi T^2{}_2 = \dot{q}Y + q^2 - f'X - f^2 + \frac{\dot{U}Y}{R} + \frac{U}{R} q - \frac{Z'X}{R} - \frac{Z}{R} f \quad , \tag{1.15}$$

$$8\pi K = \frac{2}{R}(U'X - Zq) = \frac{2}{R}(\dot{Z}Y - Uf) \quad . \tag{1.16}$$

These equations are not yet in a form convenient for physical interpretation. We can simplify them as follows. From equation (1.16)

$$\frac{2U}{R}\,q = \frac{2UU'X}{ZR} - \frac{8\pi KU}{Z} \quad .$$

Using this equation (1.13) becomes

$$8\pi R^2(T^0{}_0 Z + KU)b = [R\,(1 + U^2 - Z^2)]' \quad . \tag{1.17}$$

Similarly, we may rewrite equation (1.14) as

$$8\pi R^2(T^1{}_1 U - KZ)a = [R\,(1 + U^2 - Z^2)]^{\cdot} \quad . \tag{1.18}$$

Equations (1.17) and (1.18) suggest that we define a quantity $M$ such that

$$Z^2 = 1 + U^2 - 2M/R \quad . \tag{1.19}$$

Then

$$M'X = 4\pi R^2(T^0{}_0 Z + KU) \quad , \tag{1.20}$$

and

$$\dot{M}Y = 4\pi R^2(T^1{}_1 U - KZ) \quad . \tag{1.21}$$

If the geometry is to be locally Euclidean at $R = 0$, we must have $Z = 1$ there to insure that the ratio of circumference to proper radius of a small circle about the origin is $2\pi$. Other boundary conditions which follow from this are that $U = 0$ and $M = 0$ at $R = 0$.

We will now consider the effect of coordinate transformations of $r$ and $t$ which maintain the form of the metric (1.01) in order to

explore the non-trivial degree of freedom left in our choice of coordinate system. Obviously R is invariant under these transformations, and we will show that M is also invariant. Let

$$v = b_1\left(\frac{\partial r_1}{\partial t_2}\right)_{r_2} Y_2 \quad : \tag{1.22}$$

Then

$$(1 + v^2)^{\frac{1}{2}} = a_1\left(\frac{\partial t_1}{\partial t_2}\right)_{r_2} Y_2 \quad . \tag{1.23}$$

v is the "momentum per unit mass" of a particle at rest in coordinate system 2 relative to coordinate system 1.

From equation (1.19)

$$Z_1 = \frac{\partial R}{\partial r_1} X_1 = (1 + U_1^2 - 2M_1/R)^{\frac{1}{2}} \quad , \tag{1.24}$$

and

$$Z_2 = \frac{\partial R}{\partial r_2} X_2 = (1 + U_2^2 - 2M_2/R)^{\frac{1}{2}} \quad . \tag{1.25}$$

An immediate consequence of (1.22) and (1.23) is

$$U_2 = U_1(1 + v^2)^{\frac{1}{2}} + Z_1 v \quad , \tag{1.26}$$

and this together with (1.24) gives

$$(1 + U_2^2 - 2M_1/R)^{\frac{1}{2}} = Z_1(1 + v^2)^{\frac{1}{2}} + U_1 v \quad . \tag{1.27}$$

The transformation equations for the metric tensor can be used to show

$$b_1\left(\frac{\partial r_1}{\partial r_2}\right)_{t_2} X_2 = (1 + v^2)^{\frac{1}{2}} \quad , \tag{1.28}$$

$$a_1\left(\frac{\partial t_1}{\partial r_2}\right)_{t_2} X_2 = v \quad . \tag{1.29}$$

Thus

$$Z_2 = Z_1(1 + v^2)^{\frac{1}{2}} + U_1 v \quad . \tag{1.30}$$

Using equation (1.30) to compare (1.25) and (1.27), we see that $M_1 = M_2$; that is, the quantity $M$ is invariant under transformations which preserve the diagonal form of the metric.

Equations (1.26) and (1.30) are identical to those of a covariant transformation in special relativity. $U$ corresponds to the time component and $Z$ to the space component of the gradient of the invariant $R$. The proper time and radial derivatives of any other invariant transform in the same way as $U$ and $Z$.

The transformation from Schwarzschild coordinates, in which $r = R$, $U = 0$, and $Z = (1-2M/R)^{\frac{1}{2}}$, is of particular interest. Let $v$ be the momentum per unit mass of a particle at constant $r$ measured by an observer at constant $R$. $U$ and $Z$ in the "comoving" coordinates are related to $v$ by

$$v = U/(1-2M/R)^{\frac{1}{2}} \quad , \tag{1.31}$$

$$(1 + v^2)^{\frac{1}{2}} = Z/(1-2M/R)^{\frac{1}{2}} \quad . \tag{1.32}$$

The transformation becomes singular when $2M/R \geq 1$, or $|U| > Z$. This is due to the well-known fact that if $2M/R > 1$, the line $R,\theta,\varphi$ constant in space-time is space-like and thus cannot be the world line of a particle or the time axis of a coordinate system.

The quantity $M$ can be interpreted as the total amount of energy inside the radius $r$ at time $t$, as will become clearer in the next chapter.

## II. Comoving Coordinates

We want to apply the equations of Chapter I to calculations of the dynamics of matter in a spherically symmetric gravitational field. This implies a particular choice of coordinate system. The one which is most convenient for our purposes and which allows the most direct physical interpretation of the equations is the comoving coordinate system. The variable $r$ becomes the label of a given mass shell in a Lagrangian form of hydrodynamics.

An important advantage of comoving coordinates is that the metric does not become singular unless the components of the energy-momentum tensor in the rest frame of the matter have become singular. The equations behave perfectly smoothly when $2M/R$ becomes greater than one; the Schwarzschild coordinate system, with $r = R$, breaks down there.

The definition of the comoving coordinate system is unambiguous only when the matter is made up of just one kind of particle, the number of particles being conserved. In general, however, there are several different kinds of particles present whose masses and numbers may change. The definition of the comoving coordinate system may be made precise in one of the following ways:

1) such that the amount of some quantity known to be exactly conserved, such as baryon number, inside a given $r$ be independent of $t$;

2) such that the energy flux $K$ in this coordinate system be zero, or equivalently that the time-like eigenvector of the energy-

momentum tensor point in the direction of the t-axis;

3) such that the "pressure" be isotropic; that is, $T^1_1 = T^2_2$ .

The most convenient definition in that the resulting equations are most directly related to the thermodynamics of the matter is the first. We shall define our comoving coordinates, then, so that the number of baryons in the mass shell between $r$ and $r + dr$ is independent of the time, t.

We may define a set of local observers by introducing an orthonormal tetrad of vectors $\lambda_{(a)}{}^i$ at each point in space-time. One member of the tetrad, $\lambda_{(0)}{}^i$, is time-like and equal to the four-velocity of the local observer. The other three are unit vectors along three mutually perpendicular axes in the locally Euclidean three-space normal to $\lambda_{(0)}{}^i$. The projection of a tensor or vector on the tetrad gives the values of the components of the tensor or vector as measured by a local observer. A full discussion of the construction of a local frame of reference may be found in Synge (17).

The tetrad index may be raised and lowered by the special relativistic metric tensor $\eta_{(a)(b)}$ with signature -2. The orthonormality conditions are

$$\lambda_{(a)}{}^i \lambda^{(b)}{}_i = \delta_{(a)}{}^{(b)} \quad , \tag{2.01}$$

and conversely,

$$\lambda_{(a)}{}^i \lambda^{(a)}{}_j = \delta^i{}_j \quad . \tag{2.02}$$

The projections of a given vector at a given point in space-time on two different tetrads at that point are related by a Lorentz transformation.

A convenient choice for the tetrad of an observer at fixed $r$ in our comoving coordinates is

$$\lambda_{(0)}{}^{i} = Y\delta^{i}{}_{0} \quad , \quad \lambda_{(1)}{}^{i} = X\delta^{i}{}_{2} \quad ,$$
$$\lambda_{(2)}{}^{i} = \frac{1}{R}\,\delta^{i}{}_{2} \quad , \quad \lambda_{(3)}{}^{i} = \frac{1}{R\sin\theta}\delta^{i}{}_{3} \quad . \tag{2.03}$$

The projection of the energy-momentum tensor on this tetrad is

$$T^{(0)(0)} = T^{0}{}_{0} \quad , \quad T^{(0)(1)} = bY\,T^{1}{}_{0} = K \quad ,$$
$$T^{(1)(1)} = -\,T^{1}{}_{1} \quad , \quad T^{(2)(2)} = -\,T^{2}{}_{2} \quad . \tag{2.04}$$

The local flatness of space-time allows some results of classical or special relativistic thermodynamics, statistical mechanics, and kinetic theory to be applied to the calculation of local properties of the matter, such as the local projections of the energy-momentum tensor, if the mean free path of the matter particles is small compared to the scale on which the gravitational fields change. Of course, it must be remembered that in general the comoving system is not an inertial system; a local observer feels an acceleration.

The magnitude of this acceleration is the same as the acceleration of a test particle falling from rest in the comoving coordinate system and being acted upon only by gravitational forces. That is, the test particle follows a time-like geodesic. If $s$ is the proper time

the test particle its four-velocity is

$$v^0 = \frac{dt}{ds} \quad , \qquad v^1 = \frac{dr}{ds} \quad .$$

The other components are zero since the acceleration must be in the radial direction. The momentum per unit mass seen by a local observer is $v = b\frac{dr}{ds}$;

$$v^0 = (1 + v^2)^{\frac{1}{2}} \quad , \qquad v^{(1)} = v \quad .$$

There is only one independent geodesic equation in the case of purely radial motion, which can be written as

$$\frac{dv}{ds} = - f\,(1 + v^2) - q\,v(1 + v^2)^{\frac{1}{2}} \quad .$$

In the limit $v \to 0$,

$$\frac{dv}{ds} = - f \quad .$$

Thus $-f$ is the acceleration, or "gravitational force per unit mass", felt by a local observer. By Newton's third law $f$ must be the sum of the non-gravitational forces acting on the local observer per unit mass.

An exact description of the behavior of the matter involves some form of kinetic theory, such as that of Tauber and Weinberg (9). However, an often valid approach is to assume that the particles collide often enough so that we may define a local temperature which is the same for the different kinds of particles making up the matter. Deviations from thermal equilibrium can still be treated by kinetic theory, as we do for electromagnetic radiation in Chapter IV. The effect of the non-inertiality of the matter rest frame on the energy-momentum tensor will

be small if the fractional change in velocity of a particle in one mean free path is much less than one. Sometimes it may be a good approximation to completely neglect all deviations from adiabatic local thermodynamic equilibrium, in which case the energy flux $K$ is zero and the pressure is isotropic for the comoving observer. This is usually called the ideal fluid approximation.

The ideal fluid, or adiabatic, approximation will be our point of departure in discussions of the energy-momentum tensor. We will use the symbols $L_0$, $L_1$, $L_2$, and $K$ to denote the terms in the energy-momentum tensor due to deviations from an ideal fluid or to the effects of non-thermal forms of energy such as a neutrino flux. $E$ will be the thermal energy density, including the rest masses of the particles, and $P$ will be the isotropic thermal pressure. Thus the components of the energy-momentum tensor will be

$$\begin{aligned} T^0{}_0 &= E + L_0 \ , & T^1{}_1 &= -P - L_1 \ , \\ T^2{}_2 &= T^3{}_3 = -P - L_1 \ , & & K \ . \end{aligned} \tag{2.05}$$

The only deviation from an ideal fluid completely consistent with local thermodynamic equilibrium is an energy flux $K$, as will be seen from our discussion of radiative transfer.

Using the energy-momentum tensor in the form (2.05) we may arrange the Einstein equations as follows. With

$$z^2 = 1 + U^2 - 2M/R \ , \tag{2.06}$$

we have

$$M' = 4\pi R^2 b\,(EZ + L_0 Z + KU) \quad , \tag{2.07}$$

$$\dot{M} = -4\pi R^2\,[P\dot{R} + (L_1 U + KZ)a] \quad , \tag{2.08}$$

$$\dot{Z} = f\dot{R} + 4\pi KRa \quad , \tag{2.09}$$

$$q = \frac{\dot{b}}{b}Y = U'X/Z - 4\pi KR/Z \quad , \tag{2.10}$$

$$\dot{U}Y = Zf - M/R^2 - 4\pi(P + L_1)R \quad . \tag{2.11}$$

Only four of the equations are independent, since we have not used the $T^2_2$ Einstein equation. Equations (2.09) and (2.10) are just different arrangements of the $T^1_0$ Einstein equation, and equations (2.08) and (2.11) are both based less directly on the $T^1_1$ Einstein equation. We find it convenient to complete our set of equations with those arising from the zero covariant divergence of the energy-momentum tensor. These two equations are intimately connected with the properties of matter, and their discussion will be postponed until we have a better foundation for interpreting them.

Comoving coordinates allow a semi-Newtonian interpretation which views gravitation as a force rather than a property of the geometry of space-time. R is the geometrical radius of a mass shell, and U corresponds to a special relativistic momentum per unit mass since it is the rate of change of R with proper time for a mass shell. U is not the actual momentum seen by an observer at constant R, however, it remains finite and real when $2M/R \geq 1$.

Equation (2.11) looks like the Newtonian force equation, and it is natural to call $\dot{U}Y$ the acceleration of a mass shell. Strictly speaking, since $f$ is the true non-gravitational force per unit mass, $\dot{U}Y/Z$ should be called the acceleration. However, we will call $\dot{U}Y$ the acceleration and call $Zf$ the effective non-gravitational force per unit mass. The "gravitational force" per unit mass is then the usual $M/R^2$ term plus a force due to the radial stress at the mass shell which is related to the deflection of light passing near the sun being twice the Newtonian value.

An immediate result of equation (2.06) is that as long as $2M/R$ is greater than one $U$ cannot change sign, since $Z$ would be imaginary if $U$ were equal to zero. This is taken care of in the equation for the rate of change of $U$ by the factor $Z$ in front of $f$. If $f$ is a large repulsive force and $U < 0$, by equation (2.09) $Z$ decreases and as $Z \to 0$ the effect of $f$ on $\dot{U}Y$ goes to zero. As will be discussed further in the last two chapters of this paper, if $f$ is large enough $Z$ may become less than zero, in which case a positive (repulsive) $f$ actually causes an acceleration <u>toward</u> $R = 0$, increasing the absolute value of $U$.

The quantity $Z$, which is geometrically the ratio of the change in $R$ between two mass shells to the proper radial distance between them at a given time $t$, also has the interpretation, from equation (2.06) clearly valid in the Newtonian limit, of being the sum of the mass, kinetic, and gravitational potential energy per unit mass of the fluid or of an observer moving with the fluid. This inter-

pretation is born out by equation (2.07), which says that except for a term involving $K$ the contribution of a mass shell to the total energy $M$ is given by the internal energy of the mass shell times $Z$. The energy flux term may be understood as a correction due to the fact that our comoving coordinate system is not really comoving with respect to the local energy density if $K \neq 0$. Note that if $Z < 0$ a positive internal energy density makes a negative contribution to the total energy.

The rate of change of $Z$ is related by equation (2.09) to the work done by the non-gravitational force, again except for a term involving $K$. This energy flux term is necessary if there is to be no gravitational force associated with the direct convection of energy from one mass shell to another. Energy can also be transferred through the work done by the radial stress, as is seen in equation (2.08).

The semi-Newtonian interpretation is useful in that it allows many of the concepts of classical hydrodynamics to be carried over into general relativity with only slight modification. However, it is difficult to see how it can be carried over into non-spherically symmetric systems since a mass element can no longer be located in a simple way geometrically.

The amount of proper volume between two spherical shells at $r$ and $r + dr$ at a certain time $t$ is $4\pi R^2 b\ dr$. For any quantity the amount of which inside $r$ is a function only of the number of baryons inside $r$ we may define a specific volume $V$ such that $V$ times the amount of the quantity between $r$ and $r + dr$ is the proper volume between $r$ and $r + dr$. The important property of $V$ is that

$$\frac{\dot{V}}{V} = \frac{\dot{b}}{b} + \frac{2\dot{R}}{R} \quad . \tag{2.12}$$

Such a quantity might be the mass of a proton times the baryon number. As long as the total number of particles of each kind associated with a given number of baryons is independent of $t$ we can, in particular, define $V$ so it is the amount of proper volume occupied by a unit amount of "rest mass", defined as the sum of the rest mass energies of all the particles present. We will call the mass associated with $V$ the invariant mass. The total amount of invariant mass inside a given $r$, denoted by $M_o$, is by definition independent of time. The invariant mass density is $1/V$.

We could take $r = M_o$, but this is awkward near the origin. Usually we will take $r = R_o$, the initial value of $R$ at the mass shell. Then

$$\frac{dM_o}{dr} = 4\pi R_o^2 b_o / V_o$$

$$= 4\pi R^2 b/V \quad . \tag{2.13}$$

Equivalently, we have for a given $M_o(r)$ the equation

$$V = \frac{\partial}{\partial M_o}\left(\frac{4\pi R^3}{3}\right)_t \quad . \tag{2.14}$$

Using $V$ in the equations arising from the zero covariant divergence of the energy-momentum tensor, we can write them as

$$[(\dot{E}V)Y + P\dot{V}Y]/V = -(L_o\dot{V})Y/V - L_1\frac{\dot{b}}{b}Y - L_2\frac{2U}{R}$$

$$- (R^2Ka^2)'X/(R^2a^2) \quad , \tag{2.15}$$

and

$$(E+P+L_0+L_1)\,f = -P'X - L_1'X - \frac{2Z}{R}(L_1 - L_2) - \dot{K}Y - 2K\left(\frac{\dot{b}}{b} + \frac{\dot{R}}{R}\right)Y \quad . \qquad (2.16)$$

The first equation gives the rate of change of the internal energy per unit invariant mass, or specific internal energy, and the second equation can be used to determine the non-gravitational force per unit mass f as it depends on the energy-momentum tensor. From f the quantity $a = 1/Y$ can be obtained by integration. That the results are what one would expect from the viewpoint of a local observer using special relativity can be checked for various assumptions about the physics of the energy-momentum tensor. For the case of an ideal fluid the force per unit mass is consistent with the force per unit volume being given by the pressure gradient if the inertial mass per unit volume is taken to be E+P. The pressure contributes to the inertial mass because of the transfer of energy associated with the work done by the pressure. In the chapter on radiative transfer we will analyze these equations as they relate to the local interaction of the radiation with the matter.

## III. Thermodynamics and Hydrodynamics of an Ideal Fluid

We now concentrate our attention on the general relativistic hydrodynamic equations for an ideal fluid, occasionally introducing non-adiabaticity in the form of a small energy flux K. This model is valid if all forms of energy not bound up in the matter are in local thermodynamic equilibrium with one another.

Except at extremely high temperatures when all particles are relativistic it is convenient to separate the internal energy density E into a rest mass density A/V and a thermal energy density W/V. We write

$$EV = A + W \quad . \tag{3.01}$$

The precise way the energy is split up is not of critical importance; however, the irreversible changes in nuclear masses associated with non-thermal nuclear reactions should be associated with A. At relatively low temperatures A might include the rest masses of all the particles present, while at high temperatures such that, say, electron-positron pairs are being produced, it may be more convenient to include the rest masses of the pairs in the thermal energy. W is the usual thermodynamic specific internal energy. Nuclear energy generation will tend to decrease A and increase W; in the absence of nuclear reactions A will be independent of time. At one particular time it is possible to normalize V so A = 1 at all mass shells. At subsequent times 1 - A will be the fraction of the initial rest mass energy turned to heat by nuclear reactions.

The total rest mass is equal to $\int \dot{A}\, dM_o$. However, what we will call the binding energy is the quantity $M_o - M$, what may or may not actually correspond to the amount of energy needed to disperse the mass to infinity. In all the numerical calculations in this paper $\dot{A}$ is zero and $A$ is normalized to unity; so, $M_o$ is the total rest mass energy. $M_o$ has the useful property of being an invariant of the motion for a given mass shell, and thus the change in $M_o - M$ is always equal to the negative of the change in $M$.

According to the first law of thermodynamics

$$\dot{W} + P\dot{V} = T\dot{S} + \sum_i \mu_i \dot{n}_i \quad , \tag{3.02}$$

where $T$ is the temperature and $S$ the specific entropy. The chemical potential terms will be lumped with $\dot{A}$ in what follows. Equation (2.15), with $L_o = L_1 = L_2 = 0$, becomes

$$\dot{Q}Y = (\dot{A} + T\dot{S})Y = -\frac{1}{a^2}\frac{\partial}{\partial M_o}(4\pi R^2 K a^2) \quad . \tag{3.03}$$

$\dot{Q}$ is the rate at which heat is being added to the mass shell, per unit invariant mass. The gradient of $a$ associated with the divergence of the energy flux is due to the gravitational red shift of the energy being transported.

In equation (2.08) we may define

$$\dot{M}_K = -4\pi R^2 K Z a \tag{3.04}$$

to be the non-adiabatic rate of change of the total energy inside a mass shell. Substituting in equation (3.03),

$$\dot{Q} = Y\frac{\partial}{\partial M_o}\left(\frac{\dot{M}_K}{YZ}\right) \quad . \tag{3.05}$$

Inverting the equation,

$$\dot{M}_K = YZ \int \frac{\dot{Q}Z}{YZ} \, dM_o \quad . \tag{3.06}$$

Equation (2.07) for the total energy can be written

$$M = \int (A + W)Z \, dM_o \quad , \tag{3.07}$$

assuming that $K \ll P$ and that $U$ is much less than the speed of sound so the term involving $K$ in (2.07) can be neglected in what follows. We call $(A + W)Z$ the local energy per unit invariant mass. Equation (3.06) may now be interpreted as saying that in a quasi-static situation the effect of a non-adiabatic change in local energy, $\delta QZ$, at radius 1 on the total energy inside a larger radius 2 is reduced by a factor $(YZ)_2/(YZ)_1$. The ordinary gravitational red shift for local observers is just $Y_2/Y_1$. The factor $Z$ includes the red shift to infinity, where in effect the total energy is measured.

For an ideal fluid, with $K = 0$, the non-gravitational force per unit mass $f$ is given by

$$f = - \frac{P'X}{P+E} \quad , \tag{3.08}$$

and

$$\frac{Y'}{Y} = \frac{P'}{P+E} \quad . \tag{3.09}$$

If

$$\frac{E'}{P+E} = - \frac{V'}{V} \quad , \tag{3.10}$$

we may obtain an explicit expression for $Y$,

$$Y = (P+E)V \quad , \tag{3.11}$$

within a scale transformation of the time.

Equation (3.10) is equivalent to

$$A' + TS' = 0 \tag{3.12}$$

if $V$ is normalized so the invariant mass per baryon is independent of radius. If $S$ and $A$ are independent of radius in this normalization, we say that the mass is isentropic. If $\dot{A} = 0$ everywhere this isentropic condition continues to hold as long as no shock waves arise, and $Y$ is given by equation (3.11).

In our discussion of the hydrodynamics of the ideal fluid it is convenient to define

$$\Gamma_1 - - \frac{V}{P}\left(\frac{\partial P}{\partial V}\right)_S \quad , \tag{3.13}$$

$$\Gamma_2 - 1 = - \frac{V}{T}\left(\frac{\partial T}{\partial V}\right)_S \quad . \tag{3.14}$$

For either a completely non-relativistic or a completely relativistic perfect gas, $\Gamma_1 = \Gamma_2$.

Taking $U$ and $V$ as the basic dependent variables, we can write the hydrodynamic equations, which are a hyperbolic system of partial differential equations, in a form which displays the characteristics. We use

$$P' = - \Gamma_1 P \frac{V'}{V} + (\Gamma_2 - 1) \frac{TS'}{V} \quad . \tag{3.15}$$

This equation is only valid if $V$ changes radially the same way it does in time; the amount of invariant mass per baryon must be independent of radius. If the rest mass per baryon varies with radius, $A'$ will not be zero. After some manipulation, we obtain

$$(\dot{U}_Y \pm CU'_X) \mp 2C\left(\frac{\dot{V}}{V}_Y \pm C\frac{V'}{V}X\right)$$

$$= -2(\Gamma_2 - 1)\frac{TS'_X}{(P+E)V} \mp 2C\,\frac{2U}{R} - M/R^2 - 4\pi PR \quad , \tag{3.16}$$

and

$$\dot{A} + T\dot{S} = 0 \quad . \tag{3.17}$$

Usually the adiabaticity of the ideal fluid approximation will also involve taking $\dot{A} = 0$. The quantity $C$, which is the velocity at which weak low frequency disturbances propagate through the fluid as measured by an observer comoving with the fluid, the speed of sound, is given by

$$C = \left(\frac{\Gamma_1 P}{P+E}\right)^{\frac{1}{2}} \quad . \tag{3.18}$$

This must be less than one to avoid conflict with causality.

A general feature of solutions of the hydrodynamic equations is that there may exist surfaces across which some of the components of the energy-momentum tensor are discontinuous. There are two types of discontinuity surfaces possible in classical hydrodynamics, the contact discontinuity and the shock wave. The contact discontinuity is stationary with respect to the matter and the pressure is the same on both sides. The shock wave is characterized by discontinuities in the pressure and velocity of flow. The surface of a star not losing mass would be a contact discontinuity on a scale of analysis which neglects the structure of the atmosphere, and the jump conditions for going from just inside the surface of the star to essentially empty space determine the outer boundary conditions on a solution of the hydrodynamic equations inside the star.

In comoving coordinates in general relativity the basic difference between the two types of discontinuity is the behavior of the metric tensor in the region of the discontinuity, which is considered a hypersurface in space-time. We can require that the coordinates $r$ and $t$ be continuous across the discontinuity. The continuity of $r$ is related to the assumption that no baryons can be created or destroyed, which we have used in the definition of comoving coordinates. The continuity of $t$ is assured by the freedom of choice of coordinate time scale. However, it is clear from the time dilation of special relativity that if the velocity of fluid flow is discontinuous, the proper time rates and therefore $a$ and $Y$ are discontinuous. Thus for a shock wave $a$ is discontinuous, while for a contact discontinuity $a$ is continuous.

Synge (18) discusses jump conditions which continue a solution across a hypersurface of discontinuity on the basis of the concept of admissable coordinates. It is assumed that there always exists some coordinate system, the admissable coordinates, in which the components of the metric tensor and their first derivatives are continuous across the hypersurface, although second derivatives may be discontinuous. Let $g(x^i) = 0$ be the equation of the hypersurface of discontinuity. Then if admissable coordinates exist, in any coordinate system in which the components of the metric tensor are continuous the set of quantities

$$T_i^{\ k} \frac{\partial g}{\partial x^k} \tag{3.19}$$

are continuous across the hypersurface.

The function $g$ can be written in comoving coordinates as

$$g(r,t) = r - r_s(t) = 0 \quad . \tag{3.20}$$

$r_s(t)$ is the position of the discontinuity as a function of time. Since $r$ and $t$ are continuous, $dr_s/dt$ is the same on both sides. The jump conditions (3.19) can be applied only if $a$ is continuous and $dr_s/dt = 0$; then the scale of $r$ can be chosen on the two sides of the discontinuity to make $b$ continuous even though $V$ is not. The jump conditions become, in the presence of radiation, that

$$K \tag{3.21}$$

and

$$P + L_1 \tag{3.22}$$

be continuous.

Of particular interest is the boundary condition at the surface of a mass. Outside the mass we have $E = P = 0$, but $L_1$ and $K$ will not be zero if the mass is emitting radiation. If the amount of this radiation is negligible, or if the transport of the radiation near the surface is such that $L_-$ as well as $K$ is continuous across the surface, the boundary condition at the surface becomes simply $P = 0$. The energy density $E$ may be discontinuous at the surface.

In order to obtain the jump conditions for a shock wave in comoving coordinates, we shall prove that the conditions (3.19) apply in Schwarzschild coordinates and then transform back to comoving coordinates separately on the two sides of the shock. We will denote the Schwarzschild time and radius coordinates by $T$ and $R$, and let

$g_{00} = A^2$, $g_{11} = -B^2$, $g_{22} = -R^2$. In these coordinates

$$B = 1/Z = (1 - 2M/R)^{-\frac{1}{2}} . \tag{3.23}$$

The intrinsic geometry of the hypersurface of discontinuity is specified completely by the first and second fundamental forms,

$$\Phi_1 = g_{ik}\, dx^i\, dx^k \tag{3.24}$$

and

$$\Phi_2 = n_{i;k}\, dx^i\, dx^k . \tag{3.25}$$

The displacements are constrained to lie in the hypersurface, and $n_i$ is the unit four-vector normal to the hypersurface. The requirement that the intrinsic geometry of the hypersurface be the same viewed from either side, or that $\Phi_1$ and $\Phi_2$ be continuous across the shock for arbitrary displacements in the hypersurface, is equivalent to requiring the existence of a coordinate system in which the metric tensor components and their first derivatives are continuous (19).

In Schwarzschild coordinates the first fundamental form is, with $R_s$ the radius of the shock as a function of time,

$$\Phi_1 = A^2\, dT^2 - B^2 \left[\frac{dR_s}{dT}\right]^2 dT^2 - R^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) . \tag{3.26}$$

Let

$$v_s = \frac{dR_s}{dT}\,\frac{B}{A} , \tag{3.27}$$

the proper velocity of the shock front as seen by a local observer at constant R. dT, $d\theta$, and $d\varphi$ are independent; so, we obtain the jump

conditions that

$$A^2 (1 - v_s^2) \tag{3.28}$$

and

$$R^2 \tag{3.29}$$

be continuous. We have used the freedom of scale transformations in time to make $dT$ the same on both sides of the shock. From (3.29) the circumferential radius $R$ is the same on both sides of the shock.

The unit normal to the shock hypersurface has the non-zero components

$$n_0 = Av_s/(1 - v_s^2)^{\frac{1}{2}} , \tag{3.30}$$

$$n_1 = - B/(1 - v_s^2)^{\frac{1}{2}} . \tag{3.31}$$

We only need the continuity of $\Phi_2$ for a displacement in the $\theta$ direction:

$$n_{2;2} = - \frac{R}{B} (1 - v_s^2)^{-\frac{1}{2}} . \tag{3.32}$$

Combining this with (3.28) we get that $A/B$ is continuous, which implies that $v_s$ is continuous across the shock and thus that $A$ and $B$ are separately continuous. We have now shown that we may apply the jump conditions (3.19) in Schwarzschild coordinates. Also, from the equation (3.23) for $B$ we see that $M$ is continuous across the shock. The relaxing of the requirement on admissable coordinates to allow discontinuity of the first derivatives of the metric tensor in maximally continuous coordinates would be necessary to obtain infinite energy density and finite change in $M$ in the shock, as discussed

in general terms by Papapetrou and Treder (20). However, this is not physically reasonable in a hydrodynamic shock.

We will write the shock wave jump conditions assuming the energy-momentum tensor is that of an ideal fluid. If $v$ is the momentum per unit mass of the fluid in the Schwarzschild coordinates, from (3.19) the expressions

$$[E + (P + E)v^2]v_s - (P + E)v(1 + v^2)^{\frac{1}{2}} \tag{3.33}$$

and

$$(P + E)v(1 + v^2)^{\frac{1}{2}}v_s - P - (P + E)v^2 \tag{3.34}$$

are continuous across the shock.

The transformation to comoving coordinates is discussed in Chapter I. The velocity of the shock according to an observer comoving with the matter is

$$v_c = bY\frac{dr_s}{dt} \quad . \tag{3.35}$$

The time scales are chosen so $dr_s/dt$ is the same on both sides of the shock. From the transformation equations (1.22), (1.23), (1.28), and (1.29),

$$v_s = \frac{U + Zv_c}{Z + Uv_c} \quad . \tag{3.36}$$

Since $v_s$ is continuous one jump condition is that

$$\frac{U + Zv_c}{Z + Uv_c} \tag{3.37}$$

be continuous. When (3.36) is substituted into (3.33) and (3.34) some

algebra gives jump conditions requiring the continuity of

$$\frac{EZv_c - PU}{Z + Uv_c} \tag{3.38}$$

and

$$\frac{EUv_c - PZ}{Z + Uv_c} \quad . \tag{3.39}$$

The jump condition (3.37) can be rewritten in terms of $(1 - v_s^2)^{-\frac{1}{2}}$ or $v_s/(1 - v_s^2)^{\frac{1}{2}}$ to obtain the equivalent conditions that

$$\frac{Z + Uv_c}{(1 - v_c^2)^{\frac{1}{2}}} \tag{3.40}$$

$$\frac{U + Zv_c}{(1 - v_c^2)^{\frac{1}{2}}} \tag{3.41}$$

be continuous. The continuity of R and t implies the continuity of

$$\frac{dR_s}{dt} = \dot{R} + R' \frac{dr_s}{dt} = a\,(U + Zv_c) \quad . \tag{3.42}$$

Combining (3.41) and (3.42) we obtain

$$a\,(1 - v_c^2)^{\frac{1}{2}} \quad , \tag{3.43}$$

which within continuous factors is identical to the jump condition that

$$\frac{1}{V} \frac{v_c}{(1 - v_c^2)^{\frac{1}{2}}} \tag{3.44}$$

be continuous. Condition (3.44) is related to the conservation of mass in the classical limit.

If nuclear reactions take place at a finite rate in the shock, A is continuous, and we can use (3.44) to eliminate the rest mass energy from (3.38). Our final forms for the jump conditions corresponding to conservation of energy and momentum are, by way of (3.40),

$$\frac{E(Z - 1)v_c + v_c W/V - PU}{(1 - v_c^2)^{\frac{1}{2}}} \tag{3.45}$$

and

$$\frac{EUv_c - PZ}{(1 - v_c^2)^{\frac{1}{2}}} \quad . \tag{3.46}$$

Four independent jump conditions which determine the continuation of the hydrodynamic equations across a shock and remain non-singular when $2M/R > 1$ are (3.41), (3.44), (3.45), and (3.46).

The jump condition (3.43) can be used to show that if a is continuous either $dr_s/dt = 0$ or b is also continuous. The first alternative is a contact discontinuity; the second implies no discontinuity exists.

From (3.42) if $2M/R > 1$ in collapse, so $-U > Z$, the geometrical radius of even an outward moving shock decreases in time. The absolute value of $v_c$ must be less than one from causality.

In numerical calculation the only use of the jump conditions would be to check the results of an artificial viscosity, since direct use of the jump conditions is very awkward in a difference equation framework.

## IV. Radiative Transfer

The kinetic theory of matter in general relativity has been investigated in some detail by Tauber and Weinberg (9). However, their treatment is based on a phase space density function defined in the rest frame of each group of particles. When the particles are massless and travel on null rather than time-like geodesics this formulation is not convenient. Another more recent approach to transport problems has been that of Misner and Sharp (7), who have considered thermal diffusion in general relativity from the point of view of thermodynamics. Our treatment of radiative transfer is designed to make the description of the interaction of radiation with matter as simple and direct as possible. The radiative transfer equation we derive is applied in different approximations to the transport of neutrinos and electromagnetic radiation in spherically symmetric masses. In the case of local thermodynamic equilibrium we reproduce the results of Misner and Sharp.

The physical assumptions underlying our discussion of radiative transfer are that a large part of the energy present is in the form of "matter" whose particles have short mean free paths compared to those of the radiation particles. In our applications the matter will be treated as an ideal fluid. The radiation is composed of "particles" which may interact with matter by means of short range forces, but whose interactions with each other and long

range interactions with the matter are limited to the smooth averaged effects of gravitational fields, as determined by a macroscopic energy-momentum tensor. That is, the particles of the radiation travel along geodesics, null geodesics for massless particles, between scatterings. The scattering or absorption of a particle of radiation by the matter takes place in a region of space-time infinitesimal on the scale of the overall dynamics of the matter and radiation and can be represented by absorption and differential scattering cross-sections. Coherent scattering effects are assumed to be negligible; the "index of refraction" of the matter is taken to be unity. This picture of radiative transfer corresponds to the usual classical one, and is adequate for the treatment of photons and neutrinos in many astrophysical applications.

The presence of the matter means that one has a natural frame of reference in radiative transfer, that of observers comoving with the matter. It is for these observers that the description of the interaction between radiation and matter is simplest. We represent a comoving observer by an orthonormal tetrad as defined in Chapter II. The time-like component of the tetrad, $\lambda_{(0)}{}^{i}$, is the velocity four-vector of the matter, $v^{i}$. Since each mass element has an orthonormal tetrad associated with it, the $\lambda_{(a)}{}^{i}$ comprise a set of four vector fields, which are taken to be continuous functions of position.

If $p^i$ is the momentum four-vector of a particle of radiation, its projection on the tetrad of a comoving observer gives the energy

$$p^{(0)} = p^i \lambda(0)_i = p^i v_i \tag{4.01}$$

and momentum

$$p^{(\alpha)} = p^i \lambda(\alpha)_i \tag{4.02}$$

of the particle as seen by the local observer. If $m_o$ is the rest mass of the particle, assumed to be constant between scatterings,

$$p^i p_i = p^{(a)} p_{(a)} = m_o^2 \quad . \tag{4.03}$$

For a photon the energy $p^{(0)}$ is proportional to the frequency.

We describe the radiation by a number density of particles in phase space. Because of the role of the comoving observer there is no need to define a covariant eight-dimensional phase space, as has been done by Tauber and Weinberg (9). We define our phase space to have a volume element which is the product of a volume element $\Delta S$ in the three-space normal to the velocity four-vector of the matter, $v^i$, and a volume element in the momentum space of the comoving observer,

$$\Delta p = dp^{(1)} dp^{(2)} dp^{(3)} \quad . \tag{4.04}$$

Let

$$N(x^i, p^{(\alpha)})\ \Delta S\ \Delta p \tag{4.05}$$

be the number of radiation particles of a given type in the three-volume element $\Delta S$ about $x^i$ and in the momentum range $\Delta p$ about

$p^{(\alpha)}$. Each type of radiation particle (neutrino, photon, etc.) should be treated independently by a separate $N$. This is consistent with our assumption that the radiation interacts only with the matter, not with itself or other types of radiation.

The contribution of the radiation to the energy-momentum tensor may be expressed in terms of $N(x^i, p^{(\alpha)})$ using the construction of Synge (21) based on the particle flux four-vector $N_i$ defined by requiring $N_i v^i \Delta S$ to be the number of world lines crossing the hypersurface element $\Delta S$ with unit normal $v^i$. $N_i$ is independent of the direction of $v^i$ and is given by

$$N_i = \int N p_i \frac{dp^{(1)} dp^{(2)} dp^{(3)}}{p^{(0)}} \quad . \tag{4.06}$$

$\Delta p/p^{(0)}$ is invariant under Lorentz transformations of the reference tetrad, and thus $N$ is an invariant description of the radiation. The flux of a quantity across the hypersurface element $\Delta S$ if each particle carries an amount $q$ is given by $Q_i v^i \Delta S$,

$$Q_i = \int N p_i \, q \frac{\Delta p}{p^{(0)}} \quad , \tag{4.07}$$

if all particles cross the hypersurface in the same direction (going forward in time). In particular, if $q$ is the momentum four-vector $p_i$ of the particle we obtain the energy-momentum tensor

$$T_{ij} = \int N p_i \, p_j \frac{\Delta p}{p^{(0)}} \quad , \tag{4.08}$$

of the radiation. The energy-momentum tensor projected on the tetrad of the local observer is

$$T^{(a)\,(b)} = \int N\, p^{(a)} p^{(b)} \frac{\Delta p}{p^{(0)}} \quad , \tag{4.09}$$

and can be evaluated if $N$ is known as function of $p^{(\alpha)}$.

The propagation of waves in curved space is discussed by Synge (22) and the propagation of massless elementary particles by Pagels (23). From each point of view, as long as coherent scattering is unimportant photons and neutrinos travel along null geodesics. The momentum vector $p^i$ of a particle is tangent to the geodesic it is following, and the parameter $u$ along the geodesic can be normalized such that

$$p^i = \frac{dx^i}{du} \quad . \tag{4.10}$$

The equation of the geodesic is that the absolute derivative of the momentum four-vector is zero;

$$\frac{\delta p^i}{\delta u} = p^i{}_{;k} \frac{dx^k}{du} = 0 \quad . \tag{4.11}$$

In deriving the radiative transfer equation we will not insist on complete rigor and generality of the argument in order to discuss from the viewpoint of a local observer the behavior of the particles in phase space.

At a point $x^i$ in space-time consider a pencil of radiation which intersects the hypersurface normal to the four-velocity $v^i$ of

the local observer in the three-volume element $\Delta S$ and which has a range of momentum $\Delta p$ about a three-momentum $p^{(\alpha)}$ as measured by the local observer. The particles occupy the volume element $\Delta S \Delta p$ in phase space and the number of particles is $N \Delta S \Delta p$. The volume element in momentum space is given by (4.04). We will explicitly construct the three-volume element $\Delta S$ as a rectangular solid with corners at $x^i$, $x^i + \Delta x^{(1)} \lambda_{(1)}{}^i$, $x^i + \Delta x^{(2)} \lambda_{(2)}{}^i$, $x^i + \Delta x^{(3)} \lambda_{(3)}{}^i$, $x^i + \Delta x^{(1)} \lambda_{(1)}{}^i + \Delta x^{(2)} \lambda_{(2)}{}^i$, etc. The numerical value of $\Delta S$, since the $\lambda_{(\alpha)}{}^i$ are three mutually perpendicular directions in the locally Euclidean hypersurface normal to $v^i$, is $\Delta x^{(1)} \Delta x^{(2)} \Delta x^{(3)}$.

We set the parameter $u$ of the geodesics of the particles equal to zero when the particles' world lines intersect $\Delta S$. The pencil of radiation defined above intersects the hypersurface normal to the four-vector $v^i(x^i + \frac{dx^i}{du} u_o)$ at the point $x^i + \frac{dx^i}{du} u_o$ in space-time, forming a three-volume element $\Delta S'$, and the particles have a new range of momenta $\Delta S'$ about the three-momentum $p^{(\alpha)} + \frac{dp^{(\alpha)}}{du} u_o$. The number of particles intersecting $\Delta S'$ in the momentum range $\Delta p'$ is

$$N(x^i + \frac{dx^i}{du} u_o,\ p^{(\alpha)} + \frac{dp^{(\alpha)}}{du} u_o)\ \Delta S'\ \Delta p' \quad . \tag{4.12}$$

Let the net rate of addition, due to scattering, emission, and absorption, of particles to the three-volume element $\Delta S$ about $x^i$ in the momentum range $\Delta p$ about $p^{(\alpha)}$ be

$$Q(x^i,\ p^{(\alpha)})\ \Delta S\ \Delta p \tag{4.13}$$

per unit proper time of the local observer whose four-velocity is the normal to $\Delta S$. The source function $Q$ under our assumptions depends only on local properties of the matter and radiation and can be taken over directly from classical radiative transfer theory.

The difference in proper time between two approximately parallel hypersurfaces at $u = 0$ and $u = u_o$ is the component of the displacement in space-time along the geodesic, $\frac{dx^i}{du} u_o$, along the normal to the hypersurfaces, $v^i$. Thus the net number of particles added to the pencil of radiation between $u = 0$ and $u_o$ is approximately

$$Q \, \Delta S \, \Delta p \quad v_i \frac{dx^i}{du} u_o \quad .$$

This is equal to the difference between (4.12) and (4.05), and in the limit that $u_o$ is very small we obtain

$$\frac{dN}{du} + N \frac{\Delta S' - \Delta S}{u_o \Delta S} + N \frac{\Delta p' - \Delta p}{u_o \Delta p} = Q \, v_i \frac{dx^i}{du} \quad , \tag{4.14}$$

where

$$\frac{dN}{du} = \frac{\partial N}{\partial x^i} \frac{dx^i}{du} + \frac{\partial N}{\partial p(\alpha)} \frac{dp^{(\alpha)}}{du} \quad . \tag{4.15}$$

Equation (4.14) is the equation of transfer; we now need to evaluate the rate of dispersion of the pencil in phase space. This will be broken into two parts. First consider $(\Delta S' - \Delta S)/u_o \Delta S$, using our explicit construction of $\Delta S$ as a rectangular solid. The volume element $\Delta S'$ in the new hypersurface is calculated assuming that all the particles have the same momentum $p^{(\alpha)}$. The new position of the

reference corner is

$$x^i + \frac{dx^i}{du}(x^i)\, u_o \quad , \tag{4.16}$$

and of the $\alpha$th neighboring corner,

$$x^i + \Delta x^{(\alpha)}\lambda_{(\alpha)}{}^i + \frac{dx^i}{du}(x^i + \Delta x^{(\alpha)}\lambda_{(\alpha)}{}^i\, u_\alpha) \quad , \tag{4.17}$$

$\alpha = 1,2,3$. $\frac{dx^i}{du}$ depends on the initial position of the particle because of the dependence of the tetrad vector fields on position. $u_\alpha$ is determined by requiring the new displacements between corners,

$$\Delta x^{(\alpha)}\lambda_{(\alpha)}{}^i + u_o\, \Delta x^{(\alpha)}\lambda_{(\alpha)}{}^k \frac{\partial}{\partial x^k}\left(\frac{dx^i}{du}\right)_p (1,2,3)$$

$$+ (u_\alpha - u_o)\frac{dx^i}{du} \quad , \tag{4.18}$$

to be orthogonal to the new value of $v_i$,

$$v_i + \frac{\partial v_i}{\partial x^k}\frac{dx^k}{du}\, u_o \quad . \tag{4.19}$$

When the new displacement of the $\alpha$th corner is projected on the $\beta$th space-like member of the tetrad at $x^i + \frac{dx^i}{du}u_o$, a three by three matrix is formed whose determinant is the value of the new volume element $\Delta S'$. Using the orthonormality relations for the tetrads and the relations between $\frac{dx^k}{du}$, $p^k$, and $p^{(a)}$ based on (4.10) and (4.11), we can simplify the result to

$$\frac{\Delta S' - \Delta S}{u_o \Delta S} = p^{(a)}\lambda_{(a)}{}^k{}_{;k} \quad . \tag{4.20}$$

Similarly, the rate of dispersion in momentum space can be

calculated assuming the initial momentum space volume element is a "rectangular solid" with one corner at the reference momentum $p^{(\alpha)}$. All the particles are assumed to start at the same point in space-time $x^i$, but because of their different momenta they will intersect the new hypersurface at different space-time points. Requiring the new hypersurface to be orthogonal to the new $v^i$ of the reference particle determines the values of the geodesic parameters at the new hypersurface for the "corners" of the volume element in momentum space. We obtain at first

$$\frac{\Delta p' - \Delta p}{u_o \Delta p} = \frac{\partial}{\partial p^{(\alpha)}} \frac{dp^{(\alpha)}}{du} - \frac{1}{p^{(0)}} \frac{dp^{(0)}}{du} \quad ; \tag{4.21}$$

using

$$\frac{dp^{(a)}}{du} = p^i \lambda^{(a)}_{i;k} p^k \quad , \tag{4.22}$$

$$\frac{\partial p^{(0)}}{\partial p^{(\alpha)}} = p^{(\alpha)}/p^{(0)} \quad , \tag{4.23}$$

and rearranging terms,

$$\frac{\Delta p' - \Delta p}{u_o \Delta p} = - p^{(a)} \lambda_{(a)}{}^{k}{}_{;k} \quad . \tag{4.24}$$

The result is, then, that the dispersion in ordinary space is canceled by the dispersion in momentum space; so,

$$\frac{dN}{du} = Q v_i \frac{dx^i}{du} \quad . \tag{4.25}$$

If there is no scattering of the radiation particles the volume in phase space occupied by a given group of particles remains constant

as the particles move along geodesics in a gravitational field. General relativity obeys Liouville's theorem; gravitation in general relativity is a "conservative" field.

We wish to apply equation (4.25) to the study of radiative transfer in a spherically symmetric mass. Because of the spherical symmetry a local observer can only distinguish one spatial direction, the radial direction. The phase space distribution function can only depend on the coordinates $t$ and $r$, the magnitude of the momentum,

$$p = ( - p^{(\alpha)} p_{(\alpha)} )^{\frac{1}{2}} \quad , \tag{4.26}$$

and the angle the momentum makes with the radial direction as measured by a local observer, $\theta_p$. In comoving coordinates the tetrad describing the local comoving observers as a function of $r$ and $t$ is most conveniently chosen as in (2.03). Then

$$p^{(1)}/p = \cos \theta_p \quad . \tag{4.27}$$

The symmetry in momentum space makes it desirable to change the momentum space coordinates from the $p^{(\alpha)}$ to $p$, $x_p = \cos \theta_p$, and $\varphi_p$ is the azimuthal angle about the radial direction. The volume element becomes

$$\Delta p = p^2 \, dp \, d\Omega_p \quad ; \tag{4.28}$$

after integrating over $\varphi_p$ the solid angle element is

$$d\Omega_p = 2\pi \, dx_p \quad . \tag{4.29}$$

The types of radiation we will be discussing, neutrinos and photons, are characterized by massless particles; so, from now on we

assume

$$p = p^{(0)} \quad . \tag{4.30}$$

To conform with the usual conventions of radiative transfer we will define, using (4.30),

$$F(t,r,\nu,x_p) = h\, p^3\, N(t,r,p,x_p) \quad . \tag{4.31}$$

For massless particles, $F\, d\nu\, d\Omega_p$ is the amount of energy per unit volume due to radiation directed in solid angle $d\Omega_p$ with frequency $\nu = p/h$ in the range $d\nu$. To be consistent we also will redefine our source function $Q$ as

$$\Sigma = h\, p^3\, Q(t,r,p,x_p) \quad ; \tag{4.32}$$

so, $\Sigma$ will be the amount of energy emitted per unit volume per unit solid angle per unit frequency range per unit proper time.

The transfer equation in terms of $F$ and $\Sigma$ is

$$\frac{dF}{du} - 3F \frac{1}{\nu} \frac{d\nu}{du} = a \frac{dt}{du} \Sigma \quad . \tag{4.33}$$

The geodesic equations

$$\frac{d}{du}\left(\frac{dx^i}{du}\right) + \Gamma^i_{jk} \frac{dx^j}{du} \frac{dx^k}{du} = 0 \tag{4.34}$$

may be rewritten in terms of $\nu$ and $x_p$ using, for massless particles,

$$a \frac{dt}{du} = h\nu \quad , \quad \frac{b}{a} \frac{dr}{dt} = x_p \quad ,$$

$$\frac{1}{a}\left[\left(R \frac{d\theta}{dt}\right)^2 + \left(R \sin\theta \frac{d\varphi}{dt}\right)^2\right]^{\frac{1}{2}} = \left(1 - x_p^{\,2}\right)^{\frac{1}{2}} \quad . \tag{4.35}$$

The result is

$$\frac{Y}{\nu}\frac{d\nu}{dt} = -\left[f x_p + q x_p^2 + \frac{U}{R}(1 - x_p)^2\right] , \tag{4.36}$$

$$Y\frac{dx_p}{dt} = (1 - x_p^2)\left[\frac{Z}{R} + \frac{U}{R}x_p - f - q x_p\right] . \tag{4.37}$$

f and q are given by (1.05) and (1.06). The terms in equation (4.36) for the rate of change of frequency along a geodesic arise from the gravitational energy shift between neighboring mass shells ( $fx_p$ ), the Doppler shift due to radial expansion ( $qx_p^2$ ), and the Doppler shift due to the divergence of mass elements following different radial lines ( $\frac{U}{R}(1 - x_p^2)$ ).

When (4.36) and (4.37) are substituted into (4.33) we obtain the useful form of the radiative transfer equation,

$$\dot{F}Y + x_p F'X - \nu\frac{\partial F}{\partial \nu}\left[fx_p + qx_p^2 + \frac{U}{R}(1 - x_p^2)\right] ,$$

$$+ \frac{\partial F}{\partial x_p}(1 - x_p^2)\left[\frac{Z}{R} + \frac{U}{R}x_p - f - qx_p\right]$$

$$+ 3F\left[fx_p + qx_p^2 + \frac{U}{R}(1 - x_p^2)\right] = \Sigma . \tag{4.38}$$

The contribution of the radiation to the energy-momentum tensor in terms of F is

$$L_0 = \int_0^\infty d\nu \int_{-1}^{1} 2\pi dx_p\, F , \tag{4.39}$$

$$L_1 = \int_0^\infty d\nu \int_{-1}^{1} 2\pi dx_p\, x_p^2\, F , \tag{4.40}$$

$$L_2 = \int_0^\infty d\nu \int_{-1}^{1} 2\pi dx_p\, \tfrac{1}{2}(1 - x_p^2)\, F , \tag{4.41}$$

$$K = \int_0^\infty d\nu \int_{-1}^{1} 2\pi dx_p \, x_p \, F \quad . \tag{4.42}$$

If we define

$$\sigma = \int_0^\infty \Sigma \, d\nu \tag{4.43}$$

to be the net energy emitted from the matter per unit proper time per unit volume per unit solid angle, we may write the integral of equation (4.38) over all energies and angles as

$$\dot{L}_0 Y + qL_0 + \frac{2U}{R}L_0 + qL_1 + \frac{2U}{R}L_2$$

$$+ K'X + 2fK + \frac{2Z}{R}K = \int \sigma \, d\Omega_p \quad , \tag{4.44}$$

using the identity that

$$L_0 = L_1 + 2L_2 \quad . \tag{4.45}$$

If equation (4.38) is multiplied by $x_p$ before integration the result is

$$\dot{K}Y + 2K(q + \frac{U}{R}) + L_1'X + \frac{2Z}{R}L_1 - \frac{2Z}{R}L_2$$

$$+ fL_0 + fL_1 = \int \sigma \, x_p d\Omega_p \quad . \tag{4.46}$$

Now $\int \sigma \, d\Omega_p$ is just the net rate per unit volume that the matter is losing energy. Since $c = 1$ in our units, $\int \sigma \, x_p d\Omega_p$ is equal to the net rate per unit volume at which momentum is being emitted by the matter as measured by a local comoving observer. Thus it is the negative of the force per unit volume on the matter due to radiation reaction, or radiation pressure. When (4.44) is combined with equation (2.15) for the covariant divergence of the energy-momentum tensor of an ideal

fluid plus radiation, we obtain

$$\left[(\dot{E}V)Y + P\dot{V}Y\right]/V = -\int \sigma \, d\Omega_p \quad . \tag{4.47}$$

Equations (4.46) and (2.16) give

$$f = -\frac{P'X}{P+E} - \frac{\int \sigma \, x_p \, d\Omega_p}{P+E} \quad . \tag{4.48}$$

The result (4.47) just says that energy put into the radiation field is lost from the matter in a way consistent with the thermodynamics of an ideal fluid. Equation (4.48) confirms the identification of $P + E$ as the inertial mass per unit volume of the ideal fluid. The combined result of the two equations is to demonstrate that the local energetics and the local acceleration of the matter as measured by a comoving observer are affected by the presence of the radiation only if the radiation is scattered, emitted, or absorbed by the matter. Special relativistic calculations in the frame of the comoving observer are adequate to describe the local interaction of the radiation with the matter.

The usual assumption of classical radiative transfer theory is that the unstimulated emission of radiation from a mass element of locally isotropic matter is isotropic. The rates of absorption and stimulated emission have no angular dependence other than that of the flux $F$ incident on the mass element. The rate of scattering of the radiation will in general depend on the angle between the incident ray and the scattered ray but will also be proportional to the incident flux. All of these considerations are as valid in general relativity as they

are classically. The usual convention (24) is to write

$$\Sigma = (j_\nu - \varkappa_\nu F)/V \quad . \tag{4.49}$$

$j_\nu$ is the emission coefficient and $\varkappa_\nu$ is the absorption coefficient. Both are normalized per unit invariant mass since usually the rate of emission and absorption will be proportional to the number of particles of the matter. In general, $j_\nu$ and $\varkappa_\nu$ are functions of the frequency and the internal properties of the matter; they are isotropic if the matter is locally isotropic and in the presence of scattering or stimulated emission if the radiation field is also isotropic.

Even approximate solution of the equation of radiative transfer is in general a very complicated numerical problem, especially in the dynamic situations likely when general relativistic effects are important. We address ourselves only to two special cases: The radiation is not scattered or absorbed after being emitted, as for neutrinos at all except very high densities of the matter; the radiation is in local thermodynamic equilibrium with the matter.

When scattering of the radiation can be neglected $(\varkappa_\nu = 0)$ the invariance of the phase space particle density function along a geodesic means that if the geodesics are known the problem is solved in principle. However, it is not likely to be practical to keep track of all the geodesics passing through each point in space-time. A simple approximation which crudely follows the escape of the radiation from the mass while treating the non-gravitational interaction of the radiation with the matter exactly is to assume the radiation field is the sum of an isotropic component G and a purely radial outgoing component

K such that $L_0 = G + K$, $L_1 = \frac{1}{3}G + K$, $L_2 = \frac{1}{3}G$, and $K = K$. Letting $\int \sigma \, d\Omega_p = H$ and requiring $\int \sigma \, x_p \, d\Omega_p = 0$ (isotropic emission) gives the two equations

$$\dot{G}Y + \frac{4}{3}G\,\frac{\dot{V}}{V}Y = H + \frac{1}{3}(Ga^4)'X/a^4 \quad . \tag{4.50}$$

$$(\dot{K}V)Y/V + (R^2Ka^2)'X/(R^2a^2) = -\frac{1}{3}(Ga^4)'X/a^4 \quad . \tag{4.51}$$

Energy is first injected into the isotropic part of the radiation field and then converted into outward moving radiation at the rate $-\frac{1}{3}(Ga^4)'X/a^4$ per unit volume. The boundary conditions on the equations are $G = 0$ outside the neutrino producing region and $K = 0$ at $R = 0$. The component $G$ is propagated inward at one-third the speed of light; the component $K$ is propagated outward just at the speed of light. The approximation can be generalized by considering additional components of the radiation field having the angular dependence of the first, second, third, etc. Legendre polynomials in $x_p$. As it stands it can be used in numerical calculation of gravitational collapse to treat the gravitational effects of neutrino loss in a crude way.

If the radiation scatters many times over a distance in which the pressure and therefore the temperature vary only slightly, one might expect some sort of local thermal equilibrium to be set up between the radiation and the matter. The precise statement of what is meant by local thermodynamic equilibrium is that the ratio of the emission coefficient to absorption coefficient, B, be a function only of the frequency and temperature,

$$\Sigma = \frac{\kappa_\nu}{V} [B(\nu,T) - F] \quad . \tag{4.52}$$

For photons $B(\nu,T)$ is the Planck function. The only properties of $B(\nu,T)$ relevant to our discussion are

$$T \frac{\partial B}{\partial T} + \nu \frac{\partial B}{\partial \nu} = 3B \quad , \tag{4.53}$$

$$\int_0^\infty B \, d\nu = aT^4/4\pi \quad ; \tag{4.54}$$

a is here the Stefan-Boltzmann constant.

The equation of transfer can be written

$$\frac{V}{\kappa_\nu} \gamma \frac{d}{dt} \left( \frac{F}{\nu^3} \right) = - \frac{F}{\nu^3} + \frac{B}{\nu^3} \quad , \tag{4.55}$$

where we have modified $\kappa_\nu$ to include stimulated emission, as discussed in Chandrasekhar (24). The solution of this equation proceeds in exactly the same manner as the solution of the classical equation of transfer. If we define the parameter $\eta$ along a geodesic by

$$\frac{d}{d\eta} = \frac{V}{\kappa_\nu} \gamma \frac{d}{dt} \quad , \tag{4.56}$$

The solution of (4.55) can be written in the form

$$\frac{F}{\nu^3}(\eta_o) = \int_{-\eta_o}^\infty [B(-\eta)/\nu(-\eta)^3] \, e^{-\eta_o - \eta} \, d\eta \quad . \tag{4.57}$$

If $\kappa_\nu$ is large enough so the integrand is cut off by the exponential before B and $\nu$ have changed very much, we may evaluate the integral by expanding $B/\nu^3$ in a Taylor series and integrating term by term. If

$\eta_o$ is zero, to first order in $\eta$,

$$\underline{B}_3(-\eta) = \underline{B}_3(0) - \frac{\eta}{\nu 3}\frac{V}{\varkappa_\nu}\left[T\frac{\partial B}{\partial T}\left(\frac{\dot{T}}{T}Y + x_p\frac{\dot{T}'}{T}X\right) + \nu\frac{\partial B}{\partial \nu}\frac{Y}{\nu}\frac{d\nu}{dt}\right]$$

$$+ \frac{\eta}{\nu 3}\frac{V}{\varkappa_\nu} 3B\frac{Y}{\nu}\frac{d\nu}{dt} + O(\eta^2) \quad . \tag{4.58}$$

Using (4.58) the result for F is

$$F = B - T\frac{\partial B}{\partial T}\frac{V}{\varkappa_\nu}[fx_p + qx_p^2 + \frac{U}{R}(1 - x_p^2)]$$

$$- T\frac{\partial B}{\partial T}\frac{V}{\varkappa_\nu}\left[\frac{\dot{T}}{T}Y + x_p\frac{T'}{T}X\right] \quad . \tag{4.59}$$

When this is integrated over frequency and angle it is useful to define what is called the Rosseland mean opacity, $\varkappa_o$, by

$$\int_0^\infty \frac{T}{\varkappa_\nu}\frac{\partial B}{\partial T}\,d\nu \equiv \frac{1}{\varkappa_o}\int_0^\infty T\frac{\partial B}{\partial T}\,d\nu = aT^4/(\pi\varkappa_o) \quad . \tag{4.60}$$

From equations (4.39) - (4.42) the energy-momentum tensor to first order in $V/\varkappa_o$ is then

$$L_0 = aT^4 - \frac{V}{\varkappa_o}\left[(a\dot{T}^4)Y + \frac{4}{3}aT^4\frac{\dot{V}}{V}Y\right] \quad , \tag{4.61}$$

$$L_1 = \frac{1}{3}L_0 - \frac{16}{45}\frac{V}{\varkappa_o}aT^4(q - \frac{U}{R}) \quad , \tag{4.62}$$

$$L_2 = \frac{1}{3}L_0 + \frac{8}{45}\frac{V}{\varkappa_o}aT^4(q - \frac{U}{R}) \quad , \tag{4.63}$$

$$K = -\frac{V}{\varkappa_o}\left(\frac{aT^4}{3}\right)'X - \frac{4}{3}\frac{V}{\varkappa_o}aT^4 f \quad . \tag{4.64}$$

$V/\varkappa_o$ in distance is an average mean free path for the radiation, and in time is the time for the radiation to go one mean free path. The expansion in $V/\varkappa_o$ is in either time or distance an expansion in the ratio of the mean free path to the interval over which the temperature and density change by a sizeable fraction.

To zeroth order in $V/\varkappa_o$ the energy-momentum tensor of the radiation is that of a perfect fluid. If we let $E_R = aT^4$ and $P_R = \frac{1}{3}aT^4$ be the equilibrium energy density and pressure of the radiation, denoting the energy density and pressure of the matter by $E_G$ and $P_G$,

$$\int \sigma \, d\Omega_p = (\dot{E}_R V)Y/V + P_R \frac{\dot{V}}{V} Y = \dot{Q}_R Y \tag{4.65}$$

and

$$\int \sigma \, x_p d\Omega_p = f\,(E_R + P_R) + P_R{}'X \tag{4.66}$$

to zeroth order in $V/\varkappa_o$. When (4.66) is substituted into (4.48) we obtain the expected result for f,

$$f = -\frac{(P_G + P_R)'X}{(E_G + P_G + E_R + P_R)} \,. \tag{4.67}$$

When this expression is substituted into equation (4.64) for the energy flux K,

$$K = -\frac{V}{\varkappa_o} \frac{(E_G + P_G)(E_R + P_R)}{(E_G + P_G + E_R + P_R)} \left[ \frac{P_R{}'X}{E_R + P_R} - \frac{P_G{}'X}{E_G + P_G} \right] \,. \tag{4.68}$$

The radiation pressure gradient term dominates the gas pressure gradient term as long as the gas particles are non-relativistic, $P_G \ll E_G$.

However, in the limit that the particles of the gas become relativistic, implying $P_G \rightarrow \frac{1}{3}E_G$, the two terms cancel if the ratio $P_R/P_G$ is independent of radius, and the energy flux due to the radiation in the comoving system goes to zero.

The existence of a thermal diffusion energy flux $K$ as found above is consistent with the concept of local thermodynamic equilibrium at least as long as $K \ll P_G$. Even if $P_R \gg P_G$ as it is in massive stars (2,3) one can imagine local thermal equilibrium being set up when the matter is static. However, the validity of this assumption when the density and temperature are varying in time is much more than doubtful.

The expressions (4.61) - (4.63) for the diagonal components of the energy-momentum tensor are essentially special relativistic in character since they only involve Doppler shifts due to the relative velocity of neighboring mass elements in the expanding or contracting gas. The special relativistic corrections to the thermal equilibrium values for $L_0$, $L_1$, and $L_2$ first order in $V/\mathcal{H}_0$ have been discussed by a number of authors (25, 26) under the heading of radiative viscosity. However, if we try to interpret these corrections in the context of local thermodynamic equilibrium we run into inconsistencies. For example, we may write

$$L_0 = aT^4 - \frac{V}{\mathcal{H}_0}\dot{Q}_R Y \quad . \tag{4.69}$$

The local energy density of the radiation is less than the thermal equilibrium value by an amount equal to the energy transferred from the matter to the radiation in the time for the radiation to go one

mean free path. It is as if energy lost by the matter is withheld from the thermal radiation field for one mean free path time, although there is no place in which this energy can be, consistent with the idea of local thermodynamic equilibrium. Our conclusion is that the so-called radiative viscosity terms are indications of the degree of breakdown of the local thermodynamic equilibrium assumption, but that their inclusion in any calculation without the inclusion of other deviations from local thermal equilibrium, in particular those due to B being an explicit function of time not exactly equal to the Planck function, is not justified. A necessary condition for the validity of local thermodynamic equilibrium is that the fractional change in the energy density of the radiation in one mean free path time be much less than one.

The gravitational effects of the radiation on the matter have been considered by us only in so far as the radiation contributes to the overall energy-momentum tensor. The radiation contributes to the total energy M through $L_0$ and K and to the radial stress gravitational force through $L_1$. Misner (27) considers the gravitational effects of a purely radial flux of radiation (neutrinos), for which $L_0 = L_1 = K$, and calls the attractive force due to $L_1$ an "induction" force associated with the rate of change of M due to the energy flux K. However, if the radiation is not purely radial $L_1$ does not equal K and the interpretation breaks down. The term involving K in equation (2.09) for $\dot{Z}$ is associated by Misner with the gravitational force due to $L_1$, while, as is discussed in Chapter II, this term is more precisely connected with the absence of a gravitational force due to K. Misner

(27) and Lindquist and Schwartz (28) also discuss in some detail the exterior solution for a radiating star and the transport of the energy to infinity, which we have done in this paper only for the quasi-static case (see Chapter III).

## V. Equilibrium and Stability

The construction of a model of a spherically symmetric mass in equilibrium in general relativity involves a pressure $P$, a specific thermal energy $W$, a specific volume $V$ (the inverse of the invariant mass density), an invariant mass $M_o$, a total energy $M$, and the circumferential radius $R$. There will also be a specific rest mass $A$ depending on the normalization of $V$. At times we will allow the presence of a small energy flux $K$, $K \ll P$, to study evolutionary development along a sequence of quasi-static equilibrium models. Any deviations from an ideal fluid in the energy-momentum tensor will be assumed to have a completely negligible effect on the hydrostatic equilibrium.

The equilibrium equation is obtained from equation (2.11) by letting $U$ and $\dot{U}$ go to zero. Inserting $f$ as given by (3.08) for an ideal fluid and using $M_o$ as the independent variable,

$$Y\dot{U} = -\frac{4\pi R^2 Z}{A+W+PV}\frac{\partial P}{\partial M_o} - M/R^2 - 4\pi PR = \Phi = 0 \quad , \tag{5.01}$$

$$Z = (1 - 2M/R)^{\frac{1}{2}} = X \quad , \tag{5.02}$$

if $r = R$.

The symbol $\Phi$ will be used as a shorthand for the algebraic form of (5.01) when we study pertubations away from the hydrostatic equilibrium. A convenient equation for $M$, exact if $K = 0$ and within the quasi-static approximation if $K \neq 0$, is equation (3.07),

$$M = \int (A + W)\, dM_o \quad . \tag{5.03}$$

An equation of state connects W, P, and V and may depend on how V is normalized. The proper thermodynamic normalization is such that the number of baryons per unit invariant mass is constant. A computationally convenient normalization is such that A = 1. In the models we calculate the two normalizations are equivalent.

Another relation between the thermodynamic variables as a function of radius is in general needed to specify a model. This second relation determines how the specific entropy varies as function of radius. The usual classical stellar models are characterized by a balance between energy generation and luminosity; one requires that the flux of energy across a spherical shell equal the rate of energy generation by nuclear reactions inside this shell. The energy transport is due to radiation if the temperature gradient is less than the adiabatic one,

$$-\frac{V}{P}\frac{dP}{dV} < \Gamma_1 \tag{5.04}$$

Convection becomes dominant if the opacity is too high for radiation to do the job at sub-adiabatic temperature gradients. If convection is efficient the temperature gradient is equal to the adiabatic one and the entropy is constant throughout the convective region. However, if convection cannot, because of low matter density or too high a required energy flux, transport the energy, the temperature gradient will become super-adiabatic and radiative transfer may again be important. The turbulent velocities associated with convection must be much

less than the thermal velocities (velocity of sound) of the gas if the convection is to be efficient and the presence of convection is not to disturb the hydrostatic equilibrium (29).

In general relativity convection is different from thermal diffusion in that it is not a local phenomenon; the distances over which mass elements move in convection are usually not small compared with the distances over which the gravitational potential energy changes significantly. However, if the turbulent pressure is a negligible fraction of the thermal pressure, in a quasi-static situation the energy flux due to the convection may be handled in the same way as the energy flux due to thermal diffusion. The effect of the turbulent pressure and energy density compared to that of the energy flux in the energy transfer equation (2.15) goes as the ratio of turbulent pressure to thermal pressure or the ratio of the product of the turbulent velocity and the overall velocity of the matter to the square of the speed of light, whichever is larger.

Exact treatment of the radiative equilibrium of a star causes computational difficulties and introduces physical complications into models which we wish to keep as simple as possible in order to be able to understand the effects of general relativity on their structure. Also, for models in quasi-static equilibrium undergoing Kelvin-Helmholtz contraction the precise distribution of entropy as a function of radius depends on the past history of energy loss since there is no longer a steady state in energy flux at a given mass shell. Thus the models we calculate will be one or both of the following types:

entropy per baryon constant as a function of radius or a polytrope of index n with

$$P = P_c \left(\frac{V_c}{V}\right)^{\frac{n+1}{n}} \quad . \tag{5.05}$$

If constant entropy is interpreted to mean the presence of convection the resultant mixing will keep A independent of radius and thus normalizable to A = 1 through the whole mass. In this sense constant entropy is equivalent to isentropy as defined in Chapter III. If $\Gamma_1$ is independent of radius an isentropic model will be a polytrope of the type (5.05) with

$$\Gamma_1 = \frac{n+1}{n} \quad . \tag{5.06}$$

Isentropic models have an additional importance besides their association with convection in that a close relation between binding energy and stability exists for them.

Once the equation of state and the variation of entropy with radius or polytropic index are specified the integration of the hydrostatic equilibrium equation (5.01) may be started at the center with given values of the central invariant mass density $(1/V)_c$ and the central ratio of pressure to invariant mass density $(PV)_c$ and continued until P drops to zero. Other parameters characterizing a model are the total energy M, the total invariant mass $M_o$, equal to the total rest mass if A = 1, and the radius R of the surface. Of particular interest are the dimensionless combinations of these parameters: $(PV)_c$; the ratio of the gravitational radius to radius, $2M/R$; and the fractional binding energy, $(M_o-M)/M_o$ .

At zero temperature (degenerate matter) only one of these parameters is independent, and might be taken to be $M_o$, or the total number of baryons. Such models have been calculated recently by Misner and Zapolsky (30) and Harrison (31) as well as in the classic paper of Oppenheimer and Volkov (32). Note that zero temperature implies zero entropy and thus constant entropy as a function of radius. At non-zero temperatures a one-parameter sequence of models may be constructed by letting $(1/V)_c$ vary monotonically but keeping the total number of baryons (or $M_o$) constant. Of course, the entropy distribution must be specified for each model. It is the latter type of sequence that we discuss in this paper. If A is kept constant such a sequence might correspond to a mass undergoing a generalized Kelvin-Helmholtz contraction in quasi-static equilibrium in which increase as well as decrease of total energy is allowed. An increase of the total energy is unphysical, of course, but there is a close relationship between a sequence with increasing M and constant A and a sequence with constant M and uniformly decreasing A due to, say, nuclear energy generation, if the fractional binding energy is much less than one.

Iben (33) has calculated both isentropic and non-isentropic equilibrium models based on an equation of state which is that of a non-relativistic gas plus radiation in local thermodynamic equilibrium. His non-isentropic models are polytropes defined in the same way as ours. In a sequence of isentropic models with constant $M_o$ he found a maximum of binding energy which was at progressively lower central temperatures

as $M_o$ was increased from $10^2 M_\odot$ to $10^9 M_\odot$

Tooper (34) constructs polytrope models which differ from ours in that it is the logarithmic derivative of pressure with respect to total energy density instead of specific volume which is held constant,

$$P = P_c \left( \frac{E}{E_c} \right)^{\frac{n+1}{n}} . \qquad (5.07)$$

The difference between the polytropes (5.07) and (5.06) is small when $PV \ll 1$, but is very significant when $PV > 1$. Tooper's models are physically unrealistic when the ratio $P/E$ approaches and exceeds unity. If $\Gamma_1 > 1$, we see from equation (3.18) that the speed of sound becomes greater than the speed of light in the limit $P/E \to \infty$. Another serious difficulty is that one expects an internal energy density of the same order of magnitude as the pressure, and as $P/E \to \infty$ this can be true only if the rest mass density $A/V$ becomes less than zero, which is nonsense. The usual limit that is put on $P/E$ is $P/E \leq 1/3$.

No such difficulties arise with our polytropes (5.06) since even as $PV \to \infty$ a suitable equation of state for $E$ can keep $P/E$ finite and the speed of sound less than the speed of light. Of course, the need for specifying an equation of state in addition to the polytrope is an increase in arbitrariness in the model over Tooper, although a necessary one if a binding energy is to be calculated.

A characteristic feature of both zero temperature and finite temperature sequences of models is the existence of oscillations in the binding energy as a function of central density in the region $(PV)_c \gtrsim 1$.

The interpretation of these oscillations for finite temperature isentropic models in terms of stability is one of the main results of this chapter.

We are interested in the general properties of equilibrium models, not the details associated with a particular equation of state. Therefore, our calculations have been made with equations of state of the type

$$W = \frac{PV}{\Gamma_1 - 1} \quad , \tag{5.08}$$

in which $\Gamma_1$ is the adiabatic index defined by (3.13) and is a constant for a given sequence of models. $V$ is normalized so $A = 1$.

An example of a physical system for which this type of equation of state is a good approximation in a range of pressures and densities for which general relativistic effects are of interest is an isentropic large mass in which radiation pressure is much larger than the gas pressure (2,3,33). If $\beta$ is the ratio of gas pressure to total pressure,

$$W = 3(1 - \tfrac{1}{2}\beta)PV \tag{5.09}$$

and

$$\Gamma_1 = \frac{4}{3} + \frac{\beta}{3}\frac{4 - 3\beta}{8 - 7\beta} \quad . \tag{5.10}$$

When $\beta \ll 1$,

$$\frac{PV}{\Gamma_1 - 1} = 3(1 - \tfrac{1}{2}\beta + \frac{3}{16}\beta^2 - \ldots)PV \quad . \tag{5.11}$$

Along an adiabat

$$\frac{V}{\beta}\left(\frac{\partial\beta}{\partial V}\right)_S = \frac{\beta/8}{1 + \beta/8} \quad . \tag{5.12}$$

Thus the fractional change of $\beta$ while $V$ changes by many orders of magnitude is much less than one if $\beta \ll 1$. To first order in $\beta$, which is important for stability although not for the structure of an isentropic mass, $W$ is given by (5.08) and

$$\Gamma_1 \simeq \frac{4}{3} + \frac{\beta}{6} \quad . \tag{5.13}$$

The quantity of critical importance for stability is

$$\frac{3}{4}\Gamma_1 - 1 \simeq \beta/8 \tag{5.14}$$

for these models of large masses.

We shall call our models calculated with equation of state (5.08) "constant $\Gamma_1$" models. The $\Gamma_1 = 4/3$ model is the limiting model one expects when all particles are relativistic ($PV \gg 1$) or when all of the pressure and thermal energy are due to zero mass particles. Note that the $\Gamma_1 = 4/3$ model satisfies $P = \frac{1}{3}E$ in the limit $PV \gg 1$. As long as $\Gamma_1 \leq 2$ the speed of sound does not exceed the speed of light as $PV \to \infty$. An isentropic constant $\Gamma_1$ model is a member of the class of polytropes defined by (5.05).

The equation of state (5.08) has the important computational advantage that it depends only on dimensionless quantities; since the equilibrium equation is also independent of the scale of mass, which determines the scale of length and time, the structure of a constant $\Gamma_1$ model depends on $(PV)_c$, but is independent of the precise value of $(1/V)_c$. There is an adjustable scale factor $\alpha$ relating masses in

in the ratio $\alpha$, radii in the ratio $\alpha$, densities and pressures in the ratio $\alpha^{-2}$, and specific volumes in the ratio $\alpha^{2}$. This greatly simplifies the numerical computation of a sequence of models with the same $M_o$ since an initial guess at the central density for a given $(PV)_c$ can be corrected by such a scale factor. Dimensionless quantities such as $PV$, $(M_o - M)/M_o$, and $2M/R$ are independent of the scale factor and thus independent of the mass of the model.

The results of the numerical integration are most strikingly presented as a plot of the fractional binding energy against the surface value of $2M/R$. Figure 1 is such a plot for a sequence of isentropic models with $\Gamma_1 = \frac{n+1}{n} = 2$ and is typical of such sequences with $\Gamma_1 > 4/3$. The binding energy rises to a maximum and then becomes negative, spiralling in toward a limit point as the central density and pressure become infinite. A similar plot is given in Figure 2 and 3 for $\Gamma_1 = \frac{n+1}{n} = 4/3$. Classically the binding energy is zero for a $\Gamma_1 = 4/3$ model and a post-Newtonian analysis such as that made by Fowler (5) shows that as long as $2M/R \ll 1$ the fractional binding energy should be negative and proportional to $(2M/R)^2$. The log-log plot of Figure 2 compares the exact curve with the straight line of the post-Newtonian theory, with the result that the post-Newtonian approximation is quite good up to $(PV)_c = 1 - 2 \times 10^{-2}$, corresponding to $2M/R = 2 - 4 \times 10^{-2}$. The spiral which is formed when $(PV)_c \gtrsim 1$ is displayed in Figure 3. The behavior of a sequence of non-isentropic models can be quite different; in Figure 4, with $\frac{3}{4}\Gamma_1 = 1.3$ and $\frac{3}{4}\frac{n+1}{n} = 1.0$, the spiral curves in the opposite direction and the binding energy remains positive as

$(PV)_c$ becomes large.

The internal structure of all the models when $(PV)_c > 1$ is characterized by a very rapid decrease of the density and pressure away from the origin, as is necessary if $2M/R$ is not to become greater than one inside the mass. In fact, the maximum value of $2M/R$ inside the models calculated is about one-half. The small high density core is surrounded by a comparatively tenuous envelope which actually contains most of the mass. An interesting feature of the core is that the value of $2M/R$ may oscillate several times near its maximum value before decreasing monotonically in the envelope to its value at the surface. As $(PV)_c$ becomes infinite a limiting model is approached which has infinite density and pressure at the center; only a small region near the center differs from this limiting model when $(PV)_c \gg 1$.

The stability of an equilibrium model in general relativity may be studied either by varying the hydrostatic equilibrium equation, as done by Chandrasekhar (12,13) and independently by the author, or by taking the second order adiabatic variation of the total energy under virtual displacements, as done by Dyson (35) for Newtonian gravitation. Our main interest is in the relation between a stationary point in the total energy along a sequence of equilibrium models and the stability of the radial modes of oscillation as a function of position along the sequence. We will find that in a sequence of isentropic models some radial mode of oscillation is in neutral equilibrium precisely at a stationary point in energy. From just a plot of $(M_o - M)/M_o$ against

$2M/R$ for such a sequence it is possible to say for a given model which modes of oscillation are stable and which unstable. Thus it will later be profitable to discuss in more detail the behavior of sequences of equilibrium models as displayed in Figures 1, 2, 3, and 4.

The radial modes of oscillation and their stability may be determined by considering adiabatic perturbations of the equilibrium equation (5.01) first order in the change in radius, $\delta R$, of a mass shell. To understand the relation of stability to binding energy it is helpful to consider first order non-adiabatic perturbations such as might be involved in quasi-static evolution along a sequence of equilibrium models. We will also need the second order adiabatic perturbations of certain quantities to relate the stability to virtual changes in the total energy under adiabatic perturbations.

The perturbations of such quantities as $M$, $Z$, $W$, $P$, and $V$ must be found in terms of the Lagrangian perturbation in $R$, $\delta R$. In general $\delta R$ will be a function both of radius $r$ and time $t$. It is convenient to let $r$ for a mass shell be the value of $R$ for this mass shell in the unperturbed equilibrium model. The first order perturbation of a quantity $F$ will be denoted by

$$\delta_1 F = \left(\frac{\partial F}{\partial R}\right)_r \delta R \qquad (5.15)$$

and the second order perturbation by

$$\delta_2 F = \tfrac{1}{2}\left(\frac{\partial^2 F}{\partial R^2}\right)_r (\delta R)^2 \quad . \qquad (5.16)$$

The unperturbed value of the quantity will not be represented by a distinguishing subscript. Thus to second order in $\delta R(r,t)$ a quantity F will have the value $F(R) + \delta_1 F(r,t) + \delta_2 F(r,t)$.

The non-adiabaticity in the first order perturbations will be introduced by an energy flux K in the form of the non-adiabatic change in total energy interior to a mass shell,

$$\delta M_K = - \int^t 4\pi R^2 K Z a \, dt$$

$$\simeq - 4\pi R^2 Z a \int^t K \, dt \qquad (5.17)$$

(see equation (3.04) ). In a general process of energy transfer there may be non-thermal contributions to the diagonal components of the energy-momentum tensor the same order of magnitude as K. As long as the time scale of the transfer of energy is sufficiently long, however, these terms will be negligible compared to the thermal energies and pressures, and in equation (2.15) have an effect down by a factor of the velocity of the matter over the speed of light compared to that of K. Thus the discussion in Chapter III of non-adiabatic energy transfer may be applied in all quasi-static situations.

We have, then, from equation (2.08),

$$\delta_1 M = - 4\pi P R^2 \delta R + \delta M_K \ . \qquad (5.18)$$

Using (5.02) for Z,

$$\delta_1 Z/Z = (M/R^2 + 4\pi P R)\delta R/Z^2 - \delta M_K/(Z^2 R)$$

$$= - \frac{Y'}{Y} \delta R - \delta M_K/(Z^2 R) \ . \qquad (5.19)$$

Thermodynamics gives

$$\delta_1 P = - \Gamma_1 P \frac{\delta_1 V}{V} + (\Gamma_2 - 1) \frac{T\delta S}{V} , \qquad (5.20)$$

where $\delta S$ is the change in the specific entropy. Another thermodynamic result required is

$$\begin{aligned} \delta_1(W+PV) &= T\delta S + V\delta_1 P \\ &= - \Gamma_1 PV \frac{\delta_1 V}{V} + \Gamma_2 T\delta S . \end{aligned} \qquad (5.21)$$

If there are nuclear reactions taking place, there will be a change $\delta A$ in the specific rest mass. Since $M_o$ is invariant under the pertubation,

$$\delta M_K = YZ \int \frac{(\delta A + T\delta S)}{Y} dM_o \qquad (5.22)$$

from equation (3.06). Finally, an expression for $\delta_1 V/V$ in terms of $\delta R$ may be extracted from equation (2.14),

$$\begin{aligned} \frac{\delta_1 V}{V} &= (R^2 \delta R)'/R^2 - \delta_1 Z/Z \\ &= (R^2 Y \delta R)'/(YR^2) + \delta M_K/(Z^2 R) . \end{aligned} \qquad (5.23)$$

We will first study sinusoidal adiabatic perturbations in order to obtain an equation for the radial modes of oscillation. Thus we neglect all terms in (5.18) - (5.23) involving $\delta M_K$, $\delta S$, or $\delta A$, and use the resulting equations to find the first order perturbation of the equilibrium equation (5.01) with

$$\delta R(r,t) = \delta R(r) e^{i\omega t} . \qquad (5.24)$$

The sinusoidal time dependence is in terms of the coordinate time $t$ instead of the proper time $\int a\,dt$ because of the "red" and "blue" shifts of the outward and inward traveling sound waves, the superposition of which forms the standing wave of a normal mode. The period of oscillation of a normal mode measured in the proper time of a local observer is different in different places in the mass.

To first order in $\delta R$,

$$\dot{U}Y = -\omega^2 Y^2 \delta R(r) e^{i\omega t} \quad . \tag{5.25}$$

The perturbation of the equilibrium equation, omitting the factor of $e^{i\omega t}$, is

$$= -Z^2 \frac{P'}{P+E}\left[\frac{2\delta R}{R} + \frac{\delta_1 Z}{Z} + \frac{\Gamma_1 P}{P+E}\frac{\delta_1 V}{V}\right] + \frac{Z^2}{P+E}\left[\Gamma_1 P \frac{\delta_1 V}{V}\right]' + \frac{2M}{R^2}\frac{\delta R}{R} + 4\pi R \Gamma_1 P \frac{\delta_1 V}{V} \quad . \tag{5.26}$$

Using the fact that

$$(YZ)'/(YZ) = -4\pi(P+E)R/Z^2 \tag{5.27}$$

and the zeroth order equilibrium equation we can write the equation for the radial modes of oscillation as

$$\frac{d}{d\Omega}\left[\frac{\Gamma_1 P}{Y^3 Z}\frac{dg}{d\Omega}\right] - \left[\frac{4}{9}\frac{RY'}{Y}\left(1 - \frac{1}{4}\frac{RY'}{Y}\right) + \frac{8\pi}{9}\frac{PR^2}{Z^2}\right]\frac{P+E}{Y^3 Z\,\Omega^2} g$$

$$= -\frac{\omega^2}{9}\frac{(P+E)R^2}{YZ^3\,\Omega^2} g = \frac{P+E}{Y^2 Z^3 R^2}\delta\Phi \quad , \tag{5.28}$$

in which the eigenfunction $g = R^2 Y \delta R(R)$ and $\Omega = R^3/3$. The boundary conditions to be imposed on $g$ are $g/\Omega$ finite at $R = 0$ and $g$

finite at the surface, where $P = 0$. This is not quite the standard Sturm-Liouville problem, but equation (5.28) is mathematically similar to the corresponding classical equation, for which it has been shown (36) that if $P$ behaves reasonably smoothly near the surface the eigenvalues and eigenfunctions have the usual Sturm-Liouville properties. There is a discrete set of eigenvalues, $\omega_i^2$, with a lowest eigenvalue $\omega_o^2$; the corresponding eigenfunctions $g_i(R)$ are mutually orthogonal with respect to the weight function $\frac{(P+E)R^2}{YZ^3\Omega^2}\frac{d\Omega}{dR}$. The eigenfunction of the fundamental mode, $g_o$, has no nodes, and the eigenfunction corresponding to the nth eigenvalue above the fundamental has $n$ nodes. Any reasonably smooth function of the radius between $R = 0$ and the surface obeying the proper boundary conditions may be expanded as an infinite sum over the eigenfunctions, which form a complete set.

An immediate result of equation (5.28) is that

$$\omega^2 = \frac{\int d\Omega \frac{P+E}{Y^3Z}\left[\frac{\Gamma_1 P}{P+E}\left(\frac{dg}{d\Omega}\right)^2 + \left(\frac{4}{9}\frac{RY'}{Y}\left(1 - \tfrac{1}{4}\frac{RY'}{Y}\right) + \frac{8\pi}{9}\frac{PR^2}{Z^2}\right)\left(\frac{g}{\Omega}\right)^2\right]}{\int d\Omega \frac{1}{9}\frac{(P+E)R^2}{YZ^3}\left(\frac{g}{\Omega}\right)^2} \qquad (5.29)$$

is stationary with respect to variations of $g$ about an eigenfunction and is an absolute minimum when $g = g_o$, giving $\omega^2 = \omega_o^2$. Thus (5.29) is a variational test for stability. If the minimum of (5.29) with respect to variation of parameters in a trial function for $g$ is less than zero, the mass is unstable in that the amplitude of the fundamental mode in the expansion for an arbitrary perturbation of equilibrium will grow

exponentially with time, since $\omega$ is imaginary in (5.24).

The sign of $\omega^2$, which gives the stability, is the same as the sign of the integral in the numerator of the variational expression (5.29), since the denominator integral is always positive and just normalizes the trial function. A necessary and sufficient condition for dynamic stability of a mass, then, is that the integral

$$2\pi \int d\Omega \frac{P+E}{Y^3 Z}\left[\frac{\Gamma_1 P}{P+E}\left(\frac{dg}{d\Omega}\right)^2 + \left(\frac{4}{9}\frac{RY'}{Y}\left(1 - \tfrac{1}{4}\frac{RY'}{Y}\right) + \frac{8\pi}{9}\frac{PR^2}{Z^2}\right)\left(\frac{g}{\Omega}\right)^2\right] \qquad (5.30)$$

be greater than zero for all functions $g(R)$ satisfying the usual boundary conditions. In the classical limit this integral is just the second order virtual change in total energy under a virtual adiabatic perturbation away from equilibrium, and it is interesting to see how this integral may be obtained in general relativity by similar considerations.

The natural procedure to calculate the virtual change of the total energy, virtual meaning that the velocity $U$ is kept zero, under an adiabatic perturbation is to use the equation (5.03) for $M$ at the surface ($r = r_s$) to write the change in $M$, $\delta M$, as

$$\delta M = \int_0^{r_s} [\delta_1 W\, Z + (1+W)\,\delta_1 Z + \delta_2 W\, Z + \delta_1 W\,\delta_1 Z + (1+W)\,\delta_2 Z]\, dM_o \quad . \qquad (5.31)$$

$M_o$ and $A$ are constant in the perturbation, and we have normalized $A$ to one.

$Z$ is given by (5.01) to all orders. To find $\delta_2 Z$ we need $\delta_1 M$, the actual first order adiabatic change in $M$ at a mass shell

from equation (5.18) with $\delta M_K = 0$, and $\delta_2 M$, the actual second order adiabatic change in M,

$$\delta_2 M = \tfrac{1}{2} \frac{\partial}{\partial R}(-4\pi PR^2)_r (\delta R)^2 = -4\pi PR(\delta R)^2 - 2\pi R^2 \delta R\, \delta_1 P \quad . \tag{5.32}$$

By thermodynamics

$$\delta W = -P\delta V + \tfrac{1}{2}\Gamma_1 PV\left(\frac{\delta V}{V}\right)^2 \quad . \tag{5.33}$$

In the first term of (5.33), $\delta V = \delta_1 V + \delta_2 V$, and

$$Z\,\delta_2 V + \delta_1 Z\,\delta_1 V + V\,\delta_2 Z = \frac{d}{dM_o}\left[4\pi R(\delta R)^2\right] \quad . \tag{5.34}$$

In terms of $\delta_1 V$ and $\delta_2 V$,

$$\delta_1 W = -PV\frac{\delta_1 V}{V} \quad , \tag{5.35}$$

$$Z\,\delta_2 W = -P\frac{d}{dM_o}\left[4\pi R(\delta R)^2\right] + \tfrac{1}{2}VZ\Gamma_1 P\left(\frac{\delta_1 V}{V}\right)^2 + P\,\delta_1 V\,\delta_1 Z + PV\,\delta_2 Z \quad . \tag{5.36}$$

Substituting into (5.31) and rearranging terms,

$$\delta M = \int_o^{r_s}\left[-PVZ\frac{\delta_1 V}{V} + EVZ\frac{\delta_1 Z}{Z} - P\frac{d}{dM_o}\left[4\pi R(\delta R)^2\right] + (P+E)VZ\frac{\delta_2 Z}{Z} + \tfrac{1}{2}\Gamma_1 PVZ\left(\frac{\delta_1 V}{V}\right)^2\right] dM_o \quad . \tag{5.37}$$

We will denote the integrand of (5.37) by $\delta[(1+W)Z]$. After considerable integration by parts, etc., using equations (5.18) - (5.20) and (5.23) for $\delta_1 M$, $\delta_1 P$, $\delta_1 Z$, and $\delta_1 V$ with the non-adiabatic terms set equal to zero, the integrand can be written in the form

$$\delta[(1+W)Z] = \frac{d}{dM_o}\left[\delta_1 M + \delta_2 M\right] + 2\pi \frac{d\Omega}{dM_o} YZ\, g \left[ - \frac{d}{d\Omega}\left(\frac{\Gamma_1 P}{Y^3 Z}\frac{dg}{d\Omega}\right)\right.$$

$$\left. + \left(\frac{4}{9}\frac{RY'}{Y}\left(1 - \tfrac{1}{4}\frac{RY'}{Y}\right) + \frac{8\pi}{9}\frac{PR^2}{Z^2}\right)\frac{P+E}{Y^3 Z^2 \Omega^2} g\right] . \qquad (5.38)$$

Since $\delta_1 M$ and $\delta_2 M$ are the actual first and second order changes in $M$ at a given mass shell, the first term in (5.38) is the actual adiabatic change in $(1+W)Z$ in the perturbation and integrates to zero, since $\dot{M} = 0$ at the surface. The second term is the virtual change in $(1+W)Z$, $\delta_v[(1+W)Z]$, and is second order in $\delta R$. Because of the extra factor of $YZ$ relative to the integrand of (5.30), the integral of $\delta_v[(1+W)Z]$ over $M_o$ is not positive definite for a stable equilibrium model. The factor $YZ$ means that if $g$ is expanded as a sum over normal mode eigenfunctions the cross terms in (5.38) do not integrate to zero and (5.31) contains terms first order in the deviation of $g$ from the eigenfunction of a mode of radial oscillation.

The proper evaluation of the virtual change in the total energy seems to be

$$\delta_v M = (YZ)_s \int_0^{r_s} \frac{\delta_v[(1+W)Z]}{YZ}\, dM_o \ , \qquad (5.39)$$

since it is this integral which is the stability integral (5.30). The integral (5.39) has the same form as the integral (3.06) which gives the change in the total energy due to a non-adiabatic change in "local energy" per unit invariant mass, including the gravitational red shift in the transport of the energy to infinity. Thus one may interpret

(5.39) as saying that it is the sum of the virtual changes in local energy after they are carried to infinity, including the gravitational red shift, that determines the stability of a mass.

A variational test for stability may be obtained directly from (5.39), following the method of Dyson (35) for the Newtonian case. The equation

$$\frac{d}{d\Omega}\left[\frac{\Gamma_1 P}{Y^3 Z}\frac{dg}{d\Omega}\right] = \gamma\frac{P+E}{Y^3 Z\,\Omega^2}\left[\frac{\Omega}{Y}\frac{dY}{d\Omega}\left(1-\tfrac{1}{4}\frac{RY'}{Y}\right)+\frac{2\pi}{3}\frac{PR^2}{Z^2}\right]g \qquad (5.40)$$

with the boundary conditions $g/\Omega$ finite at $R = 0$ and $g$ finite at $P = 0$ is a Sturm-Liouville problem with a lowest eigenvalue $\gamma = \gamma_0$. A necessary and sufficient condition that (5.30) be positive definite is that $\gamma_0 > 4/3$. A variational estimate of $\gamma_0$ can be obtained from the fact that $\gamma_0$ is the minimum over all $g$ of

$$\left[\int d\Omega\,\frac{\Gamma_1 P}{Y^3 Z}\left(\frac{dg}{d\Omega}\right)^2\right]\Bigg/-\int d\Omega\,\frac{P+E}{Y^3 Z}\left[\frac{\Omega}{Y}\frac{dY}{d\Omega}\left(1-\tfrac{1}{4}\frac{RY'}{Y}\right)+\frac{2\pi}{3}\frac{PR^2}{Z^2}\right]\left(\frac{g}{\Omega}\right)^2 . \qquad (5.41)$$

However, the direct variational estimate of $\omega_o^2$ using (5.29) is easier to evaluate accurately numerically, especially if $\Gamma_1$ is only slightly greater than $4/3$, the classical dividing point between stability and instability.

Our numerical calculations of the stability have used both the variational principle (5.29) with a trial function of the form

$$g = Y\Omega(1 + CR^2) \ , \qquad (5.42)$$

and direct integration of equation (5.28), guessing at a number for the

eigenvalue until the integration of (5.28) reaches the surface with $g$ finite. The trial function (5.42) seems to give slightly better results for constant $\Gamma_1$ models near the instability point of the fundamental mode than similar trial functions in which the factor $Y$ is replaced by $YZ$, $Y^{3/2}$, or unity. If the constant $C$ is set equal to zero instead of being varied to make (5.29) a minimum, a factor of $Y^{3/2}$ seems to give the best value of $\omega_o^2$ for these models.

The most accurate method of determining the stability of a model is to integrate equation (5.28) with $\omega^2$ set equal to zero, starting with $g/\Omega = 1.0$, say, at the center. If in the course of the integration $g$ goes through zero, the model is unstable; $g$ with $\omega^2 = 0$ is trying to be a mixture of the fundamental and higher modes, so $\omega_o^2 < 0$. If $g$ remains positive and becomes indefinitely large as the surface is approached, the model is stable, $\omega_o^2 > 0$. This integration is also the starting point of a method for obtaining both the eigenfrequency and eigenfunction of any mode of radial oscillation to any desired accuracy. For the nth mode, if $g$ goes through zero $n+1$ times before the integration reaches the surface, $\omega_n^2 < 0$. If $g$ blows up before going through zero $n+1$ times, $\omega_n^2 > 0$. A previous estimate of the absolute value of $\omega_n^2$, perhaps obtained from the variational method, is given the appropriate sign and used in (5.28) for the second trial integration. Whether the true value of $\omega_n^2$ is greater than or less than the trial value of $\omega_n^2$ is determined in the same way as the sign of $\omega_n^2$. If the absolute value of $\omega_n^2$ is greater

than that of the second trial value, a new trial value is obtained by multiplying the old trial value by a factor such as 1.5. This process is repeated until one has a trial value greater in absolute value than the true value of $\omega_n^2$. Then one can begin a process of interpolation between the last trial $\omega_n^2$ and the closest previous trial $\omega_n^2$ below or above it as indicated by the behavior of $g$ in the last integration. We cut off each integration when the absolute value of $\varphi = \Omega \frac{d}{d\Omega}\left(\frac{g}{\Omega}\right)$ becomes greater than a limit such as 2.0 for the fundamental mode, with higher limits for higher modes. The value of $V_c/V$ at the cutoff point is stored and used as a weighting factor in interpolating between trial values of $\omega_n^2$. We have only calculated models for which $1/V$ as well as $P$ is zero at the surface. With the cutoff $\varphi$ it must be noted that if $\varphi/g$ at cutoff is less than zero, $g$ would go through zero once more before going to infinity. Thus $\varphi/g < 0$ at cutoff is equivalent to $g$ going through zero $n+1$ times. The limit on $\varphi$ must be set high enough so $\varphi$ does not exceed it prematurely.

If a reasonable estimate is made of the absolute value of $\omega_n^2$, ten integrations usually suffice to obtain $\omega_n^2$ to three or four significant figures. Using the density ratio to weight the interpolation helps mainly in correcting for a rather poor initial guess of $\omega_n^2$ or in taking advantage of a particularly lucky trial value.

The variational method for determining the eigenvalue of the fundamental mode is accurate only if a reasonably good guess can be

made of the shape of the eigenfunction. Classically the eigenfunction of the fundamental mode when it becomes unstable is $g_o = \Omega$ if $\Gamma_1$ is constant throughout the mass. This is the motivation for the choice (5.42), which should give good results for constant $\Gamma_1$ models near the instability point if $(PV)_c \ll 1$ and $2M/R \ll 1$ there. The exact $g_o/\Omega$ decreases away from the origin when $\omega_o^2 = 0$. In any case, the variational method result for $\omega_o^2$ is sufficiently good to use as an estimate of its magnitude in the trial eigenvalue method. The value of $CR^2$ at the surface obtained in the variational method is an indication of the correctness of the trial function and thus of the accuracy obtained for $\omega_o^2$. The calculations both of stability and equilibrium were made on an IBM 7094 computer using a subroutine which integrates simultaneous first order differential equations by a Runge-Kutta method, adjusting the step size to give a specified accuracy.

Before discussing the results of our stability analysis, we will analyze what we should expect in the way of correlation of binding energy with stability. We consider two neighboring models in a sequence of equilibrium models with constant number of baryons and constant $M_o$. The invariant mass per baryon at corresponding mass shells is the same. Define a displacement $\delta R(r)$ between mass shells in the two models having the same number of baryons inside them. Both models satisfy equation (5.01), and thus the difference of $\Phi$ between corresponding mass shells in the two models is $0 - 0 = 0$. In the limit that the two models approach each other along the sequence, $\delta R(r)$ goes to zero, this difference of $\Phi$ can be written in terms of the first order pertur-

bations of the quantities contained in $\Phi$. These perturbations are given by equations (5.18) - (5.23), including the non-adiabatic terms.

If all the non-adiabatic terms happen to be zero, $\delta\Phi$ between the two models is the same as $\delta\Phi$ in the adiabatic perturbation of a single model, and since $\delta\Phi = 0$ the function $\delta R(r)$ between the two models is an eigenfunction of equation (5.28) with eigenvalue $\omega^2 = 0$. That is, $\delta R(r)$ has the shape of one of the normal modes of oscillation, and this normal mode of oscillation is in neutral equilibrium.

At a maximum or minimum in the binding energy along a sequence of equilibrium models, since $M_o$ is by definition constant, the first order change in the total energy of the mass, $\delta M_T$, between neighboring models is zero. From equation (5.22) and the fact that $\delta M_T$ equals $\delta M_K$ evaluated at the surface of the mass,

$$\delta M_T = (YZ)_s \int_0^{r_s} \frac{\delta Q}{Y}\, dM_o \quad . \tag{5.43}$$

If $\delta Q$ has the same sign at every mass shell, $\delta M_T$ can be zero only if $\delta Q$ is zero at every mass shell. One case in which $\delta Q = \delta A + T\delta S$ does have the same sign everywhere is if $\delta A$ is zero and if the specific entropy is independent or radius in each model, so for any pair of models $\delta S$ has a constant value throughout the mass. Then $\delta M_T = 0$ implies $\delta Q = T\delta S = 0$ at every mass point and the perturbation between models at a maximum or minimum in binding energy are completely adiabatic. As discussed above, this means that for a series of isentropic models some mode of radial oscillation is in neutral equilibrium at a stationary point in the binding energy as a function of central density.

When an even mode, which has an even number of nodes, is in neutral equilibrium, the fact that $\delta R$ between models has the same shape as the eigenfunction of the mode implies that the radius of the mass decreases as the central density increases. Near the point where an odd mode is in neutral equilibrium the radius of the mass increases as the central density increases. If at successive maxima and minima of the binding energy progressively higher modes of radial oscillation become unstable the result is a spiral in the plot of binding energy against the surface value of $2M/R$, as is observed for the isentropic constant $\Gamma_1$ models. This spiral should be typical of all sequences of isentropic models for which $\Gamma_1$ is more or less constant or monotonically decreasing as the density and temperature increase. If $\Gamma_1$ increases sharply at some value of the density, the highest unstable mode might become stable again, in which case the curve would oscillate before resuming the spiral.

A plot of binding energy against $2M/R$ for a sequence of isentropic models starting from zero central density, where the binding energy is zero, can be used to read off the stability of the modes of radial oscillation. If the binding energy increases from zero the fundamental mode is stable initially; if it decreases from zero, as is the case for $\Gamma_1 = 4/3$, the fundamental mode is unstable initially. As $(1/V)_c$ and $(PV)_c$ increase a clockwise (right-hand) curvature at a maximum or minimum in the binding energy means the lowest stable mode has become unstable. A counter-clockwise (left-hand) curvature means the highest unstable mode has become stable. The relevant Sturm-Liouville

property is the strict increase of the eigenvalues with the number of nodes in the corresponding eigenfunctions.

The requirement $\delta A = 0$ is a necessary one if the above properties are to hold, because otherwise even if $\delta Q$ were zero everywhere, $\delta S$ would not be zero and $\delta P$ would not be adiabatic. A maximum in binding energy may coincide with the instability point of the fundamental mode in a sequence of non-isentropic models, but the specific entropy must be a minimum at every mass shell if $\delta R$ is to have the shape of the fundamental mode there. For the purposes of this section of the paper isentropy need only imply $S' = 0$; $A'$ need not equal zero, as in a model with constant entropy per baryon but varying rest mass per baryon as a function of radius. For any particular equilibrium model it is possible to normalize $V$ so $S' = 0$, but $V$ may only be normalized once in a sequence of models. $S'$ must equal zero in all models near a stationary point in the binding energy if the above arguments are to go through.

In constructing a sequence of non-isentropic models one may vary either $(PV)_c$ or the structure, represented by the index $n$ in polytropic models. Perhaps the most natural type of sequence is one in which $n$ is held constant and $(PV)_c$ varied. Calculations with constant $\Gamma_1$ polytropes show that if $\frac{n+1}{n} < \Gamma_1$, corresponding to a sub-adiabatic temperature gradient, the value of $(PV)_c$ at which the fundamental mode becomes unstable is less than that at which a sequence with $\frac{n+1}{n} = \Gamma_1$ becomes unstable. There is a strong tendency for a maximum in binding energy to occur very close to the instability point,

but comparing $P V^{\Gamma_1}$, a function only of the entropy, at the center of the models one finds that the entropy there is not a minimum at the maximum of the binding energy.

Our most extensive calculations have been made with models in which $\frac{3}{4}\Gamma_1 - 1.3$. A sequence with $\frac{3}{4}\frac{n+1}{n} - 1.3$ (isentropic case) becomes unstable when $(PV)_c = 0.1836$ at a maximum fractional binding energy of $5.2608 \times 10^{-2}$. A sequence with $\frac{3}{4}\frac{n+1}{n} = 1.25$ becomes unstable when $(PV)_c = 0.1808$, still to the accuracy of the calculation at the same point as the maximum binding energy, $(M_o-M)/M_o = 5.2643 \times 10^{-2}$. For $\frac{3}{4}\frac{n+1}{n} = 1.0$ the instability point at $(PV)_c = 0.165$ is slightly past the binding energy maximum of $5.2371 \times 10^{-2}$ at $(PV)_c = 0.1635$. An interesting feature of the last sequence, displayed in Figure 4, is that $2M/R$ reaches its maximum before the maximum in binding energy. Thus $\delta R(r)$ between models actually has a node when the fundamental mode becomes unstable, emphasizing the non-adiabatic nature of the displacement.

A different type of sequence is one with constant $(PV)_c$ and varying n. Increasing n or decreasing $\frac{n+1}{n}$ corresponds to increasing central density. Such a sequence with $(PV)_c = 0.184$, starting from a just slightly unstable isentropic model, has a fractional binding energy maximum of $5.2653 \times 10^{-2}$ at $\frac{3}{4}\frac{n+1}{n} = 1.18$ before decreasing to $5.2023 \times 10^{-2}$ at $\frac{3}{4}\frac{n+1}{n} = 1.0$. A sequence with $(PV)_c = 0.170$ starts with $(M_o-M)/M_o = 5.2464 \times 10^{-2}$ at $\frac{3}{4}\frac{n+1}{n} = 1.3$, has a maximum $(M_o-M)/M_o = 5.2731 \times 10^{-2}$ at $\frac{3}{4}\frac{n+1}{n} = 1.09$, and decreases to

$(M_o-M)/M_o = 5.2335 \times 10^{-2}$ at $\frac{3}{4}\frac{n+1}{n} = 1.0$. All of the modes of the first sequence are unstable; the second becomes unstable slightly past the maximum in binding energy.

These calculations show that one may construct sequences of non-isentropic models with monotonically increasing central density in which the maximum in binding energy occurs either before or after the instability point of the fundamental mode. In most cases the two points are quite close to each other, however. It is always possible to construct a sequence of non-isentropic models which has adiabatic displacements near the instability point of the fundamental mode, but this is only one among an infinite number of equally possible sequences.

Wright (37), working in the post-Newtonian approximation, assumes that displacements between neighboring models are adiabatic at the instability point of the fundamental mode and shows that as a consequence the binding energy has a stationary point. However, as we have just seen, his assumption that displacements between neighboring models are adiabatic is not valid in general.

In Table I we give the values of $(PV)_c$, $2M/R$, and $(M_o-M)/M_o$ at the instability point of the fundamental mode for certain isentropic constant $\Gamma_1$ models. The parameter $\alpha = \frac{3}{4}\Gamma_1 - 1$ is the convenient representation of $\Gamma_1$ in referring to stability properties, particularly when $\alpha \ll 1$. $\bar{V} = \frac{4\pi}{3}R^3/M_o$.

Table I

| $\alpha$ | $\bar{V}/V_c$ | $(PV)_c/\alpha$ | $(2M/R)/\alpha$ | $(M_o-M)/M_o\alpha^2$ |
|---|---|---|---|---|
| 0.001 | 53.28 | 0.5072 | 1.187 | 1.770 |
| 0.01 | 46.08 | 0.5131 | 1.1975 | 1.692 |
| 0.10 | 16.46 | 0.5589 | 1.204 | 1.135 |
| 0.30 | 5.55 | 0.6120 | 1.029 | 0.5845 |
| 0.50 | 3.30 | 0.636 | 0.8576 | 0.3588 |

Chandrasekhar (13) has found the instability point in the post-Newtonian approximation, valid when $(PV)_c \ll 1$, for several of Tooper's polytropes. The $n = 3$ polytrope corresponds to an isentropic constant $\Gamma_1$ model in the limit $\alpha \ll 1$. The instability point is at $(PV)_c = 0.5065\ \alpha$, consistent with our results.

The behavior of the binding energy as a function of central temperature for $n = 3$ polytrope models of large masses has been analyzed in the post-Newtonian approximation by Fowler (5). It is possible to compare his results with ours with the aid of the following conversion factors. If in our natural units the unit mass is taken to be $\lambda$ solar masses, $1.985 \times 10^{33}\ \lambda$ gm, the unit length is $1.47 \times 10^{5}\ \lambda$ cm, the unit time is $4.90 \times 10^{-6}\ \lambda$ sec, and the unit density is $6.26 \times 10^{17}\ \lambda^{-2}$ gm/cm$^{-3}$. These units, plus a unit temperature of one degree Celsius, give a gas constant $R = 9.24 \times 10^{-14}$ and a Stefan-Boltzmann constant $a = 1.34 \times 10^{-21}\ \lambda^{-2}$.

We have discussed some aspects of the gas plus radiation pressure equation of state earlier; equation (5.14) gives the relation

between $\alpha$ and $\beta$, the ratio of gas pressure to total pressure, when $\beta \ll 1$. For the classical $n = 3$ polytrope the mass $M$ is a function only of $\beta$ and the molecular weight $\mu$,

$$M/M_\odot = 18.3/(\mu\beta)^2 \quad . \tag{5.44}$$

Taking $\mu = \frac{1}{2}$, the value for a pure hydrogen composition,

$$M/M_\odot = 1.147\ \alpha^{-2} \quad . \tag{5.45}$$

Since the fractional binding energy at the instability point is proportional to $\alpha^2$ when $\alpha \ll 1$, the maximum binding energy of isentropic models of large masses is, as noted by Iben (33) and explained in the post-Newtonian approximation by Fowler (38), independent of $\alpha$ and thus of the mass. Using (5.45) and the fractional binding energy for $\alpha = 0.001$ we obtain a binding energy of $2.03\ M_\odot$, which checks closely with the result of Fowler (38). The central temperature at the instability point is, according to Fowler, $T_c = 1.25 \times 10^{13} \frac{M_\odot}{M}/\mu\ ^\circ K$. This may be correlated with our value for $(PV)_c$ by

$$(PV)_c = 9.24 \times 10^{-14}\ T_c/(\mu\beta) \quad , \tag{5.46}$$

$$(PV)_c/\alpha = 0.404 \times 10^{-13}\ \mu\ T_c\ \frac{M}{M_\odot} \quad . \tag{5.47}$$

The post-Newtonian approximation works very well when $\alpha = 0.001$, but some deviation is apparent when $\alpha = 0.01$

The most important result of this chapter is the correlation of binding energy and stability for isentropic models. Fowler (38) has made a rough argument that a mass should be stable before a maximum in

the binding energy and unstable after. When the total energy is decreasing as the radius of the equilibrium models decreases, an adiabatic pertubation inward gives more internal energy and therefore more pressure than the corresponding equilibrium model, causing a reexpansion. An adiabatic expansion outward gives too little pressure, and the mass is stable. When the total energy is increasing as the radius increases, an adiabatic pertubation inward gives too little internal energy and pressure for equilibrium, and the collapse accelerates. This argument fails for the higher maxima and minima of the binding energy because the pertubation between neighboring equilibrium models no longer has a shape like that of the fundamental mode, and it is the fundamental mode that determines the stability.

## VI. Dynamics Near Equilibrium

Our study of equilibrium models and their stability provides a foundation for the study of the dynamics of spherically symmetric masses. We work with the hydrodynamic equations of an ideal fluid as discussed in Chapters II and III. In general, a difference equation approximation to the general relativistic hydrodynamic equations is necessary, and we will discuss possible ways of setting up such equations for numerical integration on a high speed computer and the results for a particular example later in this chapter. However, near equilibrium small errors in the difference approximation to $P'$ in equation (3.09) for the non-gravitational force can introduce large errors in the value obtained for the acceleration $Y\dot{U}$ due to the almost total cancellation of $f$ and the gravitational force. Another problem with difference equation methods is that typically the time step $\Delta t$ is limited by the Courant condition (39),

$$\Delta t \lesssim Y_b \Delta r/c \quad , \tag{6.01}$$

where $c$ is the speed of sound, given by (3.18). If fluid velocities are much less than the speed of sound, as they are near equilibrium, this condition means that a large amount of computing time is used to calculate small changes in the pressure, density, radius, etc. An implicit difference equation method can avoid the latter difficulty, but we have explored a different approach which avoids difference approximations to gradients completely and allows arbitrarily large time steps in

calculating the initial stages of adiabatic collapse or expansion.

Instead of calculating from the hydrodynamic equations the new values of all the physical variables after a given time step, we assume we know beforehand how the change in $R$, $\delta R$, in the time step varies as a function of $r = R_o$, the value of $R$ at the mass shell in the equilibrium model. The new value of $\dot{R}$ can be found by requiring that the total energy of the mass be constant during the adiabatic collapse or expansion. The magnitude $\delta t$ of the time step corresponding to $\delta R(r)$ can be estimated from the average of the initial and final velocities.

The method is only as good as the choice of the shape of $\delta R(r)$. Referring to our normal mode analysis of the equilibrium model, if $\omega_o^2 < 0$ the fundamental mode is unstable and the component of the initial pertubation having the shape of $g_o$ grows exponentially with time constant $(-\omega_o^2)^{-\frac{1}{2}}$. Particularly if the higher modes are stable, the shape of the pertubation will tend to that of the fundamental mode as the deviation from the equilibrium model grows. Even if $\omega_o^2 > 0$ it may be of interest to find out how much initial kinetic energy it takes to surmount the energy barrier blocking collapse; the energy barrier should be lowest for displacements having more or less the shape of the fundamental mode.

The shape of the displacement from equilibrium would be expected to begin to deviate from the shape of the fundamental mode in any actual collapse once $\delta R/R$ becomes larger than a few per cent. This puts a limit on how far a choice of $\delta R$ based on the fundamental

mode of the equilibrium model may legitimately be extended, although a rough idea of the growth of the kinetic energy can be obtained out to larger $\delta R/R$. For an unstable model the kinetic energy should grow initially as $(\delta R)^2$. For a model in neutral equilibrium under first order pertubations of the equilibrium equation having the shape of the fundamental mode, we expect $\ddot{R} \sim (\delta R)^2$, giving a growth of kinetic energy in one direction (collapse) proportional to $(\delta R)^3$ but stability in the other direction (expansion).

In detail, our method consists of assuming that the shape of

$$\psi = R^n \delta R \tag{6.02}$$

as a function of the initial radius $R_o$ is maintained. $n$ is a parameter which determines how the initial shape of $\delta R$ is modified away from equilibrium, to the extent that $\delta R$ is not simply proportional to $R_o$. As one would expect from the fact that $g_o \sim R^2 \delta R$, $n = 2$ or 3 seems to give the best results. We store at a large number of points equally spaced in $R_o$ the values of $R$, $V$, $Z$, $M$, $\dot{R}$, plus $\psi$ and its radial derivative in the form

$$\varphi = 4\pi R_o^{2-n} \frac{\partial}{\partial M_o} (R^n \delta R) \quad . \tag{6.03}$$

Initially $\psi$ and $\varphi$ are given by

$$\psi = R^{n-2} g_o / Y \quad , \tag{6.04}$$

$$\omega = VZ \left[ \frac{dg_o}{d\Omega} + \left[ n - 2 + \frac{1}{Z^2} (M/R + 4\pi P R^2) \right] \frac{1}{3} \frac{g_o}{\Omega} \right] \quad , \tag{6.05}$$

where $g_o$ can be calculated to arbitrary accuracy by the trial eigenvalue method of the last chapter. $\psi$ and $\varphi$ are normalized by a common overall factor such that $\sigma = \delta R/R$ is some given small fraction of unity near the center of the mass.

To go a step in time from $R = R_1$ to $R = R_2$ we calculate at each storage point

$$\delta R = \psi/R_1^{\,n} \quad , \tag{6.06}$$

$$\sigma = \delta R/R_1 \quad , \tag{6.07}$$

$$R_2 = R_1 + \delta R \quad . \tag{6.08}$$

In order to calculate $V_2$, from which we can obtain the other thermodynamic quantities, we need to know $Z_2$. The calculation is considerably simplified if we can estimate $Y_2$ and $M_2$ from $Y_1$ and $M_1$ directly. An estimate of $M_2$ can be obtained from equation (2.08), expanding $\delta M$ as a power series in $\delta R/R$. To third order in $\delta R/R$, assuming $\Gamma_1$ is constant along the adiabat,

$$M_2 = M_1 - 4\pi P_1 R_1^{\,2}\delta R \left[1 + \left(1 + \frac{1}{3}\frac{\delta_1 P}{P_1} + \frac{1}{3}\Gamma_1\frac{4\pi P_1 R_1^{\,2}}{Z_1^{\,2}}\sigma\right)\left(\sigma + \tfrac{1}{2}\frac{\delta_1 P}{P_1}\right)\right.$$
$$\left. - \left(\tfrac{1}{2}\Gamma_1 - \frac{1}{3}\right)\sigma^2 + \frac{1}{6}\Gamma_1\left(R_1\frac{d\sigma}{dr_1}\right)^2 - \frac{1}{3}\Gamma_1\left(\frac{\delta_1 Z}{Z_1}\right)^2\right] \quad , \tag{6.09}$$

in which

$$\frac{\delta_1 P}{P_1} = -\Gamma_1\frac{\delta_1 V}{V_1} \quad , \tag{6.10}$$

$$\frac{\delta_1 V}{V_1} = (2 - n)\sigma + \left(\frac{R_1}{R_o}\right)^{2-n} V_1 Z_1 \varphi - \frac{\delta_1 Z}{Z_1} \quad , \tag{6.11}$$

$$R_1 \frac{d\sigma}{dR_1} = -(n+1)\sigma + \left(\frac{R_1}{R_o}\right)^{2-n} V_1 Z_1 \varphi \quad , \tag{6.12}$$

$$\frac{\delta_1 Z}{Z_1} = (M_1/R_1 + 4\pi P_1 R_1^2)\sigma/Z_1^2 \quad . \tag{6.13}$$

In computing (6.09) and (6.13) we assume that $U^2 \ll 2M/R$ and that the acceleration $\dot{Y}\dot{U}$ is much less than the free fall acceleration; that is, in computing the pertubations of $Z$ we assume that $Z = (1-2M/R)^{\frac{1}{2}}$. It is only in the first few steps away from equilibrium that the higher order terms in (6.09) involving pertubations of $Z$ are important, and here the assumption is certainly justified. (6.09) is accurate only if $\delta R/R$ in a step is kept fairly small.

The difference of $Y_2$ from $Y_1$ in what follows is not important at least as long as $PV \ll 1$, since it is only a small percentage correction to the change in kinetic energy. For an isentropic model $Y$ is related to $V$ by (3.11) and the equation of state; in general $Y_1$ must be found by integrating (3.09) at each time step.

Denoting by $Z_a$ the value of $Z$ obtained assuming $\dot{R}$ remains constant,

$$Z_a = (1 + Y_2^2 \dot{R}_1^2 - 2M_2/R_2)^{\frac{1}{2}} \quad , \tag{6.14}$$

$Z_2$ is approximately given by

$$Z_2 = Z_a(1 + \delta K) = Z_a\left[1 + \tfrac{1}{2}\frac{Y_2^2}{Z_a^2}(\dot{R}_2^2 - \dot{R}_1^2)\right] \quad . \tag{6.15}$$

$\delta t$ is a parameter which can be used to adjust the magnitude of $\dot{R}_2$ and thus the change in the total kinetic energy to keep the total energy

of the mass constant. The actual time for the displacement to take place is equal to $\delta t$ only if the acceleration is approximately constant during the time step, which it is not in the first few steps away from equilibrium. The actual time for the first step is infinite if one starts in equilibrium with zero velocity.

From equation (2.14) we can obtain an equation for $V_2$ in terms of $\psi$ and $\varphi$, avoiding a difference approximation to $\partial R/\partial M_o$. We have

$$V_2 = (1+\sigma)^2\left[V_1 Z_1(1-n\sigma) + \varphi\left(\frac{R_1}{R_o}\right)^{2-n}\right]Z_2 \quad , \tag{6.17}$$

$$= V_a(1 - \delta K)$$

to first order in $\delta K$. $V_a$ is known explicitly and can be used to calculate the corresponding pressure $P_a$ and specific thermal energy $W_a$. For our constant $\Gamma_1$ model,

$$P_a = P_o\left(\frac{V_o}{V_a}\right)^{\Gamma_1} \quad , \tag{6.18}$$

$$W_a = P_a V_a/(\Gamma_1 - 1) \quad . \tag{6.19}$$

To first order in $\delta K$,

$$W_2 = W_a[1 + (\Gamma_1 - 1)\delta K] \quad . \tag{6.20}$$

The equation (3.07) for the total energy of the mass is used with (6.15) and (6.20) to apply the condition that the total energy be constant in the adiabatic collapse. In terms of $\delta t$,

$$\delta K - 2\frac{Y_2^2(\delta R)^2}{Z_a^2}(\delta t)^{-2} - 2\frac{Y_2^2\dot{R}_1\delta R}{Z_a^2}(\delta t)^{-1} \quad , \tag{6.21}$$

and with the specific rest mass $A = 1$,

$$\int [Z_a - Z_1 + W_a Z_a - W_1 Z_1] \, dM_o$$
$$- 2 \int \frac{Y_2^2 \dot{R}_1 \delta R}{Z_a} (1 + \Gamma_1 W_a) dM_o \, (\delta t)^{-1}$$
$$+ 2 \int \frac{Y_2^2 (\delta R)^2}{Z_a} (1 + \Gamma_1 W_a) dM_o \, (\delta t)^{-2} = 0 \, , \qquad (6.22)$$

an equation for $(\delta t)^{-1}$. The first integral in (6.22) is the critical one to evaluate accurately; classically it is minus the change in kinetic energy of the mass, $-\delta T$. If initially $\dot{R} = 0$, we will call the running sum of the $\delta T$, corrected to replace $Y_2^2 \dot{R}_1^2$ in $\delta K$ by $Y_1^2 \dot{R}_1^2$, the total kinetic energy $T$.

The integrations in (6.22) are done by Simpson's rule, with several hundred radial points sufficient to give reasonable accuracy. The integrand of the integral for $\delta T$ in the first step away from equilibrium is given to second order in $\delta R/R$ by equation (5.38). The integrand has terms first order in $\delta R/R$ which should integrate to zero. The numerical integration must be sufficiently accurate that the error in integrating this part of the integrand does not exceed the result one is looking for, especially starting from neutral equilibrium when this is the order of $(\Gamma_1 - 4/3)(\delta R/R)^2$ times the integral of the absolute value of the integrand. It may be necessary to extrapolate back to $R - R_o = 0$ to obtain the behavior of the kinetic energy there. One must also be careful to calculate the shape of the fundamental mode accurately. As discussed in Chapter V a deviation from the fundamental mode appears to first order in the terms proportional to $(\delta R/R)^2$, and

even a rather small deviation can make a significant difference if $T$ should vary initially as $(R - R_o)^3$.

These numerical problems arise only in the first few time steps. As the kinetic energy grows the changes in kinetic energy are evaluated more accurately, so that runs with different numbers of radial points, different fractional changes in $R$ per step, etc., converge rapidly to a common numerical solution.

The best test we have found of the basic assumption of the method, that the displacements have in some sense the shape of the fundamental mode of the initial equilibrium model, is to compare at interior mass shells the binding energy calculated from the initial binding energy and the change in $M$ according to (6.09) with the binding energy calculated by direct integration of equation (3.07). The two values will be equal at every mass point if and only if the change in $U = \dot{Y}\dot{R}$ has been consistent with acceleration equation (2.11) at every mass shell. If the difference of the two values at each mass shell is a small fraction of the kinetic energy inside the mass shell, one can certainly trust the value obtained for the total kinetic energy. This is a very sensitive test, especially near the center of the mass where the kinetic energy is only a small fraction of the change in $M_o - M$ from the initial model.

The most interesting application of our method is to the calculation of the initial collapse of a mass from neutral equilibrium. As one would expect from the positive binding energy, we find that a mass at the maximum in binding energy of an isentropic sequence is

is unstable to collapse and stable to expansion. To present our results, consider the expansion of the kinetic energy $T$ as a power series in $x = (R-R_o)/R_o$ evaluated for a mass shell near the center. For our constant $\Gamma_1$ models the ratio $T/(M_o-M)$ is independent of $\alpha = (3/4)\Gamma_1 - 1$ in the limit $\alpha \ll 1$, and thus is a function only of $x$. Both the kinetic energy at a given $x$ and the maximum binding energy are proportional to the first order general relativistic corrections to the potential energy, or to $\alpha^2$, as was noted by Fowler (38) from his post-Newtonian analysis. We have calculated the coefficients $a_3$ and $a_4$ in the expansion

$$T/(M_o-M) = a_3 x^3 + a_4 x^4 + \cdots , \tag{6.23}$$

using an extrapolation to $x = 0$ based on the change in the kinetic energy in a step rather than the total kinetic energy since the latter is affected by errors in calculating the change in kinetic energy in the first one or two steps.

The time scale associated with the collapse can be found by integrating the equation

$$\dot{x} = \frac{A}{R_T}\left[-a_3 \frac{M_o-M}{M_o}\right]^{\frac{1}{2}} x^{3/2} ; \tag{6.24}$$

the time for $x$ to go from $x_o \ll 1$ to a value of $x$ such that $x_o/x \ll 1$ is, according to (6.24),

$$t_c = \frac{2R_T}{A}\left[-a_3 \frac{M_o-M}{M_o}\right]^{\frac{1}{2}} x_o^{-\frac{1}{2}} . \tag{6.25}$$

The dimensionless constant A depends only on the structure of the mass and the shape of the fundamental mode,

$$A = R_T \left|(\dot{R}/R_o)_c\right| (T/M_o)^{-\frac{1}{2}} \quad ; \tag{6.26}$$

$R_T$ denotes the total radius of the equilibrium model. For a classical $n = 3$ polytrope and $g_o = \Omega$, $A = 4.2$. Let

$$t_f = (V_c/6\pi)^{\frac{1}{2}}(M_o/M)^{\frac{1}{2}} \tag{6.27}$$

be a sort of "free fall" time for the center of the mass. The quantity

$$t_c x_o^{\frac{1}{2}}/t_f = \frac{3}{A}(2M/R)^{\frac{1}{2}}(\bar{V}/V_c)^{\frac{1}{2}}\left[- a_3 \frac{M_o - M}{M_o}\right]^{-\frac{1}{2}} \tag{6.28}$$

is given along with $a_3$ and $a_4$ in Table II as a measure of the degree of instability to collapse of several constant $\Gamma_1$ isentropic models at the maximum in binding energy. As in Table I $\alpha = \frac{3}{4}\Gamma_1 - 1$.

Table II

| $\alpha$ | $a_3$ | $a_4$ | $t_c x_o^{\frac{1}{2}}/t_f$ |
|---|---|---|---|
| 0.001 | - 0.35 | 0.7 | 230. |
| 0.01 | - 0.359 | 0.8 | 71.4 |
| 0.10 | - 0.580 | 1.4 | 24.6 |
| 0.30 | - 1.266 | 3.2 | 7.75 |

There is some uncertainty in the last digit of most of these numbers.

In the limit $\alpha \ll 1$ we again make contact with the gas plus radiation model of large masses. $t_c x_o^{\frac{1}{2}}/t_f$ varies as $\alpha^{-\frac{1}{2}}$ in this limit, and thus from (5.45) as $(M/M_\odot)^{\frac{1}{4}}$. The free fall time $t_f$ may be found from the expression for $\beta$ in the limit $\beta \ll 1$,

$$\beta \simeq 3 \frac{R}{\mu a} T_c^{-3} V_c^{-1} \quad . \tag{6.29}$$

The time scale of adiabatic collapse from the instability point of a large mass is then

$$\begin{aligned} t_c x_o^{\frac{1}{2}} &= 1.76 \times 10^{-5} \mu^2 (M/M_\odot)^2 \text{ seconds} \\ &= 0.57 \mu^2 (M/10^6 M_\odot)^2 \text{ years} \quad . \end{aligned} \tag{6.30}$$

The higher order terms in (6.23) tend to accelerate the collapse.

A large mass loses energy by radiation at the rate, as worked out by Hoyle and Fowler (2,3), of $2 \times 10^{38} M/M_\odot$ erg $\text{sec}^{-1}$. This gives a Kelvin-Helmholtz contraction time to the instability point of $580 (10^6 M_\odot/M)$ years if $\mu = \frac{1}{2}$. The mass will actually start collapsing before the adiabatic instability point is reached because the binding energy will not be able to supply the energy loss near its maximum as a function of central density. The relative importance of (6.30) and the energy loss in determining the time scale of collapse can be estimated very roughly by saying the acceleration due to the energy loss is the fractional change in the initial pressure due to the energy loss in the time $t_c$ given by (6.30) times the free fall acceleration. The time scale associated with this energy loss acceleration must be much greater than $t_c$. When the numbers are put in, the condition that the

collapse, except very near equilibrium, be approximately adiabatic is that $M \ll 10^{11}\ M_\odot$. Fowler (38) has shown that because of the low central temperature and density at the instability point nuclear energy generation cannot stop the collapse for $M$ much greater than $10^6\ M_\odot$. Thus (6.30) is an adequate description of the time scale of spherically symmetric gravitational collapse when $M$ is in the range $10^6 - 10^9\ M_\odot$. (6.30) may be compared with Fowler's treatment (38) of the same problem, which considers only the center of the mass in the post-Newtonian approximation. When formulated in the same way as (6.30), Fowler's result is $t_c x_o^{\frac{1}{2}} = 0.24\ \mu^2\ (M/10^6 M_\odot)^2$ years, a little less than half our value.

The behavior of the dynamics near equilibrium is now considered in more detail for constant $\Gamma_1$ models with $\alpha = 0.001$. This value of $\alpha$ is equivalent to a mass of $10^6\ M_\odot$ in the gas plus radiation type of model, and this value of the mass is used to convert the time units to seconds. The fractional binding energy and eigenfrequency of the fundamental mode are given for equilibrium models near the instability point in Figure 5. Equilibrium models with $(PV)_c > 0.5072 \times 10^{-3}$ are unstable to both expansion and collapse. The growth of the kinetic energy away from equilibrium according to the pertubation analysis of Chapter V is

$$T \simeq \tfrac{1}{2}(-\omega_o^2) \int \frac{Y^2(\delta R)^2}{Z^2}\, dM_o \quad . \qquad (6.31$$

Let

$$a_2 = (T/M_o)/[(R - R_o)/R_o]_c^2 = (-\omega_o^2)R_T^2\, A^{-2} \quad , \qquad (6.32)$$

where $A$ is defined by (6.26). The equilibrium model at $(PV)_c = 0.6 \times 10^{-3}$ has $\omega_o^2 R_T^2 = -6.72 \times 10^{-6}$, and thus $a_2 = 3.8 \times 10^{-7}$.

At $(R-R_o)/R_o = + 0.04$, $a_2 = 3.4 \times 10^{-7}$, and at $(R-R_o)/R_o = - 0.04$, $a_2 = 4.6 \times 10^{-7}$. In both cases extrapolation to $R-R_o = 0$ is consistent with the value of $a_2$ from the pertubation analysis. One expects rapid deviation from parabolic behavior of the kinetic energy as a function of $(R-R_o)/R_o$ starting from an equilibrium model near the instability point.

In Figure 6 we present the kinetic energy T as a function of the logarithm of the radius of the mass starting from $(PV)_c = 0.507 \times 10^{-3}$, $(PV)_c = 0.7 \times 10^{-3}$, and $(PV)_c = 0.85 \times 10^{-3}$. The first model is unstable only to collapse, for the second we have plotted just the expansion, and for the third both expansion and collapse. The expansion cannot continue indefinitely because of the positive binding energy of these models, but the precise point at which the expansion stops is not given accurately by our method since our ansatz for the shape of the velocity as a function of r will not remain valid so far from equilibrium. The times given in Figure 6 are rough estimates of those needed to come from near equilibrium to the points indicated.

A stable model may be made to collapse by giving it enough kinetic energy of motion in an inward direction. By our method, an equilibrium model at $(PV)_c = 0.4 \times 10^{-3}$ with $(M_o-M)/M_o = 1.663 \times 10^{-6}$ needs a kinetic energy per unit mass $T/M_o = 2.66 \times 10^{-8}$ to top the barrier preventing collapse. The kinetic energy reaches its minimum value at $(R-R_o)/R_o = - 0.37$. For the same reason as before, these numbers are not expected to be extremely accurate.

In any actual calculation of the calculation of the collapse from equilibrium, the fundamental mode method is useful in starting the numerical solution, but in order to continue the solution beyond the point where the fluid velocities are becoming comparable the speed of sound, to include non-adiabatic effects such as energy generation and shock waves, or to consider other initial conditions than an unstable equilibrium model, it is necessary to go to a difference equation approximation to the full set of hydrodynamic equations. We have not conducted any extensive program of calculation and have not exhaustively tested any particular way of setting up the difference equations. Our feeling has been that this would be worthwhile only in the context of a particular physical model believed to be relevant to some particular astrophysical situation, where general relativity will not necessarily dominate the dynamics.

Non-adiabatic effects such as heat flow by radiative diffusion and neutrino loss will not be considered explicitly in our discussion of the difference equations. The direct effect on the thermodynamics of the matter and on the non-gravitational force of energy transfer by radiation is discussed in Chapter IV, where it is shown that these effects, within factors relating proper times and radii to coordinate times and radii, can be treated much as they are classically. In most cases the gravitational effects of non-adiabatic terms in the energy-momentum tensor will be negligible. Energy generation can be handled trivially by noting that $-\dot{Y}A$ is the function of temperature and density which is the local rate of energy generation per unit invariant mass.

An artificial viscosity, which can be introduced as a contribution $Q$ to $L_1$, allows shock waves to be handled without the difference equations breaking down. Except perhaps in the ultra-relativistic limit as $R \to 0$ a form for $Q$ similar to that of Richtmeyer (39, p. 216) should be satisfactory, although we have made no calculations requiring it. Having $Q$ in $L_1$, but not in $L_2$, is probably most suitable for spherically symmetric collapse (40).

Three possible groupings of the hydrodynamics equations deserve attention as bases of difference equations. Common to them all are

$$M = M_o + \int_0^r [(A+W)Z - 1]\, dM_o \quad , \tag{6.33}$$

$$a'/a = - \frac{P' + (R^2Q)'/R^2}{P + E + Q} \quad , \tag{6.34}$$

$$\dot{W} + P\dot{V} = - VQ \frac{\dot{b}}{b} - (\dot{Y}A)a + \dot{H} \quad , \tag{6.35}$$

where $\dot{H}$ represents the net non-adiabatic transfer of energy into the mass shell as given in (2.15). Of course, an equation of state relating $W$, $P$, and $V$ is necessary. The first group of equations consists of

$$U = a\left[ - \frac{4\pi R^2 Z}{(P+E+Q)V} \frac{\partial P}{\partial M_o} - \frac{4\pi Z \;\; \partial(R^2Q)/\partial M_o}{(P+E+Q)V} \right.$$

$$\left. - M/R^2 - 4\pi(P+Q)R \right] \quad , \tag{6.36}$$

$$\dot{R} = Ua \quad , \tag{6.37}$$

$$V = \frac{1}{Z} \frac{\partial}{\partial M_o} \left[ \frac{4\pi}{3} R^3 \right] \quad , \tag{6.38}$$

$$Z = (1 + U^2 - 2M/R)^{\frac{1}{2}} \quad , \tag{6.39}$$

$$b = V \frac{dM_o}{dr} / (4\pi R^2) \quad . \tag{6.40}$$

The second group consists of (6.36) and (6.39), but

$$\dot{b}/b = U'X/Z \quad , \tag{6.41}$$

$$R = \int_0^r bZ \, dr \quad , \tag{6.42}$$

$$V = 4\pi R^2 b \frac{dr}{dM_o} \quad . \tag{6.43}$$

The third group is more radically different:

$$\dot{b} = a\left[ Z' + \frac{A+W}{R} \frac{dM_o}{dr} - Mb/R^2 \right] \Big/ U \quad , \tag{6.44}$$

$$\dot{Z} = - \frac{P'X + (R^2Q)'X/R^2}{P + E + Q} Ua \quad , \tag{6.45}$$

$$U = - (Z^2 - 1 + 2M/R)^{\frac{1}{2}} \quad , \tag{6.46}$$

plus (6.42) and (6.43) for $R$ and $V$.

Each of these groupings has the property that no two quantities, one of which can be determined in terms of the other at a given time, are calculated independently by time derivatives. This is important for the accuracy and stability of the difference equations.

The first two groupings are suitable for use outside the gravitational radius, $2M/R < 1$, since $Z$ cannot change sign there. The first set of equations corresponds to the usual classical formulation of the Lagrangian difference equations as given by Richtmeyer

(39, p. 200). There are centering problems with the general relativistic factors; in particular we must calculate $U(I+\frac{1}{2}, N+\frac{1}{2})$ and $R(I+\frac{1}{2}, N+1)$ before we can find $V(I, N+1)$ and thus $a(I+\frac{1}{2}, N+1)$, which is needed to center $a$ in (6.37). $I$ denotes the $I$th radial point and $N$ the $N$th time step. These centering problems are not critical as long as $a \simeq 1$. When $a \neq 1$ it may be necessary to reiterate each time step to keep (6.37) centered.

The third set is the one to use when $2M/R > 1$. There $U$ cannot change sign, but shock waves may create steep enough pressure gradients to make $Z < 0$, and when $Z = 0$ some of the equations in the first two sets become singular. The natural centering is to calculate $b$ and the thermodynamics quantities at $(I, N)$ and $Z$ at $(I+\frac{1}{2}, N+\frac{1}{2})$. However, $U$ must be known both at $(I+\frac{1}{2}, N)$ and $(I, N+\frac{1}{2})$.

The one example we present of a numerical difference equation solution is the collapse of a mass of $1.09 \times 10^{8} M_{\odot}$, relative to an explicit gas plus radiation equation of state with $\mu = 1.33$ and $\beta = 2.67 \times 10^{-4}$. The collapse is started from a highly unstable equilibrium model with a central temperature $T_c = 10^{8}$ K. The binding energy is negative, $(M_o - M)/M_o = -3.9 \times 10^{-3}$, and the time constant for the growth of the fundamental mode is $(-\omega_o^2)^{-\frac{1}{2}} = 172$ in natural units with unit mass of $10^{8} M_{\odot}$. However, all other modes of radial oscillation are stable. The time unit is about 490 sec and the distance unit about one astronomical unit. This choice of equilibrium model to start the collapse was made so as to be able to reach the gravitational radius in a reasonable number of steps. The initial equilibrium is necessary to

avoid trouble with shock waves in the collapse, since we did not include an artificial viscosity in our difference equations.

The difference equations are based on the second set of hydrodynamic equations; the close relation of their form of the equation to the form displaying the characteristics (equations (3.16) ) allowed the introduction of stabilizing terms in the difference equations related to approximate integration along the characteristics of the equations (3.16). These stabilizing terms, which tend to replace $U(I+\frac{1}{2}, N-\frac{1}{2})$ by $\frac{1}{2}[U(I-\frac{1}{2}, N-\frac{1}{2}) + U(I+\frac{3}{2}, N-\frac{1}{2})]$ in calculating $U(I+\frac{1}{2}, N+\frac{1}{2})$, are necessary near the center of the mass when a relatively large number of radial points are used.

Our sample integration calculates U at 51 radial points and V at 53 radial points. The radial points are spaced evenly in the initial radius out to $R_o = 30.0$ at 0.981 of the total invariant mass $M_o$, and then at even intervals in $M_o$ to the surface of the mass at $R_o = 44.3$ in order to save computing time. With only about 8 points in V and U to cover one-third of the radius, the dynamics of this part of the mass will only be very sketchily represented by the difference equations, but due to the small fraction of the mass involved errors here should not significantly affect the dynamics of the interior. The time step was chosen to be one-half of the Courant condition (6.01), or if necessary, small enough to keep the fractional change in radius per step less than two per cent.

In Figure 7 we show U and R for every fourth $(I+\frac{1}{2})$ as a function of time in the initial stages of collapse. Some of the

points are labelled at $t = 0$ with the fractional invariant mass inside them. The collapse is started by giving the core only an initial inward velocity. Because only the fundamental mode is unstable the core cannot start collapsing until a rarefaction wave, travelling at the speed of sound, reaches the surface of the mass. By $t = 311.8$ the shape of the velocity as a function of $R_o$ has become that of the fundamental mode, represented by the solid curve, except near the surface, thus tending to confirm the assumptions of the fundamental mode method of this chapter.

Later stages of the collapse are shown in Figure 8 for the central region of the star. At $t \simeq 500$, with the maximum value of $2M/R = 0.2$, the matter inside the position of this maximum value starts collapsing increasingly faster than the outer part of the mass. At $t \simeq 560$, $2M/R$ first becomes greater than one at a fractional invariant mass of 0.15. The velocity is very sharply peaked around the position of the maximum value of $2M/R$. It is likely that this behavior is typical of spherically symmetric gravitational collapse from equilibrium. Near the center $Z$ decreases to the point where the pressure gradient term in (6.36) can no longer keep up with the $M/R^2$ term. However, this "collapsing core" does not separate from the outer part of the mass; the density is increasing everywhere. As is shown in Figure 9, the density just increases faster in the core.

White and May (14) have studied in some detail the collapse of masses inside the gravitational radius. An interesting result of theirs is that a collapse starting not too far outside the Schwarzschild radius from an initial condition far from hydrostatic equilibrium

generates shock waves which do indeed make $Z < 0$ in the vicinity of a finite value of $r$. The specific volume and pressure remain finite. $Z < 0$ then implies $R' < 0$; so, a local minimum develops in the function $R(r)$. The fact that the circumferential radius of a mass shell is less than that of some mass shells inside it does not mean that the matter has interpenetrated. The proper radius $\rho = \int_0^r b\,dr$ is still a monotonically increasing function of $r$. Rather, the space has become bent back upon itself. In fact, an analysis of the asymptotic solution as $R \to 0$ of equations (6.44) - (6.46), assuming $P \propto (1/V)^{4/3}$, shows that

$$b \propto R^{-\frac{1}{2}} ,$$

$$1/V \propto R^{-\frac{3}{2}} ,$$

$$Z \propto R^{\frac{1}{2}} . \qquad (6.47)$$

Since $R' = bZ$ is constant in the limit $R \to 0$ the shape of $R(r)$ is maintained. If it has a local minimum, the mass shell at this minimum will reach $R = 0$ before the mass shells inside it do. The matter inside this mass shell is pinched off. At $R = 0$ both the energy-momentum tensor and the geometry are singular and the solution cannot be continued.

Of course, all of these effects are not accessible to an outside observor since they take place well inside the gravitational radius. Communication is even cut off between neighboring mass shells since in the limit $R \to 0$ $b$ and thus the proper distance between them becomes infinite.

## VII. Conclusion

We have set up the hydrodynamic equations of general relativity in a form, convenient for numerical computation, which allows classical concepts of energy and force to be applied in their interpretation. This Lagrangian, or comoving form of the equations is particularly convenient in studying the role of the internal properties of the matter in determining its motion. We have not tried to investigate in detail the consequences of any particular physical model for the matter, using in our numerical calculations simple mathematical forms for the equation of state and polytropic structure. However, our constant $\Gamma_1$ polytropes are, for some values of $\Gamma_1$, physically reasonable in the whole range of values of PV, in contrast to the models of Tooper (34), and are more easily interpreted in terms of specific physical conditions than those of Gratton (41) and those of Harrison in (42). Our models are not limited to constant entropy (isentropy) or zero temperature, and this accounts for the difference between our analysis of the relation of stability to virtual changes in the total energy and that of Thorne in (42), which depends on equation (3.11) being valid for Y. The analysis of the relation between binding energy and stability of isentropic equilibrium models is independent of the equation of state, although it does depend on the matter being approximately describable as an ideal fluid.

We have tried to study the effects of general relativity, as opposed to peculiarities of the internal properties of the matter. Successively higher modes of radial oscillation tend to become unstable as the central pressure increases, and at the neutral point of the fundamental mode the mass is unstable to collapse, as opposed to expansion. The point at which the instabilities arise and the subsequent behavior of the collapse, such as whether it can be stopped, depend, of course, upon the properties of the matter as much as on general relativity. The study of any except vacuum solutions of the Einstein equations brings in the rest of physics in the energy-momentum tensor. General relativity is not self-contained.

Only in the chapter on radiative transfer do we consider a specific kinetic theory model for a portion of the energy present. However, the main intention is to demonstrate the advantage of formulating kinetic theory in terms of a comoving observor, and to generate specific examples of how the equations arising from the zero covariant divergence of the energy-momentum tensor may be interpreted as governing the local transfer of energy and momentum if $P + E$ is taken as the inertial mass per unit volume.

In a typical dynamical situation with velocities the order of magnitude of the speed of sound the general relativistic terms in the equations will only be important near the gravitational radius, where $2M/R \sim 1$ and $P/E \sim 1$. If the mass is not to collapse immediately inside the gravitational radius, where it can no longer affect the outside universe, the matter must be "stiff" enough to provide a more

or less stable equilibrium with $2M/R \sim 1$. We have seen from our equilibrium models that this is only possible if $\Gamma_1$ is significantly greater than four-thirds when $P/E$ is not too much less than one. This is the case for nuclear matter, as found in a neutron star or a supernovae core. Thus, one area of application of the general relativistic hydrodynamic equations is to the calculation of the "bounce" of supernovae cores (6) or to the construction of neutron star models. However, a serious difficulty is that the equation of state of the matter is not at all well-known for densities higher than nuclear densities.

A situation where the equation of state of the matter is much better known is in large masses supported by gas and radiation pressure. Here general relativity can be important even when $2M/R \ll 1$, if the mass is very near hydrostatic equilibrium. Since $\frac{3}{4}\Gamma_1 - 1 \ll 1$, small general relativistic terms can seriously affect the stability of the almost neutral classical equilibrium, as was pointed out by Feynman (43). This extended importance of general relativity near equilibrium is why we have done most of our numerical work there. Of course, when $2M/R \ll 1$ the post-Newtonian approximation can be used. Thus, our numerical results for equilibrium models can be used to justify and to show the limitations on the use of this approximation. The implications of these results for equilibrium models of quasars have been extensively discussed in the papers of Fowler (5,38,44). Another astronomical application of the stability analysis when $2M/R \ll 1$ is the derivation of

a minimum radius for white dwarf stars much greater than the gravitational radius, since as the degenerate electrons become relativistic, $\Gamma_1 \rightarrow 4/3$ (45).

The assumption of complete spherical symmetry is not a realistic one in that any actual astronomical object is expected to have some rotation. If $\frac{3}{4}\Gamma_1 - 1 \ll 1$ a very small amount of rotation can have an important effect on the stability of the mass and significantly modify the conclusions as to the central temperature at the instability point for a given size mass. Without rotation, our discussion of Chapter V and that of Fowler (5,44) shows that the largest mass which is dynamically stable when the central temperature is high enough for a significant rate of nuclear energy generation is about $10^5 M_\odot$. Only nuclear energy seems able to give a steady supply of energy over the lifetime of a quasar, $10^3 - 10^6$ years, in a model which can explain the apparent regular fluctuations in brightness noted by Smith and Hoffleit (46). We may analyze the effect of a small rotation by considering the rotation and general relativity as small independent first order pertubations on a spherically symmetric classical polytrope. The effect of the rotation on the frequency of the fundamental mode is treated by Chandrasekhar and Lebovitz (47). If $\alpha = \frac{3}{4}\Gamma_1 - 1$, the unperturbed term in the expansion of $\omega_o^2/4\pi\rho_c$, $\rho_c$ the central density, is proportional to $\alpha$ and independent of the radius or central temperature. We are interested in the case $\alpha \ll 1$; then the first order general relativistic term, proportional to $1/R$, and the first order rotational term, also proportional to $1/R$ if

angular momentum is conserved, can be larger than the zeroth order term while only perturbing the structure of the mass slightly. The general relativistic term is negative, and the rotational term is positive. If the rotational term is larger initially in a sequence of models with the same polytropic index it will remain larger, and the mass will be stable until the first order pertubation analysis breaks down. The ultimate limit on the amount of rotation is that the centrifugal acceleration be less than the gravitational acceleration at the surface of the mass. For uniform rotation of a $n = 3$ polytrope there is only a very slight increase in the size of the mass that can be kept stable until nuclear burning temperatures are reached. However, Fowler (48) has found that with non-uniform rotation the mass limit may be raised from a few times $10^5 M_\odot$ to $10^8 M_\odot$, which is the right order of magnitude for a model of a quasar. To treat the rotation even in the post-Newtonian approximation is a complex mathematical problem, although Chandrasekhar (49) has formulated the basic equations. The exact solution of the equations of general relativity in the presence of rotation, which is necessary to understand what goes on near the gravitational radius, is much more difficult and little progress has been made thus far.

The extended extra-galactic radio sources (4) seem to involve the explosive release of very large amounts of energy, and it has been proposed (3,5,50,51,52,53) as one explanation that gravitational collapse to near the gravitational radius may release a sizeable fraction of the gravitational potential energy, which is there the same order

of magnitude as the total rest mass energy. Mechanisms departing in an essential way from spherical symmetry, such as gravitational radiation, are outside the scope of this paper and will not be discussed here, nor will more exotic general relativistic models such as ones with singular initial conditions or negative energy fields. However, we would like to analyze critically the possibility of releasing energy in spherically symmetric collapse.

In order to obtain the relativistic electrons of the radio source, there must be ejection of particles and radiation from the mass or reversal of the collapse of some part of the mass. If $2M/R > 1$, it is clear from (6.44) that there can be no reversal of the overall hydrodynamic collapse. Since the forward light cone lies completely in the direction of decreasing $R$, no radiation can escape. As $R$ decreases to the gravitational radius, the red shift of radiation escaping to a distant observer becomes greater and greater, decreasing the amount of energy released. Furthermore, radiation emitted near the center of the mass must pass each mass shell before $2M/R$ becomes greater than one at the mass shell. For large masses, where high temperatures are attained only near the gravitational radius, this time delay cuts the amount of neutrino emission drastically (54), even assuming the neutrino energy can be converted into relativistic electrons. Light, which must diffuse outward through the mass, is just swept in with the matter. Michel (55) has proposed a mechanism for releasing the internal energy of the envelope of a mass whose core is collapsing. He suggests that if a large fraction of the mass of the core can be emitted as

neutrinos, the resultant reduction of the gravitational force on the envelope will allow it to explode. The argument fails for two reasons. First, due to the red shift and the short time scale of the collapse in comoving coordinates, the fractional change in the mass of the core will be very small. Second, the effect of any change in the mass of the core is blanketed by the amount of matter reached by the rarefaction wave proceeding outward in the envelope before the neutrinos pass by. One cannot expect a really sharp separation between the core and the envelope.

Release of a sizeable amount of energy in the collapse is only possible if the collapse of the core, at least, can be stopped. In masses of greater than $10^6 M_\odot$ the temperatures and densities are too low for generation of nuclear energy to do the job, and there is no degenerate pressure or hard core as in the case of collapsing supernovae cores.

It is interesting to consider from our point of view the reversibility of the collapse once $2M/R$ has become greater than one at a mass shell. A large repulsive non-gravitational force just bends the space around (makes $Z < 0$) so the mass shell is accelerated <u>toward</u> $R = 0$, as is clear from equations (2.09) and (2.11). The boundary conditions at the origin keep $Z > 0$ for small $R$ as long as the geometry remains non-singular. From equation (2.06) the only way $U$ can go through zero at a mass shell is if $2M/R$ becomes less than one again. Since $R$ is decreasing, a way must be found to decrease $M$. It might be thought that an outward flux of radiation would do this.

However, a radial flux of neutrinos has $L_1 = K$ and from equation (2.08), when $2M/R > 1$ $(-U > Z)$ M actually increases.

To have a chance of stopping the collapse we must turn to a repulsive gravitational force, such as that caused by a negative stress, $L_1 < 0$, or a negative total energy M. The basic requirement is $L_1 < 0$ since $L_1 < 0$, besides being a repulsive force in (2.11), decreases M in (2.08). A radial electrostatic field has $L_1 < 0$, and in fact if one solves the Einstein equations for the collapse of a charged dust $(P = 0)$ one finds, following a particular mass shell with fixed charge and rest mass inside it, that the collapse is indeed reversed. The non-gravitational electrostatic force is strong enough to make $Z < 0$, thus turning it into an effective attractive force, but the repulsive force from $L_1$ dominates. The problem is that a mass shell cannot reexpand without crossing (in r) other mass shells. This crossing is associated with the density becoming infinite in a continuum model for the matter, and thus by any reasonable equation of state with the pressure becoming infinite. The infinite pressure is a positive radial stress which cancels the negative stress of the electrostatic field, thus allowing the collapse to continue until the metric becomes singular at $R = 0$. Allowing the matter to interpenetrate means the charge and rest mass inside a mass shell is no longer constant. If the collapse could be reversed in this way, causality would be violated. A negative energy field with negative stress, such as the C-field or Hoyle and Narkikar (55), can stop the collapse without these difficulties, but is not consistent with what we know of the rest of physics.

Our conclusion is that within the context of ordinary positive energy density classical physics, it is impossible to stop the collapse short of the singularity that develops when the density and pressure become infinite at $R = 0$, once $2M/R$ has become greater than one. This has been proved in greater generality by Penrose (57). It is possible that a quantum theory of gravity may in some way avoid this singularity, as discussed by Wheeler (58), but so far this is just speculation.

If indeed the extended extra-galactic radio sources are the result of gravitational collapse near the gravitational radius, which may not be the case at all, the collapse must involve rotation in an essential way. The question of whether some of the simplicities of interpretation of the spherically symmetric Einstein equations can be extended to rotating masses is still an open one. Unfortunately, the key to the spherically symmetric equations is the identification of the spatial location of a mass point by a coordinate $R$ with invariant geometrical significance. The lack of such coordinates in the rotational problem plus the many new degrees of freedom opened up makes progress likely to be very slow in understanding any process of energy release in which general relativity is important.

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# Appendix I

The Riemann-Christoffel symbols are evaluated for the metric (1.01),

$$\Gamma^{i}_{jk} = \tfrac{1}{2}\left[\frac{\partial g_{ij}}{\partial X^{k}} + \frac{\partial g_{ik}}{\partial X^{j}} - \frac{\partial g_{jk}}{\partial X^{i}}\right] .$$

$$\Gamma^{4}_{44} = \frac{\dot{a}}{a} , \qquad \Gamma^{4}_{14} = \frac{a'}{a} , \qquad \Gamma^{1}_{44} = aa'X^{2} ,$$

$$\Gamma^{1}_{14} = \frac{\dot{b}}{b} , \qquad \Gamma^{4}_{11} = b\dot{b}Y^{2} , \qquad \Gamma^{1}_{11} = \frac{b'}{b} ,$$

$$\Gamma^{2}_{24} = \Gamma^{3}_{34} = \frac{\dot{R}}{R} , \qquad \Gamma^{2}_{21} = \Gamma^{3}_{31} = \frac{R'}{R} ,$$

$$\Gamma^{4}_{22} = R\dot{R}Y^{2} , \qquad \Gamma^{4}_{33} = R\dot{R}Y^{2}\sin^{2}\theta ,$$

$$\Gamma^{1}_{22} = - RR'X^{2} , \qquad \Gamma^{1}_{33} = - RR'X^{2}\sin^{2}\theta ,$$

$$\Gamma^{3}_{32} = \cot\theta , \qquad \Gamma^{2}_{33} = - \sin\theta\cos\theta .$$

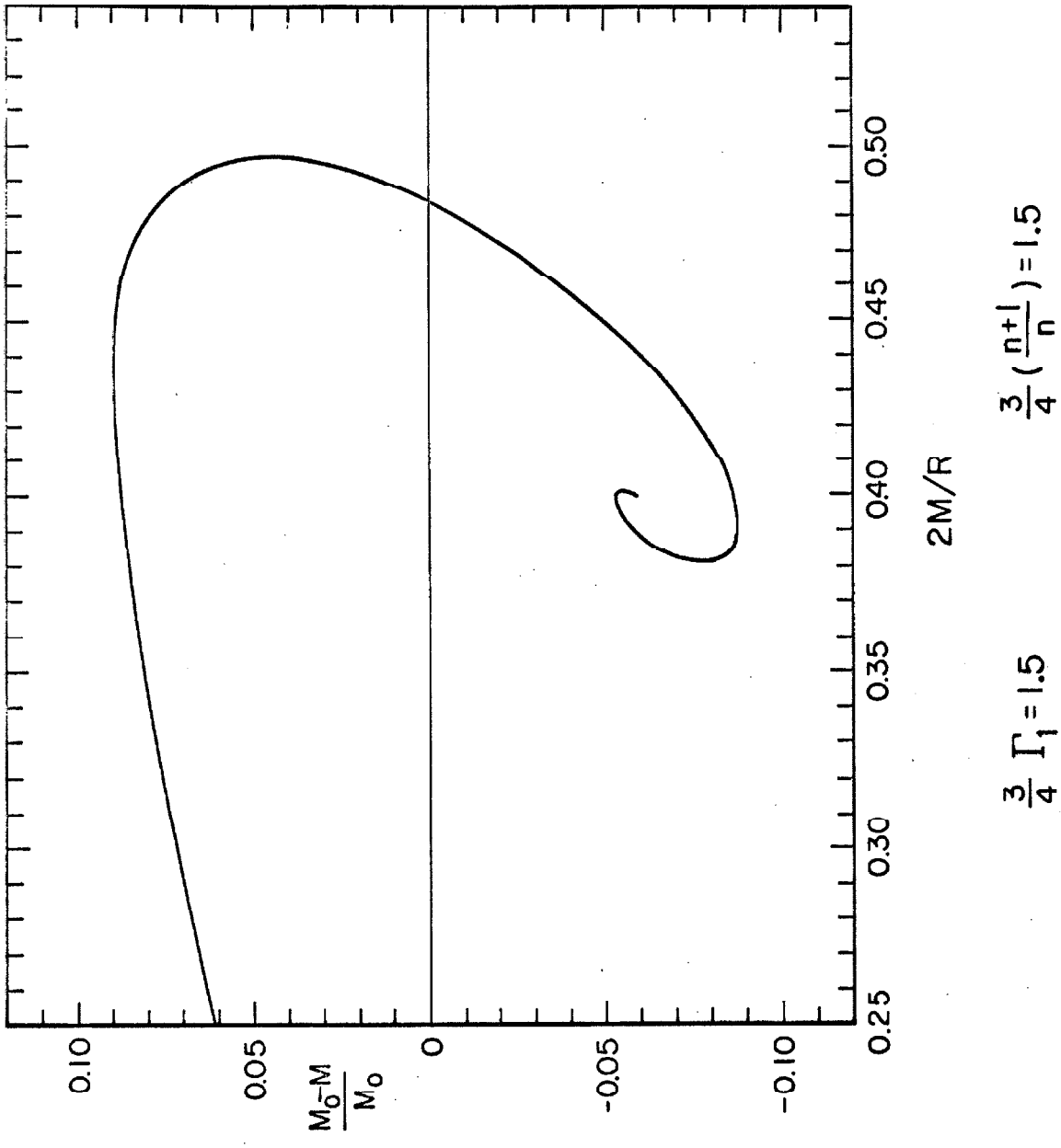

0.10
0.05
0
-0.05
-0.10
$\frac{M_0-M}{M_0}$
0.25
0.30
0.35
0.40
0.45
0.50
2M/R
$\frac{3}{4}\Gamma_1=1.5$
$\frac{3}{4}(\frac{n+1}{n})=1.5$

FIGURE 1

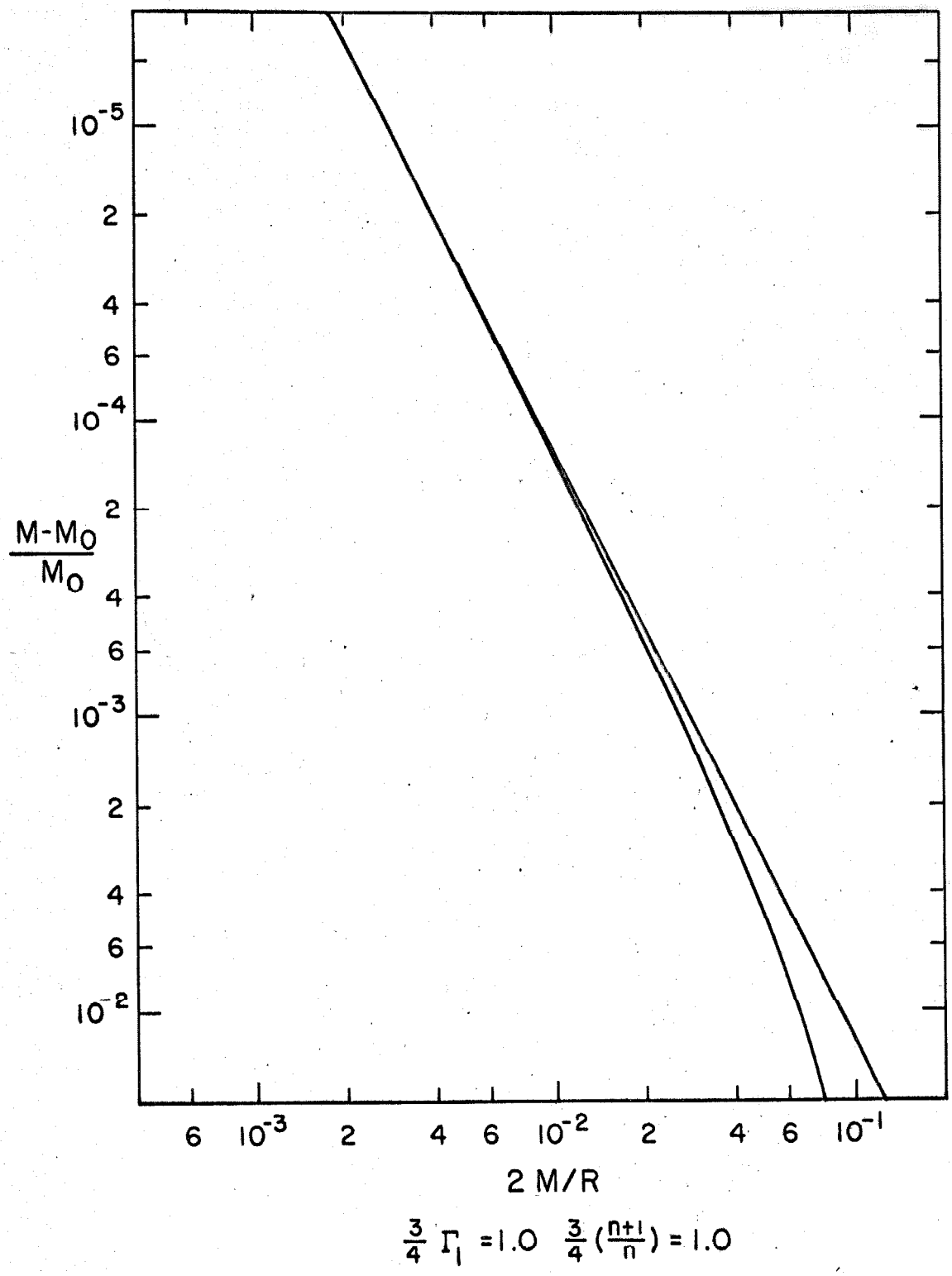

$10^{-5}$
2
4
6
$10^{-4}$
2
4
6
$10^{-3}$
2
4
6
$10^{-2}$
$\frac{M-M_0}{M_0}$
6
$10^{-3}$
2
4
6
$10^{-2}$
2
4
6
$10^{-1}$
2 M/R
$\frac{3}{4}\Gamma_1 = 1.0$
$\frac{3}{4}\left(\frac{n+1}{n}\right) = 1.0$

FIGURE 2

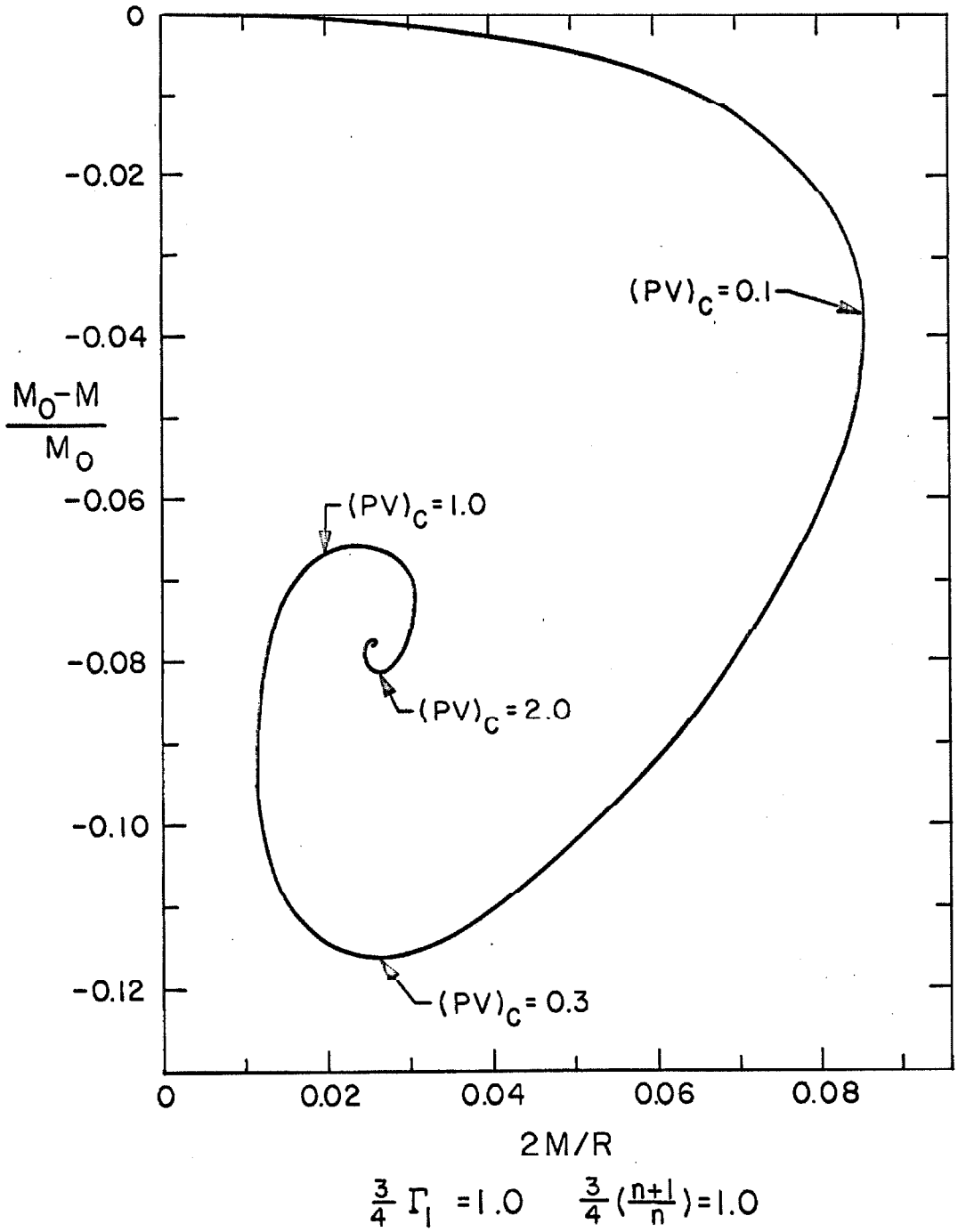

0
-0.02
-0.04
-0.06
-0.08
-0.10
-0.12
$\frac{M_0-M}{M_0}$
$(PV)_c = 0.1$
$(PV)_c = 1.0$
$(PV)_c = 2.0$
$(PV)_c = 0.3$
0
0.02
0.04
0.06
0.08
2M/R
$\frac{3}{4}\Gamma_1 = 1.0$
$\frac{3}{4}(\frac{n+1}{n}) = 1.0$

FIGURE 3

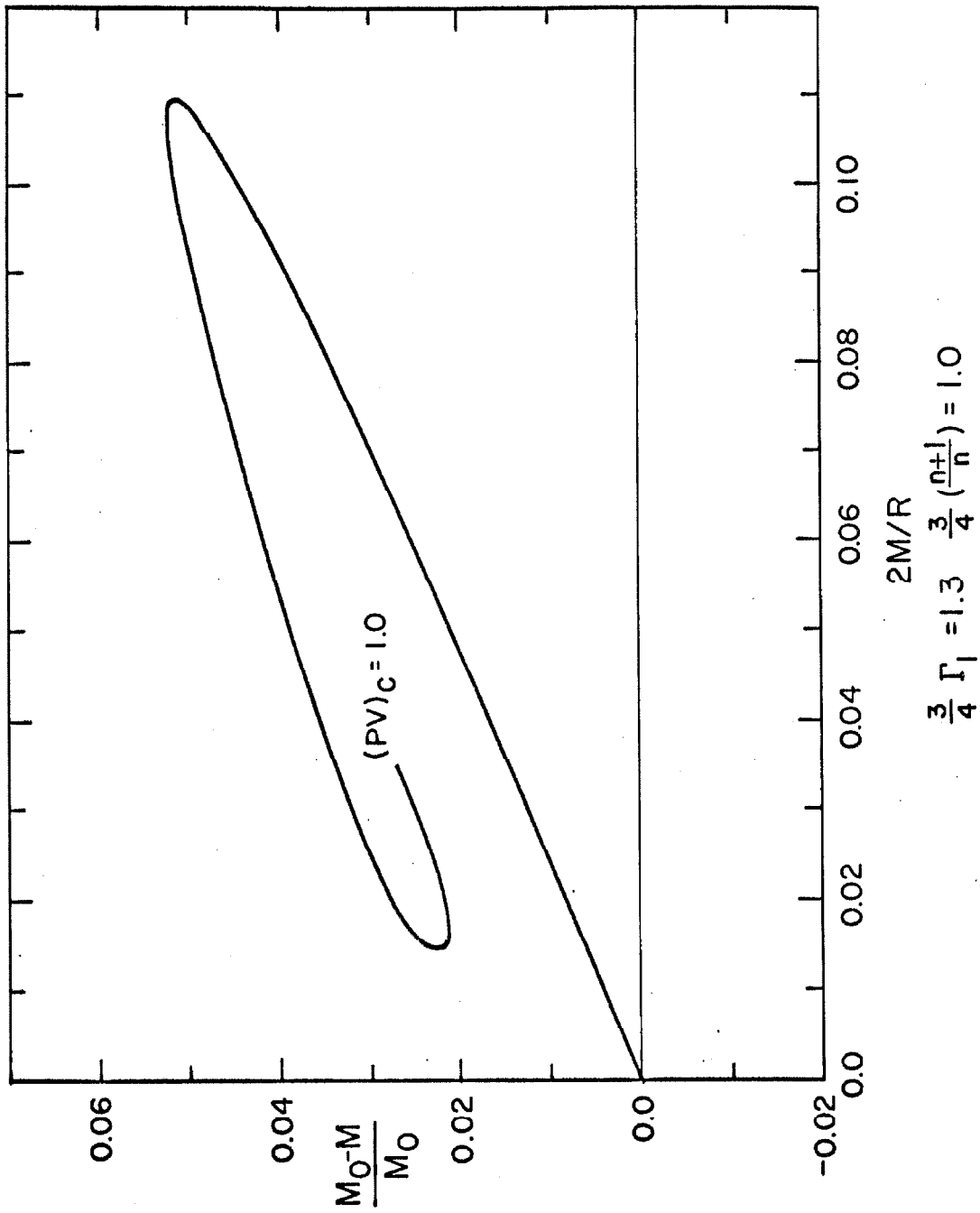

0.06
0.04
0.02
0.0
-0.02
$\frac{M_0-M}{M_0}$
0.0
0.02
0.04
0.06
0.08
0.10
2M/R
$(PV)_c = 1.0$
$\frac{3}{4}\Gamma_1 = 1.3 \quad \frac{3}{4}\left(\frac{n+1}{n}\right) = 1.0$

FIGURE 4

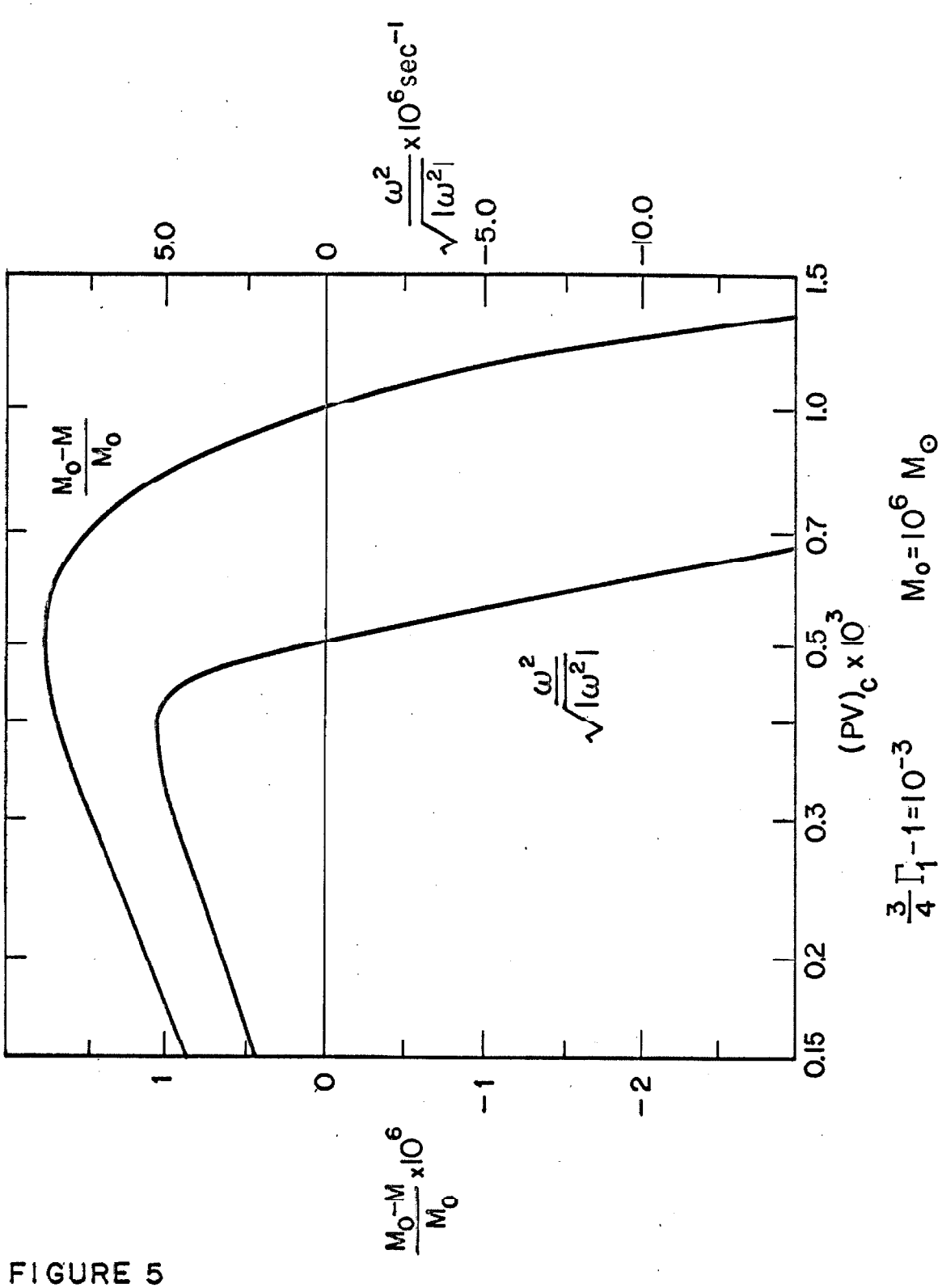

$\frac{M_0-M}{M_0}$
$\frac{\omega^2}{\sqrt{|\omega^2|}}$
$\frac{M_0-M}{M_0}\times10^6$
$\frac{\omega^2}{\sqrt{|\omega^2|}}\times10^6\,sec^{-1}$
5.0
0
-5.0
-10.0
1
0
-1
-2
0.15
0.2
0.3
0.5
0.7
1.0
1.5
$(PV)_c \times 10^3$
$\frac{3}{4}\Gamma_1-1=10^{-3}$
$M_0=10^6\ M_\odot$

FIGURE 5

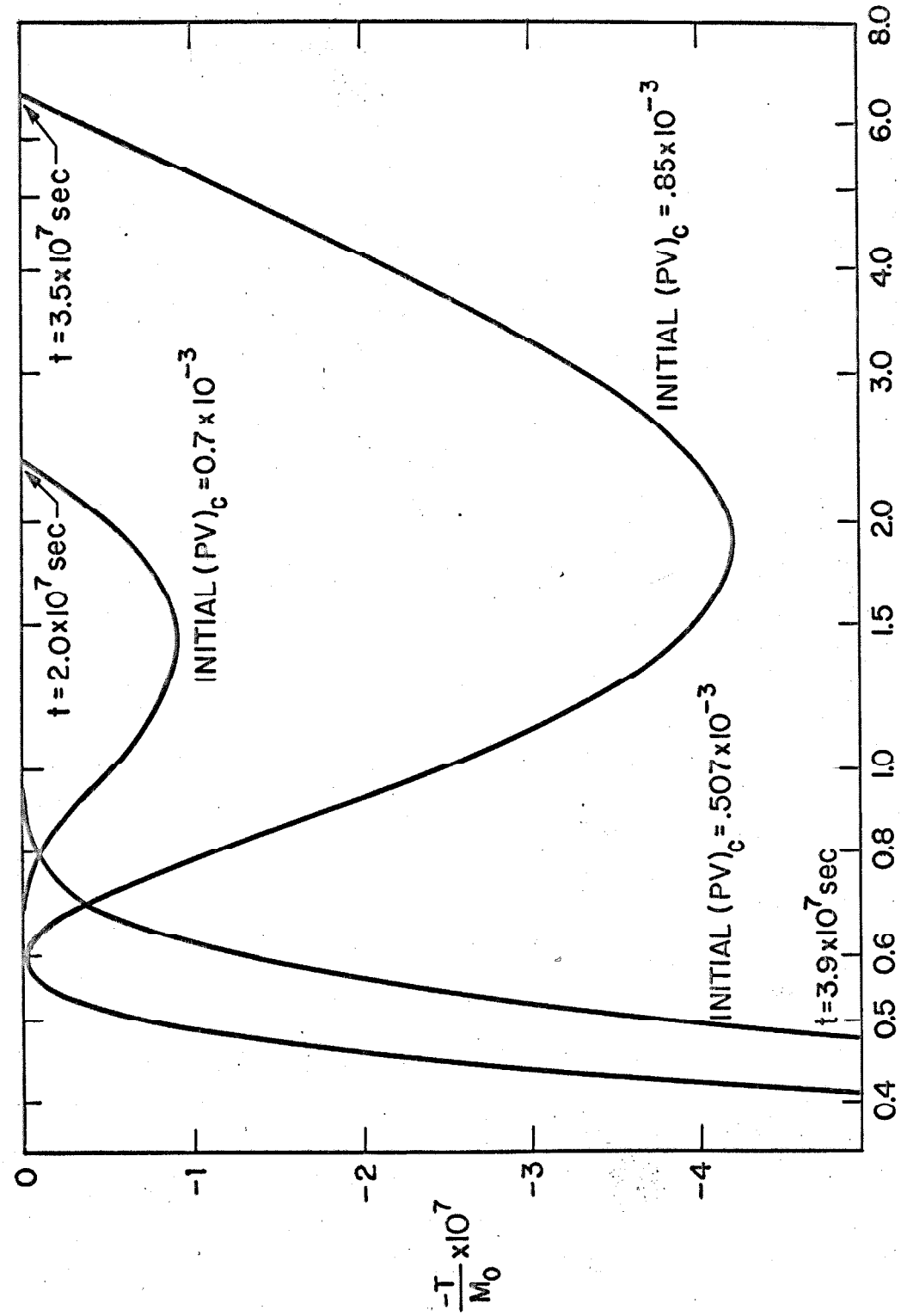

t = 2.0x10^7 sec
t = 3.5x10^7 sec
INITIAL (PV)c = 0.7 x 10^-3
INITIAL (PV)c = .85x10^-3
INITIAL (PV)c = .507x10^-3
t=3.9x10^7sec
0
-1
-2
-3
-4
0.4
0.5
0.6
0.8
1.0
1.5
2.0
3.0
4.0
6.0
8.0
-T/M0 x10^7
RATIO OF RADIUS TO RADIUS OF NEUTRAL EQUILIBRIUM MODEL

FIGURE 6

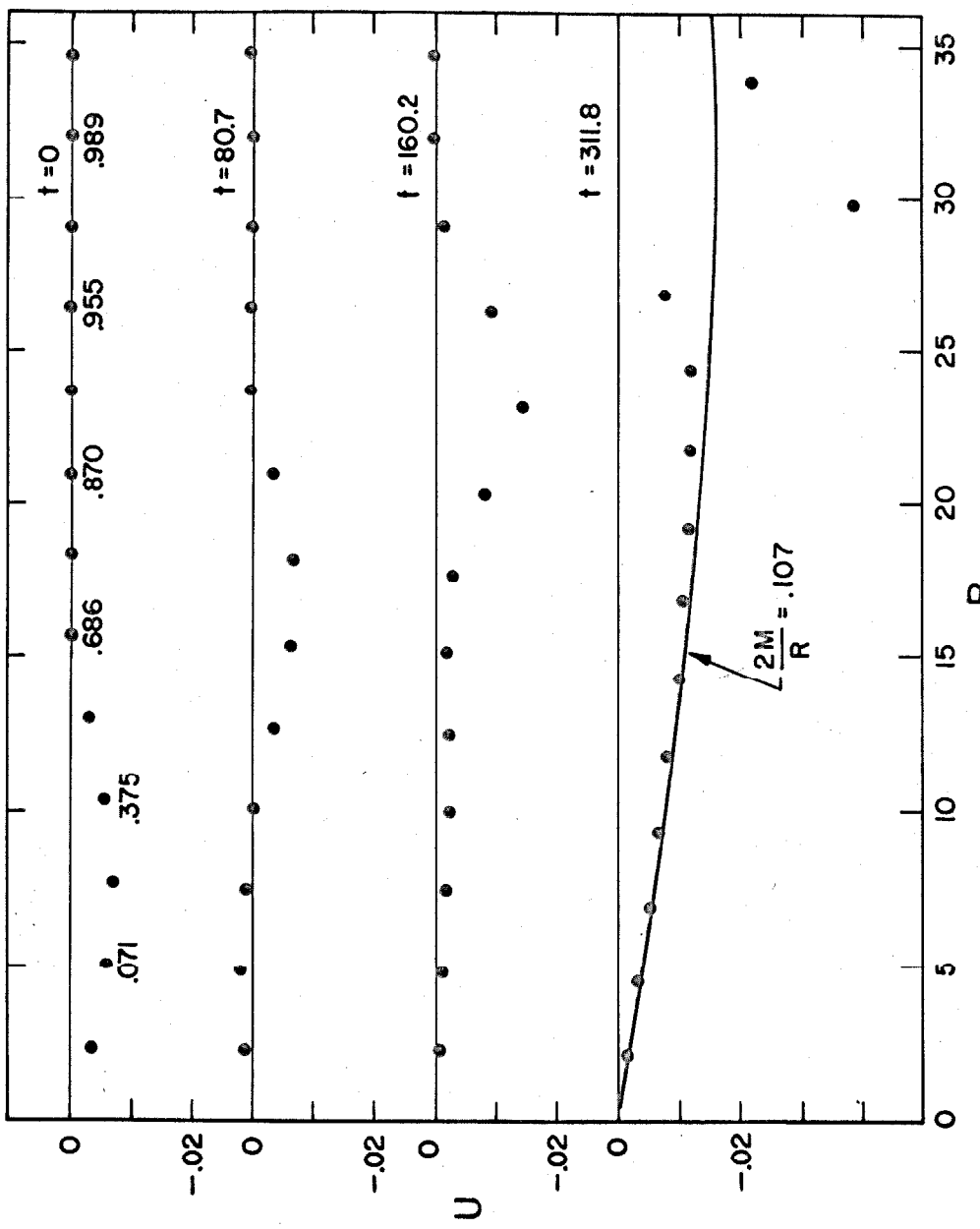
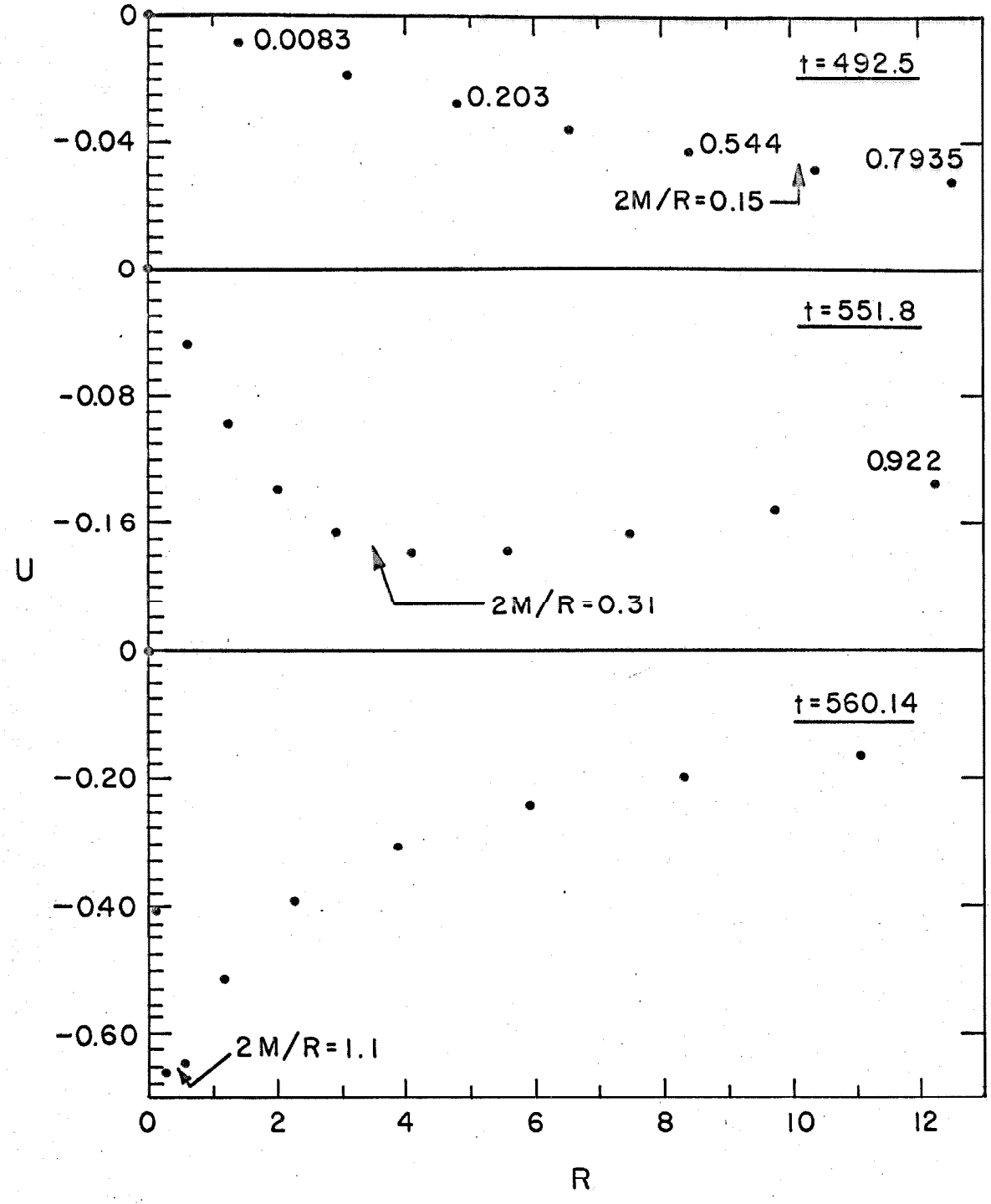
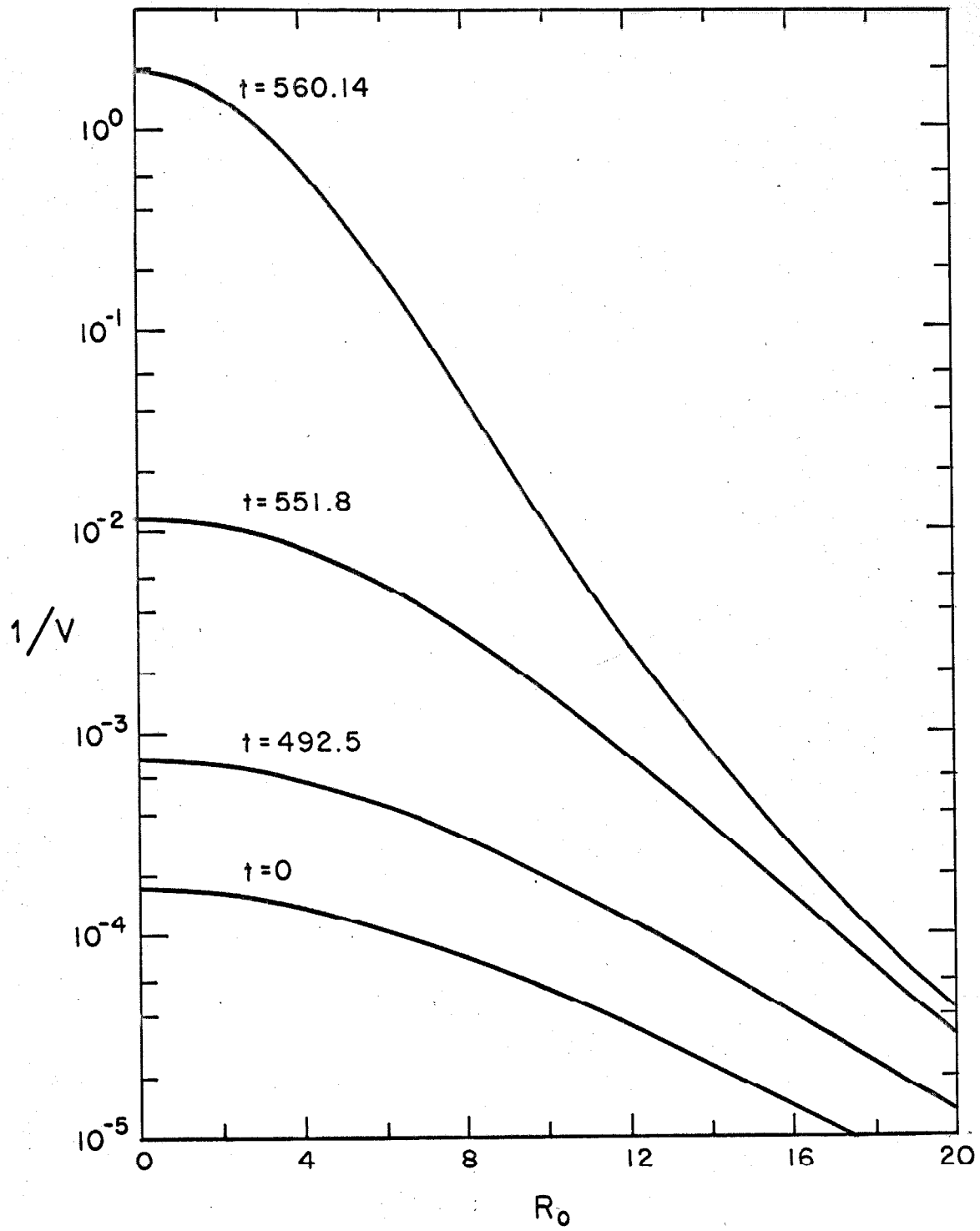

t=0
.989
.955
.870
.686
.375
.071
t=80.7
t=160.2
t=311.8
2M/R = .107
U
R
0
-.02
5
10
15
20
25
30
35
COLLAPSE OF LARGE MASS
μ=1.33 β=2.67x10⁻⁴ M₀=1.09x10⁸ M⊙ (T_c)₀=10⁸°K

FIGURE 7




$M_0 = 1.09 \times 10^8\ M_\odot$ $\qquad$ $(T_c)_0 = 10^8\ ^\circ K$

FIGURE 8




$M_o = 1.09 \times 10^8\ M_\odot$ . $(T_c)_o = 10^8\ ^\circ K$

FIGURE 9